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Existence and non-existence results of H-surfaces into 3-dimensional Riemannian manifolds

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This paper investigates the existence of solutions of generalized H-surface equations for 3-dimensional target manifolds. A class of equations which contains the constant mean curvature equation for compact target manifolds is considered and a variational structure for these equations is formulated. By this variational consideration, a sufficient condition for the existence of a solution of the equation for negatively curved target manifolds is obtained. A non-existence result is also obtained, when the target manifold is a flat 3-torus.

0. Introduction.

Let Σ be a two dimensional closed Riemannian manifold and N a 3-dimensional closed Riemannian manifold.

Definition 0.1. For a smooth 3-form α on N, we define the (2,1)-tensor H by

$$(0.1) \qquad \alpha_p(U,V,W) := \langle U, H(p)(V,W)\rangle \quad for \quad p \in N, \quad U,V,W \in T_pN.$$

A map $u \in C^2(\Sigma; N)$ is called an H-surface of Σ into N, if u satisfies

$$(0.2) \qquad trace(\nabla du) = 2H(u)(\nabla u \wedge \nabla u)$$

where right hand side of (0.1) stands for

$$H(u)(\nabla u \wedge \nabla u) := \sigma^{-2} H(u)(u_*(\frac{\partial}{\partial x^1}), u_*(\frac{\partial}{\partial x^2}))$$

where $z = x^1 + \sqrt{-1}\ x^2$ denotes an isothermal coordinate and the metric tensor of Σ is written as $\sigma^2((dx^1)^2 + (dx^2)^2)$.

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Note that a solution u of (0.2) gives a parametrization (at regular points) of a surface with the mean curvature prescribed by the tensor $H(\cdot)$ if it satisfies the conformality condition;

$$|u_{x^1}|^2 - |u_{x^2}|^2 = \langle u_{x^1}, u_{x^2} \rangle \equiv 0.$$

H-surfaces defined by Definition 0.1 generalizes that given in section 0 of [8]. In [8], an H-surface is defined as an extremal of conformally invariant functional;

$$I_\omega(u) := \frac{1}{2}\int_\Sigma |\nabla u|^2 dV_\Sigma + 2\int_\Sigma u^*\omega \tag{0.3}$$

defined for 2-form ω on N, which corresponds to the case $\alpha = d\omega$ (i.e. α is exact) in the present definition. The present generalized definition has an advantage that we can deal with the constant mean curvature equation which corresponds to the case $\alpha = Hvol_N$ for some constant H in Definition 0.1. We remark that for compact target manifold N, the volume form of N can never be an exact form. On the other hand, an H-surface in the sense of Definition 0.1 can not be considered as a solution of the Euler-Lagrange equation of functional (0.3). However, equation (0.2) can be considered as the Euler-Lagrange equation of the multi-valued functional which will be defined and considered in detail in section 1. To write down the statement of the main theorems, we need the following definition.

Let γ be a given homotopy class of C^∞−maps of Σ to N. And fix any $u_0 \in \gamma$. For $u \in \gamma$, choose a homotopy $f(\cdot, t) \in C^\infty(\Sigma \times [0,1]; N)$ with

$$f(x,0) = u_0(x), \quad f(x,1) = u(x). \tag{0.4}$$

Then, we define quantity $I(u, u_0; f)$ by

$$I(u, u_0; f) := \frac{1}{2}\int_\Sigma |\nabla u|^2 dx - \frac{1}{2}\int_\Sigma |\nabla u_0|^2 dx + 2\iint_{\Sigma\times[0,1]} f^*\alpha. \tag{0.5}$$

Set

$$|H| := \sup\{|H(u)|; u \in N\} = |\alpha| \quad \text{(the comass of 3-form } \alpha)$$

for the tensor H defined by (0.1).

In section 2, we will show that if N is a manifold of negative sectional curvature, $I(u, u_0; f)$ defined by (0.5) is globally defined in a given homotopy class γ of maps of Σ to N independent of choice of homotopy f. In this case, we can define the functional $I_H(u) := I(u, u_0; f)$ fixing $u_0 \in \gamma$.

In section 3, we shall prove the following existence theorem.

Theorem A. *Assume that the sectional curvature κ_N of N is bounded from above by $-K^2$ for some $K > 0$. Moreover, assume that*

$$|\alpha| < K. \tag{0.6}$$

Then, in any homotopy class γ of maps of Σ to N, there exists a solution of (0.2) which is a global minimizer of $I_H(\cdot)$ in γ.

Moreover, if $\alpha = Hvol_N$ for some constant H with $|H| < K$, a solution of (0.2) is unique in any homotopy class unless its image is a closed geodesic of N or constant map. If the image is a closed geodesic, every solution in the homotopy class is obtained by the rotation of the geodesic.

Finally, in section 4, a simple argument implies the following non-existence theorem.

Theorem B. *Let N be a flat 3-torus and γ a homotopy class of maps of Σ to 3-torus N. Then, γ contains no H-surface with $\alpha = Hvol_N$ for a constant $H \neq 0$, unless 2-homology class $u_*(\Sigma) \in H_2(N;\mathbb{Z})$ induced by homotopy class γ vanishes.*

Remark.

1. Theorem A can be considered as a H-surface version of theorem of Eells-Sampson [2] and Hartman [4] which states the existence and uniqueness of harmonic maps into manifolds of non-positive sectional curvature.(These corresponds to the case $H = 0$ in Theorem A.) On the other hand, Theorem B shows us that, in general, the condition that the sectional curvature is strictly negative is necessary for the existence of H-surface in the present sense.
2. Let us compare Theorem A and Theorem B with Theorem 0.1 in [8] which ensures the existence of local minimizer of functional (0.3) for sufficiently small $|d\omega|$ without the curvature bound on target manifold N. Theorem A is an improved version of Theorem 0.1 in [8] in the sense that Theorem A gives us an effect of geometry of N, while the contrast between the non-existence result of Theorem B and the existence result of Theorem 0.1 in [8] shows us the essential difference between the present definition of H-surface and that of [8].
3. An existence results for a solution of equation (0.2) with small range was obtained by Gulliver [3].

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reminding him of the result by Gulliver [3]. He also thanks Prof.T.Sunada who reminds him of paper [6] by Novikov which also proposes multi-valued functional in connection with dynamical systems.

1. Variational problem for a multi-valued functional.

To investigate the dependency of $I(u, u_0; f)$ defined in (0.5) on the choice of homotopy f, we shall compute $I(u, u_0; f) - I(u, u_0; g)$ for two homotopies f and g with (0.4) as follows. Since N is compact, it is possible to decompose α as

$$\alpha = avol_N + d\beta \tag{1.1}$$

for some constant a and 2-form β on N. And set

$$F(x,t) := \begin{cases} f(x,2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ g(x,2(1-t)) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{1.2}$$

Since f and g satisfy (0.4), F can be considered as a Lipschitz map of $\Sigma \times S^1$ to N. Thus, (0.5) and (1.1) imply

$$\begin{aligned} &I(u, u_0; f) - I(u, u_0; g) \\ =&2a \left\{ \iint_{\Sigma\times[0,1]} f^*vol_N - \iint_{\Sigma\times[0,1]} g^*vol_N \right\} \\ =&2a \int_{S^1\times\Sigma} F^*vol_N = 2adeg(F)V_N \end{aligned} \tag{1.3}$$

where V_N denotes the volume of N.

This computation shows us;

(1.4) The quantity $I(u, u_0; f)$ depends only on *the homotopy class* (*relative to boundary condition* (0.4)) *of homotopy* f as a map of $\Sigma \times [0, 1]$ to N.

(1.5) $I(u, u_0; \cdot)$ is defined modulo $2aV_N$. In particular, if $a = 0$, $I(u, u_0; f)$ is uniquely determined and it coincides with functional (0.3) essentially.

As for the global situation, (1.4) and (1.5) state that we can define the multi-valued functional $\{I(u)\}$ by

$$\{I(u)\} := \{I(u, u_0; f);\ f \text{ is a homotopy with } (0.4)\}$$

whose value is determined modulo $2KV_N$ depending on the choice of u_0. On the other hand, (1.5) enables us to define a single valued functional in some neighbourhood U of $u \in \gamma$ (with respect to the C^1-topology, for example). In fact, taking U sufficiently small, we can connect any map $v \in U$ to u by a homotopy g with

$$|\int_{\Sigma\times[0,1]} g^*\alpha| \leq \frac{a}{4}V_N. \tag{1.6}$$

(For example, for C^1- section X of $T^*\Sigma \otimes u^*TN$ with sufficiently small $|X|$ and $|\nabla X|$, we can cover some neighborhood of u in $C^1(\Sigma, N)$ by map $v := exp_u(X)$ and taking a homotopy $g(x,t) := exp_u(tX)$, we can establish (1.6).) In view of (1.5), $I(v,u;g)$ is well-defined in the neighbourhood U of u. Fix a homotopy f_0 with (0.4). Then, we can define single valued functional $I_{f_0}(v)$ on U by

$$I_{f_0}(v) := I(v,u_0;G) = I(v,u;g) + I(u,u_0;f_0)$$

where G is a homotopy between u_0 and v defined by

$$G(x,t) := \begin{cases} f_0(x,2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ g(x,2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases}$$

$I_{f_0}(v)$ is defined up to a constant depending on choice of homotopy f_0.

It is easy to check that (0.1) is a Euler-Lagrange equation for the functional I_{f_0} on U independent of the choice of u_0.

In a word, our problem is local variational problem. Thus, all the local concept in the calculus of variation can be defined for this problem. Here, the term "local "means "local in the space of admissible maps ". For example, stability (or equivalently locally minimizing property) and instability can be defined.

2. Functional for manifolds of negative sectional curvature.

Throughout section 2 and 3, we assume that the sectional curvature κ_N of N is bounded from above by $-K^2$ for some $K > 0$. We work in smooth maps or at least Lipschitz maps, since we do not have to analyze in a larger space of admissible maps because of the strong smoothing effect of the heat flow under our assumption.

2.1. Global well definedness of functional.

Lemma 2.1. *Let γ be a given homotopy class of maps of Σ to N and h a harmonic map in γ. Then, we have*

(1) *For any smooth map* $u \in \gamma$, $I(u,h;f)$ *defined by (0.5) is independent of choice of homotopy* f.

(2) *For any smooth map* $u \in \gamma$, *there is the homotopy* f *connecting* u *and* h *with*

$$f(\cdot,0) = h(\cdot), \tag{2.1}$$

$$f(\cdot,1) = u(\cdot), \tag{2.2}$$

$$t \mapsto f(x,t) \text{ is a geodesic for any } x \in \Sigma. \tag{2.3}$$

Proof. Let f and g be two homotopies connecting u and h. And define the map $F : \Sigma \times S^1 \to N$ by (1.5). We will consider the heat flow equation;

$$\partial_t w_\xi = trace(\nabla d w_\xi), \tag{2.4}$$

$$w_\xi(\cdot,0) = F(\cdot,\xi). \tag{2.5}$$

By the results of Eells-Sampson [2] and Hartman [4], the solution of equation (2.4)-(2.5) exists for $0 \le t < \infty$ and $w_\xi(\cdot,t)$ converges uniformly with respect to ξ as $t \to \infty$ to 1-parameter family of harmonic maps $\xi \mapsto h_\xi := h(\cdot,\xi)$. Thus, F and $h(\cdot,\cdot)$ are homotopic as maps of $\Sigma \times S^1$ to N. Again by the result of Hartman [4], a harmonic map in γ is unique unless its image is a closed geodesic; in this case every harmonic map in γ is obtained by the rotation of the geodesic, or it is a constant map; in this case every harmonic map in γ is constant map. In every case, we obtain $deg(h(\cdot,\cdot)) = 0$. Hence, from (1.3), $I(u,h;f) = I(u,h;g)$. This proves assertion (1). Assertion (2) is a direct consequence of the Hadamard-Cartan theorem and the lifting argument. □

Lemma 2.1 (1) implies that we can consider the functional $I_H(u) := I(u,h;f)$ as long as we restrict ourselves to maps in a given homotopy γ. Moreover, by Lemma 2.1 (2), we can always use the homotopy with (2.1)-(2.3) to estimate the functional. In the sequel, we fix a homotopy class γ and consider functional I_H.

2.2. Estimates of volume functional.

To estimate the volume functional $V_\alpha(u,h) := \int_{\Sigma\times[0,1]} f^*\alpha$ where f is a homotopy between u and h, we need some Jacobi field estimates. For the definition and further details on Jacobi fields, see [1] for example.

Let $c_i(s,t)(i=1,2)$ be two families of geodesics parametrized by t and $J_i(s) := \frac{\partial c_i}{\partial t}(s,0)(i=1,2)$ corresponding Jacobi fields. J_i^T and J_i^N denote the tangential and normal component of J_i. We assume $c(s) := c_1(s,0) = c_2(s,0)$. In the sequel, "$'$" and "$\cdot$" denote derivatives with respect to s and t respectively. We adopt the normalization;

$$|c'| = d := \text{length of } c.$$

By this normalization, parameter s moves in the interval [0,1].

Set

$$p(s) := vol_N(J_1, J_2, c'), \tag{2.6}$$

$$q(s) := \frac{1}{2}\left\{\langle J_1, J_1\rangle + \langle J_2, J_2\rangle\right\}. \tag{2.7}$$

Lemma 2.2. *There holds*

$$|p(s) - p(0)| \leq \frac{1}{2K}\left\{q'(s) - q'(0) - s\sum_{i=1}^{2}|J_i^{T'}(0)|^2\right\} \tag{2.8}$$

for any $s \in [0,1]$

Proof. Differentiating $p(s)$ and $q(s)$, we have

$$p'(s) = vol_N(J_1', J_2, c') + vol_N(J_1, J_2', c'), \tag{2.9}$$

$$q'(s) = \sum_{i=1}^{2}\langle J_i, J_i'\rangle, \tag{2.10}$$

$$\begin{aligned} q''(s) &= \sum_{i=1}^{2}\left\{|J_i'|^2 + \langle J_i, J_i''\rangle\right\} \\ &= \sum_{i=1}^{2}\left\{|J_i'|^2 + \langle R^N(c', J_i)c', J_i\rangle\right\} \end{aligned} \tag{2.11}$$

where R^N denotes the curvature tensor of N. Since we assume $R^N \leq -K^2$, (2.11) implies

$$q''(s) \geq \sum_{i=1}^{2}\left\{|J_i'|^2 + K^2 d^2 |J_i^N|^2\right\}. \tag{2.12}$$

On the other hands, (2.9) implies

$$|p'|(s) \leq d\left\{|J_1^{N'}||J_2^N| + |J_1^N||J_2^{N'}|\right\}. \tag{2.13}$$

By (2.12), (2.13) and Schwarz inequality, we have

$$|p'|(s) \leq \frac{1}{2K}\left\{q''(s) - \sum_{i=1}^{2} |J_i^{T'}(s)|^2\right\}. \tag{2.14}$$

Since $|J_i^{T'}(s)| \equiv |J_i^{T'}(0)|$, we obtain the desired result by integrating (2.14) with respect to s. □

Lemma 2.3 (Jäger-Kaul). *Suppose J is a Jacobi field along c and satisfies*

$$\langle J, c' \rangle = 0.$$

Then

$$\langle J, J' \rangle(s) - \langle J, J' \rangle(0) \geq \frac{s'_{Kd}(s)}{s_{Kd}(s)}(|J(0)|^2 + |J(s)|^2) - \frac{2}{s_{Kd}(s)}|J(0)||J(s)| \tag{2.15}$$

where $s_{Kd}(s) = \frac{1}{Kd}\sinh(Kds)$.

Proof. Noting that we adopt the normalization $|c'| = d$, the proof of [5] implies the inequality (2.15). □

Set

$$q_N^i(s) := |J_i^N|^2, \qquad q_N := \sum_{i=1}^{2} q_N^i(s). \tag{2.16}$$

Lemma 2.4. *For any $\theta \in (0,1)$, there exist constants $d(K,\theta)$ and $C(K,\theta)$ such that if $d > d(K,\theta)$, the following inequality holds;*

(2.17)

$$q_N(1) - q_N(0) - q_N'(0) \geq 2K\theta \int_0^1 \{|p(s)| + |p(0)|\}\, ds - C(K,\theta)\left(\frac{q(1)}{d} + q(0)\right).$$

Proof. Applying (2.15) to J_i^N, we have

(2.18)

$$q_N^{i\,\prime}(s) - q_N^{i\,\prime}(0) \geq Kd\frac{\cosh(Kds)}{\sinh(Kds)}(q_N^i(0) + q_N^i(s)) - \frac{2Kd}{\sinh(Kds)}\sqrt{q_N^i(0)q_N^i(s)}.$$

Estimating (2.18) by Schwarz inequality and summing up with respect to i, we obtain

$$q_N{}'(s) - q_N{}'(0) \geq Kd\frac{\cosh(Kds) - 1}{\sinh(Kds)}(q_N(0) + q_N(s)). \tag{2.19}$$

Integrating (2.19) with respect to s, we have

$$(2.20) \quad q_N(1) - q_N(0) - q'_N(0) \geq Kd \int_0^1 \frac{\cosh(Kds) - 1}{\sinh(Kds)} (q_N(0) + q_N(s)).$$

For $\theta \in (0,1)$, let $M(\theta)$ be a positive number such that

$$\frac{\cosh(\xi) - 1}{\sinh(\xi)} \geq \theta,$$

if $\xi \geq M$.

Set $d(K,\theta) = \frac{M}{2K}$. Noting that $|p(s)| \leq \frac{1}{2} d q_N(s)$ and $0 \leq \frac{cosh(\xi)-1}{sinh(\xi)} < 1$ for any ξ, from (2.20), we obtain

$$(2.21) \qquad q_N(1) - q_N(0) - q'_N(0) \geq 2\theta K \int_{\frac{M}{kd}}^1 \{|p(s)| + |p(0)|\}\, ds.$$

Since $q(s)$ is a convex function of s by (2.12), we have

$$(2.22) \qquad q(s) \leq sq(1) + (1-s)q(0).$$

Thus, by (2.21) and (2.22), we obtain

(2.23)

$$\begin{aligned} & q_N(1) - q_N(0) - q'_N(0) \\ \geq & 2\theta K \int_0^1 \{|p(s)| + |p(0)|\}\, ds - 2K\theta d \int_0^{\frac{M}{Kd}} \{q(s) + q(0)\}\, ds \\ \geq & 2\theta K \int_0^1 \{|p(s)| + |p(0)|\}\, ds - K\theta d \left[s^2 q(1) - (s-1)^2 q(0)\right]_0^{\frac{M}{Kd}} - 2\theta M q(0) \\ = & 2\theta K \int_0^1 \{|p(s)| + |p(0)|\}\, ds - \frac{M^2\theta}{Kd} q(1) - \theta\left(\frac{M^2}{Kd} - 4M\right) q(0). \end{aligned}$$

This proves the desired result. □

Lemma 2.5. *For $\theta \in (0,1)$, there exists constants $d(K,\theta)$ and $C(K,\theta)$ such that if $d > d(K,\theta)$, there holds*

$$(2.24) \quad q(1) - q(0) - q'(0) \geq \frac{4K\theta}{1+\theta} \int_0^1 |p(s)| ds - C(K,\theta) \left(\frac{q(1)}{d} + q(0)\right).$$

Proof. Integrating (2.8) with respect to s, we have

$$q(1) - q(0) - q'(0) - \frac{1}{2}\sum_{i=1}^{2}|J_i^{T\prime}(0)|^2 \geq 2K\int_0^1 (|p(s)| - |p(0)|)\,ds. \tag{2.25}$$

Multiplying (2.25) by θ and summing up both sides with (2.17), we obtain

(2.26)
$$q_N(1) - q_N(0) + \theta\,(q(1) - q(0)) - q_N'(0) - \theta q'(0) - \frac{\theta}{2}\sum_{i=1}^{2}|J_i^{T\prime}(0)|^2$$
$$\geq 4\theta K\int_0^1 |p(s)|ds - C(K,\theta)\left(\frac{q(1)}{d} + q(0)\right).$$

Since

$$q'(0) - q_N'(0) = \sum_{i=1}^{2}\langle J_i^T(0), J_i^{T\prime}(0)\rangle \leq \sum_{i=1}^{2}\left\{\frac{1}{2\theta}|J_i^T(0)|^2 + \frac{\theta}{2}|J_i^{T\prime}(0)|^2\right\},$$

(2.26) implies

$$q(1) - q(0) - q'(0) \geq \frac{4K\theta}{1+\theta}\int_0^1 |p(s)|ds - C(K,\theta)\left(\frac{q(1)}{d} + q(0)\right).$$

Thus, we obtain the desired inequality. □

Now, we are in position to estimate the volume functional V_α defined by

$$V_\alpha(u,h) := \int_{\Sigma\times[0,1]} f^*\alpha.$$

By Lemma 2.1, we may assume f is a homotopy with property (2.1)-(2.3). For $\Omega \subset \Sigma$, Set

$$D(u;\Omega) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx.$$

Proposition 2.6. *For any $\theta \in (0,1)$, there exists a constant $C(K,\theta,h)$ such that*

$$|V_\alpha(u,h)| \leq \frac{|\alpha|}{2\theta K}\left(D(u,\Sigma) - D(h,\Sigma) + C(K,\theta,h)\right). \tag{2.27}$$

Proof. Let $e_i \in T_x\Sigma (i = 1,2)$ be a orthonormal basis at $x \in \Sigma$. First, note that

$$|V_\alpha(u,h)| \leq |\alpha| \int_{\Sigma\times[0,1]} |vol_N(\nabla_{e_1} f, \nabla_{e_2} f, \partial_s f)| dV_\Sigma ds.$$

So, we only have to estimates the term;

$$V := \int_{\Sigma\times[0,1]} |vol_N(\nabla_{e_1} f, \nabla_{e_2} f, \partial_s f)| dV_\Sigma ds. \tag{2.28}$$

For $x \in \Sigma$, we define Jacobi fields $J_i(x,s) (i = 1,2)$ along the geodesic $c_x : s \mapsto f(x,s)$ so that

$$J_i(x,0) = \nabla_{e_i} h(x), \qquad J_i(x,1) = \nabla_{e_i} u(x).$$

Let $p(x,s), q(x,s)$ and $q_N(x,s)$ denote the functions defined by (2.6),(2.7) and (2.16) for $J_i(x,s)$. And set $d_x :=$ length of c_x. In these notations,

$$D(u,\Sigma) = \int_\Sigma q(x,1) dV(x),$$
$$D(h,\Sigma) = \int_\Sigma q(x,0) dV(x),$$
$$V = \int_{\Sigma\times[0,1]} |p(x,s)| dV(x) ds.$$

Setting $\Sigma_R := \{x \in \Sigma; d_x \geq R\}$ for sufficiently large $R > 0$ and integrating (2.24) over Σ_R, we have

(2.29)

$$D(u,\Sigma_R) - D(h,\Sigma_R) - \int_{\Sigma_R} \langle \nabla h, \nabla \partial_s f(0) \rangle dV$$
$$\geq \frac{4K\theta}{1+\theta} \int_{\Sigma_R\times[0,1]} |p(x,s)| dV ds - C(K,\theta) \left(\frac{D(u,\Sigma_R)}{R} + D(h,\Sigma_R) \right).$$

On the other hand, integrating (2.8) over $(\Sigma \setminus \Sigma_R) \times [0,1]$, we obtain

(2.30)

$$D(u,\Sigma \setminus \Sigma_R) - D(h,\Sigma \setminus \Sigma_R) - \int_{\Sigma\setminus\Sigma_R} \langle \nabla h, \nabla \partial_s f(0) \rangle dV$$
$$+KRD(h,\Sigma \setminus \Sigma_R) \geq 2K \int_{(\Sigma\setminus\Sigma_R)\times[0,1]} |p(x,s)| dV ds$$

Since h is a harmonic map, there holds

$$\int_\Sigma \langle \nabla h, \nabla \partial_s f(0)\rangle dV = 0.$$

Thus, summing up (2.29) and (2.30), we obtain

$$\left(1+\frac{C(K,\theta)}{R}\right)(D(u,\Sigma)-D(h,\Sigma)) \geq \frac{4\theta K}{1+\theta}V - C(K,\theta,h,R).$$

Taking θ sufficiently close to 1 and R sufficiently large, we obtain the desired result. □

3. Completion of Proof of Theorem A.

To prove Theorem A, we use the consequence of Proposition 2.6, minimizing process in [8] and the technic due to Eells-Sampson [2] and Hartman [4].

We consider the heat flow equation associated to the functional I_α defined in the previous section ;

$$\partial_t u = trace(\nabla du) - 2H(u)(\nabla u \wedge \nabla u), \tag{3.1}$$

$$u(\cdot,0) = u_0(\cdot) \tag{3.2}$$

for given initial data $u_0 \in C^\infty(\Sigma, N)$.

In the computation of section 3 and 4, we shall use the notations in the tensorial calculus. We adopt the Einstein summation convention. Fix the orthonormal frame e_1, e_2 near $x \in \Sigma$. ∇_i denotes the covariant derivative in the direction of e_i. For example,

$$\nabla_i u = \nabla_{e_i} u,$$
$$\nabla_i \nabla_j u = \nabla_i(\nabla_j u) - \nabla_{\nabla_{e_i} e_j} u.$$

Let Ric^Σ denote the Ricci tensor of Σ and set $Ric^\Sigma_{i,j} := Ric^\Sigma(e_i, e_j)$.

Lemma 3.1. *Let u be a smooth solution of* (3.1) *and* (3.2). *If $|H| < K$, we have the following inequality ;*

$$\frac{1}{2}(\Delta - \frac{\partial}{\partial t})|\nabla u|^2 - Ric^\Sigma(\nabla u, \nabla u) \geq 0 \tag{3.3}$$

where $Ric^\Sigma(\nabla u, \nabla u) := Ric^\Sigma_{i,j}\langle \nabla_i u, \nabla_j u\rangle$.

Proof. Direct computation implies

$$(3.4)\qquad \frac{1}{2}(\Delta - \frac{\partial}{\partial t})|\nabla u|^2 = |\nabla^2 u|^2 - \langle R^N(\nabla_\alpha u, \nabla_\beta u)\nabla_\alpha u, \nabla_\beta u\rangle + Ric^\Sigma(\nabla u, \nabla u) + 2\langle \nabla_\alpha \{H(u)(\nabla u \wedge \nabla u)\}, \nabla_\alpha u\rangle.$$

The 2nd term of (3.4) is computed and estimated as

$$(3.5)\qquad \begin{aligned} -\langle R^N(\nabla_\alpha u, \nabla_\beta u)\nabla_\alpha u, \nabla_\beta u\rangle &= -2\langle R^N(\nabla_1 u, \nabla_2 u)\nabla_1 u, \nabla_2 u\rangle \\ &\geq 2K^2|\nabla_1 u \times \nabla_2 u|^2. \end{aligned}$$

Since $\nabla_\alpha H$ and H are skew-symmetric, we can compute and estimate the last term as

$$(3.6)\qquad \begin{aligned} &2\langle \nabla_\alpha \{H(u)(\nabla u \wedge \nabla u)\}, \nabla_\alpha u\rangle \\ =&2\langle H(u)(\nabla_\alpha(\nabla u) \wedge \nabla u + \nabla u \wedge \nabla_\alpha(\nabla u)), \nabla_\alpha u\rangle \\ =&2\langle H(u)(\nabla_1\nabla_1 u, \nabla_2 u), \nabla_1 u\rangle + 2\langle H(u)(\nabla_1 u, \nabla_2\nabla_2 u), \nabla_2 u\rangle \\ \geq& -2|H|^2|\nabla_1 u \times \nabla_2 u|^2 - \{|\nabla_1\nabla_1 u|^2 + |\nabla_2\nabla_2 u|^2\} \end{aligned}$$

Thus, combining (3.4)-(3.6), we obtain

$$\frac{1}{2}(\Delta - \frac{\partial}{\partial t})|\nabla u|^2 - Ric^\Sigma(\nabla u, \nabla u) \geq 2|\nabla_1\nabla_2 u|^2 + (K^2 - H^2)|\nabla_1 u \times \nabla_2 u|^2 \geq 0$$

□

Lemma 3.2. *Let $u_0(x;\xi)$ be a smooth 1-parameter family of initial values and $u(x,t;\xi)$ the corresponding solution of* (3.1)-(3.2). *If $H(\cdot)$ is a tensor defined by (0.2) for $\alpha = Hvol_N$ for some constant H, we have*

$$(3.7)\qquad \frac{1}{2}(\Delta - \frac{\partial}{\partial t})|u_\xi|^2 \geq (K^2 - H^2)\sum_{i=1}^{2}|u_\xi \times \nabla_i u|^2$$

where $u_\xi := \frac{\partial u}{\partial \xi}$.

Proof. Direct computation implies

$$\begin{aligned}&\frac{1}{2}(\Delta-\frac{\partial}{\partial t})|u_\xi|^2\\ =&|\nabla u_\xi|^2-\langle R^N(u_{xi},\nabla_i u)u_\xi,\nabla_i u\rangle+2\langle\nabla_{\frac{\partial}{\partial\xi}}\{H(u)(\nabla u\wedge\nabla u)\},u_\xi\rangle\\ \geq&|\nabla u_\xi|^2+K^2\sum_{i=1}^{2}|u_\xi\times\nabla_i u|^2+2H\langle(\nabla_1 u_\xi\times\nabla_2 u+\nabla_1 u\times\nabla_2 u_\xi),u_\xi\rangle\\ \geq&(K^2-H^2)\sum_{i=1}^{2}|u_\xi\times\nabla_i u|^2\end{aligned}$$

□

Lemma 3.3. *Let* $u\in C^2(\Sigma\times[0,T];N)$ *be a solution of* (3.1)-(3.2)*, then there holds*

$$-\int_{\Sigma\times[0,T]}|\partial_t u|^2dVdt=I_H(u).$$

Proof. Multiplying $-\partial_t u$ to (3.1) and integrating on $\Sigma\times[0,T]$, we obtain the desired result. □

Proof of Theorem A. Existence: By Proposition 2.6, we have the estimate;

$$I_H(u)\geq(1-\frac{|\alpha|}{\theta K})(D(u,\Sigma)-D(h,\Sigma))-|\alpha|C(K,\theta,h). \tag{3.8}$$

Since $|\alpha|=|H|<K$ by assumption, we may assume $1-\frac{|H|}{\theta K}>\delta>0$, taking θ sufficiently close to 1. Then, (3.8) denotes the coersiveness of $I_H(u,h)$ with respect to the Dirichlet integral. Take a minimizing sequence $\{u_i\}\subset C^\infty(\Sigma,N)$ of I_H and consider the heat flow equation;

$$\partial_t w_i=trace(\nabla dw_i)-2H(w_i)(\nabla w_i\wedge\nabla w_i), \tag{3.9}$$

$$w_i(\cdot,0)=u_i(\cdot). \tag{3.10}$$

The short time existence for the equations of this type is established by usual fixed point argument. (See, for example, Section 3 of [6]) Since the flow generated by (3.9) does not increase I_H by Lemma 3.3, $D(w_i(\cdot,t),\Sigma)$ is bounded by some $M_0(K,\theta,h)>0$ uniformly with respect to t and i by the coersiveness of I_H as long as the solution exists. At the same time, since

$I_H(w_i(\cdot,t),h)$ is bounded uniformly with respect to t and i, by Lemma 3.3, we have the uniform bound ;

$$\iint_{\Sigma\times[0,T]} |\partial_t w_i|^2 dV dt < M_1(K,\theta,h) \tag{3.11}$$

for any $T^i_{max} > T > 0$ and i where T^i_{max} denotes the maximal existence time of the solution w_i. Then, by the boundedness of Dirichlet integral and inequality (3.3), we can apply the same argument as Lemma 3.1.1 of [6] to obtain

$$\sup\left\{|\nabla w_i(x,t)|^2; i\in\mathbb{N},\quad x\in\Sigma,\quad 0\le t<T^i_{max}\right\}\le C(\Sigma,M_0). \tag{3.12}$$

(3.11), (3.12) and usual parabolic boot strap argument implies that $T^i_{max} = \infty$ and there is a sequence $t^i_j \to \infty$ as $j\to\infty$ with

$$w_i(\cdot,t^i_j)\longrightarrow v_i(\cdot)\quad \text{in } C^\infty(\Sigma,N)$$

where v_i is a solution of equation (0.1). Since v_i enjoys the same gradient estimate (3.12), routine elliptic estimate implies

$$v_i(\cdot)\longrightarrow v(\cdot)\quad \text{in } C^\infty(\Sigma,N)$$

and $v(\cdot)$ is a desired minimizer satisfying equation (0.1).

Uniqueness for constant mean curvature equation: This is a direct consequence of Lemma 3.2 and the argument of Hartman [4]. □

Remark. 1. If harmonic map h in the homotopy class γ maps Σ to a closed geodesic in N or it is a constant map, harmonic map h itself is a global minimizer in the existence part of Theorem A. Moreover, since $\nabla_1 u\times\nabla_2 u\equiv 0$ in this case, h satisfies equation (0,1) for any H. To see the minimizing property, we only need Lemma 2.2. Since $p(0)=0$ in this case, the integration of (2.8) over $\Sigma\times[0.1]$ implies

$$|V|\le\frac{1}{2K}\left\{D(u;\Sigma)-D(h;\Sigma)\right\}$$

where V is the quantity defined by (2.28). This leads us to the estimate

$$I_H(u)\ge(1-\frac{|H|}{K})\left\{D(u;\Sigma)-D(h;\Sigma)\right\}.$$

Thus, h is the unique minimizer up to rotation of the geodesics, if $|H| < K$ and, even in the case $|H| = K$, h is still a minimizer.

2. Suppose $\alpha = Hvol_N$ for some constant H. Lemma 3.2 and the argument in [4] implies that, even in the case $|H| = K$, *if there exists an H-surface u_0 in γ*, any H-surface u in γ is minimizing functional I_H and there is a homotopy $u(x,s)$ between u_0 and u with the properties;

(1) $u(\cdot, s)$ is a H-surface for any s,

(2) $c_x : s \mapsto u(x,s)$ is a geodesic with $|c'_x| = 1$.

4. Proof of Theorem B.

Notations. Since $\pi_2(T^3) = 0$, we may assume $g =$ genus of $\Sigma \geq 1$ to prove Theorem B. Let $\{a_i, b_i\}_{i=1\ldots g}$ be a canonical generator of $\pi_1(\Sigma)$ with single relation $\Pi_{i=1\ldots g} a_i b_i a_i^{-1} b_i^{-1} = 1$.

$\Omega(\Sigma)$ denotes a fundamental region of Σ in the universal covering $\widetilde{\Sigma}$ of Σ which is bounded by a curve $\Pi_{i=1\ldots g} \tilde{a}_i \tilde{b}_i \tilde{a}_i^{-1} \tilde{b}_i^{-1}$. Here, $\tilde{a}_i, \tilde{b}_i$ denote curves obtained by lifting a_i, b_i.

Suppose flat torus N is represented as a quotient space $\mathbb{R}^3/\Gamma$ by lattice Γ in $\mathbb{R}^3$. Then, by lifting the map, solutions of the constant mean curvature equation of Σ into N with prescribed homotopy class γ can be identified with a class of solutions of the constant mean curvature equation of $\widetilde{\Sigma}$ into $\mathbb{R}^3$. The class consists of $u \in C^\infty(\widetilde{\Sigma}, \mathbb{R}^3)$ satisfying

$$\Delta u = 2H \nabla_1 u \times \nabla_2 u, \tag{4.1}$$

$$u(cx) = u(x) + \theta(c) \quad \text{for any } c \in \pi_1(\Sigma) \tag{4.2}$$

where $c \in \pi_1(\Sigma)$ acts $\widetilde{\Sigma}$ as the deck transformation and θ denotes the conjugacy class $\pi_1(\Sigma) \to \pi_1(N) = \Gamma \subset \mathbb{R}^3$ induced by homotopy class γ.

Proof of Theorem B. Integrating the first component of equation (4.1) over $\Omega(\Sigma)$, we obtain

$$\int_{\Omega(\Sigma)} \Delta u^1 dV = 2H \int_{\Omega(\Sigma)} \left(\nabla_1 u^2 \nabla_2 u^3 - \nabla_1 u^3 \nabla_2 u^2\right) dV. \tag{4.3}$$

Let $P^1(u)$ denotes the parallel field on N induced by the vector field $\frac{\partial}{\partial u_1}$ in $\mathbb{R}^3$. Then we have

$$\Delta u^1 = div\langle \nabla u, P^1 \rangle. \tag{4.4}$$

Since the representation of right hand side of (4.4) is defined on Σ, the left hand side of (4.3) vanishes by Stokes Theorem. On the other hand, since

$$(\nabla_1 u^2 \nabla_2 u^3 - \nabla_1 u^3 \nabla_2 u^2)\, dV = \frac{1}{2} d(u^2 du^3 - u^3 du^2)$$

where d denotes the exterior differential of forms, we obtain

$$H \int_{\partial\Omega(\Sigma)} (u^2 du^3 - u^3 du^2) = 0.$$

Since $\partial\Omega(\Sigma) = \Pi_{i=1\dots g} \tilde{a}_i \tilde{b}_i \tilde{a}_i^{-1} \tilde{b}_i^{-1}$,

$$\begin{aligned} &\int_{\partial\Omega(\Sigma)} (u^2 du^3 - u^3 du^2) \\ &= \sum_{i=1}^{g} \Big\{ \int_{\tilde{a}_i} (u^2 du^3 - u^3 du^2) + \int_{\tilde{b}_i} (u^2 du^3 - u^3 du^2) \\ &\quad + \int_{\tilde{a}_i^{-1}} (u^2 du^3 - u^3 du^2) + \int_{\tilde{b}_i^{-1}} (u^2 du^3 - u^3 du^2) \Big\}. \end{aligned} \tag{4.5}$$

Since we $\tilde{a}_i$ and $(\tilde{a}_i^{-1})^{-1}$ only differ by the deck transformation induced by b_i, we compute as

$$\int_{\tilde{a}_i} u^2 du^3 + \int_{\tilde{a}_i^{-1}} u^2 du^3 = -\theta^2(b^i)\theta^3(a_i). \tag{4.6}$$

Applying the same computation to all terms in (4.5), we obtain

$$2H \int_{\partial\Omega(\Sigma)} (u^2 du^3 - u^3 du^2) = 2H \sum_{i=1}^{g} \left(\theta^2(a_i)\theta^3(b_i) - \theta^3(a_i)\theta^2(b_i)\right) = 0$$

Applying the same argument for 2nd and 3rd component of equation (4.1), we obtain

$$2H \sum_{i=1}^{g} \theta(a_i) \times \theta(b_i) = 0. \tag{4.7}$$

Thus homotopy classes (which is uniquely determined by conjugacy class in this case) which admit a H-surface except for harmonic maps should satisfy

$$\sum_{i=1}^{g} \theta(a_i) \times \theta(b_i) = 0. \tag{4.8}$$

(4.8) is equivalent to $u_*(\Sigma) = 0 \in H_2(N; \mathbb{Z})$. Thus we obtain the desired results. □

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