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A sharp sup+inf inequality for a nonlinear elliptic equation in $\mathbb{R}^2$

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1. Introduction.

In this paper we are concerned with the equation

$$\Delta u + K(x)e^u = 0 \text{ in } \mathbb{R}^2 , \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^2$ and K satisfies

$$a \leq K(x) \leq b \tag{1.2}$$

for some positive constants a and b. This equation appears in the problem of finding a metric conformal to the standard Euclidean metric in $\mathbb{R}^2$ such that $\frac{1}{2}K(x)$ is the Gaussian curvature of the new metric. For a solution u of (1.1), the total curvature is defined by

$$\alpha = \int_{\mathbb{R}^2} K(x)e^{u(x)}\, dx \tag{1.3}$$

Suppose $K(x)$ satisfies (1.2), one interesting question is how to estimate the total curvature α in terms of the constants a and b. Our first result concerning the equation (1.1) is

Theorem 1.1. *Assume K satisfies (1.2) and u is a solution of (1.1). Then*

(1) *the total curvature $\alpha \geq 4\pi\left(1 + \sqrt{\frac{a}{b}}\right)$; and,*

(2) *if $\alpha = 4\pi\left(1 + \sqrt{\frac{a}{b}}\right)$, then, after a translation,*

$$K(x) = \begin{cases} b & \text{if } |x| \leq r_0 , \\ a & \text{if } |x| > r_0 , \end{cases} \tag{1.4}$$

and
(1.5)

$$u(x)=\begin{cases} 2\log\frac{2\sqrt{\frac{a}{b}}}{1+\left(\frac{|x|}{r_0}\right)^2}-2\log r_0+\log 2 & \text{if } |x|\le r_0\\ 2\log\left(\frac{2\sqrt{\frac{a}{b}}\left(\frac{|x|}{r_0}\right)^{\sqrt{\frac{a}{b}}-1}}{1+\left(\frac{|x|}{r_0}\right)^{2\sqrt{\frac{a}{b}}}}\right)-2\log r_0+\log 2 & \text{if } |x|>r_0\end{cases}$$

hold for almost everywhere x and for some $r_0>0$.

If we consider radial solutions only, then we can estimate the upper bound of the total curvature provided that the total curvature is finite.

Theorem 1.2. *Assume that K satisfies (1.2) and both $K(x)$ and $u(x)$ are radially symmetric with respect to 0. Suppose the total curvature of u is finite, then*

(1) $\alpha\le 4\pi\left(1+\sqrt{\frac{b}{a}}\right)$*; and*

(2) *if* $\alpha=4\pi\left(1+\sqrt{\frac{b}{a}}\right)$, *then*

$$K(x)=\begin{cases} a & \text{if } |x|\le r_0\\ b & \text{if } |x|>r_0,\end{cases}\tag{1.6}$$

and
(1.7)

$$u(x)=\begin{cases} 2\log\left(\frac{2\sqrt{\frac{b}{a}}}{1+\left(\frac{|x|}{r_0}\right)^2}\right)-2\log r_0+\log 2 & \text{if } |x|\le r_0,\\ 2\log\left(\frac{2\sqrt{\frac{b}{a}}\left(\frac{|x|}{r_0}\right)^{\sqrt{\frac{b}{a}}-1}}{1+\left(\frac{|x|}{r_0}\right)^{2\sqrt{\frac{b}{a}}}}\right)-2\log r_0+\log 2 & \text{if } |x|\ge r_0\end{cases}$$

for some $r_0>0$.

As an application of Theorem 1.1, we can derive an interior estimate for a solution u of

$$\Delta u+K(x)e^{u(x)}=0 \ \text{ in } \Omega\tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and K satisfies (1.2). Let A be a compact set of Ω. It was proved in [BM] that for any solution u of (1.8), the inequality

$$\sup_A u \le C(a,\, b,\, A,\, \Omega,\, \inf_\Omega u\,)$$

holds. It was conjectured in [BM] that the dependence of C in terms of $\inf_\Omega u$ is linear. In [S], the conjecture was proved to be true, that is, there exist constants $C_1 \le 1$ and C_2 such that

$$C_1 \sup_A u + \inf_\Omega u \le C_2 \tag{1.9}$$

where $C_2 = C_2(a,\, b,\, A,\, \Omega\,)$ and C_1 depends on $\frac{a}{b}$ only.

From (1.5), we know that for (1.9) to hold true, C_1 must be less than or equal to $\sqrt{\frac{a}{b}}$. The main result of this paper is

Theorem 1.3. *Assume that K satisfies (1.2) and u is a solution of (1.8). Then, for any compact subset A of Ω, there exists a constant $C = C(a,\, b,\, A,\, \Omega\,)$ such that the inequality,*

$$\sqrt{\frac{a}{b}}\, \sup_A u + \inf_\Omega u \le C\ , \tag{1.10}$$

holds. More generally, if we assume that there are $\frac{1}{2} \ge \rho > 0$, $\sigma \ge 1$ and $B \ge 0$ such that for $|x-y| \le \rho$,

$$\frac{K(y)}{K(x)} \le \sigma + \frac{B}{|\log|x-y||} \tag{1.11}$$

holds, then any solution u of (1.8) satisfies

$$\sqrt{\frac{1}{\sigma}}\, \sup_A u + \inf_\Omega u \le C\ , \tag{1.12}$$

where C depends on a, b, A, Ω, ρ, σ, and B.

Obviously, if K is Hölder continuous, that is,

$$|K(x) - K(y)| \le B|x-y|^\beta \tag{1.13}$$

for $x, y \in \overline{\Omega}$ and for some constants B, $0 < \beta < 1$, then K satisfies (1.11) with $\sigma = 1$. Hence, as a corollary of Theorem 1.3, we answer a question asked in [BLS].

Corollary 1.4. *Suppose that K satisfies (1.2) and (1.13). Then any solution u of (1.8) satisfies*

$$\sup_A u + \inf_\Omega u \leq C$$

where $C = C(a, b, A, \Omega, B, \beta)$.

We will first prove both Theorem 1.1 and Theorem 1.2 in Section 2. The proof of Theorem 1.3 will be given in Section 3.

2. Proofs of Theorem 1.1 and Theorem 1.2.

In this section, we begin with a proof of Theorem 1.1.

Proof of Theorem 1.1. If $\alpha = +\infty$, the theorem holds trivially. Hence we may assume $\alpha < +\infty$. By a result in [CL](See Theorem 1.2 in [CL]), for any $\epsilon > 0$, there $R(\epsilon) > 0$ such that

$$\frac{-\alpha}{2\pi} \log|x| - c \leq u(x) \leq \left(\frac{-\alpha}{2\pi} + \epsilon\right) \log|x| \tag{2.1}$$

holds for $|x| \geq R(\epsilon)$, where c is a constant independent of ϵ. In particular, we have

$$\lim_{|x|\to+\infty} u(x) = -\infty\ , \tag{2.2}$$

and

$$\alpha > 4\pi\ . \tag{2.3}$$

For each $t \in \mathbf{R}$, let $\Omega_t = \{x \in \mathbb{R}^2 |\, u(x) > t\}$. By (2.2), $|\Omega_t|$ is finite, where $|E|$ denotes the area of a measurable set E in $\mathbb{R}^2$. Let u^* be the Schwartz symmetrization of u, that is, $u^*(x) = u^*(|x|)$ is nonincreasing in $|x|$ and $\Omega_t^* = \{x |\, u^*(x) > t\}$ is the ball $B_r(0)$ with the radius $r = (\frac{1}{\pi}|\Omega_t|)^{\frac{1}{2}}$. Since u is locally Lipschitz, we have u^* is locally Lipschitz also. We also note that, since u satisfies equation (1.1), $|\{x |\, u(x) = t\}| = 0$ for any $t \in \mathbb{R}$. Therefore, $|\Omega_t|$ is strictly increasing in t and then $u^*(x)$ is strictly decreasing in $|x|$.

Set

$$F(r) = \int_{\Omega_{u^*(r)}} K(x) e^{u(x)}\, dx\ , \tag{2.4}$$

and

$$\overline{K}(r) = \frac{F'(r)}{2\pi r} e^{-u^*(r)}\ , \tag{2.5}$$

where $F'(r)$ denotes the derivative of F with respect to r. Since $|\Omega_{u^*(r)}| = \pi r^2$, we have, for $r \geq s$,

$$\left|\Omega_{u^*(r)} \setminus \Omega_{u^*(s)}\right| = \pi(r^2 - s^2) = \pi(r+s)(r-s) .$$

Hence

$$F(r) - F(s) = \int_{\Omega_{u^*(r)} \setminus \Omega_{u^*(s)}} K(x) e^{u(x)}\, dx \leq c(r-s)$$

for some constant c depending on r. Thus $F'(r)$ exists for almost everywhere r and $\overline{K}(r)$ is defined for almost everywhere r. For such r, we have

(2.6)

$$\begin{aligned} 2\pi r a e^{u^*(r)} &\leq \lim_{h \to 0+} a e^{u^*(r+h)} \frac{\pi(r+h)^2 - \pi r^2}{h} \\ &\leq \lim_{h \to 0+} \frac{1}{h} \int_{\Omega_{u^*(r+h)} \setminus \Omega_{u^*(r)}} K(x) e^{u(x)} dx = F'(r) \\ &\leq 2\pi r b e^{u^*(r)} . \end{aligned}$$

By (2.5), we have

$$a \leq \overline{K}(r) \leq b \tag{2.7}$$

for almost everywhere r.

We want to derive a differential inequality for $F(r)$. First, we let $J_1 = \{\rho \geq 0 \mid \frac{du^*}{dr}$ does not exists at $r = \rho\}$, $J_2 = \{\rho \geq 0 \mid \frac{du^*}{dr} = 0$ at $r = \rho\}$, and $E = u^*(J_1 \cup J_2) = \{t | \, t = u^*(r)$ for some $r \in J_1 \cup J_2\}$. Since u^* is locally Lipschitz, it is not difficult to see $H_1(E) = 0$, where H_1 denotes the one dimensional Hausdorff measure of E. For $t \notin E$ and r satisfying $u^*(r) = t$, we have $\frac{du^*(r)}{dr} \neq 0$ and

$$-\frac{d}{dt}|\Omega_t| = \frac{d}{dr}|\Omega_t| \cdot \left(-\frac{du^*}{dr}\right)^{-1} = 2\pi r \left(-\frac{du^*}{dr}\right)^{-1} . \tag{2.8}$$

It is well-known that for almost everywhere t, the inequality

$$-\frac{d}{dt}|\Omega_t| \geq \int_{\{u=t\}} |\nabla u|^{-1}\, dH_1 , \tag{2.9}$$

holds. (See the Lemma in §2.3 in [BZ]). For such t, by (2.8) and the isoperimetric inequality, we have

$$
\begin{aligned}
(2\pi r)^2 = H_1(\partial\Omega_t^*)^2 &\le \left(\int_{\partial\Omega_t} dH_1\right)^2 \\
&\le \left(\int_{\partial\Omega_t} \frac{1}{|\nabla u|}\,dH_1\right)\left(\int_{\partial\Omega_t} |\nabla u|\,dH_1\right) \\
&\le -\left(\frac{du^*(r)}{dr}\right)^{-1}(2\pi r)\int_{\partial\Omega_t}|\nabla u|\,dH_1 \,.
\end{aligned}
\tag{2.10}
$$

Thus

$$
\begin{aligned}
-\frac{du^*(r)}{dr}(2\pi r) &\le \int_{\partial\Omega_t}|\nabla u|\,dH_1 = -\int_{\partial\Omega_t}\frac{\partial u}{\partial\nu}\,dH_1 = -\int_{\Omega_t}\Delta u\,dx \\
&= \int_{\Omega_t} Ke^u\,dx = F(r)\ ,
\end{aligned}
\tag{2.11}
$$

where $t = u^*(r)$. Let $\tilde{E} = \{t \notin E \mid (2.9) \text{ does not hold}\}$, and $J_3 = \{r \mid u^*(r) \in \tilde{E}\}$. Since $H_1(\tilde{E}) = 0$ and $\frac{du^*}{dr}(r) \neq 0$ for any $r \in J_3$, we have $H_1(J_3) = 0$. If $r \in J_2$, (2.11) holds trivially. Therefore, we conclude that (2.11) holds for almost everywhere r. By (2.5) and (2.11), we have for almost everywhere r, that is, $r \notin J_1 \cup J_3$,

$$
\begin{aligned}
\frac{d}{dr}\left(\frac{rF'(r)}{\overline{K}(r)}\right) &= 4\pi r e^{u^*(r)} + 2\pi r^2 e^{u^*}\frac{du^*}{dr} \\
&\ge \frac{2F'(r)}{\overline{K}(r)} - \frac{F'(r)F(r)}{2\pi\overline{K}(r)} \\
&= \frac{2F'}{\overline{K}}\left(1 - \frac{F}{4\pi}\right) \\
&\ge \begin{cases} \dfrac{2F'}{a}\left(1 - \dfrac{F(r)}{4\pi}\right) & \text{if } F(r) > 4\pi, \\[2ex] \dfrac{2F'}{b}\left(1 - \dfrac{F(r)}{4\pi}\right) & \text{if } F(r) \le 4\pi\ . \end{cases}
\end{aligned}
\tag{2.12}
$$

Since $\alpha = F(\infty) > 4\pi$, $F(r_0) = 4\pi$ for some r_0. Therefore, by (2.12) and noting that $\frac{rF'(r)}{\overline{K}(r)}$ is Lipschitz, we have for $r \geq r_0$,

(2.13)
$$\begin{aligned}\frac{rF'(r)}{\overline{K}(r)} &\geq \frac{1}{b}\int_0^{r_0} 2F'(s)\left(1-\frac{F(s)}{4\pi}\right)ds + \frac{1}{a}\int_{r_0}^{r} 2F'(s)\left(1-\frac{F(s)}{4\pi}\right)ds \\ &= \frac{1}{4\pi a}\left(F(r) - 4\pi\left(1-\sqrt{\frac{a}{b}}\right)\right)\left(4\pi\left(1+\sqrt{\frac{a}{b}}\right) - F(r)\right).\end{aligned}$$

Since we assume $F(\infty) < +\infty$, there is $r_k \to +\infty$ such that $\lim\limits_{k\to+\infty} r_k F'(r_k) = 0$. By (2.13), we have

$$F(\infty) \geq 4\pi\left(1+\sqrt{\frac{a}{b}}\right)$$

since $F(\infty) > 4\pi \geq 4\pi(1-\sqrt{\frac{a}{b}}\,)$. The proof of part (i) is complete.

Suppose $\alpha = 4\pi\left(1+\sqrt{\frac{a}{b}}\right)$. Then inequalities in (2.9), (2.10), (2.11) and (2.12) must become equalities for almost everywhere $r \geq 0$. In particular, we have for almost everywhere $r \geq 0$,

$$\frac{du^*(r)}{dr} \neq 0 \tag{2.14}$$

and

$$\frac{du^*(r)}{dr} = \frac{F(r)}{2\pi r} \tag{2.15}$$

Since both u^* and $F(r)$ are Lipschitz, we have $\frac{du^*(r)}{dr} \equiv \frac{F(r)}{2\pi r}$ for $r > 0$. From (2.10), we also have for almost everywhere t,

$$\Omega_t \text{ is a ball and } |\nabla u| \equiv -\frac{du^*(r)}{dr} \text{ on } \partial\Omega_t \tag{2.16}$$

where $t = u^*(r)$.

Applying the coarea formula for u, we have

$$\int_{\mathbb{R}^2} |\nabla u|^2 = \int_{-\infty}^{M_0}\int_{u^{-1}(s)} |\nabla u|\, dH_1\, ds$$

where $M_0 = \max_{\mathbb{R}^2} u$ and $u^{-1}(s) = \{x|\, u(x) = s\}$. Let $s = u^*(\rho)$. By (2.16), the coarea formula implies

$$
\begin{aligned}
\int_{\mathbb{R}^2} |\nabla u|^2 &= \int_{-\infty}^{M_0} \left(\frac{-du^*(\rho)}{d\rho}\right)(2\pi\rho)\, ds \\
&= 2\pi \int_0^\infty \left(\frac{du^*(\rho)}{d\rho}\right)^2 \rho \;\, d\rho \qquad (2.17)\\
&= \int_{\mathbb{R}^2} |\nabla u^*(x)|^2\, dx\ .
\end{aligned}
$$

By (2.14), (2.17) and Theorem 1.1 in [BZ], we conclude that $u(x) = u^*(x + x_0)$ for some $x_0 \in \mathbb{R}^2$. Without loss of generality, we may assume $x_0 = 0$. By the equation (1.1), $K(x) = -\Delta u e^{-u}$ is also radially symmetric with respect to 0. By the equality in (2.12), we have

$$
K(x) = \overline{K}(|x|) = \begin{cases} b & \text{if } \ 0 \le |x| \le r_0 \\ a & \text{if } \ |x| \ge r_0 \end{cases}
$$

holds for a.e x, where r_0 satisfies $F(r_0) = 4\pi$. Hence (1.1) reduces to the ordinary differential equation

$$
\begin{cases} u''(r) + \dfrac{u'(r)}{r} + be^{u(r)} = 0 & \text{for } \ 0 \le r \le r_0, \\ u''(r) + \dfrac{u'(r)}{r} + ae^{u(r)} = 0 & \text{for } \ r \ge r_0 \end{cases} \qquad (2.18)
$$

with $F(r_0) = 4\pi$. By elementary calculations, we can show that u has the form of (1.5). Thus the proof of Theorem 1.1 is completely finished. □

Proof of Theorem 1.2. Let u^* and $\overline{K}$ be defined as in the proof of Theorem 1.1. Since u and K are radially symmetric we have $u = u^*$ and $K = \overline{K}$ for a.e. x. By (1.1)

$$
-2\pi r \frac{du}{dr} = -\int_{B_r(0)} \Delta u\, dx = F(r)\ .
$$

The inequality (2.12) now becomes

$$
\begin{aligned}
\frac{d}{dr}\left(\frac{rF'(r)}{\overline{K}(r)}\right) &= \frac{2F'}{\overline{K}}\left(1 - \frac{F}{4\pi}\right) \\
&\le \begin{cases} \dfrac{2F'}{b}\left(1 - \dfrac{F}{4\pi}\right) & \text{if } \ F > 4\pi\ , \\ \dfrac{2F'}{a}\left(1 - \dfrac{F}{4\pi}\right) & \text{if } \ F \le 4\pi\ . \end{cases} \qquad (2.19)
\end{aligned}
$$

Let r_0 satisfy $F(r_0) = 4\pi$. Again for $r > r_0$, we integrate to obtain

$$\frac{rF'(r)}{\overline{K}(r)} \leq \frac{1}{b}\left[\left(\frac{b}{a} - 1\right)\cdot 4\pi + 2F(r) - \frac{F^2(r)}{4\pi}\right]. \tag{2.20}$$

Since $F(\infty) < \infty$, there is $\{r_k\}$ such that $\lim_{k\to\infty} r_k = \infty$ and $\lim_{k\to\infty} r_k F'(r_k) = 0$. Let $r = r_k$ in (2.20) and let $k \to \infty$, we conclude

$$0 \leq b^{-1}\left[\left(\frac{b}{a} - 1\right)\cdot 4\pi + 2F(\infty) - \frac{F^2(\infty)}{4\pi}\right]$$

and

$$F(\infty) \leq 4\pi\left(1 + \sqrt{\frac{b}{a}}\right).$$

If $\alpha = 4\pi\left(1 + \sqrt{\frac{b}{a}}\right)$, then the equality in (2.19) must hold. Thus $K(x) = a$ for $0 \leq |x| \leq r_0$ and $K(x) = b$ for $|x| > r_0$ hold for a.e. x. The form (1.7) then follows from the corresponding ordinary differential equation to (1.1). □

3. The sup+inf Type Inequality.

For the proof of Theorem 1.3, we need the following lemma, which was proved in [Su].

Lemma 3.1. *Let u be a Lipschitz function defined in $B_R(0)$ and satisfy $\Delta u + \lambda e^u \leq 0$ in $B_R(0)$. Then*

$$u(0) \leq \frac{1}{2\pi r}\int_{\partial B_r(0)} u\, ds - 2\log\left\{1 - \frac{1}{8\pi}\int_{B_r(0)} \lambda e^u\, dx\right\}_+ \tag{3.1}$$

holds for $0 < r < R$ where $\{\cdot\}_+ = \max\{\cdot, 0\}$.

Proof of Theorem 1.3. Since (1.10) is a special case of (1.12), it suffices to prove (1.12) . Assume the conclusion does not hold, that is, there are u_i and K_i which satisfy (1.2), (1.8) and (1.11) for some ρ, σ and B such that

$$\frac{1}{\sqrt{\sigma}}\sup_A u_i + \inf_\Omega u_i \to \infty \tag{3.2}$$

as $i \to \infty$. Then $\lim_{i\to\infty} (\sup_A u_i) = \infty$ and there is $\{x_i\} \subset A$ such that $u_i(x_i) = \sup_A u_i$ and $\lim_{i\to\infty} u_i(x_i) = \infty$.

Step 1. We will employ a blow up argument to show that if we rescale the functions u_i, then there is a subsequence of these functions which converges to a solution w on entire $\mathbb{R}^2$ with a minimal total curvature α_w. That is, $\alpha_w = 4\pi(1+\sqrt{\frac{a}{b}})$ and w has the form (1.5).

Let $d(S,T)$ denote the distance between two sets S and T. When S is a one-element set $\{c\}$, we will also use $d(c,T)$ to denote $d(\{c\},T)$. Let $d_0 = \frac{1}{4}d(A,\partial\Omega)$, $\Omega_1 = \{x \in \Omega \,|\, d(x,\partial\Omega) > d_0\}$ and

$$g_i(x) = u_i(x) + 2\log(d(x,\partial\Omega_1)) \ .$$

Then $\lim_{i\to\infty} g_i(x_i) = \infty$ and the maximum of g_i occurs at some point $\overline{x}_i \in \Omega_1$. We have

$$u_i(\overline{x}_i) \geq u_i(x_i) - C(\Omega)$$

with $C(\Omega)$ depending on the diameter of Ω. Let $M_i = u_i(\overline{x}_i)$, $\overline{L}_i = \frac{1}{2}d(\overline{x}_i, \partial\Omega_1)e^{\frac{M_i}{2}}$, and

$$v_i(y) = u_i(e^{-\frac{M_i}{2}}y + \overline{x}_i) - M_i \ .$$

Then

$$\Delta v_i + \overline{K}_i(y)e^{v_i} = 0 \quad \text{for } |y| \leq \overline{L}_i \ , \tag{3.3}$$

where $\overline{K}_i(y) = K_i(e^{-\frac{M_i}{2}}y + \overline{x}_i)$ and $\lim_{i\to\infty} \overline{L}_i \geq \frac{1}{2}\lim_{i\to\infty} \exp[\frac{1}{2}g_i(\overline{x}_i)] = \infty$. For $|y| \leq \overline{L}_i$,

$$\begin{aligned} v_i(y) &\leq g_i\left(\overline{x}_i + e^{-\frac{M_i}{2}}y\right) - g_i(\overline{x}_i) + 2\log\frac{d(\overline{x}_i,\partial\Omega_1)}{d(\overline{x}_i + e^{-\frac{M_i}{2}}y, \partial\Omega_1)} \\ &\leq 2\log 2 \ . \end{aligned}$$

Let (r,θ) be the polar coordinate in $\mathbb{R}^2$, then

$$\left|\int_0^{2\pi} \frac{\partial v_i}{\partial r} r\,d\theta\right| = \left|\int_{|y|\leq r} \overline{K}_i(y)e^{v_i}dy\right| \leq cr^2$$

and

$$\begin{aligned}\int_0^{2\pi} |v_i(r,\theta)|d\theta &\le \left|\int_0^{2\pi} v_i(r,\theta)d\theta\right| + 4\pi(2\log 2)\\ &\le \int_0^r \left|\int_0^{2\pi} \frac{\partial v_i}{\partial s}(s,\theta)d\theta\right| ds + 4\pi(2\log 2)\\ &\le c(1+r^2)\end{aligned}$$

Thus for $|y| \le l$ with any fixed l

$$\begin{aligned}v_i(y) &= \int_{|\eta|\le 2l} G(y,\eta)\overline{K}_i(\eta)e^{v_i(\eta)}d\eta - \int_{|\eta|=2l} \frac{\partial G}{\partial r} v_i ds\\ &\ge -c\left[\int_{|\eta|\le 2l} G(y,\eta)d\eta + \int_0^{2\pi} |v_i(2l,\theta)|d\theta\right]\\ &\ge -c(1+l^2)\end{aligned}$$

where $G(y,\eta)$ is the Green function of $-\Delta$ on $B_{2l}(0)$. By the elliptic theory, we can pass to subsequences of $\{v_i\}$ and $\{\overline{K}_i\}$ and assume on any compact set of $\mathbb{R}^2$,

$$\begin{cases} v_i \to w & \text{in } C^{1,s} \text{ for any } 0 < s < 1\\ \overline{K}_i \overset{*}{\rightharpoonup} K_0 & \text{weakly}^* \text{ in } L^\infty\end{cases}$$

as $i \to \infty$. Moreover, $a \le K_0(y) \le b$ and

$$\Delta w + K_0(y)e^w = 0 \quad \text{in } \mathbb{R}^2 .$$

By (1.11)

$$\frac{\overline{K}_i(y)}{\overline{K}_i(z)} \le \sigma + \frac{B}{\left|\log\left(e^{-\frac{M_i}{2}}|y-z|\right)\right|} \to \sigma$$

as $i \to \infty$ if $|y|, |z| \le l$. Thus the essential supremum and infimum of K_0 satisfy

$$\frac{\text{ess. sup}\, K_0}{\text{ess. inf}\, K_0} \le \sigma . \tag{3.4}$$

By theorem 1.1,

$$\alpha_w \equiv \int_{\mathbb{R}^2} K_0(y)e^w\, dy \ge 4\pi\left(1 + \sqrt{\frac{\text{ess. inf}\, K_0}{\text{ess. sup}\, K_0}}\right) \ge 4\pi\left(1 + \frac{1}{\sqrt{\sigma}}\right)$$

Recall ρ is the number in (1.11). Let $\overline{\rho} = \frac{1}{2}\min(\rho,\, d(\partial\Omega_1,\, \partial\Omega)\,)$, $L_i = \overline{\rho}e^{\frac{M_i}{2}}$ and $G(x,z)$ be the Green function of $-\Delta$ on $B_{\overline{\rho}}(\overline{x}_i)$. Then

$$
\begin{aligned}
M_i &= \int_{|z-\overline{x}_i|\le\overline{\rho}} K_i(z)e^{u_i(z)}G(\overline{x}_i,z)dz + s_i \\
&\ge \int_{|y|\le L_i}\left(\frac{M_i}{4\pi} - \frac{\log\left|\frac{y}{\rho}\right|}{2\pi}\right)\overline{K}_i(y)e^{v_i(y)}dy + s_i\ ,
\end{aligned}
\tag{3.5}
$$

where $s_i = \frac{1}{2\pi\overline{\rho}}\int_{|z-\overline{x}_i|=\overline{\rho}} u_i(z)ds$. First, we claim that

$$
\alpha_w = 4\pi\left(1+\sqrt{\frac{1}{\sigma}}\right)\ .
$$

For if $\alpha_w > 4\pi\left(1+\sqrt{\frac{1}{\sigma}}\right)$, then there are $\epsilon_1 > 0$ and large l such that

$$
\int_{|y|\le l} K_0(y)e^{w}\,dy \ge (1+2\epsilon_1)4\pi\left(1+\sqrt{\frac{1}{\sigma}}\right)\ .
$$

By (3.5)

$$
\begin{aligned}
M_i &\ge \int_{|y|\le l}\left(\frac{M_i}{4\pi} - \frac{\log|\frac{l}{\rho}|}{2\pi}\right)\overline{K}_i(y)e^{v_i}\,dy + s_i \\
&\ge \left(1-\frac{\epsilon_1}{1+2\epsilon_1}\right)\frac{M_i}{4\pi}\int_{|y|\le l}\overline{K}_i e^{v_i}\,dy + s_i \\
&\ge (1+\epsilon_1)4\pi\left(1+\sqrt{\frac{1}{\sigma}}\right)\cdot\frac{M_i}{4\pi} + s_i
\end{aligned}
\tag{3.6}
$$

for large i. Thus

$$
0 \ge \sqrt{\frac{1}{\sigma}}M_i + s_i
$$

for large i. Since $s_i \ge \inf_{\Omega} u_i$ and $M_i \ge u_i(x_i) - C(\Omega)$, it contradicts to (3.2). Hence $\alpha_w = 4\pi\left(1+\sqrt{\frac{1}{\sigma}}\right)$ must hold. By Theorem 1.1, w is radially symmetric with respect to some point $y_0 \in \mathbb{R}^2$. After a translation, we may assume $y_0 = 0$.

Step 2. We will find a sequence $l_i \leq L_i$ such that

$$\int_{|y|\leq l_i} \overline{K}_i e^{v_i}\, dy \geq 4\pi\left(1+\sqrt{\frac{1}{\sigma}}\right) - C_1 M_i^{-1} \tag{3.7}$$

and

$$\int_{|y|\leq l_i} \log\left|\frac{y}{\rho}\right| \overline{K}_i e^{v_i}\, dy \leq C_2 \tag{3.8}$$

for C_1 and C_2 independent of i as $i \to \infty$.

Let

$$l_i = \sup\left\{ l \leq L_i \,\middle|\, \int_{|y|\leq l} \overline{K}_i(y) e^{v_i}\, dy \leq 4\pi\left(1+\frac{1}{\sqrt{\sigma}}\right)\right\}. \tag{3.9}$$

Since v_i uniformly converges to w in any compact set of $\mathbb{R}^2$ and w satisfies

$$\int_{\mathbb{R}^2} K_0(y) e^w\, dy = 4\pi\left(1+\frac{1}{\sqrt{\sigma}}\right),$$

we have

$$\lim_{i\to+\infty} l_i = +\infty .$$

Let ϵ be a small positive number which will be chosen later. Then for large i, there exist r_1^i and r_1 such that

$$\int_{|y|\leq r_1} K_0(y) e^w dy = 4\pi\left(1+\frac{1}{\sqrt{\sigma}}\right) - \epsilon ,$$

and

$$\int_{|y|\leq r_i^i} \overline{K}_i(y) e^{v_i(y)}\, dy = 4\pi\left(1+\frac{1}{\sqrt{\sigma}}\right) - \epsilon . \tag{3.10}$$

Obviously, $\lim_{i\to+\infty} r_1^i = r$. Hence,

$$\int_{r_1^i\leq|y|\leq l_i} \overline{K}_i e^{v_i}\, dy \leq \epsilon . \tag{3.11}$$

Since $0 < a \leq K_i \leq b$, we have $\Delta v_i + b e^{v_i} \geq 0$ in $B_{l_i}(0)$. Thus, if ϵ is small enough, we can apply Lemma 3.1 to v_i. By Lemma 3.1, the inequality

$$\begin{aligned} v_i(x) &\leq \frac{1}{2\pi r}\int_{\partial B_r(x)} v_i\, ds - 2\log\left\{1 - \frac{1}{8\pi}\int_{B_r(x)} b e^u\, dx\right\}_+ \\ &\leq \frac{1}{2\pi r}\int_{\partial B_r(x)} v_i\, ds + \log 4 . \end{aligned}$$

holds for $r \le \frac{|x|}{2}$ and $2r_1^i \le |x| \le \frac{l_i}{2}$ if ϵ is sufficiently small in (3.11) such that

$$b \int_{r_1^i \le |y| \le l_i} e^{v_i(y)}\, dy \le 4\pi \ . \tag{3.12}$$

Thus,

$$v_i(x) \le \frac{1}{\pi r^2} \int_{B_r(x)} v_i(y)\, dy + \log 4 \ .$$

By Jensen inequality and letting $r = \frac{|x|}{2}$, we have

$$\begin{aligned} e^{v_i(x)} &\le 4 \exp\left(\frac{1}{\pi r^2} \int_{B_r(x)} v_i \right) \le \frac{4}{\pi r^2} \int_{B_r(x)} e^{v_i(y)}\, dy \\ &\le \frac{4}{a\pi r^2} \int_{r_1^i \le |y| \le l_i} \overline{K}_i(y) e^{v_i(y)}\, dy \\ &\le \frac{16\epsilon}{\pi a} |x|^{-2} = C_1 \epsilon |x|^{-2} \ . \end{aligned} \tag{3.13}$$

for $2r_1^i \le |x| \le \frac{l_i}{2}$. Let

$$\overline{v}_i(r) = \frac{1}{2\pi r} \int_{|y|=r} v_i(y)\, dy \ .$$

Then for $r \le l_i$,

$$\begin{aligned} \frac{d}{dr} \overline{v}_i(r) &= -\frac{1}{2\pi r} \int_{|y| \le r} \Delta v_i\, dy = -\frac{1}{2\pi r} \int_{|y| \le r} \overline{K}_i e^{v_i}\, dy \\ &\ge -2 \left(1 + \frac{1}{\sqrt{\sigma}} \right) \frac{1}{r} \ . \end{aligned}$$

Integrating the inequality above gives

$$\overline{v}_i(r) \ge -2 \left(1 + \frac{1}{\sqrt{\sigma}} \right) \log r + C_2 \tag{3.14}$$

for some constant C_2 and $r \le l_i$. We want to apply the Harnack inequality to obtain a lower bound for v_i. To see this, we employ (3.13) and have for $4r_1^i \le r \le \frac{l_1}{4}$,

$$v_i(x) + 2 \log r \le \log C_1 + \log \epsilon \le \frac{1}{2} \log \epsilon < 0 \quad \text{for } \frac{r}{2} \le |x| \le 2r \tag{3.15}$$

provided ϵ is small enough. From now on ϵ is a fixed small positive number such that (3.12) and (3.15) hold. Let

$$\tilde{v}_i(x) = v_i(rx) + 2\log r$$

for $\frac{1}{2} \le |x| \le 2$. Then $\tilde{v}_i(x) \le \frac{1}{2}\log\epsilon$ in $\frac{1}{2} \le |x| \le 2$ and $\tilde{v}_i$ satisfies

$$\Delta\tilde{v}_i + \overline{K}_i(rx)e^{\tilde{v}_i} = 0 \quad \text{in } \frac{1}{2} \le |x| \le 2 \; .$$

Since $\frac{\overline{K}_i(rx)e^{\tilde{v}_i}}{|\tilde{v}_i|} \le C_1\frac{\epsilon}{|\log\epsilon|}$ by (3.13) and (3.15), the Harnack inequality can be applied to $-\tilde{v}_i$. Hence there exists a constant $C \ge 1$ such that for $|x| = 1$

$$-\tilde{v}_i(x) \le -C\int_{|x|=1} \tilde{v}_i \; .$$

Going back to the function v_i, we have from (3.14)

$$v_i(x) \ge -2\log|x| - \frac{2C}{\sqrt{\sigma}}\log|x| + C_3$$

for $4r_1^i \le |x| \le \frac{l_i}{4}$. Therefore, letting $\delta = \left(2 + \frac{2C}{\sqrt{\sigma}}\right)^{-1}$, we have for $4r_1^i \le |x| = R^\delta \le \frac{l_i}{4}$

$$\begin{aligned} v_i(x) &\ge -2\log R + C_3 \\ &\ge \sup_{\frac{l_i}{2} \ge |y| \ge R} v_i(y) \end{aligned} \tag{3.16}$$

Obviously, $v_i(x) \ge \sup\limits_{\frac{l_i}{2}\ge|y|\ge R} v_i(y)$ holds true also for $|x| = R^\delta \le 4r_1^i$ since $w(x) = w(|x|)$ is strictly decreasing in $|x|$ and v_i converges to w on any compact set.

Let $m_i(r) = \max\limits_{|y|=r} v_i(y)$ and $t_0 = m_i\left(\frac{l_i}{2}\right)$. For $t > t_0$, let $\Omega_t^i = \left\{y \mid |y| \le \frac{l_i}{2}\right.$ and $v_i(y) > t\}$. Obviously, the closure $\overline{\Omega}_t^i$ of Ω_t^i is always contained in the open ball $B_{\frac{l_i}{2}}(0)$. Let $v_i^*(x) = v_i^*(|x|)$ denote the Schwartz symmetrization, that is, $v_i^*(|x|)$ is nonincreasing in $|x|$ and $\{x|\, v_i^*(x) > t\}$ is the ball $B_r(0)$ with $r = \left(\pi^{-1}|\Omega_t|\right)^{\frac{1}{2}}$ for $t > t_0$. Since v_i satisfies the equation (3.3), $|\{y|\,|y| \le l_i,\; v_i(y) = t > t_0\}| = 0$. Hence $|\Omega_t|$ is strictly decreasing in t and then $v_i^*(|x|)$ is strictly decreasing in r. We also have $v_i^*(|x|)$ is locally Lipschitz. As in Section 2, we let

$$F_i(r) = \int_{\Omega_{v_i^*(r)}} \overline{K}_i(y)e^{v_i}\,dy \; , \tag{3.17}$$

and

$$\hat{K}_i(r) = \frac{F_i'(r)}{2\pi r e^{v_i^*(r)}} \tag{3.18}$$

for all r such that $v_i^*(r)$ is well-defined. As before, we can prove that $F_i(r)$ is locally Lipschitz, therefore, $\hat{K}_i(r)$ is defined for almost everywhere r.

We note that an immediate consequence of (3.16) is, for $R \leq \frac{l_i}{4}$:

$$B_{R^{\delta^2}}(0) \subseteq \Omega^i_{m(R^\delta)} \subseteq B_R(0) \ . \tag{3.19}$$

The first part of (3.19) is easily seen from (3.16). For the second part of (3.19), let $z \in \Omega_{m(R^\delta)}$. Then by (3.16), we have

$$v_i(z) > \max_{|y|=R^\delta} v_i(y) \geq \max_{\frac{l_i}{2} \geq |y| \geq R} v_i(y) \ .$$

Hence $z \in B_R(0)$, and (3.19) is proved. From (3.19), we immediately have for $R \leq \frac{l_i}{4}$,

$$\int_{|y|\leq R} \overline{K}_i(y) e^{v_i(y)}\, dy \geq F\left(\left(\pi^{-1}\left|\Omega_{m(R^\delta)}\right|\right)^{\frac{1}{2}}\right) \geq F\left(R^{\delta^2}\right) \ . \tag{3.20}$$

Let $R_i = \left(\pi^{-1}\left|\Omega_{m((\frac{l_i}{4})^{\frac{\delta}{2}})}\right|\right)^{\frac{1}{2}}$. Then

$$\left(\frac{l_i}{4}\right)^{\frac{\delta^2}{2}} \leq R_i \leq \left(\frac{l_i}{4}\right)^{\frac{1}{2}} \ . \tag{3.21}$$

Thus, $\lim_{i\to+\infty} R_i = +\infty$. Obviously, $v_i^*(r)$ is defined for all $r \leq R_i$. By (3.18) and (2.6), (3.19) implies for $r \leq R_i$,

$$a_i \leq \hat{K}_i(r) \leq b_i \tag{3.22}$$

where

$$a_i = \operatorname{ess.inf}_{|y|\leq(\frac{l_i}{4})^{\frac{1}{2}}} \overline{K}_i(y) \ , \tag{3.23}$$

and

$$b_i = \operatorname{ess.sup}_{|y|\leq(\frac{l_i}{4})^{\frac{1}{2}}} \overline{K}_i(y) \ . \tag{3.24}$$

Let $r_0^i < r_1^i$ satisfy $F_i(r_0^i) = 4\pi$. As in section 2, we can derive a similar differential inequality for $F_i(r)$ as in (2.13), that is, the inequality

$$\frac{rF_i'(r)}{\hat{K}_i(r)} \geq \frac{-1}{4\pi a_i}\left[F(r) - 4\pi\left(1 - \sqrt{\frac{a_i}{b_i}}\right)\right]\left[F(r) - 4\pi\left(1 + \sqrt{\frac{a_i}{b_i}}\right)\right] \tag{3.25}$$

holds true for $r_0^i \leq r \leq R_i$ almost everywhere. Let $\tilde{R}_i = \sup\left\{r \leq R_i \middle| \, F_i(r) \leq 4\pi\left(1 + \sqrt{\frac{a_i}{b_i}}\right)\right\}$. By (3.25), we have

$$\frac{F_i'(r)}{F_i(r) - 4\pi\left(1 - \sqrt{\frac{a_i}{b_i}}\right)} + \frac{F_i'(r)}{4\pi\left(1 + \sqrt{\frac{a_i}{b_i}}\right) - F(r)} \geq 2\sqrt{\frac{a_i}{b_i}}\, r^{-1}$$

for $r_0^i \leq r \leq \tilde{R}_i$. Integrating the differential inequality gives

$$\log \frac{4\pi\left(1 + \sqrt{\frac{a_i}{b_i}}\right) - F_i(r)}{F_i(r) - 4\pi\left(1 - \sqrt{\frac{a_i}{b_i}}\right)} \leq -2\sqrt{\frac{a_i}{b_i}} \log \frac{r}{r_0^i}\,,$$

which implies

$$F_i(r) \geq 4\pi\left(1 + \sqrt{\frac{a_i}{b_i}}\right) - Cr^{-2\sqrt{\frac{a_i}{b_i}}} \tag{3.26}$$

for $r_0^i \leq r \leq \tilde{R}_i$, where C is a positive constant independent of i. Trivially, (3.26) holds for $\tilde{R}_i \leq r \leq R_i$ also.

By (3.23) and (3.24), we have

$$\frac{b_i}{a_i} \leq \sup_{|y|,|z| \leq (\frac{l_i}{4})^{\frac{1}{2}}} \frac{\overline{K}_i(y)}{\overline{K}_i(z)} \leq \sigma + C_0 \left|\log\left(l_i^{\frac{1}{2}} e^{-\frac{M_i}{2}}\right)\right|^{-1} \leq \sigma + C_1 M_i^{-1}\,.$$

Therefore,

$$\sqrt{\frac{a_i}{b_i}} \geq \frac{1}{\sqrt{\sigma}} - C_2 M_i^{-1}\,. \tag{3.27}$$

Combined with (3.26), it yields

$$F_i(r) \geq 4\pi\left(1 + \frac{1}{\sqrt{\sigma}}\right) - Cr^{-2\sqrt{\frac{1}{\sigma}}} - C_2 M_i^{-1} \tag{3.28}$$

for $r_0^i \le r \le R_i$. By (3.20) and (3.28), we have

$$(3.29)\quad \int_{|y|\le R} \overline{K}_i e^{v_i}\,dy \ge F\left(R^{\delta^2}\right) \ge 4\pi\left(1+\sqrt{\frac{1}{\sigma}}\right) - CR^{-2\sqrt{\frac{1}{\sigma}}\delta^2} - C_2 M_i^{-1}$$

for $R \le \left(\frac{l_i}{4}\right)^{\frac{1}{2}}$. Hence,

$$(3.30)\qquad \int_{R\le|y|\le l_i} \overline{K}_i e^{v_i}\,dy \le CR^{-2\sqrt{\frac{1}{\sigma}}\delta^2} + C_2 M_i^{-1}\,.$$

Applying Lemma 3.1 and (3.13) again, there exists $C_3 > 0$ such that

$$(3.31)\quad e^{v_i(x)} \le \frac{4}{a\pi r^2}\int_{\frac{1}{2}|x|\le y\le l_i} \overline{K}_i e^{v_i}\,dy \le C_3\left(|x|^{-2-2\sqrt{\frac{1}{\sigma}}\delta^2} + M_i^{-1}|x|^{-2}\right)$$

for $|x| \le \frac{1}{2}\left(\frac{l_i}{4}\right)^{\frac{1}{2}}$. Choose l_i^* satisfy $(\log l_i^*)^2 = \log l_i$. Obviously, $l_i^* \le \frac{1}{2}\left(\frac{l_i}{4}\right)^{\frac{1}{2}}$ for large i. Then, by (3.30) and (3.31), we have

$$\begin{aligned}
&\int_{|y|\le l_i} \log\frac{|y|}{\rho}\overline{K}_i(y)e^{v_i(y)}\,dy \\
&\le \int_{|y|\le l_i^*} \log\frac{|y|}{\rho}\overline{K}_i(y)e^{v_i(y)}\,dy + \log\frac{l_i}{\rho}\int_{l_i^*\le|y|\le l_i} \overline{K}_i(y)e^{v_i(y)}\,dy \\
&\le C_4\left[\left(1+M_i^{-1}(\log l_i^*)^2\right) + (\log l_i)\left(l_i^{*\,-2\sqrt{\frac{1}{\sigma}}\delta^2} + M_i^{-1}\right)\right] \\
&\le C_5\,,
\end{aligned}$$

which obviously yields (3.8). If $l_i < L_i$, then (3.7) holds trivially. If $l_i = L_i$, then by letting $R = \left(\frac{l_i}{4}\right)^{\frac{1}{2}}$ in (3.29), it yields

$$\int_{|y|\le L_i} \overline{K}_i e^{v_i}\,dy \ge 4\pi\left(1+\frac{1}{\sqrt{\sigma}}\right) - C_3 l_i^{-\delta^2\sqrt{\frac{1}{\sigma}}} - C_2 M_i^{-1}$$

which (3.7) follows immediately.

Step 3. To obtain a contradiction to (3.2), we note that by (3.5), (3.7) and (3.8),

$$\begin{aligned}
M_i &\ge \int_{|y|\le l_i}\left(\frac{M_i}{4\pi} - \frac{\log\frac{|y|}{\rho}}{2\pi}\right)\overline{K}_i(y)e^{v_i(y)}\,dy + s_i \\
&\ge \left(1+\frac{1}{\sqrt{\sigma}}\right)M_i + \inf_{\Omega} u_i - C_6\,,
\end{aligned}$$

where C_6 is a constant. Thus, we have

$$\frac{1}{\sqrt{\sigma}} M_i + \inf_{\overline{\Omega}} u_i \leq C_6 \ ,$$

which obviously leads to a contradiction to (3.2). The proof of Theorem 1.3 is completely finished. □

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