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A finiteness property of representations of π_1 of algebraic surfaces

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0. Introduction

Let X be a compact complex algebraic smooth surface and $\pi_1(X)$ be the fundamental group of X. In this article we shall prove the following

Theorem 1 *Suppose that X contains a numerically nonzero divisor D of rational curves (possibly singular) and the self intersection $D^2 = 0$. Then any n-dimensional reductive representation ρ of $\pi_1(X)$ is either finite, or factors through a surjective morphism $f: X \to C$ onto an algebraic curve C of connected fibers after passing to some finite etale covering and blowing of X.*

Recently, B. Lasell and M. Ramachadran [14] proved that if X contains a divisor D of rational curves such that $D^2 > 0$, then any n-dimensional reductive representation of $\pi_1(X)$ is finite. Their theorem gives a positive answer in the representation case of the following question has been studied by Nori.

Question A *Let X be an algebraic surface. Suppose that there is a rational curve R (possibly singular) on X, with $R^2 > 0$. Is $\pi_1(X)$ finite?*

Gurjar has related this question to the Shafarevitch conjecture [17] which asks: is $\tilde{X}$ holomorphically convex ? F. Campana and J. Kollar have studied, for examples, non negativity of a variant Kodaira dimension [3] and some nonvanishing properties for plurigenera [12]. The author [23] has shown recently that the Shafarevitch varieties $sh(X)$ in the semisimple representations case are always of general type and all nonconstant morphisms form all curves of genus ≤ 1 into $sh(X)$ must be contained in a proper subvariety of $sh(X)$. In a sense, Theorem 1 can be thought as a sharp version of this property.

Let us recall a part of this conjecture, so called *Remmert-Stein reduction,* it says that there should exists a proper holomorphic map $sh : \tilde{X} \rightarrow \tilde{Y}$ such that every compact subvariety is contained in some fibers of sh. And this is enough to solve Question A. Let us say a few words about that. If $\pi_1(X)$ is infinite, then f is not constant. So, a connected component of preimage of R (which is always compact) is therefore mapped to point. But, this will lead a contradiction to $D^2 > 0$.

Combining this conjecture and Theorem 1, one could similarly ask the following question

Question B *Let X and D be the same as in Theorem 1. Is the following true? either $\pi_1(X)$ is finite, or there exists a fibration $f : X \rightarrow C$ such that D is contained in a fiber of f and the image $\pi_1(f^{-1}(c)) \rightarrow \pi_1(X)$ is finite for each fiber of f.*

Originally, Bogomolov informed the author very recently that he has constructed a series of examples. Such a surface is a fibration of curves over a curve, it contains a configuration D of rational curves as a fiber such that image $\pi_1(D) \rightarrow \pi_1(X)$ is infinite. So, these examples give a negative answer of Question B. It might also lead a negative answer to the Shafarevitch conjecture.
Applying Theorem 1 to Bogomolov's examples, we see that any n-dimensional reductive representation of π_1 of such a surface must have a infinite kernel.

It is easy to check Theorem 1 is also valid in the following situation: A smooth projective variety X of dimension d, with an effective divisor on E such that $\pi_1(E_i)$ of each irreducible component of E has a finite image in $\pi_1(X)$ and $E^2H^{d-2} \geq 0$ for an ample divisor H.

The method involved in the proof is, so called, *equivariant (archimedian + p−adic)-pluriharmonic maps and nonabelian Hodge theory.* Roughly saying, if all these maps are constant, then ρ is unitary and discrete, hence, is finite. It could also be a general method to treat infinite dimensional representations in some cases. Because of the classification theory of surface, we only need to consider surfaces of minimal model and of general type. A significant property of such a surface is, $\chi(\mathcal{O}_X) > 0$. Therefore, by a theorem due to Gromov [7], the nature representation of $\pi_1(X)$ in the Hilbert-space of L_2- holomorphic 2-forms is *almost faithful.* (I thank F. Campana for telling me this fact.) Once a good theorem of harmonic maps is developed for infinite dimensional representations in archimeian and p-adic cases (for example, maps into infinite dimensional symmetric spaces and infinite dimensional Bruhat-Tits building), then our method should work also in this case.

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During a short visit in Nancy university, the author made a very interesting discussion with F. Campana on this subject. I thank him for drawing my attention to this problem. I like to thank F. Bogomolov for very useful communications. In fact, his examples and questions give me a motivation to prove Theorem 1 and write down this paper. Finally, I should like to thank the referee for many valuable and clarifying remarks.

1. Pluriharmonic maps in archimedian case, $p-$adic case and non-abelian Hodge theory

Let X be a compact Kähler manifold and $\rho : \pi_1(X) \to GL_n(\mathrm{C})$ be a complex representation. One considers an equivariant map u from the universal covering $\tilde{X}$ of X to the symmetric space $N = GL_n(\mathrm{C})/U_n$. The invariant Riemannian metric on N and the Kähler metric on $\tilde{X}$ together defines a metric $<,>$ on $T^*_X \otimes u^{-1}T_N$. The energy of u is therefore defined by $E(u) = \int_X < du, du > dV$ where dV is the volume form induced by the Kähler form. An equivariant map u is called harmonic if u is a critical point of $E(.)$, and called pluriharmonic if the restriction of u to any complex submanifold of X is again harmonic. The following theorem proved by to Hitchin, Donaldson and Corlette ([6] [5] [4]) is a special case of a general theorem of existence of equivariant harmonic maps proved independently by Jost-Yau [10] and Labourie [13].

Theorem 1.1 *There exists a $\rho-$equivariant pluriharmonic map (in fact, a unique one), if and if ρ is reductive.*

Here, ρ called reductive, if either $\rho\pi_1(X)$ is not contained in a parabolic subgroup of $SL_n(\mathrm{C})$, or if it stabilizes (i.e. maps to itself) a totally geodesic flat subspace of N.

As an interpretation of equivariant pluriharmonic maps, Hitchin and Simpson have constructed a holomorphic object, so called *Higgs bundle* [6], [18]. Let us briefly describe their construction. Let V be the flat vector bundle associated to ρ with the flat connection D. Given a metric g on V, (or equivalent to say, given a ρ−equivariant map $u : \tilde{X} \to N$) then there is an unique decomposition $D = (D'_g + D''_g) + (\theta + \bar{\theta})$ such that the connection $D'_g + D''_g$ respects to g and $\theta + \bar{\theta} \in A^{1,0}(\mathrm{End}\, V) \oplus A^{0,1}(\mathrm{End}\, V)$ is self-adjoint with respect to g. The metric g is harmonic (equivalent to u is harmonic) if $(D''_g + \theta)^2 = 0$. Hence, from this equation we get a holomorphic vector bundle $E = (V, D''_g)$ and a holomorphic section $\theta \in H^0(X, \Omega^1(\mathrm{End}\, E)$. This pair (E, θ) is called Higgs bundle corresponding to ρ. It is easy to see that ρ is an unitary representation if and only if $\theta = 0$.

There is a variant semi negativity (positivity) theorem due to Mok, so called *semi Kähler structure.* In a sense, it is stronger than Lemma 1.3. Let us briefly describe it. Considering a compact Kähler manifold X and let ρ be a noncompact reductive representation of $\pi_1(X)$. Then by Theorem 1.1 we have a non constant ρ− equivarient pluriharmonic map $u : \tilde{X} \to N$. The (1, 0)-part of the complexified differential $d'u$ is holomorphic, and therefore it defines a meromorphic foliation $\mathcal{F}$ on $\tilde{X}$. The following is a part of Mok′s theorem ([16] (1.2), Theorem 3, (1.3), Theorem 4 and p.574). We need it in the proof of Theorem 1.

Lemma 1.2 *Let h be the canonical Riemannian metric on N. Then the (1,1)-part of the pull back $u^*(h)$ is a possibly degenerated Kähler metric g on X. Moreover, denote by β the closed (1,1)- Kähler form on X, associated to g, then β is semipositive definite, and the Kernel of the semipositive Hermitian form of β on $T^{1,0}_x(X)$ coincides with $T^{1,0}_x(\mathcal{F})$.*

Now we consider the similar question in the p−adic case. Let ρ be a representation of $\pi_1(X)$ into a semisimple p−adic algebraic group $G(K_p)$ where K_p is a complete field with a discrete valuation p. We denote the valuation ring by $\mathcal{O}_p$. Instead of the symmetric space in the archimedian case, here we take the Bruhat-Tits building $\triangle$ of $G(K_p)$ [2]. It is a $rk_{K_p}G$-dimensional contractible simplicial complex. One defines a metric on $\triangle$ by glueing the Euclidean metrics of simplicial together, and it make $\triangle$ to be nonpositive curved space in the global sense. $G(K_p)$ is then the isometry group of $\triangle$. In the same way we consider a ρ equivariant map $u : \tilde{X} \to \triangle$ A point $x_0 \in \tilde{X}$ is called a *regular* point of u if there exists a ball $B(x_0, \sigma_0)$ of radius $\sigma_0 > 0$ and a $\mathrm{rk}_{K_p}G$−flat $F \subset \triangle(G)$ with $B(x_0, \sigma_0) \subset F$. (see

[8], page 68). And $S(u) = \tilde{X} \setminus \tilde{X}^r$ is called the singular set of u. In the same way We define the energy of u by $E(u) := \int_X < du, du > dV$. The following theorem is proved in [8]in the $p-$adic number field case and in the function field case in [11]

Theorem 1.3 *There exists a $\rho-$equivariant pluriharmonic map (in fact, it is unique) if and only if ρ is reductive.*

Remark. Note that the harmonic map in the function field case will be used in the case where some unbounded representation will be produced from nonrigid representations (see the second construction before of Lemma of 2.6). It makes the argument shorter. Nevertheless, our main application will be in the $p-$adic number field case. In fact, harmonic maps in the $p-$adic number field case suffice for all applications given in the present paper (see the first construction before lemma 2.6).

Instead by Higgs bundle, the similar holomorphic object here is a collection of equivariant holomorphic 1-forms. ([JZ]) We start with an apartment $A \simeq \mathrm{R}^{n-1}$ of $\triangle$. Let $W = Z \times \bar{W}$ be the affine Weyl group of $\triangle(G)$, here, $\bar{W}$ is the usual Weyl group of $G(K_v)$. which operates on A as a finite linear subgroup generated by reflections, and Z^r acts on A as the usual translations. Let $\mathcal{R} = \{\beta_1, ..., \beta_l\}$ be the roots system of $\bar{W}$, where β_i are normalized vectors in R^r and $\beta^\perp$ are the reflection hyperplanes. We can consider β_i as coordinate functions on A by orthogonal projection from A to β_i. Now, by taking the differential, we get a collection of differential 1-forms $\{d\beta_1, ..., d\beta_l\}$ on A. One can glue these collections of 1-forms on all apartments together. It turns out, however, a $l-$valued 1-form ω, since on the common part of two apartments these two collections coincide as set, but the orders of the 1-forms in these two collections are differed by a permutation from $\bar{W}$. Here, a differential form on $\triangle$ means that its restriction to each apartment is a usual differential form. Equivalently, by taking a base of the invariant polynomials, one gets a collection of single valued differential forms $\alpha_1, ..., \alpha_l$ in symmetric tensor product such that the $l-$valued 1-form ω is the roots of the polynomial $\alpha^l + \alpha_1 t^{l-1} + ... + \alpha_l$. (see also [19] and [8] for SL_2 case). By taking the complexified pull back $u^{*c}(\omega)$ via the differential du, its (1,0)-part is a $l-$valued a holomorphic 1-form on $\tilde{X}$ which we call again ω. It can be seen as follows: First, ω is holomorphic on $\tilde{X}^r = \tilde{X} \setminus S$. Since u is Lipschitz, du is bounded near S. Noting that $\operatorname{codim} S \geq 2$, we extend ω over S. Since ω is $\pi_1(X)-$invariant, it descends a $l-$valued holomorphic 1-form that we call again ω on X. The image $\omega(X)$ in the

total space of holomorphic cotangent bundle X is a subvariety, and can be regarded as a $l-$fold covering of X under the nature projection. There is an equivalent way to see that, pulling back the single valued differential forms $\alpha_1, .., \alpha_l$ via the complexified differential $d^c u$, this gives the single valued holomorephic forms which we call again $\alpha_1, ..., \alpha_l$ of symmetric tensor on X. Then the polynomial $t^l + \alpha_1 t^{l-1} + ... + \alpha_l$ defines the suvariety $\omega(X)$ in T^*_X.
Now by taking the Galois closure of the function field extension $K(\omega(X))/K(X)$, we get a Galois covering $\sigma : X^s \to X$ and $\sigma^*\omega$ splits into l single valued 1-form $\omega_1, ..., \omega_l$. They are just $d'(u\sigma)\beta_1, ..., d'(u\sigma)\beta_l$. The Galois group is a subgroup of $\bar{W}$. We obtain ([11])

Lemma 1.4 *There exists a finite ramified Galois covering $\sigma : X^s \to X$ so that differentials $d'(u\sigma)\beta_1, ..., d')(u\sigma)\beta_l$ of the coordinate functions on all $\mathrm{rk}_{K_p} G$-flats chosen as above piece together and yield l single valued holomorphic 1-forms $\omega_1, ..., \omega_l$ on X^s*

2. Proof of Theorem 1

What we really need to do in Theorem 1 is to show either ρ is finite, or there exists a fibration $f : X \to C$ such that the restriction $\rho|_{f^{-1}}$ to a generic fiber of f is finite. This implies already that after passing to a finite etale covering and blowing up the pulled back of ρ factors through the Stein factorization of f, if one uses the fact that $\rho\pi_1(X)$ is residually finite and the same argument in [12], Theorem 4.5.

First we prove Theorem 1 the abelian case. This is an easy consequence form a sharper version of Bogomolov's lemma

Lemma 2.1 *Let X be an algebraic surface. Suppose that X contains a configuration of curves $i : B \hookrightarrow X$ such that there is a nonzero holomorphic 1-form ω on on X with the pulled back $i^*\omega = 0$. If there is a numerically nonzero divisor D supported on B with $D^2 = 0$. Then there exists a fibration $f : X \to C$ of connected fibers over a curve C such that ω factors though f and D is a fiber of f.*

Proof Consider the Albanese map $\Phi : X \to Alb(X)$. By taking the quotient of $Alb(X)$ divided by the abelian subvariety generated by the images of integral subvarieties of ω, we obtain a surjective map of connected fibers

$f : X \to Y$. We notice that ω factors through f, so, f is not constant. We claim Y must be a curve. It can be seen as follows. Suppose Y is a surface. Since $f(D)$ is points, we find an ample curve $H \subset Y$ such that it does not intersect with $f(D)$, hence, $f^*(H)D = 0$. Now Hodge index theory implies $D^2 < 0$. This is a contradiction. So, we show that Y is an algebraic curve and D is contained in some fiber of f. Since $D^2 = 0$, D is a fiber of f. Lemma 2.1 is proved.

Corollary 2.2 *Let X and D be the same as in Theorem 1, then either $H_1(X, Z)$ is finite, or there exists a fibration $f : X \to C$ over a curve such that the image $H_1(f^{-1}(c), Z) \to H_1(X, Z)$ is finite for every fiber of f, and D is a fiber of f.*

The following lemma is a nonabelian version of Cor.2.2 for discrete Zariski dense representation into a semisimple Lie group of noncompact type.

Lemma 2.3 *Let X and D be the same as in Theorem 1. Suppose there exists a discrete Zariski dense representation ρ of $\pi_1(X)$ into a semisimple Lie group G of noncompact type, then there exists a fibration $f : X \to C$ over a curve C such that ρ factors through f and D is a fiber of f.*

Proof Since ρ is not unitary, u is non constant. We claim that the foliation $\mathcal{F}$ on $\tilde{X}$ defined by $d'u$ is a 1-dimensional family of curves. This can be seen as follows. If the generic leaves of $\mathcal{F}$ are points, then the associated semi Kähler form β of g in Lemma 1.2 is strictly positive on an open subset of X. Hence, $\int_X \beta \wedge \beta > 0$. Noting that u maps every rational curve P^1 from D to point and g is a pull back via u, we obtain $\int_{P^1} \beta = 0$. This implies also the intersection of the cohomology classes $\beta \wedge c_1(\mathcal{O}_X(D)) = 0$. Applying Hodge index theorem, we obtain either $D^2 < 0$, or $c_1(\mathcal{O}_X(D)) = 0$. The both cases lead contradictions. So, we proved that $\mathcal{F}$ is a 1-dimensional family of curves on $\tilde{X}$. Since ρ here satisfies the condition in Mok′s factorization theorem ([16]), it descends a fibration on a blowing up $f : \widehat{X} \to C$, such that ρ factors through f. In fact, f descends a fibration on X in the surface case. It can be seen as follows, suppose it does not, then there exists a exceptional curve on $\widehat{X}$ which maps onto C. This implies that ρ is trivial. But, this is a contradiction. D is contained in a fiber, since every P^1 from R is mapped by u to point. D is a fiber of f, since $D^2 = 0$. Lemma 2.3 is proved.

Now we are in a position to prove Theorem 1. Let ρ be a reductive representation into some $GL_n(\mathrm{C})$. Taking its Zariski closure, we get a Zariski dense representation $\rho : \pi_1(X) \to \overline{\rho\pi_1(X)}$. Since ρ is reductive, $\overline{\rho\pi_1(X)} = T \times \prod_i G_i$ is an almost direct product of a torus with some almost simple algebraic groups G_i over C. Clearly, if $\dim \overline{\rho\pi_1(X)} = 0$, then we get the conclusion that ρ is finite. Suppose $\dim \overline{\rho\pi_1(X)} > 0$. We consider the projection into each factor and prove at first Theorem 1 for the representation induced by the projection.

Lemma 2.4 *Let $\rho : \pi_1(X) \to T$ be a Zariski dense representation in to a complex torus T of positive dimension. Then there exists a fibration $f : X \to C$ such that $\rho|_{f^{-1}}$ is finite and D is a fiber of f.*

Proof It is clear that ρ factors through to a representation $\tau : H_1(X, \mathrm{Z}) \to T$. Since τ is a Zariski dense representation into a torus of positive dimension, τ is infinite. Hence $H_1(X, \mathrm{Z})$ is infinite. Applying Cor. 2.2 we complete the proof.

Now we consider Zariski dense representations of $\pi_1(X)$ into an almost simple complex linear algebraic group G. We may assume G is defined over some number field L after a conjugation in $GL_n(\mathrm{C})$. So, the space $R(G)$ of representations of $\pi_1(X)$ into G and the moduli space $M(G)$ are also algebraic varieties defined over L, since $\pi_1(X)$ is a finitely presented group. We divide $R(G)$ into the following two types

Type A 1) ρ is rigid in $G(\mathrm{C})$, i.e. $[\rho]$ is an isolated point in $M(G)$. Hence ρ is valued in some number field K after a conjugation.
2) ρ is $p-$bounded for every prime ideal p from $\mathcal{O}_K$, the ring of the algebraic integers of K.

Type B Either ρ is non rigid in $G(\mathrm{C})$, or ρ is valued in a number field K and is $p-$unbounded for a prime ideal p from $\mathcal{O}_K$.

Lemma 2.5 *Suppose that ρ is of Type A. Then there exists a fibration $f : X \to C$ such that D is a fiber of f and $\rho|_{f^{-1}}$ is finite.*

Proof If ρ is of Type A, then ρ is more or less closer to a discrete representation, and we may apply Lemma 2.3. More precisely, by condition 2) in Type A $\rho\pi_1(X) \cap G(\mathcal{O}_K)$ is a subgroup of $\rho\pi_1(X)$ of finite index ([20], page 120-121). So, after passing to a finite etale covering of X, we may assume

that $\rho\pi_1(X) \subset G(\mathcal{O}_K)$. We now need to consider *Restriction of Scalars* ([20], page 116-120). Let $\sigma_1(= \text{id.}), ..., \sigma_d$ be distinct embeddings of K into C, $R_{K/\mathrm{Q}}(G) = \prod_{i=1}^d G^{\sigma_i}$ and $\alpha : G(K) \to R_{K/\mathrm{Q}}(G)$ be the diagonal embedding. It is well known that $R_{K/\mathrm{Q}}(G)$ is (isomorphic to) an algebraic Q−group, such that $\alpha G(K) = R_{K/\mathrm{Q}}(G)(\mathrm{Q})$ and $\alpha G(\mathcal{O}_K) = R_{K/\mathrm{Q}}(G)(\mathrm{Z})$. The Zariski closure $H := \overline{\alpha\rho\pi_1(X)} \subset R_{K/\mathrm{Q}}(G)$ is then an algebraic Q−group, and is semisimple. This can be argued as follows: The projection $p_i : R_{K/\mathrm{Q}}(G) \to G^{\sigma_i}$ satisfies $p_i\alpha = \sigma_i$. Since $\sigma_i\rho\pi_1(X)$ is Zariski dense in G^{σ_i}, $p_i(H) = G^{\sigma_i}$. The radical of H must be trivial. Otherwise, it would be projected to a G^{σ_i} by p_i with the positive dimensional image. So, the image is a nontrivial algebraic solvable normal subgroup of G^{σ_i}. A contradiction to the fact that G^{σ_i} is semisimple.
The diagonal embedding α induces now a discrete and Zariski dense representation

$$\tau = \alpha\rho : \pi_1(X) \to H(\mathrm{Z}).$$

In order to make Lemma 2.3 applicable, we decompose $H(\mathrm{R}) = F_1 \times F_2$ into a noncompact factor F_1 and a compact factor F_2. The projection $q_1\tau$ into F_1 is again Zariski dense and discrete, since F_2 is compact.
Applying Lemma 2.3 to $q_1\tau : \pi_1(X) \to F_1$ we get a fibration $f : X \to C$ so that $q_1\tau|_{f^{-1}}$ is finite. In particular, this implies that $\tau|_{f^{-1}}$ is compact. Since $\tau|_{f^{-1}}$ is also discrete, $\tau|_{f^{-1}}$ is finite. Because $\tau = \alpha\rho$ is the diagonal embedding of ρ, $\rho|_{f^{-1}}$ is also finite. Lemma 2.5 is completed.

Suppose now ρ is of Type B. Namely, either ρ is nonrigid in $G(\mathrm{C})$, or there exists a p−adic number field K_p such that $\rho\pi_1(X) \subset G(K_p)$ is p−unbounded.
For a nonrigid ρ we may associate ρ to some unbounded representations in the following two different ways.

The first construction It avoid to involve representations over function field. Choosing an integral structure for G, then $M(G)$ is a variety defined over $Spec(\mathcal{O}_K)$. Taking the completion $K_p \supset K$ for a prime ideal p from $\mathcal{O}_K$, we may assume $M(G)$ is defined over $Spec(\mathcal{O}_{K_p})$. The $\mathcal{O}_{K_p}$−integral points of $M(G)$ correspond to p−bounded representations into $G(\mathcal{O}_{K_p})$. Because ρ is nonrigid, we may find an algebraic curve $T \subset M(G)$ passing through ρ and contains infinitely many non integral points $\{\rho_t\}_{t\in T_0}$. Those points correspond to unbounded representations into $G(E_p)$, where

$E_p \supset K_p$ are some finite extensions of K_p. Since all the Zariski dense representations into a semisimple group G forms a Zariski open subset of $R(G)$, we may assume that $\{\rho_t\}_{t\in T_0}$ are Zariski dense.

The second construction This is due to Simpson [18] which goes back to M. Culler and P.B. Shalen [1]; see also [15] for a nice exposition. Since $R(G)$, $M(G)$ are affine varieties and $\dim_{[\rho]} M(G) > 0$, we may find an affine curve S in the space of representations passing through ρ for which the image in M_B is again an affine curve. Let $\bar{S} = S \cup \infty$ be the completion of S at infinity, and $K(S)_\infty$ be the completion of the function field of $\bar{S}$ w.r.t. the valuation of a point in ∞. The family of the representations $\{\rho_s\}_{s\in S}$ induces tautologically a Zariski dense representation $\rho_S : \pi_1(X) \to G(K(S)_\infty)$ such that $\rho_S|_s = \rho_s$. Clearly, ρ_S is ∞−unbounded, since the image of S in $M(G)$ goes to infinity and hence there are some G−invariant functions on S which are unbounded at infinity.

Lemma 2.6 *Suppose that ρ is of Type B. Then there exists a fibration $f : X \to C$ such that D is fiber of f and $\rho|_{f^{-1}}$ is finite.*

Proof The main idea is to apply harmonic maps for unbounded representations into Bruhat-Tits building. First we consider ρ is a unbounded Zariski dense representation into $G(K_p)$. Applying Theorem 1.3 in the p−adic number field case, we get a nonconstant ρ−equivariant pluriharmonic map u into the corresponded Bruhat-Tits building. Hence by Lemma 1.2 the (1,0)-part $d'u$ gives rise a collection of non zero holomorphic 1-forms $\{\omega_1, ..., \omega_l\}$ on a finite branched Galois covering $\sigma : X^s \to X$. For every rational curve $P^1 \subset D$, the restriction of the pulled back representation $\sigma^*\rho|_{\sigma^* P^1}$ is again trivial. This implies that the pulled back of all ω_i to $\sigma^{-1}P^1$ are zero. Applying Lemma 2.1 we get a collection of fibrations $\{f_i : X^s \to C_i\}$ such that ω_i factors through f_i. Since all these fibrations contain a common fiber $\sigma^* D$, they coincide with a fibration $f^s : X^s \to C^s$. Because the Galois group of the covering $\sigma : X^s \to X$ permuts $\{\omega_1, ..., \omega_l\}$, f^s is equivariant. Hence it descends to a fibration $f : X \to C$ so that u factors through it.

Claim 2.7 *Suppose ρ is a unbounded Zariski dense representation into an almost simple algebraic group G. and If the ρ−equivarinat harmonic map u into Bruhat-Tits building factors through a fibration $f : X \to C$. Then $\rho|_{f^{-1}}$ is finite.*

Proof of Claim 2.7 Since $u : \tilde{X} \to \Delta$ is unbounded, the convex subcomplex of Δ which is generated by $u(\tilde{X})$ contains at least one geodesic line L. Fixing a generic point $c_0 \in C$, then $\rho\pi_1(f^{-1}(c_0))$ fixes c_0. In fact, it fixes L, since for any two generic points $c_1, c_2 \in C$, the images of $\pi_1(f^{-1}(c_1)), \pi_1(f^{-1}(c_2))$ in $\pi_1(X)$ coincide, thus $\pi^*\rho\pi_1(f^{-1}(c_0))$ fixes $u(\tilde{X})$, hence L. This implies that $\rho|_{f^{-1}}$ is contained in the normalizer of some torus in G.
Now let $f : f^{-1}(C^0) \to C^0$ the regular part of the fibration such that all fibers over C^0 are smooth and connected. Since X is smooth, the inclusion $f^{-1}(C^0) \hookrightarrow X$ induces the surjective map of the fundamental groups $\pi_1(f^{-1}(C^0)) \to \pi_1(X)$, so $\rho|_{f^{-1}(C^0)}$ is again Zariski dense in G.
Now we consider the homotopy exact sequence

$$\pi_1(f^{-1}(c_0)) \longrightarrow \pi_1(f^{-1}(C^0)) \longrightarrow \pi_1(C^0) \longrightarrow 1,$$

in particular, $\pi_1(f^{-1}(c_0))$ is normal in $\pi_1(f^{-1}(C^0))$. So, the Zariski closure $\overline{\rho\pi_1(f^{-1}(c_0))}$ is a normal algebraic subgroup of $\overline{\rho\pi_1(f^{-1}(C^0))} = G$. We have shown already that $\rho\pi_1(f^{-1}(c_0))$ is contained in the normalizer of some torus in G, and since G is almost simple, so $\rho\pi_1(f^{-1}(c_0))$ must be finite. The claim is proved.

By Claim 2.7 we proved Lemma 2.6 for unbounded representations.

Proof for the nonrigid case Now we consider nonrigid Zariski dense representations into G. We state two proofs here according to the above different constructions of unbounded representations.

The first proof Let $\{\rho_t\}_{t\in T_0}$ be the family of unbounded representations in the first construction. Applying Theorem 1.3 in the p–adic number field case and using the same argument as the above, we obtain a family of fibrations $\{f_t : X \to C_t\}_{t\in T_0}$ such that $\rho_t|_{f_t^{-1}}$ is finite. Since all f_t have a common fiber D, all f_t coincide with a fibration $f : X \to C$. Hence $\rho_t|_{f^{-1}}$, $t \in T_0$ are finite. In particular, they are not Zariski dense. Since T_0 is a Zariski open subset of the algebraic curve $T \subset M(\pi_1(f^{-1}) \to G)$ and the subset of Zariski dense representation is a Zariski open subset of $M(\pi_1(f^{-1}) \to G)$, $\rho|_{f^{-1}}$ is also not Zariski dense. Finally, applying the homotopy exact sequence as above, we show that $\rho|_{f^{-1}}$ is finite. The first proof is completed.

The second proof Applying Theorem 1.3 in the function field case to the unbounded representation ρ_S in the second construction, we get a nonconstant ρ_S–equivariant pluriharmonic map u into the corresponded Bruhat-Tits building. Using the same argument in the unbounded case, we obtain a fibration $f : X \to C$ such that $\rho_S|_{f^{-1}}$ is finite. This implies that $\rho|_{f^{-1}}$ is also finite. Lemma 2.6 is completed.

Proof of Theorem 1 This is a direct consequence from Lemmas 2.4-2.6. Since all fibrations appear in Lemmas 2.4-2.6. have a common fiber D, they are all the same.

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