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The Dirichlet Problem for Complex Monge-Ampère Equations and Regularity of the Pluri-Complex Green Function

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1. Introduction.

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^∞ boundary $\partial\Omega$. In this paper we are concerned with the Dirichlet problem for complex Monge-Ampère equations

$$(1.1) \qquad \det(u_{z_j\bar{z}_k}) = \psi(z, u, \nabla u) \quad \text{in} \quad \Omega, \quad u = \varphi \quad \text{on} \quad \partial\Omega$$

and related questions.

When Ω is a strongly pseudoconvex domain, this problem has received extensive study. In [4]-[6], E. Bedford and B. A. Taylor established the existence, uniqueness and global Lipschitz regularity of generalized pluri-subharmonic solutions. S.-Y. Cheng and S.-T. Yau [8], in their work on complete Kähler-Einstein metrics on non-compact complex manifolds, solved (1.1) for $\psi = e^u$ and $\varphi = +\infty$, obtaining a solution in $C^\infty(\Omega)$. In 1985, L. Caffarelli, J. J. Kohn, L. Nirenberg and J. Spruck [7] proved the existence of classical pluri-subharmonic solutions of (1.1) for the non-degenerate case $\psi > 0$, under suitable conditions on ψ. The degenerate case $\psi \geq 0$ has also attracted a lot of attention, and counterexamples have been found showing that there need not be a C^2 solution (see [3], [11]). It is of interest in complex analysis to ask whether $C^{1,1}$ regularity holds for the degenerate case; see [1] for related results and further references. In [20], S.-Y. Li studied the Neumann problem for complex Monge-Ampère equations.

In this paper we treat the Dirichlet problem (1.1) for general domains which are not necessarily pseudoconvex. We shall prove

Theorem 1.1. *Let φ, ψ be real-valued smooth functions, $\psi > 0$. Suppose there exists a strictly pluri-subharmonic subsolution $\underline{u} \in C^2(\overline{\Omega})$ of (1.1), that* *is,*

$$\det(\underline{u}_{z_j\bar{z}_k}) \geq \psi(z, \underline{u}, \nabla \underline{u}) \quad in \quad \Omega, \qquad u = \varphi \quad on \quad \partial\Omega. \tag{1.2}$$

Then there exists a strictly pluri-subharmonic solution $u \in C^\infty(\overline{\Omega})$ of (1.1) with $u \geq \underline{u}$.

In [13] and [12] J. Spruck and the author treated the Dirichlet problem for the real Monge-Ampère equations in non-convex domains. As in the real case, the point here is that no restrictions (other than being bounded and smooth) to the underlying domain Ω are needed. As complex Monge-Ampère equations are closely related to certain problems in geometry and complex analysis, it seems reasonable to expect such a result to find interesting applications. In this paper we will apply Theorem 1.1 to prove the $C^{1,\alpha}$ regularity of the pluri-complex Green function for a strongly pseudoconvex domain. We recall that, given a domain $\Omega \subset \mathbb{C}^n$ and a point $\zeta \in \Omega$, the function

$$\begin{aligned} g_\zeta(z) = \sup\{v(z) : \quad & v \text{ is pluri-subharmonic on } \Omega, \quad v < 0 \\ & \text{and } v(z) \leq \log|z - \zeta| + O(1)\} \end{aligned}$$

is called the pluri-complex Green function on Ω with logarithmic pole at ζ (see [9], [15] and [18]). In the case that Ω is smooth, bounded and strictly convex, Lempert [18] has shown that $g_\zeta \in C^\infty(\overline{\Omega} - \{\zeta\})$. In the strongly pseudoconvex case, however, E. Bedford and J.-P. Demailly [2] have found counterexamples which show that g_ζ in general does not belong to $C^2(\overline{\Omega} - \{\zeta\})$. We will prove

Theorem 1.2. *Let Ω be a smooth bounded strongly pseudoconvex domain and $\zeta \in \Omega$. Then $g_\zeta \in C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$ for any $0 < \alpha < 1$.*

In [23], S. Semmes developed a theory of generalized Riemann mappings that is closely related with the pluri-complex Green functions (see Theorem 2.2 of [23]). Using the work of Lempert [18], he proved the existence of smooth Riemman mappings with given smooth strictly convex images in $\mathbb{C}^n$. Theorem 1.2 has the following consequence: if $\rho : B_n \to \mathbb{C}^n$ is a Riemann mapping whose image is a smooth strongly pseudoconvex domain in $\mathbb{C}^n$, where B_n denotes the unit ball in $\mathbb{C}^n$, then ρ is $C^{1,\alpha}$ in $\overline{B}_n - \{0\}$ for any $0 < \alpha < 1$.

A fundamental property of the pluri-complex Green function is that it is a week solution of the following problem

$$\begin{cases} u \text{ is pluri-subharmonic} & \text{in } \Omega - \{\zeta\} \\ \det(u_{z_j \bar z_k}) = 0 & \text{in } \Omega - \{\zeta\} \\ u = 0 & \text{on } \partial\Omega \\ u(z) = \log|z-\zeta| + O(1) & \text{as } z \to \zeta. \end{cases} \tag{1.3}$$

We will prove Theorem 1.2 by showing that the above problem has a unique solution in $C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$ if Ω is a smooth bounded strongly pseudoconvex domain. The proof, which is contained in Section 4, involves deriving interior estimates for the Laplacian of solutions to the approximate nondegenerate equations. We formulate the estimates in the following more general form due to its own interest.

Theorem 1.3. *Let $u \in C^4(\Omega) \cap C^1(\overline{\Omega})$ be a strictly pluri-subharmonic solution of (1.1). Assume that there exists a strictly pluri-subharmonic function $v \in C^2(\overline{\Omega})$ with $v = \varphi$ on $\partial\Omega$. Then*

$$|u_{z_j \bar z_k}(z)| \le \frac{C}{(\operatorname{dist}(z, \partial\Omega))^N}, \quad \text{for } z \in \Omega, \tag{1.4}$$

where C and N are constants depending on n, Ω, $\|u\|_{C^1(\overline{\Omega})}$, $\|v\|_{C^2(\overline{\Omega})}$, ψ up to its second derivatives, and a lower bound $\psi_0 > 0$ of $\psi(x, u, \nabla u)$, which in turn depends on $\|u\|_{C^1(\overline{\Omega})}$.

This may be regarded as an analogue of Pogorelov's C^2 interior estimates for the corresponding real Monge-Ampère equations (see [21]). For $\psi = \psi(z,u)$ and $\varphi = 0$ (in this case, Ω must be strongly pseudoconvex, and one may take $v \equiv 0$, though it is not strictly pluri-subharmonic), this result was proved by F. Schulz [22] whose proof uses the integral method approach of N. M. Ivochkina [14] to the real Monge-Ampère equations. S.-Y. Cheng and S.-T. Yau [8] also obtained similar estimates which in addition depend on $\sup_\Omega \sum u^{jk} u_{z_j} u_{\bar z_k}$. Our proof, which is presented in Section 3, is an extension to the complex case of Pogorelov's original argument. We also should point out that, as we will see in Section 4, Theorem 1.3 can not be directly used in the proof of Theorem 1.2 as our problem (1.3) is degenerate.

In Section 2 we will establish *a priori* bounds on the boundary of Ω for the second derivatives of strictly pluri-subharmonic solutions of (1.1), thereby proving Theorem 1.1. Section 3 contains the proof of Theorem 1.3. In Section 4 we prove the solvability of (1.3) in $C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$.

Notation. Let $z_1, \dots, z_n$ be complex coordinates in $\mathbb{C}^n$, $z_j = x_j + iy_j$ and $z = (z_1, \dots, z_n)$. If u is a C^2 function on an open set of $\mathbb{C}^n$ we use the notation

$$u_j = u_{z_j} = \partial_j u, \quad u_{\bar{j}} = u_{\bar{z}_j} = \partial_{\bar{j}} u, \quad u_{j\bar{k}} = u_{z_j \bar{z}_k} = \partial_j \partial_{\bar{k}} u, \text{ etc,}$$

and

$$\nabla u = (u_{z_1}, u_{\bar{z}_1}, \dots, u_{z_n}, u_{\bar{z}_n}),$$

where

$$\partial_j = \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \partial_{\bar{j}} = \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

A notable property of the complex Monge-Ampère operator is that under a holomorphic change of variable $z \mapsto w$ we have

$$\det(u_{z_j \bar{z}_k}) = |\det(w_z)|^2 \det(u_{w_j \bar{w}_k}).$$

We also recall that a real-valued function $u \in C^2(\Omega)$ is *strictly pluri-subharmonic* if the complex Hessian matrix $\{u_{z_j \bar{z}_k}\}$ is positive definite in Ω. We denote by $\{u^{j\bar{k}}\}$ the inverse matrix of $\{u_{z_j \bar{z}_k}\}$ when it is invertible.

2. Boundary estimates for second derivatives.

It is now well known, through the work [10], [16] and [7], that the solvability of the Dirichlet problem (1.1) depends upon the establishment of global *a priori* estimates, up to the second derivatives, for prospective solutions. Let $u \in C^4(\overline{\Omega})$ be a strictly pluri-subharmonic solution of (1.1) with $u \geq \underline{u}$. Our goal of this section is to derive a bound

$$\|u\|_{C^2(\overline{\Omega})} \leq C. \tag{2.1}$$

Theorem 1.1 then may be proved by the method of continuity and degree theory as in the real case in [12].

From [7], we have the global C^1 estimate

$$|u| + |\nabla u| \leq K \quad \text{in} \quad \Omega. \tag{2.2}$$

It follows that $\psi(x, u, \nabla u)$ is bounded below from zero. We set

$$\psi_0 \equiv \min_{|z|+|p| \leq K,\, x \in \overline{\Omega}} \psi(x, z, p) > 0, \quad \psi_1 \equiv \max_{|z|+|p| \leq K,\, x \in \overline{\Omega}} \psi(x, z, p).$$

It is also shown in [7] how to derive global bounds on $\overline{\Omega}$ for the second derivatives from (2.2) and *a priori* estimates on the boundary

$$\max_{\partial\Omega} |\nabla^2 u| \leq C. \tag{2.3}$$

The rest of this section is devoted to deriving (2.3).

At any point $0 \in \partial\Omega$, we may choose coordinates $z_1, \dots, z_n$ with origin at 0 and such that the positive x_n axis is the interior normal direction to $\partial\Omega$ at 0. For convenience we set $t_1 = x_1, t_2 = y_1, \dots, t_{2n-3} = x_{n-1}, t_{2n-2} = y_{n-1}, t_{2n-1} = y_n, t_{2n} = x_n$, and $t' = (t_1, \dots, t_{2n-1})$. Near 0, we may represent $\partial\Omega$ as a graph

$$x_n = \rho(t') = \frac{1}{2} \sum_{\alpha,\beta<2n} B_{\alpha\beta} t_\alpha t_\beta + O(|t'|^3). \tag{2.4}$$

Since $(u - \underline{u})(t', \rho(t')) = 0$, we have

$$(u - \underline{u})_{t_\alpha t_\beta}(0) = -(u - \underline{u})_{x_n}(0) B_{\alpha\beta}, \quad \alpha, \beta < 2n. \tag{2.5}$$

It follows that

$$|u_{t_\alpha t_\beta}(0)| \leq C, \quad \alpha, \beta < 2n. \tag{2.6}$$

Next we proceed to estimate $u_{t_\alpha x_n}(0)$ for $\alpha \leq 2n$. Consider the linearized operator $L = u^{j\bar{k}} \partial_j \partial_{\bar{k}}$ where $\{u^{j\bar{k}}\}$ is the inverse matrix of $\{u_{j\bar{k}}\}$. For any first order differential operator D of constant coefficients, we have

$$L(Du) = D(\log \psi(z, u(z), \nabla u(z))).$$

Set $\mathcal{L} = L - f_{p_j} \partial_j - f_{\bar{p}_j} \partial_{\bar{j}}$ where $f \equiv \log \psi$. We will employ a barrier function of the form

$$v = (u - \underline{u}) + t(h - \underline{u}) - N d^2, \tag{2.7}$$

where h is the harmonic function in Ω with $h|_{\partial\Omega} = \varphi$, d is the distance function from $\partial\Omega$, and t, N are positive constants to be determined. We may take $\delta > 0$ small enough so that d is smooth in $\Omega_\delta = \Omega \cap B_\delta(0)$. The key ingredient is the following

Lemma 2.1. *For N sufficiently large and t, δ sufficiently small,*

$$\mathcal{L} v \leq -\frac{\epsilon}{4} \left(1 + \sum u^{k\bar{k}}\right) \quad \text{in } \Omega_\delta, \quad v \geq 0 \text{ on } \partial\Omega_\delta,$$

where $\epsilon > 0$ is a uniform lower bound of the eigenvalues of $\{\underline{u}_{j\bar{k}}\}$ on $\overline{\Omega}$.

Proof. It follows from $u^{j\bar{k}}(u_{j\bar{k}} - \underline{u}_{j\bar{k}}) \le n - \epsilon \sum u^{k\bar{k}}$ that

$$\mathcal{L}(u - \underline{u}) \le C_0 - \epsilon \sum u^{k\bar{k}}. \tag{2.8}$$

Next, since $\Delta \underline{u} \ge n\epsilon > 0$,

$$(h - \underline{u})(x) \ge c_0 d(x), \quad \text{for } x \in \Omega$$

for some uniform constant $c_0 > 0$. Moreover, we have

$$\mathcal{L}(h - \underline{u}) \le C_1 \left(1 + \sum u^{k\bar{k}}\right),$$

for some constant $C_1 > 0$ under control. Thus

$$\mathcal{L}v \le C_0 + tC_1 + (tC_1 - \epsilon) \sum u^{k\bar{k}} - 2N(d\mathcal{L}d + u^{j\bar{k}} d_j d_{\bar{k}}) \quad \text{in } \Omega_\delta.$$

It is easy to see that

$$\mathcal{L}d \ge -C_2 \left(1 + \sum u^{k\bar{k}}\right).$$

Furthermore, since $\{u^{j\bar{k}}\}$ is positive definite and $d_{t_\beta}(0) = 0$ for all $\beta < 2n$, $d_{t_{2n}}(0) = 1$, we have, for δ sufficiently small,

(2.9)
$$u^{j\bar{k}} d_j d_{\bar{k}} \ge u^{n\bar{n}} d_n d_{\bar{n}} + \sum_{k<n} (u^{n\bar{k}} d_n d_{\bar{k}} + u^{k\bar{n}} d_k d_{\bar{n}}) \ge \frac{u^{nn}}{2} - C_3 \delta \sum u^{k\bar{k}} \quad \text{in } \Omega_\delta.$$

Let $\lambda_1 \le \cdots \le \lambda_n$ be the eigenvalues of $\{u_{j\bar{k}}\}$. We have $\sum u^{k\bar{k}} = \sum \lambda_k^{-1}$ and $u^{n\bar{n}} \ge \lambda_n^{-1}$. By the inequality for arithmetic and geometric means,

$$\frac{\epsilon}{4} \sum u^{k\bar{k}} + N u^{n\bar{n}} \ge \frac{n\epsilon}{4} (N \lambda_1^{-1} \cdots \lambda_n^{-1})^{\frac{1}{n}} \ge \frac{n\epsilon}{4(\psi_1)^{1/n}} N^{\frac{1}{n}} \equiv c_1 N^{\frac{1}{n}}.$$

Now we fix $t > 0$ sufficiently small so that $tC_1 \le \frac{\epsilon}{4}$ and fix N so that $c_1 N^{1/n} \ge C_0 + \epsilon$. We obtain

$$\mathcal{L}v \le -\frac{\epsilon}{4} \left(1 + \sum u^{k\bar{k}}\right) \quad \text{in } \Omega_\delta$$

if we require δ to satisfy $2(C_2 + C_3) N \delta \le \frac{\epsilon}{4}$ in Ω_δ.

Next we examine the value of v on $\partial\Omega_\delta$. On $\partial\Omega \cap B_\delta(0)$ we have $v = 0$. On $\Omega \cap \partial B_\delta(0)$,

$$v \ge tc_0 d - Nd^2 \ge (tc_0 - N\delta) d \ge 0,$$

if we require, in addition, $N\delta \leq tc_0$. Now we can fix δ sufficiently small to complete the proof of Lemma 2.1. □

To estimate $u_{t_\alpha x_n}(0)$ for $\alpha < 2n$ we consider in $\Omega \cap B_\sigma(0)$ the real, linear operator

$$T = \frac{\partial}{\partial t_\alpha} + \rho_{t_\alpha} \frac{\partial}{\partial x_n}.$$

Since T is a tangential operator on $\partial\Omega$,

$$T(u - \underline{u}) = 0 \quad \text{on} \quad \partial\Omega \cap B_\sigma(0).$$

Moreover, we have

$$T(u - \underline{u}) \leq C \quad \text{in } \Omega \cap B_\sigma(0),$$

and (see [7])

$$\mathcal{L}\left(\pm T(u - \underline{u}) - (u_{y_n} - \underline{u}_{y_n})^2\right) \leq C\left(1 + \sum F^{k\bar{k}}\right) \quad \text{in} \quad \Omega \cap B_\sigma(0).$$

We note that, since on $\partial\Omega$ near 0,

$$(u - \underline{u})_{y_n} = -(u - \underline{u})_{x_n} \rho_{y_n},$$

by (2.4) and (2.2),

$$(u_{y_n} - \underline{u}_{y_n})^2 \leq C|z|^2.$$

Now by Lemma 2.1 we may choose $A \gg B \gg 1$ so that

$$\mathcal{L}\left(Av + B|z|^2 - (u_{y_n} - \underline{u}_{y_n})^2 \pm T(u - \underline{u})\right) \leq 0 \quad \text{in} \quad \Omega \cap B_\sigma(0),$$

and

$$Av + B|z|^2 - (u_{y_n} - \underline{u}_{y_n})^2 \pm T(u - \underline{u})) \geq 0 \quad \text{on} \quad \partial(\Omega \cap B_\sigma(0)).$$

Consequently, from the maximum principle,

$$Av + B|z|^2 - (u_{y_n} - \underline{u}_{y_n})^2 \pm T(u - \underline{u})) \geq 0 \quad \text{in} \quad \Omega \cap B_\sigma(0).$$

It follows that

$$|u_{t_\alpha x_n}(0)| \leq Av_{x_n}(0) + |\underline{u}_{t_\alpha x_n}(0)| \leq C, \qquad \alpha < 2n. \tag{2.10}$$

It remains to establish the estimate

$$|u_{x_n x_n}(0)| \leq C. \tag{2.11}$$

Since we have already derived

$$|u_{t_\alpha t_\beta}(0)|,\ |u_{t_\alpha x_n}(0)| \le C, \quad \alpha, \beta < 2n, \tag{2.12}$$

it suffices to prove

$$|u_{n\bar{n}}(0)| \le C. \tag{2.13}$$

Solving equation (1.1) for $u_{n\bar{n}}(0)$ we see that (2.13) follows from (2.12) provided that

$$\sum_{\alpha,\beta<n} u_{z_\alpha \bar{z}_\beta}(0)\xi_\alpha \bar{\xi}_\beta \ge c_0 > 0 \tag{2.14}$$

for any unit vector $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{C}^{n-1}$.

Proposition 2.2. *There exists $c_0 = c_0(\psi_0, \varphi, \underline{u}, \partial\Omega)$ such that (2.14) holds.*

Proof. Without loss of generality, it suffices to show that

$$u_{1\bar{1}}(0) \ge c_0 > 0. \tag{2.15}$$

We may also assume $u(0) = u_{t_j}(0) = 0, j \le 2n-1$. We have, similar to (2.5),

$$(u - \underline{u})_{z_\alpha \bar{z}_\beta}(0) = -(u - \underline{u})_{x_n}(0)\rho_{z_\alpha \bar{z}_\beta}(0), \quad \alpha, \beta < n. \tag{2.16}$$

In particular,

$$u_{1\bar{1}}(0) = \underline{u}_{1\bar{1}}(0) - (u - \underline{u})_{x_n}(0)\rho_{1\bar{1}}(0). \tag{2.17}$$

It follows that if $\rho_{1\bar{1}}(0) \le \frac{1}{4K}\underline{u}_{1\bar{1}}(0)$ (where K as in (2.2), so $0 \le (u - \underline{u})_{x_n}(0) \le 2K$), then $u_{1\bar{1}}(0) \ge \frac{1}{2}\underline{u}_{1\bar{1}}(0) > 0$. So we may assume $\rho_{1\bar{1}}(0) \ge \frac{1}{4K}\underline{u}_{1\bar{1}}(0) > 0$. The function $\tilde{u} = u - \lambda x_n$, where $\lambda = \underline{u}_{x_n}(0) + \underline{u}_{1\bar{1}}(0)/\rho_{1\bar{1}}(0)$, satisfies

$$\det(\tilde{u}_{j\bar{k}}) = \det(u_{j\bar{k}}) \ge \psi_0, \tag{2.18}$$

and

$$\left(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2}\right) \tilde{u}(t', \rho(t')) = 0 \quad \text{at } 0. \tag{2.19}$$

On $\partial\Omega$, $\tilde{u}$ is expanded in a Taylor series

$$\tilde{u}|_{\partial\Omega} = \frac{1}{2}\sum_{\alpha,\beta<2n} \gamma_{\alpha\beta}t_\alpha t_\beta + q(t') + O(|t'|^4) \tag{2.20}$$

where $q(t')$ is a cubic polynomial. We may assume $\gamma_{11} = \gamma_{12} = \gamma_{22} = 0$. For in view of (2.19), we have $\gamma_{11} + \gamma_{22} = 0$, and hence (2.18), (2.19) still hold if we replace $\tilde{u}$ by

$$\tilde{u} - \frac{1}{2}\left(\gamma_{11}x_1^2 + 2\gamma_{12}x_1y_1 + \gamma_{22}y_1^2\right).$$

We claim that, after subtracting the real part of a holomorphic polynomial (this does not affect (2.18) and (2.19)), we may assume

$$\tilde{u}|_{\partial\Omega} \le \operatorname{Re}\sum_{1<j\le n} a_j z_1 \bar{z}_j + C\sum_{1<j\le n} |z_j|^2, \tag{2.21}$$

for suitable $a_j \in \mathbb{C}$. To see this we first note that

$$\sum_{\alpha=1}^{2}\sum_{\beta=3}^{2n-1} \gamma_{\alpha\beta}t_\alpha t_\beta = \operatorname{Re}\sum_{j=2}^{n-1} z_1(a_{1j}z_j + a_{1\bar{j}}\bar{z}_j) + \operatorname{Re}(cz_1y_n).$$

Thus

$$\frac{1}{2}\sum_{\alpha,\beta<2n} \gamma_{\alpha\beta}t_\alpha t_\beta = \operatorname{Re}\sum_{j=2}^{n-1} z_1(a_{1j}z_j + a_{1\bar{j}}\bar{z}_j) + \operatorname{Re}(cz_1y_n) + O(t_3^2 + \cdots + t_{2n-1}^2).$$

Next, in $q(t')$, the cubic in (t_1, t_2) has a unique decomposition $\operatorname{Re}(az_1^3 + bz_1|z_1|^2)$, the terms that are quadratic in (t_1, t_2) can be written in the form

$$\operatorname{Re}\sum_{j=2}^{n-1} z_1^2(a'_{1j}z_j + a'_{1\bar{j}}\bar{z}_j) + \operatorname{Re}\sum_{j=2}^{n-1} c_j z_j |z_1|^2,$$

and all the other terms are bounded by $C\sum_{3\le\beta<2n} t_\beta^2$. Finally, with the aid of (2.4) we may replace $|z_1|^2$ by $(\rho_{1\bar{1}}(0))^{-1}x_n$ modulo a holomorphic polynomial and an error controlled by $C\sum_{1\le\beta\le n} |z_\beta|^2$, if we change the coefficients $a_{1\bar{j}}$ and c appropriately. So we have verified (2.21).

Now we assume (2.18), (2.19), (2.21) hold simultaneously and consider the barrier function

$$h = -\epsilon x_n + \delta|z|^2 + \frac{1}{2B}\sum_{1<j\le n} |a_j z_1 + Bz_j|^2 \tag{2.22}$$

in $\Omega \cap B_\sigma(0)$ for $\sigma > 0$ sufficiently small. The smallest eigenvalue of $\{h_{j\bar{k}}\}$ is 2δ and the largest eigenvalue is bounded above by CB with C a controlled constant, independent of δ.

We will choose $0 < \epsilon \ll \delta$ so that $\delta|z|^2 - \epsilon x_n > 0$ on $\partial(\Omega \cap B_\sigma(0))$ and B so large (independent of δ) that $h \geq \tilde{u}$ on $\partial(\Omega \cap B_\sigma(0))$. Having thus fixed B, we can choose δ small enough that $\det(h_{j\bar{k}}) \leq \psi_0$ in $\Omega \cap B_\sigma(0)$. These choices now determine ϵ.

Thus h is an upper barrier for $\tilde{u}$. That is, by the maximum principle, $\tilde{u} \leq h$ in $\Omega \cap B_\sigma(0)$. Consequently, since $\tilde{u}(0) = h(0)$, $\tilde{u}_{x_n}(0) \leq h_{x_n}(0) = -\epsilon$. By (2.18),

$$\tilde{u}_{1\bar{1}}(0) = -\tilde{u}_{x_n}(0)\rho_{1\bar{1}}(0) \geq \epsilon \, \frac{\underline{u}_{1\bar{1}}(0)}{2K}.$$

Thus (2.15) holds with $c_0 = \frac{\epsilon}{2K}\underline{u}_{1\bar{1}}(0) > 0$. □

We have established (2.1). Thus the proof of Theorem 1.1 is complete.

3. Interior estimates for second derivatives.

The main purpose of this section is to prove Theorem 1.3. We start with the following lemma which we will also need in Section 4.

Lemma 3.1. *Let u be a pluri-subharmonic solution of (1.1) and $L = u^{j\bar{k}}\partial_j\partial_{\bar{k}}$ the linearized operator. Let η be a positive function in Ω and set*

$$W = \max_{z \in \overline{\Omega}} \max_{|\xi|=1, \xi \in \mathbb{C}^n} \eta^N \sum_{j,k} u_{j\bar{k}}(z)\xi_j\bar{\xi}_k \exp\left\{a|\nabla u(z)|^2\right\}, \tag{3.1}$$

where $a \geq 0$ and $N \geq 2$ are constant. Suppose W is achieved at an interior point $z^0 \in \Omega$ with $\xi = (1, 0, \dots, 0)$ and $u_{j\bar{k}}(z^0) = 0$ for $j \neq k$. Then, at z_0,

$$\begin{aligned} \frac{Nu_{1\bar{1}}}{\eta}L(\eta) - N\left|\frac{\eta_1}{\eta}\right|^2 + \frac{au_{1\bar{1}}}{2}\left(\sum_j u_{j\bar{j}} + \sum_{j,k}\frac{|u_{jk}|^2}{u_{j\bar{j}}}\right) \\ + (f)_{1\bar{1}} + 2au_{1\bar{1}}Re\sum_k u_{\bar{k}}L(u_k) \leq 0, \end{aligned} \tag{3.2}$$

when $N \geq 8a\max_{z\in\overline{\Omega}}|\nabla u(z)|^2$.

Proof. Since the function $N\log\eta + \log u_{1\bar 1} + a|\nabla u|^2$ attains a maximum at z^0, we have, at that point

$$N\frac{\eta_j}{\eta} + \frac{u_{1\bar 1 j}}{u_{1\bar 1}} + a\sum_k (u_k u_{\bar k j} + u_{\bar k} u_{kj}) = 0, \tag{3.3}$$

and

$$N\frac{\eta_{j\bar j}}{\eta} - N\left|\frac{\eta_j}{\eta}\right|^2 + \frac{u_{1\bar 1 j\bar j}}{u_{1\bar 1}} - \left|\frac{u_{1\bar 1 j}}{u_{1\bar 1}}\right|^2 + a\left(u_{j\bar j}^2 + \sum_k |u_{kj}|^2\right) + 2\mathrm{Re}\sum_k u_{\bar k} u_{kj\bar j} \le 0. \tag{3.4}$$

From (3.3) we have for $j \ge 2$,

$$N\left|\frac{\eta_j}{\eta}\right|^2 \le \frac{2}{N}\left|\frac{u_{1\bar 1 j}}{u_{1\bar 1}}\right|^2 + \frac{4a^2|\nabla u|^2}{N}\left(u_{j\bar j}^2 + \sum_k |u_{kj}|^2\right). \tag{3.5}$$

Differentiating equation (1.1), we obtain

$$\sum_{j,k} u^{j\bar k} u_{j\bar k l} = (f)_l,$$

$$\sum_{j,k} u^{j\bar k} u_{j\bar k 1\bar 1} - \sum_{j,k,l,m} u^{j\bar m} u^{l\bar k} u_{j\bar k 1} u_{l\bar m\bar 1} = (f)_{1\bar 1},$$

where $f = \log\psi$. For $N \ge 2$,

$$\sum_{j,k} \frac{|u_{1\bar k j}|^2}{u_{k\bar k}\, u_{j\bar j}} - \left|\frac{u_{1\bar 1 1}}{u_{1\bar 1}}\right|^2 - \left(1 + \frac{2}{N}\right)\sum_{j>1}\frac{|u_{1\bar 1 j}|^2}{u_{1\bar 1}\, u_{j\bar j}} \ge 0. \tag{3.6}$$

At z^0 we have $L = \sum u_{j\bar j}^{-1}\partial_j\partial_{\bar j}$; consequently, multiplying (3.4) by $u_{1\bar 1}u_{j\bar j}^{-1}$ and summing over j, we obtain (3.2) with the aid of (3.5) and (3.6). □

Proof of Theorem 1.3. Without loss of generality, we may assume $\det(v_{i\bar j}) \le \psi_0/2$ in $\overline{\Omega}$, where

$$\psi_0 \equiv \min_{x\in\overline{\Omega}} \psi(x, u(x), \nabla u(x)) > 0,$$

which depends on $\|u\|_{C^1(\overline{\Omega})}$. (If this is not satisfied we may apply Theorem 1.1 using v, which is strictly pluri-subharmonic, as a subsolution to obtain a strictly pluri-subharmonic function $\tilde{v}$ satisfying $\det(\tilde{v}_{i\bar{j}}) \leq \psi_0/2$ in $\overline{\Omega}$, and $\tilde{v} = \varphi$ on $\partial\Omega$. We then may replace v by $\tilde{v}$.) By a standard barrier argument one sees that

$$(v-u)(z) \geq \epsilon_0 \text{dist}(z, \partial\Omega) \quad \text{for } x \in \Omega \tag{3.7}$$

for some uniform constant $\epsilon_0 > 0$. Moreover, since v is pluri-subharmonic, we have

$$L(\eta) = L(v) - L(u) \geq -L(u) = -n. \tag{3.8}$$

In order to derive (1.4) we take $\eta \equiv v - u$ in (3.1); it suffices to derive a bound for W. Since $\eta = 0$ on $\partial\Omega$, W is achieved at some interior point $z^0 \in \Omega$ and for some $\xi \in \mathbb{C}^n$. After a holomorphic change of coordinates we may assume $\xi = (1, 0, \ldots, 0)$ and $u_{j\bar{k}}(z^0) = 0$ for $j \neq k$. So we can apply Lemma 3.1.

By a straightforward calculation, we obtain (see also [7])

$$(f)_{1\bar{1}} \geq 2\text{Re} \sum_j f_{p_j} u_{j1\bar{1}} - C\left(1 + u_{1\bar{1}}^2 + \sum_j |u_{1\bar{j}}|^2\right).$$

From (3.3) we have

(3.9)

$$\begin{aligned} 2\text{Re}\sum_j f_{p_j} u_{j1\bar{1}} &= -2a u_{1\bar{1}} \text{Re} \sum_j f_{p_j} \left(u_j u_{j\bar{j}} + \sum_k u_{\bar{k}} u_{kj}\right) \\ &\quad - \frac{2N u_{1\bar{1}}}{\eta} \text{Re} \sum_j f_{p_j} \eta_j \\ &\geq -2a u_{1\bar{1}} \text{Re} \sum_j u_{\bar{k}} L(u_k) - C u_{1\bar{1}} \left(a + \frac{N}{\eta}\right). \end{aligned}$$

Now, multiplying (3.2) by η^2, combined with (3.8) and the above two inequalities, we obtain

$$(a - C)\eta^2 (u_{1\bar{1}}^2 + \sum_j |u_{1\bar{j}}|^2) - C(a+N)\eta u_{1\bar{1}} - CN \leq 0.$$

Choosing $N \gg a \gg 1$ then yields a bound for $\eta u_{1\bar{1}}$ and hence a bound for W. Finally, for any $z \in \Omega$ we have

$$\max_{|\xi|=1, \xi \in \mathbb{C}^n} \sum_{j,k} u_{j\bar{k}}(z) \xi_j \bar{\xi}_k \leq \frac{W}{\eta^N} \exp\left\{-a|\nabla u(z)|^2\right\}.$$

In view of (3.7), this completes the proof of Theorem 1.3. □

4. The regularity of the pluri-complex Green function.

In this section we prove the following theorem which implies Theorem 1.2.

Theorem 4.1. *Let Ω be a smooth bounded strongly pseudoconvex domain and $\zeta \in \Omega$. Then there exists a unique weak solution of (1.3) in $C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$.*

Proof. The uniqueness is a easy consequence of the minimum principle of Bedford-Taylor [4] as in [19]. In the following we prove the existence. Without loss of generality, we may assume that $B_1(\zeta) \subset \Omega$. According to [7], there exists a unique strictly pluri-subharmonic solution $v \in C^\infty(\overline{\Omega})$ to the Dirichlet problem

$$\det(v_{j\bar{k}}) = 1 \quad \text{in } \Omega, \qquad v = -\log|z-\zeta| \quad \text{on } \partial\Omega.$$

Let $\underline{u} \equiv v + \log|z-\zeta| \in C^\infty(\overline{\Omega} - \{\zeta\})$. We see that $\underline{u}$ satisfies

(4.1)
$$\begin{cases} \underline{u} \text{ is strictly pluri-subharmonic} & \text{in } \Omega - \{\zeta\} \\ \det(\underline{u}_{j\bar{k}}) \geq \varepsilon_0 & \text{in } \Omega - \{\zeta\}, \text{for some } 0 < \varepsilon_0 < 1 \\ \underline{u} \leq 0, \quad \text{and } \underline{u}|_{\partial\Omega} = 0 & \\ \underline{u}(z) = \log|z-\zeta| + O(1) & \text{as } z \to \zeta. \end{cases}$$

For each positive $\varepsilon \leq \varepsilon_0$, set $\Omega_\varepsilon = \Omega - \overline{B}_\varepsilon(\zeta)$ and consider the Dirichlet problem

$$\det(u_{j\bar{k}}) = \varepsilon \quad \text{in } \Omega_\varepsilon, \qquad u = \underline{u} \quad \text{on } \partial\Omega_\varepsilon. \tag{4.2}$$

Note that, since $\underline{u}$ is a subsolution, by Theorem 1.1 there exists a unique strictly pluri-subharmonic solution $u^\varepsilon \in C^\infty(\overline{\Omega}_\varepsilon)$ of (4.2). By the maximum principle, we see that

$$\underline{u} \leq u^\varepsilon \leq u^{\varepsilon'} \leq \log|z-\zeta| \quad \text{in } \Omega_\varepsilon \ \text{ if } \varepsilon' \leq \varepsilon. \tag{4.3}$$

Thus the limit

$$u(z) \equiv \lim_{\varepsilon \to 0} u^\varepsilon(z)$$

exists for all $z \in \overline{\Omega} - \{\zeta\}$. We want to show that $u \in C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$ for any $0 < \alpha < 1$. It suffices to establish the following *a priori* estimate: for any compact subset $K \subset \overline{\Omega} - \{\zeta\}$, and for ε sufficiently small that $K \subset \overline{\Omega}_\varepsilon$

$$\|u^\varepsilon\|_{C^{1,\alpha}(K)} \leq C = C(K) \quad \text{independent of } \varepsilon. \tag{4.4}$$

First, from (4.3) we have,

$$\max_K |u^\varepsilon| \leq C_0 \quad \text{independent of } \varepsilon.$$

We now estimate the derivatives on $\partial\Omega$. Let h be the harmonic function in Ω_{ε_0} with boundary value $h|_{\partial\Omega} = 0$ and $h|_{\partial B_{\varepsilon_0}(\zeta)} = \log \varepsilon_0$. Then, for $\varepsilon \leq \varepsilon_0$,

$$\underline{u} \leq u^\varepsilon \leq h \quad \text{in } \Omega_\varepsilon.$$

It follows that

$$0 < c_1 \leq |\nabla u^\varepsilon| = u^\varepsilon_\nu \leq C_1 \quad \text{on } \partial\Omega, \quad \text{independent of } \varepsilon, \tag{4.5}$$

where ν in the unit outer normal to $\partial\Omega$. For the second derivatives at a point on $\partial\Omega$, we observe that the proof of the estimates (2.10) for the mixed normal-tangential second derivatives in Section 2 still works, while the double normal derivative estimate follows from (using notation in Section 2)

$$u_{z_\alpha \bar{z}_\beta} = -u_{x_n}(0)\rho_{z_\alpha \bar{z}_\beta},$$

together with (4.5) and the strong pseudoconvexity of Ω. Thus, we have

$$|\nabla^2 u^\varepsilon| \leq C_2 \quad \text{on } \partial\Omega, \quad \text{independent of } \varepsilon. \tag{4.6}$$

Next we derive a bound

$$\Delta u^\varepsilon \equiv \sum_{j=1}^n (u^\varepsilon_{x_j x_j} + u^\varepsilon_{y_j y_j}) \leq C \quad \text{in } K \text{ independent of } \varepsilon. \tag{4.7}$$

Choose $\varepsilon_1 > 0$ sufficiently small that $K \subset \overline{\Omega} - B_{\varepsilon_1}(\zeta)$. For $\varepsilon < \varepsilon_1$, set

$$M_\varepsilon = \max_{z \in \overline{\Omega}_\varepsilon} \max_{|\xi|=1, \xi \in \mathbb{C}^n} \eta^2 \sum_{j,k} u^\varepsilon_{j\bar{k}}(z)\xi_j \bar{\xi}_k,$$

where $\eta = |z - \zeta|^2 - \varepsilon_1^2$. We want to estimate M_ε, which implies a uniform bound for $|u_{j\bar{k}}|$ on K as in the proof of Theorem 1.3. (We should point out here that we can not apply Theorem 1.3 directly for two reasons: one is because we need a bound which does not depend on ϵ; the other reason is that we do not have a priori bounds for the gradient.) If M_ε is achieved on $\partial\Omega$, then a bound for M_ε follows from (4.6). Assume M is attained at some point in $\Omega - \overline{B}_{\varepsilon_1}(\zeta)$. By Lemma 3.1 (in (3.2), take $N = 2$, $a = 0$, $f \equiv \log \epsilon$) we obtain at that point,

$$M_\varepsilon \sum_j \frac{1}{u^\varepsilon_{j\bar{j}}} \leq C.$$

By the arithmetic-geometric mean inequality, we have

$$\sum_j \frac{1}{u^\varepsilon_{j\bar{j}}} \geq n \det(u^\varepsilon_{j\bar{k}})^{-1/n} = n\varepsilon^{-1/n}.$$

It follows that $M_\varepsilon \leq C\varepsilon^{\frac{1}{n}}$. Consequently, we have derived a bound for M_ε and therefore (4.7) since $\Delta u = 4 \sum u_{j\bar{j}}$. Finally, (4.4) follows from (4.7) by the standard regularity theory. This proves that $u \in C^{1,\alpha}(\overline{\Omega} - \{\zeta\})$ and therefore solves (1.3), completing the proof of Theorem 4.1. □

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