COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 6, Number 2, 317-329, 1998

Two index theorems in odd dimensions

DANIEL S. FREED[1]

A recent paper of Hořava and Witten [HW]—part of the current flurry of activity in string theory—contains an anomaly computation for $S^1/\langle\tau\rangle\times\mathbb{R}^{10}$, where $\langle\tau\rangle$ is the cyclic group of order two generated by a reflection. It was well established 10 years ago (e.g. [AS1], [F1]) that anomalies measure nontriviality in the determinant line bundle of a family of Dirac operators, and so can be computed topologically from the Atiyah-Singer index theory. The novelty in the Hořava-Witten computation is a nontrivial index in *odd* dimensions of a type not seen in standard index theory. We abstract two general theorems which imply the Hořava-Witten result. (Naturally, we replace $\mathbb{R}^{10}$ by a compact manifold Y^{10}). Theorem A is a Lefschetz formula for an *orientation-reversing* isometric involution on an odd dimensional manifold. The Atiyah-Bott-Segal-Singer applications of Lefschetz theory [AB1], [ASe], [AS2] all deal with *orientation-preserving* isometries for which there is no nontrivial Lefschetz formula in odd dimensions [AS2, Proposition 9.3]. Alternatively, we can consider $[0,1]\times Y$ in place of $S^1/\langle\tau\rangle\times Y$ and then the Hořava-Witten anomaly computation is a boundary value problem with *local* boundary conditions. Theorem B generalizes this situation and is closely related to the boundary value problem in the original proof of the Atiyah-Singer index theorem [P].

Our proofs use standard techniques, except for a small trick used to prove Theorem A. For simplicity we discuss the standard complex Dirac operator; the theorems are true for any Dirac operator. Our language refers mostly to a single operator, though the results hold for families of Dirac operators as required by the anomaly problem. In this regard we remark that Theorem A only holds modulo 2-torsion in the K-theory of the parameter space, whereas Theorem B holds exactly in K-theory. For the anomaly problem this means that Theorem A may not be adequate to detect all *global* anomalies. (In the general situation of Theorem A, there is probably no fixed-point formula for the exact index.)

[1]The author is supported by NSF grant DMS-9307446, Presidential Young Investigators award DMS-9057144, and by the O'Donnell Foundation.

I thank Edward Witten for bringing this anomaly computation to my attention and for discussions.

1. A Lefschetz formula for orientation-reversing isometries.

Let X be a compact odd dimensional spin manifold. Suppose $\tau : X \to X$ is an orientation-reversing isometric involution. Assume there exists a lift $\tilde{\tau} : S_X \to S_X$ to the complex spinor bundle S_X on X such that

$$D_X\tilde{\tau} = -\tilde{\tau}D_X, \tag{1.1}$$

where D_X is the Dirac operator. It follows from Lemma 1.5 below that $\tilde{\tau}^2$ is locally constant, so dividing by a square root of that locally constant function we may assume

$$\tilde{\tau}^2 = 1. \tag{1.2}$$

Then the ±1-eigenspaces of $\tilde{\tau}$ give a splitting of the spinor fields

$$S(X) \cong S^+(X) \oplus S^-(X), \tag{1.3}$$

and the Dirac operator interchanges $S^+(X)$ and $S^-(X)$. Our problem is to compute

$$\operatorname{index}\bigl[D_X : S^+(X) \longrightarrow S^-(X)\bigr]. \tag{1.4}$$

The simplest example is $X = S^1 = \mathbb{R}/\mathbb{Z}$ with τ the reflection $x \mapsto -x$. The spinor fields may be identified with the complex functions, the Dirac operator with $i\frac{d}{dx}$, and the splitting (1.3) is the splitting into even and odd functions. Here the index is 1. The Hořava-Witten example is the product with a fixed even dimensional manifold Y, in which case the index is $\operatorname{index} D_Y$.

The lift $\tilde{\tau}$, if it exists, is almost unique.

Lemma 1.5. *Suppose X is an odd dimensional spin manifold, and $\theta : S_X \to S_X$ a bundle map such that $D_X\theta = \theta D_X$. Then θ is a locally constant multiple of the identity.*

If $\tilde{\tau}_1, \tilde{\tau}_2$ are two lifts of τ satisfying (1.1) and (1.2), set $\theta = \tilde{\tau}_1\tilde{\tau}_2$ to conclude that $\tilde{\tau}_1 = \pm\tilde{\tau}_2$ on each component of X.

Proof. Fix $x \in X$ and choose a local oriented orthonormal framing $\{e_i\}$ near x. Then if ψ is a spinor field with $\psi(x) = 0$, an easy computation shows

$$0 = (D_X\theta - \theta D_X)\psi(x) = \left[c(e^i), \theta(x)\right] \nabla_{e_i}\psi(x), \tag{1.6}$$

where $\{e^i\}$ is the dual coframing, $c(\cdot)$ is Clifford multiplication, and ∇ is the Levi-Civita covariant derivative. Fix an index i. Choose a set of spinor fields $\{\psi^{(\alpha)}\}_\alpha$ so that $\psi^{(\alpha)}(x) = 0$, the derivatives $\nabla_{e_j}\psi^{(\alpha)}(x) = 0$ for $j \neq i$, and $\{\nabla_{e_i}\psi^{(\alpha)}(x)\}_\alpha$ span the fiber $(S_X)_x$. Then (1.6) implies $\left[c(e^i), \theta(x)\right] = 0$ for all i, and since the spin representation is irreducible in odd dimensions, $\theta(x)$ is a scalar. Then for any spinor field ψ,

$$0 = (D_X\theta - \theta D_X)\psi = c(d\theta)\psi,$$

from which $d\theta = 0$ so that θ is locally constant. □

Concerning the existence of $\tilde{\tau}$, we recall that in odd dimensions the spin representation $\mathbf{S}$ extends to an ungraded module for the Clifford algebra on which the volume form, suitably normalized, acts as $+1$. In particular, $\mathbf{S}$ is a representation of the Pin group. Now the isometry τ lifts to the bundle of orthonormal frames $O(X)$ of X. The spin structure induces a pin structure $\mathrm{Pin}(X)$—a principal Pin bundle which double covers $O(X)$—and it is a topological question about covering spaces to determine if τ acting on $O(X)$ lifts to $\mathrm{Pin}(X)$. If so, the lift may have order 4. In any case the spinor bundle S_X is associated to $\mathrm{Pin}(X)$, and the lift induces a map $\tilde{\tau}$ on spinor fields. But Clifford multiplication is *not* a map of Pin representations—there is a sign for elements which reverse the orientation—and so the Dirac operator does not extend simply extend to the Pin bundle. Rather, the sign means that the lift $\tilde{\tau}$ of an orientation-reversing isometry anticommutes with the Dirac operator as in (1.1).

We turn now to the index (1.4). The general Lefschetz formulas of Atiyah-Bott-Segal-Singer [AB1], [ASe], [AS2] apply to an elliptic operator $D : C^\infty(E) \to C^\infty(F)$ acting between two vector bundles E, F with endomorphisms $\hat{\tau}_E, \hat{\tau}_F$ such that

$$D\hat{\tau}_E = \hat{\tau}_F D. \tag{1.7}$$

Our problem concerns the Dirac operator $D_X : C^\infty(S_X) \to C^\infty(S_X)$, but the given lift $\tilde{\tau}$ satisfies (1.1), not (1.7). Here is the trick: Define

$$\hat{\tau} = \begin{cases} \tilde{\tau}, & \text{on the domain copy of } S_X; \\ -\tilde{\tau}, & \text{on the codomain copy of } S_X. \end{cases} \tag{1.8}$$

Now $\hat{\tau}$ satisfies (1.7)! The Lefschetz number is

$$\begin{aligned} L(\hat{\tau}, D_X) &= \operatorname{Trace}\hat{\tau}\big|_{\operatorname{Ker} D_X} - \operatorname{Trace}\hat{\tau}\big|_{\operatorname{Coker} D_X}, \\ &= 2 \operatorname{index}\big[D_X : S^+(X) \longrightarrow S^-(X)\big], \end{aligned}$$

twice the index we would like to compute.

The generalized Lefschetz formulas compute this index in terms of the fixed point set $\operatorname{Fix}(\tau)$ of τ. In our situation each component F of $\operatorname{Fix}(\tau)$ is an even dimensional manifold. The Atiyah-Segal formula [ASe, Theorem 2.12] applies in general; we first state the result with the vastly simplifying assumption that the normal bundle N_F to each component of $\operatorname{Fix}(\tau)$ is trivial. See Remark 1.10 following the statement of Theorem A for the formula when N_F is only assumed orientable.

Theorem A. *Let X be an odd dimensional spin manifold, $\tau : X \to X$ an orientation-reversing isometric involution, and $\tilde{\tau} : S_X \to S_X$ a lift to spinors which anticommutes with the Dirac operator D_X and satisfies $\tilde{\tau}^2 = 1$. Then D_X exchanges the ± 1-eigenspaces $S^{\pm}(X)$ of $\tilde{\tau}$ operating on spinor fields. Assume that each component F of the fixed point set* $\operatorname{Fix}(\tau)$ *has trivial normal bundle. The sum over these components appears in the index formula*

$$\operatorname{index}\big[D_X : S^+(X) \longrightarrow S^-(X)\big] = \sum_F \frac{\operatorname{index} D_F}{2^{r(F)+1}}. \tag{1.9}$$

Here $\operatorname{codim} F = 2r(F) + 1$ *and $D_F : S^+(F) \to S^-(F)$ is the chiral Dirac operator on F relative to an orientation chosen compatibly with $\tilde{\tau}$.*

The orientation is explained in the proof (see (1.20)). We make several remarks before proceeding to the proof.

Remark 1.10. More generally, suppose only that each component F of the fixed point set has *orientable* normal bundle. Then (1.9) is replaced by

$$\operatorname{index}\big[D_X : S^+(X) \longrightarrow S^-(X)\big] = \frac{1}{2}\sum_F \frac{\hat{A}(F)}{\operatorname{ch}\Delta(N_F)}[F], \tag{1.11}$$

where $\hat{A}$ is the usual characteristic class associated to Dirac, $\operatorname{ch}\Delta$ is the Chern character of the spin bundle, and the orientation of F is determined below. (One does not need a spin structure to define $\operatorname{ch}\Delta$.) See [AS2, §5] for a similar result. This formula only holds rationally in families.

Remark 1.12. For the Hořava-Witten example $X = S^1 \times Y$, τ is reflection on the S^1 factor, and Theorem A computes

$$\text{index}\big[D_X : S^+(X) \longrightarrow S^-(X)\big] = \text{index}\, D_Y, \tag{1.13}$$

which agrees with [HW]. Here Y is a compact even dimensional spin manifold. According to Remark 1.15 below this only holds modulo 2-torsion in families. In the next section we show that in fact this result holds exactly (see (2.11)).

Remark 1.14. For a single operator we can use the heat kernel approach to the Lefschetz formula (see [R], [BGV] for example) to derive (1.9). We write

$$\text{index}\big[D_X : S^+(X) \longrightarrow S^-(X)\big] = \int_{X\times X} \text{Trace}\big(\tilde{\tau}(x,y) e^{-tD_X^2}(y,x)\big)\, dy\, dx,$$

valid for any t, and let $t \to 0$. The integral then localizes on the fixed point set. As always in index theory, this heat kernel approach does not generalize to the integral K-theory index of a family of Dirac operators.

Remark 1.15. Theorem A applies to families of Dirac operators, but only gives a result in $K(Z)[\frac{1}{2}]$, where Z is the parameter space. (Below we use [ASe, Theorem 2.12]. Although this theorem is stated for K-theory$\otimes\mathbb{C}$, in our situation the localization of the global symbol of Dirac only involves denominators which are powers of 2.)

Remark 1.16. Theorem A also applies to (families of) real Dirac operators and Dirac operators coupled to other vector bundles. The Hořava-Witten example is actually for the *real* Dirac operator coupled to the tangent bundle. The quantity of interest is the *square root* of the determinant line bundle, which is computed in KO-theory. (See [F2, §3] for an explanation of this square root.)

Proof of Theorem A. We apply [ASe, Theorem 2.12] which asserts

$$L(\hat{\tau}, D_X) = \sum_F \text{index}\left\{ \frac{\iota_F^* \sigma(D_X)(\hat{\tau})}{\lambda_{-1}(N_F \otimes \mathbb{C})(\hat{\tau})} \right\}, \tag{1.17}$$

where $\iota_F : F \hookrightarrow X$ is the inclusion, $\sigma(D_X) \in K_G(TX)$ is the symbol of Dirac, and

$$\lambda_{-1}(N_F \otimes \mathbb{C}) = \sum (-1)^i \textstyle\bigwedge^i (N_F \otimes \mathbb{C}) \in K_G(F).$$

Here $G = \langle\hat{\tau}\rangle$, the cyclic group generated by $\hat{\tau}$. Evaluation on $\hat{\tau}$ is the homomorphism

$$K_G(F) \xrightarrow{\cong} K(F) \otimes R(G) \longrightarrow K(F)$$

which evaluates a virtual character on $\hat{\tau}$. (For the cyclic group of order two the virtual characters are real-valued.)

We work on a fixed component F of codimension $2r(F)+1 = 2r+1$. Since N_F is assumed trivial, we have an isomorphism $N_F \cong L^{\oplus(2r+1)}$ in $K_G(F)$, where L is the trivial real line bundle with $\hat{\tau}$ acting as -1. It follows easily that

$$\lambda_{-1}(N_F \otimes \mathbb{C})(\hat{\tau}) = 2^{2r+1}. \tag{1.18}$$

Recall that the symbol $\sigma(D_X)$ evaluated on a cotangent vector θ is Clifford multiplication $c(\theta) : S_X \to S_X$. We need to compute this for θ a cotangent vector to F. First, note that F is orientable, since N_F is trivial. We fix the orientations of F and N_F below. Let N_F have the trivial spin structure. This, together with the spin structure on X, induces a spin structure on F. Then, letting S_F, S_{N_F} denote the spin bundles on the tangent and normal bundles to F, we have

$$S_X\big|_F \cong S_F \otimes S_{N_F} \cong S_F^{\oplus(2^r)} \tag{1.19}$$

since the normal bundle is trivial. Therefore, $\iota_F^*\sigma(D_X)$ is Clifford multiplication on 2^r copies of S_F.

To compute the action of $\hat{\tau}$ we fix an equivariant tubular neighborhood of F, which is diffeomorphic to $F \times \mathbb{R}^{2r+1}$, and introduce a product metric. (This computation is local, so does not use the triviality of N_F.) Let e_i be the standard orthonormal basis of $\mathbb{R}^{2r+1}$, x^i the standard coordinates on $\mathbb{R}^{2r+1}$, and f a coordinate on F. Then

$$D_X = D_F + c(e^i)\nabla_{e_i}.$$

We claim

(1.20)

$$\begin{aligned}\tau(f;x^1,\dots,x^{2r+1}) &= \langle f;-x^1,\dots,-x^{2r+1}\rangle\\ (\tilde{\tau}\psi)(f;x^1,\dots,x^{2r+1}) &= \pm i^{r+1}c(e^1)\dots c(e^{2r+1})\psi(f;-x^1,\dots,-x^{2r+1})\\ &= i^m c(\omega_F)\psi(f;-x^1,\dots,-x^{2r+1}),\end{aligned}$$

where ψ is a spinor field, $\dim F = 2m$, and ω_F is a real volume form on F with $c(\omega_F)^2 = (-1)^m$. A routine computation shows that the first expression for $\tilde{\tau}$ satisfies (1.1) and $\tilde{\tau}^2 = 1$, whence the remark following Lemma 1.5 implies that this is the correct expression (with one of the signs). The second expression for $\tilde{\tau}$ follows from a simple computation with Clifford algebras. It determines ω_F uniquely.

Now we fix the orientation on F so that ω_F is an oriented volume form. Then $S_F^{\pm}$ are the $\pm(i^{-m})$-eigenspaces of $c(\omega_F)$ acting on S_F, which by (1.20) are the ± 1-eigenspaces of $\tilde{\tau}$. Use (1.8) and (1.19) to conclude that

$$\iota_F^* \sigma(D_X)(\hat{\tau}) = 2^{r+1} \sigma(D_F). \tag{1.21}$$

The desired result (1.9) follows from (1.17), (1.21), and (1.18). □

2. An index theorem for manifolds with boundary.

Let X be a compact odd dimensional spin manifold with boundary. The orientation on X determines an orientation on ∂X and so a splitting

$$S_X\big|_{\partial X} \cong S_{\partial X} \cong S_{\partial X}^+ \oplus S_{\partial X}^- \tag{2.1}$$

of the spin bundle on the boundary. This splitting leads to local boundary conditions $P^{\pm}$ for the Dirac operator D_X: the domain of $(D_X, P^{\pm})$ is the set of spinor fields ψ on X with

$$\left(\psi\big|_{\partial X}\right)^{\pm} = 0,$$

where $\phi = \phi^+ + \phi^-$ is the decomposition of a spinor field $\phi \in S(\partial X)$ relative to (2.1). These local boundary value problems are a key ingredient in the original proof of the Atiyah-Singer index theorem. Indeed [P, §17], [BW, §21]

$$\operatorname{index}(D_X, P^{\pm}) = 0. \tag{2.2}$$

This is used to show that the index of the chiral Dirac operator on the boundary vanishes:

$$\operatorname{index} D_{\partial X} = 0. \tag{2.3}$$

Equation (2.3) is the assertion that the index is a bordism invariant.

We consider a mixture of these boundary conditions. Namely, we independently choose P^+ or P^- on each component of the boundary.

Theorem B. *Let X be a compact odd dimensional spin manifold with boundary, and $\partial X = \sqcup_i Y_i$ the decomposition of the boundary into components. For each i choose $\epsilon_i = +$ or $\epsilon_i = -$ and consider the Dirac operator (D_X, P^ϵ) whose domain is the set of spinor fields ψ such that*

$$(\psi|_{Y_i})^{\epsilon_i} = 0. \tag{2.4}$$

Then

$$\operatorname{index}(D_X, P^\epsilon) = \sum_{\substack{i \text{ with} \\ \epsilon_i = -}} \operatorname{index} D_{Y_i} = -\sum_{\substack{i \text{ with} \\ \epsilon_i = +}} \operatorname{index} D_{Y_i} \tag{2.5}$$

Note that the last equality follows directly from (2.3). Also, if all $\epsilon_i = +$ or all $\epsilon_i = -$, then (2.5) reduces to (2.2) in view of (2.3). As is evident from the proof below, Theorem B is a direct consequence of well-known facts about boundary-value problems for Dirac operators.

Remark 2.6. Theorem B also holds in families; then (2.5) is an *exact* equation in $K(Z)$, where Z is the parameter space. (Contrast with Theorem A which only holds in $K(Z)\left[\frac{1}{2}\right]$.) As with Theorem A (see Remark 1.16), Theorem B holds for (families of) real Dirac operators and Dirac operators coupled to other vector bundles.

Remark 2.7. Consider $X = \left[0, \frac{1}{2}\right] \times Y$, where Y is a closed even dimensional spin manifold. We use the product metric. Then $\partial X = Y_0 \sqcup Y_{\frac{1}{2}}$, where $Y_{\frac{1}{2}} \cong Y$ and $Y_0 \cong -Y$. Here '$-Y$' denotes Y with the opposite orientation. Let $\epsilon_0 = +$ and $\epsilon_{\frac{1}{2}} = -$. Then (2.5) gives

$$\operatorname{index}(D_X, P^\epsilon) = \operatorname{index} D_Y. \tag{2.8}$$

Let $\tilde{D} : H^+ \to H^-$ be the Dirac operator in the Hořava-Witten example (1.13). Here we are working on $S^1 \times Y$ and $\psi \in H^\pm = S^\pm(S^1 \times Y)$ is an $S(Y)$-valued function on $S^1 = \mathbb{R}/\mathbb{Z}$ satisfying

$$\psi(-x) = \pm i^m c(\omega_Y)\psi(x), \tag{2.9}$$

where ω_Y is a volume form on Y and $\dim Y = 2m$ (cf. (1.20)). We now give an *a priori* argument that

$$\operatorname{index}(D_X, P^\epsilon) = \operatorname{index} \tilde{D}, \tag{2.10}$$

even in families. This is consistent with the computations (1.13) and (2.8) from Theorem A and Theorem B for single operators, and gives the exact result

$$\operatorname{index} \tilde{D} = \operatorname{index} D_Y \tag{2.11}$$

in families. (This was previously proved modulo 2-torsion.)

To prove (2.10) note that relative to the splitting $S(Y) \cong S^+(Y) \oplus S^-(Y)$ equation (2.9) asserts that $\psi \in H^\pm$ satisfies

$$\begin{aligned} \psi^+(-x) &= \pm\psi^+(x), \\ \psi^-(-x) &= \mp\psi^-(x). \end{aligned}$$

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^1[H^+] & \longrightarrow & C^1[S([0,\tfrac{1}{2}]\times Y,\epsilon)] & \xrightarrow{p} & C^0[S^+(Y)] \oplus C^0[S^+(Y)] & \longrightarrow & 0 \\ & & \big\downarrow \tilde{D} & & \big\downarrow (D_{X,\epsilon}) & & \big\downarrow \mathrm{id} & & \\ 0 & \longrightarrow & C^0[H^-] & \longrightarrow & C^0[S([0,\tfrac{1}{2}]\times Y)] & \xrightarrow{q} & C^0[S^+(Y)] \oplus C^0[S^+(Y)] & \longrightarrow & 0 \end{array}$$

where '$C^{1,\delta}[\cdot]$' and '$C^{\delta}[\cdot]$' denote spaces of Hölder functions for some $0 < \delta < 1$; '$S([0,\frac{1}{2}] \times Y, \epsilon)$' denotes the space of spinor fields satisfying (2.4), which in this case is $\psi^-(0) = \psi^-\left(\frac{1}{2}\right) = 0$; the first horizontal arrows are restriction maps; and

$$\begin{aligned} p(\psi) &= \left\langle -i\dot{\psi}^+(0), -i\dot{\psi}^+\left(\frac{1}{2}\right)\right\rangle, \\ q(\psi) &= \left\langle \psi^+(0), \psi^+\left(\frac{1}{2}\right)\right\rangle. \end{aligned}$$

(Here $\dot{\psi} = \frac{d\psi}{dx}$.) A routine check shows that the rows are exact and the diagram commutes. Now (2.10) is a consequence of the following lemma. (See [S] for a more general discussion.)

Lemma 2.12. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \big\downarrow T'_z & & \big\downarrow T_z & & \big\downarrow T''_z & & \\ 0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \end{array} \tag{2.13}$$

be a commutative diagram with exact rows, where V', V, V'', W', W, W'' are Banach spaces and T'_z, T_z, T''_z are Fredholm operators depending continuously on a parameter $z \in Z$. Then

$$\text{index}(T) = \text{index}(T') + \text{index}(T'') \in K(Z). \tag{2.14}$$

The Banach spaces are allowed to vary continuously; we omit this from the notation for convenience.

Proof. The short exact sequence of chain complexes (2.13) induces a long exact sequence in cohomology:

(2.15)

$$0 \to \operatorname{Ker} T'_z \to \operatorname{Ker} T_z \to \operatorname{Ker} T''_z \to \operatorname{Coker} T'_z \to \operatorname{Coker} T_z \to \operatorname{Coker} T''_z \to 0.$$

The exactness of (2.15) proves (2.14) for a single operator. For a family it suffices to prove (2.14) for Z compact. Then [AS3, §2] we can find $w'_1(z), \dots, w'_{N'}(z) \in W'$ and $w_1(z), \dots, w_N(z) \in W$ so that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' \oplus \mathbb{C}^{N'} & \longrightarrow & V \oplus \mathbb{C}^{N'} \oplus \mathbb{C}^N & \longrightarrow & V'' \oplus \mathbb{C}^N & \longrightarrow & 0 \\ & & \big\downarrow S'_z & & \big\downarrow S_z & & \big\downarrow S''_z & & \\ 0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \end{array}$$

satisfies the hypotheses of the lemma and in addition S'_z, S_z, S''_z are surjective. Here

$$S_z(v; \lambda^i; \mu^j) = T_z(v) + \lambda^i w'_i(z) + \mu^j w_j(z)$$

and S'_z, S''_z are the corresponding induced maps. We have

$$\text{index}\, S' = \text{index}\, T' + [Z \times \mathbb{C}^{N'}] \in K(Z)$$

with similar formulas for the indices of S and S''. Now the exactness of (2.15) (with all cokernels vanishing) proves (2.14); the extra trivial bundles cancel out. □

Proof of Theorem B. The proof is based on analysis by Calderón and Seeley [P, §17]; we rely on the account in [BW]. We remark that the index with *local* boundary conditions is a topological invariant; in fact, it has an interpretation in K-theory [AB2]. So, for example, we can deform the metric to a metric which is a product near the boundary.

Consider first a single operator. Let

$$\hat{\mathcal{K}} = \operatorname{Ker}\bigl[D_X : S(X) \longrightarrow S(X)\bigr]$$

and $\mathcal{K} \subset S(\partial X)$ the image of $\hat{\mathcal{K}}$ under restriction to the boundary. We use the Sobolev completions H^1 of $S(X)$ and $H^{1/2}$ of $S(\partial X)$. Then $\mathcal{K}$ is a closed infinite dimensional subspace of $S(\partial X)$. Let

$$P^\epsilon : S(\partial X) \longrightarrow \bigoplus_i S^{\epsilon_i}(Y_i)$$

be the projection defined by the boundary condition (2.4). The first result [BW, Theorem 20.12] is that

$$\operatorname{index}(D_X, P^\epsilon) = \operatorname{index}\bigl[P^\epsilon_{\mathcal{K}} : \mathcal{K} \longrightarrow \bigoplus_i S^{\epsilon_i}(Y_i)\bigr], \tag{2.16}$$

where '$P^\epsilon_{\mathcal{K}}$' denotes the restriction of P^ϵ to $\mathcal{K}$. This applies in particular to P^+ (which is P^ϵ with all $\epsilon_i = +$), and so [BW, Theorem 21.2]

(2.17)
$$\begin{aligned}\operatorname{index}(D_X, P^\epsilon) - \operatorname{index}(D_X, P^+) &= \operatorname{index}(P^\epsilon_{\mathcal{K}}) - \operatorname{index}(P^+_{\mathcal{K}}) \\ &= \operatorname{index}(P^\epsilon_{\mathcal{K}}) + \operatorname{index}\bigl(P^+_{\mathcal{K}}\bigr)^* \\ &= \operatorname{index}\bigl[P^\epsilon_{\mathcal{K}}(P^+_{\mathcal{K}})^* : \bigoplus_{\substack{i \text{ with}\\ \epsilon_i = -}} S^+(Y_i) \longrightarrow \bigoplus_{\substack{i \text{ with}\\ \epsilon_i = -}} S^-(Y_i)\bigr].\end{aligned}$$

The final step is the assertion (see [BW, Theorem 21.5]) that $P^\epsilon_{\mathcal{K}}(P^+_{\mathcal{K}})^*$ is a pseudodifferential operator of order 0 whose symbol—up to a factor and after restriction to the sphere bundle—is the symbol of the Dirac operator $\sum_{\substack{i \text{ with}\\ \epsilon_i = -}} D_{Y_i}$. (This is the brunt of the argument; it depends on properties of the *Calderón projector*.) Then the first equality in (2.5) follows directly from (2.17) and (2.2).

We briefly consider how to modify this argument for a family of Dirac operators parameterized by $z \in Z$. It suffices to consider Z compact for index computations. Then as in the proof of the lemma above we can find a finite number of spinor fields $\psi_1(z), \dots, \psi_N(z)$ so that

$$\begin{aligned}\bigl(T(z), P^\epsilon(z)\bigr) : S_{P^\epsilon(z)}(X) \oplus \mathbb{C}^N &\longrightarrow S(X) \\ \langle \psi; \lambda^i \rangle &\longmapsto D_X(z)\psi + \lambda^i \psi_i(z)\end{aligned} \tag{2.18}$$

is surjective. Here $S_{P^\epsilon(z)}(X) \subset S(X)$ is the subspace of spinor fields satisfying the boundary condition $P^\epsilon(z)$. Then

$$\text{index}(T, P^\epsilon) = \text{index}(D_X, P^\epsilon) + [Z \times \mathbb{C}^N] \in K(Z). \tag{2.19}$$

Now $T(z)$—the operator (2.18) extended to all of $S(X) \oplus \mathbb{C}^N$—is also surjective. Thus the orthogonal complement to the kernel of $T(z)$ varies continuously in z (using $T(z)$ as an isomorphism to the continuously varying codomains), whence the kernel $\hat{\mathcal{K}}(z)$ of $T(z)$ varies continuously as well. So does its image $\mathcal{K}(z)$ in $S(\partial X)$ Equation (2.16) is replaced by

$$\text{index}(T, P^\epsilon) = \text{index}\big[P^\epsilon : \mathcal{K} \longrightarrow \bigoplus_i S^{\epsilon_i}(Y_i)\big]. \tag{2.20}$$

This follows simply by identifying the kernel bundle of the families of operators on each side; the cokernels vanish. By adding more $\psi_i(z)$ we can ensure that (2.18) is also surjective for $\big(T(z), P^+(z)\big)$ and repeat (2.19) and (2.20) for P^+ replacing P^ϵ. Then equation (2.17) holds—the auxiliary trivial bundle cancels out—and the proof concludes as before. □

References.

[AB1] M.F. Atiyah and R. Bott, *The Lefschetz fixed point theorem for elliptic complexes I*, Ann. of Math. **88** (1967), 374–407; *The Lefschetz fixed point theorem for elliptic complexes II*, Ann. of Math. **88** (1968), 451–491.

[AB2] M.F. Atiyah and R. Bott, *The index theorem for manifolds with boundary*, in *Differential Analysis*, Proc. Bombay Symposium 1964, Oxford University Press, 175–86.

[AS1] M.F. Atiyah and I.M. Singer, *Dirac operators coupled to vector potentials*, Proc. Nat. Acad. Sci. **81** (1984), 2597.

[AS2] M.F. Atiyah and I.M. Singer, *The index of elliptic operators, III*, Ann. of Math. **87** (1968), 546–604.

[AS3] M.F. Atiyah and I.M. Singer, *The index of elliptic operators, IV*, Ann. Math. **93** (1971), 119–138.

[ASe] M.F. Atiyah and G. B. Segal, *The index of elliptic operators, II*, Ann. of Math. **87** (1968), 531–545.

[BGV] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer-Verlag, Berlin 1992.

[BW] B. Booss-Bavnbek and K. P. Wojciechowski, *Elliptic Boundary Problems for Dirac Operators*, Birkhäuser, Boston 1993.

[F1] D.S. Freed, *Determinants, torsion, and strings*, Commun. Math. Phys. **107** (1986), 483–513.

[F2] D.S. Freed, *On determinant line bundles*, in *Mathematical Aspects of String Theory*, ed. S. T. Yau,World Scientific Publishing 1987.

[HW] P. Hořava and E. Witten, *Heterotic and type I string dynamics from eleven dimensions*, Nucl. Phys. B, **460** (1996), 506–524.

[P] R.S. Palais, ed., *Seminar on the Atiyah-Singer index theorem*, Princeton University Press, Princeton 1965, Ann. of Math. Studies 57.

[R] J. Roe, *Elliptic operators, topology and asymptotic methods*, Longman Scientific & Technical, Essex, England 1988, Pitman Research Notes in Mathematics Series 179.

[S] G. Segal, *Fredholm complexes*, Quart. J. Math. **21** (1970), 385—402.

RECEIVED MAY 6, 1996.

UNIVERSITY OF TEXAS,
AUSTIN, TX 78712
E-mail address: DAFR@MATH.UTEXAS.EDU