COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 6, Number 3, 485-510, 1998

Holomorphic functions of polynomial growth on abelian coverings of a compact complex manifold

ALEXANDER BRUDNYI[1]

Let M be a compact complex manifold and $p : M_G \longrightarrow M$ be a regular covering over M with a free abelian transformation group G. We prove under some constraints on G that the space $\mathcal{P}_l(M_G)$ of holomorphic functions on M_G of l-polynomial growth is finite dimensional and is linearly imbedded into the corresponding space $\mathcal{P}_l(\mathbf{C}^N)$ of holomorphic polynomials on $\mathbf{C}^N$ with $N := \frac{1}{2}\operatorname{rank}(G)$.

1. Introduction.

The classical Liouville theorem asserts that every holomorphic function on $\mathbf{C}^n$ of polynomial growth l is a holomorphic polynomial of degree at most l. The result has many deep generalizations related to solutions of a linear or non-linear elliptic differential equation on a covering over a Riemannian or complex manifold (see, in particular, [CM], [Gu], [LS], [K], [L], [Li], [AL], [MS]). In this paper we discuss the problem of the description of the space of holomorphic functions of polynomial growth on a regular covering over a compact complex manifold. To its formulation we henceforth denote by M a compact complex manifold and by M_G a regular covering $p : M_G \longrightarrow M$ over M with a finitely generated transformation group G. Fix a minimal set of generators $e_1, ..., e_k$ of G and introduce the distance ρ on G by

$$\rho(g,h) := \min\left\{\sum_i \mid \alpha_i \mid;\ gh^{-1} = \prod_i e_i^{\alpha_i}\right\}.$$

Let $V \subset M_G$ be a fundamental compact with respect to the action of G, i.e., $M_G = \bigcup_{g\in G} gV$.

[1]The work was supported in part by the NSERC.

Definition 1.1. A holomorphic function f on M_G is said to be of l-growth if there exists a constant C such that

$$\sup_{z\in V} \mid f(gz) \mid \leq C\rho(g,1)^l.$$

The linear space of these functions will be denoted by $\mathcal{P}_l(M_G)$.

In the similar way one can introduce the space $\mathcal{H}_l(M_G)$ of harmonic functions of l-polynomial growth on a regular covering over a compact Riemannian manifold M with transformation group G. Recall that a function f on M_G is harmonic if it is a solution of the equation $\Delta f = 0$, where Δ is the pullback of the Laplacian on M.

The main question is to find out the conditions on G under which the spaces $\mathcal{P}_l(M_G)$ (or $\mathcal{H}_l(M_G)$) are finite dimensional. In harmonic case the problem is closely related to a conjecture of Yau [Y] on finite-dimensionality of the space of harmonic functions of polynomial growth defined on a complete Riemannian manifold with non-negative Ricci curvature.

It was firstly proved by Guivarch ([Gu]) and independently by Lyons and Sullivan ([LS]) that in the case of nilpotent G the space $\mathcal{H}_0(M_G)$ consists of constants only. Then Kaimanovich ([K]) proved that a similar result holds in the case of a polycyclic group G. Using a different approach V. Lin ([L]) established that $\mathcal{P}_0(M_G)$ consists of constants in the case where G is a nilpotent group. His method gives also a new proof of the Lyons-Sullivan result. Further, Avallaneda and F.-H. Lin showed that in the case $M_G = \mathbf{R}^n$ is a covering over a Riemannian manifold M diffeomorphic to a real torus, the space $\mathcal{H}_l(M_G)$ is finite dimensional. They gave also an explicit description of the space. A new proof of this result and its generalization to the case of nonlinear elliptic operators see in the paper of Moser and Struve ([MS]).

A further development in this subject was done at the midst of 1996 due to Colding and Minicozzi. In a series of papers they studied the space of harmonic functions of polynomial growth and eventually proved Yau's conjecture (see, e.g. [CM] for references of all their works). Moreover, their methods allowed them to prove finite-dimensionality of $\mathcal{H}_l(M_G)$ for any nilpotent G and give an effective estimate of the dimension. In fact, Colding and Minicozzi proved a more general statement. They considered complete manifolds satisfying the volume doubling property and the Poincaré inequality and proved finite-dimensionality of spaces of harmonic functions of polynomial growth defined on such manifolds. Finally, P. Li ([Li]) gave a deep generalization of the results of Colding and Minicozzy. He established finite-dimensionality of spaces of harmonic functions of polynomial growth with better estimates of their dimensions under rather general assumptions on a manifold. Namely, he assumed only the fulfilment of a mean value inequality for nonnegative subharmonic functions and a weak volume comparison condition.

The results described lead naturally to the following conjecture for holomorphic functions of polynomial growth posed by V.Lin.

Let G be a nilpotent group and M be a compact complex manifold. Then $\mathcal{P}_l(M_G)$ is finite dimensional. Moreover, every $f \in \mathcal{P}_l(M_G)$ (resp., $\mathcal{H}_l(M_G)$) is annihilated by any k-difference $\Delta_{g_1,...,g_k}$ of a fixed order $k = k(l)$. Here

$$\Delta_{g_1,...,g_k} := \Delta_{g_1} \circ ... \circ \Delta_{g_k} \quad (g_1, ..., g_k \in G)$$

and $\Delta_g(f)(x) := f(gx) - f(x)$.

In this paper we consider the case of a regular covering M_G over a compact complex manifold M with *a free abelian transformation group* G. Our approach is based on Algebraic Geometry technique and different from the methods of the above mentioned papers.

To present our results introduce

Definition 1.2. A regular covering $p : M_G \longrightarrow M$ over a compact complex manifold M with a free abelian transformation group G is said to be of the class $\mathcal{PF}$ if all $\mathcal{P}_l(M_G)$ are finite dimensional.

The following result describes the structure of the space $\mathcal{P}_l(M_G)$ in the case $M_G \in \mathcal{PF}$. To its formulation we need a few notions.

Let $\Omega^1_d(M)$ be the space of d-closed holomorphic 1-forms on M. Define the subspace $\Omega^1_d(M; G) \subset \Omega^1_d(M)$ as the set of all 1-forms ω such that $p^*\omega$ determines the trivial element of the cohomology class $H^1(M_G, \mathbf{C})$. Let $\{\omega_i\}_{i=1}^s$ be a basis in this subspace. Consider the family

$$\left\{ f_i(z) := \int_{z_0}^{z} p^*\omega_i \right\}_{i=1}^{s}$$

of holomorphic functions on M_G and denote by $\mathcal{P}_l(f_1, ..., f_s)$ the subspace of the space $\mathbf{C}[f_1, ..., f_s]$ consists of polynomials of degree at most l.

Theorem 1.1. *Let $M_G \in \mathcal{PF}$. Then the space $\mathcal{P}_l(M_G)$ coincides with $\mathcal{P}_l(f_1, ..., f_s)$. Moreover, $\mathcal{P}_l(M_G)$ is annihilated by any $(l+1)$-difference.*

To state the result in a sense converse to the previous one assumes that

M admits a holomorphic mapping A in a complex torus $\mathbf{CT}^r$ such that the induced homomorphism $A_* : H_1(M, \mathbf{Z}) \longrightarrow H_1(\mathbf{CT}^r, \mathbf{Z})$ is surjective.

Consider now the regular covering M_G over M with the transformation group $G = H_1(M, \mathbf{Z})/\operatorname{Ker} A_*$ ($\cong \mathbf{Z}^{2r}$). According to the covering homotopy theorem there exists the covering to A holomorphic mapping $A' : M_G \longrightarrow \mathbf{C}^r$.

Theorem 1.2. *Under the previous assumption $M_G \in \mathcal{PF}$ and for every $f \in \mathcal{P}_l(M_G)$ there exists a holomorphic polynomial p on $\mathbf{C}^r$ of degree at most l such that $f = (A')^*(p)$.*

Corollary 1.3. *Let M be a compact Kähler manifold and M_G be a regular covering over M with a free abelian group G. Then $M_G \in \mathcal{PF}$ and $\mathcal{P}_l(M_G)$ is linearly isomorphic to a subspace of the space $\mathcal{P}_l(\mathbf{C}^N)$ of holomorphic polynomials on $\mathbf{C}^N$ of degree at most l, where $N = \frac{1}{2}\dim_{\mathbf{C}} H_1(M, \mathbf{C})$.*

To formulate the second corollary denote by $\tau(M)$ the transcendency degree of the field of meromorphic functions on M.

Corollary 1.4. *Let M be a compact complex manifold with $\tau(M) \geq \dim_{\mathbf{C}} M - 1$ and G be a free abelian group. Then $M_G \in \mathcal{PF}$.*

Corollary 1.5. *Let M be a compact complex manifold of dimension 1 and genus g and M_G be a regular covering over M with a free abelian group G. Then the space $\mathcal{H}_l(M_G)$ is finite dimensional and linearly isomorphic to the subspace of harmonic polynomials on $\mathbf{C}^g$ of degree at most l that are sums of holomorphic and antiholomorphic ones.*

The author is indebted to Professor V.Lin who raised the question discussed in this paper.

2. Proof of Theorem 1.3 .

Let τ be the canonical action of G on the space $\mathcal{O}(M_G)$ of holomorphic functions on M_G induced by the action of G on M_G. By definition $\mathcal{P}_l(M_G)$ is invariant with respect to τ.

Lemma 2.1. *τ is a unipotent representation, i.e., all eigenvalues of $\tau(g)$ equal 1 for every $g \in G$.*

Proof. Assume, to the contrary, that there exist a nonzero $g \in G$ and $f \in \mathcal{P}_l(M_G)$, and $\lambda \neq 1$ such that

$$\tau(g)f = \lambda f.$$

Firstly prove that $|\lambda| = 1$. Otherwise we can assume without loss of generality that $|\lambda| > 1$. In the case of a free abelian group the word metric ρ is clearly satisfies $\rho(g^n) = |n| \rho(g)$. Therefore

$$|\lambda^n f(z)| = |(\tau(g^n)f)(z)| \leq C(n\rho(g))^l$$

and, hence,

$$|f(z)| \leq \lim_{n\to\infty} \frac{C(n\rho(g))^l}{|\lambda|^n} = 0 \quad (z \in V).$$

Since the fundamental compact V has a nonempty interior, $f = 0$ on M_G. This contradiction implies that $|\lambda| = 1$. Now we prove that there exists a nonzero function $w \in \mathcal{P}_l(M_G)$ such that

(i) $\tau(h)(w) = \lambda(h)w$ for every $h \in G$;

(ii) $\tau(g)(w) = \lambda w$.

Really, let W be the minimal subspace of $\mathcal{P}_l(M_G)$ containing the f and invariant with respect to the action τ. Since W is the linear span of $\tau(G)f$, every $w \in W$ is an eigenvector for $\tau(g)$ with eigenvalue λ. As every finite dimensional representation of a free abelian group of finite rank, $\tau|_W$ has logarithm (see, e.g., [G], ch. 8). Therefore it can be extended to a representation of the abelian Lie group $G \otimes \mathbf{R}$ into $GL(W)$. Indeed, let $\{e_i\}_{i=1}^k$ be a basis of G and $\{f_i := log(\tau(e_i))\}_{i=1}^k$. Then the required extension is given by

$$\sum_{i=1}^k a_i f_i \mapsto \exp(\sum_{i=1}^k a_i f_i) \quad (a_i \in \mathbf{R}).$$

By the Lie theorem there exists a common eigenvector w for the extended representation. It is clear that w satisfies conditions (i) and (ii).
Repeating the arguments of the first part of the proof we conclude that $|\lambda(h)| = 1$ for every $h \in G$. This implies that the plurisubharmonic function $|w(z)|$ is invariant with respect to the action of G on M_G. Since M is a compact complex manifold, the function has to be a constant. From here it follows that w is a constant too and therefore $\lambda = 1$. □

Lemma 2.2. *Let* $\dim_{\mathbf{C}} \mathcal{P}_l(M_G) = n$. *Then for every* $f \in \mathcal{P}_l(M_G)$ *and* $g_1, ..., g_n \in G$

$$\Delta_{g_1,...,g_n}(f) = 0.$$

Proof. Similarly to the construction of the previous proof we can define the extension of τ to a representation $\tau' : G \otimes \mathbf{R} \longrightarrow GL_n(\mathbf{C})$. By Lemma 2.1 τ' is unipotent. In virtue of the Lie theorem (see, e.g., [OV]), τ' is equivalent to a representation τ'' into the nilpotent matrix subgroup $N \subset GL_n(\mathbf{C})$ of upper triangular matrices with units on the diagonal. From here it follows that in an appropriate basis of $\mathcal{P}_l(M_G)$

$$\Delta_{g_1,...,g_n} = \prod_{i=1}^{n} (\tau''(g_i) - I)$$

Now observe $\tau''(g_i) - I$ is an upper triangular matrix with zero diagonal. So the product of n such matrices equals 0. □

Let $e_1, ..., e_k$ be a basis for $G \cong \mathbf{Z}^k$. Denote by $S_l \subset G$ the *l-simplex along the group G*, that is,

$$S_l := \left\{ g = \sum_{i=1}^{k} s_i e_i;\ s_i \geq 0,\ \sum_{i=1}^{k} s_i \leq l \right\}, \tag{2.1}$$

Lemma 2.3. *Let* $x \in M_G$ *be a fixed point. Assume that a function* $f \in \mathcal{P}_l(M_G)$ *equals* 0 *on the set* $S_{n-1}x \subset M_G$. *Then* $f = 0$ *identically.*

Proof. Let to the contrary $f \neq 0$. Then Lemma 2.2 implies the existence of a difference $\Delta_{g_1,...,g_l}$ of the maximal order l such that

(i) $f_l := \Delta_{g_1,...,g_l}(f) \neq 0$;

(ii) $\Delta_g(f_l) = 0$ for every $g \in G$.

Easy computation shows that

$$\Delta_{g_1,...,g_l} = \sum_{i \in I} \pm \tau(h_i) \circ \Delta_{g_1^i,...,g_l^i},$$

where every g_s^i coincides with one of $e_1, ..., e_k$. Therefore without loss of generality we can assume that in condition (i) every element g_i belongs to the basis of G. Because of the maximality of l the condition (ii) is also fulfilled for this choice of g_i. But $f_l(x)$ is a linear combination of $\tau(s_j)f(x)$ with $s_j \in S_{n-1}$. So $f_l(x) = 0$ by the assumption of lemma. Moreover, in virtue of (ii) the holomorphic function f_l is invariant with respect to the action of G and, therefore, is a constant. Hence, $f_l = 0$ on M_G and we obtain the contradiction with the maximality of l. □

Lemma 2.4. *$\mathcal{P}_l(M_G)$ is annihilated by any $(l+1)$-difference.*

Proof. Let x be a fixed point of M_G. Henceforth we identify the orbit $\{Gx\}$ with the lattice $\mathbf{Z}^k$ of integer points in $\mathbf{R}^k$. Let $r_x : \mathcal{P}_l(M_G) \longrightarrow l_\infty(S_{n-1})$ be the restriction to $S_{n-1}(x)$. According to Lemma 2.3 r_x is injective. We define now the linear mapping i from $l_\infty(S_{n-1})$ into the space $\mathcal{P}_{n-1}(\mathbf{R}^k)$ of complex-valued polynomials of degree at most $n-1$ as follows. Since $S_{n-1} \subset G$ is the set of uniqueness for $\mathcal{P}_{n-1}(\mathbf{R}^k)$ and $\dim_{\mathbf{C}} l_\infty(S_{n-1}) = \dim_{\mathbf{C}} \mathcal{P}_{n-1}(\mathbf{R}^k)$, the operator $p \mapsto p|_{S_{n-1}}$, where $p \in \mathcal{P}_{n-1}(\mathbf{R}^k)$, is invertible. Its inverse is the required operator i.

Let us choose now an arbitrary $f \in \mathcal{P}_l(M_G)$ and denote by f_x the restriction of f to $\{Gx\} = \mathbf{Z}^k$. We show that

$$(i \circ r_x)(f_x)\,|_{\mathbf{Z}^k} = f_x.$$

Let $\phi := (i \circ r_x)(f_x)\,|_{\mathbf{Z}^k} - f_x$. Since $(i \circ r_x)(f_x) \in \mathcal{P}_{n-1}(\mathbf{R}^k)$, this polynomial is annihilated by any n-difference. According to Lemma 2.2 f_x is also annihilated by any n-difference and therefore the same is valid for ϕ. Using the arguments from the proof of Lemma 2.3 we can state that ϕ is uniquely determined by its values on S_{n-1}. Since $\phi\,|_{S_{n-1}} = 0$ by the definition of i, this function equals 0 identically.

Let us consider now the polynomial $p_{f,x} := (i \circ r_x)(f_x)$ and prove that its degree less than or equal to l. From here will follow that this polynomial is annihilated by any $(l+1)$-difference and therefore f_x will be also annihilated by such differences. Because of the arbitrariness of x this will prove the lemma.

Assume, to the contrary, that $deg(p_{f,x}) > l$. By the definition of $\mathcal{P}_l(M_G)$ the function $p_{f,x}\,|_{\mathbf{Z}^k} = f_x$ has l-polynomial growth at infinity. Let $s_{f,x}$ be the homogeneous part of $p_{f,x}$ of maximal degree. Then there exists a line $L := \{(a_1 t, ..., a_k t) \in \mathbf{R}^k; t \in \mathbf{R},\ a_i \in \mathbf{Z},\ i = 1, ...k\}$ such that $s_{f,x}\,|_L \neq 0$. Otherwise $s_{f,x} = 0$ on $\mathbf{Z}^k$ and hence it equals 0 on $\mathbf{R}^k$. According to our assumption $(s_{f,x}\,|_L)(t) = ct^d$ with $d > l$ whereas this polynomial goes to infinity as $|t|^l$ in integer points of L. Therefore $c = 0$ and this contradiction proves that $deg(p_{f,x}) \le l$. $\square$

We now are in a position to finish the proof of Theorem 1.1 by induction on l. Namely, we introduce the set $\mathcal{D}_l(M_G)$ of holomorphic on M_G functions annihilated by any $(l+1)$-difference and prove by induction that any element of this set can be represented as a polynomial in $f_1, ..., f_s$ of degree at most l. Since every such polynomial belongs to $\mathcal{P}_l(M_G)$, this result together with Lemma 2.4 will complete the proof of the theorem.
The basis of the induction is

Lemma 2.5. *Let $f \in \mathcal{D}_1(M_G)$. Then f is a linear polynomial in $f_1, ..., f_s$.*

Proof. Since f is annihilated by any 2-difference, $f(gx) - f(x)$ is a constant for a fixed $g \in G$. Therefore df is a G-invariant d-closed holomorphic 1-form on M_G. Hence there exists a form $\eta \in \Omega^1_d(M;G)$ such that its pullback to M_G coincides with df. From here it follows that f up to an additive constant can be represented as a linear combination of the functions f_i, i.e., $f \in \mathcal{P}_1(f_1, ..., f_s)$. $\square$

Assume now that the statement of the induction has already been proved for $\mathcal{D}_{l-1}(M_G)$ and prove it for $\mathcal{D}_l(M_G)$.

Let F be the maximal free abelian subgroup of the homology group $H_1(M, \mathbf{Z})$ and $R \subset F$ determined by $F/R \cong G$. Since $\int_\gamma \omega = 0$ for every $\omega \in \Omega^1_d(M;G)$ and $\gamma \in R$, we get

$$\dim_{\mathbf{C}} \Omega^1_d(M;G) \le \frac{1}{2}\,\mathrm{rank}(G).$$

In addition, $\Omega^1_d(M;G)$ can be identified with a subspace of the space dual to $F \otimes \mathbf{C} \cong H_1(M, \mathbf{C})$. Therefore there exists a subgroup $A \subset F$ of

$$\mathrm{rank}(A) = 2\dim_{\mathbf{C}} \Omega^1_d(M;G)$$

such that the set Γ of the vectors $\left(\int_\alpha \omega_1, ..., \int_\alpha \omega_s\right) \in \mathbf{C}^s \quad (\alpha \in A)$ satisfies the condition

(2.2) $\qquad\Gamma$ is a lattice of rank $2s$ in $\mathbf{C}^s$.

Since $A\bigcap R = \{0\}$, the natural surjection $F \longrightarrow F/R \cong G$ imbeds A into G. So we can consider A as a subgroup of G. Denote by E the quotient group G/A and consider the regular covering $p_E : M_E \longrightarrow M$ with the transformation group E. Because of the choice of E there exists a regular covering $p_1 : M_G \longrightarrow M_E$ with the transformation group A. Condition (2.2) implies that the multivalued function

$$t(z) := \left(\int_{z_0}^{z} p_E^*\omega_1, ..., \int_{z_0}^{z} p_E^*\omega_s\right) \quad (z \in M_E),$$

where $z_0 \in M_E$ is fixed, determines the holomorphic mapping into the complex torus $\mathbf{C}^s/\Gamma$. Then one naturally defines the mapping $t' : M_G \longrightarrow \mathbf{C}^s$, which covers t and is clear to be equivariant with respect to the actions of A on M_G and Γ on $\mathbf{C}^s$. Therefore there exists the set $\{g_1, ..., g_s\}$ of $\mathbf{C}$-linear independent holomorphic polynomials of degree 1 on $\mathbf{C}^s$ such that

$$(t')^*(g_i) = f_i := \int_{y_0}^{y} p^*\omega_i \quad (1 \le i \le s).$$

Here $y, y_0 \in M_G$ and y_0 is fixed. Using an affine transformation we can regard g_i as the coordinate functions of $\mathbf{C}^s$. Assume that the similar relation is valid for every f annihilated by any l-difference, that is, there exists a holomorphic polynomial $p \in \mathcal{P}_{l-1}(\mathbf{C}^s)$ of degree at most $l-1$ such that

$$(t')^*(p) = f.$$

In view of the previous relation this means that f is a polynomial in f_i $(1 \le i \le s)$ of degree at most $l-1$.

To finish the proof it remains therefore to establish the corresponding statement for $f \in \mathcal{D}_l(M_G)$. To this end choose an arbitrary $x \in M_G$ and consider the orbit $\{Ax\} \subset M_G$, which we can identify with the lattice $\mathbf{Z}^{2s} \subset \mathbf{R}^{2s}$. Consider now the k-simplex S_k along the group $A \cong \mathbf{Z}^{2s}$, see (2.1) for its definition. In this definition $e_1, ..., e_{2s}$ is the standard basis for $\mathbf{Z}^{2s}$. As a consequence of the statement of induction for $(l-1)$ we get S_{l-1} is a set of uniqueness for $\mathcal{D}_{l-1}(M_G)$. Then S_l is a set of uniqueness for $\mathcal{D}_l(M_G)$. For if $h \in \mathcal{D}_l(M_G)$ equals 0 on S_l then clearly the function $h_i(z) := h(e_iz) - h(z)$ is an element of $\mathcal{D}_{l-1}(M_G)$ equals 0 on S_{l-1}. Therefore $h_i = 0$ for every i and thus h is A-invariant. The same is clear to be correct for $h^g(z) := h(gz) - h(z)$ $(g \in G)$. Moreover, $h^g \in \mathcal{D}_{l-1}(M_G)$ and therefore by induction there exists a polynomial $p^g \in \mathcal{P}_{l-1}(\mathbf{C}^s)$ such that

$$(t')^*(p^g) = h^g.$$

Then for every $a \in A$ we get

$$(t')^*(p^g(a+z) - p^g(z)) = h^g(az) - h^g(z) = 0,$$

that is, p^g is A-invariant. Thus p^g is a constant and so h^g is. This means that $h \in \mathcal{D}_1(M_G)$ and by Lemma 2.5 $h = 0$. By this one has proved that S_l is a set of uniqueness for $\mathcal{D}_l(M_G)$.

Using this fact we can find for the function $f \in \mathcal{D}_l(M_G)$ a polynomial p_f on $\mathbf{R}^{2s}$ of degree at most l such that

$$p_f = f \quad on \quad \mathbf{Z}^{2s},$$

see Lemma 2.4. Since t' is equivariant with respect to the actions of A on M_G and Γ on $\mathbf{C}^s$, we can identify $\mathbf{R}^{2s}$ with $\mathbf{C}^s$ and $\mathbf{Z}^{2s}$ with Γ. These identifications and the assumption of the induction applying to f get that any difference $\Delta_a(p_f) := p_f(a+z) - p_f(z)$ is a holomorphic on $\mathbf{C}^s$ polynomial of degree at most $(l-1)$. Then $(0,1)$-form $\overline{\partial} p_f$ is Γ-invariant and therefore by the Hodge decomposition

$$\overline{\partial} p_f = \sum_{i=1}^{s} a_i \overline{dz_i} + \overline{\partial} w,$$

where w is Γ-invariant and $z_1, ..., z_s$ stand for the standard coordinate functions on $\mathbf{C}^s$. This leads to the identity

$$p_f(z) = \sum_{i=1}^{s} a_i \overline{z_i} + w(z) + u(z),$$

where u is a holomorphic on $\mathbf{C}^s$ function of l-polynomial growth, that is, u is a holomorphic polynomial of degree at most l. From here and Γ-invariance it follows that w has to be a constant. Consider now the function $f - (t')^*(u)$. Because of the above identity and the interpolating property of p_f any 2-difference of the function equals 0 on the orbit $\{Ax\}$. Bearing in mind that such a difference belongs to $\mathcal{D}_{l-2}(M_G)$ and applying the statement of the induction we obtain that any 2-difference of $f - (t')^*(u)$ equals 0 on M_G. Therefore $f - (t')^*(u) \in \mathcal{D}_1(M_G)$ and by Lemma 2.5 there exists a linear holomorphic on $\mathbf{C}^s$ polynomial q such that

$$f = (t')^*(u+q).$$

This proves the statement of the induction and completes the proof of Theorem 1.1. □

3. Properties of PF Spaces.

Here we collect a few results which will be used in the sequel. Some of them are interesting in their own right. To formulate the first result consider a compact complex manifold M possessing the following property:

> There exist collections $\{V_s\}_{s\in S}$ of complex compact manifolds and holomorphic mappings $\{\phi_s : V_s \longrightarrow M\}_{s\in S}$ such that
>
> (i) $\phi_s : V_s \longrightarrow M$ induces the surjective homomorphism $(\phi_s)_* : H_1(V_s, \mathbf{Z}) \longrightarrow H_1(M, \mathbf{Z})$;
>
> (ii) the union of all $\phi_s(V_s)$ has nonempty interior.

Denote by F_s and F the maximal free abelian subgroups of $H_1(V_s, \mathbf{Z})$ and $H_1(M, \mathbf{Z})$, respectively.

Proposition 3.1. *Assume that for every s the regular covering W_s over V_s with the transformation group F_s belongs to $\mathcal{PF}$. Then the regular covering M_F over M with the transformation group F belongs to $\mathcal{PF}$.*

Proof. By assumption (i) the image $(\phi_s)_*(F_s)$ can be represented as $F \oplus G_s$ with a finite subgroup G_s contained in the torsion subgroup of $H_1(M, \mathbf{Z})$. Let $H_s := (\phi_s)_*^{-1}(F)$ and consider a subgroup E_s in F_s such that

$$H_s \bigcap E_s = \{0\}, \quad H_s + E_s \ \text{ has finite index in } F_s.$$

Let M_1 be the regular covering over M with the transformation group $H_1(M, \mathbf{Z})$. By assumption (i) and the covering homotopy theorem there is a holomorphic mapping $\phi'_s : W_s \longrightarrow M_1$ which covers ϕ_s and is equivariant with respect to the actions of F_s on W_s and $(\phi_s)_*(F_s)$ on M_1. Similarly for the covering M_F there exists the covering mapping $r : M_1 \longrightarrow M_F$ equivariant with respect to the action F on M_1 and M_F.

Let now r^*f be the pullback to M_1 of a function $f \in \mathcal{P}_l(M_F)$ and $f' := (\phi'_s)^*(r^*f)$. In virtue of the definition of E_s and the equivariance of ϕ'_s and r the function f' is invariant with respect to the action of the group E_s. Since the free abelian group F is quotient group of H_s and $f \in \mathcal{P}_l(M_F)$, the function f' has l-polynomial growth with respect to the action of H_s. In addition, $H_s \oplus E_s$ has a finite index in F_s and therefore f' has also l-polynomial growth with respect to the action of F_s. By the assumptions of the proposition $W_s \in \mathcal{PF}$. So applying the statement of Theorem 1.1 to f'

we obtain that any $(l+1)$-difference $\Delta_{h_1,\dots,h_{l+1}}$ with $h_i \in H_s$ annihilates f'. Since the image of $(\phi_s)_*(H_s)$ coincides with F, the equivariance arguments imply the restriction of f to $\phi_s'(W_s)$ is annihilated by any $(l+1)$-difference. However, from assumption *(ii)* it follows that the union of all $\phi_s'(W_s)$ has nonempty interior. So holomorphicity of f implies that any $(l+1)$-difference annihilates f, i.e., $f \in \mathcal{D}_l(M_F)$. It was proved in Theorem 1.1 that the latter vector space admits an imbedding into the space of holomorphic polynomials of degree at most l defined on some $\mathbf{C}^s$. This means that $M_F \in \mathcal{PF}$. □

Let $h : X \longrightarrow Y$ be a holomorphic surjective mapping of compact complex manifolds. Assume that fiber $h^{-1}(y)$ over the generic point $y \in Y$ is finite. Let $k := deg(h)$ be degree of h, that is, the number of points in $h^{-1}(y)$ over the generic $y \in Y$. Denote by H and G the maximal free abelian subgroups of $H_1(Y, \mathbf{Z})$ and $H_1(X, \mathbf{Z})$, respectively. Assume also that

(i) $h_* : H_1(X, \mathbf{Z}) \longrightarrow H_1(Y, \mathbf{Z})$ maps G into H;

(ii) there is a surjective homomorphism τ from H into a free abelian group F.

Put $G' := (\tau \circ h_*)(G)$.

Finally denote by $X_{G'}$ and Y_F regular coverings over X and Y with the transformation groups G' and F, respectively.

Proposition 3.2. *Let $\mathcal{P}_{kl}(Y_F)$ be finite dimensional. Then $\mathcal{P}_l(X_{G'})$ has finite dimension, as well.*

Proof. Surjectivity of h implies $h_*(\pi_1(X))$ is of finite index in $\pi_1(Y)$. In particular, G' is of finite index in F. From here and the covering homotopy theorem follow that there exists the covering to h holomorphic mapping $h' : X_{G'} \longrightarrow Y_F$ such that h' is a proper surjective mapping of the degree k equivariant with respect to the actions of G' on $X_{G'}$ and Y_F. Consider the Stein factorization of h', that is, the analytical variety V and holomorphic mappings $h_1 : X_{G'} \longrightarrow V$ and $h_2 : V \longrightarrow Y_F$ such that h_2 is a finite branched covering over Y_F of degree k and h_1 has compact connected fibers, and $h' = h_2 \circ h_1$. Let f be a holomorphic function on $X_{G'}$. Then f is clear to be constant on any fiber of h_1 and therefore there exists a holomorphic on V function f' such that $h_1^*(f') = f$.

We are now in a position to prove the proposition. To this end we construct a smooth immersion of $\mathcal{P}_l(X_{G'})$ into a finite dimensional space.

Since such immersion preserves dimension we obtain from here the required result. To construct the immersion fix a function $f \in \mathcal{P}_l(X_{G'})$. Then there exists a holomorphic on V function f' such that $h_1^*(f') = f$. Let $x \in Y_F$ be the generic point and $h_2^{-1}(x) = \{x_1, ..., x_k\}$. Consider the polynomial

$$\prod_{i=1}^{k}(t - f'(x_i)) = \sum_{i=1}^{k} s_i(f')(x)t^i.$$

The symmetric polynomials $s_i(f')$ are correctly defined outside of a divisor $D \subset Y_F$ and locally bounded in a neighborhood of every point of D. So by the Riemann theorem they can be extended to Y_F as holomorphic functions for which we preserve the same notations. Moreover, in virtue of equivariance of h' with respect to the cocompact actions of G' on $X_{G'}$ and Y_F, the function $s_i(f')$ belongs to $\mathcal{P}_{il}(Y_F)$. Let now

$$i(f) := (s_1(f'), ..., s_k(f')) \quad (f \in \mathcal{P}_l(X_{G'})).$$

Since the Gâteaux derivative of $f \mapsto s_1(f')$ is readily seen to be different from 0, i has no critical points, i.e., the smooth mapping i is an immersion into the direct product $\mathcal{P}_l(Y_F) \times ... \times \mathcal{P}_{kl}(Y_F)$. By the assumption of the proposition the latter is finite dimensional. This completes the proof. □

Remark 3.3. Literally repeating these arguments we obtain the same result in the case where $h : X \longrightarrow Y$ is a finite branched covering over a compact complex manifold Y with irreducible X.

Let $h : X \longrightarrow Y$ be a holomorphic surjective mapping of compact complex manifolds with connected fibers. Denote a fiber $h^{-1}(y)$ over the generic point $y \in Y$ by Z_y and the maximal free abelian subgroup of $H_1(Z_y, \mathbf{Z})$ by H_y. Let G and F be the maximal free abelian subgroups of $H_1(Y, \mathbf{Z})$ and $H_1(X, \mathbf{Z})$, respectively. Finally denote by Y_G, X_F and Z'_y the regular coverings over X, Y and Z_y with the transformation groups G, F and H_y, respectively.

Proposition 3.3. *Let $Y_G \in \mathcal{PF}$ and $Z'_y \in \mathcal{PF}$ for the generic y. Then $X_F \in \mathcal{PF}$.*

Proof. Let K be an arbitrary fiber of $h : X \longrightarrow Y$ and $i^K : K \longrightarrow X$ be the natural imbedding into X.

Lemma 3.4. *There exists an injective linear mapping*

$$s : H_1(Y, \mathbf{R}) \longrightarrow H_1(X, \mathbf{R})$$

such that $H_1(X, \mathbf{R})$ *is isomorphic to* $s(H_1(Y, \mathbf{R})) \oplus (i^K)_*(H_1(K, \mathbf{R}))$.

Proof. By the Sard theorem there exists a proper closed analytical subset $D \subset Y$ such that $h : X \setminus h^{-1}(D) \longrightarrow Y \setminus D$ is a C^∞ fiber bundle. To begin with let K be a fiber of this bundle. Since X is a connected complex manifold, the imbedding $X \setminus h^{-1}(D) \longrightarrow X$ induces a surjective homomorphism τ of the corresponding 1-homology groups. Moreover, the exactness of the homotopy sequence for the above constructed fiber bundle leads to existence of an injective linear mapping $s' : H_1(Y \setminus D, \mathbf{R}) \longrightarrow H_1(X \setminus h^{-1}(D), \mathbf{R})$ such that

(i)
$$H_1(X \setminus h^{-1}(D), \mathbf{R}) = s'(H_1(Y \setminus D, \mathbf{R})) \oplus (i^K)_*(H_1(K, \mathbf{R}));$$

(ii) the mapping $\tau \circ s'$ equals 0 on

$$\mathrm{Ker}\{H_1(Y \setminus D, \mathbf{R}) \longrightarrow H_1(Y, \mathbf{R})\}.$$

The mapping of the homology groups in (*ii*) is induced by the natural imbedding. It follows from (*ii*) that $\tau \circ s'$ determines the injective mapping $s : H_1(Y, \mathbf{R}) \longrightarrow H_1(X, \mathbf{R})$. In addition, (*i*) implies that $s(H_1(Y, \mathbf{R})) \bigcap (i^K)_*(H_1(K, \mathbf{R})) = \{0\}$ and direct sum of these two vector spaces coincides with $H_1(X, \mathbf{R})$. It proves the result in this case. In particular, in this situation we obtained that

$$(i^K)_*(H_1(K, \mathbf{R})) = \mathrm{Ker}(h)_*. \tag{3.1}$$

Let now K be an arbitrary fiber of h. Then the triangulation theorem for analytical sets implies that there exists an open neighborhood U of K such that K is a deformation retract of U. By compactness arguments U contains a fiber E of the above constructed C^∞ fiber bundle. Using this fact we will prove (3.1) for such K. Then the above defined imbedding s determines the required decomposition of $H_1(X, \mathbf{R})$. To prove (3.1) assume to the contrary that $(i^K)_*(H_1(K, \mathbf{R})) \neq \mathrm{Ker}(h)_*$. Since K is a deformation retract of U, from here it follows that there exist a d-closed 1-form ω on X and an element $\gamma \in (i^E)_*(H_1(E, \mathbf{R}))$ such that $\omega |_U$ is d-exact but $\int_\gamma \omega \neq 0$. However, $E \subset U$ and we get a contradiction. □

Denote now by $A(K)$ the maximal free abelian subgroup of $(i^K)_*(H_1(K,\mathbf{Z}))$. As a consequence of the lemma we obtain that $A(K) \oplus s(G)$ is of finite index in F.

Lemma 3.5. *Let H be the maximal free abelian subgroup of* $\mathrm{Ker}(h)_*$. *Then there exists a subgroup $L \subset H$ of a finite index such that for every fiber K of h*

$$L \subset A(K) \subset H.$$

Proof. Let E be a fiber of the above constructed C^∞ bundle over $Y \setminus D$. It is clear that $A(E)$ is independent of E and has a finite index in H. Denote this group by L. For an arbitrary fiber K choose as above an open neighborhood U such that K is a deformation retract of U and U contains E. Then $L = A(E)$ is to be contained in $A(K) \subset H$. □

Consider now the regular covering $p : X_E \longrightarrow X$ with the finite transformation group $E := F/(L \oplus s(G))$. Let (V, f_1, f_2) be the Stein factorization of the mapping $h \circ p : X_E \longrightarrow Y$. So V is an analytic space, $f_1 : X_E \longrightarrow V$ is a surjective holomorphic mapping with connected fibers and $f_2 : V \longrightarrow Y$ is a finite branched covering. According to definition of E the composite mapping

$$H_1(X_E,\mathbf{Z}) \stackrel{(f_1)_*}{\longrightarrow} H_1(V,\mathbf{Z}) \stackrel{(f_2)_*}{\longrightarrow} H_1(Y,\mathbf{Z}) \longrightarrow G$$

is surjective. Therefore there exists the regular covering V_G over V with the transformation group G. Then the covering to f_2 mapping $f_2' : V_G \longrightarrow Y_G$ is finite branched and equivariant with respect to the actions of G on V_G and Y_G. Note that the generic fiber of f_1 is biholomorphic to the generic fiber of $h : X \longrightarrow Y$. The same is clear to be true for the generic fiber of the covering to f_1 mapping $f_1' : X_G' \longrightarrow V_G$, where X_G' is the covering over X_E with the transformation group G. Finally, consider the covering $r : X_F \longrightarrow X_G'$ with the transformation group L. According to Lemmae 3.4 and 3.5 we get:

> $f_1' \circ r : X_F \longrightarrow V_G$ has connected fibers and L acts cocompactly on every fiber of $f_1' \circ r$.

Futher fix an $x \in Y_G$ such that its preimage $(f_2' \circ f_1' \circ r)^{-1}(x)$ in X_F is a smooth submanifold. Consider the set $S_{lk}x \subset Y_G$, where $k := deg(f_2)$ and S_{lk} is the lk-simplex along the group G (see (2.1)). Denote by S preimage of $S_{lk}x$ under the mapping f_2'. So, $S \subset V_G$ and is finite. Let R be a finite subset of X_F such that $f' |_R$ is a bijection onto S. Finally, consider the set $S_l R$, where S_l is the l-simplex along the group L. To complete the proof of the proposition it remains to establish the following result.

Lemma 3.6. *Let $f \in \mathcal{P}_l(X_F)$ equal 0 on the set $S_l R$. Then f equals 0 on X_F.*

Proof. Assume, to the contrary, that $f \neq 0$. Let $Z \subset X_F$ be the generic fiber of $f_1' \circ r$. Then as above the group L acts cocompactly on the complex manifold Z and the quotient manifold Z/L is biholomorphic to the generic fiber of $h : X \longrightarrow Y$. By the assumption of the proposition the space $\mathcal{P}_l(Z)$ of holomorphic l-polynomial growth functions with respect to the action of L is finite dimensional. In virtue of Theorem 1.1 every $(l+1)$-difference defined with the help of elements of L annihilates $\mathcal{P}_l(Z)$. Since $f |_Z$ belongs to $\mathcal{P}_l(Z)$ and union of generic fibers of $f_1' \circ r$ has nonempty interior, f is annihilated by each $(l+1)$-difference defined by elements of L. This and the arguments from the proof of Lemma 2.3 applying to the $f \neq 0$ lead to existence of a difference $\Delta_{g_1,\dots,g_s}$ $(s \leq l)$ with g_i from the set of generators of L such that

$$Q := \Delta_{g_1,\dots,g_s} f \neq 0 \ \ and \ \ \Delta_g Q = 0, \ for\ every\ g \in L.$$

Therefore for each fiber K of $f_1' \circ r$ the function $Q |_K$ is invariant with respect to the action of L. But K/L is a compact connected complex variety and so $Q |_K$ is a constant. Consequently, there exists a holomorphic on V_G function $Q' \neq 0$ such that $(f_1' \circ r)^*(Q') = Q$. Since $f_1' \circ r$ is equivariant with respect to the action of G, the function Q' belongs to $\mathcal{P}_l(V_G)$. By the assumption of the lemma $Q = \Delta_{g_1,\dots,g_s} f$ equals 0 on the set R and therefore $Q' = 0$ on S. Show now that $Q' = 0$ identically that will lead to contradiction with $f \neq 0$. To establish this consider the symmetric polynomials $s_i(Q')$ defined by

$$\prod_{i=1}^{k}(t - Q'(y_i)) = \sum_{i=1}^{k} s_i(Q')(y)t^i,$$

where $\{y_1, \dots, y_k\}$ is a fiber of $f_2' : V_G \longrightarrow Y_G$ over the generic y. As in the proof of Proposition 3.2 $s_i(Q')$ can be extended to a holomorphic on Y_G function from $\mathcal{P}_{il}(Y_G)$. Moreover, $s_i(Q')$ equals 0 on the set $S_{kl}x$. Since $Y_G \in \mathcal{PF}$, the arguments from proof of Theorem 1.1 applying to functions $s_i(Q')$ yield these functions equal 0 identically $(i \leq k)$. This implies immediately $Q' = 0$. □

To finish the proof of the proposition consider the mapping from $\mathcal{P}_l(X_F)$ into $l^\infty(S_l R)$ defined by $f \mapsto f\mid_{S_l R}$. The above lemma states that it is injective. Therefore $\dim_{\mathbf{C}} \mathcal{P}_l(M_G) < \infty$. $\square$

4. Proof of Theorem 1.4.

The proof is divided into three parts. First we prove the theorem in the case of a compact complex curve. Then we consider the case of a compact projective manifold and, finally, finish the proof.

Let M be a compact complex curve of genus g. The Albanese mapping A imbeds M into its Jacobi manifold biholomorphic to a complex torus $\mathbf{CT}^g$ and induces the surjective homomorphism of the corresponding fundamental groups. As well-known the regular covering M_G over M with the transformation group $G = A_*(\pi_1(M)) \cong \mathbf{Z}^{2g}$ is the maximal abelian covering over M. Therefore it suffices to prove the theorem for the case of M_G. Let $A' : M_G \longrightarrow \mathbf{C}^g$ be the holomorphic imbedding that covers A. Since the Jacobi manifold of the curve M is projective, there exists a very ample line bundle L over $\mathbf{CT}^g$. According to the Kodaira imbedding theorem we may think of $\mathbf{CT}^g$ as imbedded in some projective space $\mathbf{CP}^N$ and of L as the restriction to $\mathbf{CT}^g$ of the hyperplane bundle with the standard positively curved metric. Then zero loci of sections of L are hyperplane sections of $\mathbf{CT}^g$. By Bertini's theorem, the generic linear subspace of codimension $g-1$ intersects $\mathbf{CT}^g$ transversely in a smooth curve C. By the Lefschetz hyperplane theorem, C is connected and the map $\pi_1(C) \longrightarrow \pi_1(\mathbf{CT}^g)$ is surjective. Let C_G be the regular covering with the above defined transformation group $G \cong \pi_1(\mathbf{CT}^g)$. Then there exists a holomorphic imbedding of C_G into $\mathbf{C}^g$ and we will consider C_G as a submanifold of $\mathbf{C}^g$.

Proposition 4.1. *For every integer $l \geq 0$ there exists a very ample bundle L over $\mathbf{CT}^g$ such that $\mathcal{P}_{(g!)l}(C_G)$ is finite-dimensional. Moreover, for every $f \in \mathcal{P}_s(C_G)$ with $s \leq (g!)l$ there exists a holomorphic polynomial p on $\mathbf{C}^g$ of degree at most s such that its restriction to C_G coincides with f.*

Proof. Let

$$\rho(z_1, \ldots, z_g) := \left(1 + \sum_{i=1}^{g} \mid z_i \mid^2\right)^{1/2}.$$

Obviously, $f \in \mathcal{P}_l(C_G)$ if and only if

$$|f(z)| \leq C(f)\rho(z)^l \quad (z \in C_G). \tag{4.1}$$

Let $\omega := \sum_{i=1}^{g} dz_i \wedge d\overline{z_i}$ be the Kähler form on $\mathbf{C}\mathbf{T}^g$ defining the Euclidean metric and dV_C and dV be the volume forms determined by ω on C and $\mathbf{C}\mathbf{T}^g$, respectively. We preserve the same notations for the pullback volume forms on C_G and $\mathbf{C}^g$ and the pullback Kähler form on $\mathbf{C}^g$.

Lemma 4.2. *For every $f \in \mathcal{P}_s(C_G)$ the inequality*

$$\int_{C_G} |f|^2 \rho^{-t} dV_C < \infty$$

holds with $t := 2s + 2g + 1$.

Proof. Since dV_C is invariant with respect to the action of G, the integrability condition can be rewritten as

$$\sum_{g \in G} \int_{V_0} |f(gz)|^2 \rho(gz)^{-t} dV_C < \infty,$$

where V_0 is a compact fundamental domain. Because of compactness of V_0

$$C\rho(g) \leq \rho(gz) \quad (g \in G,\ z \in V_0)$$

for some constant $C > 0$. Together with (4.1) this gives for $t > 2s$:

$$\sum_{g \in G} \int_{V_0} |f(gz)|^2 \rho(gz)^{-t} dV_C \leq C' \sum_{g \in G} \rho(g)^{2s-t} vol(V_0).$$

The righthand side is finite if $2s - t < -2g$. Letting $t := 2s + 2g + 1$ we complete the proof. □

To construct the required bundle L consider a positive linear vector bundle E on $\mathbf{C}\mathbf{T}^g$ that determines an imbedding into some linear projective space. Let $\Theta(E) \in \Omega^{1,1}(\mathbf{C}\mathbf{T}^g)$ be the curvature form of E. Now observe that the function $\log(\rho)$ satisfies

(i) $d\log(\rho)$ is bounded on $\mathbf{C}^g$ with respect to the Euclidean metric;

(ii) the Levi form $\mathcal{L}(\log(\rho))$ is bounded on $\mathbf{C}^g$ with respect to the Euclidean metric.

Then there exists a postive integer $n := n(l, E)$ such that for the linear vector bundle $L := E^{\otimes n}$ the inequality

$$\Theta(L) = n\Theta(E) \geq (2(g!)l + 2g + 1)\mathcal{L}(\log(\rho)) + \epsilon\omega \tag{4.2}$$

holds on $\mathbf{C}^g$ for some $\epsilon > 0$.

Let now $f \in \mathcal{P}_s(C_G)$ and $s \leq (g!)l$. By Lemma 4.2 f belongs to the weighted space $L_2(w, C_G)$ with the weight $w = \exp(-(2s + 2g + 1)\log(\rho))$. This and inequality (4.2) show that the conditions of Lárusson's extension theorem (see [La], th.3.1) are fulfilled for f and zero loci C of L. Applying his result we can extend f to a holomorphic function $f' \in L_2(\exp(-(2s + 2g + 1)\log(\rho)), \mathbf{C}^g)$. By subharmonity of $| f' |$ from here it follows (see, e.g., [FN], p.1117) that f' is a holomorphic polynomial of degree at most s. So, f is the restriction of the polynomial f' to C_G. □

Going over to the covering M_G over the curve M consider the direct and the symmetric products $M^{\times g}$ and $SM^{\times g}$ of g-copies of M. Then $SM^{\times g}$ is the quotient manifold of $M^{\times g}$ under the action of the permutation group S_g. Therefore there exists the finite holomorphic surjective mapping $M^{\times g} \longrightarrow SM^{\times g}$. Further, $SM^{\times g}$ is birational isomorphic to $\mathbf{CT}^g$ (see, e.g., [GH]). Denote by $j : M^{\times g} \longrightarrow \mathbf{CT}^g$ the composition of these two mappings. Let $X \subset M^{\times g}$ be an irreducible component of $j^{-1}(C)$, which without loss of generality we can assume to be a smooth curve. X clearly satisfies the following conditions:

(i) $j\,|_X$ is a finite branched covering over C of degree at most $| S_g |= g!$;

(ii) there exists a holomorphic surjective mapping $h : X \longrightarrow M$ (restriction to X of projection on a factor of $M^{\times g}$).

Let G' be the image of $\pi_1(X)$ under the composite homomorphism $\pi_1(X) \xrightarrow{j_*} \pi_1(C) \longrightarrow \pi_1(\mathbf{CT}^g)$. Clearly G' is of a finite index in $\pi_1(\mathbf{CT}^g)$. Consider the regular covering $X_{G'}$ over X with the transformation group G'.

Lemma 4.3. *There exist finite branched coverings $j' : X_{G'} \longrightarrow C_G$ and $h' : X_{G'} \longrightarrow M_G$ such that*

(i) *j' and h' are equivariant with respect to the action of G':*

$$j'(gz) = gj'(z) \quad \text{and} \quad h'(gz) = gh'(z) \quad (g \in G');$$

(ii) *j' is a covering to j.*

Proof. In what follows we think of M as a subvariety of $\mathbf{CT}^g$. Let $\eta_1, ..., \eta_g$ be a basis for the space $\Omega^1(\mathbf{CT}^g)$ of holomorphic 1-forms on $\mathbf{CT}^g$. Consider g-dimensional subspaces E_j and E_h of $\Omega^1(X)$ spanned by $(j)^*(\eta_1), ..., (j)^*(\eta_g)$ and $(h)^*(\eta_1), ..., (h)^*(\eta_g)$, respectively. Denote by F_j and F_h their complements in $\Omega^1(X)$. Then there exists a $\mathbf{C}$-linear isomorphism $\phi : \Omega^1(X) \longrightarrow \Omega^1(X)$ such that

$$\phi(F_h) = F_j, \quad \phi((h)^*(\eta_i)) := (j)^*(\eta_i), \quad (1 \le i \le g). \tag{4.3}$$

If $\omega_1, ..., \omega_s$ is a basis of F_h, then $\phi(\omega_1), ..., \phi(\omega_s)$ is a basis of F_j. Define now two lattices Γ_j and Γ_h of $\mathbf{C}^{g+s}$ by

$$\Gamma_j := \left\{g = \left(\int_\gamma j^*(\eta_1), ..., \int_\gamma j^*(\eta_g), \int_\gamma \phi(\omega_1), ..., \int_\gamma \phi(\omega_s)\right); \gamma \in H_1(X, \mathbf{Z})\right\};$$
$$\Gamma_h := \left\{g = \left(\int_\gamma h^*(\eta_1), ..., \int_\gamma h^*(\eta_g), \int_\gamma \omega_1, ..., \int_\gamma \omega_s\right); \gamma \in H_1(X, \mathbf{Z})\right\}.$$

Let $T_j := \mathbf{C}^{g+s}/\Gamma_j$ and $T_h := \mathbf{C}^{g+s}/\Gamma_h$ be the corresponding complex tori. The Albanese mappings constructed by the above two bases of $\Omega^1(X)$ imbed X into T_j and T_h, respectively. Denote the corresponding image of X in T_j by X_j and in T_h by X_h. In virtue of (4.3) the linear operator ϕ induces an isomorphism of $\Omega^1(T_h)$ onto $\Omega^1(T_j)$. Since the latter two spaces are naturally isomorphic to $\mathbf{C}^{g+s}$, we determine in this way the $\mathbf{C}$-linear automorphism ϕ' of $\mathbf{C}^{g+s}$. Let $I : \mathbf{C}^{g+s} \longrightarrow \mathbf{C}^{g+s}$ be the $\mathbf{C}$-linear operator conjugate to ϕ'. Then $I : \Gamma_j \longrightarrow \Gamma_h$ and therefore I determines the biholomorphic mapping I' which maps T_j onto T_h. Moreover, I' maps X_j onto X_h. Consider the induced homomorphisms $j_* : \pi_1(X_j) \longrightarrow \pi_1(\mathbf{CT}^g)$ and $(h \circ I')_* : \pi_1(X_j) \longrightarrow \pi_1(\mathbf{CT}^g)$ and show that

$$\mathrm{Ker}(j_*) = \mathrm{Ker}((h \circ I')_*). \tag{4.4}$$

Note that $h \in \mathrm{Ker}(j_*)$ if and only if

$$\int_{j(h)} \eta_i = 0 \quad (1 \le i \le g),$$

and $l \in \mathrm{Ker}((h \circ I')_*)$ if and only if

$$\int_{(h \circ I')(l)} \eta_i = 0 \quad (1 \le i \le g).$$

So for such l we have

$$0 = \int_{(h \circ I')(l)} \eta_i = \int_{I'(l)} h^*(\eta_i) = \int_l j^*(\eta_i) = \int_{j(l)} \eta_i$$

and hence $l \in \mathrm{Ker}(j_*)$. The inverse imbedding is proved similarly.

Since $j(X_j) = C$ and $(h \circ I')(X_j) = M$, equality (4.4) leads to existence of holomorphic mappings $j' : X_{G'} \longrightarrow C_G$ and $h' : X_{G'} \longrightarrow M_G$ that cover j and $h \circ I'$, respectively. These mappings clearly satisfy the statement of the lemma. □

We are now in a position to prove the theorem for the complex curve M. By Propositions 4.1 and 3.2 $\mathcal{P}_l(X_{G'})$ is finite dimensional. Moreover, by statement (i) of the previous lemma $(h')^*$ determines imbedding of $\mathcal{P}_l(M_G)$ into $\mathcal{P}_l(X_{G'})$. Therefore $\mathcal{P}_l(M_G)$ is finite dimensional, as well. Apply now to our situation the result of Theorem 1.1. The functions f_i of the theorem in this case coincide with the pullback of linear holomorphic functions defined on $\mathbf{C}^g$ by the mapping $A' : M_G \longrightarrow \mathbf{C}^g$ which covers the Albanese mapping A. From here it follows that every $f \in \mathcal{P}_l(M_G)$ is the pullback of a holomorphic polynomial of degree at most l on $\mathbf{C}^g$.

This completes the proof in the case of curves.

Let now $M \subset \mathbf{CP}^n$ be a projective manifold. Since the Albanese mapping from M into its Picard manifold (complex torus) induces a surjective homomorphism of $\pi_1(M)$ onto the maximal free abelian subgroup G of $H_1(M, \mathbf{Z})$, it suffices to prove the theorem for the covering M_G. By Bertini's theorem the generic subspace L of codimension $\dim_{\mathbf{C}} M - 1$ intersects M transversely in a smooth curve C and the imbedding $C \subset M$ induces the surjective homomorphism of fundamental groups. Moreover, the union U of such curves C is an open subset of M. Finally, according to Theorem 1.2 in the case of curves $C_G \in \mathcal{PF}$. So the conditions of Proposition 3.1 are fulfilled for the family of curves C and therefore $M_G \in \mathcal{PF}$, that is, the theorem is proved in this case, as well.

Finally, consider the general case of a compact complex manifold M and a holomorphic mapping $A : M \longrightarrow \mathbf{CT}^r$ which induces the surjective homomorphism of the corresponding fundamental groups. Show that this case can be reduced to the case of a manifold Y' which is a desingularisation $d_Y : Y' \longrightarrow Y$ of the image $Y := A(M)$. To accomplish this note that there exist a desingularisation $d_M : M' \longrightarrow M$ and a holomorphic surjective mapping $B : M' \longrightarrow Y'$ such that

$$d_Y \circ B = A \circ d_M.$$

It is clear that M' is birational equivalent to M and therefore $\pi_1(M') \cong \pi_1(M)$. Consider regular coverings M'_G and M_G with the transformation group $G := A_*(\pi_1(M))$ over M' and M, respectively. Then there exists covering to d_M holomorphic mapping $h : M'_G \longrightarrow M_G$ which is a birational isomorphism too. From here it follows that $(h)^* : \mathcal{P}_l(M_G) \longrightarrow \mathcal{P}_l(M'_G)$ is an isomorphism. Thus it suffices to prove the theorem for M'_G.

Let now (V, f_1, f_2) be the Stein factorization of $B : M' \longrightarrow Y'$, that is, V be a compact complex variety, $f_1 : M' \longrightarrow V$ be surjective with connected fibers and $f_2 : V \longrightarrow Y'$ be a finite branched covering such that $B = f_2 \circ f_1$. Because of surjectivity of A_* one can define correctly coverings V_G and Y'_G with the same transformation group G over V and Y', respectively. Then there exist covering to f_1 and f_2 mappings $f'_1 : M'_G \longrightarrow V_G$ and $f'_2 : V_G \longrightarrow Y'_G$ equivariant with respect to the actions of G. Every fiber of f'_1 is a connected compact complex variety and therefore $(f'_1)^*$ determines an isomorphism of $\mathcal{P}_l(V_G)$ onto $\mathcal{P}_l(M'_G)$. So it remains to proof that $V_G \in \mathcal{PF}$. But we can apply Proposition 3.2 (see also Remark 3.3) to the case of the finite branched covering $f_2 : V \longrightarrow Y'$. According to this proposition $V_G \in \mathcal{PF}$ if and only if $Y'_G \in \mathcal{PF}$. So we reduce the problem to the case of a desingularisation Y' of $Y = A(M)$. To choose the required desingularisation we make use of the next statement which follows directly from theorem 10.9 of [U].

There exists a desingularisation Y' of Y, a projective manifold Z and a holomorphic surjective mapping $g : Y' \longrightarrow Z$ such that g has connected fibers and the generic fiber of g is biholomorphic to a complex torus.

Taking the Y' from the statement and applying Theorem 1.2 to the case of projective manifolds and complex tori and then applying Proposition 3.3 to the mapping g we obtain that $Y'_G \in \mathcal{PF}$.

This completes the proof of the theorem. □

5. Proof of Corollaries.

Proof of Corollary 1.3. Let M_G be a regular covering over a compact Kähler manifold M with a free abelian transformation group G. Since $\mathcal{P}_l(M_G)$ is linearly imbedded into $\mathcal{P}_l(M_F)$, where F is the maximal free abelian subgroup of $H_1(M, \mathbf{Z})$ it suffices to consider the case $G = F$. The Albanese mapping $A : M \longrightarrow \mathbf{CT}^r$, where $r := \frac{1}{2}\dim_{\mathbf{C}} H_1(M, \mathbf{C})$ induces the surjective homomorphism A_* onto F. So Theorem 1.2 with this A imposes that every $f \in \mathcal{P}_l(M_G)$ has a form $(A')^*(p)$. Here $A' : M_G \longrightarrow \mathbf{C}^r$ is the covering of A and p is a holomorphic polynomial on $\mathbf{C}^r$ of degree at most l. $\square$

Proof of Corollary 1.4. Let M_G be a regular covering over a compact complex manifold M with $\tau(M) \geq \dim_{\mathbf{C}} M - 1$ and a free abelian transformation group G. Recall that $\tau(M)$ is the transcendency degree of the field of meromorphic functions on M. As above it suffices to consider the case where G is the maximal free abelian subgroup of $H_1(M, \mathbf{Z})$. First consider the case $\tau(M) = \dim_{\mathbf{C}} M$. Then M is birational equivalent to a compact projective manifold M' (see, e.g., [Sh], ch.8, sec.3) and, in particular, $\pi_1(M) \cong \pi_1(M')$. Applying Corollary 1.3 to the projective manifold M' we get the required result.

Let now $\tau(M) = \dim_{\mathbf{C}} M - 1$. Then there exists a modification $p : M' \longrightarrow M$ such that M' admits a holomorphic surjective mapping onto a projective manifold of complex dimension $\dim_{\mathbf{C}} M - 1$ and the generic fiber of this mapping is biholomorphic to an elliptic curve (see, e.g., [Sh], ch.8, sec.4). It remains to apply Theorem 1.2 in the case of projective manifolds and elliptic curves and then to use Proposition 3.3. $\square$

Proof of Corollary 1.5. Let M_G be a regular covering over a compact complex curve M of genus g. Here G is a free abelian transformation group which without loss of generality to be assumed maximal. We have to prove that every $f \in \mathcal{H}_l(M_G)$ can be represented as $(A')^*(p)$ with a harmonic polynomial p on $\mathbf{C}^g$ which is a sum of holomorphic and antiholomorphic ones of degrees at most l. Here $A' : M_G \longrightarrow \mathbf{C}^g$ is the covering of the Albanese mapping $A : M \longrightarrow \mathbf{CT}^g$. In what follows we regard M as a submanifold of $\mathbf{CT}^g$. Since $\mathbf{CT}^g$ is a complex abelian Lie group, there exists a neighborhood $U \subset \mathbf{CT}^g$ of M such that M is a holomorphic retract of U. Let U_G be the preimage of U in $\mathbf{C}^g$ with respect to the canonical covering $\mathbf{C}^g \longrightarrow \mathbf{CT}^g$ and $r : U_G \longrightarrow M_G$ be the corresponding retraction. Since f is a harmonic function on the complex curve M_G, it satisfies the equation $\partial\overline{\partial} f = 0$. In particular, f is locally a sum of holomorphic and antiholomorphic functions. Because of holomorphicity of r, $\partial\overline{\partial}(r^* f) = 0$. In addition, $r^* f$ is an l-growth function on U_G with respect to the Euclidean metric on $\mathbf{C}^g$. Then the Cauchy inequalities for derivatives of the holomorphic part of $r^* f$ show that the d-closed holomorphic 1-form $\alpha := \partial(r^*f)$ can be represented as follows:

$$\alpha = \sum_{i=1}^{g} a_i(z)dz_i,$$

where a_i is a holomorphic function of l-growth on U_G, $1 \le i \le g$. Theorem 1.2 implies that $a_i \mid_{M_G}$ is restriction to M_G of a holomorphic polynomial of degree at most l. Therefore $\Delta_{g_1,\dots,g_{l+1}}(a_i) = 0$, where $g_1, \dots, g_{l+1}$ belong to G. Since $f = r^*f \mid_{M_G}$ we get from here that $\partial(\Delta_{g_1,\dots,g_{l+1}}f) = 0$. The similar arguments applied to $\overline{\partial}(r^*f)$ lead to the equality $\overline{\partial}(\Delta_{g_1,\dots,g_{l+1}}f) = 0$. These two equalities imply $\Delta_{g_1,\dots,g_{l+1}}f$ is a constant on M_G and so every $f \in \mathcal{H}_l(M_G)$ is annihilated by any $(l+2)$-difference. Applying the arguments of the proof of Lemma 2.3 we conclude from here that $\mathcal{H}_l(M_G)$ is finite dimensional. In addition, the induction arguments of Theorem 1.1 show that every $f \in \mathcal{H}_l(M_G)$ can be written as $f = (A')^*(p)$ with a polynomial p on $\mathbf{C}^g$ of degree at most l.

It remains to prove that the p is sum of holomorphic and antiholomorphic polynomials. To this end note that the restrictions $a_i \mid_{M_G}$ of the coefficients of the above defined holomorphic 1-form α coincide with restrictions to M_G of holomorphic polynomials on $\mathbf{C}^g$. So we can extend $\alpha \mid_{M_G}$ to a holomorphic polynomial 1-form β and therefore the polynomial p satisfies

$$\partial p - \beta = 0 \quad \text{on} \quad M_G.$$

Check that $\partial p - \beta = \sum_{i=1}^{g} p_i(z)dz_i$ equals 0 identically on $\mathbf{C}^g$. Assume to the contrary that $p_{i_0} \neq 0$ for some i_0. Then there exists a difference operator $\Delta_{g_1,\dots,g_t}$ such that $\Delta_{g_1,\dots,g_t}(\partial p - \beta) \neq 0$ but $\Delta_g(\Delta_{g_1,\dots,g_t}(\partial p - \beta)) = 0$ for any $g \in G$. From here we get

$$\Delta_{g_1,\dots,g_t}(\partial p - \beta) = \sum_{i=1}^{g} c_i dz_i$$

with some constants c_i. But $(\partial p - \beta) \mid_{M_G} = 0$ and the forms $dz_1 \mid_{M_G}, \dots, dz_g \mid_{M_G}$ are linearly independent. Hence all $c_i = 0$ and we get a contradiction to the condition $\Delta_{g_1,\dots,g_t}(\partial p - \beta) \neq 0$.

Thus $\partial p - \beta = 0$ and, in particular, ∂p is a d-closed holomorphic 1-form on $\mathbf{C}^g$. Therefore the holomorphic function

$$h_1 := \int_{z_0}^{z} \partial p$$

is correctly defined on $\mathbf{C}^g$. Moreover, the growth conditions on p show that h_1 is a holomorphic polynomial. Then $h_2 := p - h_1$ is an antiholomorphic polynomial and therefore $p = h_1 + h_2$ is harmonic. □

Remark 5.1. The latter corollary can be extended with the same proof to the case of l-polynomial growth functions f on M_G with a compact Kähler manifold M satisfying $\overline{\partial}\partial f = 0$. In the case of curves the latter condition is equivalent to the harmonicity of f.

References.

[AL] M. Avallaneda and F.-H. Lin, *Une théorème de Liouville pour des équations elliptiques à coefficients périodiques*, Compt. Rendus Acad. Sci. Paris, Sér. I, **309** (1989), 245–250.

[CM] T. Colding and W. Minicozzi, *Weyl type bounds for harmonic functions*, preprint.

[FN] C. Fefferman and R. Narasimhan, *On the polynomial-like behavior of certain algebraic functions*, Ann. Inst. Fourier, Grenoble, **44** (1994), 1091–1179.

[G] F.R. Gantmaher, *Matrix Theory*, "Nauka", Moscow, 1967.

[Gu] Y. Guivarch, *Mouvement brownien sur les revêtements d'une variété compacte*, C.R. Acad. Sci. Paris, Sér. I, **292** (1981), 851–853.

[GH] Ph. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, N.Y., 1978.

[K] V. Kaimanovich, *Brownian motion and harmonic functions on covering manifolds. An entropy approach*, Soviet Mathematics Doklady, **33** (1986), 812–816.

[La] F. Lárusson, *An extension theorem for holomorphic functions of slow growth on covering spaces of projective manifolds*, J. of Geometric Analysis, **5** (1995), 281–291.

[L] V. Lin, *Liouville coverings of complex spaces and amenable groups*, Mathematics of the USSR-Sbornik, **60** (1988), 197–216.

[Li] P. Li, *Harmonic sections of polynomial growth*, Math. Research Letters, **4** (1997), 35–44.

[LS] T. Lyons and D. Sullivan, *Function theory, random paths and covering spaces*, J. of Differential Geometry, **19** (1984), 299–323.

[MS] J. Moser and M. Struwe, *On a Liouville-type theorem for linear and nonlinear elliptic differential equations on a torus*, ETH, Zürich, 1991.

[OV] A. Onishchik and E. Vinberg, *Seminar in Lie groups and algebraic groups*, Springer-Verlag, 1992.

[Sh] I. Shafarevich, *Basic Algebraic Geometry 2. Schemes and complex manifolds*, Springer-Verlag, 1994.

[U] K. Ueno, *Classification theory of algebraic varieties and compact complex surfaces*, Lecture Notes in Math. **439**, Springer-Verlag, 1975.

[Y] S.T. Yau, *Nonlinear Analysis in Geometry*, L'Enseignement Mathématique, Série des Conférences de l'Union Mathématique Internationale, No. 8, SRO-KUNDIG, Genève, 1986.

RECEIVED SEPTEMBER 10, 1996.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TORONTO
100 ST. GEORGE ST.
TORONTO, ON M5S 1A1
CANADA
E-MAIL: BRUDNYI@MAS.TORONTO.EDU