COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 5, Number 2, 333-387, 1997

Global Existence Theorems for Harmonic Maps to Non-locally Compact Spaces

NICHOLAS J. KOREVAAR[1] AND RICHARD M. SCHOEN[2]

Introduction.

This paper is the second in our study of harmonic mapping theory into general target spaces of nonpositive curvature. Our first paper [KS] was mainly focused on the local structure of finite energy maps to metric spaces, and local existence and general regularity issues. In this paper we consider global existence issues related to the non-local compactness of the target. Specifically, we consider a discrete group Γ which is generally assumed to be the fundamental group of a compact Riemannian manifold M. Some of our results work more generally for groups which are only finitely generated. We assume that the group acts isometrically on a metric space X; i.e. there is a group homomorphism $\rho : \Gamma \to Isom(X)$. We then study the existence of energy minimizing ρ-equivariant maps from the universal covering $\tilde{M}$ to X. We assume throughout that X is an NPC space; that is, a length space of nonpositive curvature in the sense of Alexandrov. (This notion is reviewed in Section 1.) We mainly focus on existence issues in this paper. By way of history we should mention that the results of this paper may be viewed as generalizations of the work of Eells and Sampson[ES]. One should also compare the results of this paper with the work of Gromov [G2]. Our first main existence theorem, Theorem 2.1.3, asserts the existence of a harmonic equivariant map under the assumption that the action of Γ on X is *proper* in the sense that the set of points of X which are translated by any bounded amount by *all* generators in a generating set for Γ is a bounded subset of X. (Note that X is not assumed locally compact.) A similar theorem was proved by J. Jost[J] in a slightly different context. Under the assumption that X has curvature bounded from above by a negative constant, we are able to prove (Theorem 2.3.1) that either there exists an equivariant harmonic map,

[1]Supported by NSF Grant
[2]Supported by NSF Grant DMS 9504441

or the action fixes an equivalence of rays in X. In particular, this existence result holds in case X is a tree (Corollary 2.3.2).

We then develop methods which are applicable to the improper case for general NPC spaces X. We prove a very general theorem, Theorem 3.9, which asserts that for a sequence of minimizing (or almost minimizing) equivariant maps $u_k : \tilde{M} \to X_k$ into a sequence of NPC spaces X_k, there is a subsequence, still denoted $\{u_k\}$, an NPC space X, an action of Γ on X, and a minimizing equivariant map $u : \tilde{M} \to X$ such that the pullback distance functions from the X_k via u_k converge to the pullback of the distance from X via u. We show that the infinitesimal pullback metrics, and the energy densities of the u_k converge in L^1 norm to that of u. We also show that if the X_k have uniform lower curvature bounds and the property that geodesics are extendible, then X may be taken to have this property as well. The technical machinery behind these results includes a careful study in Section 1.5 of mollification of maps by ε-approximate identities, and resulting energy estimates for the mollified maps.

We then take up the study of *uniform* actions. By this we mean an action such that each point of X is translated by a fixed positive amount by a generator of Γ (for an arbitrarily chosen set of generators). We prove the general theorem that for a nonuniform action of Γ on a space X which does not have a fixed point, there exists a uniform action on a new NPC space Y arising as a limit of rescalings of X together with a (nonconstant) equivariant harmonic map from $\tilde{M}$ to Y. In case the space X has curvature bounded from below, and has the property that its geodesics are extendible, the space Y is shown to be a Hilbert space. In particular, it follows that a nonuniform action of a group Γ which satisfies Property T on such a space X has a fixed point. Another corollary of this result is that if M is Kähler, and if Γ acts nonuniformly on such a space X (for example if Γ does not satisfy Property T, then such an action exists on a Hilbert space X), then there exists a uniform action of Γ on a complex Hilbert space Y, and an equivariant holomorphic map from $\tilde{M}$ to Y.

Finally we mention that Section 1 also contains a new Poincaré inequality for maps from a Riemannian manifold M to any metric space X. The constant has sharp geometric dependence on M and is independent of X. The result generalizes and simplifies earlier work of Li and Yau[LY], and Gromov[G1].

1. Preliminaries.

1.1. Upper and lower curvature bounds.

As in our first paper, *Sobolev spaces and harmonic maps for metric space targets* [KS], we will use the abbreviation NPC for non-positively curved metric spaces. The notation CAT(0) space is also common, and is due to M. Gromov. Recall that an NPC space (X, d) is a complete metric space which satisfies the following two hypotheses:

(i) (Length Space) For any two points $x_0, x_1 \in X$ there is a rectifiable curve γ from x_0 to x_1 such that

$$d(x_0, x_1) = Length(\gamma). \tag{1.1.1}$$

We call such a curve γ a geodesic.

(ii) (Triangle Comparison) Given any three points z, x_0, x_1 in X, $\lambda \in [0, 1]$, and any geodesic γ from x_0 to x_1, let $x_\lambda := (1-\lambda)x_0 + \lambda x_1$ denote the point which is a fraction λ of the way from x_0 to x_1 along γ. Write

$$d(z, x_0) := d_0, \quad d(z, x_1) := d_1, \quad d(z, x_\lambda) := d_\lambda, \quad d(x_0, x_1) := L.$$

For an $\mathbf{R}^2$ comparison triangle of side lengths d_0, d_1, L we require that d_λ be less than or equal to the distance from the vertex corresponding to z to the point on the opposite side λ of the way from the vertex corresponding to x_0 to the one corresponding to x_1. The precise inequality is

$$d_\lambda^2 \leq (1-\lambda)d_0^2 + \lambda d_1^2 - \lambda(1-\lambda)L^2. \tag{1.1.2}$$

Uniqueness of geodesics is an immediate consequence of this definition.

Condition (i) is implied by the seemingly weaker hypotheses that any two points x_0, x_1 have a midpoint $x_{\frac{1}{2}}$ for which

$$d(x_0, x_{\frac{1}{2}}) = d(x_{\frac{1}{2}}, x_1) = \frac{1}{2}d(x_0, x_1).$$

This is because one can then use diadic subdivision to define a distance-realizing continuous path γ. Condition (ii) is implied by the seemingly weaker hypothesis that it hold just in case $\lambda = \frac{1}{2}$, as can also be checked using diadic subidivision. Another way to check the equivalence is to apply the $\lambda = \frac{1}{2}$ case as follows: Take any geodesic arc $\gamma(s)$ which is parameterized by arclength s and consider points $z, x_0 = \gamma(s - \Delta s), x_1 = \gamma(s + \Delta s)$. Applying the triangle comparison one obtains a finite-difference expression which implies that the function $f(s) := d^2(\gamma(s), z)$ satisfies

$$\frac{d^2}{ds^2} f \geq 2 \tag{1.1.3}$$

distributionally. Integrating this inequality yields (1.1.2).

We will sometimes impose a lower curvature bound for our spaces X : An NPC space will be said to satisfy $(K \geq \kappa)$, for $\kappa = -a^2$, if a triangle comparison to the hyperbolic space of curvature κ holds, i.e. if

(1.1.4)

$$cosh(ad_\lambda) \geq \frac{sinh((1-\lambda)aL)}{sinh(aL)} cosh(ad_0) + \frac{sinh(\lambda aL)}{sinh(aL)} cosh(ad_1)$$

holds for all z, x_0, x_1, λ as above. In case $a = 0$ this condition is to be interpreted as the implication of (1.1.4) as $a \to 0^+$, namely the reverse inequality to (1.1.2). The condition (1.1.4) is implied by the seemingly weaker requirement that the inequality always hold for $\lambda = \frac{1}{2}$,

$$cosh(ad_{\frac{1}{2}}) \geq \frac{cosh(ad_0) + cosh(ad_1)}{2cosh(\frac{aL}{2})}. \tag{1.1.5}$$

One way to see this equivalence is to apply (1.1.5) to a geodesic γ parameterized by arclength s, again taking points $z, x_0 = \gamma(s - \Delta s), x_1 = \gamma(s + \Delta s)$ as above. This time one obtains a finite-difference statement which implies that the function $f(s) := cosh(ad(\gamma(s), z))$ satisfies

$$\frac{d}{ds^2} f \leq a^2 f \tag{1.1.6}$$

distributionally. The statement (1.1.4) is the integrated version of this inequality.

The lower curvature bound immediately implies uniqueness of geodesic extensions: Let γ be a geodesic from x_0 to y, and suppose this arc extends geodesically past y, as arcs γ_1, γ_2. Then for some $0 < \lambda < 1$ we may find $z \in \gamma_1, x_1 \in \gamma_2$ for which y is λ of the way from x_0 to z and from x_0 to x_1. Applying (1.1.4) with this choice of points, multiplying through by $sinh(aL)$ and recalling the sinh addition formula we deduce that $cosh(ad_1) = 1$, i.e. that $x_1 = z$.

An NPC space X is said to be geodesically complete if every geodesic arc extends to a geodesic line. The following characterization of Hilbert Spaces must be well-known, but for completeness we include a proof.

Proposition 1.1. *Let X be an NPC space which is geodesically complete and which also satisfies $\kappa \geq 0$. Then X is a Hilbert Space.*

Proof. By hypothesis X satsifies the triangle comparison

$$d_\lambda^2 = (1-\lambda)d_0^2 + \lambda d_1^2 - \lambda(1-\lambda)L^2, \tag{1.1.7}$$

where we use our previous notation. One should interpret (1.1.7) as saying that any one of the four quantities (the three side-lengths and the $\lambda-$median) can be computed from the other three, and the length will agree with the corresponding Euclidean computation. We will say that a subset $S \subset X$ embeds isometrically into Euclidean space if there is a map $f : S \to \mathbf{R}^n$ so that the pull-back pseudodistance equals the restriction of d to $S \times S$. Consider the set S_1 containing S which is the set of all points on lines through pairs of points in S. Inductively construct $S \subset S_1 \subset S_2 \subset S_3, \ldots$. The geodesic completion of S is the closure of $\cup S_k$. If S is separable and $f : S \to \mathbf{R}^n$ is an isometric embedding, then f extends as an isometric embedding to the convex geodesic closure of S. This is because one can extend f to the line through s_1, s_2 by

$$f((1-\lambda)s_1 + \lambda s_2) = (1-\lambda)f(s_1) + \lambda f(s_2).$$

It follow from (1.1.7) that this extension is an isometry. One may obtain an isometry from a dense subset of the geodesic completion of S to $\mathbf{R}^n$ by a countable number of these "line-extensions", and then extend the isometry to the full geodesic completion.

It follows from the discussion above that any three non-colinear points in X have geodesic completion which is isometric to the Euclidean plane $\mathbf{R}^2$. We claim that any four non-coplanar points $\{P_1, P_2, P_3, P_4\} \subset X$ have geodesic completion which is isometric to $\mathbf{R}^3$. To see why, consider the two triangles P_1, P_2, P_3 and P_2, P_3, P_4. Construct Euclidean comparison triangles (writing $\overline{P}_i$ for the $\mathbf{R}^2$-points), so that both triangles share the arc $\overline{P}_2\overline{P}_3$, and so that $\overline{P}_1, \overline{P}_4$ lie on the same side of this arc. Pick a point $\overline{Q}$ which lies in both comparison triangles and so that the perpendicular segment from $\overline{Q}$ to $\overline{P}_2\overline{P}_3$ does as well. Write d for the length of this segment. Then $\overline{Q}$ is the isometric image of points Q_1, Q_2 in the two $X-$ triangles (via the isometric extension maps discussed above). From the triangle inequality the distance $d(Q_1, Q_2)$ is between 0 and $2d$. Consider the Euclidean plane containing the two comparison triangles as lying in $\mathbf{R}^3$ and rotate one of the two comparison triangles about the arc $\overline{P}_2\overline{P}_3$ so that the Euclidean distance between the points $\overline{Q}_1, \overline{Q}_2$ exactly equals $d(Q_1, Q_2)$. One has constructed an isometric embedding of the four points P_1, P_2, Q_1, Q_2. By our discussion above this embedding extends to the geodesic completion of these four points, which is the identical to the geodesic completion of the original points P_1, P_2, P_3, P_4. This proves the claim.

We may now deduce that X is a Hilbert space as follows: Fix any point $0 \in X$, and call it the origin. Let $U, V, W \subset X, \lambda \in \mathbf{R}$. Since the convex geodesic closure of $0, U, V$ is one of $\mathbf{R}, \mathbf{R}^2$, $U + V$ and λU are naturally defined if we consider 0 to be the origin. The vector space axioms hold because the geodesic completion of $0, U, V, W$ is isometric to (a subspace of) $\mathbf{R}^3$. One may also define the norm of "vectors", the dot product between pairs of them via this isometry, and verify that all necessary axioms hold. □

1.2. Functional Analysis Lemma.

One can interpret (1.1.2) as a statement about the uniform convexity of geodesic balls in an NPC space. Consequently, the following result, which is well-known for Banach spaces which are uniformly convex, also holds for NPC spaces.

Proposition 1.2. *Let $\{C_i\}$ be a nested (decreasing) sequence of nonempty, closed bounded convex sets in an NPC space X. Then $\cap C_i$ is non-empty.*

Proof. We may assume that some C_i is not all of X for otherwise all points of X are common to all C_i. By removing the first finite number of $C_i = X$, there is no loss of generality in assuming that C_1 is not all of X. Let Q be a point which is not in C_1, and let

$$r_i = dist(Q, C_i), \qquad i = 1, 2, \dots .$$

The sequence $\{r_i\}$ is increasing and bounded, and therefore has a limit which we denote r. Now let D_i be defined by

$$D_i = \{P \in C_i : d(Q, P) \leq r + 2^{-i}\}.$$

The sequence $\{D_i\}$ is a again a nested sequence of nonempty, closed, bounded, convex sets. We show that the diameter of D_i tends to zero. This implies that the D_i have a common point since any sequence of points $\{P_i\}$ with $P_i \in D_i$ will be a Cauchy sequence and hence convergent since X is complete. Since each D_i is closed, the limit is a common point to all of the D_i.

To bound the diameter of D_i, let $P_0, P_1 \in D_i$ be any two points, and let $P_{\frac{1}{2}}$ denote the midpoint between P_0, P_1. By triangle comparison we have

$$\begin{aligned} d^2(Q, P_{\frac{1}{2}}) &\leq \frac{1}{2}d^2(P_0, Q) + \frac{1}{2}d^2(P_1, Q) - \frac{1}{4}d^2(P_0, P_1) \\ &\leq (r + 2^{-i})^2 - \frac{1}{4}d^2(P_0, P_1). \end{aligned}$$

Since $P_{\frac{1}{2}} \in C_i$ we have $d(Q, P_{\frac{1}{2}}) \geq r_i$, so it follows that

$$d^2(P_0, P_1) \leq 4[(r + 2^{-i})^2 - r_i^2].$$

Since the right hand side goes to zero, we have shown that the diameter of D_i goes to zero, and this completes the proof of Proposition 1.2. □

1.3. Quadrilateral comparisons.

Much of the analysis related to harmonic map theory depends on strong convexity estimates for the distance function between geodesics. These estimates are consequences of quadrilateral comparisons derived first by Y.G. Reshetnyak [R]. Given any ordered quadruple $\{P, Q, R, S\} \subset X$ there is a (non-unique) quadruple $\{\overline{P}, \overline{Q}, \overline{R}, \overline{S}\} \subset \mathbf{R}^2$ so that the resulting Euclidean quadrilateral is convex and so that

$$\begin{aligned} &d(P, Q) = |\overline{P} - \overline{Q}|, \ d(Q, R) = |\overline{Q} - \overline{R}|, \\ &d(R, S) = |\overline{R} - \overline{S}|, \ d(S, P) = |\overline{S} - \overline{P}|, \\ &d(P, R) \leq |\overline{P} - \overline{R}|, \ d(Q, S) \leq |\overline{Q} - \overline{S}|. \end{aligned} \tag{1.3.1}$$

One constructs this convex "subembedding" using only the triangle comparisons (1.1.2) for various triangles determined by choices of three points from $\{P, Q, R, S\}$. By using this subembedding and the NPC length comparison (1.1.2) repeatedly for triangles whose vertices lie on the geodesic edges (or vertices) of the quadrilateral, one derives the following inequalities [R]:

Proposition 1.3 (e.g. Corollary 2.1.3 [KS]). *Abbreviate the distance function $d(T,U)$ by d_{TU}. For an ordered sequence $\{P,Q,R,S\} \subset X$, define the geodesic interpolation points $P_\lambda = (1-\lambda)P + \lambda S,$, $Q_\mu = (1-\mu)Q + \mu R$. Then for any $0 \le \alpha, t \le 1$ the following estimates hold.*

(1.3.2)
$$\begin{aligned} d^2_{P_tQ_t} \le (1-t)d^2_{PQ} &+ td^2_{RS} \\ &- t(1-t)(\alpha(d_{SP}-d_{QR})^2 + (1-\alpha)(d_{RS}-d_{PQ})^2). \end{aligned}$$

(1.3.3)
$$\begin{aligned} d^2_{Q_tP} + d^2_{Q_{1-t}S} \le d^2_{PQ} + d^2_{RS} + t(d^2_{SP} - d^2_{QR}) + 2t^2 d^2_{QR} \\ - t(\alpha(d_{SP}-d_{QR})^2 + (1-\alpha)(d_{RS}-d_{PQ})^2). \end{aligned}$$

In case $t = 1$ we obtain from (1.3.3) the quadrilateral inequality:

(1.3.4)
$$\begin{aligned} d^2_{PR} + d^2_{QS} \le d^2_{PQ} + d^2_{QR} + d^2_{RS} + d^2_{SP} \\ - \alpha(d_{SP}-d_{QR})^2 - (1-\alpha)(d_{RS}-d_{PQ})^2. \end{aligned}$$

If we set $\alpha = 0$ in (1.3.2) we conclude the convexity of the distance between geodesics,

(1.3.5) $$d_{P_tQ_t} \le (1-t)d_{PQ} + td_{RS}.$$

(This particular estimate may also be derived more directly from the triangle comparison 1.1.2.) We will also have occasion to use the special case of (1.3.4) when $\alpha = 1$,

(1.3.6) $$d^2_{PR} + d^2_{QS} \le d^2_{PQ} + d^2_{RS} + 2d_{SP}d_{QR}.$$

For lower curvature bounds we do not know of quadrilateral comparisons equal in strength to Reshetnyak's. However, in as much as the NPC estimates above are in the nature of convexity statements for various distances between geodesic arcs, lower curvature bounds imply various concavity estimates. We will write these estimates in the case the lower curvature bound $-a^2$ is -1. Because curvature scales inversely to d^2, one can deduce the more general estimates from this special case by replacing every length with a times the length, as in (1.1.4). For $a = 1$ the lower curvature estimate (1.1.4) immediately impies a concavity estimate for the distance between geodesic rays eminating from a single vertex: Consider the triangle with vertices z, x_0, x_1, $d(z, x_0) = d_0$, $d(z, x_1) = d_1$, $d(x_0, x_1) = L$ as in §1.1. For $0 \leq t \leq 1$ define L_t to be the distance between the two points x_{0t}, x_{1t} which are a fraction t of the way from z to the two x_i. If $t = \frac{1}{2}$ one can apply (1.1.5) twice, once to the intermediate triangle spanned by $z, x_{1\frac{1}{2}}, x_0$ and once to the original triangle, to deduce the estimate

$$ch(L) - 1 \leq 4ch(\frac{d_0}{2})ch(\frac{d_1}{2})(ch(L_{\frac{1}{2}}) - 1). \tag{1.3.7}$$

(We abbreviate *cosh, sinh* with *ch, sh*.) For general t one uses (1.1.4) in a similar, but significantly more tedious computation to derive the generalization

$$ch(L) - 1 \leq \frac{sh(d_0)sh(d_1)}{sh(td_0)sh(td_1)}(ch(L_t) - 1). \tag{1.3.8}$$

Now consider a quadrilateral $\{P, Q, R, S\}$, the points P_t, Q_t as in the NPC comparisons, and assume geodesic extendability and $-1 \leq K \leq 0$. We will derive an upper bound for d_{RS} in terms of $d_{PQ}, d_{P_tQ_t}, d_{PS}, d_{QR}$. Let U be the point on the geodesic extension of the ray $\overline{PQ_t}$ for which $Q_t = (1-t)P + tU$. Let $U_t = (1-t)Q + tU$. Then from the NPC triangle comparison or using (1.3.5) on the triangle UPQ one knows that

$$d(Q_t, U_t) \leq (1-t)d_{PQ}. \tag{1.3.9}$$

Applying (1.3.8) to the triangle QRU implies

$$ch(d_{RU}) - 1 \leq \frac{sh(d_{QU})sh(d_{QR})}{sh(td_{QU})sh(td_{QR})}(ch(d_{U_tQ_t}) - 1). \tag{1.3.10}$$

Using (1.3.8) on the triangle PSU implies

$$ch(d_{SU}) - 1 \leq \frac{sh(d_{PU})sh(d_{PS})}{sh(td_{PU})sh(td_{PS})}(ch(d_{P_tQ_t}) - 1). \tag{1.3.11}$$

The combination of the three inequalities above and the fact that $d_{RS} \leq d_{RU} + d_{SU}$ yield an estimate above for d_{RS}. Let D be the maximum of 1 and the pairwise distances between the four points P, Q, P_t, Q_t, and let δ be the maximum of just the two distances $d_{PQ}, d_{P_tQ_t}$. Then one uses the triangle inequality, the functional relation $f(2x) = 2f^2(x) + 4f(x)$ for $f(x) = chx - 1$, and various crude estimates to obtain

$$ch(d_{RS}) - 1 \leq 12exp(\frac{5D}{t})(ch(\delta) - 1). \tag{1.3.12}$$

1.4. A Poincaré inequality for maps to metric spaces.

In this section we prove a general version of the Poincaré inequality for maps to metric spaces. For functions this was first proven for $p = 2$ by Li and Yau[LY]. For $p = 1$ it was proven by Gromov[G1]. Note that for functions, the result for one value of p implies it for all larger p. The bound for $p = 2$ is equivalent to a lower bound on the first Neumann eigenvalue of the Laplacian. Our method of proof is substantially simpler than the original proofs, and our constants, although not sharp, are explicitly given. Recall that the p-energy is discussed in [KS], and the p-energy density was denoted $|\nabla u|_p$, so that $|\nabla u|_2 = |du|^2$. We follow this notation in this section.

Theorem 1.4.1. *Let $\Omega \subseteq (M,g)$ be a smooth compact domain with locally convex boundary. Assume that there is a constant $k \geq 0$ so that on Ω we have $Ric(g) \geq -(n-1)k \cdot g$. Define the distance between points of Ω as the infimum of path lengths for curves lying in Ω, and assume that $diam(\Omega) \leq D$ for a constant D. If $u \in W^{1,p}(\Omega, X)$ where (X,d) is a complete metric space, then we have*

$$\inf_{Q\in X} \int_\Omega d^p(u(x),Q)d\mu(x) \leq c(n,p,k,D)\int_\Omega |\nabla u|_p d\mu$$

The constant appearing in the Poincaré inequality may be taken to be $c(n,p,k,D) = c(n,p)D^p 2^{n-1} e^{\frac{n-1}{2}\sqrt{k}D}$ if $k > 0$, and $c(n,p,0,D) = c(n,p)D^p 2^{n-1}$, where $c(n,p)$ is a constant which is equal to one if X is the real line, or if $p = 2$, and X has curvature bounded from above.

Proof. We consider the double integral

$$\int_\Omega \int_\Omega d^p(u(x_0),u(x_1))d\mu(x_0)d\mu(x_1). \tag{1.4.1}$$

Now for almost all pairs $(x_0,x_1) \in \Omega \times \Omega$ there is a unique minimizing geodesic which we parametrize by constant speed, and denote $\{x_t\}_{0\leq t\leq 1}$. Thus the tangent vector $\dot{x}_t$ has length $d(x_0,x_1)$. The restriction of u to x_t is Borel measurable, and we have for almost all (x_0,x_1) that $u \circ x_t \in W^{1,p}((0,1),X)$, and

(1.4.2)

$$d^p(u(x_0),u(x_1)) \leq \int_0^1 \|u_*(\dot{x}_t)\|^p dt \leq c(n,p)d^p(x_0,x_1)\int_0^1 |\nabla u|_p(x_t)dt.$$

The constant $c(n,p)$ can be chosen to be one if X is the real line, or if X has curvature bounded from above because in this case the square of the directional derivative is bounded by the energy(see[KS]). For general image spaces X, see[KS] for a discussion of $c(n,p)$. To verify (1.4.2), first observe that we may assume that $x_0 \neq x_1$ since the diagonal of $X \times X$ is a set of measure zero, and we may also assume that $\bar{x_0}, \bar{x_1}$ are not cut points. Then, let $f(x_0, x_1)$ be a nonnegative smooth function supported in a small neighborhood of such a pair $(\bar{x_0}, \bar{x_1})$. The function $\zeta(t)$ is defined by

$$\zeta(t) = \int_\Omega \int_\Omega d(u(x_0), u(x_t)) f(x_0, x_1) d\mu(x_0) d\mu(x_1).$$

By the change of variable $(x_0, x_1) \to (x_0, x_t)$ on sees that $\zeta(t)$ is continuous on $[0,1]$, and C^1 on $(0,1)$. In particular, since $\zeta(0) = 0$, we have

$$\zeta(1) = \int_0^1 \zeta'(t) dt \tag{1.4.3}$$

Now, for a fixed x_0 and t_0, let F_t be the family of diffeomorphisms of a neighborhood of $\bar{x}_{t_0}$ given by $F_t(x_{t_0}) = x_{t_0+t}$. If we denote by Y the vector field which is the derivative of F_t at $t = 0$, then we have by a change of variables

$$\int_\Omega d(u(x_0), u(x_t)) f(x_0, x_1) d\mu(x_1) = \int_\Omega d(u(x_0, u(x_{t_0}))(F_t^{-1})^*(f_{x_0} d\mu)(x_{t_0})$$

where $f_{x_0}(y) = f(x_0, y)$. Differentiating in t and setting $t = 0$, we get the expression for the derivative of the integral on the left at $t = t_0$

$$-\int_\Omega d(u(x_0), u(x_{t_0})) \mathcal{L}_Y (f_{x_0} d\mu)(x_{t_0})$$

where $\mathcal{L}_Y$ denotes Lie differentiation along Y. Integration by parts then shows

$$\frac{d}{dt} \int_\Omega d(u(x_0), u(x_t)) f(x_0, x_1) d\mu(x_0) d\mu(x_1)$$
$$= \int_\Omega \mathcal{L}_Y d(u(x_0, u(x_t)) f(x_0, x_1) d\mu(x_1).$$

Now we have

$$|\mathcal{L}_Y d(u(x_0), u(x_t))| = |\frac{d}{dt} d(u(x_0), u(x_t))| \leq \|u_*(\dot{x}_t)\|.$$

Therefore we conclude

$$\zeta'(t) \le \int_\Omega \int_\Omega \|u_*(\dot{x}_t)\| f(x_0, x_1) d\mu(x_0) d\mu(x_1).$$

Combining this with (1.4.3) we then have

$$\int_\Omega \int_\Omega f(x_0, x_1)[d(u(x_0), u(x_1)) - \int_0^1 \|u_*(\dot{x}_t)\| dt] d\mu(x_0) d\mu(x_1) \le 0$$

for any f as above. Since $(\bar{x}_0, \bar{x}_1)$ were arbitrarily chosen points in an open set of full measure, this proves (1.4.2) with $p = 1$, and it follows for $p > 1$ by the Hölder inequality.

Using (1.4.2) to estimate the integral (1.4.1), we therefore have

$$\mathrm{Vol}(\Omega) \inf_{Q \in X} \int_\Omega d^p(u(x), Q) d\mu(x) \le \int_\Omega \int_\Omega d^p(u(x_0), u(x_1)) d\mu(x_0) d\mu(x_1)$$

$$\le c(n,p) D^p \int_\Omega \{ \int_\Omega \int_0^1 |\nabla u|_p(x_t) dt d\mu(x_1) \} d\mu(x_0).$$

Note that we have by the change of variable $(x_0, x_1) \to (x_1, x_0)$ we have

$$\int_\Omega \int_\Omega \int_0^{1/2} |\nabla u|_p(x_t) dt d\mu(x_0) d\mu(x_1) = \int_\Omega \int_\Omega \int_{1/2}^1 |\nabla u|_p(x_t) dt d\mu(x_0) d\mu(x_1).$$

Therefore the right hand side of the previous inequality may be replaced by

$$2c(n,p) D^p \int_\Omega \{ \int_\Omega \int_{1/2}^1 |\nabla u|_p(x_t) dt d\mu(x_1) \} d\mu(x_0).$$

We now use the Bishop comparison theorem to obtain for $t \in [1/2, 1]$ and $k > 0$

$$\frac{d\mu(x_1)}{d\mu(x_t)} \le \frac{\sinh^{n-1}(\sqrt{k} d(x_0, x_1))}{\sinh^{n-1}(\sqrt{k} t d(x_0, x_1))}.$$

By elementary estimation this implies

$$\frac{d\mu(x_1)}{d\mu(x_t)} \le 2^{n-1} e^{\frac{n-1}{2}\sqrt{k} D}$$

for any $t \in [1/2, 1]$. If $k = 0$, then we get

$$\frac{d\mu(x_1)}{d\mu(x_t)} \le 2^{n-1}.$$

Let $c_1(n,k)$ be $e^{\frac{n-1}{2}\sqrt{k}D}$ if $k > 0$, and $c_1(n,0) = 2^{n-1}$. Using this bound, and Fubini's theorem we then have

$$\mathrm{Vol}(\Omega) \inf_{Q\in X} \int_\Omega d^p(u(x),Q)d\mu(x) \le c(n,p)D^p c_1(n,k)\mathrm{Vol}(\Omega) \int_\Omega |\nabla u|_p d\mu$$

where we have replaced

$$\int_{\Omega_{x_0,t}} |\nabla u|_p(x_t)d\mu(x_t)$$

with

$$\Omega_{x_0,t} = \{exp_{x_0}(tv) : v \in T_{x_0}\Omega,\ d(x_0, exp_{x_0}(v)) = \|v\|\}$$

by the total integral of $|\nabla u|_p$. This establishes the desired Poincaré inequality, and completes the proof of Theorem 1.4.1.

The Poincaré inequality with a rough constant can be established for any compact connected domain from Theorem 1.4.1.

Corollary 1.4.2. *Let Ω be a compact, smooth, connected domain in a Riemannian manifold. There is a constant c depending on p and Ω such that for any $u \in W^{1,p}(\Omega, X)$ with (X,d) a complete metric space, we have*

$$\inf_{Q\in X} \int_\Omega d^p(u(x),Q)d\mu(x) \le c \int_\Omega |\nabla u|_p d\mu.$$

Proof. We first make a conformal deformation of the metric g so that $\partial\Omega$ becomes locally convex. It is an easy calculation that if we let v be a smooth function in a neighborhood of Ω, and we consider the metric $\bar{g} = e^{2v}g$, then the second fundamental form of $\partial\Omega$ is given by

$$\bar{h}_{ij} = e^v(h_{ij} - \frac{\partial v}{\partial \nu} g_{ij})$$

in a local basis on $\partial\Omega$, where ν denotes the inward unit normal vector for $\partial\Omega$ with respect to g. We see that we can choose v so that $(\Omega, \bar{g})$ has locally convex boundary. Since Ω is connected, if we choose distance to be the path distance for paths in Ω, then the diameter is finite, and by Theorem 1.4.1 we have the Poincaré inequality with respect $\bar{g}$. Since the metrics g and $\bar{g}$ are uniformly equivalent, the p-energy of a map with respect to $\bar{g}$ is bounded by a constant times the p-energy with respect to g. Therefore we have

$$\inf_{Q\in X} \int_\Omega d^p(u,Q)d\bar{\mu} \le c \int_\Omega |\nabla u|_p d\mu.$$

Since the volume forms for $g, \bar{g}$ are bounded in ratio, this implies the desired inequality. This completes the proof of Corollary 1.4.2.

1.5. Mollification.

It is useful to mollify maps to NPC spaces. Let $\eta \in C_0^\infty(B(0,1))$ be a non-negative, monotone radial test function of total integral 1. Define

$$\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\frac{x}{\varepsilon}). \tag{1.5.1}$$

For $z \in \Omega_\varepsilon$ define $u * \eta_\varepsilon(z)$ to be the center of mass of u with respect to the push-forward of the measure $\eta_\varepsilon(x)dx$ induced by the exponential map $exp_z : B(0,\varepsilon)_z \to \Omega$.

In §2.5-2.6 of our earlier paper [KS] we mollified maps (using normalized characteristic functions of geodesic balls instead of smooth approximate identitities), and derived estimates to show that mollification essentially decreases the energy of Sobolev maps; the estimates we obtained can be derived quickly in the classical case (of Euclidean domains and maps to $\mathbf{R}$) by writing

$$D(u * \eta_\varepsilon) = Du * \eta_\varepsilon. \tag{1.5.2}$$

In the classical setting the equality (1.5.2) and Jensen's inequality immediately imply

$$|D(u * \eta_\varepsilon)|^2 \leq |Du|^2 * \eta_\varepsilon.$$

It is the generalization of this estimate which we derived in [KS], at least for characteristic function mollifiers. The estimate we derive now corresponds classically to one obtainable by writing

$$D(u * \eta_\varepsilon) = u * D\eta_\varepsilon. \tag{1.5.3}$$

This present (stronger) result gives meaningful Sobolev estimates for mollifications of L^2 maps u, of the nature that the Sobolev energy of an ε-mollification of u is bounded above by a corresponding ε-approximate energy for u.

In order to derive the mollification estimate first recall elements of the Sobolev space theory in Chapter 1 of [KS]: Given a map $u \in L^2(\Omega, X)$ the (spherically averaged) ε-approximate energy density function is given by

$$e_\varepsilon(x) = \frac{1}{\omega_n} \int\limits_{S(x,\varepsilon)} \frac{d^2(u(x), u(y))}{\varepsilon^2} \frac{d\sigma(y)}{\varepsilon^{n-1}} \tag{1.5.4}$$

(for $x \in \Omega_\varepsilon$, i.e. for $d(x, \delta\Omega) > \varepsilon$). Letting ν be any Borel measure on the interval $(0, 2)$ satisfying

$$(1.5.5) \qquad \nu \geq 0, \qquad \nu((0,2)) = 1, \qquad \int_0^2 \lambda^{-2} d\nu(\lambda) < \infty,$$

one defines the approximate energy density function ${}_\nu e_\varepsilon(x)$ by averaging the spherical averages $e_\varepsilon(x)$:

$$(1.5.6) \qquad {}_\nu e_\varepsilon(x) = \int_0^2 e_{\lambda\varepsilon}(x) d\nu(\lambda),$$

for $x \in \Omega_{2\varepsilon}$ (and ${}_\nu e_\varepsilon(x) = 0$ otherwise). Then the functions ${}_\nu e_\varepsilon(x)$ are in $L^1(\Omega, \mathbf{R})$, and one defines the functionals ${}_\nu E_\varepsilon^u(f) = {}_\nu E_\varepsilon(f)$ by

$$(1.5.7) \qquad {}_\nu E_\varepsilon(f) = \int_\Omega f(x)\, {}_\nu e_\varepsilon(x)\, d\mu(x),$$

for $f \in C_c(\Omega)$. The map u is defined to be in $W^{1,2}(\Omega, X)$ if and only if (there is some ν for which)

$$(1.5.8) \qquad \sup_{\substack{f \in C_c(\Omega) \\ 0 \leq f \leq 1}} \left(\limsup_{\varepsilon \to 0} \, {}_\nu E_\varepsilon(f) \right) \equiv {}_\nu E < \infty.$$

In this case the expression above is finite (and the same) for all choices of ν and the measures ${}_\nu e_\varepsilon(x) d\mu$ all converge weakly (as $\varepsilon \to 0$) to an absolutely continuous energy density measure $|du|^2(x) d\mu$.

Let Z be a Lipschitz vector field on $\overline{\Omega}$, $Z \in \Gamma(T\overline{\Omega})$, and let $\overline{x}(x, t)$ denote the flow along Z at time t, starting at point x. For points sufficiently interior to Ω one can define the ε-energy density function (of u in the direction Z) by

$$(1.5.9) \qquad {}^Z e_\varepsilon(x) = \frac{d^2(u(x), u(\overline{x}(x, \varepsilon)))}{\varepsilon^2}.$$

For a non-negative Borel measure ν satisfying (1.5.5) we can construct other approximate directional energies ${}^Z_\nu e_\varepsilon(x)$, as in (1.5.6). If $u \in W^{1,2}(\Omega, X)$ then the measures ${}^Z_\nu e_\varepsilon(x) d\mu(x)$ all converge weakly to a directional energy, denoted by $|u_*(Z)|^2 d\mu$. For a Riemannian domain on which there is a global frame $\{e_1, \dots e_n\}$, one can identify $\omega \in S^{n-1}$ with the vector field $\omega^i e_i$, and show that

$$|du|^2(x) = \frac{1}{\omega_n} \int_{S^{n-1}} |u_*(\omega)|^2 d\sigma(\omega) \tag{1.5.10}$$

almost everywhere.

Next recall the well-known fact (e.g. Lemma 2.5.1 in [KS]):

Lemma. *Let $(\mathcal{M}, \nu)$ be a probability measure space, let (X, d) be an NPC space, and let $u \in L^2(\mathcal{M}, X)$. Then there exists a unique center of mass $\overline{u} = \overline{u}_\nu$ for u, defined as the point in X which minimizes the integral*

$$I_{u,\nu}(Q) = \int_{\mathcal{M}} d^2(u(m), Q) d\nu(m).$$

It follows from the quadrilateral comparison estimate (1.3.3) that one can estimate the distance between the center of mass $\overline{u} = \overline{u}_\nu$ of a map u with respect to the probability measure ν, and the center of mass $\overline{v} = \overline{v}_{\nu'}$ of a map v with respect to a measure ν', in terms of the L^2 distances between u, v and the difference of the measures ν, ν'. Such estimates allow one to estimate energies of mollified maps. An estimate of this form appears as Proposition 2.5.2 in [KS], and the same procedure as is employed there implies:

Proposition 1.5.1. *Let $\mathcal{M}$ be a measure space, and let ν, ν' be two probability measures on $\mathcal{M}$. Suppose u, v are in $L^2(\mathcal{M}, X)$ for both measure choices. Let $\overline{u}$ and $\overline{v}$ be the resulting centers of mass, as above. Write $\overline{v}_t = (1-t)\overline{v} + t\overline{u}$. Then*

$$d^2(\overline{u}, \overline{v}) \le d(\overline{u}, \overline{v}) \int_{\{u \ne v\}} d(u, v) d\nu + \frac{1}{2} \limsup_{t \to 0} \int_{\mathcal{M}} \frac{d^2(v, \overline{v}) - d^2(v, \overline{v}_t)}{t} (d\nu - d\nu').$$

Proof. Using the notation from the preceding lemma we have

$$I_{u,\nu}(\overline{u}) \le I_{u,\nu}(\overline{v}_{(1-t)}), \qquad I_{v,\nu'}(\overline{v}) \le I_{v,\nu'}(\overline{v}_t).$$

Hence

(1.5.11)

$$\int d^2(u, \overline{u}) + d^2(v, \overline{v}) \ d\nu \le \int d^2(u, \overline{v}_{(1-t)}) + d^2(v, \overline{v}_t) \ d\nu + \int [d^2(v, \overline{v}) - d^2(v, \overline{v}_t)](d\nu - d\nu').$$

Estimate the integrand which appears in the first term of the right side by using Euclidean distance comparison: For each $m \in \mathcal{M}$ we construct the quadrilateral with consecutive vertices $u(m), \overline{u}, \overline{v}, v(m)$ and apply (1.3.3) with $\alpha = 1$:

(1.5.12)
$$\begin{aligned} d^2(u, \overline{v}_{1-t}) + d^2(v, \overline{v}_t) \leq\ & d^2(u, \overline{u}) + d^2(v, \overline{v}) \\ & - 2td^2(\overline{u}, \overline{v}) + 2td(u, v)d(\overline{u}, \overline{v}) \\ & + 2t^2 d^2(\overline{u}, \overline{v}). \end{aligned}$$

Integrating (1.5.12) with respect to ν yields a bound for the first term on the right of (1.5.11). Divide the resulting inequality by $2t$, cancel the order zero terms, and let $t \to 0$ in order to obtain Proposition 1.5.1 □

Let $f \in C_c(\Omega_{2\varepsilon}), f \geq 0$. Define the auxillary function f_ε by

(1.5.13)
$$\begin{aligned} f_\varepsilon(x) =& f(x) + \omega(f)(x, \varepsilon) \\ \omega(f)(x, \varepsilon) =& \max_{|y-x| \leq \varepsilon} |f(y) - f(x)|. \end{aligned}$$

Theorem 1.5.2. *Let Ω be a compact Riemannian domain, and let $u \in L^2(\Omega, X)$. Then for $\varepsilon > 0$ small (depending on (Ω, g)) the map $u * \eta_\varepsilon$ is Lipschitz in Ω_ε. The Lipschitz constant depends on ε, on the L^2 oscillation of u, and on the Riemannian structure of (Ω, g). If the domain Ω is Euclidean, then for any non-negative test function f as above*

(1.5.14)
$$E^{u*\eta_\varepsilon}(f) \leq {}_\nu E^u_\varepsilon(f_\varepsilon).$$

Here the measure $d\nu(\lambda)$ is determined from the mollifier η, is absolutely continuous with repsect to $d\lambda$ and satisfies (1.5.5). In case Ω is a general Riemannian domain there is a $C > 0$ depending only on (Ω, g) so that

(1.5.15)
$$E^{u*\eta_\varepsilon}(f) \leq {}_\nu E^u_\varepsilon(f_\varepsilon) + C\varepsilon({}_\mu E_\varepsilon(f_\varepsilon)).$$

The measure ν in the non-Euclidean estimate is the same as in the Euclidean case. The measure μ also satisfies (1.5.5).

Proof. To motivate the calculations which follow consider the classical case of a real-valued function u defined on a Euclidean domain. Then we may differentiate $u * \eta$ in the x^1-direction, at the point z, and use the symmetry of η to write

$$\begin{aligned}(u*\eta)_{,1}(z) &= \int\limits_{B(0,1)} u(z-x)\eta_{,1}(x)dx \\ &= \int\limits_{B(0,1)} u(z+x)(-\eta_{,1}(x))dx.\end{aligned}$$

Since $\eta_{,1}(x)$ is an odd function with respect to reflection across the plane $x^1 = 0$, we may rewrite the integral above as

$$\begin{aligned}(u*\eta)_{,1}(z) &= \int\limits_{x^1>0} (u(z+\tilde{x}) - u(z+x))\eta_{,1}(x)dx \\ &= \int\limits_{x^1>0} \frac{(u(z+\tilde{x}) - u(z+x))}{2x^1} 2x^1\eta_{,1}(x)dx,\end{aligned}$$

where $\tilde{x}$ is the reflection of x across the $x^1 = 0$ plane. Integrate by parts to notice that the measure $-2x^1\eta_{,1}(x)$ is a probability measure on the set $x^1 > 0$. Therefore we may use Jensen's Inequality to conclude

(1.5.16)
$$|(u*\eta)_{,1}(z)|^2 \leq \int\limits_{x^1>0} \frac{|u(z+\tilde{x}) - u(z+x)|^2}{(2x^1)^2} 2x^1|\eta_{,1}(x)|dx.$$

In other words, the square of the directional derivative of the convolution is bounded above by a weighted average of approximate directional energies (at slightly displaced points). This classical estimate is the basis of the results stated in Theorem 1.5.2. Of course, in the setting of maps to NPC spaces, derivative estimates must be obtained as limits of difference estimates, and final results will be in integral, not pointwise, form.

We now begin the general proof. Consider first the case of Euclidean domain Ω. Since both sides of (1.5.14) scale identically with respect to domain homotheties it suffices to show the inequality in case $\varepsilon = 1$. In order to estimate the left side we consider $\vec{\delta} \in \mathbf{R}^n$ small and use Proposition 1.5.1 to estimate $d^2(u*\eta(z), u*\eta(z+\vec{\delta}))$. We write $w = z + \vec{\delta}$ and $\overline{u} = u * \eta$. Rotate and translate coordinates so that $z = 0$ and $\vec{\delta} = \delta\frac{\partial}{\partial x^1}$. Taking $u = v$, $d\nu(x) = \eta(x)dx$, $d\nu'(x) = \eta(x - \vec{\delta})dx$, writing

$$\overline{v}_t = (1-t)\overline{u}(w) + t\overline{u}(z)$$

we obtain

(1.5.17)

$$d^2(\overline{u}(z),\overline{u}(w)) \leq \frac{1}{2} \limsup_{t\to 0} \int_{B(0,1)} \frac{d^2(u(x),\overline{u}(w)) - d^2(u(x),\overline{v}_t)}{t} (\eta(x) - \eta(x-\vec{\delta}))dx.$$

For $x = (x^1, \dots x^n) = (x^1, y) \in \mathbf{R}^n$ write $\tilde{x} = (-x^1, y)$ for the reflection of x through the $y-$plane. Then since η is radial,

$$\eta(x) - \eta(x - \vec{\delta}) = \delta\eta_{,1}(x) + O(\delta^2)$$

is within $O(\delta^2)$ of being an odd function with respect to the reflection $x \to \tilde{x}$. We take advantage of this symmetry in estimating the right side of (1.5.17):

(1.5.18)

$$\begin{aligned} d^2(\overline{u}(z),\overline{u}(w)) \leq \limsup_{t\to 0} \frac{1}{2t} \int_{B(0,1)\cap\{x^1>0\}} & [-d^2(u(x),\overline{u}(w)) + d^2(u(x),\overline{v}_t) \\ & + d^2(u(\tilde{x}),\overline{u}(w)) - d^2(u(\tilde{x}),\overline{v}_t)][-\delta\eta_{,1}(x)]dx \\ + C\delta^2 \limsup_{t\to 0} \frac{1}{t} \int_{B(0,1)} & |d^2(u(x),\overline{u}(w)) - d^2(u(x),\overline{v}_t)|dx. \end{aligned}$$

Applying (1.3.6) with $\alpha = 1$ and $P = u(x), Q = \overline{u}(w), R = \overline{v}_t, S = u(\tilde{x})$, we estimate

(1.5.19)

$$\begin{aligned} -d^2(u(x),\overline{u}(w)) + d^2(u(x),\overline{v}_t) + d^2(u(\tilde{x}),\overline{u}(w)) - d^2(u(\tilde{x})\overline{v}_t) \\ \leq 2d(u(x),u(\tilde{x}))d(\overline{v}_t,\overline{u}(w)) \\ = 2td(u(x),u(\tilde{x}))d(\overline{u}(z),\overline{u}(w)). \end{aligned}$$

To bound the $O(\delta^2)$ error term in (1.5.18) we use the triangle inequality to estimate

(1.5.20)

$$\begin{aligned} |d^2(u(x),\overline{u}(w)) - d^2(u(x),\overline{v}_t))| \\ \leq (d(u(x),\overline{u}(w)) + d(u(x),\overline{v}_t))td(\overline{u}(z),\overline{u}(w)). \end{aligned}$$

Combining the estimates (1.5.19),(1.5.20) into (1.5.18), and dividing the result by $\delta d(\overline{u}(z),\overline{u}(w))$ yields

(1.5.21)
$$\frac{d(\overline{u}(z),\overline{u}(w))}{\delta} \leq \int\limits_{x^1>0} d(u(x),u(\tilde{x}))|\eta_{,1}(x)|dx + C\delta \int\limits_{B_1(0)} d(u,\overline{u}(w)) + d(u,\overline{u}(z))dx.$$

As noted previously, integration by parts verifies that

$$\int\limits_{x^1>0} 2x^1(-\eta_{,1})dx = 1.$$

Writing I_1, I_2 for the two integrals on the right of (1.5.21) and applying Jensen's inequality three times we estimate (for a larger C):

(1.5.22)
$$\begin{aligned}\frac{d^2(\overline{u}(z),\overline{u}(w))}{\delta^2} &\leq \frac{1}{1-\delta}I_1^2 + \frac{1}{\delta}(C\delta I_2)^2 \\ &\leq \frac{1}{1-\delta}\int\limits_{x^1>0} \frac{d^2(u(x),u(\tilde{x}))}{(2x^1)^2} 2x^1|\eta_{,1}(x)|dx \\ &\quad + C\delta \int\limits_{B_1(0)} d^2(u,\overline{u}(w)) + d^2(u,\overline{u}(z))dx.\end{aligned}$$

Notice the similarity between (1.5.22) and (1.5.16). The second integral on the right of (1.5.22) may be estimated from above very crudely via the triangle inequality and the minimizing property of convolutions, and is bounded by

$$C\int\limits_{\Omega} d^2(u(z),Q)d\mu(z)$$

for any $Q \in X$, and so is bounded for the L^2 map u. Letting $f \in C_c(\Omega_1), f \geq 0$, we integrate the pointwise estimate (1.5.22):

(1.5.23)

$$\begin{aligned}&\int_\Omega f(z)\frac{d^2(\overline{u}(z),\overline{u}(z+\vec{\delta}))}{\delta^2}d\mu(z)\\&\leq \frac{1}{1-\delta}\int_{z\in\Omega}\int_{x^1>0} f(z)\frac{d^2(u(z+\tilde{x}),u(z+x))}{(2x^1)^2}2x^1|\eta_{,1}(x)|dxd\mu(z)+C\delta|f|_\infty\\&\leq \frac{1}{1-\delta}\int_{\zeta\in\Omega} f_1(\zeta)\int_{x^1>0}\frac{d^2(u(\zeta),u(\zeta+2x^1))}{(2x^1)^2}2x^1|\eta_{,1}(x)|dx\ d\mu(\zeta)+C\delta|f|_\infty.\end{aligned}$$

The second inequality in (1.5.23) uses Fubini's Theorem and the change of variables $\zeta = z+\tilde{x}$. Notice that this last integral is an $\varepsilon = 1$ directional energy functional, for the function $f_1(\zeta)$, with approximate energy density function ${}^{\delta_1}_{\nu}e_1(\zeta)$, where we may write $dx = dx^1dy$, $2x^1 = \lambda$ to express

$$d\nu(\lambda) = \lambda\Big(\int_{x^1=\frac{\lambda}{2}} |\eta_{,1}(x)|dy\ \Big)\frac{d\lambda}{2} := \nu(\lambda)d\lambda. \tag{1.5.24}$$

Evidently ν satisfies the integrability conditions (1.5.5). Letting $\delta\to 0$ in (1.5.23) we deduce

(1.5.25)

$$\int_\Omega f(z)|(\overline{u})_*(\delta_1)|^2d\mu(z) \leq \int_\Omega f_1(z)\ {}^{\delta_1}_{\nu}e_1(z)d\mu(z).$$

(In the case of a map to $\mathbf{R}$ this estimate would have followed immediately by integrating the point-wise estimate (1.5.16).) Averaging over the unit-sphere of directions, rescaling from 1 back to ε, and applying (1.5.10), we obtain

(1.5.26)

$$\int_\Omega f(z)|d(u*\eta_\varepsilon)|^2(z)d\mu(z) \leq \int_\Omega f_\varepsilon(z)\ {}_{\nu}e_\varepsilon(z)d\mu(z),$$

which is exactly (1.5.14).

We now consider a Riemannian domain (Ω, g), and modify the Euclidean argument above accordingly. It suffices to prove (1.5.15) for a local coordinate chart. This is because Ω can be covered by a finite number of charts and so one can create a smooth subordinate partition of unity $\{\phi_i\}$, apply (1.5.15) to the functions $\phi_i f$, and add the resulting estimates to get the desired general result (for a larger value of C).

We again scale the Riemannian domain (i.e. a particular coordinate chart) so that ε corresponds to 1. Thus if we renormalize an original orthonormal frame on Ω we obtain a global frame $\{e_1, \ldots, e_n\}$ having covariant derivative of order ε. (Also, the new domain curvature is of order ε^2.) As usual we identify $x \in \mathbf{R}^n$ (resp. $\omega \in S^{n-1}$) with the vector field $x^i e_i$ (resp. $\omega^i e_i$). In this scaling the statement which corresponds to (1.5.15) is

$$E^{\overline{u}}(f) \leq {}_\nu E_1^u(f_1) + C\varepsilon \, {}_\mu E_1^u(f_1). \tag{1.5.27}$$

Fix $z \in \Omega_1$ and write $w = exp(z, \vec{\delta})$. We must first estimate $d^2(\overline{u}(z), \overline{u}(w))$. Formally,

(1.5.28)
$$\overline{u}(z) = \int_{B(0,1)_z} u(exp(z,x))\eta(x)dx, \quad \overline{u}(w) = \int_{B(0,1)_w} u(exp(w,y))\eta(y)dy.$$

To compare these centers of mass we must understand the change of coordinates map $\tau : x \to y$ between the two tangent space coordinate charts. Precisely this map is defined by $y = \tau(x)$ if and only if $exp(z,x) = exp(w,y)$. For each tangent space $T\Omega_z, T\Omega_w$ pick Euclidean coordinate directions $\{\partial_i\}$, $\{\overline{\partial}_i\}$ to be consistent with the frame $\{e_1, \ldots, e_n\}$. That is,

$$exp(z,0)_*\partial_i = e_i, \quad exp(w,0)_*\overline{\partial}_i = e_i.$$

Identifying $\{\partial_i\}$ with $\{\overline{\partial}_i\}$in the two coordinate charts we see that τ is a well-defined diffeomorphism on a domain ball of radius having order $\frac{1}{\varepsilon}$. By studying geodesic triangles and Jacobi fields it is then possible to verify that for our orientation of the two tangent-space coordinate charts one has

$$\begin{aligned} D\tau(x) &= I + O(\varepsilon\delta) \\ \tau(x) &= x - \vec{\delta} + O(\varepsilon\delta), \end{aligned} \tag{1.5.29}$$

uniformly for $x \in B(0,1)_z$. Here $D\tau$ is the Jacobian matrix of the change of variables map τ, and the second estimate in (1.5.29) follows by integrating the first one. Using the estimates above for the change of variables $y = \tau(x)$ in (1.5.28) we see that we are computing centers of mass for the map

$u(exp(z,x))$ with respect to measures

$$(1.5.30)\qquad \begin{aligned} d\nu &= \eta(x)dx \\ d\nu' &= \eta(x-\vec{\delta})dx + O(\varepsilon\delta)dx. \end{aligned}$$

Proceed now as in the earlier Euclidean estimate, through steps analogous to (1.5.17) - (1.5.21). Because of the additional error term in $d\nu'$ (1.5.30), one obtains an estimate similar to (1.5.21), except that the coefficient of the second integral on the right side of the modified estimate is now $C(\delta+\varepsilon)$ instead of just $C\delta$. Thus one deduces the analog of (1.5.22):

(1.5.31)
$$\frac{d^2(\overline{u}(z),\overline{u}(w))}{\delta^2} \leq \frac{1}{1-\delta-\varepsilon}\int\limits_{x^1>0} \frac{d^2(u(exp(z,x)),u(exp(z,\tilde{x})))}{(2x^1)^2}2x^1|\eta_{,1}(x)|dx + C(\delta+\varepsilon)\int\limits_{B(0,1)_x} d^2(u,\overline{u}(w))+d^2(u,\overline{u}(z))dx.$$

Writing

$$d(u(exp(z,x)),\overline{u}(z)) \leq d(u(exp(z,x)),u(exp(z,y))) + d(u(exp(z,y)),\overline{u}(z))),$$

multiplying by $\eta(y)$, integrating with respect to y and then x (and doing an analogous estimate for $d(u(exp(z,x)),\overline{u}(w)))$ one obtains an estimate for the second integral in (1.5.31). Because of the center of mass properties of $\overline{u}(z),\overline{u}(w)$ one can replace these two values in the estimate with $u(x)$ and maintain its validity. In this way one obtains the bound

(1.5.32)
$$C\int\limits_{B(0,1)}\int\limits_{B(0,1)} d^2(u(exp(z,x)),u(exp(z,y)))dxdy$$

for the second integral on the right of (1.5.31). Using Theorem 1.10 in [KS] we have

(1.5.33)
$$E^{\overline{u}}(f) = \lim_{\delta\to 0}\int\limits_{\Omega}\int\limits_{S^{n-1}} f(z)\frac{d^2(\overline{u}(z),\overline{u}(exp(z,\delta\omega)))}{\delta^2}\frac{d\sigma(\omega)}{\omega_n}d\mu(z).$$

Applying the estimates above we obtain (for $\tilde{x} = x - 2(x \bullet \omega)\omega$),

(1.5.34)

$$\begin{aligned} E^{\overline{u}}(f) &\leq (\frac{1}{1-\varepsilon}) \bullet \\ &\int_{\Omega} \int_{S^{n-1}} \int_{x \bullet \omega > 0} f(z) \frac{d^2(u(exp(z,x)), u(exp(z,\tilde{x})))}{(2x \bullet \omega)^2} \times \\ &\qquad \times 2 \frac{(x \bullet \omega)^2}{r} |\eta_r(x)| dx \frac{d\sigma(\omega)}{\omega_n} d\mu(z) \\ &+ C\varepsilon \int_{\Omega} \int_{B(0,1)} \int_{B(0,1)} f(z)[d^2(u(exp(z,x)), u(exp(z,y)))dxdyd\mu(z). \end{aligned}$$

Using the change of variables $\zeta = exp(z,x)$, the second term in the estimate above is easily seen to be majorized by

$$C\varepsilon \int_{\Omega} \int_{B(\zeta,2)} f(\zeta) d^2(u(\zeta), u(\psi)) d\mu(\psi) d\mu(\zeta). \tag{1.5.35}$$

Now estimate the first term on the right of (1.5.34). To this end we perform the following change of coordinates: Map the triple (z, ω, x) to (ζ, ω', x'), where $\zeta = exp(z, \tilde{x})$, and ω' is the direction of the geodesic starting at ζ and passing through $exp(z,x)$. Since ω' is close to ω there is a unique minimal Euclidean rotation $R_{\omega,\omega'}$ taking ω to ω' and leaving the orthogonal complement fixed. Let x' be the scaling of the rotation $R_{\omega,\omega'}(x)$ for which

$$exp(\zeta, 2(x' \bullet \omega')\omega') = exp(z,x).$$

By arguments similar to those of (1.5.29) this scaling factor is within $O(\varepsilon)$ of 1. Write $|x'| = r'$. Note that in the Euclidean case the transformation is given by the volume preserving map

$$\zeta = z + x - 2(x \bullet \omega)\omega, \ \omega' = \omega, \ x' = x.$$

(The fact that $\omega = \omega'$, $x = x'$ in the Euclidean case is the reason that the argument leading to (1.5.26) could be made via (1.5.25).) In our case the global change of variables map is within $0(\varepsilon)$ of being volume preserving. Dominate $f(z)$ by $f_1^0(\zeta)$. Note that

$$\frac{|\eta_r(x)|}{r} \leq \frac{|\eta_r(x')|}{r'} + C\varepsilon.$$

Combining all of these estimates gives the upper bound below for the first integral on the right of (1.5.34): by

(1.5.36)

$$(1+C\varepsilon)\int_{\Omega}\int_{S^{n-1}}\int_{x'\bullet\omega'>0} f_1(\zeta)\frac{d^2(u(\zeta)),u(exp(\zeta,(2x'\bullet\omega')\omega')))}{(2x'\bullet\omega')^2}2\frac{(x'\bullet\omega')^2}{r'}|\eta_r(x')|dx'\frac{d\sigma(\omega')}{\omega_n}d\mu(\zeta)$$

$$+C\varepsilon\int_{\Omega}\int_{S^{n-1}}\int_{x'\bullet\omega'>0} f_1(\zeta)d^2(u(\zeta)),u(exp(\zeta,(2x'\bullet\omega')\omega')))dx'\frac{d\sigma(\omega')}{\omega_n}d\mu(\zeta).$$

The order 1 term in (1.5.36) is precisely ${}_\nu E_1^u(f_1)$. The first order-ε error term is $C\varepsilon\ {}_\nu E_1^u(f_1)$, and the second order-$\varepsilon$ error term is $C\varepsilon$ times an approximate energy functional for $f_1(\zeta)$, where the energy density measure is constructed from a normalizing multiple of

$$d\alpha(\lambda)=\lambda^2(1+(\frac{\lambda}{2})^2)^{\frac{n-1}{2}}d\lambda,$$

so satisfies (1.5.5). Pick μ to be the average of the three measures which arise in the order ε error terms (the two above and the ball averaging measure in (1.5.35)). For this choice of μ we have verified (1.5.27) and the proof of Theorem 1.5.2 is complete. □

An interesting corollary of the Euclidean-domain mollification result (1.5.14) is that one can prove a model geometric rigidity theorem for harmonic maps without the use of a Bochner-type estimate.

Corollary 1.5.3. *Let $\Gamma=\mathbf{Z}^n$ act faithfully on $\mathbf{R}^n$ by translation. Let Γ also act isometrically on an NPC space X. Then any minimizing equivariant map $u:\mathbf{R}^n\to X$ is totally geodesic and flat.*

Proof. Working on the quotient torus T^n of the Euclidean action, we pick $f\equiv 1$, $\varepsilon=1$, and apply (1.5.14) to deduce

$$E^{u*\eta}\leq\ {}_\nu E_1^u.$$

The map u is minimizing (hence Lipschitz [KS]), and we can explicitly expand the expression for ${}_{\nu}E_1^u$ to obtain the following chain of inequalities:

(1.5.37)

$$\begin{aligned} E^u &\le E^{u*\eta} \le {}_{\nu}E_1^u \\ &= \frac{1}{\omega_n}\int_{S^{n-1}}\int_0^2\int_{T^n}\frac{d^2(u(x),u(x+\lambda\omega))}{\lambda^2}dx\ \nu(\lambda)d\lambda d\sigma(\omega) \\ &\le \frac{1}{\omega_n}\int_{S^{n-1}}\int_0^2\int_{T^n}\Big(\int_0^\lambda |u_*(\omega)|(x+t\omega)\frac{dt}{\lambda}\Big)^2dx\ \nu(\lambda)d\lambda d\sigma(\omega) \\ &\le \frac{1}{\omega_n}\int_{S^{n-1}}\int_0^2\int_{T^n}\Big(\int_0^\lambda |u_*(\omega)|^2(x+t\omega)\frac{dt}{\lambda}\Big)dx\ \nu(\lambda)d\lambda d\sigma(\omega) \\ &= E^u. \end{aligned}$$

Therefore all of the inequalities in (1.5.37) are actually equalities. We first deduce that for almost all (x,ω,λ) in $\mathbf{R}^n \times S^{n-1} \times supp(\nu)$ we have

$$d(u(x),u(x+\lambda\omega)) = \int_0^\lambda |u_*(\omega)|(x+t\omega)dt. \tag{1.5.38}$$

Thus for these (x,ω,λ) the image arcs $\{u(x+t\lambda\omega)\}$, $0\le t\le\lambda$, are geodesics. We also know from the application of Jensen's inequality in (1.5.37) that the function $|u_*(\omega)(x+t\omega)|$ is constant in t for almost all (x,ω,λ). Hence for almost all ω almost all image arcs in the ω-direction are constant speed geodesics. Since X is NPC the local-geodesic property implies a global geodesic property, so for almost all ω it is true that the images of almost all ω-direction lines are geodesic lines in X. But since u is Lipschitz the set of lines with geodesic image is closed, also with respect to direction ω. Thus all lines are mapped to constant-speed geodesics. Again applying the Lipschitz property of the map (to arbitrarily long arcs), we see that the speed depends only on the direction ω. Thus the map u is flat, in the sense that the pull-back metric tensor Π is constant. In fact we see that the map u can be factored as a composition of a linear equivariant map to (a possibly lower dimensional) Euclidean space, followed by a metric space isometry to its image in X. □

We remark on the technical limitations of the Bochner approach in trying to prove Corollary 1.5.3. It is immediate from the Remark 2.4.3 in [KS] that the Γ-invariant function of x, $d^2(u(x), u(x+v))$, is subharmonic and hence constant on $\mathbf{R}^n$, for any displacement vector v. Thus one deduces immediately the constancy of the pull-back tensor Π, i.e. the flatness of the map u. Proving that the map is totally geodesic is done classically by showing that the second derivatives are all zero. (Flat does not automatically imply totally geodesic, witness the cylinder-wrapping map of $\mathbf{R}^2$ to $\mathbf{R}^3$). In the NPC setting this would amount to understanding the positive terms in the calculation of (the weak estimate for) $\triangle d^2(u, u_v)$; in the finite-difference Bochner-limit which is indicated in section 2.4 of [KS], this amounts to understanding the positive terms in the calculation of $\triangle|du|^2$. Classically this is where domain and target curvature terms, and second derivative terms, enter the reasoning. The technical problem in trying to follow this approach to prove either the Eells-Sampson or other Bochner formulas is that it is technically daunting to make sense of target curvature and second derivative terms in the NPC setting. This remains a tantalizing problem on which we have made only partial progress.

In light of Corollary 1.5.3 one might wonder whether others of the known vanishing theorems (of the nature that harmonic maps are totally geodesic) have "macroscopic" proofs such as the one exhibited therein. Such proofs may be easier to generalize to the NPC-mapping generality, since they don't rely on pointwise second derivative or curvature computations.

2. Existence theory for manifolds without boundary.

In our earlier paper [KS] we constructed equivariant harmonic maps into arbitrary (NPC) spaces provided the domain manifold has a nonempty boundary, and a Dirichlet boundary condition is specified. The boundary was needed there to anchor the maps, and thus keep them from going to infinity. In this section we prove existence results for manifolds without boundary. These require hypotheses on the action of $\Gamma = \pi_1(M)$ on the (NPC) space X. First recall the terminology. Let (X, d) be an (NPC) space as defined in Section 1.1. If $M, g)$ is a complete Riemannian manifold, then an isometric action of $\Gamma = \pi_1(M)$ on X is a homomorphism $\rho : \Gamma \to \mathrm{Isom}(X)$ where $\mathrm{Isom}(X)$ denotes the isometry group of X. Assuming Γ is finitely generated with a set of generators $\gamma_1, \cdots, \gamma_p$, we let $\delta : X \to \mathbf{R}_+$ be the function

$$\delta(P) = \max_{i=1,\cdots,p} d(\rho(\gamma_i)(P), P).$$

We will call the action of Γ <u>*uniform*</u> if δ is bounded below by a positive constant; i.e. there is a $\delta_0 > 0$ so that $\delta(P) \geq \delta_0$ for all $P \in X$. We will call the action proper if the sublevel sets of the function δ are bounded in X; i.e. if $P_0 \in X$, then for any $L > 0$, there is a number $R > 0$ (depending on L, P_0) so that

$$\{P \in X : \delta(P) \leq L\} \subseteq B_R(P_0).$$

2.1. Proper actions.

We remark that the function δ is a Lipschitz convex function on X. By the triangle inequality we have

$$|\delta(P_1) - \delta(P_2)| \leq 2d(P_1, P_2).$$

To see the convexity, observe that for any $F \in \mathrm{Isom}(X)$ the function $\delta_F(P) = d(F(P), P)$ is convex because, if P_0, P_1 are given, then $F(P_\lambda)$ represents the point which is a fraction λ of the way from $F(P_0)$ to $F(P_1)$. By the quadrilateral comparison theorems of Section 1.3, $\delta_F(P\lambda)$ is a convex function of $\lambda \in [0,1]$. Since δ is a maximum of convex functions, it is a convex function. We note the following corollary of Proposition 1.2.1.

Proposition 2.1.1. *A continuous convex function on an* (NPC) *space achieves its minimum on any closed bounded convex set. In particular, if an action* $\rho : \Gamma \to \mathrm{Isom}(X)$ *is proper and fixed point free, then it is uniform.*

Proof. Let C be a closed bounded convex set, and let $f : X \to \mathbf{R}$ be a convex function. Let I denote the infimum of the function f taken over C. If $I = -\infty$, let C_i denote the subset of C for which $f \leq -2^i$. The sequence $\{C_i\}$ is then a nested sequence of closed bounded convex sets. Applying Proposition 1.2.1, let P be a point common to all of the C_i. It follows that $f(P) \leq -2^i$ for all i which is a contradiction. Thus $I > -\infty$, so we may set $C_i = \{P \in C : f(P) \leq I + 2^{-i}\}$, and a point P common to all of the C_i will satisfy $f(P) = I$ as required.

We see that the function δ achieves its minimum on any closed ball, so in case the action is proper, δ achieves its infimum on all of X. Thus if the action has no fixed point, this infimum is positive, and the action is uniform. This completes the proof of Proposition 2.1.1.

We now prove the main result of this section which is that harmonic equivariant maps exist under the condition that the action is proper. Recall that we consider finite energy maps $u : \tilde{M} \to X$, where $\tilde{M}$ is the universal covering manifold of M, which are equivariant with respect to the action of Γ

on $\tilde{M}$ by deck transformations and the action of Γ on X via ρ. For simplicity we assume that M is compact, and we consider the space $Y = L^2_\rho(\tilde{M}, X)$ of ρ-equivariant, locally L^2 maps. The space Y has a natural distance function d_2 given by

$$d_2(u,v) = (\int_{\tilde{M}/\Gamma} d^2(u(x), v(x)) d\mu(x))^{1/2}$$

where we note that $x \to d^2(u(x), v(x))$ is Γ-invariant on $\tilde{M}$, and hence defines a function on $M = \tilde{M}/\Gamma$. We first show that (Y, d_2) is an (NPC) space.

Lemma 2.1.2. *The space (Y, d_2) is a length space which satisfies* (NPC).

Proof. If $u_0, u_1 \in Y$ we may consider the curve $u_t(x)$, $0 \le t \le 1$ which is a constant speed parametrization of the geodesic from $u_0(x)$ to $u_1(x)$. It is clear that the length of the curve $t \to u_t$ is equal to $d_2(u_0, u_1)$. Thus (Y, d_2) is a length space. Since the (NPC) condition is an inequality which is linear in squares of distance functions, and a geodesic triangle is simply a family of geodesic triangles parametrized on $\tilde{M}$, we may derive (NPC) by integrating. To be precise, suppose $w, u_0, u_1 \in Y$. For almost any $x \in \tilde{M}$ we then have by (NPC) on X

$$d^2(w(x), u_{1/2}(x)) \le \frac{1}{2} d^2(w(x), u_0(x)) + \frac{1}{2} d^2(w(x), u_1(x)) - \frac{1}{4} d^2(u_0(x), u_1(x)).$$

Integrating over $\tilde{M}/\Gamma$ we then get

$$d_2^2(w, u_{1/2}) \le \frac{1}{2} d_2^2(w, u_0) + \frac{1}{2} d_2^2(w, u_1) - \frac{1}{4} d_2^2(u_0, u_1)$$

which yields (NPC) for (Y, d_2). This proves Lemma 1.2.1.

Theorem 2.1.3. *Assume M is compact and ρ is a proper action of $\Gamma = \pi_1(M)$. There exists an equivariant, Lipschitz harmonic map $u : \tilde{M} \to X$.*

Proof. By [KS] there exists a sequence $\{u_i\}$ of equivariant maps with $E(u_i) \ge E(u_{i+1})$,

$$\lim_{i \to \infty} E(u_i) = E_0, \; E_0 = \inf\{E(u) : u : \tilde{M} \to X \quad \rho-\text{equivariant}\}.$$

Moreover the sequence $\{u_i\}$ is locally uniformly Lipschitz. Let C_i, $i = 1, 2, \cdots$ be the closed convex hull in Y of the tail end $\{u_i, u_{i+1}, \cdots\}$ of the sequence. The sequence $\{C_i\}$ is then a nested sequence of closed convex subsets of Y. Since the energy is convex with respect to d_2 on the subset of Y consisting of finite energy maps (see[KS]), and is lower semicontinuous with respect to d_2 convergence, we have

$$\sup_{u \in C_i} E(u) \leq E(u_i) \ \ \forall i.$$

If there is a common point u in all of the C_i, then u is a harmonic equivariant map. In order to find such a u, we need only show (by Proposition 1.2.1) that C_1 is bounded in Y. let $P_0 \in X$ be a chosen point, and let $F \subseteq \tilde{M}$ be a fundamental domain for the action of Γ on $\tilde{M}$. Let $u_0 \in Y$ be the equivariant map such that $u_0(x) = P_0$ for $x \in F$. Because $\bar{F}$ is compact, and the $\{u_i\}$ are uniformly locally Lipschitz we have

$$d(u_i(\gamma_\alpha(x)), u_i(x)) \leq c d_{\tilde{M}}(\gamma_\alpha(x), x)$$

for $x \in F, \ \alpha = 1, \cdots, p$. By the equivariance of u_i we then have for $x \in F$

$$d(\rho(\gamma_\alpha)(u_i(x)), u_i(x)) \leq c.$$

Thus $\delta(u_i(x))) \leq c$ for $x \in F$ and for all i. By the properness of the action we then have $d_2(u_i, u_0) \leq c$ for all i. It follows that $d_2(u, u_0) \leq c$ for all $u \in C_1$, and thus C_1 is bounded as required. This completes the proof of Theorem 2.1.3.

Remark 2.1.4. The proof of Theorem 2.1.3 requires only that there be a bounded minimizing sequence in order to produce a minimizing equivariant map.

Remark 2.1.5. Theorem 2.1.3 holds also under the assumption that M is complete, ρ is proper, and there exists a finite energy ρ-equivariant map $u : \tilde{M} \to X$. It is proved by taking a compact subset $M_0 \subseteq M$, and considering $\tilde{M}_0 = \pi^{-1}(M_0)$ where $\pi : \tilde{M} \to M$ is the covering projection. The existence of the sequence $\{u_i\}$ as above is done in [KS]. Let $Y = L^2_\rho(\tilde{M}_0, X)$, and the same argument as above produces u.

2.2. Locally compact spaces and Hilbert spaces.

In this section we apply the main result of the previous section concerning proper actions to the cases when the space X is locally compact, and when it is a Hilbert space. The result we prove in the locally compact case is essentially known. The proof given here is simplified by our construction of the uniformly Lipschitz minimizing sequence which is already used in the proof of Theorem 1.2.3. We include the result since it follows from that theorem quite directly. First recall that a locally compact (NPC) space has a compactification given in terms of geodesic rays. A ray is a geodesic σ which may be parametrized by arclength on the interval $[0,\infty)$. Two geodesics σ_1, σ_2 are said to be equivalent if the Hausdorff distance between them is finite. We denote by ∂X the set of equivalence of rays, and by $\bar{X}$, the union of X with ∂X. There is a natural Hausdorff topology so that $\bar{X}$ is the closure of X, and $\bar{X}$ is compact. To describe this topology, observe that the (NPC) property implies that there is a unique representative of each equivalence class with initial point at a chosen point $Q \in X$. If σ_0 is a ray with initial point at Q, then a neighborhood basis for σ_0 consists of sets $\mathcal{O}_{\sigma_0,R}$ for $R > 1$ consisting of those equivalence classes of rays whose representative σ with initial point at Q satisfies

$$HD(\sigma_0 \cap B_R(Q), \sigma \cap B_R(Q)) < 1,$$

together with those points $P \in X - \overline{B_R(Q)}$ such that the geodesic segment $\overline{QP}$ satisfies

$$HD(\sigma_0 \cap B_R(Q), \overline{QP} \cap B_R(Q)) < 1.$$

One can check that these sets together with the balls centered at points of X form a basis for a topology which extends that of X, and such that $\bar{X}$ is compact. Furthermore, this topology is independent of the base point Q which was chosen. An important feature of this compactification of X is that if F is an isometry of X, then F extends as a homeomorphism of $\bar{X}$. Thus it makes sense to ask whether an action of Γ on X has a fixed point in ∂X. We have the following result.

Theorem 2.2.1. *If Γ is a finitely generated group and ρ is an action of Γ on a locally compact* (NPC) *space X, then either Γ has a fixed point on ∂X, or the action is proper. Furthermore, if Γ is the fundamental group of a compact Riemannian manifold M, and there is no fixed point for the action of Γ in ∂X, then there is an energy minimizing equivariant map from $\tilde{M}$ to X.*

Proof. If ρ is not proper, then there is a sequence of points $\{P_i\}$ in X with $d(P_i, Q) \to \infty$ for a chosen point Q, and with $\delta(P_i) \leq c$. We consider the geodesic segments $\overline{QP_i}$, and observe that the Hausdorff distance between $\overline{QP_i}$ and $\rho(\gamma_\alpha(\overline{QP_i}),\ 1 \leq \alpha \leq p$ is at most c because of the (NPC) condition. Because X is locally compact, we can find a subsequence, again denoted $\{P_i\}$ such that the $\overline{QP_i}$ converge on compact subsets to a ray σ with initial point at Q. We then have the Hausdorff distance between σ and $\rho(\gamma_\alpha(\sigma)$ bounded by c for $1 \leq \alpha \leq p$. This shows that the equivalence class of σ is fixed by Γ, and completes the proof of the first statement. The second statement follows directly from Theorem 2.1.3. This completes the proof of Theorem 2.2.1.

We now consider the simplest non-locally compact space, a Hilbert space H. Note first that an isometry F of H is of the form $F(v) = A(v) + v_0$ where A is a unitary transformation, an isometry which fixes the origin. An action of Γ on H then has an associated unitary representation given by the unitary part of the isometries of $\rho(\Gamma)$. We denote this unitary representation by ρ_0, and let δ_0 denote the translation function associated with ρ_0. Thus $\delta_0(\lambda v) = \lambda v$ for $\lambda > 0$. We say that ρ_0 almost fixes a unit vector if

$$\inf_{S_1(0)} \delta_0 = 0$$

where $S_1(0)$ denotes the unit sphere of H. We then have the following result.

Proposition 2.2.2. *Let Γ be a finitely generated group, and ρ an isometric action of Γ on a Hilbert space H. If ρ_0 does not have an almost fixed unit vector, then the action ρ is proper. If in addition, Γ is the fundamental of a compact Riemannian manifold M, then there is a harmonic equivariant map from $\tilde{M}$ to H.*

Proof. Let $v_\alpha,\ 1 \leq \alpha \leq p$ denote the translational part of the isometries $\rho(\gamma_\alpha)$. Let c be a number such that $\|v_\alpha\| \leq c,\ 1 \leq \alpha \leq p$. Let ϵ_0 be the infimum of δ_0 taken over the unit sphere. For any $v \in H$ we then have

$$\delta(v) \geq \delta_0(v) - c \geq \epsilon_0\|v\| - c.$$

This implies that ρ is proper, and the second assertion follows from Theorem 2.1.3. This completes the proof of Proposition 2.2.2.

Recall that a group Γ satisfies *Property T* if any unitary representation which has an almost fixed unit vector has a fixed unit vector. It is interesting to note that for a group Γ satisfying T, it is true that any isometric action of Γ on a Hilbert space is either proper or has a fixed ray (just as in the case of actions on locally compact spaces). It is also known however ([H]) that any such action has a fixed point, and therefore there is a (constant) equivariant harmonic map. We will further pursue the connection between property T and existence theory in Section 4 of this paper.

2.3. Spaces of strictly negative curvature and trees.

In this section we prove some special existence results under the assumption that X has stricly negative curvature. Without loss of generality we may assume that the curvature of X is less than or equal to -1 (see Section 1 for definitions). For simplicity, we use the standard terminology that X is a $CAT(-1)$ space. In this case we are able to show that if the action of Γ does not fix an equivalence class of rays, then there exists a harmonic equivariant map. This is similar to the locally compact case discussed in Theorem 2.2.1, except we do not prove the action is proper under the assumption that no equivalence class of rays is fixed. This does not seem to be true, since any unitary representation can appear as the linearization of the action at a fixed point, and these may have almost fixed directions.

Theorem 2.3.1. *If X is a $CAT(-1)$ space, and ρ is an isometric action of the fundamental group Γ of a compact Riemannian manifold on X, then either there is a ray in X whose equivalence class is fixed by Γ, or there is a harmonic equivariant map from $\tilde{M}$ to X.*

Proof. We consider a point $x_0 \in \tilde{M}$, and consider a ball B centered at x_0 with radius large enough that it contains a fundamental domain for the action of Γ on $\tilde{M}$. For any $\epsilon > 0$, let $\mathcal{C}_\epsilon \subseteq Y$ be the collection of locally Lipschitz equivariant maps from $\tilde{M}$ to X whose energy is bounded by $E_0+\epsilon$, where E_0 denotes the infimum of energies of equivariant maps, and whose Lipschitz constant on B is bounded by a constant c (which will be fixed throughout). The set $\mathcal{C}_\epsilon$ is nonempty for every $\epsilon > 0$ if c is chosen large enough, and we observe that $\mathcal{C}_\epsilon$ is a convex subset of Y. This is because the energy is convex, and the Lipschitz constant is as well. To see that the Lipschitz constant is convex, observe that quadrilateral comparisons imply

$$d(u_t(x), u_t(y)) \leq (1-t)d(u_0(x), u_0(y)) + td(u_1(x), u_1(y))$$

for any $x \in \tilde{M}$ and any $t \in [0,1]$. It follows that the Lipschitz constant of u_t on B is bounded by c if it is so for u_0, u_1. Note also that the set $\mathcal{C}_\epsilon$ is closed since the energy is lower semicontinuous under convergence in Y, and so is the Lipschitz constant since L^2 convergence is equivalent to uniform convergence in the presence of a Lipschitz bound, and the Lipschitz constant is lower semicontinuous under uniform convergence. Now we consider the set $C_\epsilon \subseteq X$ which is the set of values of $u(x_0)$ taken over maps u in $\mathcal{C}_\epsilon$. The set C_ϵ is then a closed convex subset of X. Fix a point $Q \in X$. The existence of a harmonic equivariant map is then equivalent to the existence of $R > 0$ such that $C_\epsilon \cap B_R(Q) \neq \phi$ for all $\epsilon > 0$. This is because, if a minimizer exists, then it lies in all of the C_ϵ, and conversely, if the C_ϵ intersect a ball of fixed radius, then we can find a minimizing sequence which is bounded in Y, and by Remark 2.1.4 we have a harmonic equivariant map.

Now suppose that there is no harmonic equivariant map. By the above discussion we see that for any $R > 0$, there is $\epsilon_0 > 0$ such that C_ϵ lies outside $B_R(Q)$ for all $\epsilon < \epsilon_0$. Now let $\{u_i\}$ be a minimizing sequence with $u_i \in \mathcal{C}_{2^{-i}}$, and consider the triangle in X with vertices $Q, u_i(x_0), u_j(x_0)$ for $i < j$. If we construct a comparison triangle in the unit disk conformal model of the hyperbolic plane with the vertex corresponding to Q at the origin, and with equal side lengths, then by the $CAT(-1)$ condition we see that the shortest distance from the origin to the opposite side is at least as large as the distance from Q to the opposite side in X. Since, for i large, this distance in X is large, we see that the origin lies far from its opposite side. It follows that the angle of the comparison triangle at the origin is small. Therefore, in X, the segments $\overline{Qu_i(x_0)}$ and $\overline{Qu_j(x_0)}$ are uniformly close on bounded subsets of X. Therefore this sequence of segments is Cauchy in the topology of uniform convergence on bounded subsets of X, and converges to a ray σ. Since the u_i are equivariant and uniformly locally Lipschitz, we have for any γ_α, $1 \leq \alpha \leq p$ that

$$d(\rho(\gamma_\alpha)(u_i(x_0)), u_i(x_0)) = d(u_i(\gamma(x_0)), u_i(x_0)) \leq c_1$$

for a constant c_1. Thus $\delta(u_i(x_0))$ is uniformly bounded, and hence δ is bounded along the segment $\overline{Qu_i(x_0)}$ for each i. Therefore δ is bounded along σ, and thus the equivalence class of σ is invariant under Γ. This completes the proof of Theorem 2.3.1.

Now we consider the case in which X is a tree. Assume that for any two points of X there is a unique embedded continuous path joining the points. Since we also assume that X is NPC, this curve must be the geodesic joining the points. We will refer to such a space X as a *tree*. We claim that a tree is a $CAT(-1)$ space. (In fact, it is $CAT(k)$ for any $k < 0$.) To see this, let A, B, C be three points in X, and consider the triangle they determine. The geodesic $\overline{AB}$ must intersect the interior of $\overline{BC}$ since otherwise $\overline{AB} \cup \overline{BC}$ would be a second embedded path from A to C. If C lies on the geodesic $\overline{AB}$, then the triangle determined by A, B, C is isometrically embedded in the real line. Since the real line can be embedded as a geodesic in the hyperbolic plane, we see that the $CAT(-1)$ inequality is satisfied with equality in this case. Therefore it must be true that the intersection of the geodesics $\overline{AB}$ and $\overline{BC}$ is a segment $\overline{DB}$ for some point D between A and B. If we parametrize the geodesic $\overline{BC}$ by arclength we have a geodesic $\sigma(s)$ for $0 \leq s \leq a$ with $\sigma(0) = B, \sigma(L) = C$, and $a = d(B, C)$. Let $s_0 \in (0, a)$ be such that $\sigma(s_0) = D$. We may then describe the function $d(s) = d(A, \sigma(s))$ explicitly. For $s \in [0, s_0]$ we have $d(s) = c - s$ where $c = d(A, B)$, while for $s \in [s_0, a]$ we have $d(s) = b - a + s$ where $b = d(A, C)$. If $f(s) = \cosh(d(s))$, then we see that the equality $f''(s) = f(s)$ for $s \in (0, s_0) \cup (s_0, a)$, and the first derivative jumps up at s_0. It follows that the inequality $f''(s) \geq f(s)$ is satisfied distributionally on $(0, a)$. This shows that X is a $CAT(-1)$ space. Therefore, the following result is a consequence of Theorem 2.3.1.

Corollary 2.3.2. *If X is a tree, and ρ is an isometric action of the fundamental group Γ of a compact Riemannian manifold on X, then either there is a ray in X whose equivalence class is fixed by Γ, or there is a harmonic equivariant map from $\tilde{M}$ to X.*

Remark 2.3.3. On might hope that for trees it would be true that there is either a fixed equivalence class of rays, or the action is proper as holds in the case of locally compact X. We describe an example which shows that this is not generally true. Consider a combinatorial tree T with a central vertex Q, and a countable number of adjoining edges $\{E_i\}$ indexed on the positive integers. Let P_i denote the remaining vertex adjoining E_i, and add two additional edges $E_{i,1}, E_{i,2}$ to P_i. Consider the automorphism F which fixes Q and all of the P_i, but maps $E_{i,1}$ to $E_{i,2}$. Choose a metric on T so that E_i has length i, and $E_{i,1}, E_{i,2}$ have infinite length. The automorphism is then an isometry, and for the group it generates we see that there is no fixed equivalence class of rays, but the action is not proper (since points arbitrarily far from Q are fixed).

3. Limit space constructions.

Let Ω be a set, let (X, d) be an NPC space, and let $u : \Omega \to X$ be a map. Denote the closed convex hull of u's image by $\mathcal{C}(u(\Omega))$. (The closed convex hull of S is the smallest closed convex set containing S.) A dense subset of $\mathcal{C}(u(\Omega))$ is parameterized naturally by a set Ω_∞ which is constructed inductively from products of Ω with intervals $I = [0, 1]$. The map u extends to

$u_\infty : \Omega_\infty \to X$ and $C(u(\Omega))$ can be reconstructed from Ω_∞ and the resulting pull-back pseudodistance d_∞. Thus one is led to a characterization of those pseudodistances on Ω which arise as pull-backs for maps to NPC spaces. Further, this leads to a useful notion of convergence for maps $u_k : \Omega \to X_k$, namely the pointwise convergence of the corresponding $d_{k,\infty}$. A similar construction allows one to consider convergence of maps to NPC spaces which are geodesically complete and which have uniform lower curvature bounds.

In the present section we examine these characterizations of maps to NPC spaces. We show local C^0 precompactness for sequences of maps having local uniform modulus of continuity control. These considerations will be used in the next section to study actions of discrete groups. Energy-minimizing properties are preserved under this convergence. Thus, for example, in an equivariant problem which cannot be solved for a harmonic mapping (because all minimizing sequences approach infinity), it is still true that locally uniformly Lipschitz minimizing sequences have subsequences which converge to a limit harmonic map, *to a limit NPC space.*

Recall that a pseudodistance on Ω is a non-negative function $\rho(x, y)$ defined on $\Omega \times \Omega$ which satisifies the triangle inequality and which is zero whenever $x = y$. By identifying points in Ω via the equivalence relation of zero pseudodistance, one obtains an actual distance function on the quotient set. Now, let X, u be as above, define Ω_0 to be Ω, let $u_0 := u$ and let d_0 be the pullback pseudodistance induced on $\Omega_0 \times \Omega_0$ by u,

$$d_0(x, y) := d(u(x), u(y)).$$

Let I be the unit interval $[0,1]$. Inductively define Ω_{i+1} to be $\Omega_i \times \Omega_i \times I$. Identify Ω_i with a subset of Ω_{i+1} via the inclusion map

$$x \mapsto (x, x, 0). \tag{3.1}$$

Extend u_i to $u_{i+1} : \Omega_{i+1} \to X$ by

$$u_{i+1}(x, y, \lambda) := (1 - \lambda)u_i(x) + \lambda u_i(y), \tag{3.2}$$

and let d_{i+1} be the corresponding pull-back pseudodistance. Note that u_i, d_i agree with u_{i+1}, d_{i+1} on the subset Ω_i defined by (3.1). In fact, letting $x, y \in \Omega_i$, $z \in \Omega_{i+1}$, $\lambda, \mu \in I$ it follows immediately that

(3.3)
$$\begin{aligned} d_{i+1}((x,x,0),(y,y,0)) =& d_i(x,y), \\ d_{i+1}((x,y,\lambda),(x,y,\mu)) =& |\lambda - \mu| d_i(x,y), \\ d_{i+1}^2(z,(x,y,\lambda)) \le& (1-\lambda)d_{i+1}^2(z,x) + \lambda d_{i+1}^2(z,y) - \lambda(1-\lambda)d_i^2(x,y). \end{aligned}$$

Define

$$\Omega_\infty := \cup \Omega_i.$$

There is a natural map $u_\infty : \Omega_\infty \to X$ given by $u_\infty := u_i$ on Ω_i, and so there is the corresponding pseudodistance d_∞ defined on Ω_∞. Construct the quotient metric space from $(\Omega_\infty, d_\infty)$, and denote its completion by (Z, d). Using only the properties (3.3) one can show that that (Z, d) is an NPC space, and deduce that in this case it is isometric to the closed convex hull $\mathcal{C}(u(\Omega))$. In fact, (3.3) characterizes those pseudodistances on Ω which arise as pullback pseudodistances from maps to NPC spaces:

Lemma 3.1. *Let Ω be a set and let ρ be a pseudodistance defined on Ω. Then ρ is a pullback pseudodistance for some map u from Ω to an NPC space if and only if $\rho = d_0$ for some pseudodistance d_∞ defined on $\Omega_\infty \times \Omega_\infty$ and satisfying (3.3). In this case, the completed quotient space (Z, d) is isometric to $\mathcal{C}(u(\Omega))$.*

Proof. The construction above shows that pullback pseudodistances from NPC spaces satisfy (3.3) on Ω_∞ so we must study the converse. The main step is to show that whenever $(\Omega_\infty, d_\infty)$ satisfies (3.3), then the completed quotient space (Z, d) is NPC.

From (3.3) we deduce that geodesic convexity (1.3.5) holds on Ω_∞. Now let $z, w \in (Z, d)$. Pick

$$\{z_i\}, \{w_i\} \subset \Omega_\infty, [z_i] \to z, [w_i] \to w.$$

Geodesic convexity applied to quadrilaterals $\{z_i, z_j, w_j, w_i\}$ immediately implies that the sequence of midpoints $\frac{z_i + w_i}{2}$ is Cauchy and must converge to the midpoint of z and w. Hence every two points in (Z, d) have a midpoint, and by standard diadic subdivision arguments there is a connecting curve of distance-realizing length.

The NPC property (for the sufficient case $\lambda = \frac{1}{2}$) follows by the same argument: Any collection of three points in (Z, d) can be approximated by a sequence of (equivalence classes of) triples in Ω_∞ For each triple the NPC inequality for the distance from the first point to the midpoint of the other two holds. Since these midpoints converge to the corresponding limit midpoint, the desired NPC estimate is the limit of these Ω_∞ inequalities.

Since (Z, d) is NPC we may consider the natural map $u : \Omega \to Z$ given by the composition of the natural inclusion of Ω into Ω_∞ (via (3.1)), followed by projection onto the completed quotient metric space Z. It is easy to see that $\rho = d_0$ is the pullback pseudometric of this map.

In case d_∞ is the pullback of a map u_∞ to an NPC space we see that the map u_∞ descends to a well-defined map from the quotient metric space, since points of zero pseudodistance have the same image point. This mapping on the quotient metric space is an isometry by construction, and so extends uniquely to be an isometry of the completion (Z, d). The image set is closed because (Z, d) is complete and the map is an isometry. The image set is convex because (Z, d) is NPC and because the map is an isometry. Thus the image set contains $\mathcal{C}(u(\Omega))$. Conversely, it follows by induction on the u_k that $u_\infty(\Omega_\infty) \subset \mathcal{C}(u(\Omega))$, so that also $u(Z) \subset \mathcal{C}(u(\Omega))$. Thus the two sets coincide. □

Now suppose that X is NPC, satisfies $K \geq -a^2$ (1.1.4), and is geodesically complete (i.e. every geodesic arc extends to a geodesic line). Then the construction above can be modified by replacing the interval I with the entire real line $\mathbf{R}$. In addition to the estimates (3.3) one has

(3.4)

$$\begin{aligned} d_{i+1}((x,y,\lambda),(x,y,\mu)) =& |\lambda - \mu| d_i(x,y), \quad \lambda, \mu \in \mathbf{R} \\ cosh(ad_{i+1}(z,(x,y,\lambda)) \geq& \frac{sinh((1-\lambda)ad_i(x,y))}{sinh(ad_i(x,y))} cosh(ad_{i+1}(z,x)) \\ &+ \frac{sinh(\lambda ad_i(x,y))}{sinh(ad_i(x,y))} cosh(ad_{i+1}(z,y)), \quad 0<\lambda<1. \end{aligned}$$

In this case we will denote the underlying set by $\Omega_\infty^{\mathbf{R}}$. We arrive at a characterization of those pseudodistances which arise as pullbacks from target spaces which are geodesically complete, NPC, and satisfy $(K \geq a^2)$:

Lemma 3.2. *Let Ω be a set and let ρ be a pseudodistance defined on Ω. Then ρ is the pullback pseudodistance for some map u from Ω to a geodesically complete NPC space satisfying $K \geq -a^2$ if and only if $\rho = d_0$ for some pseudodistance d_∞ on $\Omega_\infty^{\mathbf{R}}$, where d_∞ satisfies (3.3), (3.4). In this case, the space (Z, d) is isometric to the geodesic completion of $u(\Omega)$, i.e. the smallest geodesically complete set containing $u(\Omega)$.*

Proof. By the same reasoning as in Lemma 3.1 it suffices to show that if $(\Omega_\infty^{\mathbf{R}}, d_\infty)$ satisfies (3.3),(3.4), then the completed quotient space and the natural map to it have the desired properties. Of these, only the geodesic completeness of (Z, d) needs to be shown, since the curvature inequalities will follow by the argument of Lemma 3.1. By the usual diadic subdivision techniques it suffices to show that given any two points in (Z, d) there is a third point so that the second point is the midpoint of the first and third ones. The proof proceeds analogously to the midpoint construction in Lemma 3.1, except one uses the lower curvature bound: By hypothesis we may assume the existence of

$$\{x_i\}, \{y_i\}, \{z_i\} \subset \Omega_\infty,$$
$$\{[x_i]\} \to x, \{[y_i]\} \to y,$$

and with each y_i the midpoint of x_i and z_i. Applying the consequence (1.3.12) of (1.3.4) (or in our case of (3.4)) to the quadrilaterals $\{x_i, x_j, z_i, z_j\}$ with $t = \frac{1}{2}$ we deduce immediately that that $\{z_i\}$ is Cauchy, and hence that it converges to the desired z.

If d_∞ is the pullback pseudodistance obtained from a map u to a space satisfying the hypothesized conditions, then the argument that (Z, d) is isometric to the geodesic closure of $u(\Omega)$ follows the ideas in the proof of Lemma 3.1. □

Definition 3.3. Let $\{u_k : \Omega \to X_k\}$ be a sequence of maps to NPC spaces X_k. We will say that $u_k \to u$ in the pullback sense if the corresponding pullback pseudodistances $d_{k,\infty}$ converge pointwise to a limit d_∞, and if the map u is the natural projection of Ω onto the completed quotient space (Z, d) which is constructed from the limit d_∞. We may specify either Ω_∞ or $\Omega_\infty^{\mathbf{R}}$ in using this definition.

The following result is immediate from the considerations above:

Proposition 3.4. *Let $\{u_k : \Omega \to X_k\}$ be a sequence of maps to NPC spaces X_k. If $u_k \to u$ in the pullback sense on Ω_∞, then u is also a map to an NPC space. If each X_k is geodesically complete, has $K \geq -a^2$ and if $u_k \to u$ in the pullback sense on $\Omega_\infty^{\mathbf{R}}$, then the limit NPC space is geodesically complete and has $K \geq -a^2$.*

We now specialize to Riemannian domains (Ω, g). In this case each Ω_i has a Riemannian structure arising from the canonical product metric and we shall use the notation (Ω_i, g) to refer to this structure. We are interested in local problems, for which Ω is a compact Riemannian domain, as well as in global problems, in which Ω is the universal cover $\tilde{M}$ of a Riemannian manifold M having fundamental group Γ.

If $\Omega = \tilde{M}$ then each Ω_i is a product of intervals with copies of Ω, so the action of Γ by isometry on $\tilde{M}$ extends to one on each (Ω_i, g) by the diagonal action, i.e. via the inductive formula

$$\gamma((x,y,\lambda)) := (\gamma x, \gamma y, \lambda). \tag{3.5}$$

If Γ also acts by isometry on an NPC space X, via the homomorphism $\rho : \Gamma \to isom(X)$, and if $u : \tilde{M} \to X$ is Γ-equivariant, then it follows inductively from (3.2),(3.5) that also $u_\infty : \Omega_\infty \to X$ is Γ-equivariant. Therefore the pullback pseudodistance d_∞ on Ω_∞ is Γ-invariant. This condition characterizes those d_∞'s arising as pullbacks from equivariant mappings:

Lemma 3.5. *Let $\Omega = \tilde{M}$ as above and let ρ be a pseudodistance defined on Ω. Then ρ is a pullback pseudodistance for some equivariant map u to an NPC space if and only if $\rho = d_0$ for some Γ-invariant pseudodistance d_∞ defined on Ω_∞, where d_∞ satisfies (3.3). ρ is a pullback pseudodistance for some equivariant map u to a geodesically complete NPC space having $K \geq -a^2$ if and only if $\rho = d_0$ for some Γ-invariant pseudodistance d_∞ defined on $\Omega_\infty^{\mathbf{R}}$, where d_∞ satisfies (3.3),(3.4).*

In fact, if d_∞ is invariant (on either Ω_∞ or $\Omega_\infty^{\mathbf{R}}$), then Γ acts naturally on the completed quotient space (Z,d) and the natural projection map u from Ω to (Z,d) is equivariant, with pull-back pseudodistance ρ. Therefore if we have $u_k \to u$ in the pull-back sense (on either Ω_∞ or $\Omega_\infty^{\mathbf{R}}$), and if each u_k is an equivariant map, then so is u.

Proof. Let d_∞ be invariant. This means that the diagonal action of Γ on Ω_∞ $(\Omega_\infty^{\mathbf{R}})$ preserves d_∞, i.e. Γ acts isometrically on the pseudometric space $(\Omega_\infty, d_\infty)$ $((\Omega_\infty^{\mathbf{R}}, d_\infty))$. Any isometry on a pseudometric space descends to one on the quotient metric space. And any isometry on a metric space extends uniquely to one on its completion. Therefore Γ acts naturally on (Z,d). That the projection map $u : (\Omega, g) \to (Z,d)$ is equivariant follows from the fact that the identity may $id : (\Omega, g) \to (\Omega, d_0)$ is trivially equivariant. □

We will show precompactness theorems relative to convergence in the pullback sense, in the setting of maps with uniform local modulus of continuity control.

Definition 3.6. Suppose $u_k \to u$ in the pullback sense of Definition 3.3. We will say that the convergence is locally uniform if the convergence of $d_{k,i}$ to the limit d_i is uniform on each compact subset of $\Omega_i \times \Omega_i$.

Proposition 3.7. *Let $\{u_k : \Omega \to X_k\}$ be a sequence of maps to NPC spaces for which there is uniform modulus of continuity control. That is,* *assume for each $x \in \Omega$ and $R > 0$ there is a positive function $\omega(x, R)$ which is monotone in R, satisfying*

$$\lim_{R \to 0} \omega(x, R) = 0,$$

and so that for each $k \in \mathbf{Z}$ there holds

$$\max_{y \in B(x,R)} |d(u_k(x), u_k(y)| \leq \omega(x, R). \tag{3.6}$$

Then there is a subsequence of the $\{u_k\}$ which converges locally uniformly in the pullback sense to a limit map u, and u satisfies the same modulus of continuity estimates.

If each X_k is geodesically complete and satisfies $K \geq -a^2$ and if the convergence is for $\Omega^{\mathbf{R}}_{\infty}$, then the same result holds.

Proof. Consider the first case. If $v : \Omega \to X$ is any map to an NPC space which satisfies local modulus of control (3.6), then it follows that the extension map $v_i : (\Omega_i, g) \to X$ also has local modulus of continuity control. This follows from the triangle inequality and the inductive estimates

(3.7)
$$d(u_{i+1}(x, y, \lambda), u_{i+1}(x, y, \mu) = |\lambda - \mu| d(u_i(x), u_i(y))$$
$$d(u_{i+1}(x_1, y_1, \lambda), u_{i+1}(x_2, y_2, \lambda) \leq (1 - \lambda) d(u_i(x_1, x_2)) + \lambda d(u_i(y_1, y_2))$$

which follow from (3.3). But local modulus of continuity control for $v_i : \Omega_i \to X$ implies local modulus of continuity control for the pullback pseudodistance $d_i : \Omega_i \times \Omega_i \to \mathbf{R}$. It follows that all of the pullback pseudodistances $d_{k,i}$ obtained from the sequence $\{u_k\}$ have locally uniform modulus of continuity control. Therefore by a Cantor diagonalization with respect to both i and a compact exhaustion of each Ω_i, we obtain a subsequence of the $d_{k,\infty}$ which converges locally uniformly to a limit d_∞. Because of the pointwise convergence in particular of the (subsequence of) $d_{k,0}$ we see that the limit map u also satisfies the estimate (3.6).

In the second case of the theorem, the only part of the argument which must be modified is the second estimate in (3.7). We need a similar bound valid for any $\lambda \in \mathbf{R}$. From symmetry and the fact that (3.6) gives a suitable bound whenever $0 \leq \lambda \leq 1$, it suffices to derive one for $\lambda > 1$. The desired estimate follows immediately from (1.3.12), if we take

$$P = u_i(x_1), Q = u_i(x_2), R = u_{i+1}(x_2, y_2, \lambda), S = u_{i+1}(x_1, y_1, \lambda),$$
$$t = \frac{1}{\lambda}$$
$$\delta = max(d(u_i(x_1), u_i(x_2)), d(u_i(y_1), u_i(y_2)))$$
$$D = max\{d(z_i, z_j) | z_i, z_j \in \{u_i(x_1), u_i(x_2), u_i(y_1), u_i(y_2)\}\}.$$

The estimate is

$$ch(d_{RS}) - 1 \leq 12exp(5D\lambda)(ch(\delta) - 1). \tag{3.8}$$

This shows inductively that all of the extension maps $u_{k,i}$ obtained from the sequence $\{u_k\}$ have locally uniform modulus of continuity control, and so the argument can proceed as in the first case. □

Lemma 3.8. *Let $u_k \to u$ locally uniformly, in the pullback sense (For either construction of Ω_∞). Suppose each u_k satisfies uniform local energy bounds*

$$\int_{B(x,R)} |du_k|^2 d\mu \leq E(x, R) \tag{3.9}$$

for bounded functions E(x,R). Then the limit map is locally a $p = 2$ Sobolev mapping satisfying the same estimates (3.9). In fact, the Sobolev energy functional and the directional energy functionals are lower semicontinuous with respect to this locally uniform convergence; for any $f \in C_c(\Omega), f \geq 0$ and any smooth vector field Z on Ω there holds

$$\int f|du|^2 d\mu \leq \liminf_{k\to\infty} \int f|du_k|^2 d\mu, \tag{3.10}$$

$$\int f|u_*(Z)|^2 d\mu \leq \liminf_{k\to\infty} \int f|(u_k)_*(Z)|^2 d\mu. \tag{3.11}$$

Proof. Using, say, sphere averaging we have the ε-approximate energy density functions

$$e_\varepsilon^{u_k}(x) = \frac{1}{\omega_n} \int_{S^{n-1}(x,\varepsilon)} \frac{d_{k,0}^2(x,y)}{\varepsilon^2} \frac{d\sigma(y)}{\varepsilon^{n-1}}.$$

Because of the local uniform convergence of $\{d_{k,0}\}$ to the limit d_0 it is clear that the $e_\varepsilon^{u_k} \to e_\varepsilon^u$ locally uniformly. Integrating against $f \in C_c(\Omega), f \geq 0$ we deduce that the ε-energy functionals

$$E_\varepsilon^{u_k}(f) := \int e_\varepsilon^{u_k}(x) f(x) d\mu \to E_\varepsilon^u(f) \tag{3.12}$$

as $k \to \infty$. But by the "monotonicity" of the energy functionals [KS, Theorem 1.5.1] we have

$$E_\varepsilon^{u_k}(f) \leq E^{u_k}(f_\varepsilon^C), \tag{3.13}$$

where $f_\varepsilon^C = (1 + C_\varepsilon) f_\varepsilon$ and f_ε is defined in (1.5.13), and C depends on the support of f. Equations (3.12),(3.13) and the uniformity estimate (3.9) imply that the limit map u is locally Sobolev. Furthermore one has the inequalities

$$\begin{aligned} E_\varepsilon^u(f) &\leq \liminf_{k\to\infty} E^{u_k}(f_\varepsilon^C) \\ &\leq \liminf_{k\to\infty} E^{u_k}(f) + |f_\varepsilon^C - f|_\infty E(x, R), \end{aligned} \tag{3.13}$$

where the support of f_ε^C lies inside $B(x, R)$. The semicontinuity claim (3.10) and hence also the estimate (3.9) for u now follow by letting $\varepsilon \to 0$ in (3.14). It follows that u has finite directional energies, and one can repeat the same argument as above, using the $\varepsilon-$ approximate directional energies, to deduce (3.11). □

Finally, we wish to address harmonic map questions. We focus on the following two problems:

(I) The equivariant mapping problem, in which Ω is the universal cover $\tilde{M}$ of a compact Riemannian manifold M having fundamental group Γ.

(II) The Dirichlet Problem for a compact Riemannian domain (Ω, g).

For both of these problems a map $u : \Omega \to X$ is said to be within ε of minimizing if the $p = 2$ Sobolev energy E^u satisfies

$$E^u \leq inf E^v + \varepsilon,$$

where the v range through admissible competitors.

Theorem 3.9. *Consider the equivariant problem (I). Let $\{u_k : \tilde{M} \to X_k\}$ be a sequence of equivariant maps to NPC spaces X_k. Assume $\{u_k\}$ converges locally uniformly, in the pullback sense, to a limit map $u : \tilde{M} \to X$. Suppose that the energies of the u_k are uniformly bounded and that u_k is within ε_k of minimizing, with $\varepsilon_k \to 0$ as $k \to 0$. Then u is a minimizing equivariant map. Furthermore, the Sobolev energy density measures and the directional energy density measures of the u_k converge weakly to those of u.*

Proof. Let $M_0 \subset \tilde{M}$ be a compact, Lipschitz fundamental domain for M. Consider the equivariant harmonic map problem for the limit space X. By Theorem 2.6.4 in [KS] there is a uniformly Lipschitz minimizing sequence (and the Lipschitz constant depends only on M and on E^u). Therefore for any admissible map w (of which u is an example) and any $\delta > 0$ there is a uniformly Lipschitz equivariant map v so that

$$E^v < E^w + \delta. \tag{3.15}$$

Now fix an approximate identity η as in §1.5, and the corresponding measure ν. Fix ε sufficiently small so that

$${}_\nu E^v_\varepsilon,\ {}_B E^v_\varepsilon < E^w + \delta. \tag{3.16}$$

Because v is uniformly Lipschitz, it can be approximated arbitrarily closely in L^∞ by equivariant maps which are locally constant, with the added constraint that the image of M_0 is a finite set. Since the (equivalence classes of) points in Ω_∞ are dense in (X, d) we may assume that our piecewise-constant approximate map factors through one of the Ω_i. Call this map $\tilde{v}$. It is straightforward to check, using the modulus of continuity control on v and the properties (1.13) for the ν and ball measures, that if $\tilde{v}$ is close enough in L^∞ to v, then also

$${}_\nu E^{\tilde{v}}_\varepsilon,\ {}_B E^{\tilde{v}}_\varepsilon < E^w + \delta. \tag{3.17}$$

Because they factor through Ω_i, the maps $\tilde{v}$ determine natural maps $\tilde{v}_k$ to the pre-limit spaces X_k. Because the $d_{k,i}$ converge locally uniformly on Ω_i we deduce that for k large

$${}_\nu E^{\tilde{v}_k}_\varepsilon, {}_B E^{\tilde{v}_k}_\varepsilon < E^w + \delta. \tag{3.18}$$

Now apply the mollification result, Proposition 1.5, with $f := 1$, to deduce

$$E^{\tilde{v}_k * \eta_\varepsilon} \leq (1 + C\varepsilon)(E^w + \delta) \tag{3.19}$$

for k large. But the map $\tilde{v}_k * \eta_\varepsilon$ is a competitor for u_k, so we have the final estimate

$$E^{u_k} - \varepsilon_k \le (1 + C\varepsilon)(E^w + \delta) \tag{3.20}$$

for k large. Letting $k \to \infty$ and then noting that ε and δ were arbitrary we deduce

$$\limsup_{k \to \infty} E^{u_k} \le E^w$$

for any competitor w in the limit equivariant problem. Combining this with the semicontinuity result (3.10) yields

$$E^u \le \liminf E^{u_k} \le \limsup E^{u_k} \le E^w.$$

Hence u is minimizing, and taking $w = u$ above we also conclude that $E^{u_k} \to E^u$. Since there is no loss in the total energy of the limit map and because the Sobolev and directional energy functionals are lower semicontinuous it follows that the corresponding measures actually converge weakly. □

Corollary 3.10. *Consider the equivariant problem (I). Regardless of whether there exists a minimizing harmonic map to the original space, it is nevertheless true that any uniformly Lipschitz minimizing sequence* $\{u_k\}$ *has a subsequence which converges locally uniformly, in the pullback sense, to a limit harmonic map* u *(to a limit space). If* X *is geodesically complete and satisfies a lower curvature bound (1.1.4), then so does the limit space. The limit* L^1 *tensor* Π *and limit energy density function* $|du|^2$ *are unique, and they are* L^1 *limits of those in the sequences. In fact,*

$$\lim_{k \to \infty} \int_M \int_{S^{n-1}} \big||(u)_*(\omega)| - |(u_k)_*(\omega)|\big|^2 d\sigma(\omega) d\mu(x) = 0. \tag{3.21}$$

Proof. The existence of the limit equivariant harmonic map follows from Lemma 3.5, Proposition 3.7 and Theorem 3.9. The L^1 convergence of the tensors Π and of the energy densities follows from the statement (3.19). This statement follows from the weak convergence of the directional energy measures proven in Theorem 3.9, together with Proposition 2.6.3 in [KS], which asserts that

$$\lim_{i,j\to\infty} \int_M \int_{S^{n-1}} ||(u_i)_*(\omega)| - |(u_j)_*(\omega)||^2 d\sigma(\omega) d\mu(x) = 0$$

for any minimizing sequence. □

There is local version of Theorem 3.9, which is slightly more technical because one must deal with boundary values. Let Ω be a smooth compact Riemannian domain. Write $\Sigma := \partial\Omega$. For small non-negative t consider the subdomains Ω_t consisting of points which are at least distance t from Σ, and write $\partial\Omega_t := \Sigma_t$ for their parallel boundaries.

Theorem 3.11. *Let Ω be as above. Let $\{u_k : \Omega \to X_k\}$ be a sequence of finite energy maps to NPC spaces X_k. Assume $\{u_k\}$ converges locally uniformly, in the pullback sense, to a limit map $u : \Omega \to X$. Suppose that the energies of the u_k are uniformly bounded and that u_k is within ε_k of minimizing, with $\varepsilon_k \to 0$ as $k \to 0$. Additionally, assume that the energy densities of the u_k are uniformly controlled near Σ. Specifically, assume that there is a positive function $D(t)$ defined for small positive t, with $D(t) \to 0$ as $t \to 0$, and so that*

$$\int_{\Omega-\Omega_t} |du_k|^2 d\mu \le D(t). \tag{3.22}$$

Then u is a minimizing map (for its given boundary values). Furthermore, the Sobolev energy density measures and the directional energy density measures of the u_k converge weakly to those of u.

Proof. For each small non-negative t consider the Dirichlet Problem on Ω_t, with boundary data given by the trace of u on Σ_t. Let $E(t)$ be the energy of the minimizer w_t. (There exists a unique solution, with local interior Lipschitz continuity control, by Theorems 2.2 and 2.4.6 of [KS].) We claim that $E(t)$ is a continuous function at $t = 0$: For $t > 0$ a competitor to $w_0 := w$ is the map which equals w_t in Ω_t and u on the complement. Therefore

$$E(0) \le E(t) + \int_{\Omega-\Omega_t} |du|^2 d\mu.$$

In order to deduce a reverse inequality construct diffeomorphisms $\Psi_t : \Omega_t \to \Omega$ which are within $O(|t|)$ of being Riemannian isometries, and so that these diffeomorphisms preserve the normal directions in a tubular neighborhood of Σ. Consider the competitor for w_t obtained as follows. First reflect the values of u exterior to Ω_t through Σ_t, via the normal vector field. This defines the competitor in the exterior of Ω_{2t}. Then use the almost-isometry ψ_{2t} to reparameterize the map w from Ω to Ω_{2t}. Deduce that

$$E(t) \leq (1 + O(t))E(0) + C \int_{\Omega - \Omega_t} |du|^2 d\mu.$$

These two inequalities imply the continuity of E at $t = 0$.

Write v_t to refer to the map on Ω which is given by w_t inside Ω_t and by u in its complement. By the reasoning above we see that the total energy of v_t is continuous at $t = 0$. By the boundary regularity theorem of T. Serbinowski [S] we also know that v_t is globally C^α on $\overline{\Omega_t}$.

We are now in a position to repeat the argument of Theorem 3.9. Let $\delta > 0$. Pick $t > 0$ sufficiently small so that (3.15) holds, with $v := v_t$. Define total approximate energies ${}_\nu E^v_\varepsilon$, ${}_B E^v_\varepsilon$ as the integrals of the corresponding density functions over $\Omega_{\frac{t}{2}}$. Then it follows as before that also (3.16) holds, for ε sufficiently small.

Approximate v in the *sup* norm by a map $\tilde{v}$, as follows: In the complement of Ω_t let $\tilde{v} := v = u$. Inside Ω_t replace v with a piecewise constant map having a finite number of image points, and so that the replacement map factors through one of the Ω_i. (Note, u factors through Ω_0.) By approximating closely enough in the *sup* norm, we can satisfy (3.17).

Because the $d_{k,i}$ converge uniformly on $\Omega_{\frac{t}{2}}$ we deduce that (3.18) holds for k large. The mollification estimate (3.19) follows as before, where the energy of the mollified map is being computed on $\Omega_{\frac{t}{2}}$. The mollifed map is not a competitor for the u_k Dirichlet problem, because even though $\tilde{v}_k$ equals u_k in a boundary strip, the mollifed map does not (and is not even defined near Σ). Therefore we "bridge" the gap between u_k and $\tilde{v}_k * \eta_\varepsilon$ somewhere in the strip $\Omega_{\frac{t}{2}} - \Omega_t$. The estimate we need is the following:

Lemma 3.12. *Let Σ be a smooth compact Riemannian manifold. Let $v, w : \Sigma \to X$ be finite energy maps to an NPC space X. Then for any $\rho > 0$ the map $W : \Sigma \times [0, \rho]$ given by*

$$W(x, s) = (1 - \frac{s}{\rho})v(x) + \frac{s}{\rho}w(x)$$

has finite energy E^w bounded by

$$E^w \leq \frac{\rho}{2} \int_{\Sigma} |dv|^2 + |dw|^2 d\sigma + \frac{1}{\rho} \int_{\Sigma} d^2(v, w) d\sigma. \tag{3.23}$$

Proof. It is easy to verify that the map W is a Sobolev map. The estimate in this lemma is true because the energy density of maps from a Riemannian product is the sum of the densities relative to each factor. To estimate (approximate) energy densities on a slice $\Sigma \times s$ one uses the quadrilateral inqualities of §1.3, applied to points $v(x), v(y), w(x), w(y)$. One concludes that

$$|d_{\Sigma} W|^2((x,s)) \leq (1 - \frac{s}{\rho})|dv|^2 + \frac{s}{\rho}|dw|^2.$$

Estimating in the $\mathbf{R}$-direction, one has trivially

$$|W_*(\frac{\partial}{\partial s})|^2 = \frac{1}{\rho^2} d^2(v, w).$$

The estimate (3.23) follows by summing and integrating the two estimates above. □

In order to use the bridging lemma we estimate

$$\int_{\Omega_{\frac{2t}{3}} - \Omega_{\frac{3t}{4}}} d^2(u_k, \tilde{v}_k * \eta_\varepsilon) \leq C\varepsilon^2 \tag{3.24}$$

$$\int_{\Omega_{\frac{2t}{3}} - \Omega_{\frac{3t}{4}}} |du_k|^2 + |d\tilde{v}_k * \eta_\varepsilon|^2 \leq CD(t). \tag{3.25}$$

The estimate (3.24) holds because $\tilde{v}_k$ is the function u_k in $\Omega_{\frac{t}{2}} - \Omega_t$, so one is really estimating the L^2 distance between a Sobolev map and its ε-mollification. For any such function u abbreviate the mollification by $\overline{u}$. By the triangle inequality

$$d^2(u(x), \overline{u}(x)) \leq 2d^2(u(y), u(x)) + 2d^2(u(y), \overline{u}(x)).$$

Integrate this inequality over $B(x, \varepsilon)$, with respect to the exponential push-forward of the measure $\eta_\varepsilon(z)dz$ on TM_x. Thus

$$d^2(u(x), \overline{u}(x)) \leq 4\varepsilon^2 {}_\eta e_\varepsilon(x),$$

where the approximate energy is the one induced by η. Integrate this expression over $\Omega_{\frac{2t}{3}} - \Omega_{\frac{3t}{4}}$. Because Sobolev energy dominates ε-approximate energy we deduce (3.24), where the constant C depends on the uniform energy bounds for the sequence u_k.

The estimate 3.25 follows from the uniform energy bounds for u_k, and from the fact that mollification essentially decreases energy.

From (3.24),(3.25), the average square L^2 distance between u_k and $\overline{u}_k$ on Σ_s is bounded by $C\frac{\varepsilon^2}{t}$, for $\frac{2t}{3} < s < \frac{3t}{4}$. The average energy of the restriction of u or $\overline{u}$ to Σ_s is bounded by $C\frac{D(t)}{t}$ on the same interval. We choose a subinterval $[s, s+\mu t]$ ($\mu << 1$) on which to apply the bridging lemma, between the map u_k and the map $\overline{u}_k$. We can make the first term in the energy contribution less than δ for at least half the s in our interval, by taking μ small and choosing s carefully. (This part of the argument does not use $D(t) \to 0$.) For this given choice of μ we can specify ε to have been small enough so that for at least one of these restricted s the L^2 contribution to the energy estimate is also less that δ. Using this bridge choice we obtain a competitor to u_k, and hence an estimate analogous to (3.20), namely,

$$E^{u_k} - \varepsilon_k \leq (1 + C\varepsilon)(E^w + \delta) + 2\delta + \int_{\Omega - \Omega_t} |du_k|^2 d\mu.$$

Now, let $k \to \infty$, note that δ, ε, t can be chosen arbitrarily small, and that the estimate (3.22) holds for the energy of u_k in the strip. Hence

$$\limsup_{k \to \infty} E^{u_k} \leq E^w.$$

Combining the estimate above with semicontinuity, as in the proof of Theorem 3.9, we deduce that u is the minimizer w, and the other claims of Theorem 3.11 follow as well. □

4. Nonuniform actions.

This chapter deals with the situation in which the action of Γ on X is not uniform. This is the same as saying that the action almost fixes a point of X; that is, there is a sequence of points $\{P_i\}$ in X such that $\delta(P_i) \to 0$. Of course, the simplest case in which this would happen is if the action actually fixes a point of X. We will be interested in the case in which there is no fixed point.

4.1. A basic construction and spaces of bounded curvature.

We first describe a technical result which is basic for the work of this section and the next. Suppose X is an NPC space, and ρ is an action of a finitely generated group Γ on X. let P_1 be a point of X, and let $\epsilon_1 = r_1^{-1}\delta(P_1)$. Assume that there is no fixed point for the action in $\overline{B_{r_1}(P_1)}$ for some $r_1 > 0$. We then have the following result.

Proposition 4.1.1. *There exists a point $P_2 \in B_{r_1}(P_1)$ such that if we set $r_2 = 1/2(r_1 - d(P_1, P_2))$, then we have the following conditions satisfied*

$$r_2^{-1}\delta(P_2) \leq 4\epsilon_1, \ \ \delta(P_2) \leq 4\inf\{\delta(P) : \ P \in B_{r_2}(P_2)\}.$$

Proof. Since we are assuming that there is no fixed point in $\overline{B_{r_1}(P_1)}$, we may apply Proposition 2.1.1 to see that the function δ achieves a positive minimum value in $\overline{B_{r_1}(P_1)}$. Let $\delta_1 > 0$ denote the minimim value of δ on $\overline{B_{r_1}(P_1)}$. We then consider the function f on $B_{r_1}(P_1)$ given by

$$f(P) = (r_1 - d(P_1, P))^{-1}\delta(P).$$

We then see that the infimum of f satsfies

$$r_1^{-1}\delta_1 \leq \inf_{B_{r_1}(P_1)} f \leq \epsilon_1 \tag{4.1.1}$$

where the second inequality holds because $f(P_1) = \epsilon_1$. Since the infimum of f is positive, we may choose a point $P_2 \in B_{r_1}(P_1)$ so that

$$f(P_2) \leq 2 \inf_{B_{r_1}(P_1)} f \tag{4.1.2}$$

We now define $r_2 = 1/2(r_1 - d(P_1, P_2))$ as in the statement of the proposition. Combining (4.1.1) and (4.1.2) we have

$$f(P_2) \leq 2\epsilon_1.$$

From the definition of f and r_2, this implies $r_2^{-1}\delta(P_2) \leq 4\epsilon_1$ which is the first inequality of the desired conclusion. To prove the second inequality, we observe if $P \in B_{r_2}(P_2)$, we have by the triangle inequality

$$r_1 - d(P_1, P) \geq r_1 - d(P_1, P_2)) - d(P_2, P) \geq 1/2(r_1 - d(P_1, P_2)) = r_2.$$

Thus we have

$$\inf_{B_{r_2}(P_2)} f \leq r_2^{-1} \inf_{B_{r_2}(P_2)} \delta.$$

Since $f(P_2) = (2r_2)^{-1}\delta(P_2)$, we may combine this with (4.1.2) to conclude $\delta(P_2) \leq 4\inf\{\delta(P) : P \in B_{r_2}(P_2)\}$ as desired. This completes the proof of Proposition 4.1.1.

We now apply this result to the case in which X has a lower curvature bound, and is geodesically complete in the sense discussed in the previous section (geodesic arcs are extendable). For such a space X we have the following result.

Theorem 4.1.2. *Assume X is a geodesically complete* NPC *space with a lower curvature bound. Assume Γ is a finitely generated group and ρ is an isometric action of Γ on X. If the action is not uniform, but does not have a fixed point in X, then there is a uniform isometric action ρ_∞ of Γ on a Hilbert space H. Moreover, if Γ is the fundamental group of a compact Riemannian manifold M, then ρ_∞ may be chosen so that there is a ρ_∞-equivariant harmonic map from $\tilde{M}$ to H.*

Proof. We may assume that Γ is the fundamental group of a complete Riemannian manifold. Let $\{P_i\}$ be a sequence of points from X such that $\epsilon_i = \delta(P_i) \to 0$. We then apply Proposition 4.1.1 in a unit ball about P_i to produce a new sequence of points $\{Q_i\}$, and radii $\{r_i\}$ such that

$$r_i^{-1}\delta(Q_i) \leq 4\epsilon_i, \ \delta(Q_i) \leq 4\inf\{\delta(P) : P \in B_{r_i}(Q_i)\}.$$

Now define a new NPC space $X_i = (X, \delta(Q_i)^{-1}d)$. Denote by ρ_i the same action ρ on X_i, and let δ_i be the associated translation function. We then have $\delta_i = \delta(Q_i)^{-1}\delta$ and the ball $B_{r_i}(Q_i)$ becomes the ball $B^i_{R_i}(Q_i)$ in X_i where $R_i = \delta(Q_i)^{-1}r_i$. Thus we have

$$R_i \geq 1/4\epsilon_i^{-1}, 1 = \delta_i(Q_i) \leq 4\inf\{\delta_i(P) : P \in B^i_{R_i}(Q_i)\}.$$

Now by [KS, Proposition 2.6.1], there exists a uniformly locally Lipschitz sequence of maps $u_i : \tilde{M} \to X_i$ which map a chosen point $x_0 \in \tilde{M}$ to Q_i. We then apply Theorem 3.9 to this sequence to produce a limit NPC space X_∞, and a limit map u_∞ which is equivariant with respect to a limiting action ρ_∞. Note that the curvature of X_i is bounded from below by $-\delta(Q_i)^2\kappa$ if that of X is bounded below by $-\kappa$. Therefore from Corollary 3.10 we see that X_∞ has curvature bounded above and below by 0. Moreover, X_∞ is geodesically complete, so by Proposition 1.1.1, X_∞ is isometric to a Hilbert space H. Since $R_i \to \infty$, the action ρ_∞ is uniform; in fact, $\inf \delta_\infty \geq 1/4$. This establishes the first part of the theorem.

The second assertion follows by applying Theorem 3.9 to a minimizing sequence of ρ_∞-equivariant maps from $\tilde{M}$ to H. One then obtains a new Hilbert space with a uniform action of Γ together with a minimizing equivariant map. This completes the proof of Theorem 4.1.2.

As a corollary of this result we prove a general fixed point for actions of Property T groups on NPC spaces.

Corollary 4.1.3. *Let Γ be a finitely generated group, and let X be a geodesically complete* NPC *space with curvature bounded from below. If Γ satisfies Property T, then any isometric action of Γ on X which is not uniform fixes a point of X. If Γ does not satisfy Property T, then there exists a uniform isometric action of Γ on a Hilbert space.*

Proof. By Theorem 4.1.2, a nonuniform action of Γ on X would produce a uniform isometric action of Γ on a Hilbert space; in particular, a fixed point free action. This is impossible for a Property T group (see [H]). This proves the first statement. On the other hand, if Γ does not satisfy Property T, then there is a unitary representation ρ on a Hilbert space with an almost fixed unit vector, but no fixed unit vector. We can apply Proposition 4.1.1 in balls of radius $1/2$ about a sequence of unit vectors v_i with $\delta(v_i) \to 0$. The same argument as that above then produces a uniform isometric action on a Hilbert space. This completes the proof of Corollary 4.1.3.

We may also apply the theorem to produce harmonic and holomorphic functions in very general situations.

Corollary 4.1.4. *Let M be a compact Riemannian manifold whose fundamental Γ does not satisfy Property T. There exists a uniform affine action of Γ on a Hilbert space H together with an equivariant harmonic map from $\tilde{M}$ to H. If M is Kähler, then H may be taken to be a complex Hilbert space, and the map may be taken to be holomorphic. In particular, for any such M, the universal cover admits harmonic (holomorphic if M is Kähler) functions with linear growth.*

Proof. Corollary 4.1.3 constructs a uniform isometric action ρ of Γ on a Hilbert space H, and the existence of the harmonic map u follows as in the proof of the second part of Theorem 4.1.2. Classical vanishing theorems imply that the map is pluriharmonic. Thus on the universal cover $\tilde{M}$, we can find a pluriharmonic conjugate map $v : \tilde{M} \to H$. If we normalize u, v so that $u(x_0) = v(x_0) = 0$ for some chosen $x_0 \in \tilde{M}$, then the map v is also ρ-equivariant since for any $\gamma \in \Gamma$, $\rho(\gamma) \circ v \circ \gamma^{-1}$ is a pluriharmonic conjugate to u which maps x_0 to 0. Therefore this map agrees with v, and v is ρ-equivariant. The map $u + \sqrt{-1}v$ is then a holomorphic map to the complex Hilbert space $H \oplus \sqrt{-1}H$ which is equivariant for the action $\rho \oplus \sqrt{-1}\rho$. By the equivariance property, the gradient of the map is bounded, so it is of linear growth. This completes the proof of Corollary 4.1.4.

4.2. Application of zero vanishing theorems.

There are certain applications of this theory in which the curvature of the target is not bounded below, but for which one can prove certain vanishing theorems which can be used to replace the lower curvature bound. In particular, a vanishing formula which implies that equivariant harmonic maps are constant can generally be used to prove that an action has a fixed point without knowing the existence of a harmonic map. We will refer to such a vanishing theorem as a 'zero vanishing theorem'. In order to carry this out, one needs to have a suitable approximate version of this vanishing theorem. In this section we abstract the necessary conclusions which need to be obtained from such an approximate vanishing theorem, and use these to prove the fixed point theorem. Let $\bar{\epsilon}(t)$ be a continuous decreasing function of $t \geq 0$ with $\bar{\epsilon}(t) \geq 1$ for $t \leq 1$, and $\lim_{t\to\infty} \bar{\epsilon}(t) = 0$. Let M be a given complete Riemannian manifold, and we say that an NPC space X is in the class $\mathcal{Z}_{M,\bar{\delta}}$ if for any action ρ of $\Gamma = \pi_1(M)$ on X, and for any point $Q \in X$, and any $R > 0$ we have

$$\inf\{\delta(Q)^{-1}\delta(P) : P \in B_R(Q)\} \leq \bar{\epsilon}(\delta(Q)^{-1}R). \tag{4.2.1}$$

The main result of this section is the following

Proposition 4.2.1. *Assume Γ is a finitely generated group which is the fundamental group of a complete Riemannian manifold M. Suppose X is a space which lies in $\mathcal{Z}_{M,\bar{\delta}}$. Then any isometric action of Γ on X has a fixed point.*

Proof. Let ρ be any isometric action of Γ on X. Fixing Q in (4.2.1), and taking R large shows that the action is not uniform. If we take a point P_1 for which $\delta(P_1)$ is small, then, assuming there is no fixed point, we may apply Proposition 4.1.1 with $r_1 = 1$, and find a point P_2 and a radius r_2 for which $\delta(P_2)^{-1}r_2$ is large, and for which the infimum of $\delta(P_2)^{-1}\delta$ taken over $B_{r_2}(P_2)$ is at least 1/4. This contradicts (4.2.1) if we take $Q = P_2, R = r_2$. Therefore the action must have a fixed point; in fact, there must be a fixed point near P_1. This completes the proof of Proposition 4.2.1.

References.

[ES] J. Eells and J.H. Sampson, *Harmonic mappings of Riemannian Manifolds*, Amer.J. Math. **86** (1964), 109-160.

[G1] M. Gromov, *Paul Levy's isoperimetric inequality*, preprint, 1980.

[G2] M. Gromov *Foliated Plateau problem, part II: harmonic maps of foliations*, GAFA **1** (1991), 253-320.

[H] P. de la Harpe, *La Propriete (T) de Kazhdan pour les Groupes Localement Compacts*, Paris: Soc math de France, 1989.

[J] J. Jost, *Equilibrium maps between metric spaces*, Calc. Var. **2** (1994), 173-204.

[KS] N. Korevaar and R. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. **1** (1993), 561-659.

[LY] P. Li and S. T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Symp. Pure. Math. **36** (1980), 205-239.

[R] Y.G. Reshetnyak, *Nonexpanding maps in a space of curvature no greater than K*, Siberian Math. Journ. **9** (1968), 918-927.

[S] T. Serbinowski, *Boundary regularity of harmonic maps to nonpositively curved metric spaces*, Comm. Anal. Geom. **2** (1994), 139-154.

RECEIVED OCTOBER 14, 1996.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
233 JOHN WIDTSOE BLDG.
SALT LAKE CITY, UT 84112-1107
E-MAIL: KOREVAAR@MATH.UTAH.EDU

Department of Mathmatics
Stanford University
Stanford, CA 94305
E:mail: schoen@gauss.stanford.edu