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On the nonlinear parabolic equation $\partial_t u = F(\Delta u + nu)$ on S^n

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1. Introduction.

In this paper, we consider the following initial value problem on the unit n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$

$$(1.1) \qquad \begin{aligned} \partial_t u(x,t) &= F(\Delta u(x,t) + nu(x,t)) \qquad x \in S^n, \ t \geq 0 \\ u(x,0) &= u_0(x) \qquad x \in S^n \end{aligned}$$

where $F : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth increasing function with $F' > 0$ everywhere and Δ is the Laplace operator on S^n with respect to the standard Riemannian metric.

Technically speaking, this is a fully nonlinear second order parabolic equation, however because of its special form, it is perhaps more similar to a semilinear equation. The difficulty in studying this equation, in particular, proving long time existence, is to obtain apriori estimates strong enough to bound the solution in $C^{2,\alpha}$ norm, after which standard theory yields the existence of a smooth solution.

One importance of equation (1.1) is that it is equivalent to a geometric equation, which may be interpreted as follows. If in addition, the symmetric 2-tensor $(\nabla\nabla u + ug)(x, t_0)$ is positive definite for all $x \in S^n$ (here g denotes the standard Riemannian metric on S^n and ∇ the corresponding covariant derivative) at a given time t_0, then $u(\cdot, t_0)$ is the support function of a smooth convex hypersurface M_{t_0}.

More precisely, we can extend $u(\cdot, t_0)$ to be a smooth, homogeneous of degree one function on $\mathbb{R}^{n+1} - \{0\}$. Since $(\nabla\nabla u + ug)(x, t_0)$ is positive definite for all $x \in S^n$, $u(\cdot, t_0)$ will then be a convex function on $\mathbb{R}^{n+1} - \{0\}$. It can be shown that $u(\cdot, t_0)$ is the support function of a unique convex hypersurface M_{t_0}, which is the boundary of the convex body B_{t_0} given by

$$B_{t_0} = \cap_{x \in S^n} \{y \in \mathbb{R}^{n+1} : \langle x, y\rangle \leq u(x, t_0)\}.$$

Furthermore the eigenvalues of the 2−tensor $(\nabla\nabla u + ug)(\cdot, t_0)$ at the point $x \in S^n$ are exactly the principal radii of the convex hypersurface M_{t_0} at the point with outer unit normal x. We refer the readers to the paper by Urbas [14] for more details.

At such times, if in addition $F > 0$ on $(0, \infty)$, equation (1.1) corresponds to deforming the convex hypersurface M_{t_0} in the direction of its outward unit normal with velocity $F(H)$, where H is the inverse of the harmonic mean curvature, that is, $H = \frac{1}{\kappa_1} + \frac{1}{\kappa_2} + \cdots\cdots + \frac{1}{\kappa_n}$, where $\kappa_1, ..., \kappa_n$ are the principal curvatures of M_{t_0}.

It is well-known that there exists a unique smooth solution to (1.1) on $S^n \times [0, T)$ for some short time $T > 0$. We shall prove that the solution exists for long time. More precisely, the solution exists on $S^n \times [0, T_{\max})$,where either $T_{\max} = \infty$ or $\lim_{t \to T_{\max}} \min_{S^n \times \{t\}} |u| = \infty$. In section 2, we recall the gradient estimate of Chow-Gulliver [2]. In section 3, we prove the crucial Laplacian estimate. In section 4, we prove the long time existence theorem. This raises the question of the asymptotic behavior of the solution. For example, one can ask the following questions: Is $\|\nabla u\|_{C^k(S^n)}$ uniformly bounded for some or all $k \geq 0$? Under what conditions, does the solution satisfy $(\nabla\nabla u + ug)(x, t) > 0$ for all time (i.e. evolving hypersurfaces remain convex) if $(\nabla\nabla u_0 + u_0 g) > 0$ on S^n? Under what conditions, does the solution satisfy $(\nabla\nabla u + ug)(x, t) > 0$ (i.e., u is the support function of a convex hypersurface) for t sufficiently close to $T_{\max}$?

In section 5, 6 and 7, we impose additional hypotheses on F and answer these questions in part. In section 5 we show that the second derivative of u is uniformly bounded under suitable assumption on F. Therefore we have $(\nabla\nabla u + ug)(x, t) > 0$ if we know $u(x, t)$ becomes positively large eventually and hence $u(x, t)$ is the support function of a convex hypersurface for t sufficiently close to $T_{\max}$. In section 6 we assume $F > 0$ on $(0, \infty)$ and assume the initial data u_0 is the support function of a given smooth convex hypersurface M_0. We then investigate the long time behavior of the evolving hypersurface M_t. In section 7 we show how to rescale the hypersurface M_t so that it converges to the unit round sphere S^n in C^1 or C^2 norms.

The main innovation of this paper is that we consider certain *nonhomogeneous* curvature flows of closed convex hypersurfaces. There are numerous important works on homogeneous curvature flows. One should consult Gage-Hamilton [4], Huisken [10], and Tso [13] for contracting flows; Gerhardt [6], Huisken [9] and Urbas [14] for expanding flows; Andrews [1] for general flows. See also Chow-Tsai [3] for nonhomogeneous expanding curvature flows of closed convex plane curves.

2. The Gradient Estimate.

The first main estimate is a uniform bound for the gradient of u. This is a special case of Theorem 3.1(iv) in Chow-Gulliver [2], which provides a uniform gradient estimate for more general equations based on an Aleksandrov reflection argument.

Proposition 2.1. *Let $u(x,t)$ be the unique smooth solution to equation (1.1) on $S^n \times [0,T)$ where $F : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth increasing function with $F' > 0$ everywhere. There exists a constant C depending only on u_0 such that*

$$|\nabla u(x,t)| \leq C \qquad \text{on } S^n \times [0,T).$$

An immediate consequence of this is

$$u_{\max}(t) - u_{\min}(t) \leq C\pi \qquad \text{on } [0,T),$$

although $u_{\max}(t)$ may blow up as t approaches T.

3. The Laplacian Estimate.

In this section we obtain a crucial estimate for the Laplacian of u on $S^n \times [0,T)$, which depends on F, T, u_0 and $\sup_{S^n \times [0,T)} |u|$. To prove this estimate, we first show that $|F(\Delta u + nu)| \leq C$. To do this, we would like to apply the maximum principle to the evolution equation satisfied by $F(\Delta u + nu)$. However, there is a uncontrolled bad term which appears on the right-hand side. To overcome this, the idea is to consider the new quantity

$$Q = (|\nabla u|^2 + a)F(\Delta u + nu),$$

where a is some large constant (this idea first appears in Liou [12]). Choosing a to be a large constant depending only on the initial condition u_0, we estimate Q, which implies an estimate for $F(\Delta u + nu)$. The estimate for Q holds roughly for the following reasons. The evolution equation for $|\nabla u|^2$ introduces good (i.e., negative) term which dominates the bad term in the evolution equation for $F(\Delta u + nu)$. One also shows that any positive term introduced by $|\nabla u|^2$ is controllable by using the fact that we already know $|\nabla u|^2$ is bounded by Proposition 2.1. Once we have shown $|F(\Delta u + nu)| \leq C$, we can apply the maximum principle to the evolution equation for $\Delta u + nu$ to get an estimate for Δu. We provide the details of our argument below.

Let $\varphi = F(\Delta u + nu)$ and $H = \Delta u + nu$. Recall that $Q = (|\nabla u|^2 + a)\varphi$, where a is a constant to be chosen later. First, we compute the following evolution equations.

Lemma 3.1. *Under the evolution equation (1.1) on $S^n \times [0,T)$, we have*

$$(a)\quad \partial_t |\nabla u|^2 = F'(H)\left[\Delta |\nabla u|^2 - 2|\nabla\nabla u|^2 + 2|\nabla u|^2\right]$$

and

$$(b)\quad \partial_t Q = F'(H)\left\{\Delta Q - 2\frac{\nabla |\nabla u|^2}{|\nabla u|^2 + a}\cdot \nabla Q + R\varphi\right\}$$

where

$$R = (n+2)|\nabla u|^2 + na - 2|\nabla\nabla u|^2 + \frac{2\left|\nabla |\nabla u|^2\right|^2}{|\nabla u|^2 + a}.$$

Proof. (a) It is easy to see that

$$\partial_t |\nabla u|^2 = 2F'(H)\nabla_i(\Delta u + nu)\nabla_i u,$$

where we have followed the usual summation convention. That is, $\nabla_i(\Delta u + nu)\nabla_i u$ means $g^{ij}\nabla_i(\Delta u + nu)\nabla_j u$ and we sum over repeated indices. The formula for commuting the Laplacian and covariant derivative on a general Riemannian manifold is given by

$$\Delta\nabla_i f - \nabla_i \Delta f = R_{ij}\nabla_j f$$

where f is a smooth function and R_{ij} is the Ricci tensor. On S^n, we have $R_{ij} = (n-1)g_{ij}$ and hence the equations

$$\Delta\nabla_i u - \nabla_i\Delta u = (n-1)\nabla_i u,$$

and

$$\Delta |\nabla u|^2 = 2\Delta\nabla_i u\nabla_i u + 2|\nabla\nabla u|^2,$$

which we substitute into the right hand side of $\partial_t |\nabla u|^2$ to obtain (a).

(b) Using (a) and

$$\partial_t\varphi = F'(H)\left[\Delta\varphi + n\varphi\right],$$

we obtain

$$\partial_t Q = F'(H)\left\{\Delta Q - 2\nabla |\nabla u|^2 \cdot \nabla\varphi - 2|\nabla\nabla u|^2\varphi + 2|\nabla u|^2\varphi + n(|\nabla u|^2 + a)\varphi\right\}$$

Now (b) follows by the formula

$$\nabla |\nabla u|^2 \cdot \nabla \varphi = \frac{1}{|\nabla u|^2 + a} \left\{ \nabla |\nabla u|^2 \cdot \nabla Q - \varphi \left| \nabla |\nabla u|^2 \right|^2 \right\}.$$

□

Proposition 3.2. *Under the evolution equation (1.1) on $S^n \times [0,T)$, there exists a constant C depending on F, u_0 and $\sup_{S^n \times [0,T)} |u|$, such that*

$$|F(\Delta u + nu)| \le C \qquad \text{on } S^n \times [0,T).$$

Proof. Recall that $\varphi = F(\Delta u + nu)$ and the evolution of φ is given by

$$\partial_t \varphi = F'(H)\,[\Delta \varphi + n\varphi] = F' \circ F^{-1}(\varphi)\,[\Delta \varphi + n\varphi]\,.$$

Since the term $n\varphi F' \circ F^{-1}(\varphi)$ is nonlinear in φ, we cannot immediately apply the maximum principle to estimate φ. In fact, looking at the corresponding O.D.E.

$$\frac{d\psi}{dt} = n\psi F' \circ F^{-1}(\psi)$$

suggests that φ may blow up in finite time and before u blows up. We overcome this difficulty as follows. By Proposition 2.1, we have a uniform bound for $|\nabla u|$. Let a be a positive constant (to be chosen equal to $7 \sup_{S^n \times [0,T)} |\nabla u|^2$), we consider the quantity $Q = (|\nabla u|^2 + a)\varphi$. By Lemma 3.1 (b) we have

$$\partial_t Q = F'(H) \left\{ \Delta Q - 2 \frac{\nabla |\nabla u|^2}{|\nabla u|^2 + a} \cdot \nabla Q + R\varphi \right\}$$

where R is defined by

$$R = (n+2)\,|\nabla u|^2 + na - 2\,|\nabla \nabla u|^2 + \frac{2\left| \nabla |\nabla u|^2 \right|^2}{|\nabla u|^2 + a}$$

Using the fact that

$$\left| \nabla |\nabla u|^2 \right|^2 \le 4\,|\nabla \nabla u|^2\,|\nabla u|^2$$

and choosing $a = 7 \sup_{S^n \times [0,T)} |\nabla u|^2$, we obtain

$$\frac{2\left|\nabla |\nabla u|^2\right|^2}{|\nabla u|^2 + a} \leq |\nabla\nabla u|^2 \qquad \text{on } S^n \times [0,T).$$

This implies

$$R \leq (n+2)\,|\nabla u|^2 + na - |\nabla\nabla u|^2 \qquad \text{on } S^n \times [0,T).$$

Since

$$|\nabla\nabla u|^2 \geq \frac{1}{n}(\Delta u)^2$$

and $a = 7 \sup_{S^n \times [0,T)} |\nabla u|^2$, we have

$$R \leq (8n+2) \sup_{S^n \times [0,T)} |\nabla u|^2 - \frac{1}{n}(\Delta u)^2.$$

Hence, at any point in $S^n \times [0,T)$ where $Q \geq 0$ (this is equivalent to $\varphi \geq 0$), we have

$$\partial_t Q \leq F'(H) \left\{ \Delta Q - 2\frac{\nabla(|\nabla u|^2)}{|\nabla u|^2 + a} \cdot \nabla Q + \left[(8n+2) \sup_{S^n \times [0,T)} |\nabla u|^2 - \frac{1}{n}(\Delta u)^2 \right] \varphi \right\}$$

(The same statement holds with the inequality reversed wherever $Q \leq 0$). We may rephrase the above inequality as follows. At any point in $S^n \times [0,T)$ where

$$|\Delta u| \geq \sqrt{n(8n+2)} \sup_{S^n \times [0,T)} |\nabla u|$$

and $Q \geq 0$, we have

$$\partial_t Q \leq F'(H) \left\{ \Delta Q - 2\frac{\nabla(|\nabla u|^2)}{|\nabla u|^2 + a} \cdot \nabla Q \right\}.$$

Let p_t and q_t be points in S^n at which $\max_{S^n \times \{t\}} Q = Q(p_t, t)$ and $\min_{S^n \times \{t\}} Q = Q(q_t, t)$. Define

$$M = 8\left(\sup_{S^n \times [0,T)} |\nabla u|^2 \right) F(\sqrt{n(8n+2)} \sup_{S^n \times [0,T)} |\nabla u| + n \sup_{S^n \times [0,T)} u)$$

Since F is an increasing function and by the definition of a, we have that if $Q \geq \max\{M, 0\}$, then $|\Delta u| \geq \sqrt{n(8n+2)} \sup_{S^n \times [0,T)} |\nabla u|$. Hence, if $Q \geq \max\{M, 0\}$ at (p_t, t), then

$$\partial_t Q \leq 0.$$

Therefore

$$Q(x,t) \leq \max\left\{ \max_{x \in S^n} Q(x,0),\ M,\ 0 \right\}.$$

Similarly, one shows that

$$Q(x,t) \geq \min\left\{ \min_{x \in S^n} Q(x,0),\ m,\ 0 \right\}$$

where

$$m = 8 \left(\sup_{S^n \times [0,T)} |\nabla u|^2 \right) F(-\sqrt{n(8n+2)} \sup_{S^n \times [0,T)} |\nabla u| + n \inf_{S^n \times [0,T)} u).$$

Hence Q is bounded from above and below by constants depending only on F, $\sup_{S^n \times [0,T)} |u|$ and $\sup_{S^n \times [0,T)} |\nabla u|$, and the proposition follows from the uniform gradient estimate. □

Proposition 3.3. *Under the evolution equation (1.1) on $S^n \times [0,T)$, there exists a constant C depending on F, T, u_0 and $\sup_{S^n \times [0,T)} |u|$ such that*

$$|\Delta u(x,t)| \leq C \qquad \text{on} \quad S^n \times [0,T).$$

Proof. Recall $H = \Delta u + nu$. We compute its evolution equation

$$\partial_t H = F'(H) \Delta H + F''(H) |\nabla H|^2 + nF(H).$$

Since $|F(H)| \leq C$, by the maximum principle, H is bounded above and below depending on time T.

4. Higher Derivatives Estimate and Long Time Existence.

We shall consider the evolution equation for φ again, which is

$$\partial_t \varphi = F' \circ F^{-1}(\varphi) \left[\Delta \varphi + n\varphi \right].$$

Since H and φ are bounded, $F' \circ F^{-1}(\varphi) = F'(H)$ is bounded above and below by positive constants. Hence we may apply Theorem 4.2 in Krylov-Safonov [11] to conclude that φ is bounded in C^{α} norm[1], which implies that $F' \circ F^{-1}(\varphi)$ is bounded in C^{α} norm. By standard linear theory (See Chapter 3 of Friedman [5].) we have a bound for the $C^{2,\alpha}$ norm of φ, which implies that $F' \circ F^{-1}(\varphi)$ is bounded in $C^{2,\alpha}$ norm. In this way we can apply bootstraps argument to get bounds on all $C^{k,\alpha}$ norms of φ. Because $H = F^{-1}(\varphi)$ and F^{-1} is a smooth function, we can bound all of the $C^{k,\alpha}$ norms of H , which easily yields the bounds on all $C^{k,\alpha}$ norms of u.

From the above argument we have shown that, if $\sup\limits_{S^n \times [0,T)} |u| < \infty$, then all the $C^{k,\alpha}$ norms of u are bounded depending on F, T, u_0 and $\sup\limits_{S^n \times [0,T)} |u|$. Denote the maximal time interval on which the solution to (1.1) exists by $[0, T_{\max})$. We have □

Theorem 4.1. *(Long time existence) The solution $u(x,t)$ to the equation (1.1) exists on the maximal time interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $\lim\limits_{t \to T_{\max}} \min\limits_{S^n \times \{t\}} |u| = \infty$.*

Proof. Suppose $T_{\max} < \infty$ and $\sup\limits_{0 \le t < T_{\max}} \min\limits_{S^n \times \{t\}} |u| < \infty$, then by Proposition 2.1 we have

$$\sup_{S^n \times [0,T_{\max})} |u| = \sup_{0 \le t < T_{\max}} \max_{S^n \times \{t\}} |u| \le \sup_{0 \le t < T_{\max}} \min_{S^n \times \{t\}} |u| + C < \infty.$$

Therefore all the $C^{k,\alpha}$ norms of u are bounded on $[0, T_{\max})$ and we can extend the solution to a larger interval, which contradicts that $[0, T_{\max})$ is the maximal time interval of existence. We hence know that the solution exists on the maximal time interval $[0, T_{\max})$ such that either $T_{\max} = \infty$ or $\sup\limits_{0 \le t < T_{\max}} \min\limits_{S^n \times \{t\}} |u| = \infty$. Finally we observe that the condition $\sup\limits_{0 \le t < T_{\max}} \min\limits_{S^n \times \{t\}} |u| = \infty$ can be replaced by $\lim\limits_{t \to T_{\max}} \min\limits_{S^n \times \{t\}} |u| = \infty$, since either (a) $F(n \min\limits_{S^n \times \{t\}} u) \le 0 \le F(n \max\limits_{S^n \times \{t\}} u)$ for all time, in which case $|u|$ is bounded for all time, or (b) $F(n \min\limits_{S^n \times \{t\}} u)$ becomes positive for some t, which persists, and $\min\limits_{S^n \times \{t\}} u$ keeps increasing, or (c) $F(n \max\limits_{S^n \times \{t\}} u)$ becomes

[1] In this section, all norms are with respect to space and time.

negative for some t, which persists, and $\max_{S^n \times \{t\}} u$ keeps decreasing. The proof of the theorem is now complete. □

5. Second Derivative Estimate.

By now, it should be noted that up to Theorem 4.1 we only assume that $F : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth increasing function with $F' > 0$ everywhere and that the initial function u_0 is arbitrary and smooth. In this section we shall improve the second derivative estimate and show that, under a weak assumption on the growth rate of the function F near $\pm\infty$, the second derivative of u on $S^n \times [0, T_{\max})$ is uniformly bounded by a constant. The result is

Proposition 5.1. *If F satisfies $\lim_{|H|\to\infty} \sup \dfrac{|F''(H)|}{F'(H)} \leq C_0 < \infty$, then under the evolution equation (1.1) on $S^n \times [0, T_{\max})$, there exists a constant C depending only on u_0, C_0 and F such that*

$$|\nabla\nabla u(x,t)| \leq C \qquad \text{on } S^n \times [0, T_{\max}).$$

To prove the proposition, the idea is to obtain a bound on Δu first. After that, we shall consider a quantity involving both Δu and $|\nabla\nabla u|$, in order to bound $|\nabla\nabla u|$. We first need the following lemma.

Lemma 5.2. *Under the hypothesis of Proposition 5.1, there exists a constant C depending on u_0, C_0 and F such that*

$$|\Delta u(x,t)| \leq C \qquad \text{on } S^n \times [0, T_{\max}).$$

Proof. We compute the evolution of Δu, which is

$$\begin{aligned} (\Delta u)_t &= \Delta [F(\Delta u + nu)] \\ &= F'\Delta(\Delta u + nu) + F'' |\nabla \Delta u + n\nabla u|^2 . \end{aligned}$$

Here $F' = F'(H)$ and $F'' = F''(H)$, where $H = \Delta u + nu$. To control the second and the third terms on the right hand side, we consider the quantity

$$\chi = \Delta u + \epsilon |\nabla u|^2 ,$$

where ϵ is a constant to be chosen sufficiently small. The reason for considering χ is as follows. On the right hand side of the evolution equation for Δu there are two uncontrolled terms: $F'\Delta u$ and $F''|\nabla u|^2$. On the other hand, the evolution equation for the quantity $\epsilon|\nabla u|^2$ introduces a good term $-2\epsilon F'|\nabla\nabla u|^2$ on the right hand side which dominates these two bad terms when Δu is sufficiently large. However, to apply the maximum principle, we need to introduce a gradient term of $\epsilon|\nabla u|^2$ to match the gradient term $F''\left[|\nabla\Delta u|^2 + 2n\nabla\Delta u\cdot\nabla u\right]$. One then finds that in order to make the right hand side nonpositive when χ is sufficiently large, one needs to choose ϵ sufficiently small. We provide the details below.

The evolution equation for χ is given by

$$\begin{aligned}\chi_t &= F'\Delta(\Delta u + nu) + F''|\nabla\Delta u + n\nabla u|^2 \\ &\quad + \epsilon F'\Delta|\nabla u|^2 - 2\epsilon F'|\nabla\nabla u|^2 + 2\epsilon F'|\nabla u|^2 \\ &= F'\Delta\chi + nF'\Delta u + F''\left[|\nabla\Delta u|^2 + 2n\nabla\Delta u\cdot\nabla u\right] + n^2F''|\nabla u|^2 \\ &\quad - 2\epsilon F'|\nabla\nabla u|^2 + 2\epsilon F'|\nabla u|^2.\end{aligned}$$

We can convert the term

$$F''\left[|\nabla\Delta u|^2 + 2n\nabla\Delta u\cdot\nabla u\right]$$

into the gradient terms as

$$\begin{aligned}&F''\left[|\nabla\Delta u|^2 + 2n\nabla\Delta u\cdot\nabla u\right] \\ &= F''\left[(\nabla\chi - \epsilon\nabla|\nabla u|^2)\cdot\nabla\Delta u + 2n(\nabla\chi - \epsilon\nabla|\nabla u|^2)\cdot\nabla u\right] \\ &= F''\nabla\chi\cdot\nabla\Delta u - \epsilon F''\nabla|\nabla u|^2\cdot(\nabla\chi - \epsilon\nabla|\nabla u|^2) \\ &\quad + 2nF''\nabla\chi\cdot\nabla u - 2n\epsilon F''\nabla|\nabla u|^2\cdot\nabla u \\ &= F''\nabla\chi\cdot\nabla\Delta u - \epsilon F''\nabla\chi\cdot\nabla|\nabla u|^2 + 2nF''\nabla\chi\cdot\nabla u \\ &\quad + \epsilon^2F''\left|\nabla|\nabla u|^2\right|^2 - 2n\epsilon F''\nabla|\nabla u|^2\cdot\nabla u.\end{aligned}$$

Therefore we get

$$\begin{aligned}\chi_t &= F'\Delta\chi + nF'\Delta u + F''\nabla\chi\cdot\nabla\Delta u - \epsilon F''\nabla\chi\cdot\nabla|\nabla u|^2 \\ &\quad + 2nF''\nabla\chi\cdot\nabla u + \epsilon^2F''\left|\nabla|\nabla u|^2\right|^2 - 2n\epsilon F''\nabla|\nabla u|^2\cdot\nabla u \\ &\quad + n^2F''|\nabla u|^2 + 2\epsilon F'|\nabla u|^2 - 2\epsilon F'|\nabla\nabla u|^2\end{aligned}$$

and hence

$$\begin{aligned}(5.1)\quad \chi_t &\le F'\Delta\chi + nF'\Delta u + F''\nabla\chi\cdot\nabla\Delta u - \epsilon F''\nabla\chi\cdot\nabla|\nabla u|^2 \\ &\quad + 2nF''\nabla\chi\cdot\nabla u + C_1\epsilon^2|F''||\nabla\nabla u|^2 + C_2n\epsilon|F''||\nabla\nabla u| \\ &\quad + C_3n^2|F''| + C_4\epsilon F' - \epsilon F'|\nabla\nabla u|^2 - \frac{\epsilon}{n}F'(\Delta u)^2,\end{aligned}$$

where all the constants C_i are uniform constants depending on n and u_0. Here we have assumed that $\epsilon > 0$ and used Proposition 2.1, inequalities $\left|\nabla |\nabla u|^2\right|^2 \leq 4 |\nabla\nabla u|^2 |\nabla u|^2$ and $|\nabla\nabla u|^2 \geq \frac{1}{n} (\Delta u)^2$ in deriving (5.1). From the hypothesis of Proposition 5.1, it is easy to see that $|F''| \leq C_5 F'$ for some positive constant C_5 depending on C_0 and F. Therefore we can choose $\epsilon > 0$ so small that

$$-F'\epsilon |\nabla\nabla u|^2 + C_1 |F''| \epsilon^2 |\nabla\nabla u|^2 \leq -\frac{1}{2} F'\epsilon |\nabla\nabla u|^2 .$$

By the maximum principle we get

$$\begin{aligned} \frac{d}{dt}\chi_{\max} &\leq F'\left[-\frac{\epsilon}{n}(\Delta u)^2 + n(\Delta u) + C_4\epsilon\right] \\ &\quad -\frac{1}{2}F'\epsilon |\nabla\nabla u|^2 + C_2 C_5 n\epsilon F' |\nabla\nabla u| + C_3 C_5 n^2 F'. \end{aligned}$$

Hence there exists a constant C_6 such that $\frac{d}{dt}\chi_{\max} \leq 0$ if $\chi_{\max} \geq C_6$. This implies $\chi = \Delta u + \epsilon |\nabla u|^2 \leq C$. The inequality $\Delta u + \epsilon |\nabla u|^2 \geq -C$ can be proved analogously by simply choosing $\epsilon < 0$ with $|\epsilon|$ small. Therefore $|\Delta u| \leq C$, and the lemma is proved. □

Proof. [5.1.]We now compute the evolution of $|\nabla\nabla u|^2$, which is

$$\begin{aligned} \nabla_i\nabla_j u_t &= F'[\nabla_i\nabla_j\Delta u + n\nabla_i\nabla_j u] \\ &\quad + F''[\nabla_i\Delta u + n\nabla_i u][\nabla_j\Delta u + n\nabla_j u]. \end{aligned}$$

Using

$$\begin{aligned} \nabla_i\nabla_j\Delta u &= \Delta\nabla_i\nabla_j u - 2n\nabla_i\nabla_j u + 2\Delta u g_{ij} \\ \Delta |\nabla\nabla u|^2 &= 2\Delta\nabla_i\nabla_j u \cdot \nabla_i\nabla_j u + 2|\nabla\nabla\nabla u|^2 \end{aligned}$$

on S^n, we have

$$\begin{aligned} \left(|\nabla\nabla u|^2\right)_t &= F'\left[\Delta|\nabla\nabla u|^2 - 2|\nabla\nabla\nabla u|^2 - 2n|\nabla\nabla u|^2 + 4|\Delta u|^2\right] \\ &\quad + 2F''[\nabla_i\Delta u\nabla_j\Delta u + n\nabla_i\Delta u\nabla_j u \\ &\quad + n\nabla_j\Delta u\nabla_i u + n^2\nabla_i u\nabla_j u]\,\nabla_i\nabla_j u \end{aligned}$$

and hence

$$\begin{aligned} \left(|\nabla\nabla u|^2\right)_t &\leq F'\left[\Delta|\nabla\nabla u|^2 - 2|\nabla\nabla\nabla u|^2 + C_7|\nabla\nabla u|^2\right. \\ &\quad \left. + C_8|\nabla\nabla u||\nabla\Delta u|^2 + C_9|\nabla\nabla u||\nabla\Delta u| + C_{10}\right], \end{aligned} \tag{5.2}$$

where we have used $|F''| \leq C_5 F'$. From (5.2) it is obvious that $|\nabla\nabla u| \, |\nabla \Delta u|^2$ is the main bad term. We shall produce a good negative $|\nabla \Delta u|^2$ term by considering the evolution of the quantity

$$Q^m = (\Delta u + a)^m ,$$

where $a = \sup_{S^n \times [0,T_{\max})} |\Delta u| + 1$, $Q = \Delta u + a > 0$, and m is a large positive number to be determined later. We then get a negative $|\nabla\nabla u|^2 \, |\nabla \Delta u|^2$ term by looking at the evolution of

$$\omega = (Q^m + \beta) \, |\nabla\nabla u|^2 ,$$

where β is also a large positive number to be determined later. Let's compute the evolution for Q^m

$$\begin{aligned} (Q^m)_t &= mQ^{m-1} Q_t \\ &= F' \Delta Q^m - mQ^{m-2} \left[(m-1) F' - QF'' \right] |\nabla \Delta u|^2 \\ &\quad + mnF' Q^{m-1} \Delta u + mQ^{m-1} F'' \left[2n \nabla \Delta u \cdot \nabla u + n^2 |\nabla u|^2 \right] . \end{aligned}$$

Therefore

$$\text{(5.3)} \quad \begin{aligned} (Q^m)_t &\leq F' \Big\{ \Delta Q^m - mQ^{m-2} \left[(m-1) - Q \frac{|F''|}{F'} \right] |\nabla \Delta u|^2 \\ &\quad + C_{11} m Q^{m-1} |\nabla \Delta u| + C_{12} m Q^{m-1} \Big\} . \end{aligned}$$

Let $m = 2 + \sup_{S^n \times [0,T_{\max})} \left(Q \frac{|F''|}{F'} \right)$, then (5.3) becomes

$$(Q^m)_t \leq F' \left[\Delta Q^m - C_{13} |\nabla \Delta u|^2 + C_{14} \right] .$$

Combining (5.2) and (5.3) we have

$$\text{(5.4)} \quad \begin{aligned} \omega_t &\leq F' \Big\{ \Delta \omega - 2 \nabla Q^m \cdot \nabla |\nabla\nabla u|^2 - 2\beta |\nabla\nabla\nabla u|^2 \\ &\quad + (Q^m + \beta) \left[C_7 |\nabla\nabla u|^2 + C_8 |\nabla\nabla u| \, |\nabla \Delta u|^2 \right. \\ &\quad \left. + C_9 |\nabla\nabla u| \, |\nabla \Delta u| + C_{10} \right] \\ &\quad - C_{13} |\nabla\nabla u|^2 |\nabla \Delta u|^2 + C_{14} |\nabla\nabla u|^2 \Big\} . \end{aligned}$$

Because

$$-2 \nabla Q^m \cdot \nabla |\nabla\nabla u|^2 \leq C_{15} |\nabla \Delta u| \, |\nabla\nabla u| \, |\nabla\nabla\nabla u| ,$$

we can choose β to be a large number such that

$$-\frac{C_{13}}{2}|\nabla\nabla u|^2|\nabla\Delta u|^2 - 2\beta|\nabla\nabla\nabla u|^2 - 2\nabla Q^m\cdot\nabla|\nabla\nabla u|^2 \leq 0.$$

Thus (5.4) becomes

$$\begin{aligned}\omega_t \leq\ & F'\Big\{\Delta\omega - \frac{C_{13}}{2}|\nabla\nabla u|^2|\nabla\Delta u|^2 + C_{16}|\nabla\nabla u||\nabla\Delta u|^2\\ & +C_{17}|\nabla\nabla u||\nabla\Delta u| + C_{18} + C_{19}|\nabla\nabla u|^2\Big\}\\ \leq\ & F'\Big\{\Delta\omega + \Big[-\frac{C_{13}}{4}|\nabla\nabla u|^2 + C_{16}|\nabla\nabla u|\Big]|\nabla\Delta u|^2\\ & +\Big[-\frac{C_{13}}{4}|\nabla\Delta u|^2|\nabla\nabla u|^2 + C_{17}|\nabla\Delta u||\nabla\nabla u|\Big]\\ & +C_{18} + C_{19}|\nabla\nabla u|^2\Big\}.\end{aligned}$$

Therefore there exists C_{20}, C_{21}, C_{22} such that, if $|\nabla\nabla u| \geq C_{22}$, which is equivalent to $\omega \geq C'_{22}$ by Lemma 5.2, we have

$$\omega_t \leq F'\Big[\Delta\omega + C_{20}|\nabla\nabla u|^2 + C_{21}\Big]$$

Now let $\zeta = \omega + C_{20}|\nabla u|^2$, then by Lemma 3.1 we get

$$\zeta_t \leq F'\Big[\Delta\zeta - C_{20}|\nabla\nabla u|^2 + C_{23}\Big] \leq F'\Delta\zeta \quad \text{if } \zeta \geq C_{24}.$$

Thus, by the maximal principle we must have $\zeta \leq C$, which completes the proof of Proposition 5.1. □

6. Expanding Flows.

In this section, we shall assume $F > 0$ at least on the interval $(0,\infty)$. If the initial function u_0 is the support function of a convex hypersurface M_0, then equation (1.1) corresponds to expanding M_0 in the direction of its outward unit normal with velocity $F(H)$. As mentioned in the introduction, if the symmetric 2-tensor $\nabla\nabla u + ug$ is positive definite on S^n at some time t_0, then $u(\cdot, t_0)$ is the support function of a convex hypersurface $M_{t_0} \subset \mathbb{R}^{n+1}$. Consideration of this fact leads to the following questions.

1) Under what hypothesis on F is the inequality $\nabla\nabla u + ug > 0$ preserved under the evolution equation? That is, when is the convexity of the initial

hypersurface preserved under the curvature flow (with speed $F(H)$, where $H = \frac{1}{\kappa_1} + \cdots + \frac{1}{\kappa_n}$ is the inverse of the harmonic mean curvature)?

2) Under what hypothesis on F and u_0 will the inequality $\nabla\nabla u + ug > 0$ hold eventually? That is, when will the solution u be the support function of a convex hypersurface for t close enough to $T_{\max}$?

To answer the first question, we assume our initial hypersurface M_0 is convex. Without loss of generality, we may also assume that M_0 enclose the origin. Hence the initial function u_0 is positive on S^n. Let $h_{ij} = \nabla_i\nabla_j u + ug_{ij}$. We have $h_{ij} \geq \delta g_{ij}$ and $H = Trace\ h = \Delta u + nu \geq n\delta$ at $t = 0$, for some $\delta > 0$. The tensor h_{ij} satisfies a special useful identity $\nabla_k h_{ij} = \nabla_i h_{kj}$, which is analogous to the Codazzi equation for the second fundamental form. In fact, for any smooth function v on S^n, the symmetric 2-tensor $t_{ij} = \nabla_i\nabla_j v + vg_{ij}$ on S^n satisfies the identity $\nabla_k t_{ij} = \nabla_i t_{kj}$. The proof is based on the formula for commuting covariant derivatives on S^n, *i.e.*,

$$\nabla_k\nabla_i\nabla_j v - \nabla_i\nabla_k\nabla_j v = -R_{kijl}\nabla_l v$$

and the Riemann curvature operator R_{kijl} satisfies

$$R_{kijl} = g_{kl}g_{ij} - g_{kj}g_{il}$$

on S^n. Recall that the evolution equation of H, which is the trace of $\nabla^2 u + ug$, is given by

$$\partial_t H = F'(H)\Delta H + F''(H)\left|\nabla H\right|^2 + nF(H). \tag{6.1}$$

Since F is positive on $(0, \infty)$, the maximum principle asserts that $H \geq n\delta$ on $S^n \times [0, T_{\max})$. Therefore H has a positive uniform lower bound and, by Proposition 3.3, an upper bound depending on F, $T_{\max}$, u_0 and $\sup_{S^n\times[0,T_{\max})} |u|$. This ensures that, if the hypersurface M_t remains smooth, each principal curvature κ_i of M_t is bounded below by a positive constant. However, controlling H is far from controlling the whole tensor h_{ij}. Therefore, it is possible in principle that some κ_i becomes infinite at some point $\xi \in S^n \times [0, T_{\max})$ (this means that the tensor h_{ij} has a null eigenvector at ξ.) Geometrically speaking, some κ_i becoming infinite means that a singularity develops in the evolving hypersurfaces. To rule this out, we have

Proposition 6.1. *Assume the following*

(i) $F > 0$ *on* $(0, \infty)$,
(ii) $h_{ij} \geq \delta g_{ij}$ *at* $t = 0$, *where* $\delta > 0$ *is a constant,*
(iii) $\frac{2}{n-1}\frac{F'(H)}{H} + F''(H) \geq 0$ *for all* $H \in [n\delta, \infty)$,

then under the evolution equation (1.1) *on* $S^n \times [0, T_{\max})$, *we have*

$$h_{ij} \geq \delta g_{ij} \qquad \text{on } S^n \times [0, T_{\max}).$$

Geometrically speaking, if the initial hypersurface M_0 *is convex and enclose the origin, then the evolving hypersurface* M_t *remains smooth and convex, has positive support function* u, *and expands to infinity such that its support function* u *blows up as* $t \to T_{\max}$.

Proof. Our approach and computation is similar to Urbas [14]. The evolution equation of h_{ij} is given by

$$\begin{aligned} \partial_t h_{ij} &= F'(H)\Delta h_{ij} + F''(H)\nabla_i H \nabla_j H \\ &\quad + \left[F(H) + F'(H)H\right] g_{ij} - nF'(H)h_{ij}. \end{aligned}$$

We can not succeed by applying the maximum principle to the above equation because the bad term $F''(H)\nabla_i H\nabla_j H$ is uncontrolled.[2] To get a stronger result, we compute, instead, the evolution equation of the tensor h^{ij}, which is the inverse of h_{ij}, and obtain an upper bound for h^{ij}. Using the above equation, we obtain

$$\begin{aligned} \partial_t h^{ij} &= F'(H)\Delta h^{ij} - \left[F(H) + F'(H)H\right] h^{ik}h^{jk} + nF'(H)h^{ij} \\ &\quad - h^{ik}h^{jl}\left[F''(H)g_{pq}g_{mn} + 2F'(H)g_{pm}h^{qn}\right]\nabla_k h_{mn}\nabla_l h_{pq} \end{aligned}$$

Now let us suppose that the maximum eigenvalue of $[h^{ij}]$ over S^n at time t is attained at a point $p_t \in S^n$. By a rotation of the frame, we may assume that h^{11} is the maximum eigenvalue and $h^{ij} = 0$ for $i \neq j$. We get

$$\begin{aligned} \partial_t h^{11} &= F'(H)\Delta h^{11} - \left[F(H) + F'(H)H\right](h^{11})^2 + nF'(H)h^{11} \\ &\quad - (h^{11})^2\left[F''(H)g_{pq}g_{mn} + 2F'(H)g_{pm}h^{qn}\right]\nabla_1 h_{pq}\nabla_1 h_{mn} \end{aligned}$$

[2] However, if $F'' \geq 0$ on $(0, \infty)$, we can immediately get $h_{ij} \geq \delta g_{ij}$ on $S^n \times [0, T_{\max})$ by the maximum principle for tensors due to Hamilton [7].

at the point (p_t, t). Since $Hh^{11} \geq n$ at (p_t, t), we know

$$-\left[F(H) + F'(H)H\right](h^{11})^2 + nF'(H)h^{11} \leq 0 \qquad \text{at } (p_t, t).$$

Let $\eta_{pq} = \nabla_1 h_{pq}$. The term

$$(h^{11})^2\left[F''(H)g_{pq}g_{mn} + 2F'(H)g_{pm}h^{qn}\right]\nabla_1 h_{pq}\nabla_1 h_{mn}$$

now becomes

$$(h^{11})^2 F''(H)\left(\sum_p \eta_{pp}\right)^2 + 2(h^{11})^2 F'(H)\sum_{p,q} h^{pp}(\eta_{pq})^2.$$

Using the fact $\eta_{1q} = \nabla_1 h_{1q} = \nabla_q h_{11} = 0$ at (p_t, t) for all $q = 1, 2, 3, ..., n$, and assuming that $h^{11} \geq h^{22} \geq .... \geq h^{nn}$ at (p_t, t), we obtain

$$\begin{aligned}
(6.2)\qquad & (h^{11})^2 F''(H)\left(\sum_p \eta_{pp}\right)^2 + 2(h^{11})^2 F'(H)\sum_{p,q} h^{pp}(\eta_{pq})^2 \\
\geq\; & (h^{11})^2 F''(H)\left(\sum_p \eta_{pp}\right)^2 + 2(h^{11})^2 F'(H)h^{nn}\frac{1}{n-1}\left(\sum_p \eta_{pp}\right)^2 \\
\geq\; & \left[F''(H) + \frac{F'(H)}{H}\frac{2}{(n-1)}\right](h^{11})^2\left(\sum_p \eta_{pp}\right)^2 \\
\geq\; & 0 \quad \text{at } (p_t, t),
\end{aligned}$$

where the first inequality in (6.2) is due to that the indices p, q only run from 2 to n and the last inequality is due to our assumption. Therefore, we obtain

$$\partial_t h^{11} \leq 0 \qquad \text{at } (p_t, t).$$

By the maximum principle, we know each eigenvalue of h^{ij} is bounded above by $\frac{1}{\delta}$ on $S^n \times [0, T_{\max})$ and the theorem follows. □

Remark: (a). For $n = 2$ or 3, we can allow $F(H)$ to be the concave function $\log(H+1)$ on $(0, \infty)$. (b). For arbitrary n, we can allow $F(H)$ to be H^α, where $\alpha \geq 1 - \frac{2}{n-1}$.

To answer the second question we assume $F > 0$ on the whole real line $(-\infty, \infty)$ and apply the maximum principle to the equation (1.1) to yield

$$u(x,t) \geq u_{\min}(t) \geq u_{\min}(0) + F(nu_{\min}(0))t.$$

That is, u increases at least linearly and the solution to equation (1.1) exists until it blows up to $+\infty$. As an immediate consequence of Proposition 5.1, we have

Proposition 6.2. *Assume the following*

$$(i)\ \ F > 0 \ on\, (-\infty, \infty),$$
$$(ii)\ \ \lim_{H\to+\infty} \sup \frac{|F''(H)|}{F'(H)} \leq C_0 < \infty$$

then under the evolution equation (1.1) on $S^n \times [0, T_{\max})$ *with arbitrary initial data* u_0, *we have*

$$\nabla\nabla u(x,t) + u(x,t)g > 0$$

holds eventually. That is, the solution u *will become the support function of a convex hypersurface* M_t *for* t *close enough to* $T_{\max}$.

Remark: (a). Since $F > 0$ on $(-\infty, \infty)$, $H \in [C, \infty)$ for some constant C by equation (6.1). Therefore it suffices to impose the growth condition of F near $+\infty$ in order to apply Proposition 5.1. (b). We can replace the condition (i) by: $F > 0$ on $(0, \infty)$ and $u_0 > 0$ on S^n.

7. Rescaling the Hypersurfaces.

Our final discussion shall be on the rescaling and convergence of the evolving hypersurfaces when $F > 0$ on $(0, \infty)$ and the initial hypersurface M_0 is convex, enclosing the origin. First of all, for any solution of equation to equation (1.1), we have

Lemma 7.1. *There exists a solution to the O.D.E.*

$$\frac{d}{dt}R = F(nR)$$

on $[0, T_{\max})$ *such that*

$$0 < u_{\min}(t) \leq R(t) \leq u_{\max}(t), \quad \forall\, t \in [0, T_{\max}).$$

Proof. The proof is essentially the same as Lemma 9 in Chow-Tsai [3]. □

We will choose one $R(t)$ satisfying above lemma to rescale the solution $u(x,t)$. Define the rescaled solution $\tilde{u}(x,t)$ to be

$$\tilde{u}(x,t) = \frac{u(x,t)}{R(t)},$$

which is the support function of the rescaled hypersurface $\tilde{M}_t = \frac{M_t}{R(t)}$. We summarize Propositions 6.1 and 6.2 in the following.

Theorem 7.2. *(a). If F satisfies the assumption of Proposition 6.1, then the hypersurface M_t remains smooth and convex for all time and its rescaled support function satisfies*

$$(i)\ |\tilde{u}(x,t) - 1| \leq \frac{C}{R(t)} \quad on \quad S^n \times [0, T_{\max})$$

$$(ii)\ |\nabla \tilde{u}(x,t)| \leq \frac{C}{R(t)} \quad on \quad S^n \times [0, T_{\max}),$$

where C is a constant depending only on u_0 and $R(t) \to \infty$ as $t \to T_{\max}$.

(b). If, in addition, F satisfies the assumption (ii) of Proposition 6.2, then

$$(iii)\ |\nabla\nabla \tilde{u}(x,t)| \leq \frac{\tilde{C}}{R(t)} \quad on \quad S^n \times [0, T_{\max}),$$

where $\tilde{C}$ is a constant depending on u_0, F and the constant C_0 in Proposition 6.2.

Part (a) says that the rescaled hypersurfaces converge to the unit sphere S^n in C^1 norm, and (b) says they converge to S^n in C^2 norm.

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