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A Partial Regularity Result of Harmonic Maps from Manifolds With Bounded Measurable Riemannian Metrics

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0. Introduction.

In the last two decades,much progress has been made on the regularity theory of harmonic maps between manifolds,for instance,one can see [GG1], [GG2], [SU],[E],[B] etc. Since the methods used in these papers are variations of Schauder's device of freezing the coefficients they need assume that the metrics are at least uniformly continuous while in [SU],[E],[B] they assume $g \in C^2(M)$. If the image of a map is contained in a convex ball; the metric of the domain is only bounded measurable,Hildebrandt-Widman proved the map is Hölder continuous(see[H-W]). On the other hand,M.Gromov and R. Schoen established the existence and regularity theory of harmonic maps into singular space(see [GS]). It may be more interesting,in view of analysis,to consider the same problem when the domain is singular. In this paper, we shall consider this problem and prove partial regularity results of energy minimizing maps from manifolds with bounded measurable Riemannian metrics. We will consider the more complicated case,for example, the domain is a algebraic variety, in elsewhere.

Suppose M is an n-dimensional differentiable manifold, g is a metric on M ,then g can be represented by a positive definite matrix (g_{ij}) in each coordinate chart $(\mathcal{D}, x^i)$ of M.If g_{ij} merely belongs to $L^\infty(\mathcal{D}, \mathbf{R})$, then g is called bounded measurable Riemannian metric,or simply L^∞-metric.Let (N, h) be a smooth compact Riemannian manifold, by the Nash embedding theorem, we can always assume (N, h) is isometrically embedded in some Euclidean space $\mathbf{R}^k$.

In order to discuss harmonic maps from (M, g) to (N, h) , we work in the seperable Hilbert space $L_1^2(M, \mathbf{R}^k)$, the set of maps $u : M \longmapsto \mathbf{R}^k$, whose component functions have first derivatives in L^2. As in the smooth case, we can define the space $L^2_{1,0}(M,\mathbf{R}^k)$;energy functional $E(.)$ on the space $L^2_1(M,\mathbf{R}^k)$; E-minimizing map from (M,g) to (N,h), (see [SU],pp309).

Since we are interested in the regularity theory, we assume (M,g) is simply (B_1,g) where B_1 is the unit ball in $\mathbf{R}^n$,g is an L^∞- metric on it. For convenience,we identify the matrix (g_{ij}) representing g in standard coordinates $(x^1,\dots,x^n)$ of $\mathbf{R}^n$ with g itself. In addition we adopt the notation:

$$|\nabla u|^2=\sum_{1\le\alpha\le n}\sum_{1\le i\le k}(\frac{\partial u^i}{\partial x^\alpha})^2.$$

The main result of this paper is the following:

Theorem 1. *Suppose B_r is a ball in $\mathbf{R}^n$ with radius r, g is an L^∞-metric on B_1,with $\Lambda^{-1}I \le g\le \Lambda I$, where Λ is a positive constant, I is the unit matrix. Assume (N,h) is a smooth compact Riemannian manifold, u is an E-minimizing map from (B_1,g) to (N,h). Then there exists a $\varepsilon=\varepsilon(\Lambda)>0$,such that if $\int_{B_1}|\nabla u|^2\,dx\le\varepsilon^2$, then $u\in C^\alpha(B_{1/2},\mathbf{R}^k)$ for some $0<\alpha<1$.*

Remark 0.1. Meyers'example (see [G],pp 157-158) shows that the c^α-regularity in the theorem is optimal.

As a corollary of Theoem 1,we have:

Theorem 2. *Let Sing(u) be the set $\{x\in M : u$ is discontinous at $x\}$, and $\mathcal{H}^{n-2}$ be $(n-2)$-dimensional Hausdorff measure, then $\mathcal{H}^{n-2}($ Sing $u)=0$.*

We follow the approach of Hardt and Lin in [HL].However,there is a major difficulty in this approah when the metric is merely measurable,i.e. one cannot do the domain blow-ups and take the limit.To overcome this difficulty,we use the Green functions(cf.[H-K-W],[H-W]) to get the necessary estimate.

The rest of the paper runs as follows: In §1, we will introduce some notations and prove Lemmas.In §2 we will prove energy improvement theorem (Theorem 2.1).Once the energy improvement is obtained , one can prove Theorem 1 and 2 easily by standard technique,for the details see [SU] or [HL],we will omit them in this paper.

1. Preliminaries.

Let $a^{\alpha\beta}=g^{\alpha\beta}(\det(g_{\alpha\beta}))^{1/2}$,where $(g^{\alpha\beta})$ denotes the inverse matrix of (g_{ij}) . Without loss of generality, we may assume $\Lambda^{-1}I\le\ (a^{\alpha\beta})\le\ \Lambda I$.

We say that u is a minimizer under $a^{\alpha\beta}$ if u is an E–minimizing map from (B_1, g) to (N, h).It is easy to verify that,u satisfies the equations:

$$-\frac{\partial(a^{\alpha\beta}\dfrac{\partial u}{\partial x^\beta})}{\partial x^\alpha} = A(u)(\nabla u, \nabla u) \qquad \text{in} \quad B_1.$$

where A is the second fundamental form of N in $\mathbf{R}^k$.

Let $\mathcal{F}_\Lambda = \{(a^{\alpha\beta}) : \Lambda^{-1}I \le (a^{\alpha\beta}) \le \Lambda I \text{ in } B_1\}$. Clearly, $\mathcal{F}_\Lambda$ is invariant under rescaling and translating. i.e. $(a^{\alpha\beta}(\lambda x + y)) \in \mathcal{F}_\Lambda$ whenever $(a^{\alpha\beta}) \in \mathcal{F}_\Lambda$ and $\lambda x + y \in B_1$.

Lemma 1 ([HL],Corollary 2.3). *There exist postive constants c and q such that if* $0 < \lambda < 1/4$,*and* u *is an E–minimizer in* $L_1^2(B, N)$ *with* $E_1(u) = \int_{B_1} |\nabla u|^2 \, dx \le c^{-1}\lambda^{q/2}$,*then*

$$E_{1/2}(u) \equiv (1/2)^{2-n} \int_{B_{1/2}} |\nabla u|^2 \, dx \le \lambda E_1(u) + c\lambda^{-q} \fint_{B_1} |u - \overline{u}|^2 \, dx$$

where $\overline{u} = \fint_B u \, dx = \frac{1}{|B|}\int_B u \, dx$.

Main Lemma. *Let* $(a_k{}^{\alpha\beta}) \in \mathcal{F}_\Lambda$,$k = 1, 2, \ldots$.*Assume that* $v_k \in L_1^2 \cap L^\infty(B_1, \mathbf{R}^k)$ *and* $\int_{B_1} |\nabla v_k|^2 \, dx \le 1$.*Suppose that*

$$(*) \qquad -\frac{\partial(a_k{}^{\alpha\beta}(\dfrac{\partial v_k}{\partial x^\beta}))}{\partial x^\alpha} = h_k$$

and w_k *is the solution of:*

$$(**) \qquad \begin{cases} -\dfrac{\partial(a_k{}^{\alpha\beta}\dfrac{\partial w_k}{\partial x^\beta})}{\partial x^\alpha} = 0 \\ w_k|_{\partial B_1} = v_k|_{\partial B_1} \end{cases}$$

If h_k *converges strongly to 0 in* $L^1(B_1, \mathbf{R}^k)$ *as* $k \to \infty$, *then* $f_k \equiv w_k - v_k$ *converges weakly to* 0 *in* $L_1^2(B_1, \mathbf{R}^k)$.

Proof. Let $G_k(x, y)$ be the Green function of operator:

$$L_k(\cdot) \equiv -\frac{\partial(a_k{}^{\alpha\beta}\dfrac{\partial(\cdot)}{\partial x^\beta})}{\partial x_\alpha}$$

and $G_k^\sigma(x,y) = ⨍_{B_\sigma(y)} G_k(x,z)\ dz$, then $G_k(x,y)$ and $G_k^\sigma(x,y)$ enjoy the following properties, see [GW].

i) $0 \le G_k(x,y) \le c|x-y|^{2-n}, \quad c = c(n,\Lambda)$.

ii) $\int_{B_1} a_k{}^{\alpha\beta} \dfrac{\partial\varphi}{\partial x^\alpha} \dfrac{\partial G_k^\sigma(x,y)}{\partial x^\beta}\ dx = ⨍_{B_{\sigma(y)}} \varphi\ dx$, $\forall \varphi \in L^2_{1,0} \cap L^\infty(B_1, \mathbf{R}^k)$, $\forall B_\sigma(y) \subset B_1$.

iii) $G_k^\sigma(x,y) \in L^2_{1,0} \cap L^\infty(B_1, \mathbf{R})$,$\|G_k^\sigma(x,y)\|_{L^\infty(B_1,\mathbf{R}^k)} \le c(\sigma)$,for any fixed $y \in B_1$.

By (*) (**) we have

$$-\frac{\partial(a_k^{\alpha\beta}(\frac{\partial(v_k - w_k)}{\partial x^\beta}))}{\partial x^\alpha} = h_k.$$

One can show that $v_k - w_k \in L^2_{1,0} \cap L^\infty(B_1, \mathbf{R}^k)$ with the help of the maximum principle. Taking $G_k^\sigma(x,y)$ as a test function of the above equations and noting the propertity (ii) of $G_k^\sigma(x,y)$ we get:

$$⨍_{B_\sigma(y)} (v_k - w_k)\ dx = \int_{B_1} h_k \cdot G_k^\sigma(x,y)\ dx \qquad \forall y \in B_{1-2\sigma}$$

hence,

$$\int_{B_{1-2\sigma}} |⨍_{B_\sigma(y)} (v_k - w_k)\ dx|\ dy \le \int_{B_{1-2\sigma}} \int_{B_1} |h_k| \cdot G_k^\sigma(x,y)\ dxdy.$$

Due to property (iii) of $G_k^\sigma(x,y)$, we can obtain:

$$\int_{B_{1-2\sigma}} |⨍_{B_\sigma(y)} f_k\ dx|\ dy \le c(\sigma) \int_{B_1} |h_k|\ dx \to 0, \text{as} \quad k \to \infty.$$

However,it is easy to see, for every sequence $\{f_{k_i}\} \subset \{f_k\}$, there exists a subsequence of $\{f_{k_i}\}$ which is also denoted by $\{f_k\}$,and a $f_0 \in L^2_{1,0}(B_1, \mathbf{R}^k)$ such that f_k converges weakly to f_0 in $L^2_{1,0}(B_1, \mathbf{R}^k)$ and f_k converges strongly to f_0 in $L^2(B_1, \mathbf{R}^k)$. Thus, we arrive at:

$$\int_{B_{1-2\sigma}} |⨍_{B_\sigma(y)} f_0\ dx|\ dy = 0, \qquad \text{for every } y \in B_{1-2\sigma}.$$

Since σ is arbitrary in $(0,\delta)$, for some small positive δ, the above equality implies $f_0 \equiv 0$ in B_1, the Main Lemma follows. □

2. Proof of Energy improvement Theorem.

Theorem 2.1 (Energy-improvement). *There exists θ and ϵ with $0 < \theta = \theta(\Lambda) < 1/2, \epsilon = \epsilon(\Lambda) > 0$, such that if u is a minimizer under $a^{\alpha\beta}$,where $(a^{\alpha\beta}) \in \mathcal{F}_\Lambda$.then we have*

$$\theta^{2-n} \int_{B_\theta} |\nabla u|^2 \, dx \leq 1/2 \int_{B_1} |\nabla u|^2 \, dx.$$

Provided $\int_{B_1} |\nabla u|^2 \, dx \leq \epsilon^2$.

Proof. Were the thoerem false, there would exist, for each θ with $0 < \theta < 1/2$,a sequence $\{u_k\}$ of minimizers under $(a_k^{\alpha\beta}) \in \mathcal{F}_\Lambda$, such that:

$$\int_{B_1} |\nabla u_k|^2 \, dx = \epsilon_k^2 \to 0 \text{as } k \to \infty, \text{and } \theta^{2-n} \int_{B_\theta} |\nabla u_k|^2 \, dx \geq 1/2\epsilon_k^2$$

we may form an associated normlized sequence:

$$v_k = \frac{u_k - c_k}{\epsilon_k},$$

where c_k is a constant vector which will be determined later.

Clearly,

$$\int_{B_1} |\nabla v_k|^2 \, dx = 1$$

$$\theta^{2-n} \int_{B_\theta} |\nabla v_k|^2 \, dx \geq 1/2$$

and

$$-\frac{\partial(a_k^{\alpha\beta}(\frac{\partial v_k}{\partial x^\beta}))}{\partial x^\alpha} = \epsilon_k A(u_k)(\nabla v_k, \nabla v_k) \qquad \text{for every } k \geq 1.$$

We will prove this theorem by showing $\theta^{2-n} \int_{B_\theta} |\nabla u_k|^2 \, dx \leq 1/4{\epsilon_k}^2$ when θ is sufficently small, and k is sufficiently large.

First,as in [HL], we can apply Lemma 1 to get the estimate (see [HL] pp564):

$$\theta^{2-n} \int_{B_\theta} |\nabla u_k|^2 \, dx \leq \lambda^j (2^j\theta)^{(2-n)} \int_{B_{2^j\theta}} |\nabla u_k|^2 \, dx$$

(1)

$$+ c\lambda^{-q} \sum_{h=0}^{j-1} \lambda^h \fint_{B_{2^{h+1}\theta}} |u_k - {\overline{u}_k}^{(h+1)}|^2 \, dx$$

for sufficiently large k (may depend on θ), where $\overline{u}_k^h = \fint_{B_{2^h\theta}} u_k\, dx$, and j is a positive integer such that $1/4 \le 2^j\theta \le 1/2$.

Secondly,we want to show:

$$\fint_{B_{2^h\theta}} |u^k - \overline{u}_k{}^{(h)}|^2\, dx \ \le c(2^h\theta)^{2\alpha}\varepsilon_k{}^2, \text{provided k is large enough.}$$

where $0 \ \le h \le j$, $0 < \alpha < 1$, and c is constant independents of k.

Suppose $r \in (\theta, 1/2)$,take $c_k = \overline{u}_k{}^{(r)} \equiv \ \fint_{B_r} u_k\, dx$ in the definition of v_k,and set:

$$v_k^r = \frac{u_k - \overline{u}_k{}^{(r)}}{\varepsilon_k}.$$

Let w_k^r denote the solution of the equations

$$(2) \qquad \begin{cases} -\dfrac{\partial(a_k^{\alpha\beta}(\dfrac{\partial w_k^r}{\partial x^\beta}))}{\partial x^\alpha} = 0 & \text{in} \quad B_1 \\ w_k^r - v_k^r|_{\partial B_1} = 0 \end{cases} .$$

By the definition of v_k^r and Main Lemma,it is easy to see $f_k^r \equiv w_k^r - v_k^r$ converges weakly to 0 in $L^2_{1,0}(B_1, \mathbf{R}^k)$,$f_k^r$ converges strongly to 0 in $L^2(B_1, \mathbf{R}^k)$ and $\overline{w_k^r} = \fint_{B_r} w_k^r\, dx \to 0$ as $k \to \infty$.

Next,we will show $\int_{B_r} |v_k^r|^2\, dx \le cr^{2\alpha}$ when $k \ge k(\theta)$ is sufficiently large. Obviously we have:

$$\fint_{B_r} |v_k^r|^2\, dx \le 4(\fint_{B_r} |v_k^r - w_k^r|^2\, dx + \fint_{B_r} |w_k^r - \overline{w_k^r}|^2\, dx + |\overline{w_k^r}|^2).$$

Notice that w_k^r is the solution of (2),we can obtain (see [G],Theorem 21 pp.53):

$$|w_k^r(x) - w_k^r(y)| \le c(n, \Lambda)|x - y|^\alpha,$$

for every $x, y \in B_{1/2}$, here $\alpha = \alpha(\Lambda, n)$, then:

$$|w_k^r(x) - \overline{w_k^r}| \le \fint_{B_r} |w_k^r(x) - w_k^r(y)|\, dy \le c(n, \Lambda)(2r)^\alpha.$$

Since $v_k^r - w_k^r$ converges strongly to 0 in $L^2(B_1, \mathbf{R}^k)$ and $\overline{w_k^r} \to 0$ as $k \to \infty$; $r \ge \theta > 0$, we get:

$$\fint_{B_r} |v_k^r|^2\, dx \le cr^{2\alpha}$$

or

$$\fint_{B_r} \left|u_k^r - \overline{u_k}^{(r)}\right|^2 \, dx \le c r^{2\alpha} {\varepsilon_k}^2,$$

when k is sufficently large, where $c = c(n, \Lambda)$.

Choosing $r = 2^h\theta$,where $0 \le h \le j$,we can see that, for any fixed $\theta > 0$,if $k \ge k(\theta)$ is sufficiently large, there holds:

$$\fint_{B_{2^h\theta}} \left|u_k - \overline{u_k}^{(h)}\right|^2 \, dx \le c(2^h \cdot \theta)^{2\alpha} \cdot {\varepsilon_k}^2. \tag{3}$$

Finally,it follows by substituting (3) into (1) and letting $k \to \infty$;that

$$\theta^{2-n} \int_{B_\theta} |\nabla u_k|^2 \, dx \le c \cdot {\varepsilon_k}^2 (\lambda^j + \theta^{2\alpha} \cdot \lambda^{-q}), \text{where } c = c(n, \Lambda).$$

Choosing θ sufficiently small such that $c \cdot (\theta^{2\alpha}\lambda^{-q} + \lambda^j) \le 1/4$ (since $1/4 \le 2^j\theta \le 1/2$) we can get the estimate:

$$\theta^{2-n} \int_{B_\theta} |\nabla u_k|^2 \, dx \le 1/4{\varepsilon_k}^2;$$

which contradicts the original choice of u_i. This completes the proof of Theorem 2.1. □

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