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The infinitesimal variation of the spin abelian differentials and periodic minimal surfaces

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We study the first order variation of some abelian differentials closely related with the theory of triply periodic minimal surfaces. The main application is an existence result: We prove that there is a countable number of genus g, $g > 2$, compact minimal immersed surfaces in any flat real 3 torus. We also study the locus corresponding to proper triply periodic minimal surfaces in M_g, the moduli space of genus g compact connected Riemann surfaces: We show that its closure contains the thetanull divisor if $g > 2$ and the locus of the smooth plane quintics if $g = 6$.

0. Introduction.

The subjects of this paper are certain abelian differentials that play an important role in the theory of proper triply periodic minimal surfaces in the Euclidean space. This very classical topic (cf. [27] and [5]) was systematized more recently in (cf. [18] and [22]). Let us recall the generalized Weierstrass representation given in [18]. Let $(\omega_1, \omega_2, \omega_3)$ be a triple of abelian differentials of a compact connected Riemann surface X. We assume:

(0.1) $$\sum_i \omega_i^2 = 0.$$

(0.2) $$\mathrm{g} = \sum_i \omega_i \overline{\omega}_i > 0.$$

(0.3)
The triples of real periods $\mathrm{Re} \int_\gamma (\omega_1, \omega_2, \omega_3), \gamma \in H_1(X, \mathbb{Z})$, generate a rank 3 lattice Λ of $\mathbb{R}^3$

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Then, by choosing a fixed point p of X,

$$K(q) = \mathrm{Re}\int_{[p,q]}(\omega_1,\omega_2,\omega_3) \tag{0.4}$$

is well defined modulo the periods for any $q \in X$. Hence it gives a map

$$K : X \to T,$$

where $T = \mathbb{R}^3/\Lambda$. From (0.1) and (0.2) it follows that $K(X)$ is an immersed minimal surface with respect to the natural flat metric of T. Therefore its inverse image in $\mathbb{R}^3$ is a proper triply periodic minimal surface. Conversely any such surface is of the above type.

Let $W = \mathrm{span}(\omega_1,\omega_2,\omega_3)$ be the space generated by the ω_i. The condition (0.2) holds if and only if the linear system $|W|$ is base points free. Then, by (0.1), the image of $h : X \to |W|^* = \mathbb{CP}^2$ is the conic Q of equation $\sum_i z_i^2 = 0$. This defines a map $f : X \to Q \equiv \mathbb{CP}^1$ such that h is the composition of f with the Veronese embedding $\mathbb{CP}^1 \to \mathbb{CP}^2$:

$$(t_1,t_2) \to (t_1^2 - t_2^2, i(t_1^2 + t_2^2), 2t_1t_2), i = \sqrt{-1}.$$

Let $\mathcal{O}(1)$ be the tautological bundle on $\mathbb{CP}^1$ and $L = f^*(\mathcal{O}(1))$ its the pull-back to X. Notice that $L^2 = h^*(\mathcal{O}\quad{}_2(1))$ is the canonical bundle ω_X of X, i.e. L defines a spin structure on X. Besides, again by pull-back, we find two sections s_1, s_2 of L, such that:

$$\mathcal{W}(s_1,s_2) = (s_1^2 - s_2^2, i(s_1^2 + s_2^2), 2s_1s_2) = (\omega_1,\omega_2,\omega_3). \tag{0.5}$$

The meromorphic map s_1/s_2 can be identified with f and, up to the orientation, with the usual Gauss map of $K(X) : f : X \to \mathbb{CP}^1 \equiv Q \equiv S^2$. Letting $\omega = \omega_1 - i\omega_2$ we have:

$$K(q) = \mathrm{Re}\int_{[p,q]}(\omega_1,\omega_2,\omega_3) = \mathrm{Re}\int_{[p,q]}(1 - f^2, i(1+f^2), f)\omega. \tag{0.6}$$

Formula (0.6) is the usual Weierstrass-Enneper representation (see [18] and [25]) of the minimal surface $K(X)$ and (0.5) another well-known version of it (see [10], Vol. 1, Theorem 3, page 119 and [16]). It follows that $h^0(L) = \dim(H^0(X,L)) > 1$ is a conformal invariant of $K(X)$. We define even and odd periodic minimal surfaces according to the parity of $h^0(L)$. We recall (cf. [4] and [21]) that the parity of a spin structure is invariant under deformations. We suggest [16] for an extensive discussion about its geometric and topological meaning.

The abelian differentials satisfying (0.1) and (0.2) are the spin abelian differentials of the title. Their existence is equivalent to a spin bundle generated by its sections. This gives conformal obstructions: X is special in the sense of the Brill-Noether theory (cf. [2]). The corresponding locus in M_g, the moduli space of the compact connected Riemann surfaces of genus g, can be geometrically described (see [9], [14], [28], [29] and §2 below). This provides a Zariski open set of the theta-null divisor E_g ($g > 2$) in the even case, and of a codimension 3 locus O_g ($g > 4$) of M_g in the odd one. For instance E_3 (see also [23]) and O_5 are the hyperelliptic loci and O_6 is the closure of the locus defined by smooth plane quintics.

The conformal obstruction given by (0.3) is more subtle and depends upon the arithmetic of the Riemann matrices: let $\langle , \rangle$ be the intersection form on $H^1(X, \mathbb{C})$ and $\chi : H^1(X, \mathbb{C}) \to H_1(X, \mathbb{C})$ the induced map. Under the inclusion of $H^{1.0}(X) = H^0(X, \omega_X)$ in $H^1(X, \mathbb{C})$ we get:

$$\langle \gamma, \omega_i \rangle = \int_{\chi(\gamma)} \omega_i .$$

By fixing a (symplectic) basis, $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$, of $H^1(X, \mathbb{Z}) \subset H^1(X, \mathbb{C})$ we construct the period matrix:

$$A = \begin{pmatrix} \langle \alpha_1, \omega_1 \rangle, \langle \beta_1, \omega_1 \rangle, \dots, \langle \alpha_g, \omega_1 \rangle, \langle \beta_g, \omega_1 \rangle \\ \langle \alpha_1, \omega_2 \rangle, \langle \beta_1, \omega_2 \rangle, \dots, \langle \alpha_g, \omega_2 \rangle, \langle \beta_g, \omega_2 \rangle \\ \langle \alpha_1, \omega_3 \rangle, \langle \beta_1, \omega_3 \rangle, \dots, \langle \alpha_g, \omega_3 \rangle, \langle \beta_g, \omega_3 \rangle \end{pmatrix} \tag{0.7}$$

and its real part

$$\Sigma(X, L, s_1, s_2) = \mathrm{Re}(A) = (\langle \alpha_j, \mathrm{Re}(\omega_i) \rangle, \langle \beta_j, \mathrm{Re}(\omega_i) \rangle) \qquad i = 1, 2, 3;\ j = 1, \dots, g. \tag{0.8}$$

Let $G \ = G(3, H^1(X, \mathbb{R}))$ and $G \ = G(3, H^1(X, \mathbb{Q}))$ be respectively the Grassmannian of the 3 planes of $H^1(X, \mathbb{R})$ and of $H^1(X, \mathbb{Q})$, $G \ \subset G$. We may consider $\mathrm{Re}(\omega_i) \in H^1(X, \mathbb{R})$. Then

$$\Phi = \Phi(X, L, s_1, s_2) = \mathrm{span}(\mathrm{Re}(\mathcal{W})) = \mathrm{span}(\mathrm{Re}(\omega_1), \mathrm{Re}(\omega_2), \mathrm{Re}(\omega_3))$$

belongs to G because $\mathrm{Re} : H^{1.0}(X) \to H^1(X, \mathbb{R})$ is a real isomorphism. The group generated by the columns of $\mathrm{Re}(A)$ has rank 3 over $\mathbb{Q}(\mathbb{Z})$ if and only if there are $2g-3$ linear combinations of them with rational coefficients. Then (0.3) holds if and only if:

$$\Phi(X, L, s_1, s_2) \in G \ . \tag{0.9}$$

A main problem is to understand the role of the conditions (0.1) and (0.2) when the period matrix varies. As a first step we study the infinitesimal variation associated with the ω_i-periods. The starting point of our researches was an unexpected proof of an infinitesimal Torelli theorem in the even case and for X representing a general point $[X]$ (see 2.2) of E_g. Following a suggestion of Bert van Geemen we, roughly speaking, have found that the injectivity of the infinitesimal variational map:

$$\alpha_W : H^1(T_X) \to \mathrm{Hom}(W, H^1(\mathcal{O}_X))$$

fails only on a distinguished direction normal to E_g. The well-known Hopf quadratic differential (cf. [22] and see (2.5) and (2.11) below) is the responsible of it. Moreover letting $T_{[X],M_g}(\equiv H^1(T_X))$ and $T_{[X],E_g}$ be respectively the tangent spaces of M_g and of E_g at $[X]$ there is a natural splitting: $T_{[X],M_g} = T_{[X],E_g} \oplus N$. The proof of this, together with a rough description of the moduli spaces, which we need, is the content of the first two sections.

In §3 we set the basic notations we need to go further. Then, from a cancellation due to (0.1), we obtain (see 3.18) an easy, but quite useful formula. This provides a symmetric form on the first order deformation space of spin abelian differentials. In §4 we deal with the even case alone. The symmetry of (3.18) together with the above mentioned infinitesimal Torelli theorem gives a natural symmetric map

$$\rho : T_{[X],E_g} \to T^*_{[X],E_g},$$

where * stands for dual. The result (see theorem 4.10) is that ρ is non trivial. More precisely if we take a ramification point P of f, then the associated Schiffer variation ζ_P belongs to $T_{[X],E_g}$. By rewriting our formulas in terms of second kind differentials we show that $\rho(\zeta_P)$ is non zero. The suspect is that ρ is an isomorphism. This map should be a "second fundamental form" of the immersion of the theta-null divisor in the period domain.

In §5 we turn to the real periods. Let $(\omega_{1,t}, \omega_{2,t}, \omega_{3,t})$, $t \in]-\varepsilon, \varepsilon[\ \varepsilon \in \mathbb{R}, \varepsilon > 0$, be a family of spin abelian differentials, with varying the period matrix (0.7) $A(t) : \omega_{i,0} = \omega_i$ and $A(0) = A$. We study the stationary case: $\mathrm{Re}(A(t)) = \mathrm{Re}(A) + o(t^2)$. In terms of harmonic differentials this means:

$$\mathrm{Re}(\omega_{i,t}) = \mathrm{Re}(\omega_i) + o(t^2) \quad i = 1, 2, 3. \tag{0.10}$$

We show that (0.10) corresponds to a Jacobi field on the minimal surface obtained, after passing to the universal covering, from (0.4). Everything is invariant under deck transformations. This defines a function ϕ on X which satisfies:

$$\Delta\phi - 2K\phi = 0, \tag{0.11}$$

where Δ is the Laplace operator and K the Gauss curvature of the metric g defined in (0.2). The second order variation operator $\Delta - 2K$ is a Schrödinger operator associated to the Gauss map f (cf. [20]). It turns out that ϕ is trivial if and only if the first order deformation of our abelian differentials are actually trivial.

Many efforts have been done to estimate these Schrödinger operators. We suggest the paper of Montiel and Ros (see [20]) for an extensive exposition and bibliography on the subject. We use a result obtained by Nayatani [24]. It implies, in particular, that the solutions of (0.11) are trivial for suitable f having a unique pole. The existence of such f is stated in (6.16) and proved in the §8. Then, generically, the period map defined from (0.8) is a submersion. Set

(0.12)

$$E_g(\mathbb{Q}) = \{[X] \in E_g : \Phi(X, L, s_1, s_2) \in G \ , \ L \text{ is an even spin structure}\}$$
$$O_g(\mathbb{Q}) = \{[X] \in O_g : \Phi(X, L, s_1, s_2) \in G \ , \ L \text{ is an odd spin structure}\}$$
$$H_g(\mathbb{Q}) = \{[X] : \Phi(X, L, s_1, s_2) \in G \ , \ X \text{ hyperelliptic}\}$$

Interpreting (0.8) and (0.9) in the variational set up and recalling that G is dense in G , we obtain:

Theorem 1.

(i) *If $g > 2$ $E_g(\mathbb{Q})$ is dense in E_g.*

(ii) *If $g = 6$, then $O_6(\mathbb{Q})$ is dense in O_6.*

(iii) *If g $(g > 2)$ is odd, then $H_g(\mathbb{Q})$ is dense in H_g.*

Density provides an existence result:

Theorem 2.

(i) *Any flat 3 dimensional torus contains a countable number of distinct immersed compact even minimal surfaces if $g > 2$, odd minimal surfaces if $g = 6$ and hyperelliptic minimal surfaces if g is odd.*

(ii) *The Euclidean space contains a countable number of dimension* 6 *families of proper periodic even minimal surfaces if* $g > 2$, *odd minimal surfaces if* $g = 6$ *and hyperelliptic minimal surfaces if* g *is odd.*

Examples of six dimensional families of proper triply periodic minimal surfaces were first given by Meeks (cf. [18] and see also [19]). Theorem 2 prove the conjecture of Meeks in the hyperelliptic case and confirms part of the general conjecture stated by him (cf. [18]: (8.1)). The original requirement was the existence of *embedded* minimal surfaces of any genus (different from 0 and 2, both orientable or not) in any three dimensional torus. We doubt this could be proved by pure infinitesimal methods, for example for $g > 3$ the hyperelliptic minimal surfaces are not embedded (see [23]). However we believe that the existence of non-orientable compact periodic minimal surfaces would follow along the above lines. This would require a discussion of the moduli of real Riemann surfaces without real points and of real special spin structures. We prefer let it for the future. The proofs of Theorem 1 and 2 are completed in §6.

The density result seems to be completely new. In §7 we use it to give a negative answer (see 7.6) to a question risen in [22] about complex tori associated to triply periodic minimal surfaces.

The reader only interested on periodic minimal surfaces could read only §5, §6 and §7 going back, when necessary, to the descriptive part of §2 and §3: It is enough to know that the moduli of the spin abelian differential depends upon 6g real parameters to get most of the results. The proof of (6.16), confined in §8, requires a known existence theorem of a certain type of Weierstrass points.

We will, rarely, use the terms complete algebraic curve as a synonymous of compact Riemann surface and of algebraic variety as reduced algebraic scheme defined over the complex numbers.

I would to thank Claudio Arezzo who explained to me the very interesting connection between the Noether theorem and the stability of the minimal surfaces in tori (cf. [3]), Alberto Collino and Bert van Geemen (loc. cit.) for many very fruitful conversations, and I am grateful to Letterio Gatto who explained to me some basic results about Weierstrass points theory.

It took a long time to the author to get these results: The age of my daughter Margherita to whom this paper is dedicated.

Section 1.

First we show an algebraic lemma whose proof was basically suggested by B. van Geemen. Let X be a compact connected Riemann surface and L be a non trivial holomorphic line bundle over X generated by its global sections: $h^0(L) = \dim(H^0(L)) \geq 2$. Fix sections s_1, s_2 without common zeroes and set $V = \mathrm{span}(s_1, s_2) \subset H^0(L)$. Evaluating sections we define $h : V \otimes \mathcal{O}_X \to L : h(a(z), b(z)) = a(z)s_1(z) + b(z)s_2(z)$. The kernel of h is the image of $k : L^{-1} \to V \otimes \mathcal{O}_X, k(c) = (cs_2, -cs_1)$, where L^{-1} is the dual of L. This gives the exact sequence:

$$0 \to L^{-1} \to V \otimes \mathcal{O}_X \to L \to 0. \tag{1.1}$$

Let $e(s_1, s_2) \in \mathrm{Ext}(L, L^{-1}) \cong H^1(L^{-2})$ be the induced extension class. Next we consider the gaussian (or Wahl) map: $\mathrm{Wh} : \Lambda^2 H^0(L) \to H^0(L^2 \otimes \omega_X)$, cf. [30], defined by the rule:

$$Wh(s_1 \wedge s_2) = s_1 ds_2 - s_2 ds_1. \tag{1.2}$$

Remark that $H^0(L^2 \otimes \omega_X)$ is dual to $H^1(L^{-2})$ under Serre duality.

Lemma 1.3. $e(s_1, s_2) \cdot \mathrm{Wh}(s_1 \Lambda s_2) \neq 0$.

Proof. Set $e(s_1, s_2) = \zeta$, $\mathrm{Wh}(s_1 \Lambda s_2) = \Omega$ and let $f = (s_1, s_2) : X \to \mathbb{CP}^1$ be the induced map. The class $\xi \in \mathrm{Ext}(\mathcal{O}(1), \mathcal{O}(-1)) \cong H^1(\omega \quad {}_1) \cong \mathbb{C}$ defined by the tautological sequence on $\mathbb{CP}^1$:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1) \to 0 \tag{1.4}$$

corresponds to 1. Letting $f^* : H^1(\omega \quad {}_1) \to H^1(L^{-2})$ be the map induced by f, we have $f^*(\xi) = \zeta$: The (1.1) sequence is the pull-back of (1.4) and L^{-2} is isomorphic to $f^*\omega \quad {}_1$. Denote by R the ramification divisor of f. This is the zero divisor of Ω and defines the exact Hurwitz sequence:

$$0 \to L^{-2} \cong f^*\omega \quad {}_1 \to \omega_X \to \omega_{X|R} \to 0. \tag{1.5}$$

Taking cohomology we obtain the exact sequence:

$$0 \to H^0(\omega_X) \to H^0(\omega_{X|R}) \xrightarrow{\delta} H^1(L^{-2}) \xrightarrow{\eta} H^1(\omega_X) \to 0. \tag{1.6}$$

The map $\eta : H^1(L^{-2}) \to H^1(\omega_X)$ is the multiplication by $k\Omega$, where k is a non-zero constant: $\Omega \cdot \zeta = \Omega \cdot f^*(\xi) = k^{-1}\eta(f^*(\xi))$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
H^1(\omega \quad {}_1) & =\!= & H^1(\omega \quad {}_1) \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle df}\downarrow \\
H^1(L^{-2}) & \xrightarrow{\ \eta\ } & H^1(\omega_X)
\end{array}
$$

Under the isomorphisms $H^1(\omega \quad {}_1) \cong \mathbb{C} \cong H^1(\omega_X)$, df is the degree of f. We get $\Omega \cdot \zeta = k^{-1} df(\xi) = k^{-1} \deg(f) \neq 0$. □

We would like to obtain some consequence of (1.3). Tensoring (1.2) by $\omega_X \otimes L$ we get:

$$(1.7) \qquad 0 \to \omega_X \to V \otimes \omega_X \otimes L \to \omega_X \otimes L^2 \to 0.$$

Its cohomology sequence is:

$$(1.8)\ 0 \to H^0(\omega_X) \xrightarrow{j} V \otimes H^0(\omega_X \otimes L) \xrightarrow{\mu_V} H^0(\omega_X \otimes L^2) \xrightarrow{\delta} H^1(\omega_X) \to 0,$$

where μ_V is the multiplication and δ the cup-product with the class $\zeta = e(s_1, s_2)$. It follows $\mathrm{Im}\,(\mu_V) = \{\omega \in H^0(\omega_X \otimes L^2) : \xi \cdot \omega = 0\}$ and the splitting:

$$(1.9) \qquad H^0(\omega_X \otimes L^2) = \mathrm{Im}(\mu_V) \oplus Q_\Omega,$$

where $Q_\Omega \subset H^0(\omega_X \otimes L^2)$ is the space generated by $\Omega = \mathrm{Wh}(s_1 \wedge s_2)$. Set $T_\Omega = \{\xi \in H^1(L^{-2}) : \xi \cdot \Omega = 0\}$, then from (1.9) we get:

Corollary 1.10. *Serre duality defines a perfect pairing between T_Ω and $\mathrm{Im}(\mu_V)$, moreover $H^1(L^{-2}) = \mathrm{span}(\zeta) \oplus T_\Omega$.*

(1.11). Let W be the image of the map $V \otimes V \to H^0(L^2)$ and $\mu_W : W \otimes H^0(\omega_X) \to H^0(\omega_X \otimes L^2)$ be induced by cup product. Consider the commutative diagram given by multiplication:

$$
\begin{array}{ccc}
V \otimes V \otimes H^0(\omega_X) & \xrightarrow{\ F\ } & V \otimes H^0(\omega_X \otimes L) \\
\downarrow & & {\scriptstyle \mu_V}\downarrow \\
W \otimes H^0(\omega_X) & \xrightarrow{\ \mu_W\ } & H^0(\omega_X \otimes L^2).
\end{array}
$$

We get $\mathrm{Im}(\mu_W) = \mathrm{Im}(\mu_V \cdot F)$, then by standard argument, one proves the equalities $\dim(\mathrm{coker}(F)) = 2h^0(L) - 4$ and $\dim(\mathrm{coker}(\mu_W)) = 2h^0(L) - 3$. Dualizing the cup product $H^1(L^{-2}) \otimes H^0(L^2) \to H^1(\mathcal{O}_X)$ we define α :

$H^1(L^{-2}) \to \mathrm{Hom}^{(s)}(H^0(L^2), H^1(\mathcal{O}_X))$, (s) stands for symmetric, and by restriction:

$$\alpha_W : H^1(L^{-2}) \to \mathrm{Hom}^{(s)}(W, H^1(\mathcal{O}_X)). \tag{1.12}$$

Clearly α_W is dual to μ_W and then $\dim(\ker(\alpha_W)) = 2h^0(L) - 3$. Since always $\mathrm{Im}(\mu_W) \subset \mathrm{Im}(\mu_V)$ we have $\mathrm{Im}(\mu_W) = \mathrm{Im}(\mu_V)$ when $h^0(L) = 2$. It follows:

Corollary 1.13. *If L is generated by its sections and $h^0(L) = 2$. Then:*

(i) $H^0(\omega_X \otimes L^2) = \mathrm{Im}(\mu_W) \oplus Q_\Omega$ *and* $H^1(L^{-2}) = \ker(\alpha_W) \oplus T_\Omega$;

(ii) *Serre duality gives a perfect pairing between* T_Ω *and* $\mathrm{Im}(\mu_W)$;

(iii) *The restriction of* α_W, $b : T_\Omega \to \mathrm{Hom}^{(s)}(W, H^1(\mathcal{O}_X))$, *is injective.*

Section 2.

Let X be a compact connected Riemann surface of genus $g > 2$. We keep the notations of §1, but assume that L is a spin bundle, $L^2 = \omega_X$. Let M_g be the moduli space of genus g compact connected Riemann surfaces and S_g be the sub-locus corresponding to some special spin structure:

$$S_g = \{[X] \in M_g : \text{ there is } L,\ L^2 = \omega_X \text{ and } h^0(L) > 1\}.$$

We have $S_g = E_g \cup O_g$, $E_g = \{[X] \in S_g : L^2 = \omega_X, h^0(L) = 2n, n > 0\}$ and $O_g = \{[X] \in S_g : L^2 = \omega_X, h^0(L) = 2n+1, n > 0\}$.

(2.1). Cf. [14], [28] and [29]. The locus E_g, $g > 2$, is always an irreducible divisor of M_g and O_g, $g > 4$ (O_3 and O_4 are empty), has pure codimension 3 in M_g. In particular E_3 and O_5 are the hyperelliptic loci, the canonical model in $\mathbb{CP}^3$ of a general element of E_4 is contained in the quadric of equation (0.1): $x_1^2 + x_2^2 + x_3^2 = 0$, and O_6 is the closure of the locus defined by the smooth quintics in the projective plane.

Definition 2.2. We call a point $[X]$ of E_g (respectively of O_g) general if there is a unique spin structure L on X such that:

i) $h^0(L) = 2$ (respectively $h^0(L) = 3$);

ii) L is generated by its global sections.

Condition ii) is equivalent to the existence of two sections without common zeros. General points define (cf. [28]) a non-empty Zariski open set, that we denote by $E'_g(O'_g)$, of E_g, $g > 2$, (O_g, $g > 4$).

(2.3). The identification of $H^1(T_X)$ with the tangent space of M_g at $[X]$ is correct only if $\mathrm{Aut}(X)$ is trivial. Otherwise $H^1(T_X)$ is naturally the tangent space of a suitable covering of M_g, for instance the Teichmüller space. We will try implicitly to avoid this complication in the notation.

The map (1.2) now becomes:

$$\mathrm{Wh} : \Lambda^2 H^0(L) \to H^0(\omega_X^2).$$

Proposition 2.4. *Let $[X]$ be a point of $E'_g(O'_g)$ and $T_{E_g,[X]}$, $(T_{O_g,[X]})$ be the tangent space of $E_g(O_g)$ at X. Then:*

$$T_{E_g,[X]}(T_{O_g,[X]}) \equiv \{\xi \in H^1(T_X) : \xi \cdot \Omega = 0,\ \Omega \in \mathrm{Wh}(\Lambda^2 H^0(L))\}.$$

In particular $T_{E_g,[X]} = T_\Omega$.

Proof. (see [28]).

Remark 2.5. (cf. [22]). In minimal surface case (0.5) $\Omega = \mathrm{Wh}(s_1 \wedge s_2)$ can be obtained by writing Codazzi-Mainardi's equations in isothermal coordinates. Its zero divisor is the locus where the Gauss curvature of **g** (see 0.2) vanishes.

(2.6) Digression. (Will be used in §4). Let Ω be any non zero quadratic differential. To describe $T_\Omega = \{\xi \in H^1(T_X) : \xi \cdot \Omega = 0\}$ we consider the sheaves exact sequence (cf. 1.5): $0 \to T_X \to \omega_X \cong T_X(R) \to \omega_{X|R} \to 0$, $R \equiv \{\Omega = 0\}$. Its cohomology sequence (cf. 1.6) is:

$$H^0(T_X(R)_{|R}) \xrightarrow{\delta} H^1(T_X) \xrightarrow{\Omega} H^1(T_X(R)) \cong H^1(\omega_X) \to 0. \tag{2.7}$$

Set $T_\Omega = \mathrm{Im}(\delta)$. Fix $P \in R$ and write $R = nP + R'$, $P \in R'$. Let (U, z) be an open coordinate set of X, $P \in U$ and $z(P) = 0$. The expressions $\{\zeta_{i,P} = z^{-i}\partial/\partial z\}_{P \in Z, i \leq n}$ give a basis of $H^0(T_X(R)_{|R})$. The class $\xi_P = \delta(\zeta_{1,P})$ is customarily called the Schiffer variation of P. In Dolbeault cohomology we get

$$\xi_P = \delta(\xi_{1,P}) = \text{class of } \{z^{-1}\bar{\partial}\rho_\varepsilon \otimes \partial/\partial z\}. \tag{2.8}$$

(2.9). Coming back to spin, if $V = \mathrm{span}(s_1, s_2) \subset H^0(L)$, we define (cf. (0.5)):

$$\mathcal{W}(s_1, s_2) = (s_1^2 - s_2^2, i(s_1^2 + s_2^2), 2s_1 s_2) = (\omega_1, \omega_2, \omega_3),$$

$\omega_i \in H^0(\omega_X)$, $1 \leq i \leq 3$, $\sum_i \omega_i^2 = 0$. Set $W = \text{span}(\omega_1, \omega_2, \omega_3)$. Recall the maps (cf. (1.11), (1.12)) $\mu_W : W \otimes H^0(\omega_X) \to H^0(\omega_X^2)$ and $\alpha_W : H^1(T_X) \to \text{Hom}^{(s)}(W, H^1(\mathcal{O}_X))$. The last one is the restriction of

$$\alpha : H^1(T_X) \to \text{Hom}^{(s)}(H^0(\omega_X), H^1(\mathcal{O}_X)) \tag{2.10}$$

the infinitesimal variation of Hodge structure map (cf. [6] and 3.12 below). Hence α_W gives the infinitesimal variation of the ω_i-periods. In the even general case Corollary (1.10) provides a natural splitting:

$$T_{M_g,[X]} = T_{E_g,[X]} \oplus \ker(\alpha_W).$$

It follows that $e(s_1, s_2) \in \ker(\alpha_W)$ is normal to E_g. We have (cf. 1.13):

Proposition 2.11. *There is a natural holomorphic splitting:* $T_{M_g|E'_g} = T_{E'_g} \oplus N$. *The fiber of* N *at* $[X] \in E'_g$ *is generated by* $e(s_1, s_2)$. *Dually we have* $H^0(\omega_X^2) = T^*_{E_g,[X]} \oplus Q_\Omega = \text{Im}(\mu_W) \oplus Q_\Omega$.

Corollary 2.12 (An infinitesimal Torelli Theorem). *Let* $[X] \in E'_g$, *the map* $\beta : T_{E_g,[X]} \to \text{Hom}^{(s)}(W, H^1(\mathcal{O}_X))$ *is injective: If* $\xi \in H^1(T_X)$ *and* $\xi \cdot \omega_i = 0$ $i = 1, 2, 3$ *then* $\xi = \lambda e(s_1, s_2)$, *where* λ *is a constant and* $e(s_1, s_2) \notin T_\Omega = T_{E_g,[X]}$.

(2.13). We complete now the description of the moduli space of the quadruples $\{X, L, s_1 s_2\}$, $L^2 = \omega_X$ and $s_1, s_2 \in H^0(L)$. The theory of the linear series on moving complex algebraic curves (see [1] and [2]) proves that there exists an algebraic variety $\mathcal{G}^1_{g-1}$ whose points correspond to triples $([X], L, V)$, where X is a compact Riemann surface, L is a line bundle on X of degree $g-1$, and $V \subset H^0(X, L)$ a subspace of dimension 2: $|V|$ is a $\mathcal{G}^1_{g-1}$. Moreover (up to a base change as in 2.3) there is a rank 2 vector bundle S on $S \to \mathcal{G}^1_{g-1}$ whose fiber S_t, $t = ([X], L, V)$, is isomorphic to V. Let $\mathcal{E}_g$ and $\mathcal{O}_g$ be the subsets of $\mathcal{G}^1_{g-1}$ defined by

$$\mathcal{E}_g = \{([X], L, V) \in \mathcal{G}^1_{g-1} : L^2 = \omega_X,\ L \text{ even}\},$$
$$\mathcal{O}_g = \{([X], L, V) \in \mathcal{G}^1_{g-1} : L^2 = \omega_X,\ L \text{ odd}\}.$$

The map $\mathcal{E}_g \to E_g$ around a general point is biholomorphic and the general fiber of $\mathcal{O}_g \to O_g$ is a complex projective plane isomorphic to $\mathbb{P}(H^0(L)^*)$. Hence $\mathcal{E}_g$ and $\mathcal{O}_g$ have both complex dimension $3g-4$.

(2.14). In the sequel we will use the following notation: If $\mathbb{K}(= \mathbb{Q}, \mathbb{R}, \mathbb{C})$ is a field and $q : E \to B$ a K-vector bundle over B of rank e. If $k \leq e$ we denote by $F_k(E, \mathbb{K})$ the space of the k-frames and by $G_k(E, \mathbb{K})$ the Grassmannian of the k-spaces of E. We write only $F_k(E)$ and $G_k(E)$ when confusion should not occur. The natural map $F_k(E) \to G_k(E)$ is given by $(s_1, \ldots, s_k) \to \mathrm{span}(s_1, \ldots, s_k)$.

(2.15). Let $F_2(S)$ be the 2 frames space of $S \to \mathcal{G}^1_{g-1}$, $F_2(\mathcal{E}_g)$ and $F_2(\mathcal{O}_g)$ their restriction to $\mathcal{E}_g$ and to $\mathcal{O}_g$. A point of $F_2(\mathcal{E}_g)(F_2(\mathcal{O}_g))$ corresponds to the data $\{X, L, s_1, s_2\}$, where $\{X, L, \mathrm{span}(s_1, s_2)\} \in \mathcal{E}_g(\in \mathcal{O}_g)$. Denote by j the involution on $F_2(S)$ defined by $j((s_1, s_2)) = (-s_1, -s_2)$. Let $\mathcal{S}_{g,e} = F_2(\mathcal{E}_g)/j$ and $\mathcal{S}_{g,o} = F_2(\mathcal{O}_g)/j$ be the quotient spaces. Note that $\mathcal{W}(s,t) = \mathcal{W}(s',t')$ if and only if either $(s,t) = (s',t')$ or $(s,t) = j((s',t'))$. For this reason we will call the (disjoint) union

$$\mathcal{S}_g = \mathcal{S}_{g,e} \cup \mathcal{S}_{g,o}$$

the moduli space of the abelian spin differentials. Denote by $\mathcal{S}_g(\mathbb{Q}) = \mathcal{S}_{g,e}(\mathbb{Q}) \cup \mathcal{S}_{g,o}(\mathbb{Q})$ the sub-locus of $\mathcal{S}_g$ which corresponds to triply periodic minimal surfaces. From (0.9) we may set

$$\mathcal{S}_g(\mathbb{Q}) = \{\{X, L, s_1, s_2\} : \Phi(X, L, s_1, s_2) \in G \ = G_3(H^1(X, \mathbb{Q}))\}.$$

(2.16). A component of $\mathcal{S}_g$ will be called general if its image in M_g is a component of S_g. Remark that general points of S_g (cf. 2.2) define general components of $\mathcal{S}_g$. It turns out, since E_g is irreducible (cf. [29]), that there is only one general component of $\mathcal{S}_{g,e}$. It is not known if O_g is in general irreducible. Since dim $F_2(V) = 4$ *any general component of* $\mathcal{S}_g$ *has real dimension* $2(3g-4)+8 = 6g$.

(2.17). Non-general components of $\mathcal{S}_g$ are provided by the hyperelliptic Riemann surfaces of odd genus. Let $H_g, H_g \subset M_g$, be the hyperelliptic locus, we have $\dim(H_g) = 2g-1$. If g is odd, $g = 2k+1$, we take a Weierstrass point P of X, it follows that $L = \mathcal{O}_X(2kP)$ is a spin bundle generated by its global sections and $h^0(L) = k+1$. Consider now

$$\mathcal{H}_g = \{\{X, L, s_1, s_2\} \in \mathcal{S}_g : [X] \in H_g, h^0(L) = k+1\}. \tag{2.18}$$

The fiber of the projection $\mathcal{H}_g \to H_g$ can be identified with

$$F_2(H^0(L))/j \quad (j((s,t)) = (-s,-t)).$$

Since the dimension of $F_2(H^0(L))$ is $2(k+1) = g+1$, we get

$$\dim(\mathcal{H}_g) = \dim(H^g) + \dim(F_2(H^0(L)) = 3g.$$

It follows that $\mathcal{H}_g$ is a component of $\mathcal{S}_{g,e}$ if $g = 3 \bmod .4$ and of $\mathcal{S}_{g,o}$ if $g = 1 \bmod .4$. Moreover if $g > 5\, \mathcal{H}_g$ is not a general component. We have again that *the real dimension of $\mathcal{H}_g$ is $6g$.*

Section 3.

We first set up some notation (see [13]: §6). The new material starts at (3.13). Let $\mathcal{X}$ and B be complex varieties, $\mathcal{L}$ a line bundle on $\mathcal{X}$ and π a map

$$\pi : \mathcal{X} \to B. \tag{3.1}$$

We assume that:

(1) π is holomorphic, smooth, proper and with connected fibers;

(2) $\dim(\mathcal{X}) = \dim(B) + 1$, i.e. for any $t \in B$, $X_t = \pi^{-1}(t)$ is a connected compact Riemann surface of genus $g > 2$;

(3) the restriction of l to any fiber is a spin bundle, i.e. if $i_t : X_t = \pi^{-1}(t) \to \mathcal{X}$ is the inclusion and $i_t^*(\mathcal{L}) = L_t$ then $L_t^2 = \omega_{X_t}$;

(4) there are σ_1 and σ_2 sections of $\mathcal{L}$ such that $s_{1,t} = i_t^*(\sigma_1)$ and $s_{2,t} = i_t^*(\sigma_2)$ are independent for any t.

(3.2). Fix $b \in B$ and set $\{X_b, L_b, s_{1,b}, s_{2,b}\} = \{X, L, s_1, s_2\}$, we call $\{X_t, L_t, s_{1,t}, s_{2,t}\}$ a deformation of $\{X, L, s_1, s_2\}$. Consider the modular map $\mu : B \to \mathcal{S}_g$

$$\mu_(t) = \{X_t, L_t, s_{1,t}, s_{2,t}\}.$$

A deformation of $\{X, L, s_1, s_2\}$ is then trivial if μ is constant and *trivial up to the first order if $d\mu(b) = 0$.* Let $m : B \to M_g$, $m(t) = \{\text{moduli of } X_t\}$, be the composition of μ with the projection $\mathcal{S}_g \to M_g$. If B is connected the image of μ is contained either in $\mathcal{S}_{g,e}$ or in $\mathcal{S}_{g,o}$ (the image of m in E_g or in O_g) depending on the parity of $h^0(L_t)$. Set $B(\mathbb{Q}) = \mu^{-1}(\mathcal{S}_g(\mathbb{Q}))$ (see 2.15). Set:

$$\mu_1 = (\sigma_1)^2 - (\sigma_2)^2;\ \mu_2 = i((\sigma_1)^2 + (\sigma_2)^2);\ \mu_3 = 2\sigma_1\sigma_2,$$
$$\mu_i \in H^0(\mathcal{X}, \mathcal{L}^2), \quad i = 1, 2, 3.$$

Restriction to fibers gives rise to the abelian differentials (cf. (0.5))

$$\mathcal{W}(s_{1,t}, s_{2,t}) = (\omega_{1,t}, \omega_{2,t}, \omega_{3,t}) = (i_t^*(\mu_1), i_t^*(\mu_2), i_t^*(\mu_3)).$$

Alternatively we may consider the $\{\omega_{i,t}\}_{t\in B}$ $i = 1,2,3$, as sections of $\lambda = \pi_*(\mathcal{L}^2)$. This is a Hodge bundle of the family (3.1), which is a rank g vector bundle whose fibers are isomorphic to $H^0(X_t, \omega_{X_t}) = H^{1.0}(X_t)$.

We assume now that B is simply connected, there is therefore a C^∞ isomorphism:

$$\tau : X \times B \to \mathcal{X}, \tag{3.3}$$

inducing the identity on X. Let $T(X)$ be the real tangent space of X. The complex structure of X induces a decomposition $T(X) \otimes \mathbb{C} = T' \oplus T''$ and the variation (3.1) a C^∞-section of $\mathrm{Hom}(T'', T') = T' \otimes T''^*$, which is a family of (0.1) vector valued forms, $\theta : B \to \mathcal{A}^{0.1}(T_X)$, on X. Let z and $s = (\dots, s_i, \dots)$ be respectively holomorphic coordinates of X and B, we assume $b \equiv \{s = 0\}$ and locally we write: $\theta = \theta(s,z)\partial/\partial z d\bar{z}$. If T_b is the tangent space of B at b and $v = \Sigma_i a_i \partial/\partial s_i \in T_b$ is a tangent vector we set $\partial/\partial s_i(\theta(s,z))_{|s=o} = \theta^i$. Therefore we have:

$$d\theta(v) = (\Sigma_i a_i \theta^i)\partial/\partial z \otimes d\bar{z} \in \mathcal{A}^{0.1}(T_X). \tag{3.4}$$

Formula (3.4), by means of the Dolbeault cohomology, defines the Kodaira-Spencer map $\kappa : T_b \to H^1(T_X)$:

$$\kappa(v) = (\Sigma_i a_i \theta^i)\partial/\partial z \otimes d\bar{z} \quad \text{mod. } \bar{\partial}(\mathcal{A}^{0.0}(T_X)(= \bar{\partial}(C^\infty(T_X))). \tag{3.5}$$

Let λ and $\mathcal{H}$ be the Hodge bundles of (3.1): $\mathcal{H} \cong H^1(X, \mathbb{C}) \times B$. For $s \in B$, let $\mathcal{A}^n(X_s)$ and $\mathcal{A}^{p,q}(X_s)$ be the spaces of the C^∞ of n and $p.q$ forms of X_s, the diffeomorphism $\tau_s : X \to X_s$ induced by (3.3) gives an inclusion:

$$(\tau_s)^* : \mathcal{A}^{1.0}(X_s) \to \mathcal{A}^1(X).$$

Applying θ on $T(X)^* \otimes \mathbb{C} = T^{1.0} \oplus T^{0.1}$ we obtain the identity $(\tau_s)^*(dz(s)) = dz + \theta(s,z)d\bar{z}$. An element in the image of $(\tau_s)^*$ locally takes the form:

$$f(z,s)(dz + \theta(s,z)d\bar{z}). \tag{3.6}$$

The abelian differentials are the closed 1.0 forms:

$$(\tau_s)^*(H^{1.0}(X_s)) = \mathrm{Im}((\tau_s)^*) \cup \mathcal{A}^1(X)_{\text{closed}}.$$

If $\omega(s) \in H^{1.0}(X_s)$ then $(\tau_s)^*(\omega(s))$ is a global 1-form which locally can be written as in (3.6) and satisfies the CR equation:

$$\partial f/\bar{\partial} z - \partial(f\theta)/\partial z = 0. \tag{3.7}$$

The induced inclusions $H^{1.0}(X_s) \subset H^1(X, \mathbb{C})$ can be clear up by Hodge theory. Write:

$$(3.8) \qquad (\tau_s)^*(\omega(s)) = \beta(s) + \gamma(s) + dh(s) = (\beta + \partial h) + (\gamma + \bar{\partial} h),$$

where $\beta(s) \in H^{1.0}(X), \gamma(s) \in H^{0.1}(X)$ and $h \in C^\infty(X), \beta(s) + \gamma(s)$ is the harmonic representative of $(\tau_s)^*(\omega_(s))$. Writing locally $\beta(s) \equiv \beta(z,s)dz, \gamma(s) \equiv \gamma(z,s)d\bar{z}$ and comparing type in (3.6) one obtains:

$$(3.9) \qquad \beta(z,s) + \partial h/\partial z = f(z,s), \gamma(z,s) + \partial h/\bar{\partial} z = \theta(s,z)f(z,s).$$

Remark that (3.9) implies (3.7) because $\beta(z)$ is holomorphic and $\gamma(z)$ is antiholomorphic. The inclusions $H^{1.0}(X_s) \subset H^1(X, \mathbb{C})$ are given by means of the identification $\omega(s) \leftrightarrow \beta(s) + \gamma(s)$ defining the Hodge filtration $\lambda \subset \mathcal{H}$. The Hodge structure variation map is (cf. [12]) $V : B \to G(g, H^1(X, \mathbb{C}))$

$$V(s) = \{H^{1.0}(X_s) \subset H^1(X, \mathbb{C})\}.$$

With the above notation we study first order variation of a section $\omega(s)$ of λ : If $(\tau_s)^*(\omega(s)) = \beta(s) + \gamma(s) + dh(s)$ is the induced section of $\mathcal{H}$ is $\beta(s) + \gamma(s)$. Set $\omega(0) = \omega, b \equiv \{s = 0\}$. First order expansion at 0:

$$(\tau_s)^*(w(s)) = \omega + \left(\Sigma_i \partial/\partial s_i(\omega(s))_{|s=o}\right) s^i + o(2) = \omega + \Sigma(\beta^i + \gamma^i + dh^i)s^i + o(2),$$

$\beta^i = \partial/\partial s_i(\beta(s))_{|s=o}$, $\gamma^i = \partial/\partial s_i(\gamma(s))_{|s=o}$ and $h^i = \partial/\partial s_i(h(s))_{|s=o}$. Remark that $\gamma(0) = 0$. Set $v = \Sigma_i a_i \partial/\partial s_i$. By taking derivative in the v-direction we get:

(3.10)

$$L(v) = \Sigma_i a_i \beta^i + \Sigma_i a_i \gamma^i + \Sigma_i a_i dh^i = \beta_v + \gamma_v + dh_v = (\beta_v + \partial h_v) + (\gamma_v + \bar{\partial} h_v).$$

Writing $\omega(s) = f(z,s)(dz + \theta(s,z)d\bar{z})$, $f(z,0) = f(z)$ and $\omega = f(z)dz$, we obtain:

$$\omega(s) = f(z)dz + \Sigma_i s^i(f^i dz + f\theta^i d\bar{z}) + o(2)$$
$$L(v) = \Sigma_i a_i(f_i dz + f\theta_i d\bar{z}) = f_v dz + f\theta_v d\bar{z}$$

The (0.1) part of $L(v)$ is the contraction $d\theta(v) \cdot \omega$ (cf. 3.4):

$$(3.10') \qquad L(v)^{0.1} = \gamma_v + \bar{\partial} h_v = d\theta(v) \cdot \omega \quad (\text{loc.} = f\theta_v d\bar{z})$$
$$L(v)^{1.0} = \beta_v + \partial h_v \quad (\text{loc.} = f_v dz).$$

The first relation above will be called the "Kodaira-Spencer equation". In Dolbeault cohomology it gives (cf. 3.5):

$$(3.11) \qquad \kappa(v) \cdot \omega = \gamma_v.$$

Remark 3.12. Let ∇ denote the Gauss Manin connection (see [6] and [12]) on $\mathcal{H}$. We have: $\nabla_v(\omega(s)) = \beta_v + \kappa(v) \cdot \omega$. Moreover let

$$dV : T_b \to \mathrm{Hom}(H^{1.0}(X), H^1(X, \mathbb{C})/H^{1.0}(X)) \cong \mathrm{Hom}(H^{1.0}(X), H^{0.1}(X))$$

be the differential of $V : B \to G(g, H^1(X, \mathbb{C}))$, i.e. the infinitesimal variation of the Hodge structure map. By (3.10) (and see also [6]) we have

$$dV(v)(\omega) = (\nabla_v \omega(s))^{0.1} = \kappa(v) \cdot \omega.$$

It follows that dV factorizes, via the Kodaira-Spencer map, through $\Phi : H^1(T_X) \to \mathrm{Hom}(s)(H^{1.0}(X), H^{0.1}(X))$, where Φ is given by the cup product.

(3.13). Attached to $\mu = \omega(s)$ there are two bilinear forms defined on T_b. For v and ω in T_b we set

$$A_\mu(v, \omega) = \int_X \beta_v \wedge \gamma_w; \qquad B_\mu(v, \omega) = \int_X \partial hv \wedge \bar{\partial} h_w.$$

The second one, B_μ, possibly depending on the trivialization (3.3), is symmetric by Stokes theorem. In fact:

$$\int_X \partial h_v \wedge \bar{\partial} h_w = \int_X d(hv \wedge \bar{\partial} h_w) - \int_X h_v \wedge \partial\bar{\partial} h_w = \int_X h_v \wedge \bar{\partial}\partial h_w$$
$$= - \int_X \bar{\partial} h_v \wedge \partial h_w = \int_X \partial h_w \wedge \bar{\partial} h_v.$$

We shall take now the sections defined in (3.2) and write:

$$\omega_i(s) = f_i(z, s)(dz - \theta(s, z)d\bar{z}) = f_i(z, s)dz(s), \omega_i(0) = f_i(z)dz.$$

Set $\omega_i(s) = \beta_i(s) + \gamma_i(s) + dh_i(s)$ $i = 1, 2, 3$ and $\beta_i(s) = \beta_i(z, s)dz$. The condition (0.1) $\Sigma_i \omega_i(s)^2 = 0$ gives $\Sigma_i f_i(z, s)^2 = \Sigma_i(\beta_i(z, s) + \partial h_i(z, s)/\partial z)^2 = 0$ (see 3.9). Since $\beta_i(z, 0) = f_i(z)$, taking the directional derivatives at $s = 0$, we find:

$$\Sigma_i f_i(z)(\beta_i, v(z) + \partial h_{i,v}(z)/\partial z) = 0, \tag{3.14}$$

which, by using quadratic differentials, becomes:

$$\Sigma_i \omega_i \beta_{i,v} + \Sigma_i \omega_i \partial h_{i,v} = 0. \tag{3.14'}$$

Let $\rho : T_b \to H^0(\omega_X^2)$ be the map defined by

$$\rho(v) = \Sigma_i \omega_i \beta_{i,v} = -\Sigma_i \omega_i \partial h_{i,v}. \tag{3.15}$$

We note that $\rho(v) \in \mathrm{Im}(\mu_W : W \otimes H^0(\omega_X) \longrightarrow H^0(\omega_X^2))$ and (cf. 2.12) that $\mathrm{Im}(\mu_W) \equiv T^*_{E,[X]}$ in the even general case. Now we get the following:

Lemma 3.16. $\Sigma_i(\beta_{i,v} + \partial h_{i,v}) \wedge (\gamma_{i,w} + \bar{\partial} h_{i,w}) = 0.$

Proof. The Kodaira-Spencer equations (3.10′) in the w-direction read $\gamma_{i,w} + \bar{\partial} h_{i,w} = d\theta(w) \cdot \omega_i$. Therefore we get:

$$\Sigma_i(\beta_{i,v} + \partial h_{i,v}) \wedge (\gamma_{i,w} + \bar{\partial} h_{i,w}) = \Sigma_i(\beta_{i,v} + \partial h_{i,v}) \wedge d\theta(w) \cdot \omega_i.$$

Since locally $d\theta(w) \cdot \omega_i = \theta_w(z) f_i(z) dz$, from (3.14) we have:

$$\Sigma_i(\beta_{i,v} + \partial h_{i,v}) \wedge d\theta(w) \cdot \omega_i = \theta_w(z) \Sigma_i f_i(z)(\beta_{i,v}(z) + \partial h_i(z)/\partial z) = 0.$$

□

Let A_{μ_i} and B_{μ_i} be the quadratic forms (cf. 3.13) and define

$$(3.17) \qquad \begin{aligned} A(v,w) &= \Sigma_i A_{\mu_i}(v,w) = \Sigma_i \int_X \beta_{i,v} \wedge \gamma_{i,w} \\ B(v,w) &= \Sigma_i B_{\mu_i}(v,w) = \Sigma_i \int_X \partial h_{iv} \wedge \bar{\partial} h_{iw}. \end{aligned}$$

We have: $2A(v,w) = 2\Sigma_i \int_X \beta_{i,v} \wedge \kappa(w)\omega_i = \Sigma_i(\beta_{i,v}\omega_i) \cdot \kappa(w) = \rho(v) \cdot \kappa(w)$, where the last dot is Serre duality. We now prove the following:

Proposition 3.18. *$A = -B$, in particular A is symmetric and hence* $2A(v,w) = \rho(v) \cdot \kappa(w) = \rho(w).k(v)$.

Proof. By (3.16) $\Sigma_i(\beta_{i,v} + \partial h_{i,v}) \wedge (\gamma_{i,w} + \bar{\partial} h_{i,w})$ is the zero form. Now $\beta_{i,v}$ and $\gamma_{i,w}$ are harmonic of type (1.0) and (0.1) respectively. It follows

$$\int_X \beta_{i,v} \wedge \bar{\partial} h_{i,w} = \int_X d(\beta_{i,v} \wedge h_{i,w}) = 0 = \int_X d(h_{i,v} \wedge \gamma_{i,w}) = \int_X \partial h_{i,v} \wedge \gamma_{i,w}.$$

Hence $0 = \Sigma_i \int_X (\beta_{i,v} + \partial h_{i,v}) \wedge (\gamma_{i,w} + \bar{\partial} h_{i,w}) = \Sigma_i \int_X \beta_{i,v} \wedge \gamma_{i,w} + \Sigma_i \int_X \partial h_{i,v} \wedge \bar{\partial} h_{i,w}$. □

(3.19). We sketch a construction of the (uni)versal families to be used in §6 (see (6.10)). Let $\pi : \mathcal{X} \longrightarrow B$ be a (uni)versal family of compact connected Riemann of genus g, $g > 2$, B has dimension $3g-3$ and the modular map $m : U \longrightarrow M_g$ is surjective and has discrete fibers. For instance

B could be Teichmüller space. Let $\eta : \lambda \to B$ be the Hodge bundle of π and $\eta_3 : F_3(\lambda) \to B$ its 3-frame bundle (see (2.14)) :

$$F_3(\lambda) = \{(\omega_1, \omega_2, \omega_3) \in \lambda^3 : \dim(\mathrm{span}(\omega_1, \omega_2, \omega_3)) = 3\}.$$

Defined by (0.1) and (0.2) we have the locus:

$$\mathcal{D} = \left\{ (\omega_1, \omega_2, \omega_3) \in F_3(\lambda) : \sum_i \omega_i^2 = 0; \sum_i \omega_i \bar{\omega}_i > 0 \right\}.$$

Let $\kappa : \mathcal{D} \to B$ be the map induced by η_3 and define, by change of base, the family $\pi' : \mathcal{X}' \to \mathcal{D}$. We construct $\mathbf{h} : \mathcal{X}' \to \mathbb{CP}^2$, by

$$\mathbf{h}(p) = [(\omega_1(p), \omega_2(p), \omega_3(p))]$$

where $(\omega_1, \omega_2, \omega_3) = \pi'(p)$. The image of h is the conic $Q \equiv \{\sum_{i=1}^3 z_i^2 = 0\} \equiv \mathbb{CP}^1$. The last identification is given as in (0.5) by the Veronese embedding. Let $\mathcal{O}(1)$ be the tautological bundle of $\mathbb{CP}^1$ and set $\mathbf{L} = f^*(\mathcal{O}(1))$. The s_i, $i = 1, 2$, define sections of $\mathbf{L}$. The restriction of $\mathbf{L}$ gives spin structures on the fibers of π'. To obtain smooth connected families we perform a desingularization of the irreducible components of $\mathcal{D}$. If $\mathcal{D}'$ is such a family then $\mu(\mathcal{D}')$ (see (3.2)) is a component of $\mathcal{S}_g$.

Section 4.

In this section we will assume to be in the *even* case. We consider a general point, (see 2.2), $[X]$ of E_g, $\{X, L, s_1, s_2\} \in \mathcal{S}_{g,e}$ and the spin abelian differentials ω_i as before. We recall from (2.4) that $T_{E_g,[X]}$, the tangent space of E_g at $[X]$, was identified with T_Ω, $\Omega = \mathrm{Wh}(s_1 \lambda s_2)$. We shall prove that the bilinear form defined in §3 gives rise to a map $T_{E_g,[X]} \to T^*_{E_g,[X]}$.

Take a family $\pi : \mathcal{X} \to B$, as in §3, such that $\{X_t, L_t, s_{1,t}, s_{2,t}\}$ is a deformation of $\{X_b, L_b, s_{1,b}, s_{2,b}\} = \{X, L, s_1, s_2\}$ for some fixed $\beta \in B$. If there is $v \in T_b$ such that $\kappa(v) = \xi \in T_\Omega$ we will say that $\{X_t, L_t, s_{1,t}, s_{2,t}\}$ represents ξ. The existence of E_g proves that any element of T_Ω is represented by a deformation of $\{X, L, s_1, s_2\}$.

Let $\pi : \mathcal{X} \to B$ and $\pi' : \mathcal{X}' \to B'$ be two 1-dimensional deformations of $\{X, L, s_1, s_2\}$ with base points b and b' respectively. Assume that both represent ξ. Write $\omega_i(s)$ on B and $\omega_i'(t)$ on B'. Expansion (3.10) gives

$$\omega_i(s) = \omega_i + s(\beta_{i,v} + \xi\omega_i + dh_i) + s^2\{\}; \quad \omega_i'(t) = \omega_i + t(\beta'_{i,v'} + \xi\omega_i + dh_i') + t^2\{\},$$

where $v \in T_b$, $v' \in T_{b'}$ $\kappa(v) = \kappa(v') = \xi$. It could be that $\beta_{i,v} \neq \beta'_{i,v'}$, but the following holds:

Lemma 4.1. *If $\rho : T_b \to H^0(\omega_X^2)$ and $\rho' : T_{b'} \to H^0(\omega_X^2)$ are the maps defined in (3.15), then $\rho(v) = \rho'(v')$.*

Proof. Fix $\zeta \in T_\Omega$ and a one dimensional deformation, $p : \mathcal{Y} \to C$, of $\{X, L, s_1, s_2\}$ which represents ζ, so there are $c \in C$ and $w \in T_c$ such that $\kappa(w) = \zeta$. By taking fiber product we construct two families:

$$\mathcal{Y} \to C \times B; \qquad \mathcal{Y}' \to C \times B'$$

such that both the restrictions $\mathcal{Y} \to C \times \{b\}$ and $\mathcal{Y}' \to C \times \{b'\}$ are isomorphic to p. We asume that $\mathcal{Y} \to \{c\} \times \{B\}$ and $\mathcal{Y}' \to \{c\} \times \{B'\}$ restrict to π and respectively to π'. Let $r : T_c \times T_b \to T_\Omega$ and $r' : T_c \times T_{b'} \to T_\Omega$ be the induced maps (cf. 3.15). We have $r(0, v) = r(v)$ and $r'(0, v) = r'(v')$ and therefore by (3.18):

$$\rho(v) \cdot \zeta = r(0, v) \cdot \kappa(w, 0) = r(w, 0) \cdot \kappa(0, v) = r(w, 0) \cdot \xi$$
$$\rho'(v') \cdot \zeta = r'(0, v') \cdot \kappa(w, 0) = r'(w, 0) \cdot \kappa(0, v') = r'(w, 0) \cdot \xi.$$

Since $r(w, 0) = r'(w, 0)$ we get $(\rho(v) - \rho'(v')) \cdot \zeta = 0$. This holds for any $\zeta \in T_\Omega$ and gives $(\rho(v) - \rho'(v')) \in \mathrm{Ann}(T_\Omega) = Q_\Omega$. Since $\rho(v) - \rho'(v')$ also belongs to $\mathrm{Im}(\mu_\Omega) = T_\Omega^*$ (see 3.15) it follows from (2.12) that $\rho(v) - \rho'(v') = 0$. □

After (4.1) we can define $\rho(\xi)$ by means of any 1-dimensional deformation of $\{X, L, s_1, s_2\}$ which represents ξ. Under the identifications of section 2 the map $T_\Omega \to \mathrm{Im}(\mu_\Omega)$ is

$$\rho : T_{E_g,[X]} \to T^*_{E_g,[X]} \tag{4.2}$$

and the symmetry (3.18):

$$\xi \cdot \rho(\zeta) = \zeta \cdot \rho(\xi). \tag{4.3}$$

We may consider ρ as an element of $\mathrm{Sym}^2(T^*_{E_g,[X]})$.

(4.4). Let P be a point in the support of the zero locus of Ω and ξ_P be the Schiffer variation of P. After (2.6) we know that ξ_P belongs to T_Ω. To compute $\rho(\xi_P)$ we take a one dimensional family $\pi : \mathcal{X} \to B$ representing ξ_P. Next we choose a trivialization (3.3) in such a way to obtain (see 2.8) $d\theta(v) = z^{-1}\bar{\partial}(\rho_\varepsilon) \otimes \partial/\partial z$. We recall that $\rho_\varepsilon \equiv 1$ near P and $\equiv 0$ outside U, where (U,z) is an open coordinate neighbourhood of P and $z(P)=0$. Set $\beta_{i,v}=\beta_i$ and $\xi_P\omega_i=\gamma_i$. First order expansion gives:

$$\begin{aligned}\omega_i(s) &= \omega_i + s(\beta_i+\partial h_i)+s(\xi_P\omega_i+\bar\partial h_i)+s^2\{\} = (\text{on } U) f_i(s,z)dz(s)\\ &= f_i(z)dz + s(\beta_i(z)+\partial h_i(z)/\partial z)dz + s(\gamma_i(z)+\bar\partial h_i(z)/\partial z)d\bar z + s^2\{\}.\end{aligned}$$

The Kodaira-Spencer equations become

$$(4.5)\quad \gamma_i+\bar\partial h_i = z^{-1}\bar\partial(\rho_\varepsilon)(\partial/\partial z\cdot\omega_i) = z^{-1}f_i(z)(\partial\rho_\varepsilon/d\bar z)d\bar z = \bar\partial(z^{-1}f_i(z)\rho_\varepsilon).$$

Set $g_i = z^{-1}f_i(z)\rho_\varepsilon - h_i$, and $\eta_i=\partial(g_i)$. Since $\gamma_i=\bar\partial(g_i)$ we obtain

$$(4.6)\qquad 0=-\partial\gamma_i = -\partial(\bar\partial g_i) = \bar\partial(\partial g_i) = \bar\partial\eta_i,$$

Hence η_i is holomorphic in $X-p$. Expanding near P ($\rho_\varepsilon\equiv 1$), we find out:

$$(4.7)\quad \eta_i = (-z^{-2}f_i(0)+z^{-1}(f_i'(0)-f_i'(0))+k(z))dz = (-z^{-2}f_i(0)+k_i(z))dz$$

where $k(z)$ is a holomorphic near 0. Remark that $\eta_i\in H^0(\omega_X(2P))$ is a differential of the second kind.

Claim 4.8. $\rho(\xi_P)=\Sigma_i\omega_i\cdot\eta_i$.

Proof. Formula (3.15) allows us to write

$$\rho(\xi_P) = -\Sigma_i\omega_i\partial h_i = \Sigma_i\omega_i(\eta_i - \partial(z^{-1}f_i(z)\rho_\varepsilon).$$

Since $\Sigma_i\omega_i^2=0$, $\omega_i=f_i(z)dz$, we have $\Sigma_i f_i(z)^2 = (\Sigma_i f_i(z)^2)' = \Sigma_i f_i(z)f_i'(z) = 0$. On the locus where $\rho_\varepsilon\equiv 1$ we get:

$$\begin{aligned}\Sigma_i\omega_i\partial(z^{-1}f_i(z))\rho_\varepsilon &= \Sigma_i f_i(z)\partial(z^{-1}f_i(z))dz\\ &= -\Sigma_i\{f_i(z)^2z^{-2}+z^{-1}\Sigma_i(f_i(z)f_i'(z)\}dz^2=0.\end{aligned}$$

The equality

$$\Sigma_i\omega_i\eta_i = \rho(\xi_P) = \Sigma_i\omega_i\beta_i$$

holds near $P:\Sigma_i\omega_i\eta_i$ is holomorphic on X. Since $\rho(\xi_P)-\Sigma_i\omega_i\eta_i$ vanishes on a non empty open set it follows that $\rho(\xi_P)-\Sigma_i\omega_i\eta_i\equiv 0$. □

Coming back to the η_i we have: $\xi_P\cdot\omega_i+\eta_i=\gamma_i+\eta_i=\bar\partial g_i+\partial g_i = dg_i$.

Then, by taking cohomology classes, we deduce the equality $[\gamma_i]=-[\eta_i]$ in $H^1(X-p,\mathbb{C})\cong H^1(X,\mathbb{C})$. We recall that $\xi_P: H^{1.0}(X)\to H^{0.1}(X)$ has rank one: $\ker(\xi_P)$ is the space of the abelian differentials vanishing at P (see

[13]). It follows that the $\xi_P \cdot \omega_i$ and therefore the η_i, are all proportional. Set $a_i = f_i(0)$. Take $\eta \in H^0(\omega_X(2P))$ such that $\eta_i = a_i\eta$, locally

$$\eta = (z^{-2} + r(z))dz$$

where $r(z)$ is holomorphic: $a_i r(z) = r_i(z)$. Clearly $\eta \neq 0$. We obtain:

$$\rho(\xi_P) = \Sigma_i \omega_i \eta_i = \eta(\Sigma_i a_i \omega_i) = \eta\omega \tag{4.9}$$

where $\omega = \Sigma_i a_i \omega_i$. The forms ω_i, $i = 1, 2, 3$, are linearly independent and without common zeroes. It follows that the a_i cannot be all zero and hence that $\omega \neq 0$. This gives $\rho(\xi_P) = \eta\omega \neq 0$.

To obtain a further information we expand the ω_i around P :

$$\omega_i(z) = (a_i + b_i z + c_i z^2 + d_i z^3 + o(z^4))dz,$$
$$\omega(z) = \Sigma_i a_i \omega_i(z) = \left\{\Sigma_i a_i^2 + (\Sigma_i a_i b_i)z + (\Sigma_i a_i c_i)z^2 + (\Sigma_i a_i d_i)z^3 + o(z^4)\right\} dz$$

Since $\Sigma_i \omega_i^2 = 0$ we have $\Sigma_i a_i^2 = \Sigma_i a_i b_i = 2\Sigma_i a_i c_i + \Sigma_i b_i^2 = \Sigma_i a_i d_i + \Sigma_i b_i c_i = 0$.

Since P is a ramification point of $h : X \longrightarrow \mathbb{CP}^2$ there is a complex number t such that $(b_1, b_2, b_3) = t(a_1, a_2, a_3)$. This implies $\Sigma_i b_i^2 = t^2 \Sigma_i a_i^2 = 0$, then $\Sigma_i a_i c_i = 0$, and $\Sigma_i b_i c_i = t\Sigma_i a_i c_i = 0$ which proves that ω has a zero of order 4 and $\eta\omega$ one of order 2 at P. Note that $\xi_P \rho(\xi_P) = 0$. We have shown:

Theorem 4.10. *If ξ_P is a Schiffer variation of a point P in the ramification locus of f, then $\rho(\xi_P) = \eta\omega$ is a non trivial holomorphic quadratic differential having a zero of order 2 at P.*

Section 5.

We study now the infinitesimal variation of the real part of the spin abelian differentials. Let $\pi : \mathcal{X} \longrightarrow B$ be a family of Riemann surfaces satisfying $(1 \ldots 4)$ of (3.1). Fix $p \in X$ and $b \in B$ and write $\{X, L, s_1, s_2\}$ for $\{X_b, L_b, s_{1,b}, s_{2,b}\}$. Let $I =]-\varepsilon, \varepsilon[$ be a real open interval and $k : I \longrightarrow B$ a smooth map, $k(0) = b$. Set $v = d/dt$ and denote by $\xi = \kappa(dk(v))$ its Kodaira-Spencer class. Pull-back defines the Riemann surfaces family $\pi' : \mathcal{X}' \longrightarrow I$ and the deformation $\{X_t, L_t, s_{1,t}, s_{2,t}\}_{t \in I}$ of $\{X, L, s_1, s_2\}$. Assume that $s_{1,t}$ and $s_{2,t}$ have not common zeros and write as usual $\mathcal{W}(s_{1,t}, s_{2,t}) = (\omega_1(t), \omega_2(t), \omega_3(t))$. By means of a trivialization $X \times I \longrightarrow \mathcal{X}'$ (see 3.3 and 3.8) we write:

$$\omega_i(t) = \beta_i(t) + \gamma_i(t) + dh_i(t),$$

where $\beta_i(t)$ and $\gamma_i(t)$ are harmonic forms of type (1.0) and (0.1). We have: $\mathrm{Re}(\omega_i(t)) = \mathrm{Re}(\beta_i(t)) + \mathrm{Re}(\gamma_i(t)) + d\,\mathrm{Re}(h_i(t))$ and therefore:

$$\mathrm{Re}(\omega_i(t)) = \mathrm{Re}(\omega_i) + t(\mathrm{Re}(\beta_i) + \mathrm{Re}(\gamma_i) + d\,\mathrm{Re}(h_i)) + o(t^2). \tag{5.1}$$

The piece $I(\pi') = \mathrm{Re}(\omega_i) + t(\mathrm{Re}(\beta_i) + \mathrm{Re}(\gamma_i) + d\,\mathrm{Re}(h_i))$ will be called the first order variation of π'. If

$$\mathrm{Re}(\beta_i) + \mathrm{Re}(\gamma_i) = 0, \quad i = 1, 2, 3, \tag{5.2}$$

we say that $I(\pi')$ is *homologically trivial* (see 0.10). It means that there is no first order variation of the harmonic forms: $\nabla_v \mathrm{Re}\,\omega_i(t) = 0$ (cf. 3.12). Comparing type we see that (5.2) is equivalent to $\bar\beta_i + \gamma_i = 0$ hence to:

$$\bar\beta_i = -\gamma_i = -\xi\omega_i, \quad i = 1, 2, 3. \tag{5.2'}$$

Lemma 5.3. *If $I(\pi')$ is homologically trivial, then $iA(v, v) \geq 0$.*

Proof. By substituting the equation (5.2') in (3.17) we find out:

$$iA(v, v) = \Sigma i \int_X \beta_i \wedge \gamma_i = \Sigma - i \int_X \beta_i \wedge \bar\beta_i \geq 0.$$

□

Let D be the hyperbolic disk and $\lambda : D \to X$ the universal covering, $\lambda(O) = p$, where O is the origin of D. This induces a family of universal coverings $D \times I \to X \times I \to \mathcal{X}$. We define $\mathbf{y} : D \times I \to \mathbb{R}^3$ by

$$\mathbf{y}(Q, t) = \mathrm{Re} \int (\Omega_1(t), \Omega_2(t), \Omega_3(t))$$

where the $\Omega_i(t)$ are the pull-back to D of the $\omega_i(t)$. For fixed t, $\mathbf{y}(., t)$, defines an immersed minimal surface $\mathbf{Y}_t$ in the Euclidean space: we have assumed $s_{1,t}$ and $s_{2,t}$ without common zeros. Note that $\mathbf{Y}_t$ is closed in $\mathbb{R}^3$ if and only if it is triply periodic. If $I(\pi')$ is homologically trivial:

$$\mathbf{y}(Q, t) = \mathbf{x}(Q) + t \int (d\,\mathrm{Re}\, H_1, d\,\mathrm{Re}\, H_2, d\,\mathrm{Re}\, H_3) + o(t^2)$$

where $\mathbf{x}(Q) = \mathbf{y}(Q, 0) = \mathrm{Re} \int_{[0,Q]} (\Omega_1(0), \Omega_2(0), \Omega_3(0))$, $\mathbf{x} : D \to \mathbb{R}^3$ and $H_i = h_i \cdot \lambda$. Setting $\mathrm{Re}\, H_i = G_i$ we obtain:

$$\mathbf{y}(Q, t) = \mathbf{x}(Q) + t(G(Q) + \underline{C}) + o(t^2), \tag{5.4}$$

where $G(Q) = (G_1(Q), G_2(Q), G_3(Q))$ and $\underline{C} = (C_1, C_2, C_3)$ is a constant vector.

Let $\partial\mathbf{y} = (G_1, G_2, G_3)$ denote the flow defined by the minimal surfaces. Up to a change of the trivialization in (3.3) we may assume that $\partial\mathbf{y}$ is *normal* to the immersed minimal surface $\mathbf{x} = \mathbf{Y}_0$. We point out that a deck transformation α of λ acts, up to the first order, as a translation by the periods: $P(\alpha) = \operatorname{Re}\int_\gamma(\omega_1, \omega_2, \omega_3)$, where γ is a closed curve whose lifting connects $O \in D$ with $\alpha(O)$. The homological triviality gives $P(\alpha, t) = P(\alpha) + o(t^2)$ for all α.

Next for any fixed real numbers r and $r' : 0 < r < r' < 1$, let $\rho_{r,r'}$ be a bump function on D, $\rho_{r,r'} \equiv 1$ on $D_r = \{z \in D : |z| \leq r\}$ and $\rho_{r,r'} \equiv 0$ for $|z| > r'$. Consider the compactly supported variation of $\mathbf{x}$:

$$\mathbf{y}_{r,r'}(Q) = \mathbf{x}(Q) + t\rho_{r,r'}(G_1(Q), G_2(Q), G_3(Q)).$$

We have: $\mathbf{y} = \mathbf{y}_{r,r'} + o(t^2)$ on D_r. Normalizing we rewrite it as:

$$\mathbf{y}_{r,r'}(Q) = \mathbf{x}(Q) + t\Phi_{r,r'}(Q)n_Q + o(t^2),$$

where n_Q is the unit normal of $\mathbf{x}$ at Q. Let F_1 and F_2 be two fundamental domains of $\lambda : D \to X$ and $\alpha : D \to D$, $\alpha(F_1) = F_2$, their deck transformation. Assume that D_r contains both F_1 and F_2. Since a acts up to the first order as the period translation we get two expressions:

$$\begin{aligned}\mathbf{y}_{r,r'}(\alpha(Q)) &= \mathbf{x}(\alpha(Q)) + t\Phi_{r,r'}(\alpha(Q))n_Q + o(t^2)\\ &= \mathbf{x}(Q) + P(\alpha) + t\Phi_{r,r'}(\alpha(Q))n_Q + o(t^2)\\ \mathbf{y}_{r,r'}(\alpha(Q)) &= \mathbf{y}_{r,r'}(Q) + P(\alpha, t) + o(t^2)\\ &= \mathbf{x}(Q) + P(\alpha, t) + t\Phi_{r,r'}(Q)n_Q + o(t^2).\end{aligned}$$

Hence:

$$P(\alpha) + t\Phi_{r,r'}(\alpha(Q))n_Q = P(\alpha, t) + t\Phi_{r,r'}(Q)n_Q$$

Since $P(\alpha, t) = P(\alpha) + o(t^2)$ this implies $\Phi_{r,r'}(Q) = \Phi_{r,r'}(\alpha(Q))$ for all Q of F_1. Letting $\Phi = \lim_{r\to 1}(\Phi_{r,r'})$ then for any $Q \in D$

$$\begin{aligned}&\mathbf{y}(Q, t) = \mathbf{x}(Q) + t\Phi(Q)n_Q + o(t^2)\\ &\partial\mathbf{y}(Q) = (G_1(Q), G_2(Q), G_3(Q)) = \Phi(Q)n_Q\end{aligned} \tag{5.5}$$

where $\Phi : D \to \mathbb{R}$, is invariant under deck transformations. Therefore there is a real valued function ϕ on X such that $\Phi = \phi \cdot \lambda$.

The mean curvature of $\mathbf{Y}_t$ is identically zero. Let $H(t,Q)$ denote the mean curvature function along the osculating variation:

$$\mathbf{x} + t\Phi n. \tag{5.6}$$

It follows that $dH/dt(0,Q) = 0$ and hence by a standard computation:

$$\Delta\Phi - 2K\Phi = 0, \tag{5.7}$$

where Δ is the Laplacian and K is the Gauss curvature of the metric $\sum_i \Omega_i\overline{\Omega}_i$ on D. It means that $\Phi \cdot n$ is a Jacobi field of $\mathbf{x}$. If $\mathbf{g}$ is the metric defined in (0.2) we have: $\sum_i \Omega_i\overline{\Omega}_i = \lambda^*(\sum_i \omega_i\overline{\omega}_i) = \lambda^*(\mathbf{g})$. Then (5.7) can be written on X as in (0.11):

$$\Delta\phi - 2K\phi = 0 \tag{5.7$'$}$$

where by abuse of notation we have denoted with Δ and K the Laplacian and the Gauss curvature of $\mathbf{g}$. The second variation operator $\Delta - 2K$ is (cf. [20]) a Schrödinger operator on X. From (5.7') it follows:

$$Q_f(\phi,\phi) = \int_X |\nabla\phi|^2 + 2\int_X K\phi^2 = 0 \tag{5.8}$$

and ϕ belongs to

$$N(f) = \{\varphi : (\Delta - 2K)(\varphi) = 0\} \tag{5.9}$$

the nullity space of Q_f. The form Q_f and $N(f)$ depend (cf. [20]) only upon the conformal structure of X and on f. Translations give deck transformations invariant Jacobi fields of $\mathbf{x}$. Let $\mathbf{j} : \mathbb{R}^3 \to N(f)$ be the induced inclusion and

$$M(f) = N(f)/\mathbf{j}(\mathbb{R}^3) \tag{5.9$'$}$$

be its quotient. If the class $[\phi] \in M(f)$ is zero then, up to a change of the integration constant $\underline{C}$ in (5.4), we may assume $\phi = 0$. In this case we have $\mathrm{Re}(h_i) = G_i = 0$, $i = 1,2,3$, then $h_i = ik_i$, where the k_i are real functions. We compute the form B in (3.17):

$$\begin{aligned} iB(v,v) &= i\Sigma_i \int_X \partial h_i \wedge \bar\partial h_i = -i\Sigma_i \int_X \partial k_i \wedge \bar\partial k_i \\ &= -i\Sigma \int_X (k_{i,x})^2 + (k_{i,y})^2 dz \wedge d\bar z \\ &= 1/2\Sigma \int_X ((k_{i,x})^2 + (k_{i,y})^2) dx \wedge dy \geq 0. \end{aligned}$$

By (3.18) and (5.3) we have:

$$1/2\Sigma_i \int_X ((k_{i,x})^2 + (k_{i,y})^2) dx \wedge dy + \Sigma_i - i \int_X \beta_i \wedge \bar{\beta}_i = iA(v,v) + iB(v,v) = 0.$$

The only possibility is that $A(v,v) = B(v,v) = 0$: Hence $\phi = 0$ if and only if $\beta_i = 0 = -\bar{\beta}_i = \gamma_i$ and the h_i are constant $i = 1,2,3$. We can prove:

Proposition 5.10. *If* $[\phi] = 0$, *then the first order deformation of* $\{X, L, s_1, s_2\}$ *associated to* $\{X_t, L_t, s_{1,t}, s_{2,t}\}_{t\in I}$ *is trivial.*

Proof. Since $\gamma_{i,v} = \bar{\partial} h_{i,v} = 0$ the Kodaira-Spencer equations gives $d\theta(v)\cdot\omega_i = 0$. It implies $d\theta(v) = 0$ and hence $\xi = 0$. This could be deduced directly: The metric $\mathbf{g}(t)$ in (5.6) (and hence the conformal structure!) is fixed. In any case the first order deformation of $\{X_t\}$ and therefore of $\{X_t, L_t\}$ is trivial. Moreover the β_i are all zeros: $\omega_{i,t} = \omega_i + o(t^2)$ $i = 1,2,3$. Then we have $s_{i,t} = s_i + o(t^2)$, $i = 1,2$, and hence (cf. 3.2) $d\mu(b) = 0$. □

Remark 5.11. Reversing the process we find a one to one correspondence between homologically trivial I° order variations of $\{X, L, s_1, s_2\}$ and $M(f)$. By (5.3) $iA(v,v) \geq 0$ for $v \in M(f)$. Now $iA(v,v) = 0$ implies $\beta_i = \gamma_i = \xi\omega_i = 0$, $i = 1,2,3$. In the even case and with X general we infer by (2.12) that $\xi = 0$ then $\omega_{i,t} = \omega_i + o(t^2)$: iA defines a scalar product.

Section 6 (proof of Theorem 1 and 2).

We are ready to study the periodicity condition (0.9) by using variational methods. Let $\pi : \mathcal{X} \to B$ be a family as in (3.1), $b \in B$ and $\{X_b, L_b, s_{1,b}, s_{2,b}\} = \{X, L, s_1, s_2\}$. We assume B simply connected. Let $\mathbb{K}$, $\mathbb{K} = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$, be the constant sheaf $\mathbb{K} \times \mathcal{X} \to \mathcal{X}$ and $\mathcal{H} = R^1\pi_*(\mathbb{K})$ be its first direct image. The fibers of $\mathcal{H}$ are isomorphic to $H^1(X,\mathbb{K})$, $\mathcal{H}$ is the Hodge bundle considered in section 3. Letting $F_3(\mathcal{H}) \equiv F_3(H^1(X,\mathbb{R})) \times B$ be the 3-frames bundle (see 2.14), we get

$$\text{(6.1)}\quad \mathrm{Re}(\mathcal{W}(s_{1,t}, s_{2,t})) = (\mathrm{Re}(\omega_{1,t}), \mathrm{Re}(\omega_{2,t}), \mathrm{Re}(\omega_{3,t})) \in F_3(H^1(X,\mathbb{R})) \times B.$$

Set $F = F_3(H^1(X,\mathbb{R}))$ and define $\Sigma : B \to F$ by projection. We still write

$$\text{(6.2)}\qquad \Sigma(t) = \mathrm{Re}(\mathcal{W}(s_{1,t}, s_{2,t})).$$

Let G be the 3-planes Grassmannian of $H^1(X,\mathbb{R})$ and $\Phi : B \to G$

$$\text{(6.3)}\qquad \Phi(t) = \mathrm{Re}(W_t) = \mathrm{span}((\mathrm{Re}(\omega_{1,t}), \mathrm{Re}(\omega_{2,t}), \mathrm{Re}(\omega_{3,t}))$$

be the composition of Σ with the span map $F \rightarrow G$ (see 2.14).

(6.4). The map Σ is the variational form of the period matrix (0.8): Identifying the frames as the rows of the matrices we rewrite (6.2):

$$\begin{pmatrix} \langle\alpha_1, \mathrm{Re}\,\omega_{1,t}\rangle, & \langle\beta_1, \mathrm{Re}\,\omega_{1,t}\rangle, & \dots, & \langle\alpha_g, \mathrm{Re}\,\omega_{1,t}\rangle, & \langle\beta_g, \mathrm{Re}\,\omega_{1,t}\rangle \\ \langle\alpha_1, \mathrm{Re}\,\omega_{2,t}\rangle, & \langle\beta_1, \mathrm{Re}\,\omega_{2,t}\rangle, & \dots, & \langle\alpha_g, \mathrm{Re}\,\omega_{2,t}\rangle, & \langle\beta_g, \mathrm{Re}\,\omega_{2,t}\rangle \\ \langle\alpha_1, \mathrm{Re}\,\omega_{3,t}\rangle, & \langle\beta_1, \mathrm{Re}\,\omega_{3,t}\rangle, & \dots, & \langle\alpha_g, \mathrm{Re}\,\omega_{3,t}\rangle, & \langle\beta_g, \mathrm{Re}\,\omega_{3,t}\rangle \end{pmatrix}$$

where $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$, is a (symplectic) basis of $H^1(X, \mathbb{Z})$. Notice again that Σ gives the variation of the real periods of the ω_i. Letting $\mu : B \rightarrow \mathcal{S}_g$ be the (3.2) modular map, we recall that $B(\mathbb{Q}) = \mu^{-1}(\mathcal{S}_g(\mathbb{Q}))$ are the points of B defining periodic minimal surfaces. From (0.9) and (2.15) we have:

$$B(\mathbb{Q}) = (\Phi)^{-1}(G\ ). \tag{6.6}$$

To study the differential of Σ at b we identify the holomorphic and the real tangent spaces of B and write:

$$d\Sigma : T_b \rightarrow H^1(X, \mathbb{R}) \oplus H^1(X, \mathbb{R}) \oplus H^1(X, \mathbb{R}).$$

We have already computed it: Expansion (5.1) along the fixed direction v in T_b reads $\mathrm{Re}(\omega_i(t)) = \omega_i + t(\mathrm{Re}(\beta_{i,v}) + \mathrm{Re}(\gamma_{i,v}) + d\,\mathrm{Re}(h_{i,v})) + o(t^2)$, therefore:

$$d\Sigma(v) = (\mathrm{Re}(\beta_{1,v} + \gamma_{1,v}), \mathrm{Re}(\beta_{2,v} + \gamma_{2,v}), \mathrm{Re}(\beta_{3,v} + \gamma_{3,v})). \tag{6.7}$$

In particular, if we assume that s_1 and s_2 have not common zeros, we see that the elements of $\mathrm{Ker}(d\Sigma)$ give rise to first order homologically trivial deformations of $\{X, L, s_1, s_2\}$. As usual set $f = s_1/s_2$ and let $M(f)$ be defined as in (5.9′). Then, by taking classes, we define a map:

$$\nu : \mathrm{Ker}(d\Sigma) \rightarrow M(f). \tag{6.8}$$

¿From (5.10) we know that the kernel of ν is given by the first order deformations of $\{X, L, s_1, s_2\}$ that are actually trivial.

Definition 6.9. A family (3.1) will be called *complete* if $\mu(B)$ is a component of $\mathcal{S}_g$ and *versal* at b if the differential of $\mu : B \rightarrow \mathcal{S}_g$ at b, $d\mu : T_b \rightarrow T_{\mathcal{S}_g, \mu(b)}$, is an isomorphism.

(6.10). One constructs (as in 3.19) complete versal families. If $\pi : \mathcal{X} \rightarrow B$ is versal at b then any non zero element of T_b gives non-trivial first order deformation of $\mu(b) = \{X, L, s_1, s_2\}$. Moreover if B is complete and either $\mu(B)$ is a general component of $\mathcal{S}_g$ (2.16) or $\mu(B) = \mathcal{H}_g$ (2.18) then dim $(B) = 6g$. As a straightforward consequence of (5.10) we obtain:

Proposition 6.11. *Assume that B is versal at $b, \mu(b) = \{X, L, s_1, s_2\}$ and s_1 and s_2 without common zeros, then $n : \mathrm{Ker}(d\Sigma) \to M(f)$ is injective.*

We can now prove the following:

Proposition 6.12. *With the hypothesis of* (6.11) *we also assume $M(f) = 0$ and* dim $(B) = 6g$, *then :*

(i) *$B(\mathbb{Q})$ is dense in B;*

(ii) *$B(\mathbb{Q})$ contains a countable number of connected components of (real) dimension* 9.

Proof. i) Passing to universal covering we assume B simply connected. Next observe that Σ is a real analytic map, hence $\{t \in B : \dim(M(f_t)) > 0\}$ is a closed proper analytic set of B. Since dim $(F\ ) =$ dim $(B) = 6g$, $d\Sigma$ is an isomorphism on an open dense set B' of B. Therefore $d\Phi$ is submersive on B'. Since G is dense in G then $B(\mathbb{Q}) = (\Phi)^{-1}(G\ )$ is dense in B' and hence in B.

ii) The dimension of any fiber of a regular value is equal to dim $(F\ ) -$ dim $(G\ ) = 9$. We have $B(\mathbb{Q}) = \cup_{\Pi \in G}\ (\Phi)^{-1}(\Pi))$. □

(6.13). The submersivity of Σ gives more than the density, it proves that the image of Σ contains an open set of the lattices of some fixed real 3-space. This corresponds to an open set in the moduli of the flat conformal 3-tori. The dimension 9 of the fibers bears the same meaning: Let $\mathbf{G}$ be the group generated by the orthogonal group $O(3)$ and the non zero scalars. Let F be a general fiber of Σ, plainly $\mathbf{G}$ acts on F and $\dim(F/\mathbf{G}) = 9 - 4 = 5$, the moduli of the flat tori. The modular map $m : F \to M_g$ (see 3.2) factorizes through $F/\mathbf{G}$ and $\mathrm{Re} : H^{1.0} \to H^1(X, \mathbb{R})$ is an isomorphism. We get $\dim(m(F)) = 5$.

To complete the proof of Theorem 1, we need to know whenever the conditions of (6.12) are fulfilled. For, let $\Theta = (X, L, s_1, s_2) \in \mathcal{S}_g$ and consider the curve in $\mathcal{S}_g$ (cf. [17]):

$$\Theta(t) = ((X, L, ts_1, s_2),\ t \in \mathbb{R}.$$

We remark that the associated meromorphic function is $f_t = tf$. We have:

Theorem 6.14. *If f has only one pole then for $t > 0$ small enough $M(f_t) =$* 0.

Proof. See [24], Theorem 2, page 318.

Definition 6.15. We call a point $P = \{X, L, s_1, s_2\}$ of $\mathcal{S}_g$ "good" if f has only one pole and the pencil $\lambda s_1 + \mu s_2$ is base point free, i.e. $\deg(f) = g-1$.

Next we state our basic existence result:

Theorem 6.16.

(i) *If g is odd ($g > 2$) there is a good point $P = \{X, L, s_1, s_2\}$ of $\mathcal{H}_g$ (see 2.18);*

(ii) *If $g > 2$ there is a good point $P \in \mathcal{S}_{g,e}$;*

(iii) *If $g = 6$ there is a good point $P \in \mathcal{S}_{6,o}$.*

Proof. (see §8 below).

(6.17). It is perhaps worthwhile to spend a comment about the common zeros assumption, also contained in (2.2): Unnecessary in (6.14), which holds for arbitrary meromorphic functions on compact connected Riemann surfaces (cf. [24]), it was used to define the Jacobi field (5.7). On the other hand if a family of pencils, $\{\lambda s_{1,t} + \mu s_{2,t}\}$, acquires a fixed point then the degree of $f_t = s_{1,t}/s_{2,t}$ drops. The Schrödinger operators behaviour has not been, as far as we know, carried out.

(6.18). (Completion of the proof of Theorem 1). Take $P = \{X, L, s_1, s_2\} \in \mathcal{S}_g$ which satisfies (6.16). If B is a complete family versal at $b \equiv P$ we get by (6.12) that $B(\mathbb{Q})$ is dense in B. Then the closure of $m(B(\mathbb{Q}))$ contains E_g or O_6 if B is general (cf. 2.16), and H_g in the odd genus hyperelliptic case (cf. 2.18): E_g, H_g and O_6 are irreducible varieties. □

(6.19). (Completion of the proof of Theorem 2). i) Fix a versal simply connected and complete (even, odd, hyperelliptic) family B. Let $\Sigma : B \to F$, $\Phi : B \to G$ be the above maps. Assume there is $\Pi \in G$ and $b \in B$ such that $\Phi(b) = \Pi$ and $d(\Sigma)(b)$ is an isomorphism. It follows that $d(\Phi)(b)$ is surjective. For any $t \in \Phi^{-1}(\Pi) = F_\Pi$ let $\mu(t) = \{X_t, L_t, s_{1,t}, s_{2,t}\}$ and Λ_t be the group generated by the columns of $\Sigma(t) = \mathrm{Re}(\mathcal{W}(s_{1,t}, s_{2,t}))$ (see 6.5). As in (0.4) we define the torus $T_t = \Pi/\Lambda_t$ together with a minimal immersion:

$$K_t : X_t \to T_t.$$

Let $\mathbf{M}$ be the moduli space of the conformal flat real 3-tori and define $\psi : F_\Pi \to \mathbf{M}$ by:

$$\psi(t) = [T_t] = \{\text{isomorphism class of } T_t\}. \tag{6.20}$$

As remarked in (6.13) the image of ψ contains an open set $\sqcap$ of $\mathbf{M}$. Let T be any 3 dimensional real compact torus. We can find an isogeny $\mathbf{z} : T_t \to T$, where $\psi(t) \in \sqcap$. The composition

$$\mathbf{z} \cdot K_t : X_t \to T,$$

defines an immersed compact minimal surface in T. Varying $\Pi \in G$ one constructs a countable number of even ($g > 2$), odd ($g = 6$) and hyperelliptic (g odd) immersed minimal surfaces on T.

ii) The conformal tori depends upon 5 moduli, adding the rescaling we get our six dimensional families of proper triply periodic minimal surfaces in the Euclidean space. □

(6.21). Let $M_g(T) \subset M_g$ be the set defined by the moduli of the Riemann surfaces which can be minimally immersed in the 3 torus T. Combining Theorem 1 and the proof of Theorem 2 one sees that the closure of $M_g(T)$ contains E_g ($g > 2$) and O_6. Using even, odd or hyperelliptic families we see that (cf. 0.12):

$$E_g(\mathbb{Q}) = \cup_{\Pi \in G_Q} m((\Phi)^{-1}(\Pi))(g > 2), \quad O_6(\mathbb{Q}) = \cup_{\Pi \in G_Q} m((\Phi)^{-1}(\Pi))$$

and

$$H_g(\mathbb{Q}) = \cup_{\Pi \in G_Q} m((\Phi)^{-1}(\Pi)) \ (g > 2 \text{ odd}).$$

If $g > 2$ we get a countable number of connected components of $E_g(\mathbb{Q})$, $O_6(\mathbb{Q})$ and $H_g(\mathbb{Q})$. Most of them have real dimension 5. It would be interesting to know if there are families of bigger dimension. This is strongly related with the existence and the integrability of the Jacobi fields (5.7) and hence with the rigidity theory of the minimal surfaces.

Section 7.

Let $K : X \to T$ be a periodic minimal compact surface as in (0.4). We say that K defines a complex 3-torus T if there are a Lie group homomorphism $\alpha : T \to T$ and a holomorphic map

(7.1) $$K' : X \to T$$

such that $K = \alpha \cdot K'$. This is the case when $K(X)$ has a periodic "associated" or obtained as a covering of a genus 3 minimal Riemann surface (see [18] and [22] and (7.5) below). In [22] the following question arises:

(7.2) Does always K define a complex three torus?

We will see that in general the answer to (7.2) is *negative*: Let B be a versal simply connected and complete family (see 6.9) and $\{X, L, s_1, s_2\} = \{X_b, L_b, s_{1b}, s_{2b}\}$. Set $G' = G(6, H^1(X, \mathbb{R}))$, $G' = G(6, H^1(X, \mathbb{Q}))$ and as usual $W_t = \operatorname{span}(\mathcal{W}(s_{1,t}, s_{2,t}))$. In complete analogy with (6.3) define $\Xi : B \to G'$ by:

$$X(t) = \operatorname{Re}(W_t) \oplus \operatorname{Im}(W_t) = \operatorname{Re}(W_t) \oplus \operatorname{Re}(iW_t). \tag{7.3}$$

Set $B'(\mathbb{Q}) = \Xi^{-1}(G')$. We observe that t belongs to $B'(\mathbb{Q})$ if and only if W_t is naturally the cotangent space of a complex 3 sub-torus of the Jacobian variety of X_t. Next for any $\Pi' \in G'$ we set $F'_{\Pi'} = \Xi^{-1}(\Pi')$ and $Z_{\Pi'} = m(F'_{\Pi'}) \subset M_g$. It is known (see [7]) that F'_{Π} and $Z_{\Pi'}$ are complex varieties. Besides, since $\Xi(t)$ does not depend on the chosen complex frame of $V = \operatorname{span}(s_{1,t}, s_{2,t})$, we get dim $(F'_{\Pi'}) =$ dim $(Z_{\Pi'}) + 8$. It follows that either $F'_{\Pi'}$ is empty or dim $(F'_{\Pi'}) \geq 8$. If the image of Ξ contains an open set then $\dim(G') + 8 \leq 6g : g \leq 4$. We will role out the case $g = 4$.

Proposition 7.4. *If $g = 4$ then $d\Xi$ is generically surjective.*

Proof. Let $m : B \to M_4$ be the modular map, $m(B) = E_4$. Let $H_4 \subset E_4$ be the hyperelliptic locus and $\mathbf{H} = m^{-1}(H_4)$ be its inverse image. A point of $\mathbf{H}$ corresponds to $\{X, L, s_1, s_2\}$, where X is hyperelliptic and $L^2 = \omega_X$. It turns out that s_1 and s_2 have a Weierstrass point p as a common zero: $W = H^0(\omega_X(-p))$. From [8] it follows that $\Xi(\mathbf{H})$ is an open set of G' . The proposition follows now from the Sard lemma. □

We still let $g = 4$. By (7.4) there is $b \in B$, $\mu(b) = \{X, L, s_1, s_2\}$, such that both Ξ and Φ are submersive at b and $\Pi = \Phi(b) \in G$. Set $F_\Pi = \Phi^{-1}(\Pi)$, $\Xi(b) = \Pi'$, $F'_{\Pi'} = \Xi^{-1}(\Pi')$. There are two possibilities: *either* $\Pi' \in G'$ *or* $\Pi' \in G'$. In any case we have dim $(F'_{\Pi'}) = 12$, dim $(F_\Pi) = 9$, dim $(m(F_\Pi)) = 5$ and dim $m(F'_\Pi) = 4$. This implies that $F_\Pi \not\subset F'_\Pi$ and we can find $c \in B$ such that $\Phi(c) = \Pi \in G$, but $\Xi(c) \notin G'$. Let $K : X \to T$ be the corresponding genus 4 minimal surface on T. By construction $K(X)$ does not define a complex 3-torus. Moreover if $\mathbf{z}_1 : T_1 \to T$ is any isogeny we construct, by taking inverse image, a commutative diagram:

$$\begin{array}{ccc} X_1 & \longrightarrow & T_1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & T. \end{array} \tag{7.5}$$

where $X_1 \to X$ is an unramified covering and $K_1 : X_1 \to T_1$ a minimal immersion. This provides counter-examples to (7.2) for $g > 4$:

Theorem 7.6. *There are compact minimal surfaces on compact real* 3*-tori that do not define any complex* 3 *torus.*

Section 8.

Proof of (6.16): i) *Hyperelliptic odd genus case:*

Let X be a compact hyperelliptic Riemann X of genus $g = 2k + 1 > 2$. Set $L = \mathcal{O}_X(2kp)$, where p is a Weierstrass point of X, i.e. p is invariant under the hyperelliptic involution. Set $M = \mathcal{O}_X(2p)$ and let s and t be two global independent sections of M. We assume that the divisor of t is $2p$. Set $s_1 = s^k$, $s_2 = t^k$ and $f = s_1/s_2$. □

We need some notation from the theory of Weierstrass points on a compact Riemann surface. An increasing sequence of integers:

(8.1) $$0 < a_1 < ... < a_g < 2g, \quad g > 1,$$

is called a gap sequence if

(8.2) $$H = N - \{a_1, \ldots, a_g\}$$

is a sub-semigroup of the non-negative integers $(N, +)$. The integers a_i, $i = 1, \ldots, g$, are the gaps of H. Let X be a compact Riemann surface of genus $g > 1$ and p be a point of X. A non negative integer m (see [2], page 41) is a non-gap of p if there is a meromorphic functions on X of f degree $m \geq 0$ holomorphic on $X - p$. The non gaps define a sub-semigroup of $(N, +)$. By duality an integer $n > 0$ is a gap of p if

(8.3) $$h^0(\omega_X(-np)) = h^0(\omega_X(-n+1)p) - 1.$$

By Riemann Roch theorem and (8.3) there are exactly g gaps of p which define the Weierstrass gap sequence:

(8.4) $$0 < a_1(p) < \ldots < a_g(p) < 2g.$$

We recall that p is a Weierstrass point of X if $a_g(p) > g$. It is a classical unsolved problem (but see [11] and [26]) to determinate which of the sequences (8.1) are Weierstrass gap sequences (8.4). We have:

Theorem 8.5. *The sequence:* $1, \ldots, g-2, g, 2g-1$, $g > 3$, *is a Weierstrass gap sequence.*

Proof. See [26]: Th. (4.7), page 463, and Cor.(5.10), page 472.

Proof of (6.16) ii) *Even case*:

We assume $g > 3$ ($g = 3$ is the hyperelliptic case i)). By (8.5) there is a compact Riemann surface X of genus g and $p \in X$ having gap sequence $1, \dots, g-2, g, 2g-1$. Since $2g-1$ is a gap $\mathcal{O}_X((2g-2)p)$ (see 8.3) is isomorphic to ω_X, the canonical bundle of X. Set $L = \mathcal{O}_X((g-1)p)$. Since $g-1$ is the first non gap of p we obtain $h^0(L) = 2$. Therefore L defines an even spin structure of X and we may find a basis s_1, s_2 of $H^0(L)$ where p is the unique pole of $f = s_1/s_2$. Note that the degree of f is $g-1$ because $g-2$ is a gap of p. □

Proof of (6.16) iii) *Odd case* **g=6.**

Consider the Fermat curve: $X = \{(x, y, z) \in \mathbb{CP}^2 : x^5 + y^5 + z^5 = 0\}$. The genus of X is 6. Let L be the restriction of $\mathcal{O}_{CP^2}(1)$ to X. By adjunction $\mathcal{O}_{CP^2}(2)_{|X} = \omega_X$, hence L is a spin bundle of X. The coordinates x, y, z provide a basis of L, $h^0(L) = 3$. Set $s_1 = y + z$ and $s_2 = x + z$. Then $f = s_1/s_2$ has a unique pole of order 5 at $(1, 0, -1)$. □

Remark 8.4. Likely for $g > 5$ $(1, \dots, g-3, g+1, g+2, 2g-1)$ is a Weierstrass gap sequence. The above case iii) provides an example for $g = 6$ and see [15] for $g = 7$. If this is granted the general odd case follows as before.

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