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The p-energy minimality of $x/|x|$

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The map $\frac{x}{|x|} : \mathbf{B}^n > \mathbf{S}^{n-1}$ is p-energy minimizing whenever $p \in [n-1, n)$.

This minimality was first established for $p = 2$, $n = 3$ in [BCL] (with another proof in [ABL]), for $p = 2 < n$ in [L], and for $p \in \{2, 3, \ldots, n-1\}$ in [CG] (with another proof in [AL]). R. Musina [M] also proved the p-energy minimality of orthogonal rotations among degree 1 maps of $\mathbf{S}^{n-1}$ for $p \in [n-1, n)$. However, there remained, for non-integer p, the possibility of some non-homogenous extension of $\mathrm{id}_{\mathbf{S}^{n-1}}$ to $\mathbf{B}^n$ having less p-energy than $\frac{x}{|x|}$. The present paper, rules this out for $p \in [n-1, n)$. Our argument, which does not rely on the previous works, involves continuation in p, uses Jacobi field considerations, and starts with p near n. Behavior of singularities of p-energy minimizers as $p \uparrow n$ is the theme of [HLW], and questions on asymptotic behavior near singularities led to Lemma 2 below, which classifies normal p Jacobi fields along the identity.

Our goal is to show that, for $p \in [n-1, n)$, any map $u \in W^{1,p}(\mathbf{B}^n, \mathbf{S}^{n-1})$ with $u|\partial\mathbf{B}^n = \mathrm{id}_{\mathbf{S}^{n-1}}$ must have p-energy at least equal to the number

$$\int_{\mathbf{B}^n} \left|\nabla\left(\frac{x}{|x|}\right)\right|^p dx = \int_{\mathbf{S}^{n-1}} \int_0^1 \left[\frac{\sqrt{n-1}}{r}\right]^p r^{n-1} dr\, d\Sigma = \frac{(n-1)^{\frac{p}{2}}\sigma_n}{n-p} .$$

Here $d\Sigma$ denotes integration on the $(n-1)$-sphere with respect to Hausdorff measure $\mathcal{H}^{n-1}$ and

$$\sigma_n = \mathcal{H}^{n-1}(\mathbf{S}^{n-1})$$

denotes the volume of the $(n-1)$-sphere.

For functions or vectorfields defined on $\mathbf{S}^{n-1}$, we will follow the notation of [S] for the operators $\nabla^{\mathbf{S}}$, $\mathrm{div}_{\mathbf{S}}$, $\Delta_{\mathbf{S}} = \mathrm{div}_{\mathbf{S}} \circ \nabla^{\mathbf{S}}$ which involve differentiation only in directions tangent to $\mathbf{S}^{n-1}$. Thus, for any function $u \in C^1(\mathbf{S}^{n-1}, \mathbf{R})$ and point $a \in \mathbf{S}^{n-1}$,

$$(\nabla^{\mathbf{S}} u)(a) = (\nabla u)(a) - [a \cdot (\nabla u)(a)]a \in a^{\perp} .$$

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Also for any (not necessarily tangent) vectorfield $v \in \mathcal{C}^1(\mathbf{S}^{n-1}, \mathbf{R}^n)$, we have the formula

$$\mathrm{div}_{\mathbf{S}}\, v \;=\; \sum_{i=1}^{n} \mathbf{e}_i \cdot \left(\nabla^{\mathbf{S}}(\mathbf{e}_i \cdot v)\right) \;,$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0), \ldots, \mathbf{e}_n = (0, \ldots, 0, 1)$, and Stoke's theorem implies that

$$\text{(1)} \qquad \int_{\mathbf{S}^{n-1}} \mathrm{div}_{\mathbf{S}}\, v \, d\Sigma \;=\; 0 \;.$$

A map $w \in W^{1,p}(\mathbf{S}^{n-1}, \mathbf{S}^{n-1})$ is a *p-harmonic map* if, for any $\zeta \in \mathcal{C}^\infty(\mathbf{S}^{n-1}, \mathbf{R}^n)$,

$$\begin{aligned} 0 \;&=\; \frac{d}{dt}\bigg|_{t=0} \int_{\mathbf{S}^{n-1}} \left|\nabla^{\mathbf{S}} \left(\frac{w + t\zeta}{\|w + t\zeta|}\right)\right|^p d\Sigma \\ &=\; p \int_{\mathbf{S}^{n-1}} \left[\left(\left|\nabla^{\mathbf{S}} w\right|^{p-2} \nabla^{\mathbf{S}} w\right) \cdot \nabla^{\mathbf{S}} \zeta \;-\; \left|\nabla^{\mathbf{S}} w\right|^p w \cdot \zeta\right] d\Sigma \;. \end{aligned}$$

Thus, on $\mathbf{S}^{n-1}$, we have, in a distribution sense, the equation

$$\mathrm{div}_{\mathbf{S}} \left(\left|\nabla^{\mathbf{S}} w\right|^{p-2} \nabla^{\mathbf{S}} w\right) \;+\; |\nabla^{\mathbf{S}} w|^p w \;=\; 0 \;.$$

Note that $\mathrm{id}_{\mathbf{S}^{n-1}}$, or more, more generally, the restriction θ of any rotation in $O(n)$ is a p-harmonic map of $\mathbf{S}^{n-1}$ for any positive p because $|\nabla^{\mathbf{S}} \theta| \equiv \sqrt{n-1}$ and $\Delta_{\mathbf{S}} \theta = -(n-1)\theta$.

While one does not expect general weak solutions of this equation to be continuous, we will only be working with $\mathcal{C}^1$ solutions. These occur for example in the Sobolev range, $p > n - 1$.

Lemma 1. *For any numbers $n - 1 < q_0 < q_1 < \infty$, there exist positive α and C, depending only on q_0 and q_1, so that*

$$\|w\|_{\mathcal{C}^{1,\alpha}} \;\leq\; C \left(\int_{\mathbf{S}^{n-1}} \left|\nabla^{\mathbf{S}} w\right|^p d\Sigma\right)^{\frac{1}{p}}$$

for any p-harmonic map $w : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ with $q_0 \leq p \leq q_1$.

Proof. Sobolev embedding implies that

$$\|w\|_{\mathcal{C}^{0,\beta}} \;\leq\; C \left(\int_{\mathbf{S}^{n-1}} \left|\nabla^{\mathbf{S}} w\right|^p d\Sigma\right)^{\frac{1}{p}}$$

with $\beta = 1 - \frac{n-1}{q_0}$. The gradient Hölder continuity bound then follows from the regularity theory of degenerate elliptic systems. See, e.g. [HL1],§3. □

Suppose v is a $\mathcal{C}^1$ normal field along w ; that is, $v \in \mathcal{C}^1(\mathbf{S}^{n-1}, \mathbf{R}^n)$ and $v \cdot w \equiv 0$. To derive the p Jacobi field equation for v, we assume that

$$w_t = \frac{(1-t)w + tv}{|(1-t)w + tv|}$$

is a p-harmonic map for $|t|$ small and compute $\frac{d}{dt}|_{t=0}$ of the p-harmonic map equation of w_t. Thus

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \int_{\mathbf{S}^{n-1}} \left[\left(\left|\nabla^{\mathbf{S}} w_t\right|^{p-2} \nabla^{\mathbf{S}} w_t\right) \cdot \nabla^{\mathbf{S}} \zeta \;-\; \left|\nabla^{\mathbf{S}} w_t\right|^p w_t \cdot \zeta\right] d\Sigma \\ &= \int_{\mathbf{S}^{n-1}} \left[\left(\left|\nabla^{\mathbf{S}} w\right|^{p-2} \nabla^{\mathbf{S}} v \;+\; (p-2)|\nabla^{\mathbf{S}} w|^{p-4} (\nabla^{\mathbf{S}} w \cdot \nabla^{\mathbf{S}} v) \nabla^{\mathbf{S}} w\right) \cdot \nabla^{\mathbf{S}} \zeta \right. \\ &\quad \left. -\; \left(|\nabla^{\mathbf{S}} w|^p v \;+\; p|\nabla^{\mathbf{S}} w|^{p-2} (\nabla^{\mathbf{S}} w \cdot \nabla^{\mathbf{S}} v) w\right) \cdot \zeta\right] d\Sigma \end{aligned}$$

so that, on $\mathbf{S}^{n-1}$, we now have, in a distribution sense, the equation

(2)
$$\begin{aligned} J_w v \;:=\; & \operatorname{div}_{\mathbf{S}} \left(\left|\nabla^{\mathbf{S}} w\right|^{p-2} \nabla^{\mathbf{S}} v \;+\; (p-2) \left|\nabla^{\mathbf{S}} w\right|^{p-4} (\nabla^{\mathbf{S}} w \cdot \nabla^{\mathbf{S}} v) \nabla^{\mathbf{S}} w\right) \\ & + \left|\nabla^{\mathbf{S}} w\right|^p v \;+\; p \left|\nabla^{\mathbf{S}} w\right|^{p-2} (\nabla^{\mathbf{S}} w \cdot \nabla^{\mathbf{S}} v) w \;=\; 0\,. \end{aligned}$$

For w being $\mathrm{id}_{\mathbf{S}^{n-1}}$, or more generally a rotation, $|\nabla^{\mathbf{S}} w|$ is a constant, and (2) is nondegenerate elliptic. In fact, here global solutions have a simple characterization:

Lemma 2. *For $2 \le p \ne n-1$, any normal p Jacobi field v along $w = \mathrm{id}_{\mathbf{S}^{n-1}}$ is the restriction of a skew-symmetric linear map of $\mathbf{R}^n$. For $p = n-1 \ge 2$, v is the restriction of an infinitesimal conformal transformation of $\mathbf{R}^n$.*

Proof. Here we easily compute that $(\nabla^{\mathbf{S}} w)(x)$ is orthogonal projection onto $x^\perp$, so that

$$\begin{aligned} \left(\nabla^{\mathbf{S}} w\right)(x) \;=\; \mathrm{id} - \frac{x_i x_j}{|x|^2}\,, \; \left|\nabla^{\mathbf{S}} w\right| \equiv (n-1)^{\frac{1}{2}}\,, \; \left(\nabla^{\mathbf{S}} w \cdot \nabla^{\mathbf{S}} v\right) \;=\; \operatorname{div}_{\mathbf{S}} v\,, \\ \operatorname{div}_{\mathbf{S}} \left[\left(\nabla^{\mathbf{S}} w \cdot \nabla^{\mathbf{S}} v\right) \nabla^{\mathbf{S}} w\right] \;=\; \nabla^{\mathbf{S}} (\operatorname{div}_{\mathbf{S}} v)\,. \end{aligned}$$

Thus, (2) becomes, after dividing by $(n-1)^{\frac{p-2}{2}}$,

(3) $$\Delta_{\mathbf{S}} v \;+\; \frac{p-2}{n-1} \nabla^{\mathbf{S}} (\operatorname{div}_{\mathbf{S}} v) \;+\; (n-1) v \;+\; p (\operatorname{div}_{\mathbf{S}} v) X \;=\; 0\,,$$

where X is the injection of $\mathbf{S}^{n-1}$ into $\mathbf{R}^n$. Integrating (3) using (1) and integration by parts gives the vector equation

$$\begin{aligned} 0 &= 0 + 0 + \int_{\mathbf{S}^{n-1}} \left((n-1)v \;-\; p\left(\nabla^{\mathbf{S}} X\right)[v]\right) d\Sigma \\ &= (n-p-1)\int_{\mathbf{S}^{n-1}} v\, d\Sigma\,, \end{aligned}$$

so that v has average 0 provided $p \neq n-1$.

Next we multiply (3) by $-v$ and integrate over $\mathbf{S}^{n-1}$ to find that

$$\int_{\mathbf{S}^{n-1}} \left(\left|\nabla^{\mathbf{S}} v\right|^2 + \frac{p-2}{n-1}(\operatorname{div}_{\mathbf{S}} v)^2 - (n-1)|v|^2 + 0\right) d\Sigma = 0\,.$$

With $v = (v_1, \ldots, v_n)$, we may rewrite the latter equation as

$$(4)\quad \sum_{i=1}^{n} \int_{\mathbf{S}^{n-1}} \left(\left|\nabla^{\mathbf{S}} v_i\right|^2 - (n-1)v_i^2\right) d\Sigma + \frac{p-2}{n-1}\int_{\mathbf{S}^{n-1}} (\operatorname{div}_{\mathbf{S}} v)^2\, d\Sigma = 0\,.$$

Recall that $n-1$ is the first eigenvalue of $-\Delta_{\mathbf{S}}$ on $\mathbf{S}^{n-1}$, with eigenspace generated by linear functions. Since, for $p \neq n-1$, each component v_i has average zero, each summand in (4) is necessarily nonnegative, hence zero, and each v_i is the restriction of a linear function. That is, $v(x) = Ax$ for all $x \in \mathbf{S}^{n-1}$ and some linear map $A : \mathbf{R}^n \to \mathbf{R}^n$. Since the quadratic function $(Ax)\cdot x$ vanishes on $\mathbf{S}^{n-1}$, it is identically zero and may be differentiated twice to see that $A^T + A = 0$, i.e., A is skew-symmetric. Every such skew-symmetric A gives a distinct solution $v(x) = Ax$ to (3) because $\operatorname{div}_{\mathbf{S}} v \equiv 0$ and $\Delta_{\mathbf{S}} v \equiv -(n-1)v$.

In case $p = n-1 \geq 2$, v may not have average 0. But now, the p-energy on the sphere is conformally invariant, we may first apply a suitable infinitesimal Mobius transformation to change v to some other normal Jacobi field that has average 0, and then argue as above. □

Lemma 3. *For any number $q \in [2,\infty)$ except $n-1$, there exists a positive $\epsilon_0 = \epsilon_0(n,q)$ so that, for $|p-q| < \epsilon_0$, any p-harmonic map $w : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ that is ϵ_0 close, in C^1 norm, to an orthogonal rotation θ is itself an orthogonal rotation.*

Proof. We may assume that $\theta = \mathrm{id}_{\mathbf{S}^{n-1}}$. First note that a solution w of the p-harmonic map equation on $\mathbf{S}^{n-1}$ that is C^1 close to the identity has $|\nabla^{\mathbf{S}} w|$ pointwise close to the constant $(n-1)^{\frac{p}{2}}$. Thus w may be shown, by a standard bootstrap argument, to be $\mathcal{C}^\infty$. Similarly, for such w, the Jacobi field equation (2) is a uniformly strongly elliptic system having only smooth solutions. In particular, for any $k \in \{2, 3, \dots\}$, we have the standard Schauder estimate

$$\|v\|_{\mathcal{C}^{k,\alpha}} \le C\left(\|J_w v\|_{\mathcal{C}^{k-2,\alpha}} + \|v\|_{\mathcal{C}^{k-2,\alpha}}\right) ,$$

which is valid uniformly for all p in some neighborhood of any fixed $q \in (2, n-1) \cup (n-1, \infty)$. With this and Lemma 2, we readily verify, as in [HM],§6, that, with $w = \mathrm{id}_{\mathbf{S}^{n-1}}$, J_w is Fredholm and that the implicit function theorem implies that some $\mathcal{C}^1$ neighborhood of $\mathrm{id}_{\mathbf{S}^{n-1}}$ is a $\mathcal{C}^1$ manifold of dimension $\frac{n(n-1)}{2}$. However, the rotations $\theta \in O(n)$ near the identity already provide a full open $\frac{n(n-1)}{2}$-dimensional neighborhood of the identity. Thus some $\mathcal{C}^1$ neighborhood of $\mathrm{id}_{\mathbf{S}^{n-1}}$ consists entirely of rotations. □

Next we prove, for the reader's convenience, two lemmas which follow from the general discussion of [HLW].

Lemma 4 ([HLW],§1.1,1.3). *If $p > n-1$ and $w \in \mathcal{C}^1(\mathbf{S}^{n-1}, \mathbf{S}^{n-1})$ has degree $d > 0$, then*

$$\int_{\mathbf{S}^{n-1}} |\nabla^{\mathbf{S}} w|^p \, d\Sigma \ge d^{\frac{p}{n-1}} (n-1)^{\frac{p}{2}} \sigma_n ,$$

with equality if and only if $d = 1$ and w is an orthogonal rotation.

Proof. One combines Hölder's inequality with the relations

$$d\sigma_n = \int_{\mathbf{S}^{n-1}} \mathrm{Jac}\left(\nabla^{\mathbf{S}} w\right) d\Sigma \le \int_{\mathbf{S}^{n-1}} (n-1)^{\frac{1-n}{2}} \left|\nabla^{\mathbf{S}} w\right|^{n-1} d\Sigma .$$

If equality holds, then $|\nabla^{\mathbf{S}} w|$ is constant and each linear map $\nabla^{\mathbf{S}} w(a)$, for $a \in \mathbf{S}^{n-1}$ must be conformal with conformality factor 1 because w maps $\mathbf{S}^{n-1}$ to itself. □

Lemma 5. *For any positive number ϵ, there exists a positive $q = q(\epsilon) < n$ so that if $q \le p < n$, $w_p \in \mathcal{C}^1(\mathbf{S}^{n-1}, \mathbf{S}^{n-1})$ has positive degree,*

$$w_p\left(\frac{x}{|x|}\right) \text{ is } p-\text{energy minimizing} , \tag{5}$$

and

$$\int_{\mathbf{S}^{n-1}} |\nabla^{\mathbf{S}} w_p|^p \, d\Sigma \ \le \ \epsilon^{-1} , \tag{6}$$

then

$$\|w_p \ - \ \theta\|_{C^1(\mathbf{S}^{n-1})} \ < \ \epsilon , \tag{7}$$

for some rotation $\theta \in O(n)$.

Proof. By (5) each w_p is a p-harmonic map. Moreover, the family of all p-harmonic maps from $\mathbf{S}^{n-1}$ to itself satisfying (6) and corresponding to $p \in [n - \frac{1}{2}, n]$ is, by Lemma 1, precompact in $C^1(\mathbf{S}^{n-1})$. Thus, if Lemma 5 were false for some positive ϵ, then we could find a sequence $p_i \uparrow n$ and corresponding C^1 convergent sequence of maps $w_i = w_{p_i} \in C^1(\mathbf{S}^{n-1}, \mathbf{S}^{n-1})$ of positive degree satisfying (5) and (6) but not (7) for any rotation θ.

First we claim that $\deg w_i = 1$ for i sufficiently large. In fact, letting $d_i = \deg w_i$, we first note that $d = \sup_i d_i < \infty$. So we may chose fixed disjoint closed balls $\overline{\mathbf{B}_\delta(a_1)}, \ldots, \overline{\mathbf{B}_\delta(a_d)}$ in the open ball $\mathbf{B}$, and then define a comparison map $v_i : \overline{\mathbf{B}} \to \mathbf{S}^{n-1}$ so that $v_i | \partial \mathbf{B} = w_i | \partial \mathbf{B}$,

$$v_i(x) \ = \ \left(\frac{x - a_j}{|x - a_j|} \right) \quad \text{for } x \in \overline{\mathbf{B}_\delta(a_j)} \quad \text{and} \quad j = 1, \ldots, d_i \ ,$$

and v_i is Lipschitz on $\overline{\mathbf{B}} \setminus \cup_{j=1}^{d_i} \mathbf{B}_\delta(a_j)$ with

$$M \ = \ \sup_i \|\nabla v_i\|_{L^\infty(\overline{\mathbf{B}} \setminus \cup_{j=1}^{d_i} \mathbf{B}_\delta(a_j))} \ < \ \infty \, .$$

Thus, by Lemma 4 and the minimality of $w_i(x/|x|)$,

$$\begin{aligned} d_i^{\frac{p_i}{n-1}} (n-1)^{\frac{p_i}{2}} \frac{\sigma_n}{n - p_i} \ &\le \ \int_{\mathbf{B}} \left| \nabla w_i \left(\frac{x}{|x|} \right) \right|^{p_i} dx \ \le \ \int_{\mathbf{B}} |\nabla v_i|^{p_i} \, dx \\ &\le \ d_i (n-1)^{\frac{p_i}{2}} \frac{\sigma_n \delta^{n - p_i}}{n - p_i} \ + \ \frac{\sigma_n}{n} M^n \, . \end{aligned}$$

Multiplying by $n - p_i$ and letting $i \to \infty$ gives a contradiction, if $d_i \ge 2$ for infinitely many i.

Having established that $\deg w_i = 1$ for i large, we may now construct, for i large, another comparison map $\tilde{v}_i : \overline{\mathbf{B}} \to \mathbf{S}^{n-1}$ so that $\tilde{v}_i | \partial \mathbf{B} = w_i | \partial \mathbf{B}$,

$$\tilde{v}_i(x) \ = \ \frac{x}{|x|} \text{ for } x \in \overline{\mathbf{B}}_{\frac{1}{2}} \ ,$$

and $\tilde{v}_i$ is Lipschitz on $\overline{\mathbf{B}} \setminus \mathbf{B}_{\frac{1}{2}}$ with

$$N = \sup_i \|\nabla \tilde{v}_i\|_{L^\infty(\mathbf{B}\setminus\mathbf{B}_{\frac{1}{2}})} < \infty .$$

To define $\tilde{v}_i$ on $\overline{\mathbf{B}}_1 \setminus \mathbf{B}_{\frac{1}{2}}$, we may, for example, fix a Lipschitz map $\tilde{v} : \overline{\mathbf{B}}_{\frac{3}{4}} \setminus \mathbf{B}_{\frac{1}{2}}$ so that

$$\tilde{v}(x) = \begin{cases} \dfrac{x}{|x|} & \text{for } x \in \partial \mathbf{B}_{\frac{1}{2}}, \\ w\left(\dfrac{4}{3}x\right) & \text{for } x \in \partial \mathbf{B}_{\frac{3}{4}}, \end{cases}$$

where $w = (C^1) \lim_{i\to\infty} w_i$. Then let

$$\tilde{v}_i(x) = \begin{cases} \tilde{v}(x) & \text{for } x \in \overline{\mathbf{B}}_{\frac{3}{4}} \setminus \mathbf{B}_{\frac{1}{2}}, \\ \dfrac{(4-4|x|)w(x/|x|) + (4|x|-3)w_i(x/|x|)}{|(4-4|x|)w(x/|x|) + (4|x|-3)w_i(x/|x|)|} & \text{for } x \in \overline{\mathbf{B}}_1 \setminus \mathbf{B}_{\frac{3}{4}} . \end{cases}$$

Now, by Lemma 4 and the minimality of $w_i(x/|x|)$,

$$\begin{aligned} (n-1)^{\frac{p_i}{2}} \frac{\sigma_n}{n-p_i} &\le \int_{\mathbf{B}} \left| \nabla w_i \left(\frac{x}{|x|} \right) \right|^{p_i} dx \le \int_{\mathbf{B}} |\nabla \tilde{v}_i|^{p_i}\, dx \\ &\le (n-1)^{\frac{p_i}{2}} \frac{\sigma_n \left(\frac{1}{2}\right)^{n-p_i}}{n-p_i} + \frac{\sigma_n}{n} N^n . \end{aligned}$$

Multiplying by $n - p_i$ and letting $i \to \infty$ now shows that

$$\int_{\mathbf{S}^{n-1}} |\nabla^{\mathbf{S}} w_i|^{p_i}\, d\Sigma \to (n-1)^{\frac{n}{2}} \sigma_n \text{ as } i \uparrow \infty .$$

By Lemma 4, $w = (C^1) \lim_{i\to\infty} w_i$ must be a rotation. Thus (7) holds with $\theta = w$ for i sufficiently large, the desired contradiction. □

Theorem. *The map* $\frac{x}{|x|} : \mathbf{B}^n \to \mathbf{S}^{n-1}$ *is p-energy minimizing for all* $p \in [n-1, n)$.

Proof. For $n = 2$, the theorem was established in [C] (see also, [CH],§7.1). Here the argument involved first verifying that, for $1 < p < 2$, any p-harmonic map w from $\mathbf{S}^1$ to itself is a constant speed geodesic. Then it was shown that $w(\frac{x}{|x|})$ fails to be minimizing in case $|\deg w| > 1$. Thus, a nonconstant minimizing tangent map (which must, by the regularity theory [HL1], exist for any minimizer with nonzero degree boundary data) is necessarily of the form $\theta(\frac{x}{|x|})$ corresponding to some rotation θ of $\mathbf{R}^2$. In particular, $\frac{x}{|x|} = \theta^{-1}\left(\theta(\frac{x}{|x|})\right)$, being a rotation of a p-energy minimizer, is itself p-energy minimizing.

We now assume $n \geq 3$, and let

$$P = \left\{ p \in [2, n) \ : \ \frac{x}{|x|} \text{ is } p\text{-energy minimizing} \right\} .$$

First we observe that P is closed in $[2, n)$. In fact, if a sequence p_i in P approached a number $p_0 \in [2, n) \setminus P$, then there would exist some map $u \in W^{1,p_0}(\mathbf{B}, \mathbf{S}^{n-1})$ with $u|\partial\mathbf{B} = \mathrm{id}_{\mathbf{S}^{n-1}}$ and

$$\int_{\mathbf{B}} |\nabla u|^{p_0} \, dx \ < \ \int_{\mathbf{B}} \left| \nabla \left(\frac{x}{|x|} \right) \right|^{p_0} dx \ = \ \frac{(n-1)^{\frac{p_0}{2}} \sigma_n}{n - p_0} .$$

But then this strict inequality would continue to hold with p_0 replaced by p_i for i large, contradicting the p_i-energy minimality of $\frac{x}{|x|}$. Thus, $p_0 \in P$, and P is closed in $[2, n)$.

Second, we verify that $p \in P$ whenever $p < n$ is sufficiently close to n. In fact, suppose u_p is a p-energy minimizing map with $u_p|\partial\mathbf{B} = \mathrm{id}_{\mathbf{S}^{n-1}}$. By [HL1], there is an isolated singular point $a_p \in \mathbf{B}$ of u_p with

$$(\deg u_p)(a_p) \ := \ \lim_{\delta \to 0} \deg u_p| \left(\partial \mathbf{B}_\delta(a_p)\right) \ > \ 0 .$$

If $w_p\left(\frac{x}{|x|}\right)$ denotes a tangent map of u_p at a_p, then $\deg w_p = (\deg u_p)(a_p)$, and

$$\frac{1}{n-p} \int_{\mathbf{S}^{n-1}} |\nabla w_p|^p \, d\Sigma \ = \ \Theta_{u_p}(a_p) \ := \ \lim_{r \to 0} r^{p-n} \int_{\mathbf{B}_r(a_p)} |\nabla u_p|^p \, dx .$$

The interior monotonicity inequality [HL1],§4 gives the bound

$$\Theta_{u_p}(a_p) \ \leq \ (1 - |a_p|)^{p-n} \int_{\mathbf{B}_{1-|a_p|}(a_p)} |\nabla u_p|^p \, dx .$$

For $a_p \neq 0$, we may then combine this with the inclusion $\mathbf{B}_{1-|a_p|}(a_p) \subset \mathbf{B}_{2(1-|a_p|)}(\frac{a_p}{|a_p|})$ and the boundary monotonicity inequality [HL1],§5.6 centered at $\frac{a_p}{|a_p|}$ as well as the global energy bound

$$\int_{\mathbf{B}} |\nabla u_p|^p \, dx \ \leq \ \int_{\mathbf{B}} \left| \nabla \left(\frac{x}{|x|} \right) \right|^p dx \ = \ (n-1)^{\frac{p}{2}} \frac{\sigma_n}{n-p}$$

to conclude that

$$\int_{\mathbf{S}^{n-1}} |\nabla w_p|^p \, d\Sigma \;=\; (n-p)\Theta_{u_p}(a_p) \;\leq\; C(n) \;<\infty\;,$$

independent of $p \in [n-1, n)$. Now, for any positive $\epsilon \leq C(n)^{-1}$, we can apply Lemma 5 to find that, for all p sufficiently close to n, there is an orthogonal rotation θ_p so that

$$\|w_p \;-\; \theta_p\|_{C^1(\mathbf{S}^{n-1})} \;<\; \epsilon\;.$$

Insisting further that $\epsilon \leq \epsilon_0(n,n)$, Lemma 3 then implies that, for such p with $n-p<\epsilon$, w_p *itself is actually a rotation.* But $w_p(x/|x|)$, as a tangent map of the p-energy minimizer u_p, is p-energy minimizing [HL1],§6.4. Thus, $\frac{x}{|x|} = w_p^{-1}\left(w_p\left(\frac{x}{|x|}\right)\right)$, being a rotation of a p-energy minimizer, is itself p-energy minimizing, and $p \in P$.

Finally, to complete the proof that $P \supset [n-1, n)$, we will show that $P \cap (n-1, n)$ is open. Suppose, for contradiction, that a sequence $q_i \in (n-1,n) \setminus P$ approaches a number $q_0 \in P \cap (n-1,n)$. Let u_{q_i} be a q_i-energy minimizing map with boundary data the identity. Passing to a subsequence, we find the weak convergence in $W^{1,p}$, for all $p<q_0$, of u_{q_i} to a map $u_{q_0} : \mathbf{B} \to \mathbf{S}^{n-1}$. Using the minimality of u_{q_i}, one may check, as in [HL1]§6.4, that u_{q_0} is q_0-energy minimizing, that $u_{q_0}|\partial\mathbf{B} = \mathrm{id}_{\mathbf{S}^{n-1}}$, and that

$$\text{(8)} \qquad \int_{\mathbf{B}_\rho(a)} |\nabla u_{q_i}|^{q_i} \, dx \;\to\; \int_{\mathbf{B}_\rho(a)} |\nabla u_{q_0}|^{q_0} \, dx \quad \text{as} \quad i \to \infty$$

for any ball $\mathbf{B}_\rho(a) \subset \mathbf{B}$.

We claim that $u_{q_0}(x) \equiv \frac{x}{|x|}$. Since $q_0 \in P$, u_{q_0} has the same q_0-energy on $\mathbf{B}_1$ as $\frac{x}{|x|}$, and the expression

$$v(x) \;=\; \begin{cases} u_{q_0}(x) & \text{for } x \in \mathbf{B}_1, \\ \dfrac{x}{|x|} & \text{for } x \in \mathbf{B}_2 \setminus \mathbf{B}_1\;, \end{cases}$$

defines a q_0-energy minimizing map $v : \mathbf{B}_2 \to \mathbf{S}^{n-1}$. Suppose that $0 < \rho \leq 1$ and $v(x) = \frac{x}{|x|}$ a.e. in $\mathbf{B}_2 \setminus \mathbf{B}_\rho$. The C^0 boundary regularity $v|\mathbf{B}_\rho$ guarantees that v has no interior singularities on $\partial\mathbf{B}_\rho$. Thus, v is C^1 in some neighborhood of $\partial\mathbf{B}_\rho$. Moreover, since $|\nabla^{\mathbf{S}} v| \equiv \sqrt{n-1}$ on $\partial\mathbf{B}_\rho$, v satisfies a nondegenerate second order elliptic equation near $\partial\mathbf{B}_\rho$ and is actually infinitely differentiable there. Also $\partial\mathbf{B}_\rho$ is noncharacteristic for this equation, and $\frac{\partial v}{\partial r} \equiv 0$ on $\partial \mathbf{B}_\rho$. By the uniqueness of the Cauchy problem, the equation $v(x) = \frac{x}{|x|}$ holds in some open neighborhood of $\partial \mathbf{B}_\rho$. It follows that

$$\inf \left\{ \rho \;:\; v(x) = \frac{x}{|x|} \text{ a.e. in : } \mathbf{B}_2 \setminus \mathbf{B}_\rho \right\} \;=\; 0 \;,$$

hence, $u_{q_0}(x) \equiv \frac{x}{|x|}$.

Since $q_i > n - 1$, there is an isolated point $a_i \in \operatorname{sing} u_{q_i}$ with $(\deg u_{q_i})(a_i) > 0$. Next note that the constants in the small energy regularity theorem for p-energy minimizers ([HL1],§2-3) are independent of p for p near q_0. The point here is the uniform regularity of the blow-up equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$. This along with (8) implies that the singularities a_i approach $\operatorname{sing} u_{q_0} = \{0\}$ as $i \to \infty$. Let $w_i(\frac{x}{|x|})$ denote a tangent map of u_{q_i} at a_i. As before, Lemma 4 implies that

$$\int_{\mathbf{S}^{n-1}} |\nabla w_i|^{q_i}\, d\Sigma \;\geq\; (n-1)^{\frac{q_i}{2}} \sigma_n$$

because $\deg w_i > 0$. Moreover, since $\mathbf{B}_{1-|a_i|}(a_i) \subset \mathbf{B}_1(0)$, the interior monotonicity inequality now gives that

$$\begin{aligned} \int_{\mathbf{S}^{n-1}} |\nabla w_i|^{q_i}\, d\Sigma \;&=\; (n-q_i)\Theta_{u_{q_i}}(a_i) \\ &\leq\; (n-q_i)(1-|a_i|)^{q_i-n} \int_{\mathbf{B}_1(0)} |\nabla u_{q_i}|^{q_i}\, dx \\ &\leq\; (n-q_i)(1-|a_i|)^{q_i-n} \int_{\mathbf{B}_1(0)} \left|\nabla\left(\frac{x}{|x|}\right)\right|^{q_i} dx \\ &=\; (1-|a_i|)^{q_i-n}(n-1)^{\frac{q_i}{2}}\sigma_n \;\to\; (n-1)^{\frac{q_0}{2}}\sigma_n \;, \end{aligned}$$

as $i \to \infty$. Again, Lemma 5 shows, for each positive ϵ, that, for i sufficiently large,

$$\|w_i \;-\; \theta_i\|_{C^1(\mathbf{S}^{n-1})} \;<\; \epsilon \;,$$

for some rotation θ_i. Insisting now that $\epsilon \leq \epsilon_0(n, q_0)$, we deduce from Lemma 3 that, for such i with $|q_i - q_0| < \epsilon$, the map w_i itself is a rotation. As before, the q_i-energy minimality of $w_i(x/|x|)$ then implies q_i-energy minimality of $x/|x|$, which contradicts that $q_i \notin P$. □

Remark. J.M. Coron and R. Gulliver [CG] showed that $H(\frac{x}{|x|})$ is 2-energy minimizing where H is the standard Hopf map from $\mathbf{S}^3$ to $\mathbf{S}^2$. In [HL3], we generalize most of the results of [HLW] and use a Jacobi field calculation as in [R], to verify the p-energy minimality of $H(\frac{x}{|x|})$ for $p \in [p_0, 4)$ for some $p_0 < 4$.

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