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Minimizing fibrations and p-harmonic maps in homotopy classes from S^3 into S^2

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We prove that, contrary to the case of maps from S^3 into S^3, there exist infinitely many homotopy classes from S^3 into S^2 having a minimizing 3-harmonic map. We prove that the first eigenforms of the linear operator $\Delta^{1/2} = d*$ on $(\mathrm{Ker}\, d) \cap \wedge^{2n} S^{4n-1}$ are stable for the associated conformal invariant non-linear variational problem and we deduce, in particular, that the Hopf map from S^3 into S^2 minimizes the p-energy in it's homotopy class for $p \geqslant 4$ and that it remains true locally for $3 \leqslant p < 4$. We prove that the Hopf map minimizes the p-energy for $p \geqslant 3$ among a class of symmetric fibrations from S^3 into S^2.

I. Introduction.

Denote by g_0 the standard metric on S^n and by ω_{S^n} the associated volume form. For a map u between S^m and S^n we consider the p-energy defined by

$$E_p(u) = \int_{S^m} |\nabla u|^p_{g_0} \, dvol_{g_0}.$$

For $m = n$ denote by $\deg u$ the topological degree of u. Since we have

$$\text{(I.1)} \qquad m^{m/2} \, |u^* \omega_{S^m}| \leqslant |\nabla u|^m ,$$

we get

$$\text{(I.2)} \qquad |S^m| \; |\deg u| = \left| \int_{S^m} u^* \omega_{S^m} \right| \leqslant \frac{1}{m^{\frac{m}{2}}} \, E_m(u),$$

where $|S^m|$ denotes the standard volume of S^m and equality holds if and only if u is constant or conformal. From (I.2) one deduces that

$$\text{(I.3)} \qquad \forall d \in \mathbb{Z} \qquad E^d_m = \inf_{\deg u = d} E_m(u) = m^{\frac{m}{2}} |S^m| \times |d|$$

and, for $m > 2$, E_m^d is achieved if and only if $d = 0, -1, +1$ and u is respectively constant or conformal.

In this paper we will focus on the other case where there are infinitely many homotopy classes between spheres, that is, $m = 2n - 1$ and $n = 2p$.

Let u be a smooth map from S^{2n-1} into S^n. Since $u^*\omega_{S^n}$ is a closed form and since $H^n(S^{2n-1}) = 0$ there exists $\eta \in \wedge^{n-1}(S^{2n-1})$ such that $d\eta = u^*\omega_{S^n}$. The Hopf degree of u is the following quantity

$$H(u) = \frac{1}{|S^n|^2} \int_{S^{2n-1}} \eta \wedge u^*\omega_{S^n}. \tag{I.4}$$

$H(u)$ is an integer which does not depend on u in a given homotopy class of $\pi_{2n-1}(S^n)$. For the convenience of the reader we recall, in the preliminaries, some of the topological interpretations of H.

In the case $n = 2$ we have $\pi_3(S^2) = \mathbb{Z}$ and $H(u)$ coincides with the homotopy class of u. As in the $S^3 - S^3$ problem, we are interested with the following energy levels for maps from S^3 into S^2

$$\mathcal{E}_d^m = \inf \left\{ E_m(u); \quad u : S^3 \to S^2, \quad H(u) = d \right\}. \tag{I.5}$$

It is clear that for $m < 3$ (as in the $S^3 - S^3$ problem) we have $\mathcal{E}_d^m = 0$ (use the action of the conformal group of S^3 which is not compact).
Consider now $m \geqslant 3$ and take u such that $H(u) = d$. Let ξ be the closed 2-form such that

$$\Delta_{S^3}\xi = u^*\omega_{S^2}. \tag{I.6}$$

We have

$$H(u) = \frac{1}{|S^2|^2} \int_{S^3} d^*\xi \wedge u^*\omega_{S^2}. \tag{I.7}$$

Using classical results on elliptic operators we have

$$\int_{S^3} |d^*\xi|^3 \leqslant C \left(\int_{S^3} |u^*\omega_{S^2}|^{\frac{3}{2}} \right)^2 \tag{I.8}$$

and using the Hölder inequality we get

$$|H(u)| \leqslant C \left(\int_{S^3} |\nabla u|^3 \right)^{\frac{4}{3}}. \tag{I.9}$$

This implies clearly that for $d \neq 0$ and $m \geqslant 3$ we have $\mathcal{E}_d^m > 0$. We are tempted to compare the problem of computing the energy levels E_d^m and the energy levels $\mathcal{E}_d^m$. The first difference comes from the fact that we used a non local operation for finding $\eta = d^*\xi$ and that one cannot hope to get a local upper bound of $\eta \wedge u^*\omega_{S^2}$ by $|\nabla u|^3$ like in (I.1). This makes the second problem much more delicate than the first one. To illustrate this one has to think of the fact that, countrary to the scalar case, the best Sobolev constant in (I.8) is not known already. The second difference comes from the power $\frac{4}{3}$ that we have in the upper-bound (I.9) compare to the power 1 in (I.3). This power $\frac{4}{3} > 1$ could be not optimal at all and could be a consequence of the method we used for establishing the upper-bound . One way for getting a Hopf degree d map, u_d, from S^3 into S^2, is to collapse d exemplars of an Hopf degree 1 map u (see below) that we have contracted before using a conformal dilation (This is what we have done in the S^3-S^3 case for realizing E_3^d). For such construction we have $E_3(u_d) \geqslant E_3(u)\, d$ and one can deal sufficiently carefully in such a way to get

$$E_3(u_d) \underset{+\infty}{\simeq} E_3(u)d \underset{+\infty}{\simeq} Cd.$$

In fact this way of constructing a Hopf degree d map in view of minimizing E_3 in the S^3-S^2 problem is not optimal at all. Using maps whose coimages of points have self-linked connected components (see part III), we prove that the exponent 4/3 we got by the non-local operation (I.6), (I.7), (I.8) and (I.9) is the best that we can get. Precisely we establish the following fact

$$\text{(I.10)} \qquad \frac{\log \mathcal{E}_3^d}{\log d} \longrightarrow \frac{3}{4} \qquad \text{as } d \to +\infty$$

and contrary to the $S^3 - S^3$ problem (see (I.3)) the infimum of the conformal energy is not proportional to the degree. This fact, combined with the concentration-compactness method developped in particular in [19], yields to the following result proved in part III.

Theorem I.1. *There exists infinitely many homotopy classes of* $\pi_3(S^2)$ *having a minimizing 3-harmonic map.*

Remark I.1. The question about which homotopy class admits a minimizing 3-harmonic map is still open.

In parts IV and V of the paper we ask the question whether the Hopf map

$$H:\ S^3 \subset \mathbb{R}^4 \simeq \mathbb{C}\oplus\mathbb{C} \longrightarrow S^2 \subset \mathbb{R}^3 \simeq \mathbb{R}\oplus\mathbb{C}$$
$$(u,v) \longrightarrow \left(|u|^2-|v|^2, 2u\overline{v}\right)$$

plays, in some sense, in the S^3-S^2 problem, the role played by the identity in the S^3-S^3 problem. In part IV we give to this question a partial answer.

Theorem I.2. *The Hopf map is the only minimizer of the p-energy for $p \geqslant 4$ in its homotopy class modulo the action of the positive isometries in S^3. Moreover, there exists a neighborhood V of H for the C^1-topology such that, H is the only minimizer of the p-energy for $p \geqslant 3$ modulo the action of the positive isometries for $p > 3$ and the action of the positive conformal group of S^3 for $p = 3$.*

Remark I.2. It is clear that H is no more a local minimizer of the p-energy when $p < 3$, to see it , it suffices to compose H with dilations. (see the computation of the index of H for the 2-energy in [21], see also [6]).

Remark I.3. See [2] for other minimizing properties of the Hopf map relative to p-energies.

The idea for proving theorem I.2 is to observe, first, that for any map from S^3 into S^2 we have, for $q > 1$,

$$\text{(I.11)} \qquad \int_{S^3} |u^*\omega_{S^2}|^q \leq \frac{1}{2^q}\int_{S^3} |\nabla u|^{2q}$$

and that equality holds in (I.11) for $u = H$, that is to say that H is transversally conform. In view of the previous remark we are tempted to introduce the following variational problem with constraint for any $p \geqslant 1$ and $q \geqslant (4p-1)/2p$

(I.12)
$$\mathcal{I}_q = \inf\left\{\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} |d\eta|^q,\ \eta \in W^{1,q}\left(\wedge^{2p-1}S^{4p-1}\right),\right.$$
$$\left.\text{s.t. } \frac{2p}{|S^{4p-1}|}\int_{S^{4p-1}} \eta\wedge d\eta = 1\right\}$$

where $W^{1,q}(\wedge^{2p-1}S^{4p-1})$ is the Sobolev space of the $2p-1$-forms of S^{4p-1} having derivatives in L^q. Remark that the constraint $2p/|S^{4p-1}|\int_{S^{4p-1}}\eta\wedge d\eta = 1$ only depends on $d\eta$ like also $\int_{S^{4p-1}}|d\eta|^q$. Thus for a given $d\eta$ we

often use the "Coulomb Gauge" η verifying $d^*\eta = 0$ to make the problem coercive. For $q > (4p-1)/2p$ the existence of a minimizer is given by the compactness of the injections $W^{1,q} \hookrightarrow L^{\frac{4p-1}{2p-1}}$. For $q = (4p-1)/2p$ the existence of a minimizer is not so direct, one has to use the classical ideas of the concentration-compactness method of P.L.Lions (see [15]) to get the existence of a minimizer. Note that the problem $\mathcal{I}_{2-1/2p}$, the conformal invariant one, only depends on the conformal class of the metric we have chosen on S^{4p-1} and is in itself interesting.
Let $I = \{i_1, \ldots, i_{2p}\}$ be a choice of $2p$ integers in $\{1, \ldots, 4p\}$ and denote by η_I^+ the following $(2p-1)$-form of $\wedge^{2p-1}S^{4p-1}$

$$\eta_I^+ = \frac{1}{2p}\sum_{s=1}^{2p}(-1)^{s+1}\, x_{i_s} dx_{i_1} \wedge \cdots \wedge d\check{x}_{i_s} \wedge \cdots \wedge dx_{i_{2p}}$$

$$+ \frac{1}{2p}\sum_{s=1}^{2p}(-1)^{s+1}\, x_{j_s}\, dx_{j_1} \wedge \cdots \wedge d\check{x}_{j_s} \wedge \cdots \wedge dx_{j_{2p}}$$

where

$$dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} = *dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} \quad \text{in} \quad \mathbb{R}^{4p}.$$

We prove in part IV that theorem I.2 is a consequence of the following result that we prove in part IV also.

Theorem I.3. *The restrictions of the linear combinations of the η_I^+ to S^{4p-1} are the only minimizers of $\mathcal{I}_q$ for $q \geqslant 2$ moreover there exists a neighborhood V of $\eta_{I_0}^+$ for the C^1 norm, such that, $\eta_{I_0}^+$ minimizes $\mathcal{I}_q$ in V for $q \geqslant 2 - \frac{1}{2p}$ and it is the only minimizer of $\mathcal{I}_q$ in V modulo the linear combinations of the η_I^+ for $q > 2 - \frac{1}{2p}$ and modulo the linear combinations of the η_I^+ and their pull-back by the positive conformal diffeomorphism of S^{4p-1} for $q = 2 - \frac{1}{2p}$.*

Remark I.4. The question to know wether the η_I^+ minimize globally $\mathcal{I}_{2-\frac{1}{2p}}$ or not is still open.

Remark I.5. The proof of theorem I.2 uses essentially the decomposition of the closed $2p$-forms in the L^2 eigenbasis of $\Delta_{S^{4p-1}}$ on $\operatorname{Ker} d \cap \wedge^{2p}S^{4p-1}$ and the fact that the constraint in (I.12) has a very simple expression in this basis. Moreover we use also an identification of the second eigenspace of $\Delta_{S^{4p-1}}$ on $\operatorname{Ker} d \cap \wedge^{2p}S^{4p-1}$ with the Lie derivatives of the first eigenforms by the conformal Killing fields which are not pure (see proposition IV.3).

Remark I.6. The link between theorem I.3 and theorem I.2 is made by identifying $H^*\omega_{S^2}$ with the restriction to S^3 of $4\,(dx_1\wedge dx_2+dx_3\wedge dx_4)$ (see the end of part IV).

In part V of this paper we try to give a more geometric interpretation of the conformal invariant 3-energy in the first homotopy classes of $\pi_3(S^2)$. We give a lower bound of the 3-energy among fibrations from S^3 into S^2 in term of the Teichmüller classes of the coimages of the great circles of S^2 (see proposition IV.2). This lower bound is optimal: it is an equality for the composition of the Hopf fibration with the conformal diffeomorphims of S^3. This approach allow us to prove that the Hopf map minimizes the 3-energy among the class of symmetric fibrations that we define in part V (see theorem V.1). This enforces the conjecture that H minimizes the 3-energy in it's homotopy class.

II. Preliminaries.

We recall some basic facts concerning the Hopf degree.

In the definition of the Hopf degree (I.4) that we gave in the introduction we used the volume-form on S^n, but it can be replaced by any generators of $H^n_{dR}(S^n)$. This implies that, if ω_1 and ω_2 are two n-forms on S^n such that $\int_{S^n}\omega_1=\int_{S^n}\omega_2=1$, if u is a map from S^{2n-1} into S^n and let η_1, η_2 be two $(n-1)$-forms of $\wedge^{n-1}(S^{2n-1})$ such that $d\eta_i=u^*\omega_i$ $(i=1,2)$, we have

$$\text{(II.1)}\qquad H(u)=\int_{S^{2n-1}}\eta_1\wedge u^*\omega_2=\int_{S^{2n-1}}\eta_2\wedge u^*\omega_1.$$

See [1] page 230. It is proved also in [1] that if n is odd then $H(u)$ is allways equal to zero and in [10] one can find a proof of the fact that when n is even $H(\,.\,)$ can take infinitely many integer values. Thus, in the remain of the paper, we will always suppose that n is even, equal to $2p$.

We give now three different topological and geometrical interpretations of $H(u)$.

II.1. The Hopf degree as a n-topological degree.

Let ω_1 and ω_2 be two generators of $H^n_{dR}(S^n)$ having disjoint supports, let u be a map from S^{2n-1} into S^n, x a regular point in the support of ω_1 and Σ

a n-submanifold of S^{2n-1} such that $\partial\Sigma = u^{-1}(\{x\})$. Let $\eta_1 \in \wedge^{n-1}(S^{2n-1})$ be such that $d\eta_1 = u^*\omega_1$, it is proved in [1] page 232 that η_1 is the Poincaré dual of Σ in the compactly supported cohomology $H_c(S^{2n-1}\backslash u^{-1}(\operatorname{supp}\omega_1))$. Thus we have

$$\text{(II.2)} \qquad \begin{aligned} \int_{S^{2n-1}} \eta_1 \wedge u^*\omega_2 &= \int_{S^{2n-1}\backslash u^{-1}(\operatorname{supp}\omega_1)} \eta_1 \wedge u^*\omega_2 \\ &= \int_\Sigma u^*\omega_2. \end{aligned}$$

This says that $H(u)$ is the topological degree of the restriction of u to Σ into S^n.

II.2. The Hopf degree as a Link.

Let x and y be two distinct regular points of u in S^n and $B_\delta(x)$, $B_\delta(y)$ be two disjoint geodesic balls of center x and y. Let Π be the stereographic projection from S^{2n-1} to $\mathbb{R}^{2n-1}$ relative to a point which is not in $B_\delta(x) \cup B_\delta(y)$. We still denote u the map $u \circ \Pi^{-1}$. The link $L(x,y)$ of the coimage of x and y by u is, by definition, the degree of the map

$$\begin{aligned} L:\ u^{-1}(\{x\}) \times u^{-1}(\{y\}) \subset \mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1} &\longrightarrow S^{2n-2} \\ (\xi,\zeta) &\longrightarrow \frac{\xi-\zeta}{|\xi-\zeta|} \end{aligned}$$

It is clear that, if x' and y' are two regular points of u in $B_\delta(x)$ and $B_\delta(y)$ we have $L(x,y) = L(x',y')$. Just deform isotopically in $\mathbb{R}^{2n-1} \times \mathbb{R}^{2n-1}$ $u^{-1}(\{x\}) \times u^{-1}(\{y\})$ into $u^{-1}(\{x'\}) \times u^{-1}(\{y'\})$.

We present, now, the computations in the case $n = 2$. They can be established in the same way in the general case. By definition, for any couple (x',y') of regular points in $B_\delta(x) \times B_\delta(y)$, we have

$$\text{(II.3)} \qquad L(x',y') = \frac{1}{|S^2|} \iint_{u^{-1}(\{x'\}) \times u^{-1}(\{y'\})} \left(\frac{\xi-\zeta}{|\xi-\zeta|}\right)^* \omega_{S^2}.$$

After some computations this gives

(II.4)

$$L(x',y') = \frac{1}{|S^2|} \iint_{u^{-1}(\{x'\}) \times u^{-1}(\{y'\})} \tau_1(\xi) \wedge \tau_2(\zeta) \cdot \left(\frac{\xi-\zeta}{|\xi-\zeta|^3}\right),$$

where τ_1 and τ_2 are the tangential vectors along $u^{-1}(\{x'\})$ and $u^{-1}(\{y'\})$. If we choose the normal parametrisations of the curves, τ_1 and τ_2 are unit vectors and coincide with the vectors associated to the two forms

$$\frac{u^*\omega_{S^2}}{|u^*\omega_{S^2}|}(\xi) \qquad \text{and} \qquad \frac{u^*\omega_{S^2}}{|u^*\omega_{S^2}|}(\zeta).$$

Denote by $D(u)$ the following vector-field

$$D(u) = \sum_{i=1}^{3} \langle u^*\omega_{S^2};\, dx_{i+1} \wedge dx_{i-1}\rangle\, e_i,$$

where e_i is the canonical basis in $\mathbb{R}^3$. We have $\tau = \frac{D(u)}{|D(u)|}$.

Integrating (II.4) on $B_\delta(x) \times B_\delta(y)$ and using the coarea formula of Federer we get

(II.5)

$$\begin{aligned} &|B_\delta|^2\, L(x,y) \\ &= \frac{1}{|S^2|}\int_{u^{-1}(B_\delta(x))}\int_{u^{-1}(B_\delta(y))} D(u)(\xi)\wedge D(u)(\zeta)\,.\left(\frac{\xi-\zeta}{|\xi-\zeta|^3}\right) d\xi\, d\zeta \\ &\int_{u^{-1}(B_\delta(x))} D(u)(\xi)\,.\, \mathrm{rot}\left[\int_{u^{-1}(B_\delta(y))} -\frac{1}{4\pi}\frac{1}{|\xi-\zeta|}\, D(u)(\zeta)\, d\zeta\right]. \end{aligned}$$

Moreover one verifies that

$$\text{(II.6)} \qquad \mathrm{div}\left(\int_{u^{-1}(B_\delta(y))} -\frac{1}{4\pi}\frac{1}{|\xi-\zeta|}\, D(u)(\zeta)\, d\zeta\right) = 0.$$

Indeed this corresponds to integrate the gradient $\nabla\frac{1}{|\xi-\zeta|}$ along the coimages of the points of $B_\delta(y)$ which are closed curves. Thus we have on $B_\delta(y)$

(II.7)

$$\mathrm{rot}\,\mathrm{rot}\left[\int_{u^{-1}(B_\delta(y))} -\frac{1}{4\pi}\frac{1}{|\xi-\zeta|}\, D(u)(\zeta)\, d\zeta\right] = D(u)(\xi).$$

Combining (II.5) and (II.7) we have, replacing vector-fields by associated forms,

$$|B_\delta|^2\, L(x,y) = \int_{3} u^*\omega_1 \wedge \eta_2,$$

where η_2 is a 1-form verifying $d\eta_2 = u^*\omega_2$ and where

$$\omega_1 = \omega_{S^2} \times \Xi_{B_\delta(x)}, \qquad \omega_2 = \omega_{S^2} \times \Xi_{B_\delta(y)}$$

(Ξ_A denotes the caracteristic function of the set A). This yields the following result

$$L(x,y) = H(u). \tag{II.8}$$

II.3. The Hopf degree as a $2n-1$-topological degree for $n=2$.

Let u be a map from S^3 into S^2. Denote by $\tilde{u}$ the lift of u between the fiber bundle $u^{-1}H$, the pull-back by u of the Hopf bundle H, and the fiber bundle $H:\ S^3 \to S^2$. $u^{-1}H$ is a bundle with fiber S^1 and base S^3. Denote by Π the projection of this bundle. Such bundle is necessary trivial, $u^{-1}H \simeq S^3 \times S^1$ (because $\pi_2(S^1) = 0$), and $\tilde{u}$, restricted to $S^3 \times \{point\}$, realizes a map ϕ from S^3 into S^3 such that

$$H \circ \phi = u. \tag{II.9}$$

Let K be a $(n-1)$-form verifying $dK = H^*\omega_{S^2}$, since H has Hopf degree 1, we have

$$1 = \frac{1}{|S^2|^2}\int_{S^3} K \wedge H^*\omega_{S^2}. \tag{II.10}$$

In the other hand, because of (II.9), we have $u^*\omega_{S^2} = \phi^* H^*\omega_{S^2}$ and $\eta = \phi^* K$ verifies $d\eta = \phi^* dK = u^*\omega_{S^2}$. Thus we have

(II.11)
$$H(u) = \frac{1}{|S^2|^2}\int_{S^3} \eta \wedge u^*\omega_{S^2} = \frac{1}{|S^2|^2}\int_{S^3} \phi^*\left(K \wedge H^*\omega_{S^2}\right).$$

(II.10) implies in particular that $\frac{1}{|S^2|^2} \times K \wedge H^*\omega_{S^2}$ is a generator of $H^3_{dR}(S^3)$ and (II.11) implies

$$H(u) = \deg \phi. \tag{II.12}$$

III. Minimizing 3-harmonic maps in the homotopy classes of $\pi_3(S^2)$.

This part is devoted to the proof of theorem I.1 stated in the introduction. First of all we prove the following key lemma.

Lemma III.1. *Let $\mathcal{E}_d$ be the infimum of the 3-energy among maps from S^3 into S^2 having Hopf degree d, we have*

$$\log \mathcal{E}_d \underset{+\infty}{\sim} \frac{3}{4} \log d \tag{III.1}$$

Proof. In the introduction, using Sobolev embedding and Hölder inequality we have established that there exists C such that

$$|d| \leqslant C\,(\mathcal{E}_d)^{\frac{4}{3}}\,. \tag{III.2}$$

It suffices to prove that this inequality is asymptotically optimal in the following sense: there exists C' such that

$$\mathcal{E}_d \leqslant C'\,|d|^{\frac{3}{4}}\,. \tag{III.3}$$

Recall the following property verified by the Hopf degree. Let u_0 be a map from S^3 into S^2 having Hopf degree $H(u_0)$ and let v be a map from S^2 into S^2 having a topological degree denoted by $\deg v$, we have

$$H(v \circ u_0) = (\deg v)^2\, H(u_0)$$

Indeed, if ω denotes a generator of $H^2(S^2)$ and $\omega' = v^*\omega$ we have, by definition, $H(v\circ u_0) = \eta' \wedge u_0^*\,\omega'$ where $d\eta' = u_0^*\,\omega'$ and if $\deg v \neq 0$, $\omega' / \int_{S^2} \omega'$ is a generator of $H^2(S^2)$ and we have

$$H(u_0) = \frac{1}{\left(\int_{S^2} \omega'\right)^2} \int_{S^3} \eta' \wedge u_0^*\,\omega' = \frac{H(v\circ u_0)}{(\deg v)^2}.$$

Thus, if H denotes the complex Hopf map $H(v \circ H) = (\deg v)^2$. Using the coarea formula of Federer we have

$$\int_{S^3} |\nabla(v\circ H)|^3 = \int_{y\in S^2} \int_{H^{-1}(y)} \frac{|\nabla(v\circ H)|^3}{|H^*\omega_{S^2}|}$$

And since H is transversally conform, with uniform gradient $|\nabla H| \equiv 2\sqrt{2}$ in S^3, we get

(III.4)

$$\int_{S^3} |\nabla(v\circ H)|^3 = \int_{y\in S^2} \int_{H^{-1}(y)} |\nabla v|^3\,(y) \times 2 = 4\pi \int_{S^2} |\nabla v|^3\,.$$

We claim that there exists $C > 0$ such that

(III.5) $$\min_{\deg v = n} \int_{S^2} |\nabla v|^3 \leqslant C n^{\frac{3}{2}}.$$

Let v_0 be a degree 1 map from S^2 into S^2 constant in the south hemisphere S^2_- and denote by v_0^ε the map $v_0 \circ D^\varepsilon$ where D^ε is the dilation relative to the north pole which sends the geodesic ball $B_\varepsilon(North)$ in the north hemisphere.
We clearly have $|\nabla v_0^\varepsilon| \leqslant C/\varepsilon$. Moreover there exists $\lambda > 0$ such that, for any integer $n > 0$ there are n disjoint geodesic balls of radius $\lambda/\sqrt{n}$ included in S^2. Let v_n be the map which coincides with $v_0^{\lambda/\sqrt{n}}$ in each of those geodesic balls and constant elsewhere. We clearly have $\deg v_n = n$ and $|\nabla v_n| \leqslant C\sqrt{n}$. This implies (III.5). Combining (III.4) and (III.5) we get

(III.6) $$\min_{\deg v = n} \int_{S^3} |\nabla (v \circ H)|^3 \leqslant C n^{\frac{3}{2}}.$$

This implies

(III.7) $$\mathcal{E}_{n^2} \leqslant C n^{\frac{3}{2}} = C \left(n^2\right)^{\frac{3}{4}}.$$

Thus (III.3) is proved for $d = n^2$.

Let d be any integer and n such that $n^2 \leqslant d < (n+1)^2$, we have $d - n^2 \leqslant 2n$. Consider a map u from S^3 into S^2 having Hopf degree d, one can insert to u, at $d - n^2$ different points the negative Hopf map H^- by using less energy than

$$E_3(u) + \left(d - n^2\right) E_3\left(H^-\right) + \varepsilon$$

for any $\varepsilon > 0$. This is directly linked to the conformal invariance of the 3-energy on S^3. This implies

$$\mathcal{E}_d \leqslant \mathcal{E}_{n^2} + C\left(d - n^2\right) \leqslant C d^{\frac{3}{4}} + C d^{\frac{1}{2}} \leqslant C' d^{\frac{3}{4}}.$$

lemma III.1 is proved. □

We now prove the following proposition which is a consequence of the standard technics of concentration compactness whose reference papers are for instance [19] and [15].

Proposition III.1. *Let d be an integer. There exists a finite sequence of integers $d_1, \cdots, d_l$ such that $\sum_{i=1}^{l} d_i = d$ and a finite sequence of maps from S^3 into S^2 $v_1, \cdots, v_l$ such that*

$$\text{(III.8)} \qquad \begin{aligned} H(v_i) = d_i \qquad & E_3(v_i) = \mathcal{E}_{d_i} \\ \textit{and} \quad \mathcal{E}_d = & \sum_{i=1}^{l} E_3(v_i). \end{aligned}$$

Proof. Let u_n be a minimizer of $E_{3+\frac{1}{n}}$ among the maps from S^3 into S^2 having Hopf degree d. Such u_n exists because of the compactness of the injection of $W^{1,\frac{3}{2}+\frac{1}{2n}}(S^3, S^2)$ into $L^3(S^3, S^2)$ and the constraint $\int_{S^3} \eta \wedge u^*\omega = d$ becomes subcritical. We claim that u_n is a minimizing sequence of E_3 in the homotopy class considered. If not, then there would exist a map v in this homotopy class such that

$$\text{(III.9)} \qquad E_3(v) < \underline{\lim}\, E_3(u_n) \leqslant \underline{\lim}\, E_{3+\frac{1}{n}}(u_n)$$

Since the regular maps are dense in $W^{1,3}(S^3, S^2)$ (see [20]), we can always suppose that v is regular. We clearly have $E_{3+\frac{1}{n}}(v) \longrightarrow E_3(v)$ and this contradicts the strict inequality (III.9) because u_n minimizes $E_{3+\frac{1}{n}}$ in the d-homotopy class. Thus u_n is a minimizing sequence of E_3 in this homotopy class.

It is clear that u_n is a locally minimizing $3 + \frac{1}{n}$-harmonic map and from classical results of the regularity theory for p-harmonic map, since u_n is $C^{0,\beta}$ we have that u_n is at least $C^{1,\alpha}$ for some $0 < \alpha < 1$ (see [9]). The proof of proposition III.1 is essentially based on the following concentration compactness result for the sequence u_n whose ideas of the proof are developed first in [19] for harmonic maps in dimension 2 and are adapted to the p-harmonic map case in [17].

Theorem III.1. [19], [17]. *Let u_n be the sequence defined above. One can extract a subsequence, still denoted u_n, such that there exists a finite sequence of points $\{x_1, \ldots, x_k\}$ (possibly empty) and a finite sequence of positive real numbers $\{\mu_1, \ldots, \mu_k\}$ such that*

(i) *u_n converges to some u in the C^1-Topology on any compact set of $S^3 \setminus \{x_1, \ldots, x_k\}$.*

(ii) *The measure $|\nabla u_n|^3$ on S^3 converges weakly to $|\nabla u|^3 + \sum_{i=1}^{k} \mu_i \delta_{x_i}$.*

(iii) *u is a C^1 3-harmonic map from S^3 into S^2.*

The proof of theorem III.1 is not straightforward at all. The proof of (i) and (ii) can be carried out following step by step the arguments of the proof of theorem 2 in [17] (but for $n \leqslant p < n+1$). The fact that u is a C^1 3-harmonic map in $S^3 \setminus \{x_1, \ldots, x_k\}$ is a consequence of (i) and finally (iii) is a consequence of the previous fact and the singularity removability result for p-harmonic maps of Duzaar and Fuchs (see [3]).

From now on u_n denotes the extracted subsequence having the convergence given by theorem III.1. Let η_n be the coclosed 1-form such that $d\eta_n = u_n^*\omega$, where $\omega = \omega_{S^2}/|\omega_{S^2}|$. We claim that there exists a sequence $\{\nu_1, \ldots, \nu_k\}$ of non zero integer such that

$$(*)\eta_n \wedge d\eta_n \longrightarrow (*)\eta \wedge d\eta + \sum_{i=1}^{k} \nu_i \delta_{x_i} \tag{III.10}$$

where η is the coclosed 1-form on S^3 verifying $d\eta = u^*\omega$. Since $u_n^*\omega$ weakly converges to $u^*\omega$ in $L^{\frac{3}{2}}(S^3)$, η_n strongly converges to η in L^p for any $p < 3$. Because of the strong convergence of $d\eta_n = u_n^*\omega$ to $d\eta = u^*\omega$ in

$$L^{\infty}_{loc}\left(S^3 \setminus \{x_1, \ldots, x_k\}\right)$$

we have, for $p < 3$,

$$(*)\eta_n \wedge d\eta_n \longrightarrow (*)\eta \wedge d\eta \qquad \text{in} \quad L^p_{loc}\left(S^3 \setminus \{x_1, \ldots, x_k\}\right) \tag{III.11}$$

Thus the signed measure $(*)\eta_n \wedge d\eta_n - (*)\eta \wedge d\eta$ converges to a measure of the form $\sum_{i=1}^k \nu_i \delta_{x_i}$. We shall prove now that ν_i are non zero integers.

Let δ be a positive real number chosen sufficiently small such that $x_i \notin \overline{B}_\delta(x_1)$ for $i \geqslant 2$ where $B_\delta(x_1)$ denotes the geodesic ball of center x_1 and radius δ in S^3. Since u_n converges for the C^1-norm to u on $\partial B_\delta(x_1)$ one can modify u_n in $B_\delta(x_1)$ in such a way that the modified sequence $\tilde{u}_n$ verifies

$$\begin{cases} \tilde{u}_n = u_n & \text{in} \quad S^3 \setminus B_\delta(x_1) \\ \tilde{u}_n \to u & \text{in} \quad C^1_{loc}\left(S^3 \setminus \{x_2, \ldots, x_k\}\right) \end{cases} \tag{III.12}$$

Let $\tilde{\eta}_n$ be the coclosed 1-form on S^3 such that $d\tilde{\eta}_n = \tilde{u}_n^*\omega$. We have, like previously, the following weak convergence of the measures

$$(*)\tilde{\eta}_n \wedge d\tilde{\eta}_n \longrightarrow (*)\eta \wedge d\eta + \sum_{i=2}^{k} \tilde{\nu}_i \delta_{x_i}. \tag{III.13}$$

But we have the following elliptic equation

(III.14)
$$\begin{cases} d\tilde{\eta}_n - \eta_n = 0 & \text{in} \quad S^3 \setminus B_\delta(x_1) \\ d^*\tilde{\eta}_n - \eta_n = 0 & \text{in} \quad S^3 \setminus B_\delta(x_1) \end{cases}$$

Moreover $\tilde{\eta}_n - \eta_n$ strongly converges to 0 in $L^p(S^3)$ for $p < 3$. By classical elliptic estimates we have that $\tilde{\eta}_n - \eta_n$ converges to 0 in $C^1(S^3 \setminus B_\delta(x_1))$ thus combining this fact, (III.10) and (III.12) we get $\tilde{\nu}_i = \nu_i$ for $i \geqslant 2$.
Using the preliminaries on the Hopf degree we have

(III.15)
$$\forall n \in \mathbb{Z} \qquad \int_{S^3} \eta_n \wedge d\eta_n - \int_{S^3} \tilde{\eta}_n \wedge d\tilde{\eta}_n \in \mathbb{Z}$$

Combining (III.15), and the convergence of the measure

$$(*)\, (\eta_n \wedge d\eta_n - \tilde{\eta}_n \wedge d\tilde{\eta}_n)$$

to $\nu_1\, \delta_{x_1}$ we get $\nu_1 \in \mathbb{Z}$.

We claim that

(III.16)
$$\forall 1 \leqslant i \leqslant k \qquad \mu_i = \mathcal{E}_{\nu_i}.$$

Let us prove it for $i = 1$. Let α_n be a sequence of positive number tending to zero, chosen in such a way that

(III.17)
$$|u(x) - u(x_1)| + |\nabla u(x) - \nabla u(x_1)| \leqslant \frac{1}{n} \qquad \text{in} \quad B_{\delta_n}(x_1).$$

Let N_n be a sequence of integers chosen such that

(III.18)
$$\forall p \geqslant N_n \quad \|u_p(x) - u(x)\| \leqslant \frac{1}{n} \qquad \text{in} \quad B_\delta(x_1) \setminus B_{\delta_n}(x_1).$$

We can modify u_{N_n} in $\hat{u}_{N_n}$ such that

(III.19)
$$\begin{cases} \hat{u}_{N_n} = u_{N_n} & \text{in} \quad B_{\delta_n}(x_1) \\ \hat{u}_{N_n} = u(x_1) & \text{in} \quad S^3 \setminus B_{\delta_n + \frac{1}{n}}(x_1) \\ \text{and } |\nabla \hat{u}_{N_n}|_\infty \leqslant C & \text{in} \quad B_{\delta_n + \frac{1}{n}}(x_1) \setminus B_{\delta_n}(x_1) \end{cases}$$

The map $\hat{u}_{N_n}$ has been constructed in such a way that the Hopf degree of $\hat{u}_{N_n}$ is ν_1. Moreover, the measure $|\nabla \hat{u}_{N_n}|^3$ converges weakly to a measure having support in $\{x_1\}$. Thus there exists $\tilde{\mu_1} \geqslant 0$ such that

$$|\nabla \hat{u}_{N_n}|^3 \rightharpoonup \tilde{\mu}_1 \, \delta_{x_1}. \tag{III.20}$$

Let $\alpha < \delta$, (III.18) and (III.19) imply

$$\left| \int_{B_\beta} |\nabla u_{N_n}|^3 - |\nabla \hat{u}_{N_n}|^3 \right| \leqslant C\beta^3. \tag{III.21}$$

This yields $\tilde{\mu}_1 = \mu_1$. Since $H(\hat{u}_{N_n}) = \nu_1$, we have

$$\mathcal{E}_{\nu_1} \leqslant \mu_1 = \lim_{n \to +\infty} \int_{S^3} |\nabla \hat{u}_{N_n}|^3 \, . \tag{III.22}$$

Combining (ii) of theorem III.1 and (III.22), we get

$$\mathcal{E}_d = E_3(u) + \sum_{i=1}^{k} \mu_i \geqslant \mathcal{E}_\nu + \sum_{i=1}^{k} \mathcal{E}_{\nu_i}, \tag{III.23}$$

where ν is the Hopf degree of u. Moreover gluing together minimizing sequences of $\mathcal{E}_\nu$, $\mathcal{E}_{\nu_1}, \cdots, \mathcal{E}_{\nu_k}$ one proves easily, since $d = \nu + \sum_{i=1}^{k} \nu_i$, that

$$\mathcal{E}_d \leqslant \mathcal{E}_\nu + \sum_{i=1}^{k} \mathcal{E}_{\nu_i}.$$

Thus equality holds in (III.22) and (III.23) and we have

$$\mathcal{E}_d = E_3(u) + \sum_{i=1}^{k} \mathcal{E}_{\nu_i} = \mathcal{E}_\nu + \sum_{i=1}^{k} \mathcal{E}_{\nu_i}. \tag{III.24}$$

(Since $\mu_i = \mathcal{E}_{\nu_i} > 0$ we have $\nu_i \neq 0$).

We claim that, if $\nu = 0$, that is, if u is a constant map, we have $k \geqslant 2$. The approach used for proving this is similar to the one used in [19] for proving lemma 5.3.

Suppose u_n concentrates at the north point, that is

$$|\nabla u_n|^3 \rightharpoonup \mathcal{E}_d \, \delta_{North}. \tag{III.25}$$

We are going to prove that, for any $\varepsilon > 0$ there exists C_ε independant of n such that

$$E^{+,\varepsilon}_{3+\frac{1}{n}}(u_n) \leqslant C_\varepsilon\, E^{-,0}_{3+\frac{1}{n}}(u_n) \tag{III.26}$$

where $E^{+,\beta}_{3+\frac{1}{n}}(u_n)$ (resp. $E^{-,\beta}_{3+\frac{1}{n}}(u_n)$) is the $3+\frac{1}{n}$ energy of u_n restricted to the geodesic ball $B_{north}(\frac{\pi}{2}-\beta)$ (resp. $B_{south}(\frac{\pi}{2}-\beta)$) in S^3 for $\beta \geqslant 0$. It is clear that (III.26) contradicts (III.25) and this establishes the claim. Let us prove now (III.26). For simplicity of the notations denote by v any element of the sequence u_n and by p the corresponding exponent $3+\frac{1}{n}$. With this notations v is a C^1 p-harmonic map. Consider a perturbation of v given by an infinitesimal action of the conformal dilation of S^3 relative to the north pole, that is $\sin\phi\,\frac{\partial v}{\partial\phi}$ where ϕ is the angle function in $[0,\pi]$ such that $\cos\phi = x_4$ the forth coordinate of any point of $S^3 \subset \mathbb{R}^4$. We shall use the spherical coordinate $(\theta,\psi,\phi) \in [0,2\pi]\times[0,\pi]\times[0,\pi]$ on S^3 recall that we have

$$g_{S^3} = \sin^2\!\phi\,\sin^2\!\psi\,d\theta^2 + \sin^2\!\phi\,d\psi^2 + d\phi^2$$

Since v is p-harmonic and $v\,.\,\frac{\partial v}{\partial\phi} \equiv 0$ we have by the 1st variational formula for the p-energy functional (see the derivation of the general formula for maps between Riemannian manifolds in [22])

$$\int_{S^3} |dv|^{p-2}\left\langle dv\,.\,d\left(\sin\phi\,\frac{\partial v}{\partial\phi}\right)\right\rangle \omega_{S^3} = 0. \tag{III.27}$$

We have

$$\begin{aligned}\left\langle dv\,.\,d\left(\sin\phi\,\frac{\partial v}{\partial\phi}\right)\right\rangle = &\left(\frac{\partial v}{\partial\phi}\right)\cdot\left(\frac{\partial}{\partial\phi}\left(\sin\phi\,\frac{\partial v}{\partial\phi}\right)\right)\\ &+\left(\frac{1}{\sin^2\phi}\frac{\partial v}{\partial\psi}\right)\cdot\left(\frac{\partial}{\partial\psi}\left(\sin\phi\,\frac{\partial v}{\partial\phi}\right)\right)\\ &+\left(\frac{1}{\sin^2\phi\,\sin^2\psi}\frac{\partial v}{\partial\theta}\right)\cdot\left(\frac{\partial}{\partial\theta}\left(\sin\phi\,\frac{\partial v}{\partial\phi}\right)\right).\end{aligned}$$

where " . " denote the scalar product in TS^2 and $<,>$ the scalar product in T^*S^3 After computations we get

(III.28)

$$dv\,.\,d\left(\sin\phi\,\frac{\partial v}{\partial\phi}\right) = \frac{1}{2}\frac{\partial}{\partial\phi}|dv|^2\sin\phi + |dv|^2\cos\phi$$

Combining (III.27) and (III.28) we have

(III.29)
$$0 = \frac{1}{2}\int_0^\pi\int_0^\pi\int_0^{2\pi} \left(|dv|^2\right)^{\frac{p-2}{2}} \frac{\partial}{\partial\phi}|dv|^2 \sin^3\phi \sin\psi \, d\phi \, d\psi \, d\theta + \int_0^\pi\int_0^\pi\int_0^{2\pi} |dv|^p \cos\phi \sin^2\phi \sin\psi \, d\phi \, d\psi \, d\theta.$$

Integrating the first term of the right hand side of (III.29) by part we get

(III.30)
$$\int_{S^3} |dv|^p \cos\phi = 0.$$

This implies (III.26).

Proposition III.1 is now a consequence of (III.24) and the fact that if $\nu = 0$, $k = 2$. Indeed, take $d > 0$, apply (III.24) to d, then apply (III.24) to each ν_i and so on. We develop a tree procedure and in each branch an integer appears at most one time. Moreover, since $\mathcal{E}_p \longrightarrow +\infty$ as $p \longrightarrow +\infty$ only a finite number of integers are concerned, and finally we get (III.8). □

The author learned that a similar result than the one stated in proposition III.1 was recently established by F. Duzaar and E. Kuwert in a more general setting in a forthcoming paper [4].

Proof of theorem I.1. Suppose there is only a finite number of non zero integers such that the corresponding homotopy class admits a minimizing 3-harmonic map. Let $\mathcal{E}_{d_0}$ be the smallest of the energy of those homotopy classes and let $|d_1|$ be the largest corresponding integer of those homotopy classes. (III.8) implies

(III.31)
$$\forall d \geqslant 0 \qquad \mathcal{E}_d \geqslant \frac{d}{|d_1|} \times \mathcal{E}_{d_0}.$$

This contradicts lemma III.1 as d tends to infinity and theorem I.3 is proved. □

IV. A conformal invariant variational problem on closed forms of $\wedge^{2p}S^{4p-1}$.

IV.1. Presentation and basic properties.

As we pointed out in the introduction, the study of the conformal invariant 3-energy from S^3 into S^2 in the first homotopy class yields naturally to the following constrained minimizing problem (in the case $p = 1$)

(IV.1)

$$\mathcal{I}_\star = \inf\left\{\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} |d\eta|^{2-\frac{1}{2p}}\ ,\ \eta \in W^{1,2-\frac{1}{2p}}\left(\wedge^{2p-1}S^{4p-1}\right) \quad \text{s.t.}\quad \frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} \eta\wedge d\eta = 1\right\}$$

First we prove the following proposition.

Proposition IV.1. *The minimizing problem (IV.1) is achieved by a closed 2p-form $h_\star = d\eta_\star$ which verifies the following equation*

(IV.2)
$$d\eta_\star = \mathcal{I}_\star^{\frac{2p}{2p-1}}\ |\eta_\star|^{\frac{1}{2p-1}}\ (*)\,\eta_\star$$

where $(*)$ denotes the Hodge operator on S^{4p-1}.

Remark IV.1. Equation (IV.2) only depends on $h_\star = d\eta_\star$ and is equivalent to the following one

(IV.3)
$$d(*)\frac{h_\star}{|h_\star|^{\frac{2p-1}{2p}}} = \mathcal{I}_\star^{\frac{1}{2p-1}}\ h_\star.$$

Remark IV.2. The minimisation problem (IV.1) and consequently equation (IV.2) are invariant under the action of the non-compact conformal group of (S^{4p-1}, g_{can}) denoted $\text{Conf}\, S^{4p-1}$, i. e.

(IV.4)

$$\forall\phi\in \text{Conf}^+\left(S^{4p-1}\right),\ \forall\eta\in\wedge^{2p-1}\left(S^{4p-1}\right)\qquad \int_{S^{4p-1}} |d\phi^*\eta|^{2-\frac{1}{2p}} = \int_{S^{4p-1}} |d\eta|^{2-\frac{1}{2p}}$$

and naturally we have

$$\int_{S^{4p-1}} \phi^*\eta \wedge d\phi^*\eta = \int_{S^{4p-1}} \eta \wedge d\eta$$

that is the main reason why the proof of proposition IV.2 is not so direct but is an application of the concentration-compactness method of P.L. Lions.

Proof of proposition IV.1. We follow the arguments of [15].
Let $h_k = d\eta_k$ be a minimizing sequence of $\mathcal{I}_\star$, where η_k denotes the "Coulomb Gauge":

$$d^*\eta_k = 0. \tag{IV.5}$$

The choice of this η_k makes the problem coercive thus η_k weakly converges, up to a subsequence, in $W^{1,2-\frac{1}{2p}}(\wedge^{2p-1}S^{4p-1})$. Denote by η the weak limit of η_k, by μ the limiting measure of $|d\eta_k|^{2-\frac{1}{2p}}$ and by ν the limiting measure of $(*)\eta_k \wedge d\eta_k$. We first prove that there exists $(\nu_j)_{j\in J} \in \mathbb{R}^{*J}$ and $(x_j)_{j\in J}$ a sequence of points of S^{4p-1}, where J is some at most countable set, such that

$$\begin{cases} \nu = (*)\eta \wedge d\eta + \displaystyle\sum_{j\in J} \nu_j \delta_{x_j} \\ \text{and} \quad \mu \geqslant |d\eta|^{2-\frac{1}{2p}} + \left|S^{4p-1}\right|^{\frac{1}{4p}} \mathcal{I}_\star \displaystyle\sum_{j\in J} |\nu_j|^{1-\frac{1}{4p}} \delta_{x_j}. \end{cases} \tag{IV.6}$$

Let $\xi_k = \eta_k - \eta$. Denote by ν' the limit of $(*)\xi_k \wedge d\xi_k$ and μ' the limit of $|d\xi_k|^{2-\frac{1}{2p}}$. Let $\phi \in C^\infty(S^{4p-1})$, we clearly have $\phi\xi_k \wedge d(\phi\xi_k) = \phi^2\, \xi_k \wedge d\xi_k$ and this implies

$$\left| \frac{1}{|S^{4p-1}|} \int_{S^{4p-1}} \phi^2 \xi_k \wedge d\xi_k \right|^{\frac{1}{2}} \leqslant \frac{1}{\mathcal{I}_\star^{\frac{2p}{4p-1}}} \left(\frac{1}{|S^{4p-1}|} \int_{S^{4p-1}} |d\phi\, \xi_k|^{2-\frac{1}{2p}} \right)^{\frac{2p}{4p-1}} \tag{IV.7}$$

Using the fact that ξ_k weakly converges to 0 in $W^{1,2-\frac{1}{2p}}$ and using also Rellich Theorem we get

$$\left| \int_{S^4p-1} |d\phi\, \xi_k|^{2-\frac{1}{2p}} - \int_{S^{4p-1}} |\phi|^{2-\frac{1}{2p}} |d\xi_k|^{2-\frac{1}{2p}} \right| \longrightarrow 0.$$

Thus we have

(IV.8)

$$\left|\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}|\phi|^2\,d\nu'\right|^{\frac{1}{2}}\leqslant\frac{1}{\mathcal{I}_\star^{\frac{2p}{4p-1}}}\left(\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}|\phi|^{2-\frac{1}{2p}}\,d\mu'\right)^{\frac{2p}{4p-1}}.$$

This is the desired Hölder reverse inequality in view of applying lemma 1.2 of [15]. thus we obtain the existence of $(\nu_j)_{j\in J}$ such that

$$\text{(IV.9)}\qquad\begin{cases}\nu'=\sum_{j\in J}\nu_j\delta_{x_j}\\ \text{and}\quad\mu'\geqslant\left|S^{4p-1}\right|^{\frac{1}{4p}}\mathcal{I}_\star\sum_{j\in J}|\nu_j|^{1-\frac{1}{4p}}\,\delta_{x_j}.\end{cases}$$

Because of the fact that

$$\xi_k\wedge d\xi_k=\eta_k\wedge d\eta_k-\eta\wedge d\eta_k-\eta_k\wedge d\eta+\eta\wedge d\eta,$$

using, once again, Rellich Theorem we get $\nu'=\nu-(*)\eta\wedge d\eta$, this establishes the first line of (IV.6). Moreover the inequality

$$\mu\geqslant\left|S^{4p-1}\right|^{\frac{1}{4p}}\mathcal{I}_\star\sum_{j\in J}|\nu_j|^{1-\frac{1}{4p}}\,\delta_{x_j}$$

is a consequence of the following

(IV.10)

$$\begin{aligned}\left|\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}|\phi|^2\,d\nu\right|^{\frac{1}{2}}\leqslant\;&\frac{1}{\mathcal{I}_\star^{\frac{2p}{4p-1}}}\left(\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}|\phi|^{2-\frac{1}{2p}}\,d\mu\right)^{\frac{2p}{4p-1}}\\&+C\left(\int_{S^{4p-1}}|\eta|^{2-\frac{1}{2p}}\,|d\phi|^{2-\frac{1}{2p}}\right)^{\frac{2p}{4p-1}}\end{aligned}$$

where inequality (IV.10) is established similarly than inequality (IV.8). Finally, by lower semicontinuity of the $L^{2-\frac{1}{2p}}$-norm for the weak topology we have

$$\text{(IV.11)}\qquad\mu\geqslant|d\eta|^{2-\frac{1}{2p}}$$

and since $|d\eta|^{2-\frac{1}{2p}}$ is a measure orthogonal to the sum of the Dirac masses, we establish the second line of (IV.6).

We now modify the minimizing sequence $d\eta_k$ to garantee that the weak limit be non zero if it is not the case. We proceed as follows. Denote by $L_k = (L^i_k)_{i=1,\dots,4p}$ the following vector of $\mathbb{R}^{4p}$

$$L^i_k = \frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} x_i\eta_k \wedge d\eta_k. \tag{IV.12}$$

Let g be a point in the open unit ball B^{4p} of $\mathbb{R}^{4p}$, denote by g the following positive conformal transformation of S^{4p-1}

$$\forall x \in S^{4p-1} \qquad g(x) = \frac{x + (\mu x\,.\,g + \lambda)g}{\lambda\,(1 + x\,.\,g)} \tag{IV.13}$$

where $\lambda = \sqrt{1-|g|^2}$, $\mu = (\lambda - 1)/|g|^2$. Denote by ψ_k the following map from B^{4p} into $\mathbb{R}^{4p}$

$$\begin{aligned} \psi_k g &\longrightarrow L_k(g) = \big(L^i_k(g)\big)_{i=1,\dots,4p} \\ L^i_k(g) &= \frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} x_i\,\zeta_k(g)\wedge dg^*\eta_k. \end{aligned} \tag{IV.14}$$

where $\zeta_k(g)$ is the coclosed form of $\wedge^{2p-1}S^{4p-1}$ verifying $d\zeta_k(g) = dg^*\eta_k$. ψ_k can be extended by continuity from $\overline{B^{4p}}$ into $\mathbb{R}^{4p}$. Indeed, as $g \to \theta \in S^{4p-1}$ one verifies that the total mass of $\zeta_k(g)\wedge dg^*\eta_k$ concentrates at θ and since

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} \zeta_k(g)\wedge dg^*\eta_k = \frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} g^*\eta_k \wedge dg^*\eta_k = 1$$

one has $L_k(g) \to \theta$. Thus ψ_k restricted to S^{4p-1} is the identity map and using Brower theorem, we deduce that there exists $g_k \in B^{4p}$ such that $L_k(g_k) = 0$. Instead of considering η_k as a minimizing sequence, we take $\zeta_k(g_k)$ that we will also denote η_k.

Suppose $d\eta = 0$, combining (IV.10) for $\eta = 0$ and the last part of lemma 1.2 of [15] we deduce that $\operatorname{Card} J = 1$. Let x_0 be the point where $\eta_k \wedge d\eta_k$ concentrates we have that

$$L_k \longrightarrow x_0$$

which contradicts the fact that we have chosen a minimizing sequence such that $L_k = 0$ for all k. Thus $d\eta \neq 0$.

Inequality (IV.6) yields

(IV.15)

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} |d\eta|^{2-\frac{1}{2p}} \leqslant \mathcal{I}_\star\left(1 - \sum_{j\in J}\left(\frac{|\nu_j|}{|S^{4p-1}|}\right)^{1-\frac{1}{4p}}\right)$$

and we have

(IV.16)

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}|d\eta|^{2-\frac{1}{2p}} \leqslant \mathcal{I}_\star\left(1-\left(\sum_{j\in J}\frac{|\nu_j|}{|S^{4p-1}|}\right)^{1-\frac{1}{4p}}\right)$$
$$\leqslant \mathcal{I}_\star\left(1-\left(1-\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}\eta\wedge d\eta\right)^{1-\frac{1}{4p}}\right)$$
$$\leqslant \mathcal{I}_\star\left(\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}\eta\wedge d\eta\right)^{1-\frac{1}{4p}}.$$

η realizes the minimum $\mathcal{I}_\star$ and proposition IV.1 is proved. □

IV.2. The associated linear problem..

More generally we are interested in finding solutions of the following conformal invariant equation for $\mu \neq 0$

(IV.17)
$$\begin{cases} d\eta = \mu\ |\eta|^{\frac{1}{2p-1}}\,(*)\eta \quad \text{in} \quad \wedge^{2p-1} S^{4p-1} \\ \dfrac{2p}{|S^{4p-1}|}\displaystyle\int_{S^{4p-1}}\eta\wedge d\eta = 1 \end{cases}$$

There are solutions which are given by the associated linear problem on the coclosed forms of $\wedge^{2p-1}S^{4p-1}$:

(IV.18)
$$d\eta = \lambda\,(*)\eta,$$

which is equivalent to the following eigenvalue problem on $\operatorname{Ker} d \cap \wedge^{2p}S^{4p-1}$

(IV.19)
$$d(*)h = \lambda\, h.$$

(IV.19) is the eigenvalue problem for the square root of the Hodge laplacian, $\Delta^{\frac{1}{2}} = d(*)$, on $\operatorname{Ker} d \cap \wedge^{2p}S^{4p-1}$. The eigenvalues and the eigenspaces of the Hodge Laplacian on $\operatorname{Ker} d \cap \wedge^{2p}S^{4p-1}$ are described in [8], [11] and [13]. Those eigenvalues are $(2p)^2$, $(2p+1)^2$, $(2p+2)^2, \dots$ and the eigenspace E_i, corresponding to the eigenvalues $(n+i)^2$ are the restrictions to S^{4p-1} of the set of closed and coclosed , polynomial, homogeneous $2p$-forms in $\wedge^{2p}\mathbb{R}^{4p}$, having degree i. Thus, in particular, the eigenvalues λ in (IV.19) can only be $\lambda = \pm(n+i)$, $i \in \mathbb{N}$. $d(*)/(n+i)$ is an orthogonal symmetry in E_i for the L^2-scalar product. Moreover, if S is the symmetry on $\mathbb{R}^{4p}$

$$S : (x_1, x_2, \dots, x_{4p}) \longrightarrow (-x_1, x_2, \cdots, x_{4p})$$

we have

$$S^*(*)h = -(*)S^*h \qquad \text{in} \quad \wedge^{2p} S^{4p-1}.$$

Thus the pull-back by S realizes an isometry between

$$E_i^+ = \operatorname{Ker}\big(d(*) - (n+i)\ \mathrm{Id}\big)$$

and

$$E_i^- = \operatorname{Ker}\big(d(*) + (n+i)\ \mathrm{Id}\big),$$

this implies $\dim E_i^+ = \dim E_i^-$. In fact there is a more precise description of $E_i^\pm$.

Proposition IV.2. *The eigenspaces* $E_i^\pm = \operatorname{Ker}(d(*) - \pm(n+i)\ \mathrm{Id})$ *in* $\wedge^{2p} S^{4p-1} \cap \operatorname{Ker} d$ *are respectively the restrictions to* S^{4p-1} *of the dual and antidual closed, degree* i*, polynomial homogeneous* $2p$*-forms of* $\mathbb{R}^{4p}$.

Proof. From [8] and [13] we know that

$$\operatorname{Ker}\left(d(*) - (n+i)\right) \oplus \operatorname{Ker}\left(d(*) + (n+i)\right)$$

is the restriction to S^{4p-1} of the closed and coclosed, degree i, polynomial, homogeneous $2p$-forms of $\mathbb{R}^{4p}$.

First of all we prove that, if h is a coclosed, degree i, polynomial, homogeneous $2p$-form of $\mathbb{R}^{4p}$ we have

$$d(*)h = (2p+i) * h \qquad \text{on} \quad S^{4p-1}. \tag{IV.20}$$

Recall that $*$ and $(*)$ denote respectively the Hodge operators in $\mathbb{R}^{4p}$ and S^{4p-1}. From [11] we know that for such h we have

$$d\iota\left(r\frac{d}{dr}\right) * h = (2p+i) * h \qquad \text{on} \quad \mathbb{R}^{4p} \tag{IV.21}$$

where

$$r\frac{d}{dr} = \sum_{k=1}^{4p} x_k \frac{\partial}{\partial x_k}$$

and $\iota(X)$ denotes the interior product by X on forms. Thus for proving (IV.20) it suffices to verify that

$$\iota\left(r\frac{d}{dr}\right) * h = (*)h \qquad \text{on} \quad S^{4p-1}. \tag{IV.22}$$

Let $x \in S^{4p-1}$ and $(dr, \varepsilon_1, \ldots, \varepsilon_{4p-1})$ an orthonormal basis of $T_x^* \mathbb{R}^{4p}$ where $(\varepsilon_1, \ldots, \varepsilon_{4p-1})$ is an orthonormal basis of $T_x^* S^{4p-1}$. Decompose h in this basis

$$h = \sum_J \alpha_J\, dr \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{2p-1}} + \sum_I \beta_I\, \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{2p}},$$

where J denotes a choice

$$\{j_1, \ldots, j_{2p-1}\}$$

of $2p-1$ elements among $\{1, \ldots, 4p-1\}$ and I denotes a choice

$$\{i_1, \ldots, i_{2p}\}$$

of $2p$ elements among $\{1, \cdots, 4p-1\}$. We have

$$*h = \sum_J \alpha_J \operatorname{sgn}(J)\, \varepsilon_{j_1^*} \wedge \cdots \wedge \varepsilon_{j_{2p}^*} + \sum_I \beta_I \operatorname{sgn}(I)\, dr \wedge \varepsilon_{i_1^*} \wedge \cdots \wedge \varepsilon_{i_{2p-1}^*},$$

where $\{j_1^*, \cdots, j_{2p-1}^*\}$ is the complement of $\{j_1, \cdots, j_{2p}\}$ in $\{1, \cdots, 4p-1\}$ and where $\operatorname{sgn}(J)$ and $\operatorname{sgn}(I)$ are the signs such that

$$\begin{aligned}
\operatorname{sgn}(J) dr \wedge \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_{2p-1}} \wedge \varepsilon_{j_1^*} \wedge \cdots \wedge \varepsilon_{j_{2p}^*}& \\
&= \operatorname{sgn}(I)\, \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{2p}} \wedge dr \wedge \varepsilon_{i_1^*} \wedge \cdots \wedge \varepsilon_{i_{2p-1}^*} \\
&= \omega_{4p}.
\end{aligned}$$

We have

$$\iota\left(r\frac{d}{dr}\right) * h = \sum_I \beta_I \operatorname{sgn}(I)\, \varepsilon_{i_1^*} \wedge \cdots \wedge \varepsilon_{i_{2p-1}^*}$$

Moreover the restriction of h to S^{4p-1} is equal to

$$\sum_I \beta_I\, \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{2p}} \qquad \text{in} \quad S^{4p-1},$$

and

$$(*)h = \sum_I \beta_I \operatorname{sgn}(I)\, \varepsilon_{i_1^*} \wedge \cdots \wedge \varepsilon_{i_{2p-1}^*}.$$

Here we have used the fact that $n = 2p$ is even and

$$\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{2p}} \wedge dr = dr \wedge \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{2p}}.$$

Combining (IV.21) and (IV.22) we get (IV.20). Thus h is a closed and coclosed degree i polynomial homogeneous $2p$-form of $\mathbb{R}^{4p}$ belonging to $\mathrm{Ker}(d(*) - \pm(2p+i)\,\mathrm{Id})$ if and only if

$$*h = \pm h \qquad \text{on} \quad S^{4p-1}.$$

But since h is closed and coclosed in $\mathbb{R}^{4p}$ this implies

$$*h = \pm h \qquad \text{in} \quad \mathbb{R}^{4p}.$$

Proposition IV.2 is proved. □

For any part $I = \{i_1, \cdots, i_{2p}\}$ of $\{1, \cdots, 4p\}$ denote by $h_I^{\pm}$ the following forms in $\mathbb{R}^{4p}$

$$h_I^{\pm} = dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} \pm *(dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}}). \tag{IV.23}$$

From the discussion above we deduce that, since $h_I^{\pm}$ is respectively dual and antidual in $\mathbb{R}^{4p}$, we have

$$d(*)h_I^{\pm} = \pm\, 2p\, h_I^{\pm}\,. \tag{IV.24}$$

Moreover, one verifies that, since $\dim E_0 = C_{4p}^{2p}$ (see [13]), and since the $h_I^{\pm}$ are orthonormal for the L^2-scalar product, $h_I^{\pm}$ realize an orthonormal basis of $E_0^{\pm}$. Let η_I^{+} be the following $(2p-1)$-form in $\mathbb{R}^{4p}$

(IV.25)

$$\begin{aligned}\eta_I^{\pm} = {} & \frac{1}{2p}\sum_{s=1}^{2p}(-1)^{s+1}\, x_{i_s}\, dx_{i_1} \wedge \cdots \wedge d\check{x}_{i_s} \wedge \cdots \wedge dx_{i_{2p}} \\ & \pm \frac{1}{2p}\sum_{s=1}^{2p}(-1)^{s+1}\, x_{j_s}\, dx_{j_1} \wedge \cdots \wedge d\check{x}_{j_s} \wedge \cdots \wedge dx_{j_{2p}}\end{aligned}$$

where $*(dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}}) = dx_{j_1} \wedge \cdots \wedge dx_{j_n}$. We have

$$d\eta_I^{\pm} = h_I^{\pm}, \qquad |h_I^{\pm}|_{S^{4p-1}} = 1, \qquad |\eta_I^{\pm}|_{S^{4p-1}} = \frac{1}{2p}$$

$$\text{and} \qquad \eta_I^{\pm} \wedge h_I^{\pm} = \pm\frac{1}{2p}\omega_{S^{4p-1}},$$

thus $\eta_I^\pm = \pm\frac{1}{2p}\,(*)h_I^\pm = \pm\frac{1}{2p}\,(*)d\eta_I^\pm$ and this implies

(IV.26)
$$d\eta_I^\pm = \pm(2p)^{1+\frac{1}{2p-1}}\,|\eta_I^\pm|^{\frac{1}{2p-1}}\,(*)\,\eta_I^\pm \qquad \text{in} \quad S^{4p-1}.$$

We have constructed solutions of (IV.17) for $C = \pm(2p)^{\frac{2p}{2p-1}}$. From the above discussion we know that the space generated by the η_I^+ is the set of minimizers of

$$\text{(IV.27)} \qquad \min\left(\frac{\displaystyle\int_{S^{4p-1}}|d\eta|^2}{\displaystyle\int_{S^{4p-1}}\eta\wedge d\eta}\right) = 2p,$$

which is the variational problem associated with the eigenvalue problem for $\Delta^{\frac{1}{2}} = d(*)$ in $\operatorname{Ker} d \cap \wedge^{2p}S^{4p-1}$.

We are tempted to say that $\mathcal{I}_\star = (2p)^{\frac{4p-1}{4p}}$ but this is still an open question, we are just able to give a "local" answer which is theorem I.2 in the introduction.

IV.3. Local Minimizers.

The aim of this part is to prove theorem I.2 in the introduction.

We will need the following proposition

Proposition IV.3. *The second positive and negative eigenspaces of* $\Delta^{\frac{1}{2}} = d(*)$ *in* $\operatorname{Ker} d \cap \wedge^{2p}S^{4p-1}$, *i. e.* $E_1^\pm$, *corresponding to the eigenvalues* $\pm(2p+1)$ *is generated by the Lie derivatives along the positive conformal Killing fields which are not pure of the eigenforms of the first eigenspaces* $E_0^\pm$ *corresponding to the eigenvalues* $\pm 2p$.

By positive conformal Killing fields which are not pure, we mean the Killing fields which generate the positive conformal transformations of S^{4p-1} which are not isometric. Those Killing vector-fields are generated by the following ones:

(IV.28)
$$\forall k = 1,\ldots,4p \quad X_k(x) = e_k - x\,.\,e_k x \quad \text{for } x \in S^{4p-1}$$

where (e_k) denotes the canonical basis of $\mathbb{R}^{4p}$.

Before proving proposition IV.3 we construct a particular family of forms which generates E_1^+ (equivalentely E_1^-). Let I be still a choice $\{i_1, \ldots, i_{2p}\}$ of $2p$ integers in $\{1, \ldots, 4p\}$, denote by $J = \{j_1, \ldots, j_{2p}\}$ the complement of I in $\{1, \ldots, 4p\}$ (the indexing is chosen such that $dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} = *(dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}})$).
For any $i_t \in I$ and $j_s \in J$ denote by I_{i_t,j_s} the following subset of $\{1, \ldots, 4p\}$

$$I_{i_t,j_s} = \{i_1, \ldots, i_{t-1}, j_s, i_{t+1}, \ldots, i_{2p}\}$$

and remark that

$$dx_{j_1} \wedge \cdots \wedge dx_{j_{s-1}} \wedge dx_{i_t} \wedge dx_{j_{s+1}} \wedge \cdots \wedge dx_{j_{2p}} = - *(dx_{i_1} \wedge \cdots \wedge dx_{i_{t-1}} \wedge dx_{j_s} \wedge \cdots \wedge dx_{i_{2p}}).$$

Finally denote by $k^{\pm,I}_{i_t,j_s}$ the following $2p$-form

$$k^{\pm,I}_{i_t,j_s} = x_{i_t}\, h^\pm_I - x_{j_s}\, h^\pm_{I_{i_t,j_s}} \,. \tag{IV.29}$$

We have

(IV.30)
$$\begin{aligned} dk^{\pm,I}_{i_t,j_s} &= dx_{i_t} \wedge h^\pm_I - dx_{j_s} \wedge h^\pm_{I_{i_t,j_s}} \\ &= dx_{i_t} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} \\ &\quad + dx_{j_s} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{s-1}} \wedge dx_{i_t} \wedge dx_{j_{s+1}} \wedge \cdots dx_{j_{2p}} \\ &= dx_{i_t} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} - dx_{i_t} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} = 0 \end{aligned}$$

Moreover, we clearly have $*k^{\pm,I}_{i_t,j_s} = \pm k^{\pm,I}_{i_t,j_s}$. Thus $k^{\pm,I}_{i_t,j_s}$ is a closed and coclosed homogeneous degree 1 polynomial $2p$-form of $\mathbb{R}^{4p}$ which is selfdual, resp. anti-selfdual. From proposition IV.2 we deduce

$$d(*)k^{\pm,I}_{i_t,j_s} = \pm(2p+1)k^{\pm,I}_{i_t,j_s}. \tag{IV.31}$$

In fact we have the following

Proposition IV.4. *The $k^{\pm,I}_{i_t,j_s}$ generate $E_1^\pm$.*

Proof. Let k be in E_1^+, k is an homogeneous degree 1 polynomial $2p$-form of $\mathbb{R}^{4p}$, thus there exists $\alpha_{I,l} \in \mathbb{R}$ for $l \in \{1, \ldots, 4p\}$ and $I = \{i_1, \ldots, i_{2p}\}$ any choice of $2p$ elements in $\{1, \ldots, 4p\}$, such that

$$k = \sum_l \sum_I \alpha_{I,l}\, x_l\, dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} \tag{IV.32}$$

Since $k \in E_1^+$, by proposition IV.2, we have $*k = k$ and $\alpha_{I,l} = \alpha_{J,l}$ where $J = \{j_1, \dots, j_{2p}\}$ and

$$dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} = *(dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}}).$$

(IV.32) becomes

$$k = \sum_l \sum_I \beta_{I,l}\, x_l h_I^+ \tag{IV.33}$$

where $\beta_{I,l} = \pm\alpha_{I,l}$ and where we have restricted the sum among the (I, l) such that $l \in I$.

Let $I^o = \{i_1^o, \dots, i_{2p}^o\}$ and $l^o = i_t^o \in I^o$ such that $\beta_{I^o, i_t^o} \neq 0$. Consider the couples (I, l) such that $l \in I$ and

$$dx_{l^o} \wedge h_{I^o}^+ = \pm dx_l \wedge h_I^+.$$

We can always assume that the indexation of those I is chosen such that

$$dx_{l^o} \wedge h_{I^o}^+ = dx_l \wedge h_I^+. \tag{IV.34}$$

Let us denote $* dx_{i_1^o} \wedge \cdots \wedge dx_{i_{2p}^o} = dx_{j_1^o} \wedge \cdots \wedge dx_{j_{2p}^o}$. For any (I, l) chosen as above, there exists s in $\{1, \dots, 2p\}$ such that $l = j_s^o$ and $I = I_{i_t^o, j_s^o}^o$. Since $k \in E_1^+$ we have $dk = 0$. Combining this fact with the previous remark we get

$$\beta_{I^o, i_t^o} + \sum_{s=1}^{2p} \beta_{I_{i_t^o, j_s^o}^o, j_s^o} = 0, \tag{IV.35}$$

and

$$\begin{aligned} \beta_{I^o, i_t^o}\, x_{i_t^o}\, h_{I^o}^+ &+ \sum_{s=1}^{2p} \beta_{I_{i_t^o, j_s^o}^o, j_s^o}\, x_{j_s^o}\, h_{I_{i_t^o, j_s^o}^o}^+ \\ &= \sum_{s=1}^{2p} \beta_{I_{i_t^o, j_s^o}^o, j_s^o} \left[x_{j_s^o}\, h_{I_{i_t^o, j_s^o}^o}^+ - x_{i_t^o}\, h_{I^o}^+ \right] \\ &= \sum_{s=1}^{2p} \beta_{I_{i_t^o, j_s^o}^o, j_s^o}\, k_{i_t^o, j_s^o}^{+, I^o} \end{aligned}$$

This proves proposition IV.4. □

Proof of proposition IV.3. Let $I = \{i_1, \ldots, i_{2p}\}$ be a choice of $2p$ integers in $\{1, \ldots, 4p\}$ and $l_o = i_{t_o} \in I$. We compute $\mathcal{L}_{X_{l_o}} h_I^+$.
Using Cartan formula on Lie derivation, since h_I^+ is closed, we have

(IV.36)
$$\mathcal{L}_{X_{l_o}} h_I^+ = d\Big(\iota(X_{l_o})\left(dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} + dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}}\right)\Big)$$

where $\iota(X_{l_o})$ is the inner product by the Killing field X_{l_o} defined by (IV.28) and where $dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} = * \, dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}}$ in $\mathbb{R}^{4p}$. We compute

(IV.37)
$$\begin{aligned}
&\iota(X_{l_o})\left(dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}}\right) \\
&\quad = \sum_{t=1}^{2p} (-1)^t x_{l_o} x_{i_t} \, dx_{i_1} \wedge \cdots \wedge d\check{x}_{i_t} \wedge \cdots \wedge dx_{i_{2p}} + \\
&\quad + (-1)^{t_o+1} dx_{i_1} \wedge \cdots \wedge d\check{x}_{i_{t_o}} \wedge \cdots \wedge dx_{i_{2p}}
\end{aligned}$$

and

(IV.38)
$$\begin{aligned}
&\iota(X_{l_o})\left(dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}}\right) \\
&\quad = \sum_{s=1}^{2p} (-1)^s x_{l_o} x_{j_s} \, dx_{j_1} \wedge \cdots \wedge d\check{x}_{j_s} \wedge \cdots \wedge dx_{j_{2p}} \, .
\end{aligned}$$

Combining (IV.36), (IV.37) and (IV.38) we get

(IV.39)
$$\begin{aligned}
\mathcal{L}_{X_{l_o}} h_I^+ = & -(2p) x_{l_o} \, dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} - x_{l_o} \, dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} \\
& - (2p) x_{l_o} \, dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} \\
& - \sum_{s=1}^{2p} x_{j_s} \, dx_{j_1} \wedge \cdots \wedge dx_{j_{s-1}} \wedge dx_{l_o} \wedge \cdots \wedge dx_{j_{2p}} \, .
\end{aligned}$$

On S^{4p-1} we have

$$-x_{l_o} \, dx_{l_o} = \sum_{t \neq t_o} x_{i_t} \, dx_{i_t} + \sum_{s=1}^{2p} x_{j_s} \, dx_{j_s},$$

and (IV.39) becomes

$$\mathcal{L}_{X_{l_o}} h_I^+ = -(2p)x_{l_o}h_I^+ + \sum_{s=1}^{2p} x_{j_s}\big(dx_{i_1}\wedge\cdots\wedge dx_{i_{t_o-1}}\wedge dx_{j_s}\wedge dx_{i_{t_o+1}}\wedge\cdots dx_{i_{2p}} - dx_{j_1}\wedge\cdots\wedge dx_{j_{s-1}}\wedge dx_{i_{t_o}}\wedge dx_{j_{s+1}}\wedge\cdots dx_{j_{2p}}\big)$$

thus

(IV.40)

$$\mathcal{L}_{X_{l_o}} h_I^+ = -(2p)x_{l_o}h_I^+ + \sum_{s=1}^{2p} x_{j_s}h^+_{I_{i_{t_o},j_s}} = -\sum_{s=1}^{2p} k^{+,I}_{l_o,j_s}.$$

(IV.40) implies that $\mathcal{L}_{X_{l_o}} h_I^+ \in E_1^+$, moreover one verifies that

$$\left(\sum_{s=1}^{2p} k^{+,I}_{l,j_s}\right)_{(l,I)}$$

generates E_1^+ but we do not give the proof of it here because we will not use it bellow. Proposition IV.3 is proved. □

Before proving theorem I.2 we need a last proposition.

Proposition IV.5. *Let I and I' be two different choices of $2p$ integers among $\{1,\dots,4p\}$ such that $I\cup I'\neq\{1,\dots,4p\}$ we have*

(IV.41) $$\forall x\in S^{4p-1} \qquad \langle h_I^+(x)\,,\,h_{I'}^+(x)\rangle_{S^{4p-1}} = 0$$

where $<\,,\,>_{S^{4}p-1}$ denotes the scalar product of $2p$-forms on $\wedge^{2p}S^{4p-1}$.

Proof of proposition IV.5. Let $I=\{i_1,\dots,i_{2p}\}$ and $I'=\{i'_1,\dots,i'_{2p}\}$ we have

$$\forall x\in S^{4p-1} \qquad \langle h_I^+(x)\,,\,h_{I'}^+(x)\rangle_{S^{4p-1}} = dr\wedge h_I^+\,.\,dr\wedge h_{I'}^+(x).$$

On S^{4p-1} we have

$$dr\wedge h_I^+ = \sum_{s=1}^{2p} x_{j_s}\,dx_{j_s}\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_{2p}} + \sum_{t=1}^{2p} x_{i_t}\,dx_{i_t}\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{2p}}$$

and

$$dr \wedge h^+_{I'} = \sum_{s=1}^{2p} x_{j'_s}\, dx_{j'_s} \wedge dx_{i'_1} \wedge \cdots \wedge dx_{i'_{2p}} + \sum_{t=1}^{2p} x_{i'_t}\, dx_{i'_t} \wedge dx_{j'_1} \wedge \cdots \wedge dx_{j'_{2p}}$$

The result is straightforward if neither of the following occurs

$$\text{(IV.42)} \qquad \begin{array}{lll} \exists\, s_o, s'_o & \text{s.t.} & \{j_{s_o}\} \cup I = \{j'_{s'_o}\} \cup I' \quad \text{or} \\ \exists\, t_o, t'_o & \text{s.t.} & \{i_{t_o}\} \cup J = \{i'_{t'_o}\} \cup J' \quad \text{or} \\ \exists\, s_o, t'_o & \text{s.t.} & \{j_{s_o}\} \cup I = \{i'_{t'_o}\} \cup J' \quad \text{or} \\ \exists\, s'_o, t_o & \text{s.t.} & \{i_{t_o}\} \cup J = \{j'_{s'_o}\} \cup I' \end{array}$$

Suppose the first situation occurs. Since $I \neq I'$, there exists t_o such that $j'_{s'_o} = i_{t_o}$ and $I' = I_{i_{t_o}, j_{s_o}}$ and

$$\begin{aligned} dr \wedge h^+_I \,.\, dr \wedge h^+_{I_{t_o}, j_{s_o}} = & \\ \left(\sum_{s=1}^{2p} x_{j_s}\, dx_{j_s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} + \sum_{t=1}^{2p} x_{i_t}\, dx_{i_t} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} \right) & \\ \,.\, \Bigg(\sum_{s \neq s_o} x_{j_s}\, dx_{j_s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{t_o - 1}} \wedge dx_{j_{s_o}} \wedge \cdots \wedge dx_{i_{2p}} + & \\ + x_{i_{t_o}}\, dx_{i_{t_o}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{t_o - 1}} \wedge dx_{j_{s_o}} \wedge \cdots \wedge dx_{i_{2p}} - & \\ - \sum_{t \neq t_o} x_{i_t}\, dx_{i_t} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{s_o - 1}} \wedge dx_{i_{t_o}} \wedge \cdots \wedge dx_{j_{2p}} - & \\ - x_{j_{s_o}}\, dx_{j_{s_o}} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{s_o - 1}} \wedge dx_{i_{t_o}} \wedge \cdots \wedge dx_{j_{2p}} \Bigg) & \end{aligned}$$

Consider first the case $2p > 2$. In this case we have

$$\forall s\ \forall t \quad dx_{i_t} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{2p}} \,.\, dx_{j_s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{t_o - 1}} \wedge dx_{j_{s_o}} \wedge \cdots \wedge dx_{i_{2p}} = 0$$

and

$$\forall s\ \forall t \quad dx_{j_s} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} \,.\, dx_{i_t} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_{s_o - 1}} \wedge dx_{i_{t_o}} \wedge \cdots \wedge dx_{j_{2p}} = 0$$

This implies

$$
\begin{aligned}
&dr \wedge h_I^+ \,.\, dr \wedge h_{I_{t_o,j_{s_o}}}^+ \\
&= x_{j_{s_o}} x_{i_{t_o}}\, dx_{j_{s_o}} \wedge dx_{i_1} \cdots \wedge dx_{i_{2p}} \,.\, dx_{i_{t_o}} \wedge dx_{i_1} \cdots \wedge dx_{j_{s_o}} \wedge \cdots dx_{i_{2p}} - \\
&\quad - x_{i_{t_o}} x_{j_{s_o}} dx_{i_{t_o}} \wedge dx_{j_1} \cdots \wedge dx_{j_{2p}} \,.\, dx_{j_{s_o}} \wedge dx_{j_1} \cdots \wedge dx_{j_{s_o-1}} \wedge dx_{i_{t_o}} \cdots dx_{j_{2p}} \\
&= x_{j_{s_o}} x_{i_{t_o}} (-1)^{t_o-1+t_o} - x_{i_{t_o}} x_{j_{s_o}} (-1)^{s_o-1+s_o} = 0
\end{aligned}
$$

Remark that the two first situations of (IV.42) are equivalent and the two second also and one prove exactly inthe same way (IV.41) in the case where $\{j_{s_o}\} \cup I = \{i'_{t'_o}\} \cup J'$ (i. e. $J' = I_{i_{t_o}, j_{s_o}}$). Consider now the case $p = 1$ and compute

$$
\begin{aligned}
&\langle dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \,;\, dx_1 \wedge dx_3 + dx_4 \wedge dx_2 \rangle_{S^3} \\
&\quad = (x_4\, dx_4 \wedge dx_1 \wedge dx_2 + x_3\, dx_3 \wedge dx_1 \wedge dx_2 \\
&\qquad + x_1\, dx_1 \wedge dx_3 \wedge dx_4 + x_2\, dx_2 \wedge dx_3 \wedge dx_4) \\
&\qquad . (x_4\, dx_4 \wedge dx_1 \wedge dx_3 + x_2\, dx_2 \wedge dx_1 \wedge dx_3 \\
&\qquad + x_1\, dx_1 \wedge dx_4 \wedge dx_2 + x_3\, dx_3 \wedge dx_4 \wedge dx_2) \\
&\quad = -x_4 x_1 + x_1 x_4 - x_3 x_2 + x_3 x_2 = 0.
\end{aligned}
$$

Proposition IV.5 is proved. □

Proof of theorem I.2. Denote by F the following functional on $W^{1,2-\frac{1}{2p}}$ $(\wedge^{2p} S^{4p-1} \cap \operatorname{Ker} d)$

(IV.43)
$$
F(h) = \frac{1}{|S^{4p-1}|} \int_{S^{4p-1}} |h|^{2-\frac{1}{2p}} - \left| \frac{2p}{|S^{4p-1}|} \int_{S^{4p-1}} \eta \wedge h \right|^{1-\frac{1}{4p}}
$$

where $d\eta = h$. Let $I = \{i_1, \ldots, i_{2p}\}$ be a choice of $2p$ integer among $\{1, \ldots, 4p\}$. Denote by Q_I the following quadratic form on L^2:

(IV.44)
$$
\forall \phi \in L^2 \left(\wedge^{2p-1} S^{4p-1}\right) \qquad Q_I(d\phi) = \frac{d^2}{dt^2} F(h_I^+ + t\, d\phi)_{|_{t=0}}.
$$

This is well defined because $|h_I^+|_{S^{4p-1}} \equiv 1 > 0$ for any $x \in S^{4p-1}$. We shall prove the following lemma.

Lemma IV.1. *For any $d\phi \in L^2(\wedge^{2p}S^{4p-1})$ we have $Q_I(d\phi) \geqslant 0$ with equality if and only if $d\phi$ is a linear combination of the h_I^+, and the $\mathcal{L}_X h_I^+$ where X is any conformal Killing field of S^{4p-1}. Moreover if $d\phi$ is orthogonal to all of those previous forms for the L^2-norm we have*

$$Q_I(d\phi) \geqslant c \int_{S^{4p-1}} |d\phi|^2 \tag{IV.45}$$

where $c > 0$ is independant of ϕ.

Proof. Take $d\phi \in C^\infty(\wedge^{2p}S^{4p-1})$ after some computations we find

(IV.46)

$$Q_I(d\phi) = \frac{\left(1-\frac{1}{4p}\right)}{|S^{4p-1}|}\left[\int_{S^{4p-1}} |d\phi|^2 - \frac{1}{2p}\int_{S^{4p-1}} |d\phi\,.\,h_I^+|^2 - 2p\int_{S^{4p-1}} \phi\wedge d\phi + \frac{2p}{|S^{4p-1}|}\left(\int_{S^{4p-1}} \eta_I^+\wedge d\phi\right)^2\right]$$

Let us introduce a particular orthonormal basis for the L^2-norm in E_1^+. First of all remark that, if k and $l \in I$,

(IV.47)

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} \mathcal{L}_{X_k}h_I^+\,.\,\mathcal{L}_{X_l}h_I^+ = \frac{(2p)^2}{|S^{4p-1}|}\int_{S^{4p-1}} x_k x_l + \frac{\delta_{kl}}{|S^{4p-1}|}\int_{S^{4p-1}} \sum_{s=1}^{2p} x_{j_s}^2$$

where we have used the fact that $h_K^+\,.\,h_{K'}^+ \neq 0$ if and only if $K' = K$ or $K \cup K' = \{1,\dots,4p\}$. In the other hand we have

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} x_k x_l = \delta_{k,l}\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} x_k^2 = \frac{\delta_{k,l}}{4p},$$

and

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} \sum_{s=1}^{2p} x_{j_s}^2 = \frac{1}{2}.$$

Thus (IV.47) becomes

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} \mathcal{L}_{X_k}h_I^+\,.\,\mathcal{L}_{X_l}h_I^+ = \frac{2p+1}{2}. \tag{IV.48}$$

One verifies that this is still the case when k or l do not belong to I. Take

$$f_k^+ = \mathcal{L}_{\sqrt{\frac{2}{2p+1}}X_k} h_I^+$$

to be the $4p$ first elements of our basis. For any i_t, $i_{t'} \in I$ $(t \neq t')$ and $j_s \in J$, consider the $2p$-form

$$k_{i_t, i_{t'}}^{+, I_{i_t, j_s}} = x_{i_t}\, h_{I_{i_t}, j_s}^+ - x_{i_{t'}}\, h_{I_{t'}, j_s}^+ \tag{IV.49}$$

and for any $i_t \in L$ and j_s, $j_{s'} \in J$ $(s \neq s')$ the $2p$-form

$$k_{j_s, j_{s'}}^{+, I_{i_t, j_s}} = x_{j_s}\, h_{I_{i_t}, j_s}^+ - x_{j_{s'}}\, h_{I_t, j_{s'}}^+ \tag{IV.50}$$

Using still proposition IV.5 and the fact that

$$\frac{1}{|S^{4p-1}|} \int_{S^{4p-1}} x_k\, x_l = \frac{\delta_{k,l}}{4p}$$

one verifies easily that, in one hand, the forms defined by (IV.49) and (IV.50) are all orthogonal between themselves and that, in the other hand, they are all orthogonal with all the f_k^+ $(k \leqslant 4p)$. We complete the familly $(f_k^+)_{1 \quad k \quad 4p}$ with the forms (IV.49) and (IV.50) denoted

$$(f_k^+)_{4p+1 \quad k \quad N}$$

that we have renormalised in such a way that

$$\frac{1}{|S^{4p-1}|} \int_{S^{4p-1}} |f_k^+|^2 = 1 \qquad \text{for} \quad k \leqslant N.$$

One verifies also that $\forall t \in \{1, \ldots, 2p\}$ and $\forall s \in \{1, \ldots, 2p\}$

$$\begin{aligned} \frac{1}{2p}\left[f_{i_t}^+ + \sum_{s \neq s'} k_{j_s, j_{s'}}^{+, I_{i_t, j_s}} \right] &= -\, x_{i_t}\, h_I^+ + x_{j_s}\, h_{I_{i_t}, j_s}^+ \\ &= -\, k_{i_t, j_s}^{+, I} \end{aligned} \tag{IV.51}$$

and

$$\begin{aligned} \frac{1}{2p}\left[f_{j_s}^+ + \sum_{t \neq t'} k_{i_t, i_{t'}}^{+, I_{i_t, j_s}} \right] &= -\, x_{j_s}\, h_I^+ + x_{i_t}\, h_{I_{i_t}, j_s}^+ \\ &= -\, k_{j_s, i_t}^{+, I} \end{aligned} \tag{IV.52}$$

Consider $k^{+,I'}_{i',j'}$ which is neither of the form (IV.49), (IV.50) nor on the form (IV.51), (IV.52). We claim that

$$\text{(IV.53)} \qquad k^{+,I'}_{i',j'} \quad \text{is orthogonal to} \quad \underset{1\ k\ N}{\text{Vect}}\, f_k^+.$$

We can always assume that $i' \in I'$. (IV.53) is straightforward in the case where I' and $I'_{i',j'}$ are not on the form I_{i_t,j_s} or in the case where $I' = I$ or $I'_{i',j'}$. Suppose now $I' = I_{i_t,j_s}$. Since $i' \in I_{i_t,j_s}$, i' is equal to some $i_{t'}$ for $(t' \neq t)$ or $i' = j_s$. The case $i' = j_s$ has already been considered because in such case we have either $k^{+,I'}_{i',j'} = k^{+,I_{i_t,j_s}}_{j_s,j_{s'}}$ or $k^{+,I'}_{i',j'} = k^{+,I}_{i_s,j_s}$. Thus the only case we have to consider is

$$k^{+,I'}_{i',j'} = x_{i_{t'}}\, h^+_{I_{i_t,j_s}} - x_{j_{s'}}\, h^+_{I_{(i_t,i_{t'})\,(j_s,j_{s'})}}$$

where

$$I_{(i_t,i_{t'})\,(j_s,j_{s'})} = \big\{\{i_1,\ldots,i_{2p}\} \setminus \{i_t,i_{t'}\}\big\} \cup \{j_s,j_{s'}\}.$$

Using once again proposition IV.5 and the fact that

$$\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} x_k\, x_l = \frac{\delta_{kl}}{4p}$$

we deduce (IV.53) in the case where $I' = I_{i_t,j_s}$.

One proves exactly in the same way that (IV.53) holds in the case where $I'_{i',j'} = I_{i_t,j_s}$. Thus we have proved that (IV.53) holds in all the cases. Using this result and proposition IV.4 we complete the orthonormal familly $(f_k^+)_{1\ k\ N}$ by linear combinations of the $k^{+,I'}_{i',j'}$ which are neither of the form (IV.49), (IV.50) nor on the form (IV.51), (IV.52) in such a way to get an orthonormal basis $(f_k^+)_{1\ k\ N_p}$ of E_1^+ which verifies, using also proposition IV.5,

(IV.54)

$$\text{If} \quad \exists x \in S^{4p-1} \quad \text{such that} \quad f_k^+ \,.\, h_I^+(x) \neq 0$$
$$\text{then} \quad k \in \{1,\ldots,4p\} \ \text{and} \ f_k^+ = \mathcal{L}_{\sqrt{\frac{2}{2p+1}}X_k}\, h_I^+.$$

We are now in position to prove lemma IV.1, with the help of this basis of E_1^+.

Decompose $d\phi$ in $(E_0^+ \oplus E_1^+) \oplus (E_0^+ \oplus E_1^+)^\perp$ we have

$$\text{(IV.55)} \qquad d\phi = \alpha_I^+ h_I^+ + \sum_{I' \neq I} \alpha_{I'}^+ + \sum_{k=1}^{N_p} \beta_k^+ f_k^+ + dR$$

where $dR \in (E_0^+ + E_1^+)^\perp$. Let us compute the 4 terms in the left hand side of equality (IV.46). We have

(IV.56)
$$I_1 = \frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} |d\phi|^2 = \left(\alpha_I^+\right)^2 + \sum_{I'\neq I}\left(\alpha_{I'}^+\right)^2 + \sum_{k=1}^{N_p}\left(\beta_k^+\right)^2 + \frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} |dR|^2.$$

We have also

(IV.57)
$$I_2 = \frac{2p}{|S^{4p-1}|^2}\left(\int_{S^{4p-1}} \eta_I^+ \wedge d\phi\right)^2 = \frac{1}{2p}\left(\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} (*) h_I^+ \wedge d\phi\right)^2 = \frac{\left(\alpha_I^+\right)^2}{2p}$$

and

(IV.58)
$$I_3 = \frac{2p}{|S^{4p-1}|}\int_{S^{4p-1}} \phi \wedge d\phi = \left(\alpha_I^+\right)^2 + \sum_{I'\neq I}\left(\alpha_{I'}^+\right)^2 + \sum_{k=1}^{N_p}\frac{2p}{2p+1}\left(\beta_k^+\right)^2 + \frac{2p}{|S^{4p-1}|}\int_{S^{4p-1}} R\wedge dR.$$

Finally

(IV.59)
$$I_4 = \frac{1}{2p\,|S^{4p-1}|}\int_{S^{4p-1}} |d\phi\,.\,h_I^+|^2 = \frac{1}{2p\,|S^{4p-1}|}\int_{S^{4p-1}}\left(\alpha_I^+ - \sum_{k=1}^{4p} 2p\sqrt{\frac{2}{2p+1}}\,\beta_k^+ x_k + h_i^+\,.\,dR\right)^2$$

where we have used (IV.40) and (IV.54). We compute

(IV.60)

$$
\begin{aligned}
I_4 = & \frac{(\alpha_I^+)^2}{2p} + \sum_{k=1}^{2p} \frac{4p}{2p+1\,|S^{4p-1}|} (\beta_k^+)^2 \int_{S^{4p-1}} x_k^2 \\
& + \frac{1}{|S^{4p-1}|} \int_{S^{4p-1}} \left|h_I^+ \,.\, dR\right|^2 + \frac{2}{2p\,|S^{4p-1}|} \alpha_I^+ \int_{S^{4p-1}} h_I^+ \,.\, dR \\
& - \frac{2}{|S^{4p-1}|} \sqrt{\frac{2}{2p+1}} \sum_{k=1}^{4p} \beta_k^+ \int_{S^{4p-1}} x_k \, h_I^+ \,.\, dR
\end{aligned}
$$

where we have used the fact $\int_{S^{4p-1}} x_k = 0$. We claim that the two last terms of (IV.60) are equal to 0. $\int_{S^{4p-1}} h_I^+ \,.\, dR = 0$ because $dR \in E_0^\perp$.

We prove, now, the following

(IV.61)

$$
\forall dR \in \left(E_1^+\right)^\perp \quad \forall k \in \{1, \dots, 4p\} \qquad \int_{S^{4p-1}} x_k \, h_I^+ \,.\, dR = 0.
$$

Suppose $k \in I$, $k = i_{t_0}$ and compute

$$
\begin{aligned}
d\,(*)\left[x_k\, h_I^+\right] &= 2\,p x_k\, h_I^+ + 2p\, dx_k \wedge \eta_I^+ \\
&= 2p\, x_k\, h_I^+ + x_{i_{t_0}}\, dx_{i_1} \wedge \cdots \wedge dx_{i_{2p}} \\
&\quad + \sum_{s=1}^{2p} x_{j_s}\, dx_{j_1} \wedge \cdots \wedge dx_{j_{s-1}} \wedge dx_{i_{t_0}} \wedge \cdots \wedge dx_{j_{2p}}.
\end{aligned}
$$

Using the fact that

$$
x_{i_{t_0}} dx_{i_{t_0}} = -\sum_{t \neq t_0} x_{i_t}\, dx_{i_t} - \sum_{s=1}^{2p} x_{j_s}\, dx_{j_s} \quad \text{on} \quad \wedge^{2p} S^{4p-1}
$$

we get

$$
\begin{aligned}
d\,(*)\left[x_k\, h_I^+\right] = {} & 2p x_k\, h_I^+ + \\
& \sum_{s=1}^{2p} x_{j_s} \left[dx_{i_1} \wedge \cdots \wedge dx_{i_{t_0 - 1}} \wedge dx_{j_s} \wedge \cdots \wedge dx_{i_{2p}} \right. \\
& \qquad \left. - dx_{j_1} \wedge \cdots \wedge dx_{j_{s-1}} \wedge dx_{i_{t_0}} \wedge \cdots \wedge dx_{j_{2p}}\right].
\end{aligned}
$$

Thus

(IV.62)
$$d(*)\left[x_k\, h_I^+\right] = -\mathcal{L}_{X_k}\, h_I^+ = -\frac{1}{2p+1}\, d(*)\mathcal{L}_{X_k}\, h_I^+.$$

This implies the existence of a $(2p-1)$-form ξ such that

$$x_k\, h_I^+ = -\frac{1}{2p+1}\mathcal{L}_{X_k}\, h_I^+ + (*)d\xi.$$

We clearly have $\int_{S^{4p-1}}(*)d\xi\ .\ dR = 0$ and since $dR \in (E_1^+)^\perp$, (IV.61) is proved and I_4 is equal to

(IV.63)
$$I_4 = \frac{(\alpha_I^+)^2}{2p} + \sum_{k=1}^{2p}\frac{(\beta_k^+)^2}{2p+1} + \frac{1}{2p\,|S^{4p-1}|}\int_{S^{4p-1}}\left|h_I^+\,.\,dR\right|^2.$$

Since $Q_I(d\phi) = (1-1/4p)[I_1 - I_4 + I_2 - I_3]$, combining (IV.56), (IV.57), (IV.58) and (IV.63) we get

(IV.64)
$$\begin{aligned} Q_I(d\phi) = \left(1-\frac{1}{4p}\right)\Bigg[&\frac{1}{2p+1}\sum_{4p+1}^{N_p}(\beta_k^+)^2 + \frac{1}{|S^{4p+1}|}\int_{S^{4p+1}}|dR|^2 \\ &- \frac{1}{2p\,|S^{4p-1}|}\int_{S^{4p-1}}\left|h_I^+\,.\,dR\right|^2 - \frac{2p}{|S^{4p-1}|}\int_{S^{4p-1}}R\wedge dR\Bigg] \\ \geqslant \left(1-\frac{1}{4p}\right)\Bigg[&\frac{1}{2p+1}\sum_{4p+1}^{N_p}(\beta_k^+)^2 + \frac{2p-1}{2p}\,\frac{1}{|S^{4p+1}|}\int_{S^{4p+1}}|dR|^2 \\ &- \frac{2p}{|S^{4p-1}|}\int_{S^{4p-1}}R\wedge dR\Bigg], \end{aligned}$$

where we have used the fact that $|h_I^+|\equiv 1$ on S^{4p-1}. dR admits a decomposition $dR = dR^+ + dR^-$ in

$$\bigoplus_{i=2\ldots\infty} E_i^+ \oplus \bigoplus_{i=0\ldots\infty} E_i^-$$

and we have

(IV.65)
$$-\int_{S^{4p-1}} R\wedge dR = -\int_{S^{4p-1}} R^+\wedge dR^+ - \int_{S^{4p-1}} R^-\wedge dR^-$$
$$\geqslant -\int_{S^{4p-1}} R^+\wedge dR^+$$

Using the fact that the third positive eigenvalue of $d(*)$ on E^+ is $2p+2$, we have

(IV.66)
$$\int_{S^{4p-1}} |dR^+|^2 \geqslant (2p+2)\int_{S^{4p-1}} R^+\wedge dR^+.$$

Combining (IV.64), (IV.65) and (IV.66) we get

(IV.67)
$$Q_I(d\phi) \geqslant \left(1-\frac{1}{4p}\right)\left[\frac{1}{2p+1}\sum_{4p+1}^{N_p}\left(\beta_k^+\right)^2 + \frac{2p-1}{2p}\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}|dR^-|^2 + \right.$$
$$\left. + \frac{p-1}{p\,(2p+2)}\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}}\left|dR^+\right|^2\right].$$

(IV.67) proves Lemma IV.1 for $p>1$.

In the case $p=1$ one has to refine a little the lower bounds. We start from (IV.64). We decompose dR in $E_2^+\oplus(E_2^+)^\perp$: $dR = dR_2 + dR'$. We have

(IV.68)
$$\int_{S^{4p-1}}\left|dR\,.\,h_I^+\right|^2 = \int_{S^{4p-1}}\left|dR_2\,.\,h_I^+\right|^2 + \int_{S^{4p-1}}\left|dR'.\,h_I^+\right|^2$$
$$+2\int_{S^{4p-1}} dR_2\,.\,h_I^+\times dR'.\,h_I^+.$$

Using proposition IV.2 and proposition IV.5 we obtain that

$$dR_2\,.\,h_I^+ = \alpha\, x_k\, x_l \quad \text{for} \quad \alpha\in\mathbb{R} \text{ and } k,l\in\{1,\dots,4p\}.$$

The third term of (IV.68) becomes

$$2\alpha\int_{S^{4p-1}} x_k x_l h_I^+\,.\,dR'.$$

Let us study the Hodge decomposition of $x_k\, x_l\, h_I^+$ in $\wedge^{2p} S^{4p-1}$. We have

(IV.69)
$$d\,(*)\left[x_k\, x_l\, h_I^+\right] = x_k\, dx_l \wedge (*) h_I^+ + x_l\, d\left[x_k\,(*)\,h_I^+\right]$$
$$= -x_k\, \mathcal{L}_{X_l}\, h_I^+ - x_l\, \mathcal{L}_{X_k}\, h_I^+ - 2p\, x_l\, x_k\, h_I^+$$

where we have used (IV.62). Let P be the following homogeneous degree 2, $2p$-form in $\mathbb{R}^{4p}$

(IV.70)
$$P = x_k\left[2px_l h_I^+ - \sum_{s=1}^{2p} x_{j_s}\, h^+_{I_{j_s,l}}\right] + x_l\left[2px_k h_I^+ - \sum_{s=1}^{2p} x_{j_s}\, h^+_{I_{j_s,k}}\right] - 2px_l x_k h_I^+ .$$

We have $*P = P$ moreover P restricted to S^{4p-1} is equal to $d(*)\left[x_k\, x_l\; h_I^+\right]$, thus dP restricted to S^{4p-1} is equal to 0 and since $*dP$ is a coclosed degree 1 polynomial $2p$-form one verifies, uding corollary 6.6 of [11] that $dP = 0$ in $\mathbb{R}^{4p}$. P is a self-dual closed degree 2 homogeneous $2p$-form of $\mathbb{R}^{4p}$ thus, from proposition IV.2, $d(*)\left[x_k\, x_l\; h_I^+\right]$, the restriction of P to S^{4p-1} belongs to E_2^+ and we have

(IV.71)
$$\exists g \in E_2^+ \text{ and } s \in \wedge^{2p-1} S^{4p-1} \quad \text{s.t.} \quad x_k\, x_l\, h_I^+ = g + (*)ds$$

This implies

(IV.72)
$$\int_{S^{4p-1}} |dR\,.\,h_I^+|^2 = \int_{S^{4p-1}} |dR_2\,.\,h_I^+|^2 + \int_{S^{4p-1}} |dR'\,.\,h_I^+|^2$$

Since none element of E_2^+ is of the form $a(x)\; h_I^+$ there exists $\gamma < 1$ such that

(IV.73)
$$\max_{dR_2 \in E_2^+} \left(\frac{\displaystyle\int_{S^{4p-1}} |dR_2\,.\,h_I^+|^2}{\displaystyle\int_{S^{4p-1}} |dR_2|^2} \right) = \gamma$$

We get

(IV.74)
$$\int_{S^{4p-1}} |dR|^2 - \frac{1}{2p}\int_{S^{4p-1}} |h_I^+\,.\,dR|^2$$
$$= \int_{S^{4p-1}} |dR_2|^2 + |dR'|^2 - \frac{1}{2p}\int_{S^{4p-1}} |h_I^+\,.\,dR_2|^2 - \frac{1}{2p}\int_{S^{4p-1}} |h_I^+\,.\,dR'|^2$$
$$\geqslant \left(1 - \frac{\gamma}{2p}\right)\int_{S^{4p-1}} |dR_2|^2 + \left(1 - \frac{1}{2p}\right)\int_{S^{4p-1}} |dR'|^2$$

We decompose $dR' = dR'^{+} + dR'^{-}$ in

$$\bigoplus_{i=3\ldots\infty} E_i^{+} \oplus \bigoplus_{i=0\ldots\infty} E_i^{-}.$$

This time we have

$$\int_{S^{4p-1}} \left|dR'^{+}\right|^2 \geqslant (2p+3) \int_{S^{4p-1}} R'^{+} \wedge dR'^{+} \tag{IV.75}$$

Combining (IV.56), (IV.57), (IV.58), (IV.63), (IV.65), (IV.74) and (IV.75) we have

(IV.76) $Q_I(d\phi) \geqslant$

$$\left(1 - \frac{1}{4p}\right)\left[\frac{1}{2p+1}\sum_{4p+1}^{N_p} \left(\beta_k^{+}\right)^2 + \frac{2p-1}{2p}\frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} \left|dR'^{-}\right|^2 \right.$$
$$\left. + \frac{4p-3}{2p(2p+3)}\frac{1}{S^{4p-1}}\int_{S^{4p-1}} \left|dR'^{+}\right|^2 + \frac{(2-\gamma)p-\gamma}{2p(p+1)}\left|dR_2'^{+}\right|^2\right]$$

Since $\gamma < 1$ lemma IV.1 is proved in all the cases. □

Using the result established at the end of the proof of proposition IV.1 we have

$$\forall \psi \in \wedge^{2p-1} S^{4p-1} \quad \exists g \in \mathrm{Conf}^{+}\left(S^{4p-1}\right) \qquad \text{such that}$$
$$\int_{S^{4p-1}} x\zeta(g) \wedge dg^{*}\psi = 0,$$

where $\zeta(g)$ is the $(2p-1)$-form verifying $d * \zeta(g) = 0$ and $d\zeta(g) = g^{*}d\psi$. Since $F(g^{*}d\psi) = F(d\psi)$ we are tempted to work on M

$$M = \left\{ \begin{array}{c} h_I^{+} + d\phi \quad \text{s.t.} \quad d\phi \in L^{\infty} \text{ and } \displaystyle\int_{S^{4p-1}} x\phi + \eta_I^{+} \wedge d\phi + h_I^{+} = 0 \\ \text{where} \quad d(*)\phi = 0 \end{array} \right\}$$

Let $d\psi \in M$ such that

$$\left|h_I^{+} - d\psi\right|_{\infty} \leqslant \alpha \quad \text{and} \quad \frac{1}{|S^{4p-1}|}\int_{S^{4p-1}} |d\psi|^2 = 1,$$

decompose $d\psi$ in $E_0^+ \oplus (E_0^+)^\perp$ we have

$$d\psi = \sum_{I'} \alpha_{I'} h_{I'}^+ + d\phi = (1+\delta)h_I^+ + \mu h^+ + d\phi,$$

where $h^+ \in E_0^+$, $h^+ . h_I^+ = 0$ and $\int_{S^{4p-1}} |h^+|^2 = 1$. We claim that, for α sufficiently small, independant of $d\psi$ we have $F(d\psi) \geqslant 0$. We have

$$|\delta| = \left| \int_{S^{4p-1}} h_I^+ . \left(d\psi - h_I^+\right) \right| \leqslant \alpha \quad \text{and} \quad |\mu| = \left| \int_{S^{4p-1}} h^+ . \left(d\psi - h_I^+\right) \right| \leqslant \alpha$$

Denote $\widetilde{h}^+ = (1+\delta)\, h_I^+ + \mu\, h^+$ and we compute

$$F(d\psi) =$$

$$= \frac{\left(1 - \frac{1}{4p}\right)}{\left|\widetilde{h}^+\right| \, |S^{4p-1}|} \left[\int_{S^{4p-1}} |d\phi|^2 - \frac{1}{2p} \frac{\left|d\phi . \widetilde{h}^+\right|^2}{\left|\widetilde{h}^+\right|^2} - 2p \int_{S^{4p-1}} \phi \wedge d\phi \right] + O\left(\alpha \int_{S^{4p-1}} |d\phi|^2\right)$$

but

$$\left| \int_{S^{4p-1}} \frac{\left|d\phi . \widetilde{h}^+\right|^2}{\left|\widetilde{h}\right|^2} - \int_{S^{4p-1}} \left|d\phi . h_I^+\right|^2 \right| \leqslant C\alpha \int_{S4p-1} |d\phi|^2$$

Thus we have

$$|F(d\psi) - Q_I(d\phi)| \leqslant C\alpha \int_{S^{4p-1}} |d\phi|^2 \tag{IV.77}$$

Let us decompose $d\phi$ in

$$\mathrm{Vect}\{\mathcal{L}_{X_k} h_I^+\} \oplus \left(\mathrm{Vect}\{\mathcal{L}_{X_k} h_I^+\}\right)^\perp$$

we have

$$d\phi = \sum_{k=1}^{4p} \beta_k \mathcal{L}_{X_k} h_I^+ + dR = d\left(\frac{1}{2p+1} \sum_{k=1}^{4p} \beta_k (*) \mathcal{L}_{X_k} h_I^+ + R \right)$$

where we have chosen R such that $d(*)R = 0$. Since $d\psi$ is in M we get

$$\int_{S^{4p-1}} x \left[\widetilde{h}^+ + \sum_{k=1}^{4p} \beta_k \mathcal{L}_{X_k} h_I^+ + dR \right] \wedge \left[\widetilde{\eta}^+ + \frac{1}{2p+1} \sum_{k=1}^{4p} \beta_k (*) \mathcal{L}_{X_k} h_I^+ + R \right] . \tag{IV.78}$$

Since $\widetilde{h}^+ \wedge \tilde{\eta}^+ = \frac{1}{2p}\, \omega_{S^{4p-1}}$

(IV.79) $$\int_{S^{4p-1}} x\widetilde{h}^+ \wedge \tilde{\eta}^+ = 0.$$

From (IV.61) we have

(IV.80) $$\int_{S^{4p-1}} x\, dR \wedge \eta_I^+ = \frac{1}{2p} \int_{S^{4p-1}} x\, dR\, .\, h_I^+ = 0,$$

and

(IV.81)
$$\left| \int_{S^{4p-1}} x\, dR \wedge \tilde{\eta}^+ - \int_{S^{4p-1}} x\, dR \wedge \eta_I^+ \right| \leqslant C\alpha \left(\int_{S^{4p-1}} |dR|^2 \right)^{\frac{1}{2}}.$$

Moreover, since $d(*)R = 0$ and

$$(*)R \in \left(E_0^+ \oplus \operatorname*{Vect}_{k=1\ldots 4p} \{\mathcal{L}_{X_k} h_I^+\} \right)^{\perp}$$

we have also

(IV.82) $$\int_{S^{4p-1}} xh_I^+ \wedge R = \int_{S^{4p-1}} xh_I^+\, .\, (*)R = 0$$

and

(IV.83)
$$\left| \int_{S^{4p-1}} x\widetilde{h}^+ \wedge R - \int_{S^{4p-1}} xh_I^+ \wedge R \right| \leqslant C\alpha \left(\int_{S^{4p-1}} |dR|^2 \right)^{\frac{1}{2}}.$$

One verifies also, using (IV.40), that

(IV.84)
$$\forall k = 1, \ldots 4p \qquad \int_{S^{4p-1}} xh_I^+\, .\, \mathcal{L}_{X_k} h_I^+ = -\,2p \int_{S^{4p-1}} x_k^2\, e_k = -\,\frac{1}{2}\, \left|S^{4p-1}\right| e_k$$

where $(e_i)_{i=1\ldots 4p}$ is the canonical basis of $\mathbb{R}^{4p}$. Combining (IV.78) ... (IV.84), we have

(IV.85)
$$\forall k = 1, \ldots, 4p \qquad |\beta_k| \leqslant C\alpha \left(\int_{S^{4p-1}} |dR|^2 \right)^{\frac{1}{2}}.$$

This yields

$$\text{(IV.86)} \qquad \int_{S^{4p-1}} |d\phi|^2 \leqslant C \int_{S^{4p-1}} |dR|^2.$$

From Lemma IV.1 we have

$$Q_I(d\phi) \geqslant C \int_{S^{4p-1}} |dR|^2.$$

Combining (IV.86), the previous inequality and (IV.77), for α chosen sufficiently small, we have

$$\text{(IV.87)} \quad \forall d\psi \in M \quad \text{s.t.} \quad \left|d\psi - h_I^+\right|_\infty \leqslant \alpha$$
$$F(d\psi) \geqslant C \int_{S^{4p-1}} |d\phi|^2 \geqslant C \operatorname{dist}^2\left(d\psi\,,\, E_0^+\right)$$

Since $\int_{S^{4p-1}} x\; h_I^+ \wedge \eta_I^+ = 0$ there exists $\beta > 0$ such that

$$\forall \psi \in \wedge^{2p} S^{4p-1} \qquad \text{s.t.} \quad \left|d\psi - h_I^+\right|_\infty \leqslant \beta$$
$$\exists g \in \operatorname{Conf}^+\left(S^{4p-1}\right) \quad \text{s.t.} \quad g^* d\psi \in M, \ \left|g^* d\psi - h_I^+\right| \leqslant \alpha.$$

Thus

$$\left|d\psi - h_I^+\right|_\infty \leqslant \beta \implies F(d\psi) \geqslant 0$$

with equality if and only if

$$d\psi \in \operatorname{Conf}^+\left(S^{4p-1}\right)^* E_0^+.$$

This proves theorem I.2. □

IV.4. Stability of the complex Hopf fibration.

In this part we establish theorem I.1. As it is stated in the introduction we just have to identify $H^*\omega_{S^2}$, where H is the complex Hopf fibration, with the restriction to S^3 of $4\,h_1^+ = 4\,dx_1 \wedge dx_2 + 4\,dx_3 \wedge dx_4$.

Since H is a transversally conformal map having constant gradient, i.e. $|\nabla H| \equiv 2\sqrt{2}$ in S^3 we have $|H^*\omega_{S^2}| \equiv 4$ on S^3. Moreover the coimages of points are given in S^3 by the left multiplication of unit complex number in $\mathbb{R}^4 \simeq \mathbb{H}$

$$e^{i\theta} \,.\, (x_1, x_2, x_3, x_4) = (\cos\theta\, x_1 - \sin\theta\, x_2\,,\, \cos\theta\, x_2 + \sin\theta\, x_1\,,$$
$$\cos\theta\, x_3 - \sin\theta\, x_4\,,\, \cos\theta\, x_4 + \sin\theta\, x_3).$$

Thus the unit 1-form in $\wedge^1 S^3$ tangent to the coimages is

$$-x_2\,dx_1 + x_1\,dx_2 - x_4\,dx_3 + x_3\,dx_4.$$

This implies

$$(*) H^*\omega_{S^2} = 4(-x_2\,dx_1 + x_1\,dx_2 - x_4\,dx_3 + x_3\,dx_4) = 8\eta_1^+$$

thus $H^*\omega_{S^2} = 4\ h_1^+$ and theorem I.1 is proved. □

Remark IV.3. Denote by H and H respectively the quaternionic and the Cayley or octonionic Hopf fibrations from S^7 into S^4 and S^{15} into S^8 (see the definitions in [5] for instance). H and H are also transversally conform they have uniform gradient , i. e. $|\nabla H\ |_{S^7} \equiv 4$ and $|\nabla H\ |_{S^{15}} \equiv 4\sqrt{2}$ and they are fibrations having Hopf degree equal to 1. For all those reasons we are tempted to study the minimality of their p energy in their homotopy class using the same approach used above for the complex Hopf map but one verifies that $H^*\ \omega_{S^7}$ and $H^*\ \omega_{S^{15}}$are not in E_0^+.

We justify in the end of this part the last statement in the previous remark.

Since H and H have Hopf degree 1 we have

$$\frac{1}{|S^4|^2}\int_{S^7} H^*\,\omega_{S^4}\wedge\eta\ \ = 1 \quad\text{and}\quad \frac{1}{|S^8|^2}\int_{S^{15}} H^*\,\omega_{S^8}\wedge\eta\ \ = 1$$

where $d\eta\ \ = H^*\,\omega_{S^4}$ and $d\eta\ \ = H^*\,\omega_{S^8}$. Moreover we have

$$\int_{S^7} |H^*\,\omega_{S^4}|^2 = |\,S^7|\ 4^4 \quad\text{and}\quad \int_{S^{15}} |H^*\,\omega_{S^8}|^2 = |\,S^{15}|\ 4^8$$

Thus

$$\frac{\displaystyle\int_{S^7} |H^*\,\omega_{S^4}|^2}{\displaystyle\int_{S^7} H^*\,\omega_{S^4}} = \frac{2\times\pi^4\times\Gamma\left(\frac{5}{2}\right)^2\times 4^4}{2^2\times\pi^5\times\Gamma(4)} = 12 > 4,$$

and

$$\frac{\displaystyle\int_{S^{15}} |H^*\,\omega_{S^8}|^2}{\displaystyle\int_{S^{15}} H^*\,\omega_{S^8}\wedge\eta} = \frac{2\times\pi^8\times\Gamma\left(\frac{9}{2}\right)^2\times 4^8}{2^2\times\pi^9\times\Gamma(8)} = 1120 > 8.$$

V. Geometric lower-bounds of the 3-energy among the fibrations from S^3 into S^2.

In this part we restrict ourselves to the case $p = 1$. The maps u, that we will consider, are fibrations of the form $u = H \circ \psi$ where H is the Hopf fibration from S^3 into S^2 and ψ an orientation preserving diffeomorphism from S^3 into S^3.

Consider a closed oriented regular embedded curve Γ in S^2 such that Γ separates the north and the south pôles of S^2. Denote by Σ_N and Σ_S the two disjoint open sets, having Γ as boundary, containing respectively the north and the south pôles. One can allways assume that the orientation of Γ is chosen such that $\partial\Sigma_N = -\partial\Sigma_S = \Gamma$ where Σ_N and Σ_S are oriented like S^2. Denote by σ_N and σ_S the generators of $\pi_1(u^{-1}(\Gamma)) \simeq \pi_1(S^1 \times S^1)$ such that any representant of σ_N (resp. σ_S) corresponds to the positive generator of $\pi_1(u^{-1}(\Sigma_N)) = \pi_1(D^2 \times S^1) = \mathbb{Z}$ (resp. to the positive generator of $\pi_1(u^{-1}(\Sigma_S)) = \mathbb{Z}$) where the orientations of $u^{-1}(\Sigma_N)$ and $u^{-1}(\Sigma_S)$ are given by the orientation of S^3 and the orientation of the fibers. One verifies that u restricted to $u^{-1}(\Gamma)$ realizes a $(1,1)$ map into $\Gamma \simeq S^1$. That is, if η_N and η_S denote respectively the Poincaré duals of σ_N and σ_S in $u^{-1}(\Gamma)$ we have

$$\int_{u^{-1}(\Gamma)} u^*d\theta \wedge \eta_N = 1 = \int_{u^{-1}(\Gamma)} u^*d\theta \wedge \eta_S,$$

where $d\theta$ is a generator of $H^1_{dR}(\Gamma)$.

Let (T, g, σ) be a riemannian genus one surface where we have identified two generators σ_1 and σ_2 of $\pi_1(T)$. Such object will be called a marked genus one surface and simply denoted by T. Consider the following set

$$W(T) = \left\{v \in W^{1,2}\left(T, S^1\right) \text{ such that } \deg_\sigma V = (1,1)\right\}$$

where $W^{1,2}(T, S^1)$ is the set of map from (T, g) into S^1 having gradient in L^2, $\deg_\sigma v$ is the couple of integer

$$\left(\int_T u^*d\theta \wedge \eta_1\,,\ \int_T u^*d\theta \wedge \eta_2\right)$$

where η_1 and η_2 are the Poincaré duals of σ_1 and σ_2 and $d\theta$ is a form generating $H^1_{dR}(S^1)$.

We introduce, now, a quantity which plays a central role in this part

$$\text{(V.1)} \qquad I(T) = \inf_{v \in W(T)} \int_T |\nabla v|^2 dvol_g.$$

First observe that, if (T', g', σ') is another marked genus one surface, such that there exists a conformal diffeomorphism ϕ from T into T' preserving the choice of generators i. e.

$$\exists f \;\; T \to \mathbb{R} \qquad \text{s.t.} \qquad g = e^f \phi^* g' \quad \text{and} \quad \phi^* \sigma_i = \sigma_i' \quad \text{for} \quad i = 1, 2$$

where ϕ^* denotes the induced map by ϕ from $\pi_1(T)$ into $\pi_1(T')$. Then we have

$$I(T) = I(T'). \tag{V.2}$$

This corresponds to say that $I(T)$ only depends on the Teichmüler class of T (see for instance [12]). The Teichmüler Space in the case of genus 1 is given by the flat Tori $\mathbb{C}/\Gamma_\tau$ where γ_τ is a lattice group of the form

$$\Gamma_\tau = \left\{ m + n\tau \qquad \text{s.t.} \quad (m, n) \in \mathbb{Z}^2 \right\}$$

for any $\tau \in H = \{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$. Thus we just have to compute I for such tori. We have the following proposition.

Proposition V.1. *Let T be a marked genus 1 surface and let $\tau = a + i\,b$ be it's Teichmüller class in H we have*

$$I(T) = 4\pi^2 \left[b + \frac{(1-a)^2}{b} \right]. \tag{V.3}$$

Proof. T is identified with it's corresponding representant $\mathbb{C}/_{\Gamma_\tau}$. Let v be a map of $W(T)$. Consider the following diffeomorphism ψ which sends $\mathbb{C}/_{\Gamma_\tau}$ into $\mathbb{C}/_{\Gamma_1}$

$$\psi\left((x, y) \right) = \left(x - \frac{a}{b} y \,,\, \frac{y}{b} \right)$$

and let $w = v \circ \psi^{-1}$ we have

$$\int_T |\nabla v|^2 = \int_0^1 \int_0^1 b \left| \frac{\partial w}{\partial x} \right|^2 + \frac{1}{b} \left| \frac{\partial w}{\partial y} - a \frac{\partial w}{\partial x} \right|^2 \tag{V.4}$$

W is $(1,1)$ from $\mathbb{C}/\Gamma_1$ into S^1 thus we can write w in the form

$$w = e^{2\pi i\, (x + y + f(x,y))}$$

where f is a $\mathbb{Z}^2$ periodic function in $W^{1,2}_{loc}(\mathbb{C})$. (V.4) becomes

$$\int_T |\nabla v|^2 = 4\pi^2 \int_0^1 \int_0^1 b \left(1 + \left| \frac{\partial f}{\partial x} \right|^2 \right) + \frac{1}{b} \left((1-a)^2 + \left| \frac{\partial f}{\partial y} - a \frac{\partial f}{\partial x} \right|^2 \right).$$

The minimum is achieved for $f = cte$ and this yields the desired result. □

The following proposition illustrates the link between the function I and the 3-energy on S^3.

Proposition V.2. *Let u be a fibration from S^3 into S^2, we have*

$$\text{(V.5)}\qquad \begin{aligned}\int_{S^3}|\nabla u|^3 &\geqslant \frac{\sqrt{2}}{\pi}\int_{S^2} I\left(u^{-1}(S_y)\right)dy\\ &= 4\sqrt{2}\pi\int_{S^2}\frac{b^2(y)+(1-a(y))^2}{b(y)}\,dy\end{aligned}$$

where S_y is the great circle perpendicular to y and $a+ib(y)$ is the Teichmüler class of $u^{-1}(S_y)$ for the generators of π_1 mentioned above. Equality holds in (V.5) when u is the composition of any conformal map of S^3 with the Hopf fibration: in this case $a(y)\equiv 1$ and $b(y)\equiv 1$.

Remark V.1. The two sides of inequality (V.5) are invariant under the action of the conformal group of S^3, i. e. when we replace u by $u\circ\psi$ where $\psi\in \text{Conf}(S^3)$.

Remark V.2. It is a natural question to ask wether we have to deal in proposition V.2 with all the $a+ib$ in the upper half complex plane. Indeed, from [7] (see also the proof of [18]) we know that each conformal class of flat tori admits a conformal embedding in S^3 but it is still an open question to know whether each marked flat torus can be conformally embedded in S^3 such that there exists an isotopy of S^3 which diffeomorphically deforms our embedding into the the standard Clifford torus and which sends the chosen generators of our embedded torus into the standard generators of the Clifford torus.

For proving proposition V.2 we will use an inequality stated in the following lemma. Let $y\in S^2$, denote by ρ_y the distance function relative to y on S^2 and by $d\theta_y$ the 1-form perpendicular to $d\rho_y$ such that $d\rho_y\wedge d\theta_y = \frac{1}{\sin\rho_y}\,\omega_{S^2}$ we have

Lemma V.1. *Let u be a $W^{1,3}(S^3,S^2)$ map, the following inequality holds*

$$\text{(V.6)}\qquad \int_{y\in S^2}\int_{S^3}|u^*\,d\rho_y|^2\,|u^*\,d\theta_y| \leqslant \frac{\pi^2}{\sqrt{2}}\int_{S^3}|\nabla u|^3.$$

Equality holds in (V.6) if and only if u is transversally conform.

Proof. Let $x \in S^3$ and let $\theta_{u(x)}$ be an angle function in $S^2 \setminus \{\pm u(x)\}$ with value in $\mathbb{R}/2\pi\,\mathbb{Z}$ corresponding to the choice of an orthonormal basis in $T_{u(x)}S^2$. Using the coarea formula of Federer we have, for any x in S^3,

(V.7)

$$\int_{y\in S^2} |u^*\, d\rho_y|^2\, |u^*\, d\theta_y| = \int_{\theta=0}^{2\pi} d\theta \int_{\theta_{u(x)}=\theta} |u^*\, d\rho_y|^2 \frac{\left|u^*\, d\theta_y\right|}{\left|d\theta_{u(x)}\right|}$$

but $|\sin\rho_{u(x)}(y)|\ |d\theta_{u(x)}| = 1$ and $|\sin\rho_{u(x)}(y)| = |\sin\rho_y(u(x))|$. Moreover for $\theta_{u(x)}$ constant equal to θ, $d\rho_y$ and $\sin\rho_y\ d\theta_y$ are constant equal to $\cos\theta\, dx_1 + \sin\theta\, dx_2$ and $-\sin\theta\, dx_1 + \cos\theta\, dx_2$ where we have fixed an orthonormal basis $(dx_1\,,\, dx_2)$ of $T^*_{u(x)}S^2$ corresponding to the choice of $\theta_{u(x)}$. Thus (V.7) becomes

$$\text{(V.8)}\quad \int_{y\in S^2} |u^*\, d\rho_y|^2\ |u^*\, d\theta_y|$$
$$= \pi \int_{\theta=0}^{2\pi} |u^* \cos\theta\, dx_1 + \sin\theta\, dx_2|^2\ |u^* - \sin\theta\, dx_1 + \cos\theta\, dx_2|\, d\theta$$

Take $x \in S^3$ such that $\operatorname{rank} du(x) = 2$ (otherwise the left hand side of (V.8) is 0) and choose an orthonormal basis (e_1, e_2) of $(\operatorname{Ker} du(x))^\perp$ such that

$$\frac{\partial u}{\partial e_1} = \lambda\varepsilon_1 \qquad \text{and} \qquad \frac{\partial u}{\partial e_2} = \mu(\cos\alpha\,\varepsilon_1 + \sin\alpha\,\varepsilon_2)$$

where ε_i is the dual basis of dx_i. (V.8) is equivalent to

$$\text{(V.9)}\quad \int_{y\in S^2} |u^*\, d\rho_y|^2\, |u^*\, d\theta_y|$$
$$= \pi \int_{\theta=0}^{2\pi} \left(\lambda^2\, \cos^2\theta + \mu^2\, \cos^2(\theta+\alpha)\right) \sqrt{\lambda^2\, \sin^2\theta + \mu^2\, \sin^2(\theta+\alpha)}$$

After some computations we get

(V.10)

$$\int_0^{2\pi} \left(\lambda^2\, \cos^2\theta + \mu^2\, \cos^2(\theta+\alpha)\right)^2 \left(\lambda^2\, \sin^2\theta + \mu^2\, \sin^2(\theta+\alpha)\right)$$
$$= \frac{\pi}{8}\left[\lambda^6 + \mu^6 + \left[\lambda^2\mu^4 + \lambda^4\mu^2\right] \times \left(3\cos^2\alpha + 7\sin^2\alpha\right)\right]$$
$$\leqslant \frac{\pi}{8}\left[\left(\lambda^2+\mu^2\right)^3 + 4\lambda^2\mu^2\left(\lambda^2+\mu^2\right)\right] \leqslant \frac{\pi}{4}\left(\lambda^2+\mu^2\right)^3$$

with equality if and only if $\alpha = \pm \frac{\pi}{2}$ and $\lambda = \mu$ Combining (V.9) and (V.10) we get

$$\int_{y\in S^2} |u^* \, d\rho_y|^2 \, |u^* \, d\theta_y| \leqslant \frac{\pi^2}{\sqrt{2}} \left(\left|\frac{\partial u}{\partial e_1}\right|^2 + \left|\frac{\partial u}{\partial e_1}\right|^2 \right)^{\frac{3}{2}}$$

This yields the desired result and lemma V.1 is proved. □

Proof of proposition V.2. Let $y \in S^2$, using the coarea formula of Federer we have

$$\text{(V.11)} \quad \begin{aligned} \int_{S^3} |u^* \, d\rho_y|^2 \, |u^* \, d\theta_y| &= \int_{\theta=0}^{2\pi} d\theta \int_{\theta_y \circ u = \theta} |u^* \, d\rho_y|^2 \\ &= \int_{\theta=0}^{\pi} d\theta \left[\int_{\theta_y \circ u = \theta} |u^* \, d\rho_y|^2 + \int_{\theta_y \circ u = \theta + \pi} |u^* \, d\rho_y|^2 \right] \end{aligned}$$

where we have chosen an angular function $\theta_y \in \mathbb{R}/_{2\pi}$ corresponding to an orthonormal basis $(\varepsilon_1, \varepsilon_2)$ of $T_y S^2 \subset \mathbb{R}^3$. Let S_{y_θ} be the great circle of S^2 orthogonal to $\cos\theta \, \varepsilon_1 + \sin\theta \, \varepsilon_2$, we have

$$\{x \quad \text{s.t.} \quad \theta_y \circ u = \theta \quad \text{or} \quad \theta_y \circ u = \theta + \pi\} = u^{-1}(S_{y_\theta})$$

and since $d\rho_y$ represents, up to a sign, the volume form of S_{y_θ}, we have

$$\text{(V.12)} \qquad \int_{S^3} |u^* \, d\rho_y|^2 \, |u^* \, d\theta_y| \geqslant \int_{\theta=0}^{\pi} I\left(u^{-1}(S_{y_\theta})\right) d\theta$$

Thus combining (V.12) and inequality (V.6) of lemma V.1 we get

(V.13)

$$\frac{\pi^2}{\sqrt{2}} \int_{S^3} |\nabla u|^3 \geqslant \int_{y\in S^2} \frac{1}{2} \int_{S_y} I\left(u^{-1}(S_\xi)\right) d\mathcal{H}^1 \lfloor S_y(\xi).$$

We claim that $\forall f \in L^1_+(S^2)$ we have

$$\text{(V.14)} \qquad \int_{y\in S^2} \int_{\xi \in S_y} f(\xi) \; d\mathcal{H}^1 \lfloor S_y(\xi) = 2\pi \int_{S^2} f$$

Let χ be the following cut-off function on $\mathbb{R}$: $\chi \equiv 1$ for $x \in [-1, 1]$, $\chi \equiv 0$ elsewhere. Denote by $\chi_\varepsilon(\xi, y)$ the following function on $S^2 \times S^2$:

$$\chi_\varepsilon(\xi, y) = \frac{1}{2\varepsilon} \chi \left(\frac{\arccos \xi \, . \, y}{\varepsilon} \right).$$

Let g be a regular positive function on S^2 and $y \in S^2$ we clearly have

$$\lim_{\varepsilon \to 0} \int_{\xi \in S^2} \chi_\varepsilon(\xi\,, y)\, g(\xi) = \int_{S_y} g. \tag{V.15}$$

This yields

$$\int_{y \in S^2} \int_{\xi \in S_y} f(\xi)\ d\mathcal{H}^1 \lfloor S_y(\xi) = \lim_{\varepsilon \to 0} \int_{y \in S^2} \int_{\xi \in S^2} \chi_\varepsilon(\xi, y)\, f(\xi).$$

Using Fubini's theorem and (V.15) for $g \equiv 1$ we get (V.14). Finally combining (V.14) and (V.13) we get the desired result and inequality (V.5) is proved.

One can verify that inequality (V.5) is optimal for the Hopf fibration, using the fact that $|\nabla H|^3 \equiv 16\sqrt{2}$, $|S^3| = 2\pi^2$ and that the coimage of any great circle by the Hopf map is a Clifford torus (i. e. a flat square torus in S^3) see [18]. Proposition V.2 is proved. □

In the remainder of this part we will use the function I for proving that the complex Hopf fibration minimizes the 3-energy among a particular class of fibration.

An axially symmetric torus in S^3 is the rotation of a smooth closed embedded curve Γ in

$$S^2_+ = \left\{ (x_1, 0, x_3, x_4) \in \mathbb{R}^4 \middle|\ x_1 > 0 \right\} \cap S^3$$

around the 2-plane $\{x_1 = 0\} \wedge \{x_2 = 0\}$. It can be parametrized in the following form

$$\Big(\gamma(s) \cos 2\pi t\,;\ \gamma(s) \sin 2\pi t\,;\ \alpha(s)\,;\ \beta(s)\Big) \tag{V.16}$$

for $(t, s) \in [0,1] \times [0, b]$ where α, β, γ are 3 b-periodic functions such that $\gamma^2 + \alpha^2 + \beta^2 = 1$ and $\gamma > 0$. (α, β, γ) is the embedding of Γ in S^2_+. Such an axially symmetric torus is said to be "well centered" if $\int_\Gamma \frac{\alpha}{\gamma} = \int_\Gamma \frac{\beta}{\gamma} = 0$.

A symmetric fibration from S^3 into S^2 is, by definition, a fibration u such that the coimage of the north and the south poles are respectively the circles $(\{x_3 = 0\} \wedge \{x_4 = 0\}) \cap S^3$ and $(\{x_1 = 0\} \wedge \{x_2 = 0\}) \cap S^3$ and such that the coimage of each circle in S^2, parallel to the plane xOy, is a "well centered" axially symmetric torus.

We have the following theorem

Theorem V.1. *The Hopf fibration minimizes the 3-energy among the symmetric fibrations.*

Proof. Let ρ be the distance function on S^2 relative to the north pole and $d\theta$ be the 1-form orthogonal to $d\rho$ such that $d\rho\wedge d\theta=\frac{1}{\sin\rho}\,\omega_{S^2}$. $\sin\rho$ will be often denoted by r. We have

$$\begin{aligned}\int_{S^3}|\nabla u|^3&=\int_{S^3}\left(|u^*\,d\rho|^2+|u^*r\,d\theta|^2\right)^{\frac{3}{2}}\\&\geqslant\frac{1}{\sqrt{2}}\int_{S^3}\left(|u^*\,d\rho|^2+|u^*r\,d\theta|^2\right)\left(|u^*\,d\rho|+|u^*r\,d\theta|\right)\\&=\frac{1}{\sqrt{2}}\int_{S^3}|u^*d\rho|^3+|u^*d\rho|\,|u^*r\,d\theta|^2+|u^*r\,d\theta|\left(|u^*d\rho|^2+|u^*r\,d\theta|^2\right)\end{aligned}$$

Using the fact that $|u^*d\rho|^2+|u^*r\,d\theta|^2\geqslant 2\,|u^*d\rho|\,|u^*r\,d\theta|$ we finally get

(V.17)
$$\int_{S^3}|\nabla u|^3\geqslant\frac{1}{\sqrt{2}}\int_{S^3}|u^*\,d\rho|^3+\frac{3}{\sqrt{2}}\int_{S^3}|u^*\,d\rho|\,|u^*r\,d\theta|^2.$$

Remark that equality holds in (V.17) if and only if u is transversally conform. We give first a lower bound for the second term of the right hand side of (V.17). Using the coarea formula of Federer we have

(V.18)
$$\begin{aligned}\int_{S^3}|u^*\,d\rho|\,|u^*\,r\,d\theta|^2&=\int_{s=0}^{\pi}ds\int_{\rho ou=s}|u^*\,r\,d\theta|^2\\&\int_{s=0}^{\pi}\sin^2 s\,ds\int_{\rho ou=s}|u^*\,d\theta|^2.\end{aligned}$$

Since u is a symmetric fibration $\{x\mid\rho\circ u(x)=s\}$ is an axially symmetric torus whose Teichmüller class (for the generators of $\pi_1(u^{-1}(\rho^{-1}(s)))$ mentioned in the begining of part V) is a pure imaginary complex number denoted $i\,b(s)$ thus (V.18) implies

(V.19)
$$\begin{aligned}\int_{S^3}|u^*\,d\rho|\,|u^*\,r\,d\theta|^2&\geqslant\int_{s=0}^{\pi}\sin^2 s\,I\left(u^{-1}\left(\rho^{-1}(s)\right)\right)ds\\&=4\pi^2\int_{s=0}^{\pi}\sin^2 s\,\frac{1+b^2(s)}{b(s)}\,ds\end{aligned}$$

We establish now a lower bound for the first term of the right hand side of (V.17).

Since u is a symmetric fibration $\rho\circ u$ is uniquely determined on S^3 by it's values on $S^3\cap\{(x_1,0,x_3,x_4)\in\mathbb{R}^4 \mid x_1>0\}$. For $x\in S^3$ we denote by z the distance in $\mathbb{R}^4$ between x and the 2-plane $\{x_1=0\}\wedge\{x_2=0\}$. Let θ be the angle function (taking values in $\mathbb{R}/2\pi\,\mathbb{Z}$ on $S^3\setminus(\{x_1=0\}\wedge\{x_2=0\})$) such that $x_1+i\,x_2=z\,e^{i\theta}$. We have $|d\theta|_{S^3}=\frac{1}{z}$ and using one more time the coarea formula of Federer we get

(V.20)
$$\int_{S^3}|u^*\,d\rho|^3=\int_{\alpha=0}^{2\pi}d\alpha\int_{\{\theta=\alpha\}\cap S^3}\frac{|u^*\,d\rho|^3}{|d\theta|}=2\pi\int_{S^2_+}z\,|u^*\,d\rho|^3$$

Let f be a function on S^2_+ taking values in $[0,\pi]$, we say that f is centered if $f_{|_{\partial S^2_+}}\equiv\pi$, $f(North)=0$, any $s\in(0,\pi)$ is a regular point for f and

(V.21)
$$\forall t\in[0,\pi]\qquad \int_{f^{-1}\left([0,t]\right)}\frac{x}{z}\,|\nabla f|=\int_{f^{-1}\left([0,t]\right)}\frac{y}{z}\,|\nabla f|=0$$

For instance any f symmetric relatively to the $y\,Oz$ and $x\,Oz$ planes verifies (V.21). We will use the following lemma

Lemma V.2. *Let f be a centered function from S^2_+ into $[0,\pi]$ we have*

(V.22)
$$\int_{S^2_+}z\,|\nabla f|^3\geqslant 8\pi\left(\int_0^\pi\frac{\dfrac{1}{2\pi}\displaystyle\int_{f^{-1}(s)}\frac{1}{z}}{1+\dfrac{1}{(2\pi)^2}\left(\displaystyle\int_{f^{-1}(s)}\frac{1}{z}\right)^2}\,ds\right)^3 .$$

Equality holds if and only if f is 2 times the distance to the north pole in S^2_+.

Remark V.3. Both of the two sides of inequality (V.22) are invariant under the action of the conformal group of S^2_+.

Remark V.4. If $f^{-1}(s)=\Gamma$ is a closed regular embedded curve in S^2_+, $\frac{i}{2\pi}\int_\Gamma\frac{1}{z}$ is exactly the conformal class ib of the axially symmetric torus given by (V.16). Using this remark lemma V.2 becomes a direct application of proposition 5 in [16] (see also [14]).

Proof of lemma V.2. Using Hölder inequality we have

(V.23)

$$\int_{S^2_+} z\,|\nabla f|^3 \geqslant \frac{\left(\displaystyle\int_{S^2_+} z\,|\nabla f|\right)^3}{\left(\displaystyle\int_{S^2_+} z\right)^2} = \frac{1}{\pi^2}\left(\int_0^\pi ds \int_{f^{-1}(s)} z\right)^3 ,$$

where we have used also Federer's Coarea Formula. Remark that equality holds in (V.23) if and only if $|\nabla f| \equiv cte$. Since f is a centered function on S^2_+, for any s in $(0,\pi)$, $f^{-1}(s)$ is an embedded closed curve in S^2_+. Because of (V.21) we have

$$\int_{f^{-1}(s)} \frac{x}{z} = \int_{f^{-1}(s)} \frac{y}{z} = 0. \qquad \text{(V.24)}$$

We claim that for such a curve in S^2_+ we have

$$\int_{f^{-1}(s)} z \geqslant 4\pi^2 \frac{\displaystyle\int_{f^{-1}(s)} \frac{1}{z}}{4\pi^2 + \left(\displaystyle\int_{f^{-1}(s)} \frac{1}{z}\right)^2} . \qquad \text{(V.25)}$$

Remark that $2\pi \int_{f^{-1}(s)} z$ is the area of the axially symmetric torus given by (V.16) for $\Gamma = f^{-1}(s)$ and using remark V.3 and (V.24) inequality (V.25) is exactly the inequality of proposition 5 in [16]. Actually one can prove (V.25) directly with no reference to it's geometric interpretation and without using the spectra of Δ on flat tori just using Fourier decomposition of the coordinates of $f^{-1}(s)$ for a parametrisation of $f^{-1}(s) = (\alpha,\beta,\gamma)$ verifying $b\,\gamma^2 = \dot\alpha^2 + \dot\beta^2 + \dot\gamma^2$. Lemma V.2 is proved. □

We apply Lemma V.2 to $f = \rho \circ u$, since $\frac{1}{2\pi}\int_{f^{-1}(s)} \frac{1}{z}$ is the conformal class of $u^{-1} \circ \rho^{-1}(s)$ denoted by $b(s)$ above, combining (V.22) and (V.20) we have

$$\int_{S^3} |u^*\, d\rho|^3 \geqslant 16\,\pi^2 \left(\int_0^\pi \frac{b(s)}{1+b^2(s)}\, ds\right)^3 \qquad \text{(V.26)}$$

Combining (V.17), (V.19) and (V.26) we have the following lower bound

(V.27)

$$\int_{S^3} |\nabla u|^3 \geqslant \frac{4\pi^2}{\sqrt{2}} \left[4\left(\int_0^\pi \frac{b(s)}{1+b^2(s)}\, ds\right)^3 + 3\int_0^\pi \sin^2 s\, \frac{1+b^2(s)}{b(s)}\, ds \right] .$$

For simplicity denote $A(s) = \frac{b(s)}{1+b^2(s)}$. We have

$$\int_0^\pi \frac{\sin^2 s}{A(s)}\, ds \times \int_0^\pi A(s)\, ds \geqslant \left(\int_0^\pi \sin s\, ds\right)^2 = 4.$$

Let $A = \int_0^\pi A(s)\ ds$ we have

$$\int_{S^3} |\nabla u|^3 \geqslant 16\, \frac{\pi^2}{\sqrt{2}} \left[A^3 + \frac{3}{A}\right] \geqslant 2\sqrt{2} \times 16\, \pi^2 = \int_{S^3} |\nabla H|^3.$$

This is the desired result and theorem V.1 is proved. □

Remark V.5. In fact one could have proved more directly the previous result. First of all one can establish the following lower bound of the 3-energy

$$\text{(V.28)} \quad \int_{S^3} |\nabla u|^3 \geqslant \frac{3}{\sqrt{2}} \int_0^\pi \sin^2 s\, I\left(u^{-1}\left(\rho^{-1}(s)\right)\right) ds + \frac{1}{\sqrt{2}\, |S^3|^2} \left(\int_0^\pi \mathcal{H}^2\left(u^{-1}\left(\rho^{-1}(s)\right)\right) ds\right)^3$$

and use proposition V.1 above and also Proposition 5 of [16] to get directly (V.27). But we wanted to state explicitely lemma V.2 which is interesting in itself.

Remark V.6. One of the essential ingredients which makes the previous proof work is that, the restriction of H to any coimages by H of the horizontal circles in S^2 (which are tori of Teichmüller class $i\,b$) is exactly an harmonic map into S^1 which realizes I. When b is close to 1, $I(b) = 4\pi^2\, \frac{1+b^2}{b}$ is smaller, but the area of the Torus, which is centered and equal to $4\pi^2 \frac{b}{1+b^2}$ (= the conformal volume for $1 \leqslant b \leqslant \sqrt{\frac{5}{3}}$ see [14] and [16]) is greater in this case. It is a striking result that the equilibrium between those two constraints, in view of minimizing (V.28), is just achieved for $u = H$.

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