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Braid Group Technique in Complex Geometry V: The Fundamental Group of a Complement of a Branch Curve of a Veronese Generic Projection

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In this paper we compute $\pi_1(\mathbb{C}^2 - S_3)$, S_3 the branch curve of a generic projection of the Veronese surface V_3 on $\mathbb{CP}^2$. Throughout this paper we use G for $\pi_1(\mathbb{C}^2 - S_3)$. We also have a similar result concerning $\overline{G} = \pi_1(\mathbb{CP}^2 - S_3)$.

Fundamental groups of complements of curves are very important invariants but it is very difficult to compute them. We obtained G and $\overline{G}$ using our braid group techniques.

This paper is a continuation of "Braid Group Techniques in Complex Geometry, I, II, III, IV." (BGT I, BGT II, BGT III, BGT IV for short.) In BGT I we laid the foundation of our braid monodromy techniques and applied them to line arrangements. In BGT II we dealt with the braid monodromy of almost real curves and showed how to regenerate these formulas to cuspidal curves. In BGT III we presented a series of generic projected degenerations of the Veronese surface V_3 and the branch curve S_3 of its generic projection to $\mathbb{CP}^2$ to a union of 9 planes, and a branch curve $S^{(6)}$ which is a union of lines. In BGT IV we computed the braid monodromy of $S_3 \cap \mathbb{C}^2$ using the braid monodromy of $S^{(6)}$ and the regeneration rules proved in BGT II. We obtained the factorized expression for the braid monodromy denoted by $\varepsilon(18)$. In this paper we use $\varepsilon(18)$ and the Van Kampen Theorem to compute G and $\overline{G}$. If the reader is only interested in the final results he can go directly to Chapter VI.

0. Definition of Braid Group and Presentation of the Van Kampen Theorem.

We need certain terminology in order to formulate the Van Kampen Theorem.

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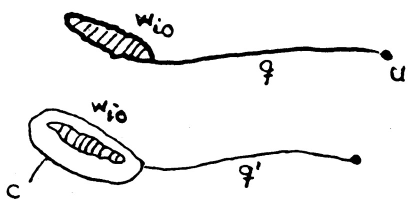


Figure 0.1.

We first recall from BGT I (Section II) the definition of a good geometric base of $\pi_1(D-K,*)$ for K a finite set in a disc D.

Definition. $\ell(\gamma)$.

Let D be a disc. Let w_i $i=1,\dots,n$ small discs in $\mathrm{Int}(D)$ s.t. $w_i \cap w_j = \emptyset$ for $i \neq j$. Let $u \in \partial D$. Let γ be a simple path connecting u with one of the w_i's, say w_{i_0}, which does not meet any other w_j, $j \neq i_0$. We assign to γ a loop (actually an element of $\pi_1(D-K,u)$) as follows. Let c be a simple loop equal to the (oriented) boundary of a small neighborhood V of w_{i_0} chosen such that $\gamma' = \gamma - V \cap \gamma$ is a simple path. Then $\ell(\gamma) = \gamma' \cup c \cup \gamma'^{-1}$. We also use the same notation $\ell(\gamma)$ for the element of $\pi_1(D-K,u)$ corresponding to $\ell(\gamma)$ (see Fig. 0.1).

Definition. *g-base (good geometric base)*

Let D be a disk, $K \subseteq D$, $\#K < \infty$. Let $u \in D-K$. Let $\{\gamma_i\}$ be a bush in (D,K,u), i.e., $\forall i,j$ $\gamma_i \cap \gamma_j = u$, $\forall i$ $\gamma_i \cap K =$ one point, and γ_i are ordered counterclockwise around u. Let $\Gamma_i = \ell(\gamma_i) \in \pi_1(D-K,u)$ be the loop around $K \cap \gamma_i$ determined by γ_i. $\{\Gamma_i\}$ is a g-base of $\pi_1(D-K,*)$.

Definition. *Braid group* $B_n = B_n[D,K]$

Let D be a closed disc in $\mathbb{R}^2$, $K \subset D$, K finite. Let B be the group of all diffeomorphisms β of D such that $\beta(K) = K$, $\beta|_{\partial D} = \mathrm{Id}_{\partial D}$. For $\beta_1, \beta_2 \in B$, we say that β_1 is equivalent to β_2 if β_1 and β_2 induce the same automorphism of $\pi_1(D-K,u)$. The quotient of B by this equivalence relation is called the braid group $B_n[D,K]$ $(n = \#K)$. We sometimes denote by $\overline{\beta}$ the braid represented by β. The elements of $B_n[D,K]$ are called braids.

Definition. $H(\sigma)$, *half-twist defined by* σ

Let D, K be as above. Let $a, b \in K$, $K_{a,b} = K - a - b$ and σ be a simple path in $D - \partial D$ connecting a with b s.t. $\sigma \cap K = \{a, b\}$. Choose a small regular neighborhood U of σ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{C}^1$ ($\mathbb{C}^1$ is taken with the usual "complex" orientation) such that $f(\sigma) = [-1, 1]$, $f(U) = \{z \in \mathbb{C}^1 \mid |z| < 2\}$. Let $\alpha(r), r \geq 0$, be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{3}{2}]$ and $\alpha(r) = 0$ for $r \geq 2$.

Define a diffeomorphism $h : \mathbb{C}^1 \longrightarrow \mathbb{C}^1$ as follows. For $z \in \mathbb{C}^1$, $z = re^{i\varphi}$, let $h(z) = re^{i(\varphi + \alpha(r))}$. It is clear that on $\{z \in \mathbb{C}^1 \mid |z| \leq \frac{3}{2}\}$, $h(z)$ is the positive rotation by 180° and that $h(z) =$ Identity on $\{z \in \mathbb{C}^1 \mid |z| \geq 2\}$, in particular, on $\mathbb{C}^1 - f(U)$. Considering $(f \circ h \circ f^{-1})|_D$ (we always take composition from left to right), we get a diffeomorphism of D which switches a and b and is the identity on $D - U$. Thus it defines an element of $B_n[D, K]$, called the half-twist defined by σ and denoted $H(\sigma)$.

Definition. *Frame of* $B_n[D, K]$

Let D be a disc in $\mathbb{R}^2$. Let $K = \{a_1, \dots, a_n\}$, $K \subset D$. Let $\sigma_1, \dots, \sigma_{n-1}$ be a system of simple paths in $D - \partial D$ such that each σ_i connects a_i with a_{i+1} and for

$$i, j \in \{1, \dots, n-1\}\ ,\ i < j \quad , \quad \sigma_i \cap \sigma_j = \begin{cases} \emptyset & \text{if } |i - j| \geq 2 \\ a_{i+1} & \text{if } j = i + 1\,. \end{cases}$$

Let $H_i = H(\sigma_i)$. We call the ordered system of (positive) half-twists $(H_1, \dots, H_{n-1})$ a frame of $B_n[D, K]$ defined by $(\sigma_1, \dots, \sigma_{n-1})$, or a frame of $B_n[D, K]$ for short.

Notation.

$[A, B] = ABA^{-1}B^{-1}$.

$\langle A, B \rangle = ABAB^{-1}A^{-1}B^{-1}$.

$(A)_B = B^{-1}AB$.

Theorem (E. Artin's braid group presentation). *B_n is generated by the half-twists H_i of a frame $\{H_i\}$ and all the relations between $H_1, \dots, H_{n-1}$ follow from*

$$\begin{aligned} [H_i, H_j] &= 1 \qquad \textit{if} \quad |i - j| > 1, \\ \langle H_i, H_j \rangle &= 1 \qquad \textit{if} \quad |i - j| = 1, \\ & 1 \leq i, j \leq n - 1. \end{aligned}$$

Proof. See proof in [MoTe4], Chapter 4. □

Proposition-Definition. Δ_n^2 $(\in B_n)$
$\Delta_n^2 = (H_1 \ldots H_n)^{n-1}$ for any frame $H_1 \ldots H_{n-1}$ of B_n.

We shall need the following definition:

Definition. *$R(\varepsilon), N(\varepsilon), G(\varepsilon)$ where ε is a factorized expression in B_m*
Let D be a disk in $\mathbb{C}$, $K \subseteq D$, $\# K = m$.
Let $\mathbb{F}_m = \pi_1(D - K)$. Consider the natural action of $B_m = B[D, K]$ on $\mathbb{F}_m$, denoted by $(\Gamma)g_i$.
Let $\varepsilon = g_1 \cdot \cdots \cdot g_t$ be a factorized expression in B_m.
Let $\Gamma_1 \cdots \Gamma_m$ be a good geometric base of $\mathbb{F}_m$.
Let $M(\varepsilon)$ be the subgroup of B_m generated by $\{g_i\}_{i=1}^t$.
Let $R(\varepsilon)$ be the subgroup of $\mathbb{F}_m$ generated by $\{(\Gamma_j)g_i \circ \Gamma_j^{-1}\}_{i=1,}^{t}{}_{j=1}^{m}$.
Let $N(\varepsilon)$ be the normal subgroup of $\mathbb{F}_m$ generated by $R(\varepsilon)$.
Let $G(\varepsilon) = \dfrac{\mathbb{F}_m}{N(\varepsilon)}$.

Lemma 0.1. (i) $N(\varepsilon) = \{(\alpha)\beta \cdot \alpha^{-1} \bigm| \alpha \in \mathbb{F}_m, \quad \beta \in M(\varepsilon)\}$.

(ii) *$G(\varepsilon) = F_m$ with the relations induced from $R(\varepsilon)$.*

Proof. Trivial. □

We recall from BGT IV [MoTe7] the definitions of Hurwitz equivalent factorizations and factorization invariant under $h \in B_m$.

Definition. *Hurwitz move*
Let $g_i \cdot \cdots \cdot g_k = h_i \cdot \cdots \cdot h_k$ be two factorized expressions of the same element in a group G. We say that $g_i \cdot \cdots \cdot g_k$ is obtained from $h_i \cdot \cdots \cdot h_k$ by a Hurwitz move if $\exists\, 1 \leq p \leq k-1$ s.t. $g_i = h_i$ $i \neq p, p+1$, $g_p = h_p h_{p+1} h_p^{-1}$ and $g_{p+1} = h_p$ or $g_p = h_{p+1}$ and $g_{p+1} = h_{p+1}^{-1} h_p h_{p+1}$

Definition. *Hurwitz equivalence of factorized expressions*
Let $g_i \cdot \cdots \cdot g_k = h_i \cdot \cdots \cdot h_k$ be two factorized expressions of the same element in a group G. We say that $g_i \cdot \cdots \cdot g_k$ is a Hurwitz equivalent to $h_i \cdot \cdots \cdot h_k$ if $h_i \cdot \cdots \cdot g_k$ is obtained from $h_i \cdot \cdots \cdot h_k$ by a finite number of Hurwitz moves. We denote it by $g_1 \cdot \cdots \cdot g_k \underset{He}{\simeq} h_i \cdot \cdots \cdot h_k$.

Definition. *Factorized expression in B_m invariant under $h \in B_m$*

We say that a factorized expression $g_1 \dots g_t$ is invariant under h if $(g_1)_h \cdot \dots \cdot (g_t)_h$ is Hurwitz equivalent to $g_1 \dots g_t$, i.e., can be obtained from $g_1 \dots g_t$ by a finite number of Hurwitz moves ($(g)_h = h^{-1}gh$).

Lemma 0.2. *If a factorized expression $\varepsilon = g_1 \dots g_t$ in B_m is invariant under $h \in B_m$, then h induces an automorphism of $G(\varepsilon)$.*

Proof. The group B_m acts on $\pi_1(D-K,u)$; thus there is a natural action of $h \in B_m$ on $\pi_1(D-k,u) = \mathbb{F}_m$. Therefore, h induces an automorphism of $\mathbb{F}_m$. Since ε is Hurwitz equivalent to $(\varepsilon)_h$, we get that $h^{-1}M(\varepsilon)h = M(\varepsilon)$ and thus $(N(\varepsilon))h \subset N(\varepsilon)$, and h induces an automorphism of $\mathbb{F}/N(\varepsilon) = G(\varepsilon)$. □

Certain factorized expressions of Δ_m^2 in B_m play an important role in the computation of the fundamental group of complements of curves, as we shall see in Theorem 0.3.

Let S be a curve in $\mathbb{CP}^2$ of degree m.

We refer the reader to BGT I, Chapter VI, [MoTe4] for the definition of a certain factorized expression in B_m related to S: **S-factorization of Δ_m^2** or **S-factorization** or **product form of Δ_m^2** or **braid monodromy factorization w.r.t. S and u**.

Theorem 0.3. Zariski-Van Kampen Theorem. (see [VK]) *Let $\overline{S}$ be a curve in $\mathbb{CP}^2$ of $\deg m$, s.t. $\overline{S}$ is transversal to the line in infinity. Let $S = \overline{S} \cap \mathbb{C}^2$. Let ε be the braid monodromy factorization w.r.t. to S and u. Let $\mathbb{C}_u = u \times \mathbb{C}$. Let $\{\Gamma_i\}$ be a g-base of $\pi_1(\mathbb{C}_u - S, u)$. Then:*

$\pi_1(\mathbb{C}^2 - S, *) = G(\varepsilon)$ and $\pi_1(\mathbb{CP}^2 - \overline{S}, *) = G(\varepsilon)$ with extra relation $\prod_{i=1}^{m} \Gamma_i = 1$.

Remark. We shall use this theorem for S_3, the branch curve of a generic projection of V_3 (the Veronese of order 3) to $\mathbb{CP}^2$. In BGT IV we computed the braid monodromy factorization related to S_3. We denote it $\varepsilon(18)$. We shall again present $\varepsilon(18)$ in the next chapter.

We are going to reformulate the Zariski-Van Kampen Theorem in a more precise form for a cuspidal curve, i.e., for a curve with only nodes and cusps.

Theorem 0.4. (Zariski). *If S is a cuspidal curve, then the related braid monodromy factorization ε is of the form $\prod\limits_{j=1}^{p} V_j^{\nu_j}$, where V_j is a half-twist and $\nu_j = 1$ or 2 or 3.*

Theorem 0.5. Zariski-Van Kampen (precise version). *Let $\overline{S}$ be a cuspidal curve in $\mathbb{CP}^2$. Let $S = \mathbb{C}^2 \cap \overline{S}$. Let ε be a braid monodromy factorization w.r.t. S and u. Let $\varepsilon = \prod\limits_{j=1}^{p} V_j^{\nu_j}$, where V_j is a half-twist and $\nu_j = 1, 2, 3$.*

For every $j = 1 \dots p$ let $A_j, B_j \in \pi_1(\mathbb{C}_u - S, u)$ be such that A_j, B_j can be extended to a g-base of $\pi_1(\mathbb{C}_u - S, u)$ and $(A_j)V_j = B_j$. Let $\{\Gamma_i\}$ be a g-base of $\pi_1(\mathbb{C}_u - S, u)$. Then $\pi_1(\mathbb{C}^2 - S, u)$ is generated by the images of $\{\Gamma_i\}$ in $\pi_1(\mathbb{C}^2 - S, u)$ and the only relations are those implied from $\left\{V_j^{\nu_j}\right\}$, as follows:

$$\begin{aligned} A_j \cdot B_j^{-1} &= 1 \quad && \text{if} \quad \nu_j = 1 \\ [A_j, B_j] &= 1 \quad && \text{if} \quad \nu_j = 2 \\ \langle A_j, B_j \rangle &= 1 \quad && \text{if} \quad \nu_j = 3 \end{aligned}$$

$\pi_1(\mathbb{CP}^2 - \overline{S}, *)$ is generated by $\{\Gamma_i\}$ with the above relations and one more relation $\prod\limits_i \Gamma_i = 1$.

Remark 0.5′. How to determine $\mathbf{A_j}$ and $\mathbf{B_j}$ from $\mathbf{V_j}$ or how to determine $\mathbf{A_V}$ and $\mathbf{B_V}$ from $\mathbf{V = H(\sigma)}$ (see formulation of Van Kampen Theorem).

To be able to use the Zariski-Van Kampen Theorem, we must know how to compute A_j if B_j for every $j = 1 \dots p$. Assume, for simplicity, that u_o is below real lines and $\{q_i\} = \mathbb{C}_u \cap S$ are real points. Assume that $\rho(\delta_j) = V_j^{\nu_j}$, where $V_j = H(\sigma)$, a half-twist corresponding to a path σ from q_1 to q_2. Take a homotopically-equivalent path σ' that passes through u_0. Let σ_1, σ_2 be the part of σ' from u_0 to q_1, q_2 respectively. Let $A_j = \ell(\sigma_1)$ $B_j = \ell(\sigma_2)$ be the loops of $\pi_1(\mathbb{C}^2 - S, u_0)$ built from σ_1, σ_2 as in the definition in the beginning of the Chapter. See Fig. 0.2 for an example how to determine A_V and B_V for $V = H(\sigma)$.

Proposition 0.6. *If an S-factorization $\Delta^2 = \prod g_i$ is invariant under $h \in B_m$ then $R((g_i)_h)$ is also a relation on $\pi_1(\mathbb{C}^2 - S, *)$.*

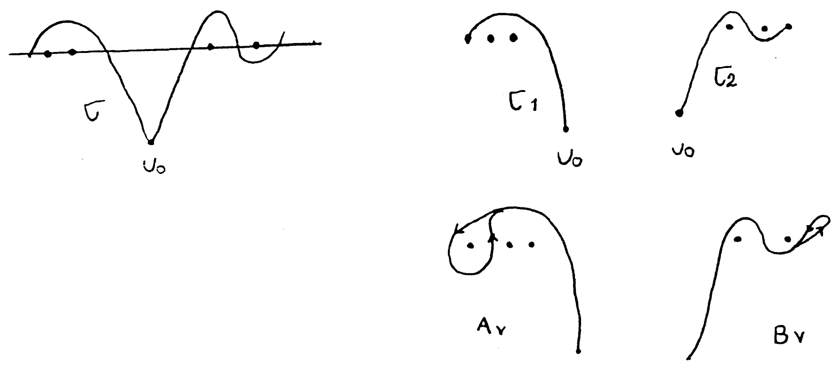


Figure 0.2.

Proof. Zariski-Van Kampen Theorem and the fact that an invariant factorization of a braid monodromy factorization, namely $\prod (g_i)_h$, is also a braid monodromy factorization (see Proposition VI.4.2 from BGT I, [MoTe4]). □

Remark. Proposition 0.6 indicates why it is important to prove invariant properties of S-factorizations. We use such properties to induce more relations on the fundamental group. □

I. The braid monodromy related to S, the branch curve of a V₃-projection.

Notation. $V_3,\ f,\ S_3, S$

Let V_3 be the Veronese surface of order 3, i.e., the following embedding of $\mathbb{CP}^2$ into $\mathbb{CP}^N$:

$$(x, y, z) \to (\dots, x^i y^j z^k, \dots)_{i+j+k=3}.$$

Let $f = f_3$ be a generic projection: $V_3 \xrightarrow{f} \mathbb{CP}^2$.
Let S_3 be its branch curve in $\mathbb{CP}^2$.
Let $\mathbb{C}^2$ be a "generic" affine piece of $\mathbb{CP}^2$.
Let $S = S_3 \cap \mathbb{C}^2$.
$\deg S = \deg S_3 = 18$.
We are interested in $\pi_1(\mathbb{CP}^2 - S_3, *)$ and $\pi_1(\mathbb{C}^2 - S, *)$.

We constructed in BGT III [MoTe7] a projective degeneration of $V_3 \xrightarrow{f} \mathbb{CP}^2$ into $Z^{(6)} \xrightarrow{f^{(6)}} \mathbb{CP}^2$ where $Z^{(6)}$ is a union of 9 planes P_j, $j = 1 \ldots 9$, the ramification curve is a union of 9 intersection lines $\hat{L}_i, i = 1 \ldots 9$, as in Fig. II.1. (Each $\hat{L}_i$ is an intersection line of 2 P_j's.) $S^{(6)}$, the branch curve of $f^{(6)}$ in $\mathbb{CP}^2$, is a union of 9 lines $L_i, i = 1 \ldots 9$. $(L_i = \pi^{(6)}(\hat{L}_i))$.

$K^{(6)} = \mathbb{C}u \cap S^{(6)}$.

$\# \, K^{(6)} = 9$.

Definition. $u, \mathbb{C}_u, K$.

Let us choose u in the x-axis of $\mathbb{C}^2$ far away from the x-projection singularities of the x-projection of S_3 and of $S^{(6)}$.

$\mathbb{C}_u = u \times \mathbb{C}$.

$K = \mathbb{C}_u \cap S$.

From the regeneration process it is obvious that for every point q_i that we had in $K^{(6)}$ we have 2 points $q_t, q_{t'}$ in K which are close to each other. Recall that we used a "real model" of $(\mathbb{C}_u, K)$. Thus, we assume that $K = \{q_i, q_{i'}\}_{i=1}^{9}$, $q_i, q_{i'}$ are real.

Remark. For arbitrary n we would get that $V_n \to \mathbb{CP}^2$ degenerated into $Z \to \mathbb{CP}^2$ where Z = a union of n^2 planes with a ramification curve which consists of $\frac{3}{2}n(n-1)$ intersection lines, and a branch curve consisting of $\frac{3}{2}n(n-1)$ lines. Thus, S_n, the branch curve of $V_n \to \mathbb{CP}^2$, is of order $3n(n-1)$.

In BGT IV we computed a braid monodromy factorization of S_3 denoted $\varepsilon(18)$.

We also proved those invariance properties of $\varepsilon(18)$ and invariance under complex conjugation. We shall repeat these results here. For this we have to recall some notations.

Notations.

$\underset{\sim}{z}_{ij} = z_{ij}$ = a path below the real line from q_i to q_j.

$$Z_{ij} = H(z_{ij})$$

$\overline{z}_{ij}$ = a path above the real line from q_i to q_j.

$$\overline{Z}_{ij} = H(\overline{z}_{ij})$$

$\overset{(a)}{\underset{\sim}{z}_{ij}}$ = a path above a and below the real line elsewhere from q_i to q_j.

$$\overset{(a)}{\underline{Z}_{ij}} = H(\underline{z}_{ij})$$

$\underset{(a)(b)}{\overline{z}_{ij}}$ = a path below a and b above the real line elsewhere

from q_i to q_j.

$$\underset{(a)(b)}{\overline{Z}_{ij}} = H(\underset{(a)(b)}{\overline{z}_{ij}})$$

ρ_i $= Z_{ii'} = H(z_{ii'}) =$ half-twist corresponds to the shortest line between q_i and $q_{i'}$

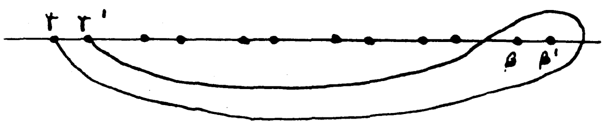
$$Y^{(2)}_{i,jj'} = \prod_{m=0}^{1} (Y^2_{ij})\rho_j^m$$

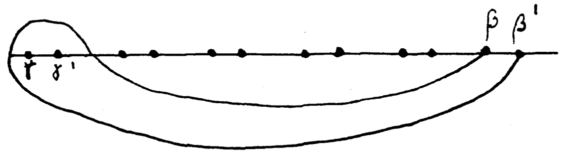
$$Y^{(2)}_{ii',jj'} = \prod_{\ell=0}^{1}\prod_{m=0}^{1} (Y^2_{ij})\rho_i^{\ell}\rho_j^m$$

$$Y^{(3)}_{i,jj'} = \prod_{m=-1}^{1} (Y^3_{ij})\rho_j^m$$

For $\beta > \gamma$

$\tilde{\underline{z}}_{\gamma\gamma'(\beta)} =$





$\tilde{z}_{\beta\beta'(\gamma)} =$

With the above notations we recall the braid monodromy factorization of $S = S^{(0)}$, denoted $\varepsilon(18)$ and some invariance properties.

Theorem I.1. $\varepsilon(18) = \prod_{\nu=7}^{1} C_\nu \; H_\nu$
where H_i, the braid monodromy of S around v_i factors as follows:

$H_1 = Z^{(3)}_{11',2} \cdot \tilde{Z}_{22'(1)}$

$H_2 = Z^{(3)}_{11',3} \cdot \tilde{Z}_{33'(1)}$

$H_3 = Z^{(3)}_{44',6} \cdot \tilde{Z}_{66'(4)}$

$H_5 = Z^{(3)}_{55',9} \cdot \tilde{Z}_{99'(5)}$

$H_6 = Z^{(3)}_{6',77'} \cdot \tilde{Z}_{66'(7)}$

$H_7 = Z^{(3)}_{8',99'} \cdot \tilde{Z}_{88'(9)}$

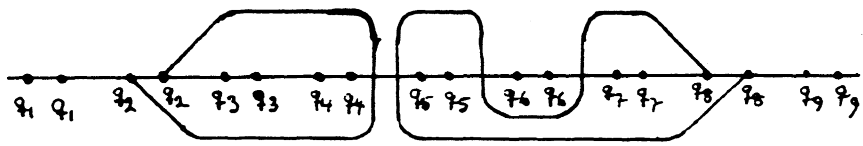


Figure I.0(a).

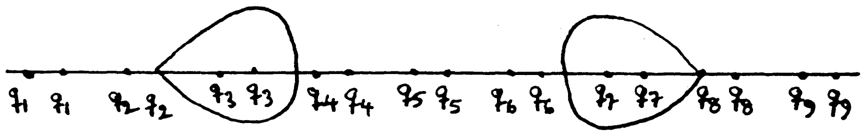


Figure I.0(b).

$$H_4 = Z^{(3)}_{2',33'} \tilde{Z}_{88'} Z^{(2)}_{44',8'} \left(Z^{(2)}_{33',8}\right)^{\bullet} \overline{Z}^{(3)}_{55',8} \left(Z^{(2)}_{44',8}\right)^{\bullet} \left(Z^{(2)}_{33',8}\right)^{\bullet} \hat{F}_1 (\hat{F}_1)_{\rho^{-1}}$$
$$\cdot Z^{(3)}_{77',8} \underset{\sim}{Z}^{2}_{2',i}, i = 8,8,7',7,5',5 \quad \overline{Z}^{(3)}_{2',44'} Z^2_{2i} \quad i = 8',8,7',7,5',5 \quad \tilde{Z}_{22'}$$

where $\tilde{Z}_{22'}$, $\tilde{Z}_{88'}$ *correspond to the paths* $\tilde{z}_{22'}$, $\tilde{z}_{88'}$ *described in Fig.* I.0(a).

$\bullet$ *denotes conjugation by a braid* $b^{\bullet}$ *induced from the motion described in Fig.* I.0(b).

(In fact, $b^{\bullet} = Z^2_{2'3'} Z^2_{2'3'} Z^{-2}_{7'8} Z^{-2}_{78}$*.)*

$$\hat{F}_1 = Z^{(3)}_{3,44'} Z^{(3)}_{55',7} \alpha^{(1)} \overset{(4)}{Z^2_{3'7}} \overline{Z}^2_{3'7'}$$

$$\hat{F}_2 = \left(Z^{(3)}_{3,44'}\right) \rho^{-1} \left(Z^{(3)}_{55',7}\right)_{\rho^{-1}} \left(\alpha^{(1)}\right)_{\rho^{-1}} \left(\overset{(4)}{\underline{Z}^2_{3'7}}\right)_{\rho^{-1}} \left(\overline{Z}_{3'7'}\right)^2 \rho^{-1}.$$

$\alpha^{(1)} = \alpha_1 \alpha_2$

where α_1 *and* α_2 *are the curves described in Fig.* I.0(c), I.0(d), *respectively.*

$\rho = \rho_7 \rho_3$

$(\alpha^{(1)})\rho = (\alpha^{(1)})\rho_3^{-1}\rho_7^{-1} = (\alpha_1)(\rho_3^{-1}\rho_7^{-1}) \cdot (\alpha_2)(\rho_3\rho_7)^{-1}$

and

$$C_1 = C_2 = I_d$$
$$C_3 = Z^{(2)}_{11',44'} \prod_{i=1,2,3,5} Z^{(2)}_{ii',66'}$$
$$C_4 = \overline{Z}^{(2)}_{11',55'} \underset{(6)}{\overline{Z}}{}^{(2)}_{11',77'} \overline{Z}^{(2)}_{11',88'} \overline{Z}^{(2)}_{66',88'}$$

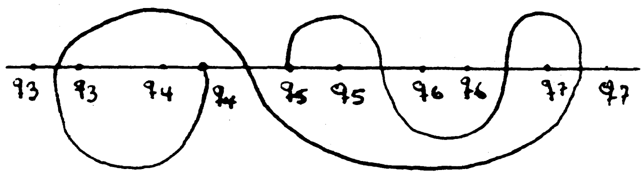


Figure I.0(c).

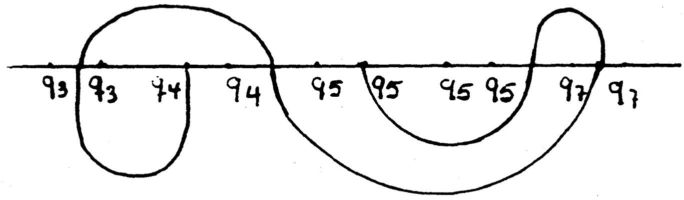


Figure I.0(d).

$$C_5 = \prod_{\substack{i=1 \\ i\neq 5}}^{7} Z^{(2)}_{ii',99'}$$

$$C_6 = C_7 = I_d.$$

The following remark gives an explicit description of a half-twist conjugated by some ρ_j and will help us later to deduce relations from $\varepsilon(18)$ using the Van Kampen method.

Remark I.1.

(i) $Z^{(3)}_{i,jj'} = \overset{(j)}{\underline{Z}^3_{ij'}}\underline{Z}^3_{ij}\underline{Z}^3_{ij'}$ (Fig. I.1(a))

(ii) $Z^{(2)}_{ii',jj'} = Z^2_{ij}Z^2_{ij'}Z^2_{i'j}Z^2_{i'j'}$. (Fig. I.1(b))

(iii) Let $Y_{ij} = H(y_{ij})$ where y_{ij} is a path connecting q_i or $q_{i'}$ with q_j or $q_{j'}$. The following graph (see Fig. I.1(c)) indicates the conjugation of Y_{ij} by $\rho_j, \rho_j^{-1}, \rho_i, \rho_i^{-1}$ for different types of y_{ij}. In the graph we only indicate the action of ρ_j and ρ_j^{-1} on the "head" of y_{ij} within a small circle around q_j and $q_{j'}$ and the action of ρ_i and ρ_i^{-1} on the "tail" of y_{ij} in a small circle around q_i and $q_{i'}$. The "body" of y_{ij} is not changing under $\rho_j^{\pm 1}$ and $\rho_i^{\pm 1}$.

Theorem I.2. Invariance Theorem (BGT IV, Proposition 18, [MoTe7])

Let $\rho = \rho_{m_1 \ldots m_4, m_6, m_9} = \rho_1^{m_1}(\rho_2\rho_8)^{m_2} \cdot (\rho_3\rho_7)^{m_3} \cdot (\rho_4\rho_5)^{m_4} \cdot \rho_6^{m_6} \cdot \rho_9^{m_9}$ *then,* $\varepsilon(18)$ *is invariant under* ρ *for every* $m_i \in \mathbb{Z}$. $\rho_i = Z_{ii'}$).

Theorem I.3. Complex Conjugation Theorem (BGT IV, Proposition 19, [MoTe7])

$\varepsilon(18)$ *is invariant under complex conjugation.*

A finite set of generators for $\pi_1(\mathbb{C}^2 - S, u_0)$ and $\pi_1(\mathbb{CP}^2 - S_3, u_0)$.

Let us choose $u_0 \in \mathbb{C}_u, u_0$ below the real line. Let $\{\Gamma_i, \Gamma_{i'}\}$ be a g-base of $\pi_1(\mathbb{C}_u - S, u_o)$. When considered as elements of $\pi_1(\mathbb{C}^2 - S, u_0)$ and of $\pi_1(\mathbb{CP}^2 - S_3, u_o)$, they generate (not freely) the groups. Thus we have $\{\Gamma_i, \Gamma_{i'}\}_{i=1}^9$, a set of generators for $\pi_1(\mathbb{C}^2 - S, u_0)$ and $\pi_1(\mathbb{CP}^2 - S_3, u_0)$.

We want to compute $G = \pi_1(\mathbb{C}^2 - S, u_0)$ and $\overline{G} = \pi_1(\mathbb{CP}^2 - S_3, u_0)$.

By the Zariski-Van Kampen Theorem, $G \simeq G(\varepsilon(18))$ and $\overline{G} \cong \dfrac{G(\varepsilon(18))}{\prod_{i=1}^{9} \Gamma_i\Gamma_{i'}}$.

Corollary I.4. $G = \pi_1(\mathbb{C}^2 - S, u_0)$ *satisfies all the relations induced in* $R(\varepsilon(18))$, *all the relations induced from* $(R(\varepsilon(18))\rho_{m_1,\ldots,m_4,m_6,m_9}$ *and all the relations induced from the complex conjugation of* $\varepsilon(18)$, *where* $\rho_{m_1 \ldots m_4, m_6, m_9} = \rho_1^{m_1}(\rho_2\rho_8)^{m_2} \cdot (\rho_3\rho_7)^{m_3} \cdot (\rho_4\rho_5)^{m_4} \cdot \rho_6^{m_6} \cdot \rho_9^{m_9}$ *and* $\rho_i = Z_{ii'}$. *Moreover,*

$$G \sim G(\varepsilon(18))$$
$$\overline{G} = G / \prod_i \Gamma_i\Gamma_{i'}.$$

Proof. Theorems I.1, I.2, I.3 and the Van Kampen Theorem. □

Corollary I.5. *Let* $G = \pi_1(\mathbb{C}^2 - S, u_0)$. *If* R *is any relation in* G *then* $(R)\rho_{m_1,\ldots,m_4,m_6,m_9}$ *is also a relation in* G, *where* $(R)\rho_{m_1,\ldots,m_4,m_6,m_9}$ *is the relation induced from* R *by replacing* Γ_i *and* $\Gamma_{i'}$ *with* $(\Gamma_i)\rho_i^{m_i}$ *and* $(\Gamma_{i'})\rho_i^{m_i}$, $i = 1,2,3,4,6,9$, *respectively and replacing* $\Gamma_8, \Gamma_{8'}, \Gamma_7, \Gamma_{7'}, \Gamma_5, \Gamma_{5'}$ *with* $(\Gamma_8)\rho_8^{m_2}, (\Gamma_{8'})\rho_8^{m_2}, (\Gamma_7)\rho_7^{m_3}, (\Gamma_{7'})\rho_7^{m_3}, (\Gamma_5)\rho_5^{m_4}, (\Gamma_{5'})\rho_5^{m_4}$, *respectively.*

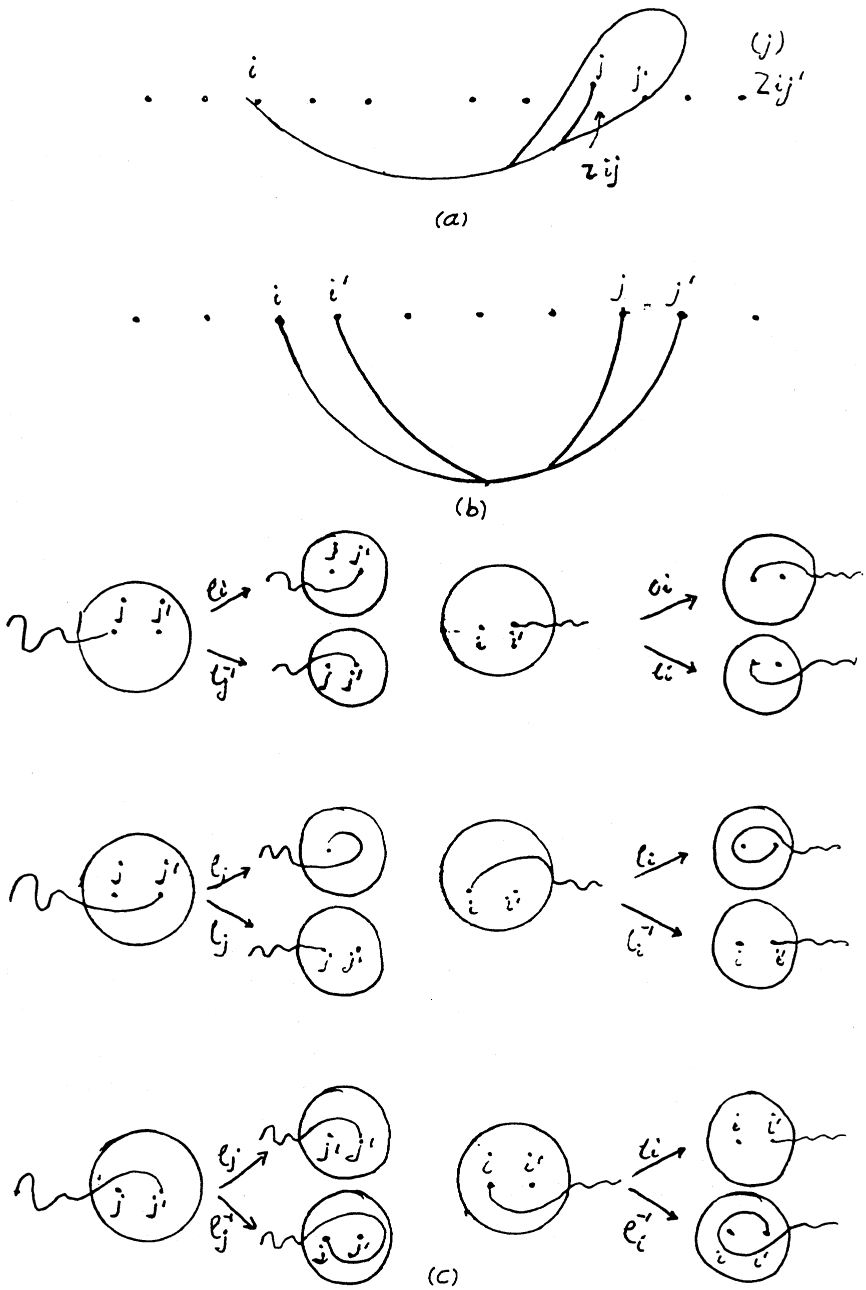


Figure I.1.

Proof. Proposition 0.6 and Theorem I.2. □

Remark. In other words, $\rho_{m_1 \ldots m_4, m_6, m_9}$ defines an automorphism of G.

Notation.

$\Gamma_{\underline{i}}$ = any element of the set $\{(\Gamma_i)\rho_i^{m_i}\}_m \in \mathbb{Z}$.

Corollary I.6. (Complete invariance in 3-points)

Let $(\alpha, \beta) = (1,2)$ *or* $(1,3)$ *or* $(4,6)$ *or* $(5,9)$ *or* $(6,7)$ *or* $(8,9)$. *Any relation* R *in* G *that involves only* Γ_α *and* Γ_β *is true when* $\Gamma_{\underline{\alpha}}$ *replaces* Γ_α *amd* $\Gamma_{\underline{\beta}}$ *replaces* Γ_β.

Proof. Without loss of generality assume $(\alpha, \beta) = (1,2)$. Let $\Gamma_{\underline{1}} = (\Gamma_1)\rho_1^{m_1}$ $\Gamma_{\underline{2}} = (\Gamma_2)\rho_2^{m_2}$. By Corollary I.5, $(R)\rho_{m_1, m_2, 0,0,0,0}$ is also a relation on R, and differs from R by replacing $\Gamma_{\underline{1}}$ by Γ_1 and $\Gamma_{\underline{2}}$ by Γ_2. □

Corollary I.7. *Let* R *be a relation in* G *that involves at most one index of each of the pairs* $(2,8)$, $(3,7)$, $(4,5)$. *For every index* i *that appears in* R *choose some* $\Gamma_{\underline{i}}$. *Then* R *is true when we simultaneously replace* Γ_i *by* $\Gamma_{\underline{i}}$ *(or* E_i*) and* Γ_i' *by* $(\Gamma_{\underline{i}})\rho_i$ *(or* $(E_{\underline{i}})\rho_i$*, respectively), for* $i \in \{$*indices that appear in* $R\}$.

Proof. We can assume w.l.o.g. that $\Gamma_8, \Gamma_{8'}, \Gamma_7 \Gamma_{7'}, \Gamma_5, \Gamma_{5'}$ appear in R and $\Gamma_2, \Gamma_{2'}, \Gamma_3, \Gamma_{3'}, \Gamma_4, \Gamma_{4'}$ do not appear in R. Let i be s.t. Γ_i or $\Gamma_{i'}$ appears in R $(i \in 1,5,6,7,8,9)$. There exist m_i s.t. $\Gamma_{\underline{i}} = (\Gamma_i)\rho_i^{m_i}$. Let $\rho = (\prod_{i=1,5,6,8,9} \rho_i^{m_i})\rho_2^{m_8}\rho_3^{m_7}\rho_4^{m_5}$. By Corollary I.5 $(R)\rho$ is also a relation in G. Since $\Gamma_2, \Gamma_{2'}, \Gamma_3, \Gamma_{3'}, \Gamma_4, \Gamma_{4'}$ do not appear in R the relation $(R)\rho$ is actually equal to $(R)\prod_{i=1,5,6,8,9} \rho_i^{m_i}$ and it differs from R by replacing Γ_i with $(\Gamma_i)\rho_i^{m_i}$ and $\Gamma_{i'}$ by $(\Gamma_{i'})\rho_i^{m_i}$ for $i = 1,5,6,7,8,9$. But $(\Gamma_i)\rho_i^{m_i} = \Gamma_{\underline{i}}$ and $(\Gamma_{i'})\rho_i^{m_i} = (\Gamma_i)\rho_i\rho_i^{m_i} = (\Gamma_i)\rho_i^{m_i}\rho_i = (\Gamma_{\underline{i}})\rho_i$, so we get the corollary. □

Remark. We can replace every Γ_i that appear in R by any $\Gamma_{\underline{i}} = (\Gamma_i)\rho_i^{m_i}$ (i.e., different m_i's for different Γ_i's) since both Γ_3 and Γ_7 (Γ_2 and Γ_8, Γ_5 and Γ_4, respectively) do not appear in R. If both Γ_3 and Γ_7 appear in R, then we could only replace Γ_3 by $(\Gamma_3)\rho_3^m$ and Γ_7 by $(\Gamma_7)\rho_7^m$ for the same m.

II. List of relations in G.

We are going to describe G using a different set of generators than those introduced in Chapter I. We use all the notations from Chapter I, all the relations induced from the braid monodromy factorization $\varepsilon(18)$ (Theorem I.1), the complex conjugation (Theorem I.3), and Corollaries I.5 and I.6.

Remark II.0. *First set of generators for G*

Let $\{\Gamma_i, \Gamma_{i'}\}$ be a g-base of $\pi_1(\mathbb{C}_u - S, u_0)$. Considered. as elements of $G = \pi_1(\mathbb{C}^2 - S, u_0)$, $\{\Gamma_i, \Gamma_{i'}\}$ generates G.

Definition.

$$E_i = \begin{cases} \Gamma_i & i \neq 2,7 \\ \Gamma_{i'} & i = 2,7 \end{cases}$$

$$E_{i'} = (E_i)\rho_i.$$

Notations.

$\rho_i = Z_{ii'}$ the half-twist corresponding to the shortest path between q_j and $q_{j'}$.

$E_{\underline{i}} =$ an element of $\{(E_i)\rho_i^m\}_m \in \mathbb{Z}$.

$\Gamma_{\underline{i}} =$ an element of $\{(\Gamma_i)\rho_i^m\}_m \in \mathbb{Z}$.

In order to use the invariance theorem we need the following lemma.

Lemma II.0. (i) $(\Gamma_j)\rho_i = \Gamma_j \quad (\Gamma_{j'})\rho_i = \Gamma_{j'}$ *for* $i \neq j$.

(ii) $(\Gamma_i)\rho_i = \Gamma_{i'} \quad (\Gamma_{i'})\rho_i = \Gamma_{i'}\Gamma_i\Gamma_{i'}^{-1} \quad (\Gamma_i)\rho_i^{-1} = \Gamma_i^{-1}\Gamma_{i'}\Gamma_i$.

(iii) *Let* $\rho = \rho_{m_1,\dots,m_4,m_6,m_9} = \rho_1^{m_1}(\rho_2\rho_8)^{m_2} \cdot (\rho_3\rho_7)^{m_3} \cdot (\rho_4\rho_5)^{m_4} \cdot \rho_6^{m_6} \cdot \rho_9^{m_9}$. *Then* $(\Gamma_i)\rho = \Gamma_{\underline{i}}$.

(iv) $\Gamma_{\underline{i}} \in \langle \Gamma_i, \Gamma_{i'} \rangle$.

Proof. Geometric observation (Fig. II.0(a) and (b)) or BGT I, Section II, §2. □

Lemma II.1. (i) $E_{i'} = \begin{cases} \Gamma_{i'} & i \neq 2,7 \\ \Gamma_{i'}\Gamma_i\Gamma_{i'}^{-1} & i = 2,7 \end{cases}$

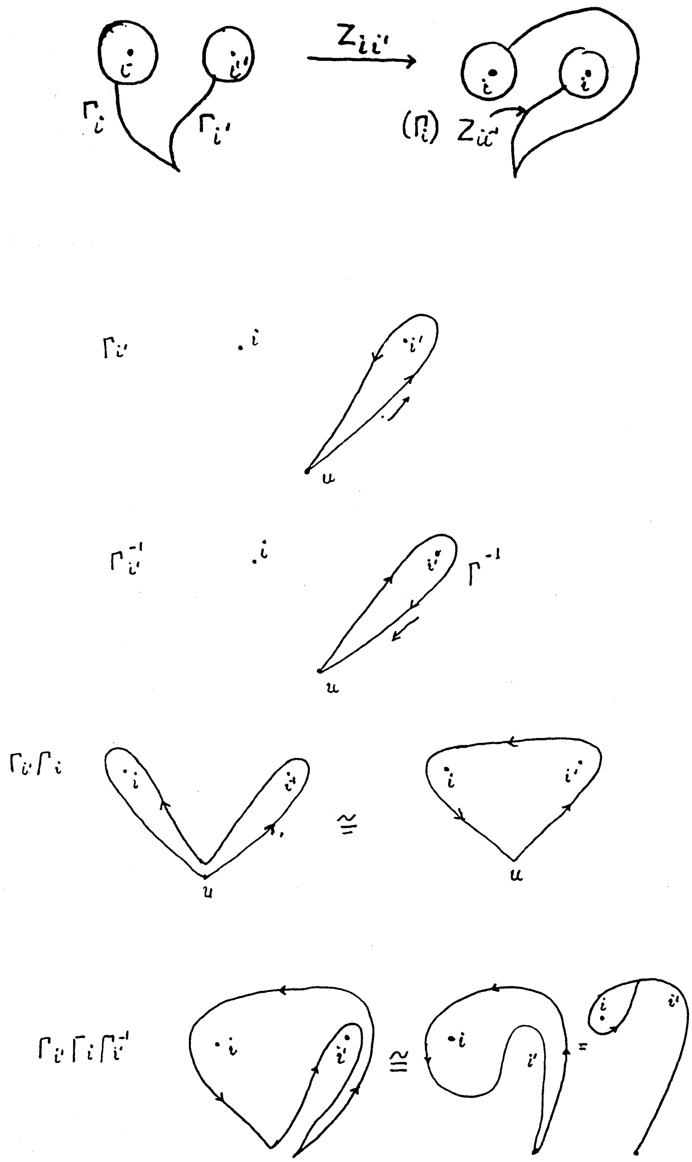


Figure II.0.

(ii) $\Gamma_{i'}\Gamma_i = E_{i'}E_i$

(iii) $(E_i)\rho_i = E_{i'}$

(iv) $(E_{i'})\rho_i = E_{i'}E_iE_{i'}^{-1}$

(v) $$\Gamma_i = \begin{cases} E_i & i \neq 2,7 \\ E_i^{-1}E_{i'}E_i & i = 2,7 \quad (= (E_{i'})\rho_i^{-2} = (E_i)\rho_i^{-1}). \end{cases}$$

Proof. Trivial. □

Lemma II.2. $\{E_i, E_{i'}\}$ *generate* G.

Proof. Trivial. □

Remark II.2. *A second set of generators for* G

We start with a set of generators $\Gamma_i, \Gamma_{i'}$ and exchange it for a set of generators $E_i, E_{i'}$.

Definition. Let ψ be the classical monodromy homomorphism from G to the symmetric group of order 9 induced by the projection $V_3 \to \mathbb{CP}^2$.

Lemma II.3. $\psi(\Gamma_i) = \psi(\Gamma_{i'}) = \psi(E_i) = \psi(E_{i'}) = (k_i, \ell_i)$ *where* $\hat{L}_i = P_{k_i} \cap P_{\ell_i}$, *and* $\{L_i\}$ *and* $\{P_j\}$ *are arranged as in Fig.* II.1.

Proof. Let γ_i be a path in $\mathbb{C}^2$ connecting u_0 to q_i s.t. $\Gamma_i = \ell(\gamma_i)$ (see the definition of $\ell(\gamma)$ in Chapter 0). One has to consider the degeneration of $V_3 \xrightarrow{f} \mathbb{CP}^2$ to $Z^{(6)} \xrightarrow{f^{(6)}} \mathbb{CP}^2$ and of S_3 to $S^{(6)}$, constructed in BGT III, [MoTe7]. The surface $Z^{(6)}$ is a union of 9 planes, $P_1 \cdots P_9$. The configuration of the planes and their intersection lines $\hat{L}_1, \ldots, \hat{L}_9$ are as in Fig. II.1. Let $f^{(6)}$ be a generic projection $Z^{(6)} \xrightarrow{f^{(6)}} \mathbb{CP}^2$ and $S^{(6)}$ its branch curve in $\mathbb{C}^2$. We choose V_3 to be close to $Z^{(6)}$. Let $p_i^{(6)} = P_i \cap {\pi^{(6)}}^{-1}(u_0)$. Let p_i be a point in $\pi^{-1}(u_o)$ which is close to $p_i^{(6)}$. Fix i between 1 and 9. It is clear that when we move along γ_i from u_0 to q_i, the lifted path in $Z^{(6)}$ which starts in $p_{k_i}^{(6)}$ will lie in P_{k_i} and will end on a point in $\hat{L}_i$ above q_i. The lifted path in $Z^{(6)}$

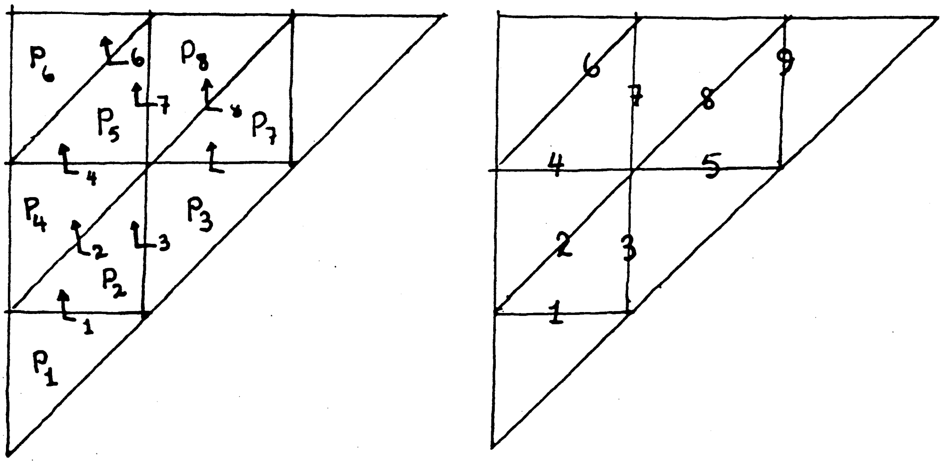


Figure II.1.

that starts in $p_{\ell_i}^{(6)}$ will lie in P_{ℓ_i} and will end in the same point in $\hat{L}_i$. Thus, in the regenerated case, the lifted path of Γ_k that starts in p_i will end in p_j. The lifted paths of γ_i in $Z^{(6)}$ that start in $p_t^{(6)}$, $t \neq k_i, \ell_i$ will be closed loops. Thus, in the regenerated case, the path obtained from lifting Γ_k that starts in p_t, $t \neq k_i, \ell_i$ is a loop. Thus, $\psi(\Gamma_k) =$ the transposition $(k_i \;\; \ell_i)$ of the symmetric group on 9 elements In the same way $\psi(\Gamma_{i'}) = (k_i \;\; \ell_i)$, and thus $\psi(E_i) = \psi(E_{i'}) = (k_i \;\; \ell_i)$. □

Corollary II.3. *The transpositions $\psi(E_i)$ are as follows:*

$\psi(E_1) = (1 \;\; 2)$
$\psi(E_2) = (2 \;\; 3)$
$\psi(E_3) = (2 \;\; 4)$
$\psi(E_4) = (3 \;\; 5)$
$\psi(E_5) = (4 \;\; 7)$
$\psi(E_6) = (5 \;\; 6)$
$\psi(E_7) = (5 \;\; 8)$
$\psi(E_8) = (7 \;\; 8)$
$\psi(E_9) = (7 \;\; 9)$

Moreover, $\psi(E_i)$ and $\psi(E_j)$ have one common index $\Leftrightarrow$ $\hat{L}_i$ and $\hat{L}_j$ are edges of some triangle in Fig. II.1.

Proof. Immediate from the previous Lemma. □

Before continuing with G, we want to prove some claims concerning an arbitrary group.

Claim II.4. (a) *If* $\langle A,B\rangle = 1$, *i.e.,* $ABAB^{-1}A^{-1}B^{-1} = 1$, *then*

$$\begin{aligned} ABA &= BAB \\ A^{-1}B^kA &= BA^kB^{-1} \\ AB^kA^{-1} &= B^{-1}A^kB \\ A^{-1}B^{-1}A^k &= B^kA^{-1}B^{-1}. \end{aligned}$$

(b) $[A,B] = [A,C] = 1 \Rightarrow [A,D] = 1$ *for* $D \in \langle B,C\rangle =$ *subgroup generated by* B,C.
(c) $[A_C, B_D] = 1,\ [C,B] = [C,D] = [A,D] = 1 \Rightarrow [A,B] = 1$.
(d) $[A, XYZ] = [A,X]\ [A,Y]_{X^{-1}}[A,Z]_{Y^{-1}X^{-1}}$.
(e) $[XY, A] = [Y,A]_{X^{-1}}[X,A]$.
(f) $[A_C, B_C] = [A,B]_C$.
(g) $[X,Z] = 1 \Rightarrow [X, Y_Z] = [X,Y]_Z$.
(h) *If* $\langle x,y\rangle = 1$, *then* $\langle Ax, y\rangle = 1 \Leftrightarrow A_{y^{-1}x^{-1}} = A^{-1}A_{y^{-1}}$. A

Proof. (a) – (g) are easy to verify.
We shall only prove (h) here.
If $\langle Ax, y\rangle = 1$, then:

$$\begin{aligned} 1 = \langle Ax, y\rangle &= Axyxy^{-1}x^{-1}A^{-1}y^{-1} \\ &= A \cdot A_{y^{-1}x^{-1}}xyxy^{-1}x^{-1}A^{-1}y^{-1}yA^{-1}y^{-1} \\ &= A \cdot A_{y^{-1}x^{-1}} \cdot 1 \cdot A^{-1}_{y^{-1}} \\ \Rightarrow A_{y^{-1}x^{-1}} &= A^{-1}A_{y^{-1}}. \end{aligned}$$

If $A^{-1}A_{y^{-1}} = A_{y^{-1}x^{-1}}$, then

$$\begin{aligned} &AA_{y^{-1}x^{-1}}A^{-1}_{y^{-1}} = 1 \\ \Rightarrow &Axy\underbrace{Ay^{-1}x^{-1}y}A^{-1}y^{-1} = 1 \\ \Rightarrow &Axy\overbrace{Axy^{-1}x^{-1}}\ A^{-1}y^{-1} = 1 \\ \Rightarrow &Ax \cdot y \cdot Ax \cdot y^{-1} \cdot (Ax)^{-1} \cdot y^{-1} = 1 \\ \Rightarrow &\langle Ax, y\rangle = 1. \end{aligned}$$

□

Lemma II.5. (a) $[\Gamma_i, \Gamma_j] = [\Gamma_{i'}, \Gamma_j] = 1 \Rightarrow [\Gamma_{\underline{i}}, \Gamma_{\underline{j}}] = 1$.
(b) $[\Gamma_i, \Gamma_j] = [\Gamma_i^{-1}\Gamma_{i'}\Gamma_i, \Gamma_j] = 1 \Rightarrow [\Gamma_{\underline{i}}, \Gamma_j] = 1$.

Proof. We only prove (a); (b) is the same argument. Since $\Gamma_{\underline{j}}$ is a product of Γ_j and $\Gamma_{j'}$, we can apply Claim II.4 (b) to get $[\Gamma_i, \Gamma_{\underline{j}}] = 1$. In particular, $[\Gamma_i, (\Gamma_j)\rho_j^{-1}] = 1$. We use invariance under $\rho_i\rho_j$ and Lemma 0.2 on it to get $[\Gamma_{i'}, \Gamma_j] = 1$. We use invariance under $\rho_i\rho_j$ and Lemma 0.2 on $[\Gamma_i, \Gamma_j] = 1$ to get $[\Gamma_{i'}, \Gamma_{j'}] = 1$. From $[\Gamma_{i'}, \Gamma_{j'}] = [\Gamma_{i'}, \Gamma_j] = 1$ we get, using Lemma II.4(b), that $[\Gamma_{i'}, \Gamma_{\underline{j}}] = 1$. From $[\Gamma_i, \Gamma_{\underline{j}}] = [\Gamma_{i'}, \Gamma_{\underline{j}}] = 1$ we get, using Lemma II.4(b), that $[\Gamma_{\underline{i}}, \Gamma_{\underline{j}}] = 1$. □

Proposition II.6. *The following relations hold in* G :

(1) $\langle E_{\underline{i}}, E_{\underline{j}} \rangle = 1$ $\forall i, j$ *s.t.* $\psi(E_i)$ *and* $\psi(E_j)$ *have exactly one common index.*

(2) $[E_{\underline{i}}, E_{\underline{j}}] = 1$ $\forall i, j$ *s.t.* $\psi(E_i)$ *and* $\psi(E_j)$ *have no common index.*

(3) $1 = (E_7E_5E_3^{-1}E_4^{-1}E_2E_4E_3E_5^{-1}E_7^{-1}E_8^{-1})(\rho_3\rho_7)^i(\rho_4\rho_5)^j \quad \forall i, j \in \mathbb{Z}$.

(4) $E_{4'}E_4E_{3'}E_3E_{2'}E_3^{-1}E_{3'}^{-1}E_4^{-1}E_{4'}^{-1} = E_2$.

(5) $E_5^{-1}E_{5'}^{-1}E_7^{-1}E_{7'}^{-1}E_8E_{7'}E_7E_{5'}E_5 = E_{8'}$.

(6)

$$E_{\beta'} = E_\alpha^{-1}E_{\alpha'}^{-1}E_\beta E_{\alpha'}E_\alpha \qquad (\alpha, \beta) = (1,2) = (1,3) = (4,6) = (5,9).$$

(7)

$$E_\alpha = E_{\beta'}E_\beta E_{\alpha'}E_\beta^{-1}E_{\beta'}^{-1} \qquad (\alpha, \beta) = (6,7) = (8,9)$$

Definitions of $\psi(E_i)$ *as in Corollary* II.3.

Proof. We divide the proof into the following 48 claims:

Claim 0. $[\Gamma_{\underline{i}}, \Gamma_{\underline{j}}] = 1$ for i, j s.t. $\hat{L}_i \cap \hat{L}_j$ do not intersect, $j \neq 9$, i.e., for $(i, j) = (1,4), (1,5), (1,6), (1,7), (1,8), (2.6), (3.6), (6.8)$.

Claim 1. $\langle \Gamma_{\underline{i}}, \Gamma_{\underline{j}} \rangle = 1$ $(i,j) = (1,2), (1,3), (4,6), (5,9), (6,7), (8,9)$

Claim 2. $\langle \Gamma_{2'}, \Gamma_{\underline{3}} \rangle = 1.$

Claim 3. $\Gamma^{\bullet}_{3'} = \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} = \Gamma_{2'}^{-1} \Gamma_3 \Gamma_{2'}.$

Claim 4. $\Gamma^{\bullet}_{3} = \Gamma_{2'}^{-1} \Gamma_{3'}^{-1} \Gamma_{3'} \Gamma_3 \Gamma_{2'}.$

Claim 5. $\langle \Gamma_{\underline{7}}, \Gamma_8 \rangle = 1.$

Claim 6. $\Gamma^{\bullet}_{7'} = \Gamma_8 \Gamma_{7'} \Gamma_7 \Gamma_{7'}^{-1} \Gamma_{8'}^{-1}.$

Claim 7. $\Gamma^{\bullet}_{7} = \Gamma_8 \Gamma_{7'} \Gamma_{8'}^{-1} = \Gamma_{7'}^{-1} \Gamma_8 \Gamma_{7'}.$

Claim 8. $\Gamma^{\bullet}_{2'} = \Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_3^{-1} \Gamma_{3'}^{-1}$
$\Gamma^{\bullet}_{8} = \Gamma_8^{-1} \Gamma_{7'}^{-1} \Gamma_8 \Gamma_{7'} \Gamma_7.$

Claim 9. $\Gamma^{\bullet}_{2} = \Gamma_2, \Gamma^{\bullet}_{8'} = \Gamma_{8'}, \Gamma^{\bullet}_{4} = \Gamma_4, \Gamma^{\bullet}_{4'} = \Gamma_{4'}, \Gamma^{\bullet}_{5'} = \Gamma_{5'}, \Gamma^{\bullet}_{5} = \Gamma_5.$

Claim 10. $[\Gamma_{\underline{2}}, \Gamma_{\underline{i}}] = 1 \qquad i = 5, 7, 8.$

Claim 11. $[\Gamma_{8'}, \Gamma_i] = 1 \qquad i = 3, 4.$

Claim 12. $[\Gamma_{\underline{8}}, \Gamma_{\underline{i}}] = 1 \qquad i = 3, 4.$

Claim 13.
$$E_{\beta'} = E_\alpha^{-1} E_{\alpha'}^{-1} E_\beta E_{\alpha'} E_\alpha \qquad (\alpha, \beta) = (1,2)$$
$$= (1,3)$$
$$= (4,6)$$
$$= (5,9)$$
$$E_\alpha = E_{\beta'} E_\beta E_{\alpha'} E_\beta^{-1} E_{\beta'}^{-1} (\alpha, \beta) = (6,7)$$
$$= (8,9)$$

Claim 14. $[\Gamma_{3'}^{\bullet}, \Gamma_{7'}^{\bullet}] = 1.$

Claim 15. $[\Gamma_3, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1.$

Claim 16. $[\Gamma_{3'}, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1.$

Claim 17. $[\Gamma_{\underline{3}}, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1.$

Claim 18. $[\Gamma_{\underline{3}}, \Gamma_{\underline{7}}] = 1.$

Claim 19. $[\Gamma_{\underline{3}}^{\bullet}, \Gamma_{\underline{7}}^{\bullet}] = 1.$

Claim 20. $\langle \Gamma_{\underline{3}}^{\bullet}, \Gamma_{\underline{4}}^{\bullet} \rangle = 1.$

Claim 21. $\langle \Gamma_{\underline{5}}^{\bullet}, \Gamma_{\underline{7}}^{\bullet} \rangle = 1.$

Claim 22. $\Gamma_7^{\bullet}\Gamma_5^{\bullet}\Gamma_7^{\bullet^{-1}} = \Gamma_{3'}^{\bullet^{-1}}\Gamma_4^{\bullet}\Gamma_{3'}^{\bullet} \; (\rho_3\rho_7)^i(\rho_4\rho_5)^j.$

Claim 23. $[\Gamma_7^{\bullet}, \Gamma_{4'}^{\bullet}\Gamma_4^{\bullet}\Gamma_{3'}^{\bullet}\Gamma_4^{\bullet^{-1}}\Gamma_{4'}^{\bullet^{-1}}] = 1.$

Claim 24. $\langle \Gamma_{\underline{3}}^{\bullet}, \Gamma_{\underline{5}}^{\bullet} \rangle = 1.$

Claim 25. $\langle \Gamma_{\underline{4}}^{\bullet}, \Gamma_{\underline{7}}^{\bullet} \rangle = 1.$

Claim 26. $\langle \Gamma_{\underline{3}}, \Gamma_{\underline{5}} \rangle = 1.$

Claim 27. $\langle \Gamma_{\underline{7}}, \Gamma_{\underline{4}} \rangle = 1.$

Claim 28. $[\Gamma_{4'}, \Gamma_5] = 1.$

Claim 29. $[\Gamma_4, \Gamma_5] = 1$

Claim 30. $[\Gamma_{\underline{4}}, \Gamma_5] = 1.$

Claim 31. $[\Gamma_{\underline{4}}, \Gamma_{\underline{5}}] = 1.$

Claim 32. $[\Gamma_{2'}, \Gamma_{7'}^{\bullet}] = 1.$

Claim 33. $[\Gamma_3, \Gamma_7^{\bullet}] = 1.$

Claim 34. $[\Gamma_7^{\bullet}, \Gamma_5^2 \Gamma_3 \Gamma_5^{-2}] = 1.$

Claim 35. $[\Gamma_{2'}, \Gamma_3^2 \Gamma_5 \Gamma_3^{-2}] = 1.$

Claim 36. $[\Gamma_{\underline{3}}, \Gamma_{\underline{4}}] = 1.$

Claim 37. $[\Gamma_{7'}^{\bullet}, \Gamma_4^2 \Gamma_{3'}^{\bullet} \Gamma_4^{-2}] = 1.$

Claim 38. $[\Gamma_{7'}, \Gamma_{3'}^{\bullet}] = 1, \qquad [\Gamma_8, \Gamma_{3'}^{\bullet}] = 1.$

Claim 39. $[\Gamma_4, \Gamma_{7'}^2 \Gamma_8 \Gamma_{7'}^{-2}] = 1.$

Claim 40. $[\Gamma_5, \Gamma_{7'}] = 1.$

Claim 41. $[\Gamma_{\underline{5}}, \Gamma_{\underline{7}}] = 1.$

Claim 42. $\langle \Gamma_{\underline{4}}, \Gamma_{\underline{2}} \rangle = 1.$

Claim 43. $\langle \Gamma_{\underline{2}}, \Gamma_{\underline{3}} \rangle = 1.$
$\langle \Gamma_{\underline{7}}, \Gamma_{\underline{8}} \rangle = 1.$

Claim 44. $\langle \Gamma_{\underline{5}}, \Gamma_{\underline{8}} \rangle = 1.$

Claim 45. $\Gamma_8 = \Gamma_{7'} \Gamma_5 \Gamma_3^{-1} \Gamma_4^{-1} \Gamma_{2'} \Gamma_4 \Gamma_3 \Gamma_5^{-1} \Gamma_{7'}^{-1}.$

Claim 46. $\Gamma_{2'} = \Gamma_{4'} \Gamma_4 \Gamma_{3'} \Gamma_3 \Gamma_{2'} \Gamma_2 \Gamma_{2'}^{-1} \Gamma_4^{-1} \Gamma_3^{-1} \Gamma_{3'}^{-1} \Gamma_4^{-1} \Gamma_{4'}^{-1}.$

Claim 47. $\Gamma_{8'} = \Gamma_5^{-1} \Gamma_{5'}^{-1} \Gamma_7^{-1} \Gamma_{7'}^{-1} \Gamma_8 \Gamma_{7'} \Gamma_7 \Gamma_{5'} \Gamma_5.$

Claim 48. $[\Gamma_{\underline{i}}, \Gamma_{\underline{9}}] = 1 \qquad i = 1, 2, 3, 4, 6, 7.$

Proofs of the Claims.

We use the braids in $\varepsilon(18)$ (see Theorem I.1) to induce relations on G via the Van Kampen Theorem (Theorem 0.5). For every factor V^{ν} in $\varepsilon(18)$, we have to find A_V and B_V to get a relation. In Remark 0.4 one can find an algorithm how to determine A_V and B_V. In $\varepsilon(18)$ we have sometimes used a compact notation for a product of a few factors. Then we use Remark I.1 to determine the factors precisely. Sometimes, instead of using a factor b in $\varepsilon(18)$, we shall use its complex conjugation $\bar{b}$. In that way, we get $R(b)$

and/or $R(\bar{b})$ a relation on G induced by the Van Kampen Theorem. We also use Corollary I.5 to get other relations using $(R)\rho_{m_1,\dots,m_4,m_6,m_9}$.

Proof of Claim 0. Taking the factors C_i' or the complex conjugate of C_i' $i = 1\dots 4$ and applying the Van Kampen method on it to produce relations on G, we get $[\Gamma_{ii'}, \Gamma_{jj'}] = 1$ $\forall i,j$ s.t. $\hat{L}_i \cap \hat{L}_j = \emptyset$ $j \neq 7,9$. By Lemma II.5 we get $[\Gamma_{\underline{i}}, \Gamma_{\underline{j}}] = 1$ $\forall i,j$ s.t. $\hat{L}_i \cap \hat{L}_j = \emptyset$ $j \neq 7,9$. For $j = 7$ we consider C_4'. In C_4' we have $\tilde{Z}_{17} = \underset{(6,6')_{17}}{\overline{Z}}$:

Its complex conjugate is $\overset{(6,6')}{\underline{Z}_{17}}$: which implies on G the relation:

$$[\Gamma_1, \Gamma_6^{-1}\Gamma_{6'}^{-1}\Gamma_7\Gamma_{6'}\Gamma_6] = 1.$$

Since we already know that $[\Gamma_{\underline{1}}, \Gamma_{\underline{6}}] = 1$ we get $[\Gamma_1, \Gamma_7] = 1$. Now, we are using the Invariance Theorem (Corollary I.6) and we get $[\Gamma_{\underline{1}}, \Gamma_{\underline{7}}] = 1$. □

Proof of Claim 1. From H_i', $i \neq 4$ and Corollary I.5. □

Proof of Claim 2. By using $Z^{(3)}_{2',33'}$ of H_4' we get $\langle \Gamma_{2'}, \Gamma_3 \rangle = \langle \Gamma_{2'}, \Gamma_{3'} \rangle = \langle \Gamma_{2'}, \Gamma_{3'}\Gamma_3\Gamma_{3'}^{-1} \rangle$. We apply on it the invariance automorphism $(\rho_3\rho_7)^{m_3}$ for all possible values of m_3 to get $\langle \Gamma_{2'}, \Gamma_{\underline{3}} \rangle = 1$. □

Proof of Claim 3. By definition of $\bullet$ (see Theorem I.1).

$$\begin{aligned}
\Gamma_{3'}^{\bullet} &= \Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}\Gamma_3\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_3^{-1} \\
&= \Gamma_{3'}\Gamma_3(\Gamma_3^{-1}\Gamma_{3'}\Gamma_3)^{-1}\Gamma_{2'}(\Gamma_3^{-1}\Gamma_{3'}\Gamma_3) \cdot \Gamma_3^{-1}\Gamma_3^{-1} \\
&\overset{\text{by Claims 2 and II.4(a)}}{=} \Gamma_3\Gamma_{2'}\Gamma_3^{-1} \\
&\overset{\text{by Claims 2 and II.4(a)}}{=} \Gamma_{2'}^{-1}\Gamma_3\Gamma_{2'}.
\end{aligned}$$

□

Proof of Claim 4. By definition of $\bullet$ (see Theorem I.1).

$$\Gamma_{3'}^{\bullet} = \Gamma_{3'}\Gamma_3\Gamma_{2'}\underbrace{\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_3\Gamma_{3'}\Gamma_3}\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_3^{-1}$$

$$\stackrel{\text{by Claims 2 and II.4(a)}}{=} \Gamma_{3'}\Gamma_3(\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_3\Gamma_{3'}\Gamma_3)^{-1}\Gamma_{2'}$$
$$(\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_3\Gamma_3\Gamma_{3'}\Gamma_3)\Gamma_3^{-1}\Gamma_{3'}^{-1}$$
$$= \Gamma_3^{-1}\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_3$$
$$\stackrel{\text{by Claims 2 and II.4(a)}}{=} \Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}\Gamma_3\Gamma_{2'}.$$

□

Proof of Claims 5,6,7. Arguments symmetric to 2, 3, 4. □

Proof of Claim 8. Easy to see from the geometric observation of the action of $b^\bullet$. □

Proof of Claim 9. $b^\bullet$ does not affect those loops. □

Proof of Claim 10. By H_4' we have $Z^2_{2',i}$ and Z^2_{2i} for $i = 8, 8', 7, 7', 5, 5'$. Thus, by the Van Kampen Theorem, $[\Gamma_{2'}, \Gamma_{ii'}] = [\Gamma_2, \Gamma_{ii'}] = 1 \quad i = 5, 7, 8$. By Claim II.5, we have $[\Gamma_{2'}, \Gamma_{\underline{i}}] = [\Gamma_2, \Gamma_{\underline{i}}] = 1 \quad i = 5, 7, 8$. By Claim II.5, again $[\Gamma_{\underline{2}}, \Gamma_{\underline{i}}] = 1$. □

Proof of Claim 11. By H_4' we have $Z^{(2)}_{44',8'}$. Using the Van Kampen method $[\Gamma_4, \Gamma_{8'}] = 1$ and $[\Gamma_{4'}, \Gamma_{8'}] = 1$ By Lemma II.5, $[\Gamma_{\underline{4}}, \Gamma_{8'}] = 1$. We have in H_4', $(Z_{33',8'})^{2\bullet}$. By the Van Kampen method we have $[\Gamma_3^\bullet, \Gamma_{8'}^\bullet] = [\Gamma_{3'}^\bullet, \Gamma_{8'}^\bullet] = 1$. By Claims 4, 3, and 9, $\Gamma_{3'}^\bullet = \Gamma_{2'}^{-1}\Gamma_3\Gamma_{2'}$, $\Gamma_3 = \Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}\Gamma_3\Gamma_{2'}$, $\Gamma_{8'}^\bullet = \Gamma_{8'}$ and thus $[\Gamma_{2'}^{-1}\Gamma_3\Gamma_{2'}, \Gamma_{8'}] = [\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}\Gamma_3\Gamma_{2'}, \Gamma_{8'}] = 1$. $[\Gamma_{3'}; \Gamma_{8'}] = 1$. Since $[\Gamma_{2'}, \Gamma_8] = 1$, (Claim 10) we get by Lemma II.4(g), $[\Gamma_{3'}, \Gamma_{8'}] = 1$ and $\Gamma_3^{-1}\Gamma_{3'}\Gamma_3, \Gamma_{8'}] = 1$. By Lemma II.5, $[\Gamma_{\underline{3}}, \Gamma_{8'}] = 1$. □

Proof of Claim 12. By Claim 11, $[\Gamma_{\underline{3}}, \Gamma_{8'}] = 1$. We apply on it Corollary I.5 with $(Z_{88'}Z_{22'})^m$ to get $[\Gamma_{\underline{3}}, \Gamma_{\underline{8}}] = 1$. Similarly, we get $[\Gamma_{\underline{4}}, \Gamma_{\underline{8}}] = 1$. □

Proof of Claim 13. We shall only prove $(\alpha,\beta) = (1,2)$. In H_1' we have $\tilde{Z}_{22'(1)}$. By the Van Kampen Theorem we get $\Gamma_2 = \Gamma_1^{-1}\Gamma_{1'}^{-1}\Gamma_2^{-1}\Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_1$. We apply on it $Z^2_{22'}$ to get

$$\Gamma_{2'}\Gamma_2\Gamma_{2'}^{-1} = \Gamma_1^{-1}\Gamma_{1'}^{-1}\Gamma_{2'}\Gamma_{1'}\Gamma_1.$$

Thus, $E_2' = E_1^{-1}E_{1'}^{-1}E_{2'}E_{1'}E_1$. The other relations are induced from $H_2', H_3', H_5', H_6', H_7'$. □

Proof of Claim 14. In $\hat{F}$ we have $\left(\overline{Z}^2_{3'7'}\right)^\bullet$. After complex conjugation this braid transforms to $\left(\underline{Z}^2_{3'7'}\right)^\bullet$. By Corollary I.4, $\left(\underline{Z}^2_{3'7'}\right)^\bullet$ implies, via the Van Kampen method, the following relation on G: $[\Gamma^\bullet_{3'}, \Gamma^\bullet_{7'}] = 1$. □

Proof of Claim 15.

$$\begin{aligned} 1 \overset{\text{Claim 14}}{=} [\Gamma^\bullet_{3'}, \Gamma^\bullet_{7'}] &= [\Gamma_{2'}^{-1}\Gamma_3\Gamma_{2'}, \Gamma_8\Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}\Gamma_8] \\ &\overset{\text{Claims 10 and II.4(g)}}{=} [\Gamma_3, \Gamma_8\Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}\Gamma_8] \\ &\overset{\text{Claims 12 and II.4(g)}}{=} [\Gamma_3, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}]. \end{aligned}$$

□

Proof of Claim 16.

By Claim 1: $[\Gamma_1, \Gamma_{\underline{7}}] = 1$. Thus, $[\Gamma_{1'}\Gamma_1, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1$.
By Claim 15: $[\Gamma_3, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1$.
Thus, $[\Gamma_1^{-1}\Gamma_{1'}^{-1}\Gamma_3\Gamma_{1'}\Gamma_1, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1$.
But by Claim 13: $\Gamma_{3'} = \Gamma_1^{-1}\Gamma_{1'}^{-1}\Gamma_3\Gamma_{1'}\Gamma_1$.
Thus, $[\Gamma_{3'}, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1$. □

Proof of Claim 17. Claim 16, Claim 15 and Claim II.4(b). □

Proof of Claim 18. We apply $(Z_{33'}Z_{77'})^m$ on $[\Gamma_{3'}, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1$ and on $[\Gamma_3, \Gamma_{7'}\Gamma_7\Gamma_{7'}^{-1}] = 1$ to get $[\Gamma_3, \Gamma_{7'}] = [\Gamma_3\Gamma_{3'}\Gamma_3^{-1}, \Gamma_{7'}] = 1$. By Claim II.4(b) $\Gamma_{\underline{3}}, \Gamma_{7'}] = 1$. We use Claim 17 and Claim II.4(b) to get $[\Gamma_{\underline{3}}, \Gamma_{\underline{7}}] = 1$. □

Proof of Claim 19. $[\Gamma_{2'}, \Gamma_8] = [\Gamma_{2'}\Gamma_7] = [\Gamma_{2'}\Gamma_{7'}]$ (Claim 10), thus $[\Gamma_{2'}, \Gamma_{7'}^\bullet] = [\Gamma_{2'}, \Gamma_7^\bullet] = 1 \Rightarrow [\Gamma_{2'}, \Gamma_{\underline{7}}^\bullet] = 1$. Now, $[\Gamma_{\underline{3}}, \Gamma_{\underline{7}}] = [\Gamma_{\underline{3}}, \Gamma_8] = 1$ (Claim 11, Claim 18). Thus, $[\Gamma_{\underline{3}}, \Gamma_{\underline{7}}^\bullet] = 1$. From $[\Gamma_{2'}, \Gamma_{\underline{7}}^\bullet] = [\Gamma_{\underline{3}}, \Gamma_7^\bullet] = 1$, we get $[\Gamma_3^\bullet, \Gamma_{\underline{7}}^\bullet] = [\Gamma_{3'}^\bullet, \Gamma_{\underline{7}}^\bullet] = 1$. Thus, $[\Gamma_{\underline{3}}^\bullet, \Gamma_{\underline{7}}^\bullet] = 1$. □

Proof of Claim 20. In $\hat{F}$ we have $\left(Z^{(3)}_{33',44'}\right)^\bullet$, which by the Van Kampen Theorem implies $\langle\Gamma_3^\bullet, \Gamma_4^\bullet\rangle = 1$. We apply on it $(Z_{33'}Z_{77'})^{m_3}(Z_{44'}Z_{55'})^{m_4}$ for all possible m_4 and m_3 to get $\langle\Gamma_{\underline{3}}^\bullet, \Gamma_{\underline{4}}^\bullet\rangle = 1 \quad (\{(\Gamma_3^\bullet)\rho^m\} = \Gamma_{\underline{3}}^\bullet)$. □

Proof of Claim 21. Same proof as Claim 20. □

Proof of Claim 22. In $\hat{F}$ we have $\alpha_2^\bullet$ where α_2 is described in Fig.I.0(d).
By the Van Kampen Theorem we get

$$\Gamma_7^\bullet \Gamma_{5'}^\bullet \Gamma_7^{\bullet^{-1}} = \Gamma_{4'}^\bullet \Gamma_4^\bullet \Gamma_{3'}^\bullet \Gamma_4^\bullet \Gamma_{3'}^{\bullet^{-1}} \Gamma_4^{\bullet^{-1}} \Gamma_{4'}^{\bullet^{-1}}.$$

Since, $\langle\Gamma_{\underline{4}}^\bullet, \Gamma_{\underline{3}}^\bullet\rangle = 1$ we can apply Claim II.4(a) to get

$$= \Gamma_{4'}^\bullet \Gamma_4^\bullet \Gamma_4^{\bullet^{-1}} \Gamma_{3'}^\bullet \Gamma_4^\bullet \Gamma_4^{\bullet^{-1}} \Gamma_{4'}^{\bullet^{-1}}$$
$$= \Gamma_{4'}^\bullet \Gamma_{3'}^\bullet \Gamma_{4'}^{\bullet^{-1}}.$$

We apply Claim II.4(a) again to get

$$= \Gamma_{3'}^{\bullet^{-1}} \Gamma_{4'}^\bullet \Gamma_{3'}^\bullet.$$

□

Proof of Claim 23. Directly from $\left(\underset{\underline{Z}^\bullet_{3'7}}{(44')}\right)^2$ in $\hat{F}$. □

Proof of Claim 24.
By Claim 20 $\langle\Gamma_{3'}^\bullet \Gamma_3^\bullet \Gamma_{3'}^{\bullet^{-1}}, \Gamma_4^\bullet\rangle = 1$.
Thus $\langle\Gamma_3^\bullet, \Gamma_{3'}^{\bullet^{-1}} \Gamma_4^\bullet \Gamma_{3'}^\bullet\rangle = 1$.
Since $[\Gamma_3^\bullet, \Gamma_7^\bullet] = 1$, $\langle\Gamma_3^\bullet, \Gamma_7^{\bullet^{-1}} \Gamma_{3'}^{-1} \Gamma_4^\bullet \Gamma_{3'}^\bullet \Gamma_7^\bullet\rangle = 1$.

By Claim 22 $\langle \Gamma^{\bullet}_{3'}, \Gamma^{\bullet}_{5} \rangle = 1$. □

Proof of Claim 25. Follows from Claim 21, as in Claim 24. □

Proof of Claim 26. Follows from Claim 24, using $[\Gamma_{2'}, \Gamma_{\underline{5}}] = 1$. □

Proof of Claim 27. From Claim 25, using $[\Gamma_8, \Gamma_{\underline{4}}] = 1$. □

Proof of Claim 28.

By Claim 23 $[\Gamma^{\bullet}_{7}, \Gamma^{\bullet}_{4'}\Gamma^{\bullet}_{4}\Gamma^{\bullet}_{3'}\Gamma^{\bullet}_{4}{}^{-1}\Gamma^{\bullet}_{4'}{}^{-1}] = 1$.

Thus $[\Gamma^{\bullet}_{4'}{}^{-1}\Gamma^{\bullet}_{7}\Gamma^{\bullet}_{4'}, \Gamma^{\bullet}_{4}\Gamma^{\bullet}_{3'}\Gamma^{\bullet}_{4}{}^{-1}] = 1$.

From Claim 27, Claim 20 and Claim II.4(a) $[\Gamma^{\bullet}_{7}\Gamma^{\bullet}_{4'}\Gamma^{\bullet}_{7}{}^{-1}, \Gamma^{\bullet}_{3'}{}^{-1}\Gamma^{\bullet}_{4}\Gamma^{\bullet}_{3'}] = 1$.

By Claim 22 $[\Gamma^{\bullet}_{7}\Gamma^{\bullet}_{4'}\Gamma^{\bullet}_{7}{}^{-1}, \Gamma^{\bullet}_{7}\Gamma^{\bullet}_{5}\Gamma^{\bullet}_{7}{}^{-1}] = 1$.

Thus, $[\Gamma^{\bullet}_{4'}, \Gamma^{\bullet}_{5}] = 1$.

Since $\Gamma^{\bullet}_{4'} = \Gamma_{4'}, \Gamma^{\bullet}_{5} = \Gamma_5$ we get the Claim. □

Proof of Claim 29.

By Claim 13 $\Gamma_{6'} = \Gamma_4^{-1}\Gamma_{4'}^{-1}\Gamma_6\Gamma_{4'}\Gamma_4$.

Thus, $\Gamma_4\Gamma_{6'}\Gamma_4^{-1} = \Gamma_{4'}^{-1}\Gamma_6\Gamma_{4'}$.

By Claim 1 and Claim II, 4(a)

$$\Gamma_{6'}^{-1}\Gamma_4\Gamma_{6'} = \Gamma_6\Gamma_{4'}\Gamma_6^{-1}.$$

Thus,

$$\Gamma_4 = \Gamma_{6'}\Gamma_6\Gamma_{4'}\Gamma_6^{-1}\Gamma_{6'}^{-1}.$$

We substitute the last equation in Claim 29 to get

$$[\Gamma_{6'}\Gamma_6\Gamma_{4'}\Gamma_6^{-1}\Gamma_{6'}^{-1}, \Gamma_5] = 1.$$

By Claim 9 $[\Gamma_5, \Gamma_{\underline{6}}] = 1$. Thus,

$$[\Gamma_{4'}, \Gamma_5] = 1.$$

□

Proof of Claim 30. By Claim 28, Claim 29 and Claim II.5(a). □

Proof of Claim 31. We apply $Z_{44'}Z_{55'}$ on $[\Gamma_{4'}, \Gamma_5] = 1$ and on $[\Gamma_4, \Gamma_5] = 1$ to get $[\Gamma_{4'}\Gamma_4\Gamma_{4'}^{-1}, \Gamma_{5'}] = [\Gamma_{4'}, \Gamma_{5'}] = 1$. We then use Lemma II.5(b) to get $[\Gamma_{\underline{4}}, \Gamma_{5'}] = 1$. Together with Claim 30, we get the Claim. □

Proof of Claim 32. By Claim 7 and Claim 10. □

Proof of Claim 33.

$$[\Gamma_3, \Gamma_7^\bullet] = [\Gamma_3, \Gamma_8\Gamma_{7'}\Gamma_8^{-1}] \overset{\text{By Claims } \underline{12} \text{ and II.4(g)}}{=} [\Gamma_3, \Gamma_{7'}]_{\Gamma^\bullet} \overset{\text{By Claim 18}}{=} 1.$$

□

Proof of Claim 34.

$[\Gamma_7^\bullet, \Gamma_5^2\Gamma_3\Gamma_5^{-2}] \overset{\text{Claims } \underline{10} \text{ and 32}}{=} [\Gamma_7^\bullet, \Gamma_5^2\Gamma_{2'}^{-1}\Gamma_3\Gamma_{2'}\Gamma_5^{-2}] \overset{\text{Claim 3}}{=} [\Gamma_7^\bullet, \Gamma_5^2\Gamma_{3'}^\bullet\Gamma_5^{-2}]$.
Thus, $[\Gamma_7^\bullet, \Gamma_5^2\Gamma_{3'}^\bullet\Gamma_5^{-2}] \Leftrightarrow [\Gamma_{5^{-1}}\Gamma_7^\bullet\Gamma_5, \Gamma_5\Gamma_{3'}^\bullet\Gamma_5^{-1}] = 1$.
Now:

$$\begin{aligned}
&[\Gamma_5^{-1}\Gamma_7^\bullet\Gamma_5, \Gamma_5\Gamma_{3'}^\bullet\Gamma_5^{-1}] \\
&\quad \overset{\text{Claim 9}}{=} [\Gamma_5^{\bullet\,-1}\Gamma_7^\bullet\Gamma_5^{\bullet\,-1}, \Gamma_5^\bullet\Gamma_{3'}^\bullet\Gamma_5^{\bullet\,-1}] \\
&\quad \overset{\text{Claims 21, } \underline{24}\text{, and II.4(a)}}{=} [\Gamma_7^\bullet\Gamma_5^\bullet\Gamma_7^{\bullet\,-1}, \Gamma_{3'}^{\bullet\,-1}\Gamma_5^\bullet\Gamma_{3'}^\bullet] = \\
&\quad \overset{\text{Claim 22}}{=} [\Gamma_{3'}^{\bullet\,-1}\Gamma_4^\bullet\Gamma_{3'}^\bullet, \Gamma_{3'}^{\bullet\,-1}\Gamma_5^\bullet\Gamma_{3'}^\bullet] = 1 \\
&\quad \overset{\text{Claim II.4(f)}}{=} [\Gamma_4^\bullet, \Gamma_5^\bullet]_{\Gamma_{3'}^\bullet} = 1 \\
&\quad \overset{\text{Claim 9}}{=} [\Gamma_4, \Gamma_5]_{\Gamma_{3'}^\bullet} \\
&\quad \overset{\text{Claim 29}}{=} 1.
\end{aligned}$$

□

Proof of Claim 35. Let $f : B_5 \to G$ be as follows:
$B_5 = \langle X_1, \ldots, X_4 \mid [X_i, X_j] = 1 \ |i - j| > 2, \ \langle X_i, X_{i+1} \rangle = 1, \ i = 1, 2, 3 \rangle$.
$f(X_1) = \Gamma_{2'}, f(X_2) = \Gamma_3, f(X_3) = \Gamma_5, f(X_4) = \Gamma_7^\bullet$.

By Claims 10, 32, 33, 2, 26, 21, f is well-defined.
Let d be the braid that satisfies $(X_i)_d = X_{5-i},\ i = 1,2,3$.
Then

$$\begin{aligned} 1 &= [\Gamma_7^\bullet, \Gamma_5^2\Gamma_3\Gamma_5^{-2}] = f[X_4, X_3^2X_2X_3^{-2}] \\ &= f[(X_1)_d, (X_2)_d^2(X_3)_d(X_2)_d^{-2}] \\ &= f([X_1, X_2^2X_3X_2^{-2}]_d) \\ &= [f(X_1), f(X_2)^2 f(X_3) f(X_2)^{-2}]_{f(d)}. \end{aligned}$$

Thus, $[f(X_1), f(X_2)^2 f(X_3) f(X_2^{-2}] = 1$.
Thus, $[\Gamma_{2'}, \Gamma_3^2\Gamma_5\Gamma_3^{-2}] = 1$. □

Proof of Claim 36.
$[\Gamma_3, \Gamma_4] \overset{\text{Claim 9}}{=} [\Gamma_3, \Gamma_4^\bullet] \overset{\text{Claim 22}}{=} [\Gamma_3, \Gamma_{3'}^\bullet\Gamma_7^\bullet\Gamma_5^\bullet\Gamma_7^{\bullet^{-1}}\Gamma_{3'}^{\bullet^{-1}}]$.
Thus,

$$\begin{aligned} &[\Gamma_3, \Gamma_4] = 1 \\ &\Leftrightarrow [\Gamma_3, \Gamma_{3'}^\bullet\Gamma_7^\bullet\Gamma_5^\bullet\Gamma_7^{\bullet^{-1}}\Gamma_{3'}^{\bullet^{-1}}] = 1 \\ &\overset{\text{Claims 18 and 19}}{\Leftrightarrow} [\Gamma_3, \Gamma_{3'}^\bullet\Gamma_5^\bullet\Gamma_{3'}^{\bullet^{-1}}] = 1 \\ &\Leftrightarrow [\Gamma_3, \Gamma_{2'}^{-1}\Gamma_3\Gamma_{2'}\Gamma_5\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{2'}] = 1 \\ &\Leftrightarrow [\Gamma_{2'}\Gamma_3\Gamma_{2'}^{-1}, \Gamma_3\Gamma_{2'}\Gamma_5\Gamma_{2'}^{-1}\Gamma_3^{-1}] = 1 \\ &\overset{\text{Claim 10}}{\Leftrightarrow} [\Gamma_{2'}\Gamma_3\Gamma_{2'}^{-1}, \Gamma_3\Gamma_5\Gamma_3^{-1}] = 1 \\ &\overset{\text{Claims 2 and II.4(a)}}{\Leftrightarrow} [\Gamma_3^{-1}\Gamma_{2'}\Gamma_3, \Gamma_3\Gamma_5\Gamma_3^{-1}] = 1 \\ &\Leftrightarrow [\Gamma_{2'}, \Gamma_3^2\Gamma_5\Gamma_3^{-2}] = 1 \end{aligned}$$

which is true by Claim 35.
We get $[\Gamma_{\underline{3}}, \Gamma_{\underline{4}}]$ by Corollary I.5. □

Proof of Claim 37. Similar to Claim 34. □

Proof of Claim 38. Claim 18, Claim 10, Claim 12, Claim 3. □

Proof of Claim 39. Let $f : B_5 \to G$ be as follows:

$$f(X_1) = \Gamma_8, f(X_2) = \Gamma_{7'}, f(X_3) = \Gamma_4, f(X_4) = \Gamma^{\bullet}_{3'}.$$

By Claim 12, Claim 38, Claim 5, Claim 27, Claim 20, f is well defined. Let d be a braid such that $(X_i)_d = X_{4-i}$, $i = 1, 2, 3, 4$.

Now:

$$\begin{aligned} 1 = [\Gamma_{7'}, \Gamma_4^2 \Gamma^{\bullet}_{3'} \Gamma_4^{-2}] &= [f(X_2), f(X_3)^2 f(X_4) f(X_3)^{-2}] \\ &= f[X_2, X_3^2 X_4 X_3^{-2}] \\ &= f[(X_3)_d, (X_2)_d^2 (X_1)_d (X_2)_d^{-2}] \\ &= [f(X_3), f(X_2)^2 f(X_1) f(X_2)^{-2}]_{f(d)} \\ &= [\Gamma_4, \Gamma_{7'}^2 \Gamma_8 \Gamma_{7'}^{-2}]_{f(d)}. \end{aligned}$$

Thus, $[\Gamma_4, \Gamma_{7'}^2 \Gamma_8 \Gamma_{7'}^{-2}] = 1$. □

Proof of Claim 40. Follows from Claim 30, like in Claim 36. □

Proof of Claim 41. Apply $(Z_{44'} Z_{55'})^{m_4}$ $(Z_{77'} Z_{33'})^{m_3}$ on $[\Gamma_5, \Gamma_{7'}] = 1$ from Claim 40. □

Proof of Claim 42. In H_4' there is $\overline{Z}^{(3)}_{2',44'}$. We take its complex conjugate $\underline{Z}^{(3)}_{2',44'}$ and apply Corollary I.4 on $\underline{Z}^{(3)}_{2',4}$ to get $\langle \Gamma_{2'}, \Gamma_4 \rangle = 1$ and then apply Corollary I.5 to get $\langle \Gamma_{\underline{2}}, \Gamma_{\underline{4}} \rangle = 1$. □

Proof of Claim 43. We apply Corollary I.5 on Claims 2 and 5. □

Proof of Claim 44. Similar to Claim 42. □

Proof of Claim 45. By Claim 22, $\Gamma^{\bullet}_7 \Gamma^{\bullet}_5 \Gamma^{\bullet-1}_7 = \Gamma^{\bullet-1}_{3'} \Gamma^{\bullet}_4 \Gamma^{\bullet}_{3'}$. Since $\langle \Gamma^{\bullet}_5, \Gamma^{\bullet}_7 \rangle = 1$ we have

$$\Gamma^{\bullet-1}_5 \Gamma^{\bullet}_7 \Gamma^{\bullet}_5 = \Gamma^{\bullet-1}_{3'} \Gamma^{\bullet}_4 \Gamma^{\bullet}_{3'}.$$

Thus, $\Gamma^{\bullet}_7 = \Gamma^{\bullet}_5 \Gamma^{\bullet-1}_{3'} \Gamma^{\bullet}_4 \Gamma^{\bullet}_{3'} \Gamma^{\bullet-1}_5$.

We substitute the formulas for $\Gamma^{\bullet}_{3'}$ and $\Gamma^{\bullet}_{7}$ to get

$$\Gamma_{7'}^{-1}\Gamma_8\Gamma_{7'} = \Gamma_5\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{2'}\Gamma_4\Gamma_{2'}^{-1}\Gamma_3\Gamma_{2'}\Gamma_5^{-1}.$$

Since $\langle\Gamma_{2'},\Gamma_4\rangle = 1$ (Claim 42)

$$\Gamma_{7'}^{-1}\Gamma_8\Gamma_{7'} = \Gamma_5\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_{2'}\Gamma_4\Gamma_3\Gamma_{2'}^{-1}\Gamma_5^{-1}.$$

Since $[\Gamma_5,\Gamma_{3'}] = 1$

$$\Gamma_{7'}^{-1}\Gamma_8\Gamma_{7'} = \Gamma_{2'}^{-1}\Gamma_5\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_{2'}\Gamma_4\Gamma_3\Gamma_5^{-1}\Gamma_{2'}.$$

Since $[\Gamma_{2'},\Gamma_8] = [\Gamma_{2'},\Gamma_{7'}] = 1$

$$\Gamma_{7'}^{-1}\Gamma_8\Gamma_{7'} = \Gamma_5\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_{2'}\Gamma_4\Gamma_3\Gamma_5^{-1}$$

and

$$\Gamma_8 = \Gamma_{7'}\Gamma_5\Gamma_3^{-1}\Gamma_4^{-1}\Gamma_{2'}\Gamma_4\Gamma_3\Gamma_5^{-1}\Gamma_{7'}^{-1}.$$

□

Proof of Claim 46. By $\tilde{Z}_{22'}$ of H'_4 we have:

$$\Gamma_2 = \Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_4^{-1}\Gamma_{4'}^{-1}.$$

We apply on it $Z_{88'}Z_{22'}$ using Corollary I.5 to get (recall that (Γ_2) $Z'_{22'}Z_{88'} = \Gamma_{2'}$ and $(\Gamma'_2)Z_{22'}Z_{88'} = (\Gamma'_2)Z_{22'} = \Gamma_{2'}\Gamma_2\Gamma_{2'}^{-1}$)

$$\Gamma_{2'} = \Gamma_{4'}\Gamma_4\Gamma_{3'}\Gamma_3\Gamma_{2'}\Gamma_2\Gamma_{2'}^{-1}\Gamma_3^{-1}\Gamma_{3'}^{-1}\Gamma_4^{-1}\Gamma_{4'}^{-1}.$$

□

Proof of Claim 47. By $\tilde{Z}_{88'}$ of H'_4 we get, using the Van Kampen Theorem:

$$\Gamma_{8'} = \Gamma_5^{-1}\Gamma_{5'}^{-1}\Gamma_7^{-1}\Gamma_{7'}^{-1}\Gamma_8\Gamma_{7'}\Gamma_7\Gamma_{5'}\Gamma_5.$$

□

Proof of Claim 48.

In C'_5 we have $\tilde{Z}^2_{i9} = H(\tilde{z}_{i9})^2$ for $i = 1,2,3,4,6,7,$ where:

$\tilde{z}_{i9} = \underset{(8,8')}{\overline{z}}_{i9}$

Case 1. $i \neq 7$

The complex conjugation of $\tilde{Z}_{i9}$ is $\overline{\tilde{Z}}_{i9} = H(\overline{\tilde{z}}_{i9})$ and $\overline{\tilde{z}}_{i9} = \overset{(8,8')}{\underline{z}_{i9}}$ which implies on G (as in Claim 0) $[\Gamma_i, \Gamma_8^{-1}\Gamma_{8'}^{-1}\Gamma_9\Gamma_{8'}\Gamma_8] = 1$.

By Claim 10, for $i = 2, 3, 4$ we have $[\Gamma_i, \Gamma_{\underline{8}}] = 1$.

By Claim 0, for $i = 1, 6$ we have $[\Gamma_i, \Gamma_{\underline{8}}] = 1$.

Thus, $[\Gamma_i, \Gamma_9] = 1$. We use Corollary I.5 to get $[\Gamma_{\underline{i}}, \Gamma_{\underline{9}}] = 1$.

Case 2. $i = 7$

In C_5' we have $\underline{Z}^2_{7,9}$ which implies $[\Gamma_7, \Gamma_9] = 1$ and thus, by Corollary I.7, $[\Gamma_{\underline{7}}, \Gamma_{\underline{9}}] = 1$.

From Case 1 and Case 2 we get Claim 48. □

Thus we have proved all the Claims.

We shall now prove the statements of the Proposition. We use the above claims, the definition of $E_i, E_{i'}$ and the facts $\{\Gamma_{\underline{i}}\} = \{E_{\underline{i}}\}$ and $E_{i'}E_i = \Gamma_{i'}\Gamma_i \ \forall i$.

(1) From Claim 1, Claim 26, Claim 27, Claim 42, Claim 43, Claim 44.

(2) From Claim 0, Claim 10, Claim 12, Claim 18, Claim 31, Claim 36, Claim 41, Claim 48.

(3) From Claim 45, definifions of E_i and Corollary I.5.

(4) From Claim 46, definitions of E_i and Corollary II.1.

(5) From Claim 47, definition of E_i and the above fact.

(6) From Claim 13.

(7) From Claim 13.

□ Proposition II.6

We need the following corollary in order to obtain in Chapter IV, §8 a smaller set of generators for G.

Corollary II.7. *Let $E_i, E_{i'}$ be as in the beginning of the chapter. Let $A_i = E_{i'}E_i^{-1}$. Then:*

(1) $A_5 = (A_4)_{E_2^{-1}E_3E_7^{-1}E_8}$

(2) $A_7 = (A_3)_{E_2^{-1}E_4E_5^{-1}E_7^{-1}}$

(3) $A_8 = (A_2)_{E_4E_3E_5^{-1}E_7^{-1}}$

(4) $A_2 = E_1^{-2}A_1^{-1}(A)_{E_2^{-1}}(E_1^2)_{E_2^{-1}}$

(5) $A_6 = E_4^{-2}A_4^{-1}(A_4)_{E_6^{-1}}(E_4^2)_{E_6^{-1}}$

(6) $A_3 = E_1^{-2}A_1^{-1}(A)_{E_3^{-1}}(E_1^2)_{E_3^{-1}}$

(7) $A_9 = E_5^{-2}A_5^{-1}(A_5)_{E_9^{-1}}(E_5^2)_{E_9^{-1}}$

(8) $(A_4)_{E_2^{-1}E_4^{-1}} = E_4^2A_3E_3^2A_2(E_3^{-2})_{E_2^{-1}}(A_3^{-1})_{E_2^{-1}}(E_4^{-2})_{E_2^{-1}}$

(9) $(A_7)_{E_6^{-1}E_7^{-1}} = E_7^2A_6(E_7^{-2})_{E_6^{-1}}$

(10) $(A_9)_{E_8^{-1}E_9^{-1}} = E_9^2A_8(E_9^{-2})_{E_8^{-1}}$

Proof. We use proposition II.6 (1), ..., (7). The claims are grouped according to the similarity of their proofs and not according to the order that we use them in Proposition IV.8.1.

Recall:

$$\begin{aligned}(E_i)\rho_i &= E_{i-1}\\ (E_{i'})\rho_i &= E_{i'}E_iE_{i'}\\ E_{i'} &= A_iE_i\\ E_{i'}E_i &= A_iE_i^2.\end{aligned}$$

For (1), (2), (3) we use (5) of Proposition II.6.

For Claims (4)–(7) we use (6) of Proposition II.6. The 4 claims are symmetric and we shall only prove the first one.

For (8) we use (4) of Proposition II.6.

For (9)–(10) we use (7) of Proposition II.6. The claims are symmetric and we shall only prove the first one.

(1) By (5) of Proposition II.6

$$E_8 = (E_4^{-1}E_2E_4)_{E_3E_5^{-1}E_7^{-1}}.$$

By (1), (2) and Claim II.4(a)

$$E_8 = (E_2E_4E_2^{-1})_{E_3E_7^{-1}E_5^{-1}}.$$

Thus,

$$E_5^{-1} E_8 E_5 = (E_4)_{E_2^{-1} E_3 E_7^{-1}}.$$

By (1) and Claim II.4(a)

$$E_8 E_5 E_8^{-1} = (E_4)_{E_2^{-1} E_3 E_7^{-1}}.$$

Thus,

$$E_5 = (E_4)_{E_2^{-1} E_3 E_7^{-1} E_8}.$$

We apply on it $\rho_4 \rho_5$ to get

$$E_{5'} = (E_{4'})_{E_2^{-1} E_3 E_7^{-1} E_8}.$$

We multiply the 2 results to get

$$A_5 = (A_4)_{E_2^{-1} E_3 E_7^{-1} E_8}.$$

(2) By (5) of Proposition II.6

$$E_7^{-1} E_8 E_7 = E_5 E_4^{-1} E_3^{-1} E_2 E_3 E_4 E_5^{-1}.$$

By (1) and Claim II.4(a)

$$E_8 E_7 E_8^{-1} = E_4 E_4^{-1} E_2 E_3 E_2^{-1} E_4 E_5^{-1}.$$

Thus,

$$E_7 = (E_9)_{E_2^{-1} E_4 E_5^{-1} E_8}.$$

We apply on this $\rho_2 \rho_3$ to get

$$E_{7'} = (E_{3'})_{E_2^{-1} E-4E-5^{-1} E_8}.$$

We multiply the results to get

$$A_7 = (A_3)_{E_2^{-1} E_4 E_5^{-1} E_8}.$$

(3) By (5) of Proposition II.6

$$E_8 = (E_2)_{E_4 E_3 E_5^{-1} E_7^{-1}}.$$

We apply on it $\rho_2 \rho_8$ to get

$$E_{8'} = (E_{2'})_{E_4 E_3 E_5^{-1} E_7^{-1}}.$$

We multiply the 2 results to get

$$A_8 = (A_2)_{E_4 E_3 E_5^{-1} E_7}.$$

(4) From (6) of Proposition II.6

$$\begin{aligned} E_{2'} &= E_1^{-1}E_{1'}^{-1}E - 2E_{1'}E_1 \\ \Rightarrow A_2E_2 &= E_1^{-2}A_1^{-1}E_2A_1E_1^2 \\ \Rightarrow A_2 &= E_1^{-2}A_1^{-1}(A_1)_{E_2^{-1}}(E_1^2)_{E_2^{-1}}. \end{aligned}$$

(8) From (4) of Proposition II.6

$$\begin{aligned} & E_2 = E_{4'}E_4E_{3'}E_3E_{2'}E_3^{-1}E_{3'}^{-1}E_4^{-1}E_{4'}^{-1} \\ & \Rightarrow E_2 = A_4E_4^2A_3E_3^2A_2E_2E_3^{-2}A_3^{-1}E_4^{-2}A_4^{-1} \\ & \Rightarrow A_4^{-1}E_2A_4 = E_4^2A_3E_3^2A_2E_2E_3^{-2}A_3^{-1}E_4^{-2} \\ & \Rightarrow A_4^{-1}(A_4)_{E_2^{-1}} = E_4^2A_3E_3^2A_2(E_3^{-2}A_3^{-1}E_4^{-2})_{E_2^{-1}} \\ & \Rightarrow A_4^{-1}(A_4)_{E_2^{-1}} = E_4^2A_3E_3^2A_2(E_3^{-2})_{E_2^{-1}}(A_3^{-1})_{E_2^{-1}}(E_4^{-2})_{E_2^{-1}}. \end{aligned}$$

By Lemma II.4(h)

$(A_4)_{E_2^{-1}E_4^{-1}} = A_4^{-1}(A_4)_{E_2^{-1}}$. Thus,

$(A_4)_{E_2^{-1}E_4^{-1}} = E_4^2A_3E_3^2A_2(E_3^{-2})_{E_2^{-1}}(A_3^{-1})_{E_2^{-1}}(E_4^{-2})_{E_2^{-1}}$.

(9) From (7)

$$\begin{aligned} E_6 &= E_{7'}E_7E_{6'}E_7^{-1}E_{7'}^{-1} \\ \Rightarrow E_6 &= A_7E_7^2A_6E_6E_7^{-2}A_7^{-1} \\ \Rightarrow A_7^{-1}E - 6A_7 &= E_7^2A_6E_6E_7^{-2} \\ \Rightarrow A_7^{-1}(A_7)_{E_6^{-1}} &= E_7^2A_6(E_7^{-2})_{E_6^{-1}}. \end{aligned}$$

By Lemma II.4(h), $A_7^{-1}(A_7)_{E_6^{-1}} = (A_7)_{E_6^{-1}E_7^{-1}}$. Thus,

$(A_7)_{E_6^{-1}E_7^{-1}} = E_7^2A_6(E_7^{-2})_{E_6^{-1}}$.

□

III. Construction theory for $\tilde{B}_n$.

Let D be a disc, $K \subset D$, $\#K = n$. Let $B_n = B_n[D, K]$ (see definition in Chapter 0).

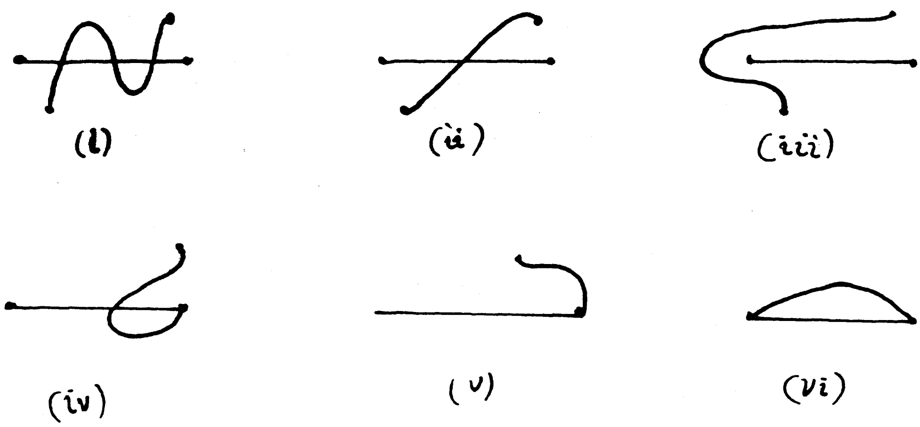


Figure III.1.0.

III.1. Definition of $\tilde{\mathrm{B}}_{\mathrm{n}}$.

Definition. Let D, K be as before. Let $H(\sigma_1)$ and $H(\sigma_2)$ be 2 half-twists in $B = B_n[D, K]$. We say that $H(\sigma_1)$ and $H(\sigma_2)$ are:

(i) weakly disjoint if $\sigma_1 \cap \sigma_2 \cap K = \emptyset$.

(ii) transversal if σ_1 and σ_2 are weakly disjoint and intersect each other exactly once (and not in any point of K), i.e., $\sigma_1 \cap \sigma_2 =$ one point, $\sigma_1 \cap \sigma_2 \cap K = \emptyset$.

(iii) disjoint if $\sigma_1 \cap \sigma_2 = \emptyset$.

(iv) adjacent if $\sigma_1 \cap \sigma_2 \cap K =$ one point.

(v) consecutive if they are adjacent and $\sigma_1 \cap \sigma_2$ do not intersect outside of K, i.e., $\sigma_1 \cap \sigma_2 =$ point $\in K$.

(vi) cyclic if $\sigma_1 \cap \sigma_2 = 2$ points $\in K$.

Claim III.1.0. *Let X, Y be 2 half-twists in B_n. Then:*

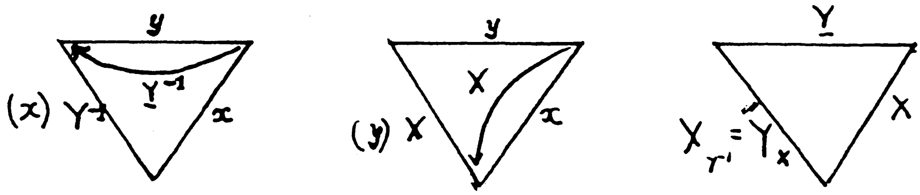


Figure III.1.1.

(i) *If X, Y are disjoint, then $[X,Y] = 1$, i.e., $XY = YX$.*

(ii) *If X, Y are consecutive, then $\langle X,Y\rangle = XYXY^{-1}X^{-1}Y^{-1} = 1$ and $XYX = YXY$ or $X_{Y^{-1}} = Y_X$. We say then that X and Y satisfy the triple relation.*

(iii) *If $X = H(x) = \overline{\beta}$, $Y = H(y)$, then $Y_X = X^{-1}YX = H((y)\beta)$.*

Proof. (i) (ii) Basic properties of a braid group. See, for example, Chapter 3 of [MoTe4].

(iii) See Fig. III.1.1 for a geometrical presentation of Y_X. The common point of x and y travels under β counterclockwise to the other end of x. Thus $(y)\beta$ connects the 2 ends of x and y, which is not a common end of either of them. By IV.3.4 of [MoTe12], $Y_X = X^{-1}YX = (Y)X^{\vee}$ (see definition or $f^{\vee}$ in [MoTe4], Chapter 4) which is equal by Claim IV.3.0 of [MoTe12] to $H((y)\beta_1)$. □

Definition. $\tilde{B}_n$

Let $\tilde{B}_n$ be the quotient of b_n, the Braid group of order n, by the subgroup generated by the commutators $[H(\sigma_1), H(\sigma_2)]$ where $H(\sigma_1)$ and $H(\sigma_2)$ are transversal half-twists.

Notations.

Let $Y \in B_n$. We denote the image of Y in $\tilde{B}_n$ by $\tilde{Y}$. When possible, we shall abuse notation and denote $\tilde{Y}$ by Y. If Y is a half-twist in B_n we call

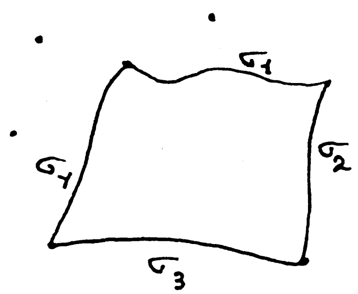


Figure III.1.2.

$\tilde{Y}$ a half-twist in $\tilde{B}_n$. We call two half-twists $\tilde{Y}, \tilde{X}$ in $\tilde{B}_n$ disjoint (or weakly disjoint, adjacent, consecutive, transversal) if Y, X are disjoint (or weakly disjoint, adjacent, consecutive, transversal). If $\{X_i\}$ is a frame of B_n, then $\{\tilde{X}_i\}$ is a frame in $\tilde{B}_n$. We also refer to $\{\tilde{X}_i\}$ as a standard base of $\tilde{B}_n$.

Definition. *Polarized half-twist, polarization*

We say that a half-twist $X \in B_n$ (or $\tilde{X}$ in $\tilde{B}_n$) is polarized if we choose an order on the end points of X. The order is called the polarization of X or $\tilde{X}$.

Definition. *Orderly adjacent*

Let X, Y be two adjacent polarized half-twists in B_n (resp. in $\tilde{B}_n$). We say that X, Y are *orderly adjacent* if their common point is the "end" of one of them and the "origin" of another.

Definition. *Good quadrangle*

Let $H(\sigma_i)$ $i = 1,2,3,4$ be 4 half-twists such that $H(\sigma_i)$ and $H(\sigma_{i+1})$ are consecutive, $H(\sigma_4)$ and $H(\sigma_1)$ are consecutive, $H(\sigma_1)$ and $H(\sigma_3)$ are disjoint and $H(\sigma_2)$ and $H(\sigma_4)$ are disjoint, and in the interior of $\bigcup\limits_{i=1}^{4} \sigma_i$ there is no point of K. We say that $\{H(\sigma_i)\}$ is a good quadrangle in B_n and $\{\widetilde{H(\sigma_i)}\}$ is a good quadrangle in $\tilde{B}_n$.

Remark III.1.1. (a) transversal, disjoint $\Rightarrow$ weakly disjoint. consecutive $\Rightarrow$ adjacent.

(b) Any two pairs of disjoint (or transversal, consecutive, cyclic) half-twists are conjugate to each other by an element $b \in B$.

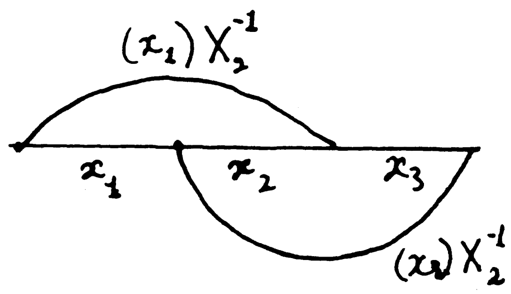


Figure III.1.3.

(c) Any two half-twists in B_n (or $\tilde{B}_n$) are conjugate to each other by an element of B_n (or $\tilde{B}_n$).

(d) Every 2 transversal or disjoint half-twists in $\tilde{B}_n$ commute. Every 2 consecutive half-twists in B_n or in $\tilde{B}_n$ satisfy the triple relation $(XYX = YXY)$.

(e) Any two good quadrangles in B_n or in $\tilde{B}_n$ are conjugate.

(f) Every 2 pairs of orderly adjacent (non-orderly adjacent) consecutive half-twists are conjugate to each other by an element $b \in B$ preserving polarization.

Proof. Geometric observation in B_n and $\tilde{B}_n$, and Claim III.1.0. □

Lemma III.1.2. *If* $\{\tilde{Y}_i\}$ $i = 1, \ldots, 4$ *is a good quadrangle in* $\tilde{B}_n$, *then* (a) $\tilde{Y}_1\tilde{Y}_3 = \tilde{Y}_3\tilde{Y}_1$, (b) $\tilde{Y}_1^2\,\tilde{Y}_3^3 = \tilde{Y}_2^2\,\tilde{Y}_4^2$.

Proof.

(a) Since they are disjoint.

(b) Let X_1, X_2, X_3 be 3 half-twists such that X_1 and X_2 are consecutive, X_2 and X_3 are consecutive and X_1 and X_3 are disjoint. Denote $X_i = H(x_i)$, $i = 1, \ldots, 4$. Clearly, $X_1, (X_3)_{X_2^{-1}}, X_3, (X_1)_{X_2^{-1}}$ is a good quadrangle (see Fig. III.1.3).

In B_n : $[X_1, X_3] = 1$ and $\langle X_1, X_2\rangle = \langle X_3, X_2\rangle = 1$. Thus, we can use Claim II.4(a).

It is clear that $(X_2)_{X_1X_3}$ is transversal to X_2. Thus, in $\tilde{B}_n$:

$$\left[\tilde{X}_2, (\tilde{X}_2)_{\tilde{X}_1\tilde{X}_3}\right] = 1.$$

Thus,

$$\begin{aligned}
1 &= [\tilde{X}_2, \tilde{X}_3^{-1}\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_1\tilde{X}_3] \\
&= \tilde{X}_2\tilde{X}_3^{-1}\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_1\underbrace{\tilde{X}_3\tilde{X}_2^{-1}\tilde{X}_3^{-1}}\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_1\tilde{X}_3 \\
\text{(By Claim II.4)} &= \tilde{X}_2\tilde{X}_3^{-1}\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_1\tilde{X}_2^{-1}\tilde{X}_3^{-1}\underbrace{\tilde{X}_2\tilde{X}_1^{-1}\tilde{X}_2^{-1}}\tilde{X}_1\tilde{X}_3 \\
\text{(By Claim II.4)} &= \qquad\qquad\qquad\qquad\qquad\qquad\overbrace{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_1}\tilde{X}_1\tilde{X}_3 \\
([X_1, X_3] = 1) &= \tilde{X}_2\tilde{X}_3^{-1}\tilde{X}_1^{-1}\tilde{X}_2\underbrace{\tilde{X}_1\tilde{X}_2^{-1}\tilde{X}_1^{-1}}\ \tilde{X}_3^{-1}\tilde{X}_2^{-1}\tilde{X}_1\tilde{X}_1\tilde{X}_3 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\overbrace{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_2} \\
&= \tilde{X}_2\tilde{X}_3^{-1}\tilde{X}_1^{-2}\ \underbrace{\tilde{X}_2\ \tilde{X}_3^{-1}\ \tilde{X}_2^{-1}}\tilde{X}_1^2\tilde{X}_3 \\
\text{(By Claim II.4)} &= \qquad\qquad\quad \overbrace{\tilde{X}_3^{-1}\tilde{X}_2^{-1}\tilde{X}_3} \\
\text{(Since } [X_1, X_3] = 1) &= \tilde{X}_2\tilde{X}_3^{-2}\tilde{X}_1^{-2}\tilde{X}_2^{-1}\tilde{X}_1^2\tilde{X}_3^2 \\
&= (\tilde{X}_3)^{-2}_{\tilde{X}_2^{-1}}\ (\tilde{X}_1)^{-2}_{\tilde{X}_2^{-1}}\tilde{X}_1^2\tilde{X}_3^2.
\end{aligned}$$

Thus, $\tilde{X}_1^2\tilde{X}_3^2 = (\tilde{X}_1)^2_{\tilde{X}_2^{-1}} \cdot (\tilde{X}_3)^2_{\tilde{X}_2^{-1}}$.

By Remark III.1.1, $\{\tilde{Y}_i\}$ is conjugate to $\{\tilde{X}_1, (\tilde{X}_3)_{\tilde{X}_2^{-1}}, \tilde{X}_3, (\tilde{X}_1)_{\tilde{X}_2^{-1}}\}$ and thus satisfies $\tilde{Y}_1^2\tilde{Y}_3^2 = \tilde{Y}_4^2\tilde{Y}_2^2$. □

III.2. $\tilde{\mathbf{B}}_{\mathbf{n}}$-groups and prime elements.

Definition. $\tilde{B}_n$ *– group.*

A group G is called a $\tilde{B}_n$-group if there exists a homomorphism $\tilde{B}_n \to \mathrm{Aut}(G)$. We denote $(g)_b$ by g_b.

Definition. *Prime element, supporting half-twist (s.h.t.) corresponding central element.*

Let G be a $\tilde{B}_n$-group.

An element $g \in G$ is called a prime element of G if there exists a half-twist $X \in B_n$ and $\tau \in \text{Center}(G)$ with $\tau^2 = 1$ and $\tau_b = \tau \ \forall\ b \in \tilde{B}_n$ such that:

(1) $g_{\tilde{X}^{-1}} = g^{-1}\tau$.

(2) For every half-twist Y adjacent to X we have:

(a) $g_{\tilde{X}\tilde{Y}^{-1}\tilde{X}^{-1}} = g_{\tilde{X}}^{-1} g_{\tilde{X}\tilde{Y}^{-1}}$

(b) $g_{\tilde{Y}^{-1}\tilde{X}^{-1}} = g^{-1} g_{\tilde{Y}^{-1}}$.

(3) For every half-twist Z disjoint from X, $g_{\tilde{Z}} = g$.

The half-twist X (or $\tilde{X}$) is called the *supporting half-twist* of g, (X is the s.h.t. of g.)

The element τ is called the *corresponding central element.*

Lemma III.2.1. *Let G be a $\tilde{B}_n$-group.*

Let g be a prime element in G with supporting half-twist X and corresponding central element τ. Then:

(1) $g_{\tilde{X}} = g_{\tilde{X}^{-1}} = g^{-1}\tau, \quad g_{\tilde{X}^2} = g$.

(2) $g_{\tilde{Y}^{-2}} = g\tau \quad \forall\ Y$ *consecutive half-twist to X.*

(3) $[g, g_{\tilde{Y}^{-1}}] = \tau \quad \forall\ Y$ *consecutive half-twist to X.*

Proof.

(1) $g_{\tilde{X}^{-2}} = (g_{\tilde{X}^{-1}})_{\tilde{X}^{-1}} = (g^{-1}\tau)_{\tilde{X}^{-1}} = (g^{-1})_{\tilde{X}^{-1}} \cdot = (g^{-1}\tau)^{-1}\tau = g \Rightarrow$ $g_{\tilde{X}} = g_{\tilde{X}^{-1}} \overset{\text{Axiom (1)}}{=} g^{-1}\tau$.

(2) $g_{\tilde{X}^{-1}\tilde{Y}^{-1}\tilde{X}^{-1}} = (g_{\tilde{X}^{-1}})_{\tilde{Y}^{-1}\tilde{X}^{-1}} = (g^{-1}\tau)_{\tilde{Y}^{-1}\tilde{X}^{-1}} = (g_{\tilde{Y}^{-1}\tilde{X}^{-1}})^{-1} \cdot \tau \overset{\text{Axiom (2)}}{=} g_{\tilde{Y}^{-1}}^{-1} \cdot g \cdot \tau$.

On the other hand:

$$g_{\tilde{X}^{-1}\tilde{Y}^{-1}\tilde{X}^{-1}} = g_{\tilde{Y}^{-1}\tilde{X}^{-1}\tilde{Y}^{-1}} = (g_{\tilde{Y}^{-1}\tilde{X}^{-1}})_{\tilde{Y}^{-1}} \overset{\text{Axiom (2)}}{=} (g^{-1}g_{\tilde{Y}^{-1}})_{\tilde{Y}^{-1}}$$
$$= g_{\tilde{Y}}^{-1} \cdot g_{\tilde{Y}^{-2}}.$$

Thus, $g_{\tilde{Y}^{-2}} = g\tau$.

(3) $g_{\tilde{X}\tilde{Y}^{-1}X^{-1}} \overset{\text{Axiom (2)}}{=} g_{\tilde{X}}^{-1} \cdot g_{\tilde{X}\tilde{Y}^{-1}} \overset{\text{by (1)}}{=} g_{X}^{-1} g_{\tilde{X}\tilde{Y}^{-1}} = g \cdot g_{\tilde{Y}^{-1}}^{-1} \cdot \tau^2.$

On the other hand:

$$\begin{aligned} g_{\tilde{X}\tilde{Y}^{-1}\tilde{X}^{-1}} &\overset{\text{by (1)}}{=} (g^{-1}\tau)_{\tilde{Y}^{-1}\tilde{X}^{-1}} \\ &\overset{\text{Axiom (2)}}{=} (g^{-1} \cdot g_{\tilde{Y}^{-1}})^{-1} \cdot \tau = g_{\tilde{Y}^{-1}}^{-1} \cdot g \cdot \tau. \end{aligned}$$

Thus, $g \cdot g_{\tilde{Y}^{-1}}^{-1} \cdot = g_{\tilde{Y}^{-1}}^{-1} \cdot g\tau$.
Thus, $g_{\tilde{Y}^{-1}} \cdot g \cdot g_{\tilde{Y}^{-1}} \cdot g^{-1} = \tau^{-1} = \tau$.
Thus, $[g_{\tilde{Y}^{-1}}, g] = \tau$. □

Lemma III.2.2. *Let G be a $\tilde{B}_n$-group. Let g be a prime element in G with s.h.t. X and corresponding central element τ. Let $b \in \tilde{B}_n$. Then g_b is a prime element with s.h.t. X_b and central element τ.*

Proof. We use the fact that $(a_b)_c = (a_c)_{b_c}$ and $(ab)_c = a_c b_c$. We have to prove 3 properties:

(1) $g_{\tilde{X}^{-1}} = g^{-1}\tau \Rightarrow (g_{\tilde{X}^{-1}})_b = (g^{-1}\tau)_b \Rightarrow g_{\tilde{X}^{-1}b} = g_b^{-1}\tau \Rightarrow g_{bb^{-1}\tilde{X}^{-1}b} = (g_b)^{-1}\tau \Rightarrow (g_b)_{\tilde{X}_b^{-1}} = (g_b)^{-1} \cdot \tau$.

(2) Let Y be a half-twist adjacent to X_b. Then $Y_{b^{-1}}$ is adjacent to X and satisfies axiom (2) of prime elements for g, X and Y_b^{-1}. Namely: $g_{\tilde{Y}_{b-1}^{-1}\tilde{X}} = g^{-1} g_{\tilde{Y}_{b-1}^{-1}}$ and $g_{\tilde{X}\tilde{Y}_{b-1}^{-1}\tilde{X}^{-1}} = g_{\tilde{X}}^{-1} g_{\tilde{X}\tilde{Y}_{b-1}^{-1}}$.

(a) $g_{\tilde{Y}_{b-1}^{-1}\tilde{X}} = g^{-1} g_{\tilde{Y}_{b-1}^{-1}} \Rightarrow$

$(g_{\tilde{T}_{b-1}^{-1}\tilde{X}})_b = (g^{-1} g_{\tilde{Y}_{b-1}^{-1}})_b \Rightarrow$

$(g_b)_{(\tilde{Y}_{b-1}\tilde{X})_b} = (g_b)^{-1}(g_b)_{\tilde{Y}^{-1}} \Rightarrow$

$(g_b)_{\tilde{Y}\tilde{X}_b} = g_b^{-1} \cdot (g_b)_{\tilde{Y}^{-1}}$.

(b) $g_{\tilde{X}\tilde{Y}_{b-1}^{-1}\tilde{X}^{-1}} = g_{\tilde{X}}^{-1} g_{\tilde{X}\tilde{Y}_{b-1}^{-1}} \Rightarrow$

$(g_b)_{\tilde{X}_b\tilde{Y}^{-1}\tilde{X}_b^{-1}} = (g_b^{-1})_{\tilde{X}_b} (g_b)_{\tilde{X}_b\tilde{Y}^{-1}}$.

(3) Let Z be a half-twist disjoint from X_b. Then $Z_{b^{-1}}$ is disjoint from X. Then $g_{\tilde{Z}_{b-1}} = g$. We conjugate $g_{\tilde{Z}_b^{-1}} = g$ by b by get: $(g_b)_{\tilde{Z}} = g_b$.

We need the following lemma on B_n to prove later a criterion for prime element in a $\tilde{B}_n$-group.

Claim III.2.3. *Let* $(X_1, \dots, X_{n-1})$ *be a frame in* $B_n = B_n[D, K]$. *Let*

$$C(X_1) = \{b \in B_n | [b, X_1] = 1\}$$

(centralizer of X_1*),*

$$C_p(X_1) = \{b \in B_n \mid (X_1)_b = X_1, \text{ preserving polarization } \}.$$

Let $\sigma = X_2 X_1^2 X_2$. *Then* $C(X_1)$ *is generated by* $\{X_1, \sigma, X_3, \dots, X_n\}$, $C_p(X_1)$ *is generated by* $\{X_1^2, \sigma, X_3, \dots, X_n\}$.

Proof of the Claim. Let $K = \{a_1, \dots, a_n\}$. Let $x_1, \dots, x_{n-1}$ be a system of consecutive simple paths in D, s.t. $X_i = H(x_i)$ ($H(x_i)$ is the half-twist corresponding to x_i; x_i connects a_i with a_{i+1}). Let $\Gamma_1, \dots, \Gamma_n$ be a free geometric base of $\pi_1(D - K, *)$ consistent with $(X_1, \dots, X_{n-1})$ (that is, $(\Gamma_{i+1})X_i = \Gamma_i$, $(\Gamma_i)X_i = \Gamma_i \Gamma_{i+1} \Gamma_i^{-1}$, $(\Gamma_j)X_i = \Gamma_j$ for $j \neq i$, $i+1$). We can assume that the x_i does not intersect the "tails" of $\Gamma_1, \dots, \Gamma_n$.

Let K_1 be a finite set of D obtained from $K \cup \{x_1\}$ by contracting x_1 to a point $\tilde{a}_2 \in x_1$. $K_1 = \{\tilde{a}_2, a_3, \dots, a_n\}$. Let $B_{n-1} = B_{n-1}[D, K_1]$. Let $Y_2, \dots, Y_{n-1}$ be a frame of B_{n-1} where Y_i can be identified with X_i for $i \geq 3$.

Let $H = \{b \in B_{n-1} \mid (\tilde{a}_2)b = \tilde{a}_2\}$. From the short exact sequence $1 \to P_{n-1} \hookrightarrow B_{n-1} \to S_{n-1} \to 1$ (see [MoTe4]) we can conclude that H is generated by $Y_3, \dots, Y_{n-1}$ and by the generators of P_{n-1}. We remove the generators of P_{n-1} that can be expressed in terms of $Y_3, \dots, Y_{n-1}$ (see [A],[B], and [MoTe4], Section IV) and conclude that H is generated by $Y_2^2, Y_3, \dots, Y_n$. The element Y_2^2 corresponds to the motion $\mathcal{M}'$ of $\tilde{a}_2, a_3, \dots, a_n$ described as follows: $\tilde{a}_2, a_4, \dots, a_n$ stays in place and a_3 is moving around $\tilde{a}_2$ in the positive direction (see Fig. III.2.(a)).

We define a homomorphism $\Phi : C_p(X_1) \to H$ as follows:

Let U be a "narrow" neighborhood of x_1 such that $\lambda = \partial U$ is a simple loop. Take $b \in C_p(X_1)$. There exists a representing diffeomorphism $\beta : D \to D$ ($\beta(K) = K$, $\beta|_{\partial D} = \text{Id}_{\partial D}$) s.t. $\beta|_{\overline{U}} = \text{Id}_{\overline{U}}$ ($\overline{U} = U \cup \lambda$).

The diffeomorphism β also defines an element of $B_{n-1}[D, K_1]$. This element is in fact in H since $\tilde{a}_2 \in x_1$ and thus $(\tilde{a}_2)\beta = \tilde{a}_2$. Denote this element by $\Phi(b)$. The map Φ constructed in this way is obviously a homomorphism, $\Phi : C_p(x_1) \to H$. Clearly, $X_3, \dots, X_{n-1} \in C_p(X_1)$. Clearly, $\Phi(X_i) = Y_i$ for $i \geq 3$. Let $\mathcal{M}$ be the following motion in (D, K): $a_1, a_2, a_4, \dots, a_n$ are stationary and a_3 goes around a_1, a_2 in the positive direction (Fig. III.2(b)). Let u be the braid in $C_p(X_1)$ induced from the motion in $\mathcal{M}$.

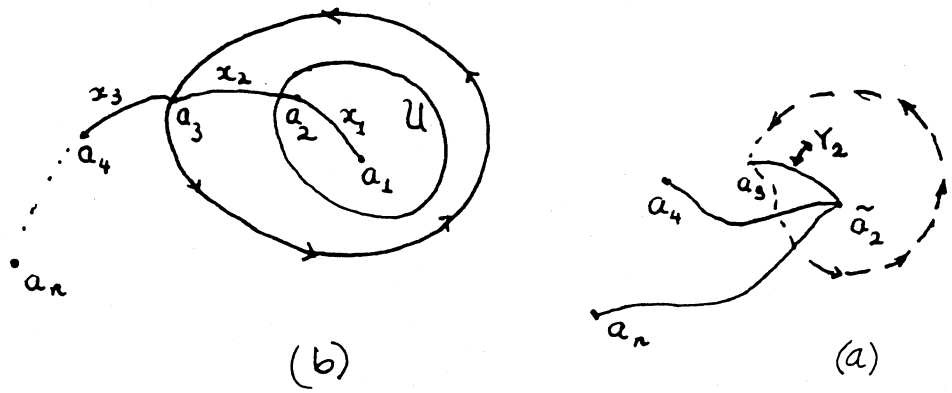


Figure III.2.

Clearly, $\Phi(u) = Y_2^2$. Thus, Φ is onto and $\Phi(u), \Phi(X_3), \ldots, \Phi(X_{n-1})$ generate h. One can check that $u = Z_{31}^2 Z_{32}^2$. But $Z_{31} = X_2 X_1 A_1^{-1}$ and $Z_{32} = X_2$. Thus $u = \sigma$. Thus $C_p(X_1)$ is generated by $\sigma, X_3, \ldots, X_{n-1}$ and a set of generators for $\ker \Phi$.

Consider $\pi_1(D - K \cup x_1, *)$. Let $\tilde{\Gamma}_2$ be the path obtained from connecting λ with $* \in \partial D$ by a simple path intersecting each of $\Gamma_3, \ldots, \Gamma_n$ only at $*$. We get a (free) geometric base $\tilde{\Gamma}_2, \Gamma_3, \ldots, \Gamma_n$ of $\pi_1(D - (K \cup x_1), *)$. It is obvious that $\tilde{\Gamma}_2 = \Gamma_1 \Gamma_2$. $\Phi(b)$ defines in a natural way an automorphism of $\pi(D_k \cup \{X_1\}, *)$ s.t. $\Phi(b)$ does not change the product $\tilde{\Gamma}_2 \Gamma_3 \ldots \Gamma_n$, and $(\tilde{\Gamma}_2)\Phi(b)$ is a conjugate of $\tilde{\Gamma}_2$.

Consider now any $Z \in \ker \Phi$. We have $(\tilde{\Gamma}_2)Z = \tilde{\Gamma}_2$ $(\tilde{\Gamma}_2 = \Gamma_1 \Gamma_2)$, $(\Gamma_j)Z = \Gamma_j$ $\forall j = 3, \ldots, n$. This implies that Z can be represented by a diffeomorphism which is the identity outside of U, that is, $Z = X_1^l$, $l \in \mathbb{Z}$. Since $Z \in C_p(X_1)$, we get $l \equiv 0 \pmod 2$.

Thus, $C_p(X_1)$ is generated by $X_1^2, \sigma, X_3, \ldots, X_{n-1}$. Clearly, $C(X_1)$ is generated by $C_p(X_1)$ and X_1. □ the Claim

Lemma III.2.4. *Let* $\{\tilde{X}_1, \ldots, X_{n-1}\}$ *be a frame in* B_n, $(\tilde{X}_1, \ldots, \tilde{X}_{n-1})$ *their images in* $\tilde{B}_n$. *Let* $u \in G$ *(*G *is a* $\tilde{B}_n$*-group) be such that*

$u_{\tilde{X}_1^{-1}} = u^{-1}\tau$ *with* $\tau^2 = 1$, $\tau \in \mathrm{Center}(G)$, $\tau_b = \tau\ \forall b \in \tilde{B}_n$;

(2_a) $u_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = u^{-1} u_{\tilde{X}_2^{-1}}$;

(2_b) $u_{\tilde{X}_1\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = u_{\tilde{X}_1}^{-1} u_{\tilde{X}_1\tilde{X}_2^{-1}}$;

(3) $u_{\tilde{X}_j} = u\ \forall j = 3, \ldots, n-1$.

Then u is a prime element in G, and $\tilde{X}_1$ is a supporting half-twist for u.

Proof. Let $Z \in B_n$ be any half-twist disjoint from X_1, $\tilde{Z}$ be the image of Z in $\tilde{B}_n$. $\exists b \in B_n$ such that $(X_1)_b = X_1$, $(X_3)_b = Z$. By Claim III.2.3, b belongs to the subgroup of B_n generated by $X_1, X_3, \dots, X_{n-1}$ and $\sigma = X_2 X_1^2 X_2$. Let $\tilde{b}$ and $\tilde{\sigma}$ be the images of b and σ in $\tilde{B}_n$. We have $u_{\tilde{\sigma}^{-1}} = u_{\tilde{X}_2^{-1}\tilde{X}_1^{-2}\tilde{X}_2^{-1}} = (u_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}})_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}} = (u^{-1}u_{\tilde{X}_2^{-1}})_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}} = (\tau u)_{\tilde{X}_2^{-1}} \cdot u_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_2^{-1}} = \tau u_{\tilde{X}_2^{-1}} \cdot u_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = \tau u_{\tilde{X}_2^{-1}} \cdot (\tau u^{-1})_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = \tau^2 u_{\tilde{X}_2^{-1}} u^{-1}_{\tilde{X}_2^{-1}} \tau(\tau u) = u$. Then $u_{\tilde{\sigma}} = u$. Now: $u_{\tilde{X}_j} = u$ for $j \geq 3$ (by assumption (3)) and $u_{\tilde{X}_1^2} = u$ (by assumption (1)). Thus, if X_1 appears in b an even number of times, then $u_b = u$. Otherwise we replace b by bX_1. The "new" b satisfies the same requirement for b, as above and $u_{\tilde{b}} = u$. Thus, we can assume $u_{\tilde{b}} = u$. We have

$$u_{\tilde{Z}} = u_{\tilde{b}^{-1}\tilde{X}_3\tilde{b}} = u_{\tilde{X}_3\tilde{b}} = u_{\tilde{b}} = u.$$

Let Y be a half-twist in B_n adjacent to X_1. $\exists b_1 \in B_n$ s.t. $(X_1)_{b_1} = X_1$, $(X_2)_{b_1} = Y$. Let $\tilde{b}_1$ and $\tilde{Y}$ be the images of b_1 and Y in $\tilde{B}_n$. As above, we can choose b_1 so that $u_{\tilde{b}_1} = u$. Applying $\tilde{b}_1$ on the assumptions (2_a) and (2_b) we get (since $u_{\tilde{b}_1} = u$, $(\tilde{X}_1)_{\tilde{b}_1} = \tilde{X}_1$, $(\tilde{X}_2)_{\tilde{b}_1} = \tilde{Y}$):

$$u_{\tilde{Y}^{-1}\tilde{X}_1^{-1}} = u^{-1}u_{\tilde{Y}^{-1}} \text{ and } u_{\tilde{X}_1\tilde{Y}^{-1}\tilde{X}_1^{-1}} = u^{-1}_{\tilde{X}_1}u_{\tilde{X}_1\tilde{Y}^{-1}}. \quad \square$$

III.3. Polarized pairs and uniqueness of coherent pairs.

Definition. *Polarized pair*

Let G be a $\tilde{B}_n$-group, h a prime element of G, X its supporting half-twist. If X is polarized, we say that (h, X) (or $(h, \tilde{X})$ is a polarized pair with central element τ, $\tau = hh_{\tilde{X}^{-1}}$.

Definition. *Coherent pairs, Anti-coherent pairs*

We say that two polarized pairs $(h_1, \tilde{X}_1)$ and $(h_2, \tilde{X}_2)$ are coherent (anti-coherent) if $\exists \tilde{b} \in \tilde{B}_n$ such that $(h_1)_{\tilde{b}} = h_2$, $(\tilde{X}_1)_b = \tilde{X}_2$, and b preserves (reverses) the polarization.

Corollary III.3.1. *Coherent and anti-coherent polarized pairs have the same central element.*

Proof. The prime elements of coherent and anti-coherent pairs are conjugate to each other. Thus by Lemma III.2.2 we get the corollary. □

We need the following Lemma to prove later the unique existence of a prime element with given s.h.t. conjugate to a given prime element.

Lemma III.3.2. *Let $h \in G$, a prime element with s.h.t. $\tilde{X}$. Let $b \in B_n$. Then: $\tilde{X}_{\tilde{b}} = \tilde{X} \Rightarrow h_{\tilde{b}} = h$.*

Proof. We can choose a set of standard generators for $B_n[D,K]$, $\{X_1, \dots, X_{n-1}\}$ with $X_1 = X$. Let $\sigma = X_2 X_1^2 X_2$. Consider $C_p(X_1)$, the centralizer of X_1 preserving polarization. By Lemma III.2.3, $C_p(X_1)$ is the subgroup of B_n generated by $X_1^2, \sigma, X_3, \dots, X_{n-1}$. Since $\tilde{X}_3, \dots, \tilde{X}_{n-1}$ are disjoint from X_1, they do not change h (by axiom(3) of prime elements). By Lemma III.2.1, $h_{\tilde{X}_1^2} = h$. Consider $h_{\sigma^{-1}} = h_{\tilde{X}_2^{-1}\tilde{X}_1^{-2}\tilde{X}_2^{-1}}$. We have:

$$h_{\tilde{\sigma}^{-1}} = h_{\tilde{X}_2^{-1}\tilde{X}_1^{-2}\tilde{X}_2^{-1}} = (h_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}})_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}}$$
$$\overset{\text{by Axiom(2) of prime element}}{=} (h^{-1}h_{\tilde{X}_2^{-1}})_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}}$$
$$= (h^{-1}_{\tilde{X}_1^{-1}} h_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}})_{\tilde{X}_2^{-1}}$$
$$\overset{\text{by Axiom(2) of prime element}}{=} (\tau h \cdot h^{-1} h_{\tilde{X}_2^{-1}})_{\tilde{X}_2^{-1}} = \tau h_{\tilde{X}_2^{-2}}$$
$$\overset{\text{by Lemma III.2.1(2)}}{=} \tau h \tau = h.$$

Thus $h_{\tilde{\sigma}} = h$. Thus, for every generator g of $C_p(X)$, $h_{\tilde{g}} = h$. Since $b \in \tilde{C}(X)$, $h_{\tilde{B}} = h$. □

Proposition III.3.3. *Let $\{h, \tilde{X}\}$ be a polarized pair, $h \in G$, $\tilde{X} \in \tilde{B}_n$. Let $\tilde{T}$ be a polarized half-twist in $\tilde{B}_n$. Then there exists a unique prime element $g \in G$ such that $\{g, \tilde{T}\}$ and $\{h, \tilde{X}\}$ are coherent.*

Proof. Let $X, T \in B_n$ be polarized half-twists representing $\tilde{X}$ and $\tilde{T}$. $\exists b \in B_n$ such that $T = X_b$ preserving polarization. Let $\tilde{b}$ be the image of b in $\tilde{B}_n$. Taking $g = h_{\tilde{b}}$ we obtain a polarized pair $\{g, \tilde{T}\}$ such that $\{g, \tilde{T}\}$ and $\{h, \tilde{X}\}$ are coherent. To prove the *uniqueness* of g, assume that $\{g_1, \tilde{T}\}$ is another polarized pair coherent with $\{h, \tilde{X}\}$. Then $\exists b_1 \in B_n$ with $g = h_{\tilde{b}_1}$ and $T = X_{b_1}$, preserving polarization. We have $T = X_{b_1} = X_b$ and $X_{b_1 b^{-1}} = X$.

Denote $b_2 = b_1 b^{-1}$, so $X_{b_2} = X$ (preserving polarization). By the previous lemma, $h_{b_2} = h$. Thus, $h_{b_1} = h_b$ or $g = g_1$. □

Definition. $L_{h,\tilde{X}}(\tilde{T})$

Let $(h, \tilde{X})$ be a polarized pair. $\tilde{T} \in \tilde{B}_n$. $L_{(h,\tilde{X})}(\tilde{T})$ is the unique prime element s.t. $(L_{(h,\tilde{X})}(\tilde{T}), \tilde{T})$ is coherent with $(h, \tilde{X})$.

From uniqueness we get a simultaneous conjugation:

Lemma III.3.4. *Assume $(h, \tilde{X})$ and $(g, \tilde{X})$ are polarized pairs. Let τ be the central element of $(g, \tilde{X})$. If $(h, \tilde{X})$ is anti-coherent to $(g, \tilde{X})$ then $h = g^{-1} \cdot \tau$.*

Proof. By assumption, $\exists b \in B_n$ s.t. $g = h_{\tilde{b}}$ and $X = X_b$, reversing polarization. Thus $X_{bX^{-1}} = X$, preserving polarization. Thus $(h_{\tilde{b}\tilde{X}^{-1}}, \tilde{X})$ is coherent with $(h_{\tilde{b}\tilde{X}^{-1}}, \tilde{X}_{\tilde{b}\tilde{X}^{-1}})$. Clearly, $(h, \tilde{X})$ is coherent with $(h_{\tilde{b}\tilde{X}^{-1}}, \tilde{X}_{\tilde{b}\tilde{X}^{-1}})$. From uniqueness, $h = h_{\tilde{b}\tilde{X}^{-1}} = g_{\tilde{X}^{-1}}$. Since τ is the central element of $(g, \tilde{X})$, $\tau = g \cdot g_{\tilde{X}^{-1}}$. Thus, $g_{X^{-1}} = g^{-1}\tau$. So $h = g^{-1}\tau$. □

Corollary III.3.5. *If $(a_i, \tilde{X})$ is coherent with $(g_i, \tilde{Y})$ $i = 1, 2$, then there exist $b \in \tilde{B}$ s.t. $(a)_b = g_i$ $i = 1, 2$.*

Proof. Let b be the element of $\tilde{B}_n$ s.t. $(a_1)_b = g_1$, $(\tilde{X})_b = \tilde{Y}$. Now, $((a_2)_b, (X)_b)$ is coherent with $(a_2, \tilde{X})$. Since $(\tilde{X})_b = \tilde{Y}$, $((a_2)_b, \tilde{Y})$ is coherent with $(a_2, \tilde{X})$. The pair $(g_2, \tilde{Y})$ is also coherent with $(a_2, \tilde{X})$. From uniqueness, $(a_2)_b = g_2$. □

III.4. $\tilde{\mathbf{B}}_{\mathbf{n}}$-action of nondisjoint half-twists.

Proposition III.4.1. *Let $\tilde{T}, \tilde{Y}$ be 2 orderly adjacent polarized half-twists in $\tilde{B}_n$, $\{h, \tilde{X}\}$ be a polarized pair, $h \in G$, $\tilde{X} \in \tilde{B}_n$. Denote by $\tilde{Y}'$ the polarized half-twist obtained from $\tilde{Y}$ by changing polarization (that is, $\tilde{T}, \tilde{Y}'$ are not orderly adjacent). Denote by*

$$L(T) = L_{\{h,\tilde{X}\}}(\tilde{T}),$$
$$L(Y) = L_{\{h,\tilde{X}\}}(\tilde{Y}),$$
$$L(Y') = L_{\{h,\tilde{X}\}}(\tilde{Y}').$$

Then

(1) $L(T)_{\tilde{T}^{-1}} = L(T)^{-1}\tau$;

(2) $L(T)_{\tilde{Y}^{-1}} = L(T)L(Y)$;

(3) $L(T)_{(\tilde{Y}')^{-1}} = L(Y)^{-1}L(T)$.

Proof.

(1) By Lemma III.2.1(1).

(2) Let $b \in B_n$ be s.t. $L(T) = h_b$, $T = X_b$, preserving polarization. Let $Y_1 = Y_{b^{-1}}$. Then $\{X, Y_1\}$ is a pair of adjacent half-twists ($X = T_{b^{-1}}$, $Y = Y_{b^{-1}}$), and so $h_{\tilde{Y}_1^{-1}\tilde{X}^{-1}} = h^{-1}h_{\tilde{Y}_1^{-1}}$. Applying $\tilde{b}$ to that equation, we get $(L(T))_{\tilde{Y}^{-1}\tilde{T}^{-1}} = L(T)^{-1}L(T)_{\tilde{Y}^{-1}}$, or

$$L(T)_{\tilde{Y}^{-1}} = L(T) \cdot L(T)_{\tilde{Y}^{-1}\tilde{T}^{-1}}$$

Let $b_1 = bY^{-1}T^{-1}$. Then $X_{b_1} = X_{bY^{-1}T^{-1}} = T_{Y^{-1}T^{-1}} = Y$ (since $T_{Y^{-1}} = Y_T$ by III.0). Using that T, Y are orderly adjacent and $X_b = T$, preserving polarization, one can easily check that actually $X_{b_1} = Y$, preserving polarization. Because of the *uniqueness* of $L(Y) = L_{\{h,\tilde{X}\}}(\tilde{Y})$, we get $L(Y) = h_{\tilde{b}_1} = h_{\tilde{b}\tilde{Y}^{-1}\tilde{T}^{-1}} = L(T)_{\tilde{Y}^{-1}\tilde{T}^{-1}}$. Together with the previous equation we get: $L(T)_{\tilde{Y}^{-1}} = L(T)L(Y)$, which is (2).

(3) Using $Y_{Y^{-1}} = Y'$ (preserving polarization) and uniqueness, we can write $L(Y') = L(Y)_{\tilde{Y}^{-1}}$. By (1), $L(Y)_{\tilde{Y}^{-1}} = L(Y)^{-1}\tau$. Thus, $L(Y') = L(Y)^{-1}\tau$. From (2) we get: $L(T)_{\tilde{Y}'^{-1}} = L(T)L(Y') = L(T)L(Y)^{-1}\tau = L(Y)^{-1}L(T)$, which is (3).

(We used $[L(T), L(T)_{\tilde{Y}^{-1}}] = \tau$, from Lemma III.2.1, which implies $\tau = [L(T), L(T)L(Y)] = [L(T), L(Y)]$.) □

Lemma III.4.2. *Let h be a prime element in G, $\tilde{X} \in \tilde{B}_n$ a supporting half-twist of h, $\tilde{Z}$ a half-twist in $\tilde{B}_n$ transversal to $\tilde{X}$. Then $h_{\tilde{Z}} = h$.*

Proof. Let X, Z be transversal half-twists in B_n, representing $\tilde{X}, \tilde{Z}$. Let x, z be 2 transversally intersecting simple paths corresponding to X, Z (see Fig. III.4).

There exists a simple path y such that the corresponding half-twist Y is adjacent to X and Z, and $Z_1 = Z_{Y^{-2}}$ is disjoint from X. Let z_1 be the path corresponding to Z_1 (see Fig. III.4). Denote by $\tilde{Y}, \tilde{Z}_1$ the images of Y, Z_1 in

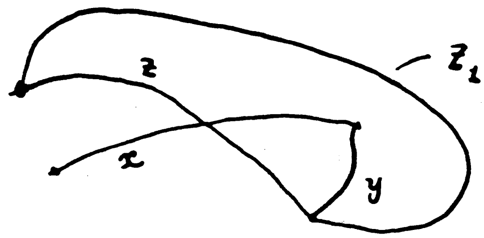


Figure III.4.

$\tilde{B}_n$. We have $h_{\tilde{Z}} = h_{\tilde{Y}^{-2}\tilde{Z}_1\tilde{Y}^2} \overset{\text{by Lemma III.2.1}}{=} (h\tau)_{\tilde{Z}_1\tilde{Y}^2} = (h\tau)_{\tilde{Y}^2} = h\tau \cdot \tau = h$.
□

III.5. Commutativity properties.

Proposition III.5.1. *Let* $\{g_1, \tilde{Y}_1\}, \{g_2, \tilde{Y}_2\}$ *be 2 polarized pairs in* G. *Assume that they are coherent or anti-coherent. Let* τ *be the corresponding central element of* $(g_1, \tilde{Y}_1)$ $(\tau = g_1(g_1)_{\tilde{Y}_1^{-1}})$.

Then

(1) *if* $\tilde{Y}_1, \tilde{Y}_2$ *are adjacent, then* $[g_1, g_2] = \tau$;

(2) *if* $\tilde{Y}_1, \tilde{Y}_2$ *are disjoint or transversal, then* $[g_1, g_2] = 1$.

Proof.

(1) Assume first that $\{g_1, \tilde{Y}_1\}, \{g_2, \tilde{Y}_2\}$ are coherent. Take $b \in \tilde{B}_n$ with $g_2 = (g_1)_b$, $\tilde{Y}_2 = (\tilde{Y}_1)_b$ (preserving polarization). Let $b_1 = \tilde{Y}_2^{-1}\tilde{Y}_1^{-1}$. Then $(\tilde{Y}_1)_{b_1} = \tilde{Y}_2$. Assume that b_1 preserves polarization of $\tilde{Y}_1, \tilde{Y}_2$. We have $\{(g_1)_{b_1}, \tilde{Y}_2\}$ and $\{g_2, \tilde{Y}_2\}$ coherent with $\{g_1, \tilde{Y}_1\}$. By Proposition III.3.3 (the *uniqueness* part) we get $(g_1)_{b_1} = g_2$. Thus we have $g_2 = (g_1)_{b_1} = (g_1)_{\tilde{Y}_2^{-1}\tilde{Y}_1^{-1}} = g_1^{-1}(g_1)_{\tilde{Y}_2^{-1}}$, and $[g_1, g_2] = [g_1, g_1^{-1}(g_1)_{\tilde{Y}_2^{-1}}] = g_1 g_1^{-1}(g_1)_{\tilde{Y}_2^{-1}} g_1^{-1}(g_1^{-1})_{\tilde{Y}_2^{-1}} g_1 = [g_1, (g_1)_{\tilde{Y}_2^{-1}}]_{g_1} \overset{\text{by Lemma III.2.1(3)}}{=} \tau_{g_1} = \tau$.

If b_1 does not preserve the polarization of $\tilde{Y}_1, \tilde{Y}_2$, consider $b_2 = b_1\tilde{Y}_2$. Then $\tilde{Y}_2 = (\tilde{Y}_1)_{b_2}$, preserving polarization. As above, we get

$$\begin{aligned} g_2 &= (g_1)_{b_2} \\ &= (g_1)_{\tilde{Y}_2^{-1}\tilde{Y}_1^{-1}\tilde{Y}_2} = (g_1)_{\tilde{Y}_1\tilde{Y}_2^{-1}\tilde{Y}_1^{-1}} = (g_1^{-1}\tau)_{\tilde{Y}_2^{-1}\tilde{Y}_1^{-1}} = \tau(g_1^{-1})_{\tilde{Y}_2^{-1}}g_1, \end{aligned}$$

and then

$$\begin{aligned} [g_1, g_2] &= [g_1, \tau(g_1^{-1})_{\tilde{Y}_2^{-1}}g_1] \\ &= [g_1, (g_1^{-1})_{\tilde{Y}_2^{-1}}] = [g_1, (g_1)_{\tilde{Y}_2^{-1}}]^{-1}_{(g_1)_{\tilde{Y}_2^{-1}}} \overset{\text{by Lemma III.2.1(3)}}{=} \tau. \end{aligned}$$

If $\{g_1, \tilde{Y}_1\}$, $\{g_2, \tilde{Y}_2\}$ are anti-coherent, denote $\tilde{Y}_2'$ the half-twist obtained from $\tilde{Y}_2$ by changing polarization. One can then check that $\{g_1, \tilde{Y}_1\}$, $\{g_{2;\tilde{Y}_2^{-1}}, \tilde{Y}_2'\}$ are coherent. We have from the above that $\tau = [g_1, (g_2)_{\tilde{Y}_2^{-1}}]$. By Corollary III.3.1 τ is also the central element of $(g_2, \tilde{Y}_2)$. Thus $\tau = g_2(g_2)_{\tilde{Y}_2^{-1}}$ which implies $(g_2)_{\tilde{Y}_2^{-1}} = g_2^{-1}\tau$. Thus $\tau = [g_1, (g_2)_{\tilde{Y}_2^{-1}}] = [g_1, g_2^{-1}\tau] = [g_1, g_2^{-1}] = [g_1, g_2]^{-1}_{g_2}$. Thus, $[g_1, g_2] = \tau_{g_2}^{-1} = \tau$.

(2) We can assume that $\{g_1, \tilde{Y}_1\}, \{g_2, \tilde{Y}_2\}$ are coherent. (Otherwise, we replace $\tilde{Y}_2$ by $\tilde{Y}_2'$ and g_2 by $(g_2)_{\tilde{Y}_2^{-1}}$ and use $[g_1, (g_2)\tilde{Y}_2^{-1}] = [g_1, g_2]^{-1}_{g_2}$ (see above).)

Consider first the case where $\tilde{Y}_1, \tilde{Y}_2$ are disjoint. We can choose a standard base of $\tilde{B}_n$, say $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_{n-1})$ such that $\tilde{X}_1 = \tilde{Y}_1$, $\tilde{X}_3 = \tilde{Y}_2$ and the given polarizations of $\tilde{Y}_1, \tilde{Y}_2$ coincide with "consecutive" polarizations of $\tilde{X}_1, \tilde{X}_3$ ("end" of $\tilde{X}_1$ = "origin" of $\tilde{X}_2$, "end" of $\tilde{X}_2$ = "origin" of $\tilde{X}_3$). Let $b_1 = \tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}$. Then $\tilde{Y}_2 = (\tilde{Y}_1)_{b_1}$, preserving polarization. From Proposition III.3.3 (*uniqueness*) it follows that

$$\begin{aligned} g_2 &= (g_1)_{b_1} \\ &= (g_1)_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}} = (g_1^{-1}(g_1)_{\tilde{X}_2^{-1}})_{\tilde{X}_3^{-1}\tilde{X}_2^{-1}} = (g_1^{-1})_{\tilde{X}_2^{-1}}(g_1)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}} \\ &= (g_1^{-1})_{\tilde{X}_2^{-1}}(g_1)_{\tilde{X}_3^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = (g_1^{-1})_{\tilde{X}_2^{-1}}(g_1)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}. \end{aligned}$$

By III.2.1(3) $[g_1, (g_1)_{\tilde{X}_2^{-1}}] = \tau$, which implies

$$[g_1, (g_1)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}] = [g_1, (g_1)_{\tilde{X}_2^{-1}}]_{\tilde{X}_3^{-1}} = \tau_{\tilde{X}_3^{-1}} = \tau.$$

We can write

$$\begin{aligned} [g_1, g_2] &= [g_1, (g_1^{-1})_{\tilde{X}_2^{-1}}(g_1)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}] \\ &= [g_1, (g_1^{-1})_{\tilde{X}_2^{-1}}] \cdot [g_1, (g_1)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}]_{(g_1)_{\tilde{X}_2^{-1}}} = \tau \cdot \tau = \tau^2 = 1. \end{aligned}$$

Assume now that $\tilde{Y}_1, \tilde{Y}_2$ are transversal. As in the proof of Lemma III.4.2, we can find a half-twist $\tilde{T} \in \tilde{B}_n$ such that $\tilde{T}$ is adjacent to $\tilde{Y}_1, \tilde{Y}_2$ and $\tilde{Y}_2' = (\tilde{Y}_2)_{\tilde{T}^{-2}}$ is disjoint from $\tilde{Y}_1$. Let $b \in \tilde{B}_n$ be such that $\tilde{Y}_2 = (\tilde{Y}_1)_b$, $g_2 = (g_1)_b$. Let $b' = b\tilde{T}^{-2}$, $g_2' = (g_1)_{b'} = (g_2)_{\tilde{T}_2^{-2}}$. Then $\{g_2', \tilde{Y}_2'\}$ is coherent, or anti-coherent, with $\{g_1, \tilde{Y}_1\}$. Since $\tilde{Y}_2', \tilde{Y}_1$ are disjoint, we get from the above $[g_1, g_2'] = 1$. By Lemma III.2.1 $g_2' = (g_2)_{\tilde{T}^{-2}} = g_2\tau$, or $g_2 = g_2'\tau$. Therefore, $[g_1, g_2] = [g_1, g_2'\tau] = [g_1, g_2'] = 1$. □

Recall that here exists a natural homomorphism $\psi_n : B_n \to S_n$. $\psi_n(X_i)$ is the transposition $(i\ i+1)$ for X_i a half-twist connecting the points q_i and q_{i+1}.

Definition. P_n

$P_n = \ker \psi_n$.

Recall from [MoTe4] that: P_n is generated by Z_{ij}^2, where:

$Z_{ij} = (X_i^2)_{X_{i+1} \dots X_{j-1}}$.

Definition. $\tilde{P}_n$

$\tilde{P}_n = \ker(\tilde{B}_n \overset{\tilde{\psi}_n}{\to} S_n)$ where $\tilde{\psi}_n$ is induced naturally from ψ_n.

Proposition III.5.2. *Assume $n \geq 4$. Let $\tilde{X}_1, \tilde{X}_2$ be 2 adjacent half-twists in $\tilde{B}_n$. Let $c = [\tilde{X}_1^2, \tilde{X}_2^2]$. Then the commutant $\tilde{P}_n'$ of $\tilde{P}_n$ is generated by c where $c_b = c\ \forall b \in \tilde{B}_n$, and $c^2 = 1$. Moreover, if $(\tilde{Y}_1, \tilde{Y}_2)$ and $(\tilde{Z}_1, \tilde{Z}_2)$ are two pairs of adjacent half-twists, then $[\tilde{Z}_1^2, \tilde{Z}_2^2] = [\tilde{Y}_1^2, \tilde{Y}_2^2] = [\tilde{Z}_1^2, \tilde{Z}_2^{-2}] = [\tilde{Z}_1^{-2}, \tilde{Z}_2^{-2}] = c$.*

Proof. Let $B_n = B_n(D, K))$. Complete $\tilde{X}_1$ and $\tilde{X}_2$ to $\tilde{X}_1, \dots, \tilde{X}_{n-1}$, a standard base of $\tilde{B}_n$, $X_i = (H(x_i)$ and $x_1, \dots, x_{n-1}$ are simple paths in D. Let $c = [\tilde{X}_1^2, \tilde{X}_2^2]$. Let $x = (x_1)_{\tilde{X}_2\tilde{X}_3}$. We have a quadrangle formed by x_1, x_2, x_3, x, (see Fig. III.5(a)).

Denote by $X \in B_n$ the half-twist defined by x. Evidently, $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}$ form a good quadrangle in $\tilde{B}_n$. Thus by Lemma III.1.2

$$\tilde{X}_1^2\tilde{X}_3^2 = \tilde{X}_2^2\tilde{X}^2. \tag{1.10}$$

Denote $y_1 = \tilde{X}_1^2$, $y_2 = \tilde{X}_2^2$, $y_3 = \tilde{X}_3^2$, $y_4 = \tilde{X}^2$, (the squares of the edges), $d_1 = \tilde{X}_1\tilde{X}_2^2\tilde{X}_1^{-1}$, $d_2 = \tilde{X}_2\tilde{X}_3^2\tilde{X}_2^{-1}$, (the squares of the diagonals), $y' = \tilde{X}_2\tilde{X}_1^2\tilde{X}_2^{-1}$ (the square of the outer diagonal)

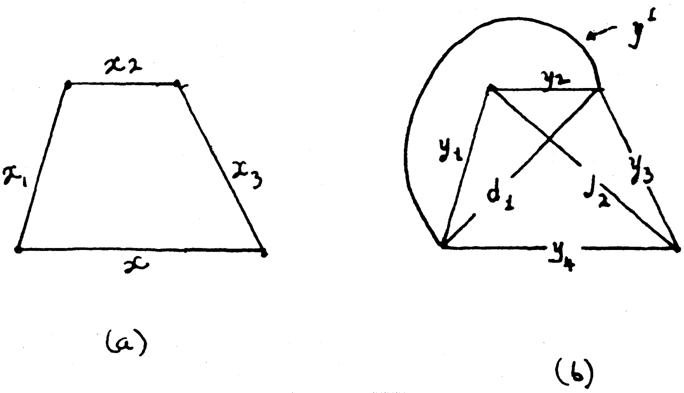


Figure III.5.

See Fig. III.5(b) where we denote the paths corresponding to the half-twists whose squares we considered here.

Clearly:

$y_1y_3 = y_3y_1$

$d_1 = (y')_{y_1^{-1}}$

$(y_3)_{x_2} = (d_2)_{x_3^{-2}} = (d_2)_{y_3^{-1}}$

$y_1y_2d_1 = y_2y_1y' = \Delta_3^2$ (a central element of P_3).

We rewrite (1.10) to get

(1.11) $$y_4 = y_1y_3y_2^{-1}$$

Conjugating (1.11) by $\tilde{X}_1$, we get:

(1.12) $$d_2 = y_1y_3y'^{-1};$$

conjugating (1.11) by $\tilde{X}_2$ we get $y_4 = d_1(d_2)_{y_3^{-1}} \cdot y_2^{-1}$. Since $d_1 = y_1y'y_1^{-1}$,

$$y_4 = y_1y'y_1^{-1}y_3d_2y_3^{-1}y_2^{-1} \overset{\text{by (1.12)}}{=} y_1y'y_1^{-1}y_3y_1y_3y'^{-1}y_3^{-1}y_2^{-1}.$$

We compare the last expression with (1.11) to get:

(1.13) $$y'y_3^2y'^{-1}y_3^{-1} = y_3, \quad \text{or} \quad [y', y_3^2] = 1.$$

Since y', y_3 are squares of two adjacent half-twists in $\tilde{B}_n$, and any two pairs of adjacent half-twists are conjugate, we conclude from (1.13) that:

(1.14) $\forall$ pairs of adjacent half-twists, say $\tilde{Z}_1, \tilde{Z}_2$ in $\tilde{B}_n : [\tilde{Z}_1^2, \tilde{Z}_2^4] = 1$,

which also implies that $[\tilde{Z}_1^2, \tilde{Z}_2^2] = [\tilde{Z}_1^{-2}, \tilde{Z}_2^2] = [\tilde{Z}_1^2, \tilde{Z}_2^{-2}] = [\tilde{Z}_1^{-2}, \tilde{Z}_2^{-2}]$.

Conjugating (1.11) by $\tilde{X}_3^{-1}$ we get:

$$d_1 = y_1y_3(d_2)_{y_3^{-1}} = y_1y_3 \cdot y_3d_2^{-1}y_3^{-1} = y_1y_3^2 \cdot d_2^{-1}y_3^{-1} \overset{(1.14)}{=} y_1d_2^{-1}y_3,$$

We have by (1.14) that $1 = [y_1^2, y_2] = [y_1, y_2]_{y_1^{-1}} \cdot [y_1, y_2]$. Denoting $c = [y_1, y_2]$, we can write:

$$(1.15) \qquad c_{y_1^{-1}} = c^{-1}, \quad \text{or} \quad c_{y_1} = c^{-1}.$$

Denote by $\tilde{P}_3$ the subgroup of $\tilde{B}_n$ generated by y_1, y_2, d_1, and by $\alpha = \Delta_3^2 = y_1y_2d_1 = y_2y_1y'$ (a central element of $\tilde{P}_3$), so that

$$(1.16) \qquad y' = y_1^{-1}y_2^{-1}\alpha.$$

So, $c_{\tilde{X}_1} = [y_1, y_2]_{\tilde{X}_1} = [y_1, y'] \overset{\text{by (1.16)}}{=} [y_1, y_1^{-1}y_2^{-1}\alpha] = y_1 \cdot y_1^{-1}y_2^{-1}\alpha \cdot y_1^{-1} \cdot \alpha^{-1}y_2y_1 = y_2^{-1}y_1^{-1}y_2y_1 = [y_2^{-1}, y_1^{-1}] = [y_2, y_1] = c^{-1}$. Thus we have $c_{\tilde{X}_1^2} = (c^{-1})_{\tilde{X}_1} = c$.

By (1.15) $c_{\tilde{X}_1^2} = c_{y_1} = c^{-1}$.

We compare the last two results to get $c = c^{-1}$ or

$$(1.17) \qquad c^2 = 1 \quad \text{and} \quad c_{\tilde{X}_1} = c.$$

Using a conjugation which sends $(\tilde{X}_1, \tilde{X}_2)$ to $(\tilde{X}_2, \tilde{X}_1)$, we obtain from (1.17)

$$(1.18) \qquad c_{\tilde{X}_2}^{-1} = c^{-1}, \quad \text{or} \quad c_{\tilde{X}_2} = c.$$

(1.17) and (1.18) show that $\forall\, z \in \tilde{B}_3$ (the subgroup of $\tilde{B}_n$ generated by $\tilde{X}_1, \tilde{X}_2$) we have

$$(1.19) \qquad c_z = c.$$

Consider now $c_{\tilde{X}_3} = [y_1, y_2]_{\tilde{X}_3} = [y_1, d_2] \overset{\text{by (1.12)}}{=} [y_1, y_1y_3y'^{-1}] = y_1y_1y_3y'^{-1}y_1^{-1}y'y_3^{-1}y_1^{-1} = y_1y_3 \cdot (y_1y'^{-1}y_1^{-1}y')y_3^{-1}y_1^{-1} \overset{\text{by (1.19)}}{=} y_1y_3cy_3^{-1}y_1^{-1} = c_{y_1^{-1}y_3^{-1}} = c_{y_1^{-3}} = c_{y_3} = c_{\tilde{X}_3^2}$, shortly $c_{\tilde{X}_3^2} = c_{\tilde{X}_3}$. This implies $c_{\tilde{X}_3} = c$.

Since $c = [\tilde{X}_1^2, \tilde{X}_2^2]$, we have $\forall \tilde{X}_j$, $j \geq 4$, $c_{\tilde{X}_j} = c$. Thus $\forall b \in \tilde{B}_n$ $c_b = c$ and $c^2 = 1$.

Let $(\tilde{Y}_1, \tilde{Y}_2)$ be a pair of adjacent half-twists. Since every 2 pairs of adjacent half-twists are conjugate in $\tilde{B}_n$, $\exists b \in \tilde{B}_n$ s.t. $[\tilde{Y}_1^2, \tilde{Y}_2^2] = [\tilde{X}_1, \tilde{X}_2^2]_b =$

c_b. Since $c_b = c \quad \forall b \in \tilde{B}_n$, $[\tilde{Y}_1^2, \tilde{Y}_2^2] = c$. Since $c^2 = 1$, $c \in \text{Center}(\tilde{B}_n)$ we also have $[\tilde{Y}_1^2, \tilde{Y}_2^{-2}] = [\tilde{Y}_1^{-2}, \tilde{Y}_2^{-2}] = c$. In particular, if $(\tilde{Z}_1, \tilde{Z}_2)$ is another pair of adjacent half-twists, $[\tilde{Z}_1^2, \tilde{Z}_2^2] = [\tilde{Y}_1^2, \tilde{Y}_2^2] = c$. Because any two disjoint and transversal half-twists of $\tilde{B}_n$ commute, and $\tilde{P}_n$ is generated by $\tilde{Z}_{ij}^2 = (\tilde{X}_i^2)_{\tilde{X}_{i+1}\cdots\tilde{X}_{j-1}}$, $1 \le i < j \le n$ (see [MoTe4]), we conclude that $\tilde{P}_n'$ is generated by c. □

III.6. $\tilde{\mathbf{P}}_{\mathbf{n}}$ as a $\tilde{\mathbf{B}}_{\mathbf{n}}$-group.

Recall: $\text{Ab}(B_n) \simeq \mathbb{Z}$ (B_n is generated by the half-twists and every 2 half-twists are conjugate).

Definition. $P_{n,0}$

$P_{n,0} = \ker(P_n \to \text{Ab}\, B_n)$ ("degree zero" pure braids).

Definition. $\tilde{P}_{n,0}$

$\tilde{P}_{n,0}$ is the image of $P_{n,0}$ in $\tilde{P}_n$.

Lemma III.6.1. *Let X_1, X_2 be 2 consecutive half-twists in B_n. Let $u = (\tilde{X}_1^2)_{\tilde{X}_2^{-1}} \tilde{X}_2^{-2}$. Then $u \in \tilde{P}_{n,0}$ u is a prime element in $\tilde{P}_n$ (considered as a $\tilde{B}_n$-group), and $\tilde{X}_1$ is the supporting half-twist of u.*

Proof. Clearly, $u \in \tilde{P}_{n,0}$. Since $X_1X_2X_1 = X_2X_1X_2$, $(X_1)_{X_2^{-1}} = (X_2)_{X_1}$ and thus, $u = (\tilde{X}_1)^2_{\tilde{X}_2^{-1}} \tilde{X}_2^{-2} = (\tilde{X}_2)^2_{\tilde{X}_1} \tilde{X}_2^{-2}$. We often use here the fact that $(X_1)_{X_2^{-1}} = (X_2)_{X_1}$ as well as the fact that $[\tilde{X}_1^{\pm 2}, X_2^{\pm 2}] = c$, i.e., $\tilde{X}_2^{-2}\tilde{X}_2^2 = c\tilde{X}_2^2\tilde{X}_1^{-2}$ and $(\tilde{X}_1^2)_{X_2^{-1}} = (\tilde{X}_1^2)_{X_2} c$ for $c \in \text{Center}(\tilde{B}_n)$, $c^2 = 1$. Complete X_1, X_2 to a frame of $B_n : X_1, \ldots, X_{n-1}$. ($\langle X_i, X_{i+1} \rangle = 1$ and $[X_i, X_j] = 1$ $|i - j| > 2$.). We shall use Lemma III.2.4, that is, we must check conditions (1), (2_a), (2_b), (3) of Lemma III.2.4.

(1) We have $u_{\tilde{X}_1^{-1}} = (\tilde{X}_2^2)_{\tilde{X}_1\tilde{X}_1^{-1}} \cdot (\tilde{X}_2^{-2})_{\tilde{X}_1^{-1}} = \tilde{X}_2^2 \cdot (\tilde{X}_1^{-2})_{\tilde{X}_2}$. Since $c = [\tilde{X}_1^2, \tilde{X}_2^2]$ (see Proposition III.5.2), $(\tilde{X}_1^{-2})_{\tilde{X}_2} = (\tilde{X}_1^{-2})_{\tilde{X}_2^{-1}} c$. Thus: $u_{\tilde{X}_1^{-1}} = \tilde{X}_2^2 \cdot (\tilde{X}_1^{-2})_{\tilde{X}_2^{-1}} c = u^{-1}c$.

(2_a) Since $[\tilde{X}_1^2, \tilde{X}_2^2] = c$, $u_{\tilde{X}_2^{-1}} = (\tilde{X}_1^2)_{\tilde{X}_2^{-2}} \cdot \tilde{X}_2^{-2} = c\tilde{X}_1^2\tilde{X}_2^{-2} = \tilde{X}_2^{-2} \cdot \tilde{X}_1^2$, and $u_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = (\tilde{X}_2^{-2})_{\tilde{X}_1^{-1}} \cdot \tilde{X}_1^2 = (\tilde{X}_1^{-2})_{\tilde{X}_2} \cdot \tilde{X}_1^2 = c(\tilde{X}_1^{-2})_{\tilde{X}_2^{-1}} \cdot \tilde{X}_1^2$.

On the other hand, $u^{-1}u_{\tilde{X}_2^{-1}} = \tilde{X}_2^2 \cdot (\tilde{X}_1^{-2})_{\tilde{X}_2^{-1}} \cdot \tilde{X}_2^{-2} \cdot \tilde{X}_1^2 = c(\tilde{X}_1^{-2})_{\tilde{X}_2^{-1}} \cdot \tilde{X}_1^2$. We get

$$u_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = u^{-1}u_{\tilde{X}_2^{-1}}.$$

(2_b) Using (1) and (2_a), we can write

$$\begin{aligned} u_{\tilde{X}_1\tilde{X}_2^{-1}} &= (u^{-1}c)_{\tilde{X}_2^{-1}} = u^{-1}_{\tilde{X}_2^{-1}} \cdot c, \\ u_{\tilde{X}_1\tilde{X}_2^{-1}\tilde{X}_1^{-1}} &= (u^{-1}c)_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = u^{-1}_{\tilde{X}_2^{-1}}u \cdot c, \\ u^{-1}_{\tilde{X}_1}u_{\tilde{X}_1\tilde{X}_2^{-1}} &= cu \cdot u^{-1}_{\tilde{X}_2^{-1}}c = u^{-1}_{\tilde{X}_2^{-1}}u \cdot c \end{aligned}$$

(we use $[u, u_{\tilde{X}_2^{-1}}] = [(\tilde{X}_1^2)_{\tilde{X}_2^{-1}} \cdot \tilde{X}_2^{-2}, \tilde{X}_2^{-2}\tilde{X}_1^2] = c \cdot c \cdot c = c$). Thus,

$$u_{\tilde{X}_1\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = u^{-1}_{\tilde{X}_1}u_{\tilde{X}_1\tilde{X}_2^{-1}}.$$

(3) Clearly, $\forall j \geq 4$, $u_{\tilde{X}_j} = u$. Consider $u_{\tilde{X}_3^{-1}} = (\tilde{X}_1^2)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}(\tilde{X}_2^{-2})_{\tilde{X}_3^{-1}}$.
Since u can also be written as $u = (\tilde{X}_1^2)_{\tilde{X}_2^{-1}} \cdot \tilde{X}_2^{-2} = \tilde{X}_2^{-2} \cdot (\tilde{X}_1^2)_{\tilde{X}_2^{-1}} \cdot c = \tilde{X}_2^{-2} \cdot (\tilde{X}_1^2)_{\tilde{X}_2}$, we have:

$$\begin{aligned} u_{\tilde{X}_3^{-1}} = u \Leftrightarrow (\tilde{X}_1^2)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} \cdot (\tilde{X}_2^{-2})_{\tilde{X}_3^{-1}} = \tilde{X}_2^{-2} \cdot (\tilde{X}_1^2)_{\tilde{X}_2} \Leftrightarrow \tilde{X}_2^2 \cdot (\tilde{X}_1^2)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} \\ = (\tilde{X}_1^2)_{\tilde{X}_2} \cdot (\tilde{X}_2^2)_{\tilde{X}_3^{-1}} \end{aligned}$$

which is true, because $\{(\tilde{X}_1)_{\tilde{X}_2}, \tilde{X}_2, (\tilde{X}_2)_{\tilde{X}_3^{-1}}, (\tilde{X}_1)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}\}$ form a good quadrangle (see Fig. III.6). □

Construction of $\underline{G}(n)$.

For $n \geq 3$ we define the group $\underline{G}(n)$ as follows:
Generators: $s_1, u_1, u_2, \ldots, u_{n-1}$.
Relations:

$$\begin{aligned} &[s_1, u_i] = 1 \quad \forall i = 1, 3, \ldots n-1; [u_i, u_j] = 1 \text{ when } |i-j| \geq 2; \\ &[s_1, u_2] = [u_i, u_{i+1}] = [u_1, u_2] \ \forall i = 2, 3, \ldots n-2; \\ &[u_1, u_2] = [u_1, u_2]_{s_1} = [u_1, u_2]_{u_i} \ \forall i = 1, 2, \ldots, n-1; \\ &[u_1, u_2]^2 = 1. \end{aligned}$$

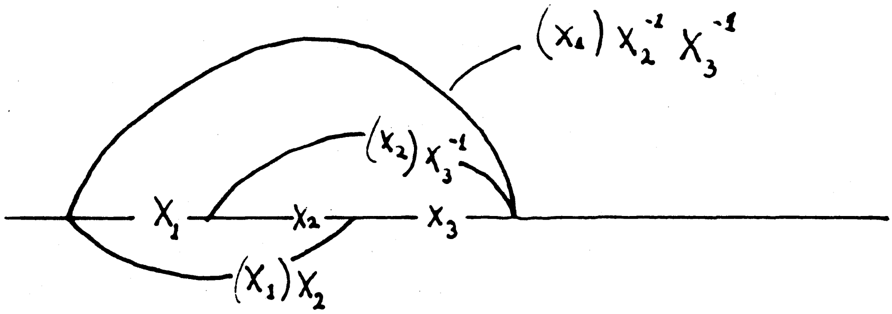


Figure III.6.

Equivalent construction of $\underline{\text{G}}$(n).

Consider a free abelian group $A(n)$ with generators $S_1, V_1, \dots, V_{n-1}$ and a skew-symmetric $\mathbb{Z}/2$- valued bilinear form $Q(x,y)$ on $A(n)$ defined by: $Q(S_1, V_i) = 0\ \forall i = 1, 3, \dots, n-1$; $Q(V_i, V_j) = 0$ when $|i-j| \geq 2$, $Q(S_1, V_2) = Q(V_i, V_{i+1}) = 1\ \forall i = 1, 2, \dots, n-2$. One can check that there exists a unique central extension G of $A(n)$ by $\mathbb{Z}/2$ with $\mathrm{Ab}(G) \simeq A(n)$, $G' \simeq \mathbb{Z}/2$ and such that $\forall x, y \in G[x,y] = Q(\overline{x}, \overline{y})$ where ($\overline{x}$ and $\overline{y}$ are the images of x, y in $A(n)$).

Claim III.6.2. (1) *The above central extension is isomorphic to* $\underline{G}(n)$.

(2) $\mathrm{Ab}(\underline{G}(n))$ *is a free abelian group with* n *generators (i.e.,* $A(n)$*) and* $\underline{G}(n)' \simeq \mathbb{Z}/2$ *(generated by* $[u_1, u_2]$*).*

(3) *The following formulas define a* $\tilde{B}_n$*-action on* $\underline{G}(n)$ *for* $(\tilde{X}_1, \dots, \tilde{X}_{n-1})$*, a standard set of generators in* $\tilde{B}_n$*, and* $\nu = [u_1, u_2]$.

$\tilde{X}_1$-*action*	$\tilde{X}_2$-*action*	$\tilde{X}_k$-*action*, $k \geq 3$
$s_1 \to s_1$	$s_1 \to u_2 s_1$	$s_1 \to s_1$
$u_1 \to u_1^{-1}\nu$	$u_1 \to u_2 u_1$	$u_{k-1} \to u_k u_{k-1}$
$u_2 \to u_1 u_2$	$u_2 \to u_2^{-1}\nu$	$u_k \to u_k^{-1}\nu$
$u_j \to u_j\ \forall j \geq 3$	$u_3 \to u_2 u_3;$	$u_{k+1} \to u_k u_{k+1}$
	$u_j \to u_j\ \forall j \geq 4$	$u_j \to u_j\ \forall j \neq k-1, k, k+1$

(4) *Let* $b \in \tilde{B}_n$, $y = (\tilde{X}_1)_b$. *Then the* y^2*-action on* $\underline{G}(n)$ *coincides with the conjugation by* $(s_1)_b$.

(5) *Let*

(1.20)

$$s_{ij} = \begin{cases} (s_1) & \text{if } (i,j) = (1,2); \\ (s_1)_{\tilde{X}_2 \dots \tilde{X}_{j-1}} \quad \left(\overset{\text{Claim III.6.2(3)}}{=} u_{j-1} \dots u_2 s_1 \right) & \text{if } i = 1,\ j \geq 2; \\ (s_1)_{\tilde{X}_2 \dots \tilde{X}_{j-1} \tilde{X}_1 \dots \tilde{X}_{i-1}} = & \\ \quad = \begin{cases} \nu \cdot u_{j-1} \dots u_1 \cdot u_{i-1} \dots u_2 s_1 & \text{if } i \geq 3,\ j > i \\ \nu \cdot u_{j-1} \dots u_1 s_1 & \text{if } i = 2,\ j > i. \end{cases} \end{cases}$$

Then:

$$[s_{ij}, s_{kl}] = \begin{cases} \nu, & \text{if } (\{i,j\} \cap \{k,l\}) = 1); \\ 1, & \text{otherwise.} \end{cases}$$

(6) *Let* $\tilde{F}_{n-1}$ *be the subgroup of* $\underline{G}(n)$ *generated by* $(s_{n-1,n}, s_{n-2,n}, \dots, s_{1n})$. $\tilde{B}_{n-1}$ *acts on* $\tilde{F}_{n-1}$ *as follows:*

$$\text{(1.21)} \qquad \begin{aligned} (s_{jn})\tilde{X}_k &= s_{jn} \qquad j \neq k, k+1; \quad k = 1 \dots n-1 \\ (s_{k,n})\tilde{X}_k &= s_{k+1,n} \\ (s_{k+1,n})\tilde{X}_k &= s_{kn}\nu = s_{k+1,n} s_{kn} s_{k+1,n}^{-1}. \end{aligned}$$

This action is a $\tilde{B}_{n-1}$*-action, where the action of the generators* $\tilde{X}_{n-2}, \dots \tilde{X}_1$ *of* $\tilde{B}_{n-1}$ *correspond to standard Hurwitz moves on*

$$(s_{n-1,n}, s_{n-1,n}, \dots, s_{1n}).$$

(See the definition of Hurwitz moves in Chapter 0.*)*

(7) *There is a natural chain of embeddings* $\underline{G}(3) \subset \underline{G}(4) \subset \dots \subset \underline{G}(n-1) \subset \underline{G}(n)$ *corresponding to the chain:* $(s_1, u_1, u_2) \subset (s_1, u_1, u_2, u_3) \subset \dots \subset (s_1, u_1, \dots u_{n-1})$.

Proof.

(1), (2), and (3) are easy to verify.

(4) Consider first the case $b = \text{Id}$. From (3) we get for the $\tilde{X}_1^2$-action:

$$s_1 \to s_1,\ u_1 \to u_1,\ u_2 \to u_2\nu,\ u_j \to u_j\ \forall j \geq 3.$$

At the same time by the first construction:

$$(s_1)_{s_1} = s_1,\ (u_1)_{s_1} = u_1,\ (u_2)_{s_1} = s_1^{-1} u_2 s_1 = u_2\nu,\ (u_j)_{s_1} = u_j\ \forall j \geq 3.$$

Thus $\tilde{X}_1^2$-action and s_1-conjugation coincide. Consider now any $b \in \tilde{B}_n$ and any $g \in \underline{G}(n)$. Let $h = g_{b^{-1}}$. We have:

$$g_{(\tilde{X}_1^2)_b} = g_{b^{-1}\tilde{X}_1^2 b} = ((h)_{\tilde{X}_1^2})_b = (h_{s_1})_b = (h_b)_{(s_1)_b} = g_{(s_1)_b}.$$

(5), (6), (7) are easy to verify. □

Lemma III.6.3. *Let $n \geq 3$.*

Let $\{X_1, \ldots, X_{n-1}\}$ be a frame of B_n.

Let

$$Z_{ij} = \begin{cases} X_1 & \text{if } (i,j) = (1,2); \\ (X_1)_{X_2 \ldots X_{j-1}} & \text{if } i = 1,\ j \geq 3; \\ (X_1)_{X_2 \ldots X_{j-1} X_1 \ldots X_{i-1}} & \text{if } i \geq 2,\ j > i. \end{cases}$$

Let $\tilde{Z}_{ij}$ be the image of Z_{ij} in $\tilde{B}_n$.

Consider $\underline{G}(n)$ as a B_n-group as in Claim III.6.2.

Then there exists a unique $\tilde{B}_n$-surjection $\Lambda_n : \tilde{P}_n \to \underline{G}(n)$ with $\Lambda_n(\tilde{X}_1^2) = s_1$ and $\Lambda_n(\tilde{Z}_{ij}^2) = s_{ij}$ for $1 \leq i < j \leq n$.

Proof. Use induction on n.

For $n = 3$, $\Lambda_3 \tilde{P}_3 \to \underline{G}(3)$ must be defined by $\lambda_3(\tilde{Z}_{ij}^2) = s_{ij}$, $1 \leq i < j \leq 3$. One can check directly that Λ_3 is well defined, and that it is a $\tilde{B}_3$-surjection. Uniqueness of such Λ_3 is evident.

Assume now that $n \geq 4$ and that the desired $\Lambda_{n-1} : \tilde{P}_{n-1} \to \underline{G}(n-1)$ exists.

Considering $(X_1, X_2) \subset (X_1, X_2, X_3) \subset \ldots \subset (X_1, \ldots, X_{n-1})$, we get a chain of embeddings $B_3 \subset B_4 \subset \ldots \subset B_n$ and the corresponding chain $P_3 \subset P_4 \subset \ldots \subset P_n$. To the latter corresponds a chain of homomorphisms: $\tilde{P}_3 \xrightarrow{i_3} \tilde{P}_4 \xrightarrow{i_4} \ldots \to \tilde{P}_{n-1} \xrightarrow{i_{n-1}} \tilde{P}_n$, where $\tilde{P}_3$ is obtained (by definition) from P_3 by adding the relations: $[\tilde{Z}_{12}^2, \tilde{Z}_{23}^2] = [\tilde{Z}_{12}^2, \tilde{Z}_{13}^2] = [\tilde{Z}_{23}^2, \tilde{Z}_{13}^2]$ and $[\tilde{Z}_{12}^2, \tilde{Z}_{23}^2]^2 = 1$.

It is known that the set $\{Z_{ij}^2, 1 \leq i < j \leq n\}$ generates P_n, and $P_n \simeq P_{n-1} \ltimes F_{n-1}$, where P_{n-1} is the subgroup of P_n generated by $\{Z_{ij}^2, 1 \leq i < j \leq n-1\}$, F_{n-1} is a free subgroup of P_n generated by $\{Z_{in}^2, 1 \leq i \leq n-1\}$, and the semi-direct product $P_{n-1} \ltimes F_{n-1}$ is defined according to the P_{n-1}-action on F_{n-1} which comes from the B_{n-1}-action by conjugation (using $B_{n-1} \subset B_n \supset P_n$). The latter coincides with the standard B_{n-1}-action on F_{n-1} (the generators $X_{n-2}, \ldots, X_1$ of B_{n-1} correspond to standard *Hurwitz moves* on $(Z_{n-1,n}^2, Z_{n-2,n}^2, \ldots, Z_{1n}^2)$ (see [MoTe4], Chapter 4).

Using canonical $P_{n-1} \to \tilde{P}_{n-1}$, we obtain from Λ_{n-1} a B_{n-1}-surjection $\hat{\Lambda}_{n-1} : P_{n-1} \to \underline{G}(n)$. For the free subgroup F_{n-1} of P_n generated by $\{Z_{in}^2,\ n-1 \leq i \leq 1\}$ define $\mu_{n-1} : F_{n-1} \to \underline{G}(n)$ by $\mu_{n-1}(Z_{in}^2) = s_{in}$. Considering P_n as $P_{n-1} \ltimes F_{n-1}$, we define $\hat{\Lambda}_n : P_n \to \underline{G}(n)$ which on P_{n-1} coincides with $\hat{\Lambda}_{n-1} : P_{n-1} \to \underline{G}(n-1) \subset \underline{G}(n)$ (see Claim III.6.2(7)) and on F_{n-1} coincides with $\mu_{n-1} : F_{n-1} \to \underline{G}(n)$. To show that such $\hat{\Lambda}_n$ exists one has to check the following:

1) The conjugation of $\mu_{n-1}(F_{n-1})$ by elements of $\hat{\Lambda}_{n-1}(P_{n-1})(\subset \underline{G}(n))$ coincides with the P_{n-1}-action defined by $P_{n-1} \subset B_{n-1} \subset B_n \to \tilde{B}_n$ and the given $\tilde{B}_n$-action on $\underline{G}(n)$. That is, $\forall f \in \mu_{n-1}(F_{n-1})$ and $\forall h$ of the form $\hat{\Lambda}_{n-1}(\tilde{Y})$ $(Y \in P_{n-1})$ we must have $h^{-1}fh = f_{\tilde{Y}}$.

2) The P_{n-1}-action on $\mu_{n-1}(F_{n-1})$ (defined by $P_{n-1} \subset B_{n-1} \subset B_n \to \tilde{B}_n$ and the given $\tilde{B}_n$-action on $\underline{G}(n)$) comes from B_{n-1}-action on $\mu_{n-1}(F_{n-1})$ in which $X_{n-2}, \ldots X_1$ correspond to the standard *Hurwitz moves* on $(s_{n-1,n}, s_{n-2,n}, \ldots s_{1n})$.

Proof of 1). Since $\forall b \in B_{n-1}$, $\hat{\Lambda}_{n-1}((X_1^2)_b) = (s_1)_b$ we see from Claim III.6.2 that $\forall f \in \mu_{n-1}(F_{n-1})$, $f_{\hat{\Lambda}_{n-1}((X_1^2)_b)} = (f_{s_1})_b = f_{(\tilde{X}_1^2)_b}$. Since P_{n-1} is generated by $\{Z_{ij}^2, 1 \leq i < j \leq n-1\}$, i.e., by $\{(X_1^2)_b, b \in B_n\}$ we get 1). □

Proof of 2). It follows immediately from Claim III.6.2(6). □

Thus, 1) and 2) are true and we can extend $\hat{\Lambda}_{n-1}$, μ_{n-1} to a homomorphism $\hat{\Lambda}_n : P_n(= P_{n-1} \ltimes F_{n-1}) \to \underline{G}(n)$ such that for $1 \leq i < j \leq n-1$ $\hat{\Lambda}_n(Z_{ij}^2) = \hat{\Lambda}_{n-1}(Z_{ij}^2) = s_{ij}$, and for $1 \leq i \leq n-1$ $\hat{\Lambda}_n(Z_{in}^2) = \mu_{n-1}(Z_{in}^2) = s_{in}$, in short $\hat{\Lambda}_n(Z_{ij}^2) = s_{ij}$ for $1 \leq i < j \leq n$.

Using induction, one can check directly that $\hat{\Lambda}_n$ is a B_n-homomorphism (recall that by Claim III.6.2 we have explicit formulas for s_{ij}'s).

Because $s_{1n} = u_{n-1} \cdot u_2 s_1$ (by 1.20), that is, $u_{n-1} = s_{1n}(u_{n-2} \ldots u_2 s)^{-1}$, we see that $\underline{G}(n)$ is generated by $\underline{G}(n-1) = \hat{\Lambda}_n(P_{n-1})$ $(= \hat{\Lambda}_{n-1}(P_{n-1}))$ and $s_{1n} = \hat{\Lambda}_n(Z_{1n}^2)$. Therefore $\hat{\Lambda}_n$ is a B_n-surjection.

Let $N = \ker(B_n \to \tilde{B}_n)$ $(= \ker(P_n \to \tilde{P}_n))$. Let $T = X_1^2 X_3^2 X_2^{-2} Z_{14}^{-2}$. Clearly, N is generated by $\{T_b, b \in B_n\}$. We have $\hat{\Lambda}_n(T) = \hat{\Lambda}_4(T) = s_1 \cdot s_{34} \cdot s_{23}^{-1} s_{14}^{-1} \overset{\text{Claim III.6.2}}{=} s_1 \cdot \eta u_3 u_2 u_1 \cdot u_2 s_1 \cdot s_1^{-1} u_1^{-1} u_2^{-1} \eta \cdot s_1^{-1} u_2^{-1} u_3^{-1} = s_1 \eta u_3 u_2 \cdot s_1^{-1} u_2^{-1} u_3^{-1} = \text{Id}$. Since $\hat{\Lambda}_n$ is a B_n-homomorphism, we get $\hat{\Lambda}_n(T_b) = \text{Id}$ $\forall b \in B_n$, and thus $\hat{\Lambda}_n(N) = \text{Id}$. Hence $\hat{\Lambda}_n$ defines canonically a $\tilde{B}_n$-surjection $\Lambda_n : \tilde{P}_n \to \underline{G}(n)$ with $\Lambda_n(\tilde{X}_1^2) = s_1$.

Uniqueness of such Λ_n follows from the fact that $\tilde{P}_n$ is generated by the B_n-orbit of $\tilde{X}_1^2$. $\square$

Theorem III.6.4. *There exists a unique $\tilde{B}_n$-isomorphism $\Lambda_n : \tilde{P}_n \to \underline{G}(n)$ with $\Lambda_n(\tilde{X}_1^2) = s_1$. In particular:*

(1) $\mathrm{Ab}\,\tilde{P}_n$ *is a free abelian group with n generators, $\tilde{P}'_n \simeq \mathbb{Z}/2$ (generated by $c = [\tilde{X}_1^2, \tilde{X}_2^2]$);*

(2) *$\tilde{P}_{n,0}$ is $\tilde{B}_n$-isomorphic to the subgroup $G_0(n)$ of $\underline{G}(n)$, generated by $u_1, \dots, u_{n-1}$, $\mathrm{Ab}\,\tilde{P}_{n,0}$ is a free abelian group with $n-1$ generators $\{\xi_1, \dots, \xi_{n-1}\}$, $\tilde{P}'_{n,0} \simeq \mathbb{Z}/2$, generated by $c = [\xi_1, \xi_2]$; ($\nu \in G_0(n)$ corresponding to $c \in \tilde{P}_{n,0}$).*

(3) *$\tilde{P}_{n,0}$ is a primitive $\tilde{B}_n$-group generated by the $\tilde{B}_n$-orbit of a prime element $u = \tilde{X}^2\tilde{Y}^{-2}$, where $\tilde{X}, \tilde{Y}$ are adjacent half-twists in $\tilde{B}_n$, $\tilde{T} = \tilde{X}\tilde{Y}\tilde{X}^{-1}$ is a supporting half-twist for u.*

Proof. Clearly, $\tilde{P}_{n,0}$ is generated by $\{\tilde{X}_1^2\tilde{Z}_{ij}^{-2}, l \le i < j \le n\}$. Because $\tilde{X}_1^2 \cdot \tilde{Z}_{ij}^{-2} = \tilde{X}_1^2\tilde{Z}_{1i}^{-2} \cdot \tilde{Z}_{1i}^2 \cdot \tilde{Z}_{ij}^{-2}$ and both $\tilde{X}_1^2\tilde{Z}_{1i}^{-2}$, $\tilde{Z}_{1i}^2 \cdot \tilde{Z}_{ij}^{-2}$ are conjugates of u, we see that $\tilde{P}_{n,0}$ is generated by the $\tilde{B}_n$- orbit of u, and since u is a prime element of $P_{n,0}$, this means that $\tilde{P}_{n,0}$ is a primitive $\tilde{B}_n$-group. By Lemma III.6.1, the s.h.t. of u is $Y^{-1}XY$. Thus, we proved (3).

Polarize each X_i (and $\tilde{X}_i$) according to the sequence $(X_1, \dots X_{n-1})$ (the "end" of X_i = the "origin" of X_{i+1}). By Proposition III.3.3 $\forall i = 1, \dots, n-1$ $\exists$ unique prime element $\xi_i = L_{\{u,\tilde{X}_1\}}(\tilde{X}_i) \in \tilde{P}_{n,0}$ such that $\{\xi_i, \tilde{X}_i\}$ is coherent with $\{u, \tilde{X}_1\}$. Clearly $\xi_1 = u$.

By Proposition III.4.1 we have $\forall i = 1, \dots, n-1$:

$$(\xi_i)_{\tilde{X}_i^{-1}} = \xi_i^{-1}c; \; (\xi_i)_{\tilde{X}_{i-1}^{-1}} = \xi_i\xi_{i-1}; \; = (\xi_i)_{\tilde{X}_{i+1}^{-1}} = \xi_i\xi_{i+1}. \tag{1.22}$$

It is clear also that $\forall j \ne i,\ i-1, i+1$

$$(\xi_i)_{\tilde{X}_j} = \xi_i. \tag{1.23}$$

We see from (1.22), (1.23) that the subgroup of $\tilde{P}_{n,0}$ generated by $(\xi_1, \dots, \xi_{n-1})$ is closed under the $\tilde{B}_n$-action. Since $\tilde{P}_{n,0}$ is generated by the $\tilde{B}_n$-orbit of $u = \xi_1$, we conclude that $\tilde{P}_{n,0}$ is generated by $(\xi_1, \dots, \xi_{n-1})$. This implies that $\tilde{P}_n$ is generated by $(\tilde{X}_1^2, \xi_1, \xi_2, \dots, \xi_{n-1})$.

We have $\xi_2 = L_{\{u,\tilde{X}_1\}}(\tilde{X}_2) = (\xi_1)_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = (\tilde{X}_1^{-2})_{\tilde{X}_2} \cdot \tilde{X}_1^2$, which implies that

$$[\tilde{X}_1^2, \xi_2] = c. \tag{1.24}$$

By Lemma III.5.1 we have

$$[\xi_i, \xi_j] = \begin{cases} c & \text{if } |i-j| = 1 \\ 1 & \text{if } |i-j| \geq 2. \end{cases} \tag{1.25}$$

Observe also that

$$(\tilde{X}_1^2)_{\tilde{X}_2} = \xi_2 \cdot \tilde{X}_1^2. \tag{1.26}$$

Formulas (1.22)–(1.26) show that we can define a $\tilde{B}_n$-homomorphism $M_n : \underline{G}(n) \to \tilde{P}_n$ with $M_n(s_1) = \tilde{X}_1^2$, $M_n(u_i) = \xi_i$, $i = 1, \ldots, n-1$. (See Claim III.6.2.)

Since $\tilde{P}_n$ is generated by the $\tilde{B}_n$-orbit of $\tilde{X}_1^2$ and $\underline{G}(n)$ is generated by the $\tilde{B}_n$-orbit of s_1, we conclude that Λ_n and M_n are inverses of each other. □

III.7. Criterion for prime element.

Proposition III.7.1. *Assume $n \geq 5$. Let G be a $\tilde{B}_n$-group,*

$$(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{n-1})$$

be a standard base of $\tilde{B}_n$. Let S be an element of G with the following properties:

(0) *G is generated by $\{S_b, b \in \tilde{B}_n\}$;*

(1_a) $S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = S^{-1} S_{\tilde{X}_2^{-1}}$;

(1_b) $S_{\tilde{X}_1\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = S_{\tilde{X}_1}^{-1} S_{\tilde{X}_1\tilde{X}_2^{-1}}$;

(2) *For $\tau = S S_{\tilde{X}_1^{-1}}$, $T = S_{\tilde{X}_2^{-1}}$ we have:*

(2_a) $\tau_{\tilde{X}_1^2} = \tau$;

(2_b) $\tau_T = \tau_{\tilde{X}_1}^{-1}$;

(3) $S_{\tilde{X}_j} = S\ \forall j \geq 3$;

(4) $S_c = S$, *where* $c = [\tilde{X}_1^2, \tilde{X}_2^2]$.

Then S *is a prime element of* G, $\tilde{X}_1$ *is a supporting half-twist of* S *and* τ *is the corresponding central element. In particular,* $\tau^2 = 1$, $\tau \in \mathrm{Center}(G)$, $\tau_b = \tau\ \forall\ b \in \tilde{B}_n$.

Proof. The proof includes several lemmas. From Theorem III.5.2, $c \in \mathrm{Center}(\tilde{B}_n)$, $c^2 = 1$. From Theorem III.6.4 it follows that $\tilde{P}'_n$ is generated by c. $\tilde{P}'_n$ is a normal subgroup of $\tilde{B}_n$. Denote by $\tilde{\tilde{B}}_n = \tilde{B}_n/\tilde{P}'_n$, $\tilde{\tilde{P}}_n = \tilde{P}_n/\tilde{P}'_n = \mathrm{Ab}\,\tilde{P}_n$. $\tilde{\tilde{P}}_n$ is a commutative group. We have $\tilde{\tilde{\psi}}_n : \tilde{\tilde{B}}_n \twoheadrightarrow S_n$. By abuse of notaion we use ψ_n for $\tilde{\tilde{\psi}}_n$. Let $Y \in B_n$. By abuse of notation we denote the image of Y in $\tilde{B}_n$ or in $\tilde{\tilde{B}}_n$ or in $\tilde{\tilde{P}}_n$ by the same symbol $\tilde{Y}$. It is clear that $\tilde{B}_n$ acts on $\tilde{\tilde{P}}_n$ (through conjugations) as the symmetric group $S_n = \tilde{B}_n/\tilde{P}_n$.

Since $S_c = S$ and $c \in \mathrm{Center}\,\tilde{B}_n$, we have $\forall b \in \tilde{B}_n$ $(S_b)_c = S_{bc} = (S_c)_b = S_b$. Since G is generated by $\{S_b, b \in \tilde{B}_n\}$ we have $\forall g \in G\ \ g_c = g$. In particular, we conclude that $\tilde{B}_n$ acts on G as its quotient $\tilde{\tilde{B}}_n$; in other words, G is a $\tilde{\tilde{B}}_n$-group.

Let (D, K) be a model for B_n, $K = \{a_1, \dots, a_n\}$, $B_n = B_n[D, K]$. Take any $a_{i_1}, a_{i_2} \in K$. Let γ_1, γ_2 be two different simple paths in $D-(K-a_{i_1}-a_{i_2})$ connecting a_{i_1} with a_{i_2}, let $H(\gamma_1), H(\gamma_2)$ be the half-twists corresponding to γ_1, γ_2, and let $\tilde{H}(\gamma_1), \tilde{H}(\gamma_2)$ be the images of $H(\gamma_1), H(\gamma_2)$ in $\tilde{\tilde{B}}_n$. □

Lemma 1. *Let* γ_1, γ_2 *be 2 simple paths in* $D - \{K - a_{i_1} - a_{i_2}\}$ *connecting* a_{i_1} *with* a_{i_2}. *Then:* $\tilde{H}(\gamma_1)^2 = \tilde{H}(\gamma_2)^2$.

Proof of Lemma 1. Choose a frame of B_n $(Y_1, \dots, Y_{n-1})$ s.t. $Y_1 = H(\gamma_1)$. Let $b \in B_n$ s.t. $\gamma_2 = (\gamma_1)b$, that is $H(\gamma_1)_b = H(\gamma_2)$. Let $\tilde{Y}_i$ be the image of Y_i in $\tilde{\tilde{B}}_n$.

Let σ_1 be the image of b in S_n. Since $(a_i)b = a_i$, $(a_j)b = a_j$, $\sigma_1 \in \mathrm{Stab}(i) \cap \mathrm{Stab}(j)$ in S_n. The subgroup of $\tilde{\tilde{B}}_n$ generated by $\tilde{Y}_3, \dots, \tilde{Y}_{n-1}$ is mapped by $\tilde{\tilde{\psi}}_n : \tilde{\tilde{B}}_n \to S_n$ onto $\mathrm{Stab}(i) \cap \mathrm{Stab}(j)$. Choose $\tilde{b}_1$ in this subgroup with its image in S_n equal to σ_1. Clearly, $(\tilde{Y}_1)_{\tilde{b}_1} = \tilde{Y}_1$. Since the image of $\tilde{b}_1^{-1}\tilde{b}$ in S_n is equal to $\sigma_1^{-1}\sigma_1 = \mathrm{Id}$, we have $\tilde{b}_1^{-1}\tilde{b} \in \tilde{\tilde{P}}_n$. Since $\tilde{\tilde{P}}_n$

is commutative when considering $\tilde{Y}_1^2$ as an element of $\tilde{\tilde{P}}_n$, $(\tilde{Y}_1^2)_{b_1^{-1}\tilde{b}} = \tilde{Y}_1^2$. Thus, we have

$$\tilde{H}(\gamma_2)^2 = \tilde{H}(\gamma_1)^2_{\tilde{b}} = (\tilde{Y}_1^2)_{\tilde{b}_1\tilde{b}_1^{-1}\tilde{b}} = (\tilde{Y}_1^2)_{\tilde{b}_1^{-1}\tilde{b}} = \tilde{Y}_1^2 = \tilde{H}(\gamma_1)^2 \quad \square \text{ Lemma 1}$$

Definition. f_{ij}

$\forall i,j \in (1,\dots,n)$, $i \neq j$, we define $f_{ij} \in \tilde{\tilde{P}}_n$ as follows: Take any simple path γ in $D-(K-a_i-a_j)$ connecting a_i with a_j. Let $f_{ij} = \tilde{H}(\gamma)^2$. Lemma 1 shows that this definition does not depend on the choice of γ. We choose for $i<j$:

$$f_{ij} = \begin{cases} (\tilde{X}_i^2)_{X_{i+1}\cdot\dots\cdot X_{j-1}} & 2<j-i \\ \tilde{X}_i^2 & i+1=j. \end{cases}$$

It is clear that for σ_1 the image in S_n of $b \in \tilde{B}_n$ we have:

$$(f_{ij})_b = f_{(i)\sigma_1,(j)\sigma_1}.$$

It is clear from our choice of γ for $f_{ij} = \tilde{H}(\gamma)^2$ that:

$$\psi_n(\tilde{H}(\gamma)) = (i,j).$$

It will be convenient to use the following notation for $g \in G$ and $b \in \tilde{\tilde{B}}_n$:

Notation. $[g,b]$: For $g \in G$, $b \in \tilde{\tilde{B}}_n$ and the action of $\tilde{\tilde{B}}_n$ on G, we denote $[g,b] = g \cdot g_{b^{-1}}^{-1}$.

One can check that:

$$\begin{aligned}
&g_b = g \Leftrightarrow [g,b]=1 \\
&[g,b]_z = g_z(g_z)^{-1}_{b_z^{-1}} \\
&[g^{-1},b] = [g,b]_g^{-1} \\
&[g,b^{-1}] = [g,b]_b^{-1} \\
&[g_1g_2,b] = [g_2,b]_{g_1^{-1}} \cdot [g_1,b] \\
&[g,b_1b_2] = [g,b_1]\cdot[g,b_2]_{b_1^{-1}}.
\end{aligned}$$

Notation. *derlineQ*$_{b,l,m}$

$\forall b \in \tilde{\tilde{B}}_n$, $\forall l,m \in (1,\dots,n)$, $l \neq m$, we denote $Q_{b,l,m} = [S_b, f_{lm}^{-1}]$.

Lemma 2. (i) *Let* $b \in \tilde{\tilde{B}}_n$ *be such that* $(\{1,2\})\psi_n(b) \cap \{l,m\} = \emptyset$. *Then* $Q_{b,l,m} = 1$.

(ii) *Let* $Q = Q_{\mathrm{Id},1,3} = [S, f_{13}^{-1}]$. *Then* $Q_{\tilde{X}_2^{-1}} = Q$.

Proof of Lemma 2.

(i) Let $\{l_1, m_1\} = (\{l,m\})\psi_n(b)^{-1}$. So $(f_{lm})_{b^{-1}} = f_{l_1 m_1}$. We have $\{1,2\} \cap \{l_1, m_1\} = \emptyset$, that is, $3 \le l_1, 3 \le m_1$. By our choice, f_{lm} is a product o X_j for $j \ge 3$. Thus, using property (3) of S we get $S_{f_{l_1,m_1}} = S$. In other words, $[S, f_{l_1,m_1}^{-1}] = 1$. We get

$$(Q_{b,l,m})_{b^{-1}} = [S_b, f_{lm}^{-1}]_{b^{-1}} = [S, (f_{lm}^{-1})_{b^{-1}}] = [S, f_{l_1,m_1}^{-1}] = 1,$$

and so $Q_{b,l,m} = 1$. □ Lemma 2(i)

(ii) From $S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = S^{-1} S_{\tilde{X}_2^{-1}}$ (assumption (1_a) of the Proposition) it follows that $S_{\tilde{X}_2^{-1}} = S S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}}$. Applying $\tilde{X}_3^{-1}$, we get

$$S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = S S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}} \tag{1.27}$$

which, after applying $\tilde{X}_2^{-1}$, gives:

$$S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}} = S_{\tilde{X}_2^{-1}} S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}}.$$

Since $S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}} = S_{\tilde{X}_3^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}$, we obtain $S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = S_{\tilde{X}_2^{-1}} S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}}$, or

$$S_{\tilde{X}_2^{-1}} = S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} S^{-1}_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}} \tag{1.28}$$

Let $b_1 = \tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}\tilde{X}_2^{-1}$. Observing that $(f_{13})_{\tilde{X}_2^{-1}} = f_{12}$, we get from (1.28): $Q_{\tilde{X}_2^{-1}} = [S_{\tilde{X}_2^{-1}}, (f_{13}^{-1})_{\tilde{X}_2^{-1}}] = [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} S_{b_1}^{-1}, f_{12}^{-1}]$. Thus:

$$Q_{\tilde{X}_2^{-1}} = [S_{b_1}^{-1}, f_{12}^{-1}]_{S^{-1}_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}} \cdot [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{12}^{-1}]. \tag{1.29}$$

Since $\psi_n(b_1) = (2 \quad 3)\ (1 \quad 2)\ (3 \quad 4)\ (2 \quad 3)$ (products of transpositions), $(\{1,2\})\psi_n(b_n) = \{3,4\}$. Since $\{3,4\} \cap \{1,2\} = \emptyset$, we get from (i) that $Q_{b_1,1,2} = 1$. Thus, $[S_{b_1}^{-1}, f_{12}^{-1}] = [S_{b_1}, f_{12}^{-1}]_{S_{b_1}}^{-1} = (Q_{b_1,1,2}^{-1})_{S_{b_1}}$. (1.29) now gives:

$$Q_{\tilde{X}_2^{-1}} = [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{12}^{-1}]. \tag{1.30}$$

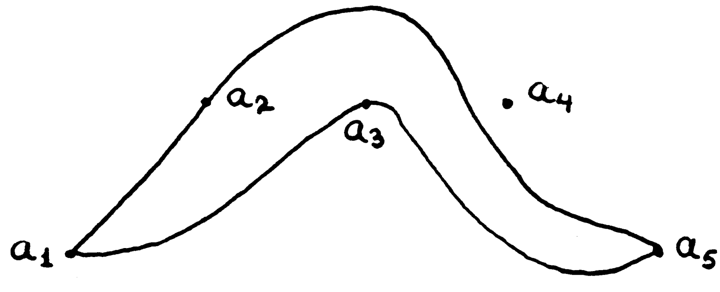


Figure III.7.1.

Consider a quadrangle formed by $\{a_1, a_2, a_3, a_5\}$, as in Fig. III.7.1. By Lemma III.1.2, we can write in $\tilde{\tilde{P}}_n$: $f_{35}f_{12} = f_{25}f_{13}$, or $f_{12} = f_{35}^{-1}f_{25}f_{13}$, $f_{12}^{-1} = f_{13}^{-1}f_{25}^{-1}f_{35}$.

From (1.30) we get:

(1.31)

$$Q_{\tilde{X}_2^{-1}} = [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{13}^{-1}f_{25}^{-1}f_{35}] = [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{13}^{-1}] \cdot [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{25}^{-1}f_{35}]_{f_{13}}$$

Consider $[S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{25}^{-1}f_{35}] = [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{25}^{-1}] \cdot [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{35}]_{f_{25}} = Q_{\tilde{X}_2^{-1}\tilde{X}_3^{-1},2,5}\cdot[S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{35}^{-1}]^{-1}_{f_{35}^{-1}f_{25}} = Q_{\tilde{X}_2^{-1}\tilde{X}_3^{-1},2,5}\cdot(Q^{-1}_{\tilde{X}_2^{-1}\tilde{X}_3^{-1},3,5})_{f_{35}^{-1}f_{25}}$. Since $\psi(\tilde{X}_2^{-1}\tilde{X}_3^{-1}) = (2\ \ 3)(3\ \ 4)$, the images of $\{1,2\}$ under it are $\{1,4\}$. But $\{1,4\} \cap \{2,5\} = \emptyset$ and $\{1,4\} \cap \{3,5\} = \emptyset$. Thus, we get by (i) that $Q_{\tilde{X}_2^{-1}\tilde{X}_3^{-1},2,5} = Q_{\tilde{X}_2^{-1}\tilde{X}_3^{-1},3,5} = 1$, and so $[S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{25}^{-1}f_{35}] = 1$. (1.31) now implies $Q_{\tilde{X}_2^{-1}} = [S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{13}^{-1}]$. By (1.27) $S_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = S \cdot S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}}$ which gives

$$\begin{aligned} Q_{\tilde{X}_2^{-1}} &= [S \cdot S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}}, f_{13}^{-1}] = [S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1}}, f_{13}^{-1}]_{S^{-1}} \cdot [S, f_{13}^{-1}] \\ &= (Q_{\tilde{X}_2^{-1}\tilde{X}_1^{-2}\tilde{X}_3^{-1},1,3})_{S^{-1}} \cdot Q. \end{aligned}$$

The value of $\psi(\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1})$ $(= (2\ \ 3)(1\ \ 2)(3\ \ 4))$ on $\{1,2\}$ is $\{2,4\}$. Since $\{2,4\} \cap \{1,3\} = \emptyset$ we get from part (i) of the Lemma that $Q_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}\tilde{X}_3^{-1},1,3} = 1$, therefore,

$$Q_{\tilde{X}_2^{-1}} = Q \square \text{ Lemma 2(ii)}$$

Lemma 3. $\tau = Q^{-1}$.

Proof. By the assumption on τ, $S_{\tilde{X}_1^{-1}} = S^{-1}\tau$. By definition of T, $T_{\tilde{X}_1^{-2}} = S_{\tilde{X}_2^{-1}\tilde{X}_1^{-2}}$. We apply assumption (1_a) twice to get, using $S_{\tilde{X}_1^{-1}} = S^{-1}\tau$, that

$(S^{-1}S_{\tilde{X}_2^{-1}})_{\tilde{X}_1^{-1}} = S^{-1}_{\tilde{X}_1^{-1}}S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = \tau^{-1}S \cdot S^{-1}S_{\tilde{X}_2^{-1}} = \tau^{-1}S_{\tilde{X}_2^{-1}} = \tau^{-1}T.$
Thus

$$T_{\tilde{X}_1^{-2}} = \tau^{-1}T,$$

or

$$\tau^{-1} = T_{\tilde{X}_1^{-2}}T^{-1}. \tag{1.32}$$

Applying $\tilde{X}_1^2$ on (1.32) and using $\tau_{\tilde{X}_1^2} = \tau$ (assumption (2_a)), we get

$$\begin{aligned}\tau^{-1} &= T \cdot T^{-1}_{\tilde{X}_1^2} = [T, \tilde{X}_1^{-2}] = [S_{\tilde{X}_2^{-1}}, f_{12}^{-1}] = [S, (f_{12}^{-1})_{\tilde{X}_2}]_{\tilde{X}_2^{-1}} = [S, f_{13}^{-1}]_{\tilde{X}_2^{-1}} \\ &= Q_{\tilde{X}_2^{-1}} \overset{\text{by Lemma 2}}{=} Q,\end{aligned}$$

that is,

(1.33)□ Lemma 3 $$\tau^{-1} = Q, \quad \text{or} \quad \tau = Q^{-1}.$$

Lemma 4. $\forall j \geq 3$, $\tau_{\tilde{X}_j} = \tau$.

Proof of Lemma 4. From $\tau = SS_{\tilde{X}_1^{-1}}$ and $S_{\tilde{X}_j} = S$ $\forall j \geq 3$ it follows that $\tau_{\tilde{X}_j} = \tau$ $\forall j \geq 3$. □ Lemma 4

Lemma 5. $\tau_{\tilde{X}_1} = \tau$.

Proof of Lemma 5. Let us use now $\tau^{-1}_{\tilde{X}_1} = \tau_T$ ((2_b) of the Proposition).

By Lemmas 2 and 3 $\tau_{\tilde{X}_2} = \tau$.

Thus, $\tau_T = \tau_{S_{\tilde{X}_2^{-1}}} = S^{-1}_{\tilde{X}_2^{-1}}\tau S_{\tilde{X}_2^{-1}} = (S^{-1}\tau S)_{\tilde{X}_2^{-1}} = (\tau_S)_{\tilde{X}_2^{-1}}$.

So $\tau_{\tilde{X}_1^{-1}} = \tau_{\tilde{X}_1} = \tau_T^{-1} = (\tau_S^{-1})_{\tilde{X}_2^{-1}}$, or

$$\tau_S = \tau^{-1}_{\tilde{X}_1^{-1}\tilde{X}_2}. \tag{1.34}$$

Since $\tau_{\tilde{X}_3} = \tau$ and $S_{\tilde{X}_3} = S$, we get $(\tau_S)_{\tilde{X}_3} = \tau_S$ and

$$\tau_{\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_3} = (\tau_S^{-1})_{\tilde{X}_3} = \tau_S^{-1} = \tau_{\tilde{X}_1^{-1}\tilde{X}_2}. \tag{1.35}$$

Applying $\tilde{X}_2^{-1}$ on (1.35), we get $\tau_{\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_3\tilde{X}_2^{-1}} = \tau_{\tilde{X}_1^{-1}}$. Since $\tau_{X_3} = \tau$ and $\langle X_2, X_3\rangle = 1$, $\tau_{\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_3\tilde{X}_2^{-1}} = \tau_{\tilde{X}_1^{-1}\tilde{X}_3^{-1}\tilde{X}_2\tilde{X}_3} = \tau_{\tilde{X}_3^{-1}\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_3} = \tau_{\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_3}$.
Thus:

(1.36). $$\tau_{\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_3} = \tau_{\tilde{X}_1^{-1}}$$

Combining formulas (1.35)-(1.36) we get $\tau_{\tilde{X}_1^{-1}} = \tau_{\tilde{X}_1^{-1}\tilde{X}_2}$. Applying it to $\tilde{X}_1$ we get $\tau = \tau_{\tilde{X}_1^{-1}\tilde{X}_2\tilde{X}_1} = \tau_{\tilde{X}_2\tilde{X}_1\tilde{X}_2^{-1}} = \tau_{\tilde{X}_1\tilde{X}_2^{-1}}$. Thus $\tau = \tau_{\tilde{X}_1\tilde{X}_2^{-1}}$, or $\tau_{\tilde{X}_1} = \tau_{\tilde{X}_2} = \tau$. □ Lemma 5

Lemma 6. $\tau_{\tilde{X}_j} = \tau \ \forall j = 1, 2, \dots, n-1$.

Proof of Lemma 6. By Lemmas 2, 3, 4, 5. □

Lemma 7. $\tau_S = \tau^{-1}$.

Proof of Lemma 7. From $\tau_{\tilde{X}_1} = \tau_{\tilde{X}_2} = \tau$ and (1.34). □

Lemma 8. $\tau_S = \tau$.

Proof of Lemma 8. Consider assumption (1_b) of the Proposition:

$$S_{\tilde{X}_1\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = S_{\tilde{X}_1}^{-1} S_{\tilde{X}_1\tilde{X}_2^{-1}}.$$

Using $S_{\tilde{X}_1^{-1}} = S^{-1}\tau$ and $\tau_{\tilde{X}_1} = \tau$, we get $S = S_{\tilde{X}_1}^{-1}\tau$, or $S_{\tilde{X}_1}^{-1} = S\tau^{-1}$, $S_{\tilde{X}_1} = \tau S^{-1}$. Assumption (1_b) now gives (using $\tau_{\tilde{X}_i} = \tau \ \forall i = 1, \dots, n-1$):

$$\tau S^{-1}_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = S\tau^{-1} \cdot \tau S^{-1}_{\tilde{X}_2^{-1}} = SS^{-1}_{\tilde{X}_2^{-1}} = ST^{-1}.$$

On the other hand, by (1_a) and (2), $S_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = S^{-1}T$. Thus

$$\tau S^{-1}_{X_2^{-1}} X_1^{-1} = \tau T^{-1} S.$$

We compare the last 2 expressions to get $\tau T^{-1} S = ST^{-1}$, or

(1.37) $$\tau = ST^{-1}S^{-1}T, \quad \text{or} \quad T^{-1}_{S^{-1}} = \tau T^{-1}.$$

By Lemmas 3 and 6

$$Q = Q_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = [S_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, (f_{13}^{-1})_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}}].$$

Thus:

(1.38) $$Q = [S_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{24}^{-1}]$$

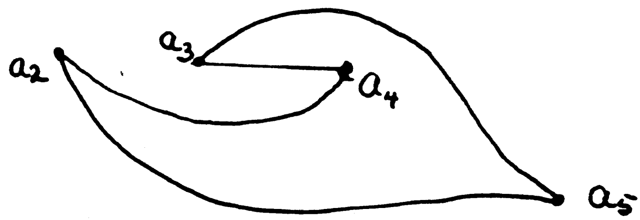


Figure III.7.2.

(we use $(\{1,3\})\psi(\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}) = (\{1,3\})(1\quad 2)(2\quad 3)(3\quad 4) = \{4,2\}$).

Considering a quadrangle formed by a_2, a_3, a_4, a_5 (see Fig. III.7.2) we can write in $\tilde{\tilde{P}}_n$ (Lemma III.2.2) $f_{35}f_{24} = f_{25}f_{34}$, or $f_{24} = f_{35}^{-1}f_{25}f_{34}$, $f_{24}^{-1} = f_{34}^{-1}f_{25}^{-1}f_{35}$. From (1.38) we get, denoting by $b = \tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}$,

(1.39)

$$\begin{aligned} Q &= [S_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{24}^{-1}] = [S_b, f_{34}^{-1}f_{25}^{-1}f_{35}] \\ &= [S_b, f_{34}^{-1}][S_b, f_{25}^{-1}f_{35}]_{f_{34}} = Q_{b,3,4}\cdot[S_b, f_{25}^{-1}]_{f_{34}}\cdot[S_b, f_{35}]_{f_{25}f_{34}} \\ &= Q_{b,3,4}\cdot(Q_{b,2,5})_{f_{34}}\cdot[S_b, f_{35}^{-1}]^{-1}_{f_{35}^{-1}f_{25}f_{34}} = Q_{b,3,4}\cdot(Q_{b,2,5})_{f_{34}}\cdot(Q_{b,3,5})^{-1}_{X}. \end{aligned}$$

Now, $(\{1,2\})\psi(b) = \{1,2\}(1\quad 2)(2\quad 3)(3\quad 4) = \{4,1\}$. Since $\{4,1\}\cap\{2,5\} = \emptyset$ and $\{4,1\}\cap\{3,5\} = \emptyset$, we get by Lemma 2 that $Q_{b,2,5} = Q_{b,3,5} = 1$, and by (1.39)

$$Q = Q_{b,3,4}.$$

We can write:

$$S_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = (S^{-1}\tau)_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}} = S^{-1}_{\tilde{X}_2^{-1}\tilde{X}_3^{-1}}\tau = T^{-1}_{\tilde{X}_3^{-1}}\tau$$

(using $\tau_{\tilde{X}_i} = \tau\ \forall i$). So

$$\begin{aligned} Q = Q_{b,3,4} = [S_{\tilde{X}_1^{-1}\tilde{X}_2^{-1}\tilde{X}_3^{-1}}, f_{34}^{-1}] &\overset{\text{by } f_{34}=\tilde{X}_3^2}{=} [T^{-1}_{\tilde{X}_3^{-1}}\tau, \tilde{X}_3^{-2}] \\ &= [\tau, \tilde{X}_3^{-2}]_{T_{\tilde{X}_3^{-1}}}\cdot[T^{-1}_{\tilde{X}_3^{-1}}, \tilde{X}_3^{-2}] = [T^{-1}_{\tilde{X}_3^{-1}}, \tilde{X}_3^{-2}] = [T^{-1}, \tilde{X}_3^{-2}]_{\tilde{X}_3^{-1}}. \end{aligned}$$

Since $Q_{\tilde{X}_3} = Q$, we get

(1.40)
$$Q = [T^{-1}, \tilde{X}_3^{-2}].$$

This implies

$$\begin{aligned}Q_{S^{-1}} &= (T^{-1}T_{\tilde{X}_3^2})_{S^{-1}} \overset{\text{assumption 3}}{=} T_{S^{-1}}^{-1} \cdot (T_{S^{-1}})_{\tilde{X}_3^2} \\ &= [T_{S^{-1}}^{-1}, \tilde{X}_3^{-2}] \overset{\text{by (1.37)}}{=} [\tau T^{-1}, \tilde{X}_3^{-2}] \\ &= [T^{-1}, \tilde{X}_3^{-2}]_{\tau^{-1}}[\tau, \tilde{X}_3^{-2}] \overset{\text{by Lemma 4}}{=} [T^{-1}, \tilde{X}_3^{-2}]_{\tau^{-1}} \overset{\text{by (1.40)}}{=} Q_{\tau^{-1}}.\end{aligned}$$

Using $Q = \tau^{-1}$ we get $\tau_{S^{-1}}^{-1} = \tau_{\tau^{-1}}^{-1}$. Thus, $\tau_{S^{-1}} = \tau$ and $\tau_S = \tau$. □ Lemma 8

We can now finish the proof of Proposition III.7.1.

By Lemma III.2.4, we only have to prove that $\tau^2 = 1$, $\tau_b = \tau$ $\forall b \in \tilde{\tilde{B}}_n$ and $\tau \in \text{Center}(G)$.

By the previous Lemma, $\tau_S = \tau$, and by Lemma 7, $\tau_S = \tau^{-1}$. Thus, $\tau = \tau^{-1}$ and $\tau^2 = 1$.

By Lemma 6, $\tau_{\tilde{X}_i} = \tau$ $\forall i \in (1, \dots, n-1)$. Thus $\tau_b = \tau$ $\forall b \in \tilde{\tilde{B}}_n$.

By Lemma 8 $\tau_S = \tau$, i.e. $[\tau, S] = 1$. Let $b \in \tilde{B}_n :$ $[\tau, S_b] = [\tau_{b^{-1}}, S]_b = [\tau, S]_b = 1$.

Thus τ commutes with S_b $\forall b \in \tilde{\tilde{B}}_n$. Since $\ker(\tilde{B}_n \to \tilde{\tilde{B}}_n)$ acts trivially on G, $\tilde{\tilde{B}}_n$ acts on G via $\tilde{B}_n$, and thus τ commutes with S_b $\forall b \in \tilde{B}_n$.

By assumption (0) of the proposition, G is generated by $\{S_b\}_{b\in\tilde{B}_n}$. Thus $\tau \in \text{Center}(G)$. □ Proposition III.7.1

IV. New Set of Generators for G.

Recall from Chapter II that $G = G(\varepsilon_{18}) = \pi_1(\mathbb{CP}^2 - S_3, *)$ is generated by E_i, E_i' and satisfies the relations listed in Theorem II.6.

In this chapter we shall introduce new generators for G, using the braid group B_9 and the quotient $\tilde{B}_9$ from Chapter III.

IV.1. New presentation of $\mathbf{B_9}$.

Definition. T_i $i = 1, \dots, 9$ $i \neq 4$

Consider a geometric model (D, k) for $\#K = 9$ as in Fig. IV.1.1.

Let $\{t_i\}_{i=1\ i\neq 4}^{0}$ be paths connecting different parts of K as in Fig. IV.1.2.

Let T_i be the half-twist corresponding to t_i $i = 1, \dots, 9$ $i \neq 4$. ($T_i = H(t_i)$)

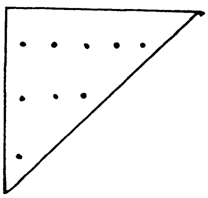


Figure IV.1.1

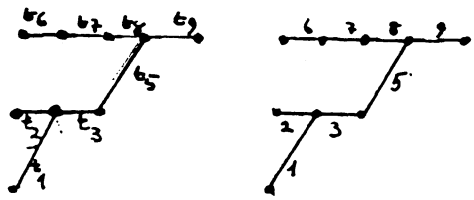


Fig. IV.1.2.

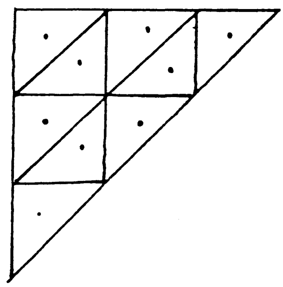


Figure IV.1.3.

Lemma IV.1.0. *T_i and T_j are adjacent for (i,j) as follows:*

$i,j \in \{1,2,3\}$

$i=5 \qquad j=3,8,9$

$i=6,7,8 \qquad j=i+1.$

T_i and T_j are disjoint for (i,j) as follows:

$i \in \{1,2,3\} \quad j \in \{6,7,8,9\}$

$i=5 \qquad j=1,2,6,7$

$i=6 \qquad j=8,9$

$i=7 \qquad j=9.$

Proof. From Fig. IV.1.2. □

Remark. The choice of the model comes from the configuration of planes in the degeneration of V_3 to the union of planes as in Fig.II.1. (We constructed this degeneration in BGT III [MoTe7].) In each of the triangles we choose a point (Fig. IV.1.3).

We choose a path connecting 2 points in neighboring triangles as in Fig. IV.1.4.

We then get a configuration which is basically equal to the one in Fig. IV.1.2.

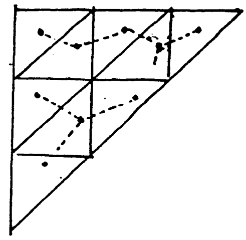


Figure IV.1.4.

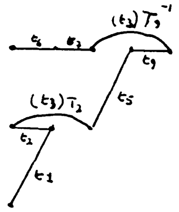


Figure IV.1.5.

Since we do not need all possible connections to get a set of generators for B_9, we skip the connection between the points in P_3 and P_5.

Lemma IV.1.1. *There exists a presentation of the braid group B_9, as follows:*

$$B_9 = \langle T_{i.} \mid i = 1, \ldots, 9, \quad i \neq 4 \rangle$$

and the following is a complete set of relations:

$$\begin{aligned} &\langle T_i, T_j \rangle = 1 && \text{if } T_i \text{ and } T_j \text{ are consecutive} \\ &[T_i, T_j] = 1 && \text{if } T_i \text{ and } T_j \text{ are disjoint} \\ &[T_1, T_2^{-1} T_3 T_2] = 1 \\ &[T_5, T_8^{-1} T_9 T_8] = 1. \end{aligned}$$

Proof. Consider the geometric model (D, K), $\#K = 9$ as in Fig. IV.1.1. We choose a frame in $B_9[D, K]$ where each half-twist in the frame corresponds to a path, as in Fig. IV.1.5.

In terms of T_i, this frame is

$$T_1, T_2, T_2^{-1} T_3 T_2, T_5, T_9, T_9 T_8 T_9^{-1}, T_7, T_6.$$

By E. Artin's presentation of the braid group (see Chapter 0), we know that B_9 is generated by the above frame and the only relations are triple

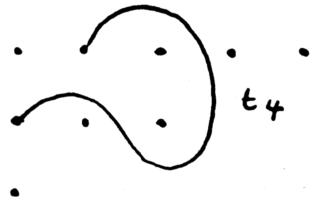


Figure IV.1.6.

relations for non-neighboring elements and commutation relations for neighboring elements. Thus, a full set of relations is:

$$\begin{aligned}\langle T_1, T_2\rangle &= \langle T_2, T_2^{-1}T_3T_2\rangle = \langle T_2^{-1}T_3T_2, T_5\rangle = \langle T_5, T_9\rangle \\ &= \langle T_9, T_9T_8T_9^{-1}\rangle = \langle T_9T_8T_9^{-1}, T_7\rangle = \langle T_7, T_6\rangle = 1\end{aligned}$$

and all possible commutation relations between other elements of the frame.

Since $T_3 = T_2\ (T_2^{-1}T_3T_2)\ T_2^{-1}$ and $T_9 = T_9^{-1}\ (T_9T_8T_9^{-1})\ T_9$, then $T_1,\ T_2,\ T_3,\ T_5\ T_9,\ T_8,\ T_7,\ T_6$ generates B_9. Translating the above relations to these generators and using simple facts about commutators, we obtain the Lemma. □

Definition. T_4.

$T_4 = T_2^{-1}\ T_3\ T_7^{-1}\ T_8\ T_5\ T_8^{-1}\ T_7\ T_3^{-1}\ T_2$. It is possible to notice that T_4 is the half-twist that corresponds to the path t_4 as in Fig. IV.1.6, and thus T_4 is adjacent to T_2 and T_6, transversal to T_7 and T_3 and disjoint from the others.

IV.2. Presentation of $\tilde{\mathbf{B}}_9$.

Let $\tilde{B}_9$ be as in Chapter III, $\tilde{B}_9 = B_9/T$, where $T = \langle [X, Y]\rangle$, and X, Y are transversal.

Let T_i be as in §1. Let $\tilde{T}_i$ be the images of T_i in $\tilde{B}_9$.

Lemma IV.2.1. *$\tilde{B}_9$ is generated by $\langle \tilde{T}_i \mid i = 1 \dots 9\rangle$ and the only relations are:*

(1) $\langle \tilde{T}_i, \tilde{T}_j\rangle = 1$ $\quad$ *T_i, T_j are consecutive $i, j \neq 4$*

(2) $[\tilde{T}_i, \tilde{T}_j] = 1$ $\quad$ *T_i, T_j are disjoint $i, j \neq 4$*

(3) $[\tilde{T}_1, \tilde{T}_2^{-1}\tilde{T}_3\tilde{T}_2] = 1$

(4) $[\tilde{T}_5, \tilde{T}_8^{-1}\tilde{T}_9\tilde{T}_8] = 1$

(5) $\tilde{T}_4 = \tilde{T}_2^{-1}\tilde{T}_3\tilde{T}_7^{-1}\tilde{T}_8\tilde{T}_5\tilde{T}_8^{-1}\tilde{T}_7\tilde{T}_3^{-1}\tilde{T}_2$.

Proof. By the previous Lemma, $\tilde{B}_9$ is generated by $\tilde{T}_i$, $i \neq 4$ with a full set of relations (1)...(4). We add one generator and express it in terms of the other relations to get $\tilde{T}_4$ and relation (5). □

Lemma IV.2.2. (1) $\langle \tilde{T}_4, \tilde{T}_2 \rangle = 1$.

(2) $\langle \tilde{T}_4, \tilde{T}_6 \rangle = 1$.

(3) $\langle \tilde{T}_7, \tilde{T}_6 \rangle = 1$.

(4) $[\tilde{T}_4, \tilde{T}_i] = 1 \qquad i = 1,\ 3,\ 5,\ 9,\ 8.$

Proof. We use Theorem III.3.1 from BGT I.

Since T_4 is consecutive to T_i, $i = 2, 6$ and T_6 is consecutive to T_7 then $\langle T_4, T_2 \rangle = \langle T_4, T_6 \rangle = \langle T_6, T_7 \rangle = 1$. Thus, $\langle \tilde{T}_4, \tilde{T}_2 \rangle = \langle T_4, T_6 \rangle = \langle \tilde{T}_6, \tilde{T}_7 \rangle = 1$.

T_4 and T_i for $i = 1, 9$ are disjoint, therefore, $[T_4, T_i] = 1$ for $i = 1, 9$. Thus, $[\tilde{T}_4, \tilde{T}_i] = 1$ for $i = 1, 9$. The half-twists T_4 and T_i $i = 3, 5, 8$ are transversal and, thus, $[\tilde{T}_4, \tilde{T}_i] = 1$ for $i = 3, 5, 8$ (Remark III.1.1). □

We need the following relations of $\tilde{T}_i$ in order to get a smaller set of generators for G.

Lemma IV.2.3.

$$(\tilde{T}_4)_{\tilde{T}_2^{-1}\tilde{T}_3\tilde{T}_7^{-1}\tilde{T}_8} = \tilde{T}_5 \quad \textit{preserving polarization}$$
$$(\tilde{T}_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\tilde{T}_7^{-1}} = \tilde{T}_8 \quad \textit{preserving polarization}$$
$$(\tilde{T}_3)_{\tilde{T}_2^{-1}\tilde{T}_4\tilde{T}_5^{-1}\tilde{T}_8} = \tilde{T}_7 \quad \textit{preserving polarization}$$

Proof. It is actually true for T_i instead of $\tilde{T}_i$. It can be verified geometrically using Fig. III.1.1 for a geometric presentation of a half-twist conjugated by another half-twist. □

Lemma IV.2.4. *Another presentation of* $\tilde{B}_9$.

Let $\tilde{T}_{1'} = \tilde{T}_2^{+2}\,\tilde{T}_1\,\tilde{T}_2^{-2}$ $\tilde{T}'_i = \tilde{T}_i$ $i \neq 1$. *Then* $\tilde{B}_9$ *is generated by* $\tilde{T}'_i$ *and the following is a complete set of relations:*

(1) $\langle \tilde{T}'_i, \tilde{T}'_j \rangle = 1$ *if* $\tilde{T}_i, \tilde{T}_j$ *are adjacent* $i, j \neq 4$.

(2) $[\tilde{T}'_i, \tilde{T}'_j] = 1$ *if* $\tilde{T}_i, \tilde{T}_j$ *are disjoint* $i, j \neq 4$.

(3) $[\tilde{T}'_1, \tilde{T}'_2 \tilde{T}'_3 \, \tilde{T}'^{-1}_2] = 1$

(4) $[\tilde{T}'_5, \tilde{T}'^{-1}_8 \, \tilde{T}'_9 \tilde{T}_8] = 1$

(5) $\tilde{T}'_4 = \tilde{T}'^{-1}_2 \tilde{T}'_3 \tilde{T}'^{-1}_7 \tilde{T}'_8 \tilde{T}'_5 \tilde{T}'^{-1}_8 \tilde{T}'_7 \tilde{T}'^{-1}_3 \tilde{T}'_2$.

Proof. Clearly, $T_i = T'_i$ $i \neq 1$ $T_1 = \tilde{T}'^{-2}_2 \tilde{T}'_1 \tilde{T}'^2_2$. We substitute these expressions in the relations of Lemma IV.2.1 to prove the Lemma. □

IV.3. $\alpha : \tilde{\mathbf{B}}_9 \to \mathbf{G}$.

We want to prove that there exists $\alpha : \tilde{B}_9 \to G$ s.t. $\alpha(\tilde{T}_i) = E_i$, for E_i that were introduced in Chapter II. For that we prove certain relations that E_i satisfy, based on Proposition II.6.

Lemma IV.3.1. (1) $\langle E_i, E_j \rangle = 1$ *if* T_i, T_j *are adjacent* $i, j \neq 4$.

(2) $[E_i, E_j] = 1$ *if* T_i, T_j *are disjoint* $i, j \neq 4$.

(3) $[E_1, E_2 E_3 E_2^{-1}] = 1$

(4) $[E_9, E_5^{-1} E_8 E_5] = 1$

(5) $E_4 = E_2^{-1} E_3 E_7^{-1} E_8 E_5 E_8^{-1} E_7 E_3^{-1} E_2$.

Proof. We use Proposition II.6, which states a list of relations satisfied by the E_i.

(1), (2) By Fig. IV.1.3, T_i, T_j $(i, j \neq 4)$ are adjacent, $\Leftrightarrow$ $\hat{L}_i$ and $\hat{L}_j$ are edges of some triangle, $\Leftrightarrow$ (by Corollary II.3) $\psi(E_i)$ and $\psi(E_j)$ have one common index. Moreover, T_i and T_j are disjoint $\Leftrightarrow$ $\psi(E_i)$ and $\psi(E_j)$ are disjoint. By (1) and (2) of Proposition II.6 we get (1) and (2) of this Proposition.

(3) By Proposition II.6 (3), $E_8 = E_7 E_5 E_3^{-1} E_4^{-1} E_2 E_4 E_3 E_5^{-1} E_7^{-1}$. By Proposition II.6 (2), $[E_3, E_4] = 1$. Thus, $E_8 = E_7 E_5 E_4^{-1} E_3^{-1} E_2 E_3 E_4 E_5^{-1} E_7^{-1}$. By Proposition II.6 (1), $E_3^{-1} E_2 E_3 = E_2 E_3 E_2^{-1}$. Thus,

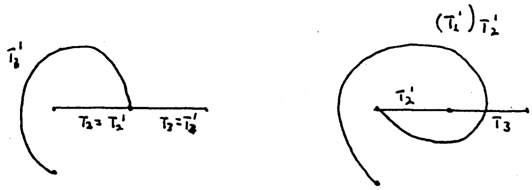


Figure IV.3.1.

$E_2\ E_3\ E_2^{-1}\ = E_4\ E_5^{-1}\ E_7^{-1}\ E_8\ E_7\ E_5\ E_4^{-1}$. By Proposition II.6 (2) E_i, for $i = 4,5,7,8$, commutes with E_1. Thus, $[E_1, E_2\ E_3\ E_2^{-1}] = 1$.

(4) By (3), $E_8 = E_7E_5E_3^{-1}E_4^{-1}E_2E_1E_3E_5^{-1}E_7^{-1}$. By (2), $[E_5, E_7] = 1$. newline Thus, $E_5^{-1}E_8E_5 = E_7E_3^{-1}E_4^{-1}E_2E_4E_3E_7^{-1}$.

For $i = 7,3,4,2$, E_i commutes with E_9. Thus, $[E_9, E_5^{-1}E_8E_5] = 1$.

(5) By Proposition II.6 (3), $E_4^{-1}E_2E_4 = E_3E_5^{-1}E_7^{-1}E_8E_7E_5E_3^{-1}$. By Proposition II.6 (2)(1), $E_4^{-1}E_2E_4 = E_2E_4E_2^{-1}$ and $[E_5, E_7] = 1$. By Proposition II.6 (1) $E_5^{-1}E_8E_5 = E_8E_5E_8^{-1}$. Thus, $E_2\ E_4E_2^{-1} = E_3E_7^{-1}E_5^{-1}E_8E_5E_7E_3^{-1}$. $E_4 = E_2^{-1}E_3E_7^{-1}E_8E_5E_8^{-1}E_7E_3^{-1}E_2$.

□

We first prove that there exist $\alpha' : \tilde{B}_9 \to G$ such that $\alpha'(\tilde{T}_i') = E_i$.

Lemma IV.3.2. *There exists a homomorphism α', $\alpha' : \tilde{B}_9 \to G$ such that $\alpha'(\tilde{T}_i') = E_i$.*

Proof. By Lemma IV.2.4, $\tilde{B}_9$ is generated by $\tilde{T}_i'$. Thus, we define $\alpha'(\tilde{T}_i') = E_i$. To prove that α' induces a homomorphism, we have to show that E_i satisfies any relation that $\tilde{T}_i'$ satisfies. In IV.2.4 we presented a full list of relations for $\tilde{T}_i'$. In Lemma IV.3.1 we proved that these relations are satisfied when $\tilde{T}_i'$ is replaced by E_i. □

Lemma IV.3.3.

$$[E_1,\ E_2^{-1}\ E_3\ E_2] = 1.$$

Proof. T_3' is transversal to $T_2'\ T_1'\ T_2^{-1}$. (See Fig. IV.3.1)

Thus, $[\tilde{T}_2'\tilde{T}_1'\tilde{T}_2^{-1}, \tilde{T}_3'] = 1$.

Thus, $\alpha'[\tilde{T}_2'\tilde{T}_1'\tilde{T}_2^{-1}, \tilde{T}_3'] = 1$.

Thus, $[E_2E_1E_2^{-1}, E_3] = 1$.
Thus, $[E_1,\ E_2^{-1}\ E_3E_2] = 1$. □

Lemma IV.3.4. *There exists* $\alpha : \tilde{B}_9 \to G$ *such that* $\alpha(\tilde{T}_i) = E_i$.

Proof. We use the presentation of $\tilde{B}_9$ from Lemma IV.2.1 where $\tilde{B}_9$ is generated by $\tilde{T}_i$. We define $\alpha(\tilde{T}_i) = E_i$. The relations listed in IV.2.1 are satisfied when $\tilde{T}_i$ is replaced by E_i by Lemma IV.3.1 and Lemma IV.3.3. Thus we can extend the definition of α to the whole of $\tilde{B}_9$ in a natural way. □

IV.4. Prime elements in $\mathbf{B_9}$.

We now recall a few results from Chapters II and III concerning the braid group B_n and the quotient group $\tilde{B}_n$. We refer the reader to Chapter III for the definition of prime element with s.h.t. (supporting half-twist) T, and central element c.

We quote here a few results from [MoTe4], Chapter II and Chapter III.

Lemma IV.4.0. *If X and Y are 2 consecutive half-twists in B_n, then*

(a) $XYX = YXY$.

(b) $XYX^{-1} = Y^{-1}XY$.

(c) $(Y)_{X^{-1}} = (X)_Y$.

(d) $(Y)_{X^{-1}}$ *is consecutive to X and to Y. It is the half-twist corresponding to a path connecting those ends of X and Y which are not a common index of X and Y.*

(e) $u = \tilde{X}^2_{\tilde{Y}^{-1}}\tilde{Y}^{-2} = \tilde{Y}^2_{\tilde{X}} \cdot \tilde{Y}^{-2}$, *is an element of $\tilde{P}_n$; u is a prime element with s.h.t. $\tilde{X}$ and central element $c = [\tilde{Y}^2, \tilde{X}^2]$ (i.e., $c^2 = 1$, $c \in$ Center$(\tilde{P}_n)$).*

(f) $[\tilde{X}'^2, \tilde{Y}'^2] = [\tilde{X}'^2, \tilde{Y}^{-2}] = [\tilde{X}'^{-2}, \tilde{Y}'^{-2}] = c\ \forall X', Y'$ *a pair of consecutive half-twists where $c^2 = 1$, $c \in$ Center(G).*

(g) $\tilde{Y}^{-2}(\tilde{Y}^2)_{\tilde{X}^{-1}} = c$ *(inverse of a prime element of $\tilde{P}_n$ with s.h.t. X).*

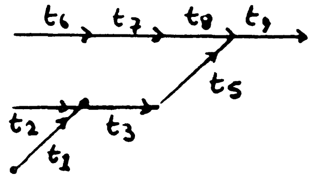


Figure IV.4.1.

(h) *If Z is transversal to X, g a prime element with s.h.t. X, then* $(g)_Z = g$.

Proof.

(a) From Lemma III.3.1 [MoTe4].

(b) From Lemma II.4 and (a).

(c) From (b).

(d) Let $Y = H(y)$, y connects a and b and $X = H(x)$ x connects b and c. Assume X corresponds to a diffeomorphism $\beta : D \to D$ s.t. $\beta(b) = c$. Then $(Y)_{X^{-1}} = H((\sigma)\beta^{-1})$. Clearly, $(\sigma)\beta^{-1}$ is a path connecting a and c. (See Claim III.1.0 and Fig. III.1.1.)

(e) Lemma III.6.1.

(f) Proposition III.5.2.

(g) We shall prove that $c(Y^{-2}(Y^2)_{X^{-1}})^{-1}$ is a prime element:

$$c(Y^{-2}(Y^2)_{X^{-1}})^{-1} = c(Y^{-2})_{X^{-1}}Y^2 = Y^2(Y^{-2})_{X^{-1}} \quad \text{(by (b))}.$$

Let $T = (Y)_{X^{-1}}$. By (e) and (c), $(T)^2_X \cdot T^{-2}$ is a prime element. But $(T)_X = ((Y)_{X^{-1}})_X = Y$. Thus $Y^2(Y)^{-2}_{X^{-1}}$ is a prime element.

(h) Lemma III.4.3. □

ξ-situation:.

Consider $\tilde{B}_9, \tilde{P}_9$ as $\tilde{B}_9$-groups by conjugation.

Let $\tilde{T}_1, \ldots, \tilde{T}_9$ be as in §1.

We choose a polarization on $\tilde{T}_i$ from smaller end index to bigger end index as shown in Fig. IV.4.1.

Let $\xi_1 = (\tilde{T}_2)^2_{\tilde{T}_1}(\tilde{T}_2)^{-2}$.

By Lemma IV.4.0 (e), ξ_1 is a prime element in $\tilde{P}_9$ with s.h.t. $\tilde{T}_1$ and central element $[T_2^2, T_1^2]$.

Let c be the corresponding central element. Thus, $c \in \text{Center}(\tilde{P}_9)$, $c^2 = 1$, $c = [T_2^2, T_1^2]$.

Let ξ_i be the unique prime element in $\tilde{P}_9$ s.t. $(\xi_i, \tilde{T}_i)$ is coherent with $(\xi_1, \tilde{T}_1)$ $i = 2, \ldots, 9$. (See Proposition III.3.3).

Claim IV.4.1. (1) *c is the corresponding central element of (ξ_i, T_i) $\forall i = 1 \ldots 9$, $c^2 = 1$.*

(2) $c \in \text{Center}(\tilde{B}_9)$.

(3) $\xi_i = (Y)^2_{\tilde{T}_i} Y^{-2}$ *for some Y half-twist adjacent to T_i.*

(4) $c = [\tilde{T}_k^2, \tilde{T}_\ell^2]$ $\forall k, \ell$ *s.t. T_k and T_ℓ are consecutive.*

(5) *ξ_i is a prime element of $\tilde{B}_9$.*

(6) *Let X, Y, be 2 half-twists, $X = H(x)$, $Y = H(y)$, $T_i = H(t_i)$ s.t. x, y, t_i make a triangle. Assume that x and y meet in v, and a counter clockwise rotation around v inside the triangle meets x before it meets y (Fig.* IV.4.2(a) *and* (b)*). Then $\tilde{T}_i$, the s.h.t. of ξ_i satisfies $T_i = XYX^{-1}$. And:* (i) *If the polarization of T_i goes from x to y, then $\xi_i = \tilde{X}^2\ \tilde{Y}^{-2}$.* (ii) *If the polarization of T_i goes from y to x, then $\xi_i = X^{-2}Y^2$.*

Proof.

(1) The pair $(\xi_i, \tilde{T}_i)$ is coherent with $(\xi_1, \tilde{T}_1)$. Thus, ξ_i is conjugate to T_1 and $\tilde{T}_i$ is conjugate to f_1 by some B_i. Then by Lemma III.3.1, their corresponding central element is equal.

(2) A priori, $c \in \text{Center}(\tilde{P}_9)$. We have to prove that $c \in \text{Center}(\tilde{B}_9)$. Consider $\tilde{P}_9$ as $\tilde{B}_9$-group. ξ_i is a prime element in $\tilde{P}_9$ as a a $\tilde{B}_9$-group where c is the central element of (ξ_i, T_i). Thus, we have $(c)b = c$ $\forall b \in \tilde{B}_9$. (( )$b$ = action of $\tilde{B}_9$ on $\tilde{P}_9$), but $(c)b = c_b$ by definition. Thus, $c_b = c$ $\forall b \in \tilde{B}_9 \Rightarrow c \in \text{Center}(\tilde{B}_9)$.

(3) The pair (ξ_i, T_i) is coherent with the pair $(\xi_1, \tilde{T}_1)$. Thus, $\exists b_i \in B$ s.t. $\xi_i = (\xi_1)_{b_i}$ and $\tilde{T}_i = (\tilde{T}_1)_{b_i}$. Denote $(\tilde{T}_2)_{b_i} = \tilde{Y}$ and apply conjugation by b_i on $\xi_1 = (\tilde{T}_2^2)_{T_1} \tilde{T}_2^{-2}$.

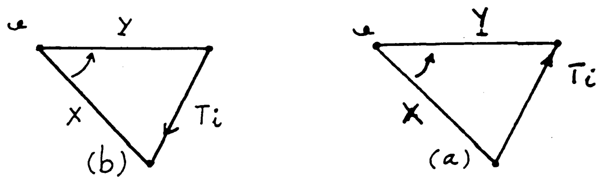


Figure IV.4.2.

(4) By Proposition III.5.2.

(5) By construction, ξ_i is a prime element of $\tilde{P}_9$ with central element c. In (2) we proved that $c \in \text{Center}(\tilde{B}_9)$. Thus, ξ_i is also a prime element of $\tilde{B}_9$.

(6) Consider the two triangles (Fig. IV.4.2(a) and (b)):

If the polarization of T_i goes from X to Y as in Fig. (a) (from Y to X as in Fig. (b)) then we take $b \in \tilde{B}_n$ s.t. $(T_1)b = \tilde{T}_i$ $(T_2)b = Y$ preserving polarization of T_1 (reversing polarization), respectively. Clearly, $((T_2)_{T_1})b = X$. Consider the polarized pair $(\xi_1, \tilde{T}_1) = ((\tilde{T}_2)^2_{\tilde{T}_1} \tilde{T}_2^{-2}, \tilde{T}_1)$. We apply b on it to get a coherent (anti-coherent) polarized pair $(\tilde{X}^2\tilde{Y}^{-2}, \tilde{T}_i)$. Recall that $(\xi_i, \tilde{T}_i)$ is coherent with $(\xi_1, \tilde{T}_1)$. Thus by Proposition III.3.3 (or by Lemma III.4.1) $\tilde{X}^2\tilde{Y}^{-2} = \xi_i$ (or $\tilde{X}^2\tilde{Y}^{-2} = \xi_i^{-1}c$), respectively. To get $\xi_i = \tilde{X}^{-2}\tilde{Y}^2$ from $\tilde{X}^2\tilde{Y}^{-2} = \xi_i^{-1}c$ we use (4) above.

□

The next Corollary is technical in nature, to be used later in order to obtain a smaller set of generators for G.

Corollary IV.4.2. (1) $\xi_1 = (\tilde{T}_2)^2_{\tilde{T}_1^{-1}} \tilde{T}_2^{-2}$.

(2) $\xi_2 = \tilde{T}_1^{-2}(\tilde{T}_1)^2_{\tilde{T}_2^{-1}}$ $\quad \xi_2 = T_4^2(T_4)^{-2}_{T_2^{-1}}$ $\quad \xi_2 = \tilde{T}_3^{-2}(\tilde{T}_3)^2_{\tilde{T}_2^{-1}}$

(3) $\xi_3 = \tilde{T}_1^2 \ (\tilde{T}_1)^{-2}_{\tilde{T}_3^{-1}}$

(4) $\xi_4 = (\tilde{T}_4)^2_{\tilde{T}_6^{-1}} \tilde{T}_6^{-2}$

(5) $\xi_5 = \tilde{T}_8^{-2}(\tilde{T}_8^2)_{\tilde{T}_7^{-1}}$

(6) $\xi_6 = (\tilde{T}_4)^{-2} \ (\tilde{T}_4^2)_{\tilde{T}_6^{-1}}$, $\quad \xi_6^{-1} = c\tilde{T}_7^2(\tilde{T}_7)^{-2}_{T_6^{-1}}$

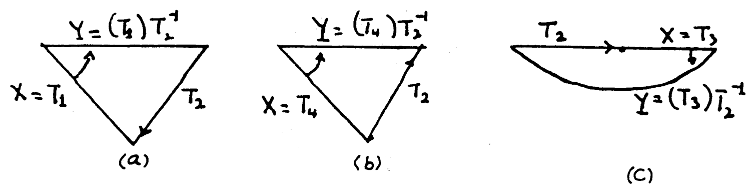


Figure IV.4.3.

(7) $\xi_7 = \tilde{T}_8^{-2}(\tilde{T}_8)^2_{\tilde{T}_7^{-1}}$

(8) $\xi_8 = \tilde{T}_9^{-2}(\tilde{T}_9)^2_{\tilde{T}_8^{-1}}$

(9) $\xi_9 = \tilde{T}_5^2(\tilde{T}_5)^{-2}_{\tilde{T}_9^{-1}}$

(10) $(\xi_4)_{\tilde{T}_2^{-1}\tilde{T}_3\tilde{T}_7^{-1}\tilde{T}_8} = \xi_5$

(11) $(\xi_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\xi_7^{-1}} = \xi_8$

(12) $(\xi_3)_{\tilde{T}_2^{-1}\tilde{T}_4\tilde{T}_5^{-1}\tilde{T}_8} = \xi_7$

Proof. In the entire proof we use Lemma IV.4.0(f) to interchange squares of consecutive half-twists, and multiplying the product by c. We also use the facts that $c^2 = 1$, $c \in \text{Center}(G)$. The main tool is Lemma IV.4.1(6). We also use Lemma IV.4.0(d) to present an edge of a triangle as a conjugation of the other 2 edges.

(1) By definition.

(2) Consider the triangle from Fig. IV.4.3(a):
Since the polarization of T_2 goes from Y to X, we apply Lemma IV.4.1(6) to get:

$$\xi_2 = X^{-2}Y^2 = \tilde{T}_1^{-2}(\tilde{T}_1)^2_{\tilde{T}_2^{-1}}.$$

Consider the triangle from Fig. IV.4.3(b).
Since the polarization of T_2 goes from X to Y, we apply Lemma IV.4.1(6) to get

$$\xi_2 = X^2Y^{-2} = \tilde{T}_4^2(\tilde{T}_4)^{-2}_{\tilde{T}_2^{-1}}.$$

Consider the "triangle" from Fig. IV.4.3(c)
Since the polarization of T_2 goes from Y to X, then:

$$\xi_2 = \tilde{T}_3^{-2}(\tilde{T}_3)^2_{\tilde{T}_2^{-1}}.$$

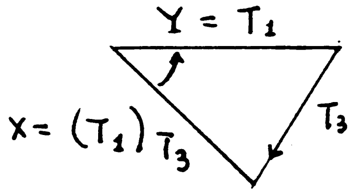


Figure IV.4.4.

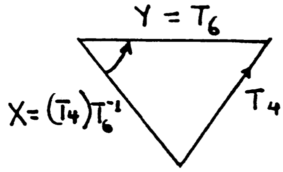


Figure IV.4.5.

(3) Consider the triangle from Fig. IV.4.4:
Since the polarization of T_3 goes from Y to X, then:

$$\xi_3 = X^{-2}Y^2 = (T_1)^{-2}_{T_3}T_1^2.$$

We apply $\tilde{T}_3^{-2}$ on the above equation to get:

$$(\xi_3)_{\tilde{T}_3^{-2}} = (\tilde{T}_1)^{-2}_{\tilde{T}_3^{-1}}(\tilde{T}_1)^2_{\tilde{T}_3^{-2}}.$$

By Lemma III.2.1, $(\xi_3)_{\tilde{T}_3^{-2}} = \xi_3$.
By Lemma IV.4.1(4), $(\tilde{T}_1^2)_{\tilde{T}_3^{-2}} = cT_1^2$.
Thus,

$$\begin{aligned}\xi_3 &= c(T_1)^{-2}_{T_3^{-1}}T_1^2\\ &= T_1^2(T_1)^{-2}_{T_3^{-1}} \quad \text{(Lemma IV.4.0(5))}.\end{aligned}$$

(4) Consider the triangle from Fig. IV.4.5:
Since the polarization of T_4 goes from X to Y

$$\xi_4 = (\tilde{T}_4)^2_{\tilde{T}_6^{-1}}\tilde{T}_6^{-2}.$$

(5) Consider the triangle from Fig. IV.4.6:
Since the polarization of T_5 goes from X to Y

$$\xi_5 = (\tilde{T}_8)^2_{\tilde{T}_5}\tilde{T}_8^{-2}.$$

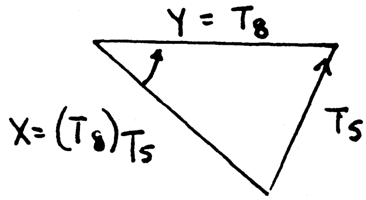


Figure IV.4.6.

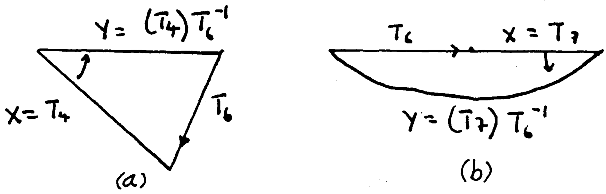


Figure IV.4.7.

(6) Consider the triangle from Fig. IV.4.7(a):
Since the polarization of T_6 goes from Y to X

$$\xi_6 = \tilde{T}_4^{-2}(\tilde{T}_4^2)_{T_6^{-1}}.$$

Consider the "triangle" from Fig. IV.4.7(b).
Since the polarization of T_6 goes from Y to X

$$\xi_6 = \tilde{T}_7^{-2}(\tilde{T}_7^2)_{T_6^{-1}} \Rightarrow \xi_6^{-1} = c\tilde{T}_7^2(\tilde{T}_7^{-2})_{\tilde{T}_6^{-1}}.$$

(7) Consider the "triangle" from Fig. IV.4.8:
Since the polarization of T_7 goes from Y to X

$$\xi_7 = \tilde{T}_8^{-2}(\tilde{T}_8^2)_{T_7^{-1}}.$$

(8) Similar to (7).
(9) Similar to (3).

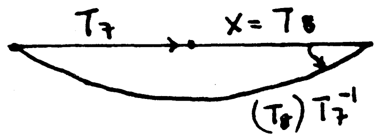


Figure IV.4.8.

For (10)–(12) we shall only prove the first assertion. The others are similar. By Lemma III.2.2, $((\xi_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\tilde{T}_7^{-1}}, (T_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\tilde{T}_7^{-1}})$ is a polarized pair coherent with $(\xi_2, \tilde{T}_2)$. By Lemma IV.2.3 $(\tilde{T}_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\tilde{T}_7^{-1}} = \tilde{T}_8$, preserving polarization. Thus $((\xi_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\tilde{T}_7^{-1}}, \tilde{T}_8)$ is coherent with $(\xi_2, \tilde{T}_2)$. But also $(\xi_8, \tilde{T}_8)$ is coherent with $(\xi_2, \tilde{T}_2)$. From uniqueness, $(\xi_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\tilde{T}_7^{-1}} = \xi_8$. □

IV.5. η-situation.

In §3 we constructed a homomorphism of groups $\alpha : \tilde{B}_9 \to G$ s.t. $\alpha(\tilde{T}_i) = E_i$.

We introduce G as a $\tilde{B}_9$-group by $(g)\tilde{Y} = g_{\alpha(\tilde{Y})} (= \alpha(\tilde{Y})^{-1} g\ \alpha(\tilde{Y}))$.

The homomorphism α here then becomes a homomorphism of $\tilde{B}_9$-groups.

Claim IV.5.1. *Let* $\mu = \alpha(c)$, $\quad c = [T_2^2, T_1^2]$, *then:*

(a) $\mu = [E_k^2, E_\ell^2]$ $\forall k, \ell$ *s.t.* T_k *and* T_ℓ *are adjacent.*

(b) *Let* ρ *be any automorphism of the type* $\rho_{m_1,m_2,m_3,m_4,m_6,m_9}$ *s.t.* $\exists j_0$ *with* $m_{j_0} \neq 0$ *and all other* $m_i = 0$. *Then* $(\mu)\rho = \mu$.

(c) $\mu \in \text{Center}(G)$.

(d) $\mu = \left[(E_k^2)\rho_k^m . (E_\ell)^2 \rho_\ell^n\right]$ $\forall k, \ell$ *s.t.* T_k *and* T_ℓ *are adjacent, and* $(k, \ell) \neq (3, 7), (2, 8), (4, 5)$.

(e) $\mu^2 = 1$.

(f) *If* $\tilde{X}$ *and* $\tilde{Y}$ *are 2 consecutive half-twists, then* $\alpha(\tilde{X})^2\alpha(\tilde{Y})^2 = \mu\alpha(\tilde{Y})^2\alpha(\tilde{X})^2$.

Proof.

(a) $c = [T_k^2, T_\ell^2]$ $\forall k, \ell$ s.t. $\tilde{T}_k$ and $\tilde{T}_\ell$ are adjacent (Claim IV.4.1 (4)). Then $\mu = \alpha(c) = \alpha[\tilde{T}_k^2, \tilde{T}_\ell^2] = [E_k^2, E_\ell^2]$ $\forall k, \ell$ such that $\tilde{T}_k$ and $\tilde{T}_\ell$ are adjacent.

(b) $\rho = \rho_1^{m_1}(\rho_2\rho_8)^{m_2}(\rho_3\rho_2)^{m_3}(\rho_4\rho_5)^{m_4}\ \rho_6^{m_6}\rho_9^{m_9}$, and $(E_i)\rho = E_i$ if ρ_i appears in ρ to the power 0. If $j_0 \neq 1, 2$, then $(E_1)\rho = E_1$ and $(E_2)\rho = E_2$. We take $\mu = [E_1^2, E_2^2]$ and apply ρ on it to get $(\mu)\rho = [E_1^2, E_2^2]\rho = [(E_1^2)\rho, (E_2^2)\rho] = [E_1^2, E_2^2] = \mu$. If $j_0 = 1$ or 2, we apply ρ on $\mu = [E_3^2, E_5^2]$ and continue similarly to get the result.

(c) Since $c \in \text{Center}(\tilde{B}_9)$, then $\mu \in \text{Center}(\alpha(\tilde{B}_9))$. Since $\alpha(\tilde{T}_i) = E_i$, we get $[\mu, E_i] = 1 \;\; \forall i$. We want to prove $[\mu, E_{i'}] = 1 \;\; \forall i$ and then we get $\mu \in \text{Center}(G)$. (G is generated by $\{E_j, E_{j'}\}_{j=1}^{9}$.) Take $\rho = \rho_1^{m_1}(\rho_2\rho_8)^{m_2}(\rho_3\rho_7)^{m_3}\,(\rho_4\rho_5)^{m_4}\rho_6^{m_6}\rho_9^{m_9}$ such that ρ_i appears in ρ to the power 1 and all other $m_i = 0$, Thus, $(E_i)\rho = E_{i'}$ and there is exactly one $m_{j_0} \neq 0$. Thus, by (*b*) $(\mu)\rho = \mu$. Thus,

$$1 = (1)\rho = [E_i, \mu]\rho = [(E_i)\rho, (\mu)\rho] = [E_{i'}, \mu].$$

(d) Since $(k, \ell) \neq (2,8), (3,7), (4,5)$, then L_k and L_ℓ are not on the same line. Let $\rho' = \rho_1^{m_1}(\rho_2\rho_8)^{m_2}(\rho_3\rho_7)^{m_2}(\rho_4\rho_5)^{m_5}\rho_6^{m_6}\rho_9^{m_6}$ where ρ_k appears to the power m and all other $m_i = 0$. In particular, ρ_ℓ appears to the power $0 \cdot ((k,\ell) \neq (2,8), (3,7), (4,5))$. Then $(E_k)\rho = (E_k)\rho_k^m, (E_\ell)\rho = E_\ell$. Let ρ' be as above s.t. ρ_ℓ appears there to the power n and all other $m_i = 0$. Then, $(E_\ell)\rho' = (E_\ell)\rho_\ell^n, (E_k)\rho' = E_k$. Thus, $\mu = (\mu)\rho\rho' = [(E_k)\rho_k^m, (E_\ell)\rho_\ell^n]$.

(e) $\mu^2 = \alpha(c)^2 = \alpha(c^2) = \alpha(1) = 1$.

(f) From Lemma IV.4.1 (4).

□

Corollary IV.5.2. $\eta_1 = \alpha(\xi_1)$ *is a prime element in* G *with s.h.t.* $\tilde{T}_1$ *and central element* μ.

Proof. ξ_1 is a prime element of $\tilde{B}_9$ with s.h.t. $\tilde{T}_1$ (see Claim IV.4.1). Thus, $\eta_1 = \alpha(\xi_1)$ is a prime element of $\alpha(\tilde{B}_9)$ with s.h.t. $\tilde{T}_1$ and central element $\mu = \alpha(c)$, from the previous lemma, $\mu \in \text{Center}(G)$. Thus, η_i is a prime element of G. □

η-situation.

Consider G as a $\tilde{B}_9$-group.

Let $\eta_1 = \alpha(\xi_1)$ be a prime element of G with s.h.t. $\tilde{T}_1$ and central element $\mu = \alpha(c)$.

Let $\eta_i = L_{\eta_1,\tilde{T}_1}(\tilde{T}_i)$ be the unique prime element of G such that $(\eta_i, \tilde{T}_i)$ is coherent with $(\eta_1, \tilde{T}_1)$.

Claim IV.5.3. *Consider the* η*-situation. Then:*

(a) $(\eta, \tilde{T}_i)$ *is a polarized pair of* G *with s.h.t.* $\tilde{T}_i$ *and corresponding element* μ. *In particular,* η_i *is a prime element with s.h.t.* $\tilde{T}_i$.

(b) $\eta_i = \alpha(\xi_i)$.

(c) *Every* η_i *is of the form* $\alpha(X^2 Y^{-2})$ *where* X, Y *are adjacent half-twists and the s.h.t.* T_i *is* XYX^{-1} $(= Y^{-1}XY)$.

(d) $\eta_1 = (E_2)^2_{E_1^{-1}} E_2^{-2}$.

$\eta_2 = E_1^{-2}(E_1)^2_{E_2^{-1}}$.

$\eta_3 = E_1^2 (E_1)^{-2}_{E_3^{-1}}$.

$\eta_4 = (E_4)^2_{E_6^{-1}} E_6^{-2}$.

$\eta_5 = (E_8)^2_{E_5} E_8^{-2}$.

$\eta_6 = (E_4)^{-2}(E_4)^2_{E_6^{-1}} \quad \eta_6^{-1} = \mu E_7^2 (E_7)^{-2}_{E_6^{-1}}$.

$\eta_7 = E_8^{-2}(E_8)^2_{E_7^{-1}}$.

$\eta_8 = E_9^{-2}(E_9)^2_{E_8^{-1}}$.

$\eta_9 = E_5^2 (E_5)^{-2}_{E_9^{-1}}$.

$\eta_5 = (\eta_4)_{\tilde{T}_2^{-1}\tilde{T}_3\tilde{T}_7^{-1}\tilde{T}_8}$

$\eta_7 = (\eta_3)_{\tilde{T}_2^{-1}\tilde{T}_4\tilde{T}_5^{-1}\tilde{T}_8}$.

$\eta_8 = (\eta_2)_{\tilde{T}_4\tilde{T}_3\tilde{T}_5^{-1}\tilde{T}_7^{-1}}$.

Proof.

(a) By the construction of η_i, the pairs $(\eta_i, \tilde{T}_i)$ are coherent polarized pairs. $(\eta_1, \tilde{T}_1)$ has μ as central element and coherent pairs have the same central element (Corollary III.3.1).

(b) ξ_1 is a prime element with s.h.t. $\tilde{T}_i$ s.t. $(\xi_i, \tilde{T}_i)$ coherent with $(\xi_1, \tilde{T}_1)$. Thus, $\alpha(\xi_i)$ is a prime element with s.h.t. $\tilde{T}_i$ s.t. $(\alpha(\xi_i), \tilde{T}_i)$ is coherent with $(\alpha(\xi_1), \tilde{T}_1) = (\eta_1, \tilde{T}_1)$. From uniqueness $\alpha(\xi_i) = \eta_i$.

(c) From Lemma IV.4.1.(6).

(d) Apply α on the formulas of Corollary IV.4.2.

□

IV.6. N-situation.

N-situation.

Let $\mathcal{G}$ be a $\tilde{B}_9$-group.

Assume there exists a $\tilde{B}_9$-homomorphism $\lambda : \tilde{B}_9 \to \mathcal{G}$ s.t.
$(g)b = g_{\lambda(b)} \quad \forall b \in \tilde{B}_9, \ \forall \ g \in \mathcal{G}$.

Let f be a prime element in $\mathcal{G}$ with s.h.t. $\tilde{T}_1$ and central element ν.

Let $f_i = L_{\{(f,\tilde{T}_1)\}}(\tilde{T}_i)\}$ be the unique prime element s.t. $(f_i, \tilde{T}_1)$ is coherent with $(f_i, \tilde{T}_1)$ $i = 1, \ldots, 9$.

By Corollary III.3.1, ν is the central element of f_i $\forall i$.

Let $\eta_1 = \lambda(\xi_1)$, η_1 is a prime element with s.h.t. $\tilde{T}_1$ and central element $\mu = \lambda(c)$.

Let $\eta_i = L_{\{\eta_1,\tilde{T}_1\}}(\tilde{T}_i)$.

It is easy to see that, similar to the situation in Claim IV.5.3, $\eta_i = \lambda(\xi_i)$.

Let $\mu = \lambda(c)$ be the central element of η_i, $\forall i$.

Let $N_1 = \{\eta_i \mid (\eta_i, \tilde{T}_i)$ be a polarized pair coherent with $(\eta_1, \tilde{T}_1)$, $i = 1 \ldots 9\}$.

Let $N_2 = \{f_i \mid (f_i, \tilde{T}_i)$ be a polarized pair coherent with $(f, \tilde{T}_1)$, $i = 1 \ldots 9\}$.

Definitions.

Let $a, b \in N_1 \cup N_2$. We say that a and b are weakly disjoint (transversal, disjoint, adjacent, consecutive or cyclic) if their s.h.t. are weakly disjoint (transversal, disjoint, adjacent, consecutive or cyclic respectively). (See definitions in the beginning of Chapter III.)

Let $a \in N_1 \cup N_2$ with s.h.t. $\tilde{X}$. Let $\tilde{Z} \in \tilde{B}_9$. We say that a and $\tilde{Z}$ are weakly disjoint (transversal, disjoint, adjacent, consecutive, cyclic) if X and Z are weakly disjoint (transversal, disjoint, adjacent, consecutive, cyclic respectively).

Lemma IV.6.1. $a \in N_1 \cup N_2$, $\tilde{Z} \in \tilde{B}_9$ $\mu = \lambda(c)$, *then:*

(i) *$a, \tilde{Z}$ are adjacent* $\Rightarrow a_{\tilde{Z}^2} = \begin{cases} a\mu & a \in N_1 \Rightarrow \\ a\nu & a \in N_2. \end{cases}$

(ii) *$a, \tilde{Z}$ are weakly disjoint or commonly supported* $\Rightarrow a_{\tilde{Z}^2} = a$.
In other words,

$$\left[\lambda(\tilde{Z}^2),\ a^{\pm 1}\right] = \left[\lambda(\tilde{Z}^{-2}),\ a^{\pm 1}\right] = \begin{cases} \mu & a \in N_1,\ a, Z \text{ adjacent} \\ \nu & a \in N_2,\ a, Z \text{ adjacent} \\ 1 & a, Z \text{ weakly disjoint or cyclic.} \end{cases}$$

Proof. Let $\tilde{X}$ be the s.h.t. of a.

(i) There exists $v \in P_9$ s.t. $Y = v^{-1}Zv$ is consecutive to X. By the definition of prime element:

$$a_{\tilde{Y}^{-2}} = \begin{cases} a\mu & a \in N_1 \\ a\nu & a \in N_2. \end{cases}$$

On the other hand, $Y^{-2} = v^{-1}Z^{-2}vZ^2Z^{-2} = \left[v^{-1}, Z^{-2}\right] Z^{-2}$. Since $v \in P_9$ and $Z^2 \in P_9$, then $\left[\tilde{v}^{-1}, \tilde{Z}^{-2}\right] \in \tilde{P}_9$. But $\tilde{P}_9'$ is generated by c (Lemma III.5.2), $c^2 = 1$, so $\tilde{Y}^{-2} = c^{\varepsilon}\tilde{Z}^{-2}$ $\varepsilon = 0, 1$. Thus, $a_{\tilde{Y}^{-2}} = a_{\lambda(\tilde{Y}^{-2})} = a_{\lambda(c^{\varepsilon})\lambda(\tilde{Z}^{-2})} = a_{\mu^{\varepsilon}\lambda(\tilde{Z}^{-2})} = a_{\lambda(\tilde{Z}^{-2})} = a_{\tilde{Z}^{-2}}$. So, $a_{\tilde{Z}^{-2}} = a_{\tilde{Y}^{-2}}$. Thus,

$$a_{\tilde{Z}^{-2}} = \begin{cases} a\mu & a \in N_1 \\ a\nu & a \in N_2. \end{cases}$$

Since $\nu, \mu \in \text{Center}(G)$

$$a_{\tilde{Z}^2} = \begin{cases} a\mu & a \in N_1 \\ a\nu & a \in N_2. \end{cases}$$

(ii) If a and Z are weakly disjoint (cyclic), then: There exists $v \in P_9$ s.t. $Y = v^{-1}\ Z\ v$ and X are disjoint (commonly supported). As above, $a_{\tilde{Y}^2} = a_{\tilde{Z}^2}$. By the definition of prime element $a_{\tilde{Y}^2} = 1$. Thus, $a_{\tilde{Z}^2} = 1$. □

Proposition IV.6.2. (i) *If $a, b \in N_1 \cup N_2$ are adjacent, then*

$$\left[a, b^{\pm 1}\right] = \begin{cases} \mu & a, b \in N_1 \\ \nu & \text{otherwise.} \end{cases}$$

(ii) *If $a, b \in N_1 \cup N_2$ are commonly supported or weakly disjoint, then*

$$\left[a, b^{\pm 1}\right] = 1.$$

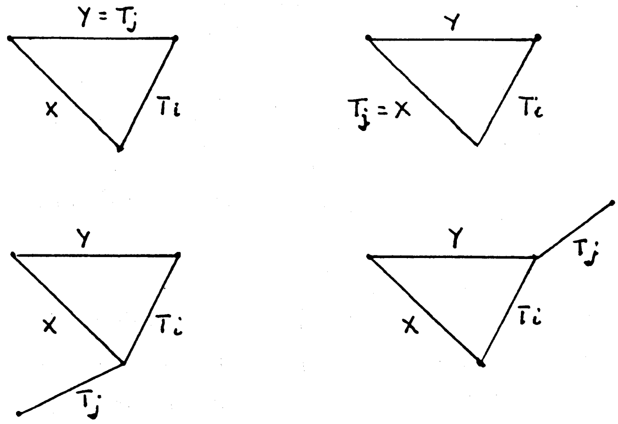


Figure IV.6.

Proof. For $a, b \in N_1$ and $a, b \in N_2$, the proof follows from Lemma III.4.2. Consider the case $a \in N_1$, $b \in N_2$. Assume $a = \eta_i$ with s.h.t. $\tilde{T}_i$, and $b \in N_2$ with s.h.t. $\tilde{T}_j$. By Claim IV.4.1(6), a is of the form $\lambda(\tilde{X}^2 \tilde{Y}^{-2})$ where X and Y are consecutive half-twists s.t. X, Y, T_i create a triangle as in Fig. IV.4.2.

Now, $\left[a, b^{\pm 1}\right] = \left[\lambda(\tilde{X}^2\, \tilde{Y}^{-1}), b^{\pm 1}\right] = \left[\lambda(\tilde{Y}^{-2}), b^{-1}\right]_{\lambda(\tilde{X}^{-2})} \cdot [\lambda(\tilde{X}^2), b^{\pm 1}]$. (by Claim II.4). By the previous Lemmas and the fact that $\nu \in \text{Center}(G)$, we get:

$$\left[\lambda(\tilde{Y}_2^{-2}), b^{\pm 1}\right]_{\lambda(\tilde{X}_2^2)} = \begin{cases} \nu & Y \text{ and } T_j \text{ are adjacent} \\ 1 & Y \text{ and } T_j \text{ are weakly disjoint or} \\ & \qquad \text{commonly supported.} \end{cases}$$

$$\left[\lambda(X_2^{-2}), b^{\pm 1}\right] = \begin{cases} \nu & X \text{ and } T_j \text{ are adjacent} \\ 1 & X \text{ and } T_j \text{ are weakly disjoint or} \\ & \qquad \text{commonly supported.} \end{cases}$$

These are the only possible values since Y (or X) are never cyclic with T_j. Since $\nu^2 = 1$, in order to get $[a, b^{\pm 1}] \neq 1$, we need one factor to be ν and the other one to be 1. Thus we need T_j adjacent to Y, and T_j is weakly disjoint or commonly supported to X, (or vice-versa). This can happen in four different cases (Fig. IV.6). But in all of the four cases T_j is adjacent to T_i (see Fig. IV.6). □

Lemma IV.6.3. *Consider the N-situation. Then:*

$$(f_i)_{\tilde{T}_k} = \begin{cases} f_i^{-1}\nu & k=i \\ f_i & T_i, T_k \quad \textit{weakly disjoint} \\ f_k f_i & T_i, T_k \quad \textit{orderly adjacent} \\ f_i f_k^{-1} & T_i, T_k \quad \textit{are not orderly adjacent.} \end{cases}$$

$$(f_i)_{\tilde{T}_k^{-1}} = \begin{cases} f_i^{-1}\nu & k=i \\ f_i & T_i, T_k \quad \textit{weakly disjoint} \\ f_i f_k & T_i, T_k \quad \textit{orderly adjacent} \\ f_k^{-1} f_i & T_i, T_k \quad \textit{are not orderly adjacent.} \end{cases}$$

Proof. Recall that f_i is a prime element with s.h.t. T_i. Moreover, $\nu \in$ Center$(G), \nu = \left[f_i^{\pm 1}, f_j^{\pm 1}\right]$ for T_i, T_j adjacent half-twists (previous Lemma).

One can see from Fig. IV.1.6 that if T_i and T_k are weakly disjoint then they are disjoint unless $k=4$ and $i=3,7$, in which case they are transversal. We shall treat separately the case where T_i and T_k are disjoint and the case where T_i and T_k are transversal.

For $k=i$ and for T_k and T_i disjoint we get from the definition of prime element that:

$$(f_i)_{\tilde{T}_k^{-1}} = f_k^{-1}\mu \quad i=k$$
$$(f_i)_{\tilde{T}_k^{-1}} = f_i \qquad \text{for } T_k \text{ and } T_i \text{ disjoint.}$$

We conjugate the above formulas by $\tilde{T}_k$ to get the correct formula for $(f_i)_{\tilde{T}_k}$ where $k=i$ or when T_i and T_k are weakly disjoint.

For T_i and T_k orderly adjacent we use Proposition III.4.1 to get:
$(f_i)_{\tilde{T}_k^{-1}} = f_i\ f_k$.
Thus, $(f_i)_{\tilde{T}_k^{-2}} = (f_i)_{\tilde{T}_k^{-1}}(f_k)_{\tilde{T}_k^{-1}} = f_i f_k f_k^{-1}\ \nu = f_i\ \nu$.
Since $\nu^2 = 1,\ \nu \in$ Center(G), we get: $f_i = \nu^{-1}(f_i)_{\tilde{T}_k^{-2}} = (f_i)_{\tilde{T}_k^{-2}}\nu$.
Thus, $(f_i)_{T_k} = (f_i)_{T_k^{-1}}\nu = f_i\ f_k\nu = f_i f_k[f_k^{-1}, f_i^{-1}] = f_i f_k f_k^{-1} f_i^{-1} f_k f_i = f_k f_i$.

For T_i and T_k be non-orderly adjacent, we use III.4.1 again to get:
$(f_i)_{\tilde{T}_k^{-1}} = f_k^{-1}\ f_i$.
Thus, $(f_k)_{\tilde{T}_k^{-2}} = (f_k)^{-1}_{\tilde{T}_k^{-1}}(f_i)_{\tilde{T}_k^{-1}} = \nu^{-1} f_k f_k^{-1} f_i = \nu f_i \Rightarrow$
$(f_i)_{T_k} = \nu\ (f_i)_{T_k^{-1}} = \nu\ f_k^{-1} f_i = [f_i, f_k^{-1}] f_k^{-1} f_i = f_i f_k^{-1} f_i^{-1} f_k f_k^{-1} f_i = f_i f_k^{-1}$.

For T_i and T_k transversal we get from Lemma III.4.2 that

$$(f_i)_{\tilde{T}_k^{-1}} = f_i/$$

We conjugate the above formula by $\tilde{T}_k$ to get the correct formula for $(f_i)_{\tilde{T}_k}$ for T_i and T_k transversal. □

IV.7. New set of generators for G : $\{\mathbf{A_j}, \mathbf{E_j}\}$.

Recall

$$E_i = \begin{cases} \Gamma_i & i \neq 2,7 \\ \Gamma_i & i = 2,7 \end{cases}$$

$$E_i' = (E_i)\rho_i$$
$$A_j = E_j' E_j^{-1}.$$

Claim IV.7.1. *$\{A_j, E_j\}_{j=1}^9$ generates G.*

Proof. By Lemma II.2, $\{E_j, E_{j'}\}_{j=1}^9$ generates G. Since $E_{j'} = A_j E_j$, $\{A_j, E_j\}_{j=1}^9$ generates G. □

Definition. $H_1 = \tilde{B}_n$-orbit of A_1.

Proposition IV.7.2. *$A = A_1$ is a prime element of H_1 with s.h.t. $\tilde{T}_1$.*

Proof. Consider the following frame of B_9: $X_1 = T_1$, $X_2 = T_3$, $X_3 = (T_5)_{T_9}$, $X_4 = T_9$. $X_5 = T_8$, $X_6 = T_7$, $X_7 = T_6$, $X_8 = (T_2)_{T_3^{-1}T_5^{-1}T_8^{-1}T_7^{-1}T_6^{-1}}$ (see Fig. IV.7.1).

In order to prove that H_1 is a prime element of A_1, we shall prove that all the necessary conditions of Proposition III.7.1 are fulfilled using the above frame of B_9. Let $\nu = A \cdot A_{\tilde{T}_1^{-1}}$. It is easy to see that $\nu = E_{1'}^2\ E_1^{-2}$. $(\nu = A \cdot A_{\tilde{T}_1^{-1}} = A \cdot A_{E_1^{-1}} = E_{1'} E_1^{-1} E_1 E_{1'} E_1^{-1} E_1^{-1} = E_{1'}^2 E_1^{-2}$.

(0) By definition of H_1, H_1 is the full orbit of $A_1 = A$.

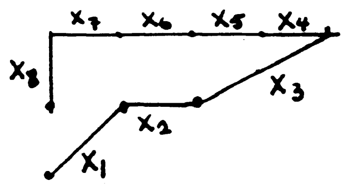


Figure IV.7.1.

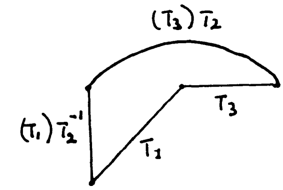


Figure IV.7.2.

(1) We have to prove $A_{\tilde{X}_2^{-1}\tilde{X}_1^{-1}} = A^{-1}A_{\tilde{X}_2^{-1}}$.

(1a) Since $AE_1 = E_1'$, we can use Proposition II.6 and Corollary II.3 to get $\langle AE_1, E_3\rangle = \langle E_1, E_3\rangle = 1$. Thus, by Lemma II.4(h) we get $A_{E_2^{-1}E_1^{-1}} = A^{-1}\cdot A_{E_3^{-1}}$. Since $A_{\tilde{X}} = A_{\alpha(\tilde{X})}$ and $\alpha(\tilde{X}_1) = \alpha(\tilde{T}_1) = E_1$, $\alpha(\tilde{X}_2) = \alpha(\tilde{T}_3) = E_3$, we get (1a).

(1b) $A_{\tilde{X}_1} = A_{\tilde{T}_1} = A_{\alpha(\tilde{T}_1)} = A_{E_1}$. Thus, $A_{\tilde{X}_1}E_1 = A_{E_1}E_1 = E_1^{-1}AE_1E_1 = E_1^{-1}E_1'E_1^{-1}E_1E_1 = E_1^{-1}E_1'E_1 = (E_1)\rho_1^{-1}$. By Proposition II.6 and Corollary II.3 we get $\langle A_{\tilde{X}_1}E_1, E_3\rangle = \langle E_1, E_3\rangle = 1$. By Lemma II.4.(h) we get $A_{\tilde{X}_1E_3^{-1}\,E_1^{-1}} = A_{\tilde{X}_1}^{-1}A_{\tilde{X}_1E_3^{-1}}$. Since $\alpha(\tilde{X}_1) = E_1$ and $\alpha(\tilde{X}_2) = \alpha(\tilde{T}_3) = E_3$, we get (1b).

(2a) We need to prove $\nu_{\tilde{X}_1^2} = \nu$. Since $\alpha(\tilde{X}_1) = \alpha(\tilde{T}_1) = E_1$, then $\nu_{\tilde{X}_1^2} = \nu_{E_1^2}$. To prove $\nu_{E_1^2} = \nu$, it is enough to prove $[E_{1'}^2, E_1^2] = 1$.

Consider the following 4 half-twists in B_9: T_1, $T_3, T_2^{-1}T_3T_2$, $T_2T_1T_2^{-1}$. Obviously, the above half-twist consists of a "good triangle" in $\tilde{B}_9$ (see Fig. IV.7.2).

By Lemma III.1.2,

$$\tilde{T}_1\cdot\tilde{T}_2^{-1}\tilde{T}_3\tilde{T}_2 = \tilde{T}_2^{-1}\tilde{T}_3\tilde{T}_2\cdot\tilde{T}_1$$
$$\text{and}\quad \tilde{T}_1^2\cdot\tilde{T}_2^{-1}\tilde{T}_3^2\tilde{T}_2 = \tilde{T}_3^2\cdot\tilde{T}_2\tilde{T}_1^2\tilde{T}_2^{-1}.$$

By Lemma IV.3.4, there exist $\alpha : \tilde{B}_9 \to G$ s.t. $\alpha(T_i) = E_i$. Thus,

$$\begin{aligned} &[E_1, E_2^{-1}E_3E_2] = 1 \\ \textit{and} \quad &E_1^2 \cdot E_2^{-1}E_3^2E_2 = E_3^2E_2 \cdot E_1^2E_2^{-1}. \end{aligned}$$

Thus, $E_1^2 = E_3^2 \cdot E_2E_1^2E_2^{-1} \cdot E_2^{-1}E_3^{-2}E_2$.

We shall find the commutator of $E_{1'}^2$ with each of the 3 factors $\alpha = E_3$, $\beta = E_2E_1^2E_2^{-1}$, $\gamma = E_2^{-1}E_3^{-2}E_2$.

To find $[E_1^2, \gamma]$ we apply on $[E_1, E_2^{-1}E_3E_2] = 1$ the Invariance Theorem (Corollary I.5) with ρ_1 ($m_1 = 1$ and all other $m_i = 0$) to get $[E_1', E_2^{-1}E_3E_2] = 1$ and thus, we get $[E_{1'}^2, E_2^{-1}E_3^2E_2] = 1$. i.e., $[E_{1'}^2, \gamma] = 1$.

$[E_{1'}^2, \alpha] = [E_{1'}^2, E_3^2] = [(E_1^2)\rho_1, E_3^2] = ([E_1^2, E_3^2])\,\rho_1 = (\mu)\ \rho_1 = \mu$ (By IV.5.1).

$$\begin{aligned} [E_{1'}^2, \beta] = [E_{1'}^2, E_2E_1^2E_2^{-1}] &= [E_{1'}^2, E_1^{-1}E_2^2E_1] = \quad \text{(Claim II.4)} \\ &= [E_{1'}^2, E_1 \cdot E_1^{-2}E_2^2E_1^2E_2^{-2} \cdot E_2^2E_1^{-1}] \\ &= [E_{1'}^2, E_1 \cdot \mu \cdot E_2^2E_1^{-1}] = (\mu \in \text{Center}(G)) \\ &= [E_{1'}^2, E_1E_2^2E_1^{-1}] \\ &= [E_1^{-1}, E_{1'}^2E_1, E_2^2]_{E_1^{-1}} \\ &= [(E_1^2)\ \rho_1^{-1}, E_2^2]_{E_1^{-1}} \\ &= \left([E_1^2, E_2^2]\ \rho_1^{-1}\right)_{E_1^{-1}} \\ &= \left((\mu)\ \rho_1^{-1}\right)_{E_1^{-1}} \quad \text{(Lemma IV.5.1)} \\ &= \mu_{E_1^{-1}} = \quad (\mu \in \text{Center}(G)) \\ &= \mu. \end{aligned}$$

To find $[E_{1'}^2, \alpha\beta\gamma]$ we use Claim II.4(d),

$$\begin{aligned} [E_{1'}^2, E_1^2] = [E_{1'}^2, \alpha\beta\gamma] &= [E_{1'}^2, \alpha]\ [E_{1'}^2, \beta]_{\alpha^{-1}} \cdot [E_{1'}^2, \gamma]_{\beta^{-1}\gamma^{-1}} \\ &= \mu \cdot (\mu)_\alpha \cdot (1)_z = (\text{since } \mu \in \text{Center}(G)) \\ &= \mu \cdot \mu \cdot 1 = \qquad (\text{since } \mu^2 = 1) \\ &= 1. \end{aligned}$$

(2b) Let $B = (A)_{\tilde{X}_2^{-1}} = (A)_{\tilde{T}_3^{-1}} = (A)_{E_3^{-1}}$. To prove (2b) we have to show $\nu_B = \nu_{E_1}^{-1}$. Now:

$$\begin{aligned}
\nu_B &= \nu_{(A)_{E_3^{-1}}} \\
&= A_{E_3^{-1}}^{-1} \cdot \nu \cdot A_{E_3^{-1}} \\
&= E_3 E_1 \underbrace{E_{1'}^{-1} E_3^{-1} \cdot E_{1'}^2}\, E_1^{-2} \cdot E_3 E_{1'} E_1^{-1} E_3^{-1} = \\
&\qquad\text{(By Claim II.4(a))} \\
&= \underbrace{E_3 E_1 E_3^2}\; E_{1'}^{-1}\, \underbrace{E_3^{-1} E_1^{-2} E_3}\; E_{1'} E_1^{-1} E_3^{-1} = \\
&\qquad\text{(By Claim II.4(a))} \\
&= E_1^2 E_3 E_1 E_{1'}^{-1}\, \overbrace{E_1 E_3^{-2} E_1^{-1}}\; E_{1'} E_1^{-1} E_3^{-1} \\
&= E_1^2 E_3 E_1^2\, \underbrace{E_1^{-1} E_{1'}^{-1} E_1}\; E_3^{-2}\, \underbrace{E_1^{-1} E_{1'} E_1}\; E_1^{-2} E_3^{-1} \\
&= E_1^2 E_3 E_1^2\, \underbrace{(E_1)\rho_1^{-1} \cdot E_3^{-2} (E_1)\rho_1^{-1}}\; E_1^{-2} E_3^{-1} = \\
&\qquad\begin{pmatrix}\text{By Claim II.4(a)} \\ \text{and Prop. II.6 (1)}\end{pmatrix} \\
&= E_1^2 E_3 E_1^{-2}\, \overbrace{E_3 (E_1^2)\rho_1^{-1} E_3^{-1}}\; E_1^{-2} E_3^{-1} \\
&= E_1^2 \cdot \underbrace{E_3 E_1^2 E_3^{-1}} \cdot \underbrace{E_3^2 (E_1^2)\rho_1^{-1} E_3^{-2}} \cdot \underbrace{E_3 E_1^{-2} E_3^{-1}} = \\
&\qquad\begin{pmatrix}\text{By Claim II.4(a) and} \\ \text{Lemmas IV.3.4, IV.4.1}\end{pmatrix} \\
&= E_1^2 \cdot \overbrace{E_1^{-1} E_3^2 E_1}\; \overbrace{\mu \cdot (E_1^2)\rho_1^{-1}} \cdot \overbrace{E_1^{-1} E_3^{-2} E_1} = \\
&\qquad(\mu \in \text{Center}(G)) \\
&= \mu E_1^2 \cdot E_1^{-1} E_3^2 E_1 \cdot E_1^{-1} E_{1'}^{-2} E_1 \cdot E_1^{-1} E_3^{-2} E_1 \\
&= \mu E_1 E_3^2 E_{1'}^{-2} E_3^{-2} E_1 \\
&= \mu E_1 \mu E_{1'}^{-2} \cdot E_1 = \\
&\qquad(\text{Since } \mu^2, \mu \in \text{Center}(G)) \\
&= E_1 E_{1'}^{-2} E_1
\end{aligned}$$

Thus, $\nu_B = E_1 E_{1'}^{-2} E_1$.

On the other hand, $\nu_{E_1}^{-1} = E_1^{-1}(E_{1'}^2 E_1^{-2})^{-1} E_1 = E_1 E_{1'}^{-2} E_1$.

(3) We have to prove $A_{\tilde{X}_j} = A \ \forall j \geq 3$. We recall: $A = E_{1'}E_1^{-1}$ and $A_{X_j} = A_{\alpha(X_j)}$. Since $X_3 = (T_5)_{T_9}$, $X_4 = T_9$, $X_5 = T_8$, $X_6 = T_7$ and $X_7 = T_6$, we get $\alpha(X_j)$ for $j \geq 3$ is a product of E_i for $i = 5, 9, 8, 7, 6$. $[E_1, E_i] = [E_{1'}, E_i] = 1$ for $i = 5, 9, 8, 7, 6$ (Proposition II.6), we get $A_{\tilde{X}_j} = A_{\alpha(\tilde{X}_j)} = A \ \forall j = 3, 4, 5, 6, 7$. Now: $\tilde{X}_8 = (\tilde{T}_2)_{\tilde{T}_3^{-1}\tilde{T}_5^{-1}\tilde{T}_8^{-1}\tilde{T}_7^{-1}\tilde{T}_6^{-1}}$. Thus, $\alpha(\tilde{X}_8) = (E_2)_{E_3^{-1}E_5^{-1}E_8^{-1}E_7^{-1}E_6'^{-1}}$. Thus, $A_{\tilde{X}_8} = A_{\alpha(\tilde{X}_8)} = A_{(E_2)_{E_3^{-1}E_5^{-1}E_8^{-1}E_7^{-1}E_6^{-1}}}$.

We first prove that $(A)_{(E_2)_{E_3^{-1}}} = A$.

Since $(T_2)_{T_3^{-1}}$ and T_1 are disjoint $[T_1, (T_2)_{T_3^{-1}}] = 1$. Thus

$$\alpha\left([\tilde{T}_1, (\tilde{T}_2)_{\tilde{T}_3^{-1}}]\right) = 1.$$

Thus, $[E_1, (E_2)_{E_3^{-1}}] = 1$. We apply on this relation the invariance Theorem (Corollary I.5) to get $[E_{1'}, (E_2)_{E_3^{-1}}] = 1$. Since $A = E_{1'}E_1^{-1}$ we get $[A, (E_2)_{E_3^{-1}}] = 1$. Thus, $A_{(E_2)_{E_3^{-1}}} = A$. Thus, $A_{\tilde{X}_8} = A_{E_5^{-1}E_8^{-1}E_7^{-1}E_6^{-1}}$. Since E for $i = 5, 8, 7, 6$ commutes with E_1, E_1' (Proposition II.6), we get $A_{\tilde{X}_8} = A$.

(4) Let $c = [\tilde{X}_1^2, \tilde{X}_2^2]$. We have to show that $A_c = A$. Since $c = [\tilde{X}_1^2, \tilde{X}_2^2] = [\tilde{T}_1^2, \tilde{T}_3^2]$, $\alpha(c) = [E_1^2, E_3^2] = \mu$, where $\mu \in \text{Center}(G)$. Thus, $A_c = A_\mu = A$.

Thus all the conditions of Proposition III.7.1 are fulfilled and A is a prime element of H_1. □

Proposition IV.7.3. *A is a prime element of G with s.h.t. $\tilde{T}_1$.*

Proof. By the previous propositions, A is a prime element of H_1 with s.h.t. T_1 and central element $\nu = A \cdot A_{\tilde{T}_1^{-1}}$, $\nu \in \text{Center}(H_1)$. By the definition of a prime element, in order to prove that A is also a prime element of G, it is enough to show that $\nu = A\ A_{\tilde{T}_1^{-1}} \in \text{Center}(G)$. Since A is a prime element of H_1 with central element ν, we get $(\nu)_b = \nu \ \forall b \in \tilde{B}_9$. In particular, $(\nu)_{\tilde{T}_i} = \nu \ \forall i = 1 \dots 9$. Thus, $\nu_{E_i} = \nu \ \forall i = 1 \dots 9$. For $i \neq 1$, we apply ρ on $\nu_{E_i} = \nu$, to get, using the Invariance Theorem, the relation $\nu_{E_i'} = \nu$ (for $i \neq 1$ $(\nu)\ \rho_i = \nu$). For $i = 1$ we use $[\nu, E_1] = 1$ (from above), and $[\nu, A] = 1$ (since $A \in H_1$ and $\nu \in \text{Center}(H_1)$) to get $[E_{1'}, \nu] = [AE_1, \nu] = 1$. Thus $[E_i, \nu] = [E_{i'}, \nu] = 1 \ \forall\ i = 1 \dots 9 \Rightarrow \nu \in \text{Center}(G)$. □

IV.8. New set of generators for G $\{\mathbf{E_j}, \mathbf{h_j}, \eta_j\}_{j=1}^{9}$.

We introduce here a new set of generators for G. In §3 we introduced a homomorphism $\alpha : \tilde{B}_9 \to G$ s.t. $\alpha(\tilde{T}_i) = E_i$ and introduced G as a $\tilde{B}_9$-group using α. In §5 we proved that $\alpha(\xi_1) = \eta_1 = (E_2^2)_{E_1} \cdot E_2^{-2}$ is a prime element of G with s.h.t. $\tilde{T}_1$ and central element $\mu = [E_2^2, E_1^2]$. We proved in §7 that $A = E_{1'}E_1$ is a prime element of G with s.h.t. $\tilde{T}_1$ and central element $\nu = A_{E_1^{-1}} = E_{1'}^2 E_1^{-2}$. In §6 we introduced the N-situation. Here we consider the N-situation for $\alpha : \tilde{B}_9 \to G$, and $h_1 = A \in G$.

Consider the *N-situation* with η_i, h_i, N_1, N_2 as follows:

h_i is the unique prime element with s.h.t. $\tilde{T}_i$ s.t. $(h_i, \tilde{T}_i)$ is coherent with $(A, \tilde{T}_1)$. The central element of h_i is ν.

η_i is the unique prime element with s.h.t. $\tilde{T}_i$ s.t. $(\eta_i, \tilde{T}_i)$ is coherent with $(\eta_{1'}\tilde{T}_1)$. The corresponding central element is μ.

$N_1 = \{\eta_i \quad i = 1 \ldots 0\}$.

$N_2 = \{h_i \quad i = 1 \ldots 0\}$.

Lemma IV.8.0. (i) *Let f be a prime element in G with s.h.t. $\tilde{T}_i$. Then f commutes with E_i^2.*

(ii) *If T_j is transversal to T_i then f commutes with E_j.*

(iii) *If T_j is consecutive to T_i then $\left[(E_i)_{E_j^{\pm 1}}, E_i^2\right] = \mu$.*

Proof.

(i) By Lemma III.2.1, $(f)_{\tilde{T}_i^2} = f$. By definition of G as a $\tilde{B}_9$-group: $(f)_{\tilde{T}_i^2} = f_{\alpha(\tilde{T}_{i'}^2)}$. Since $\alpha(\tilde{T}_i) = E_i$ (Lemma IV.3.4) we get $(f)_{E_i^2} = f$. Thus, f commutes with E_i^2.

(ii) By Lemma III.4.2 $(f)_{\tilde{T}_j} = f$. But $f_{\alpha(\tilde{T}_j)} = f_{E_j}$. Thus it commutes with E_j.

(iii) Lemma IV.5.1. □

Lemma IV.8.1. *Let $A_i = E_i' E_i^{-1}$. Let $h_i, \eta_i\mu, \nu$ be as above. Then,*

(1) $A_1 = A = h_1$.

(2) $A_2 = h_2^{-1}\eta_2$.

(3) $A_3 = h_3\eta_3^{-1}\mu\nu$.

(4) $A_4 = h_4^2\eta_4^{-2}\mu\nu$.

(5) $A_5 = h_5^2\eta_5^{-2}\mu\nu$.

(6) $A_6 = \eta_6^3\ h_6^2\mu\nu$.

(7) $A_7 = h_7\eta_7^{-1}\mu\nu$.

(8) $A_8 = h_8^{-1}\eta_8$.

(9) $A_9 = h_9^2\eta_9^{-3}\nu$.

Proof. We use the definition of prime element and Lemmas II.7, IV.5.3, IV.6.1 IV.6.2, IV.6.3, and IV.8.0. Recall that $\nu^2 = \mu^2 = 1, \nu, \mu \in \text{Center}(G)$. We use here often the following two facts: If f is a prime element in G with s.h.t. $\tilde{T}_i$, then f commutes with E_i^2 (Lemma IV.8.0(i)); $(E_i^2)_{E_j^{\pm 1}}$ and E_i^2 commute up to μ for T_i and T_j consecutive half-twists (Lemma IV.8.0(iii)).

(1) By definition of h_1.

(2) By Corollary II.7, $A_2 = E_1^{-2}A_1^{-1}(A_1)_{E_2^{-1}}(E_1^2)_{E_2^{-1}}$. Since A_1 is a prime element with s.h.t. $\tilde{T}_1$ (Proposition IV.7.3), A_1 commutes with E_1^2. Thus, $A_2 = A_1^{-1}E_1^{-2}(E_1^2)_{E_2^{-1}}(A_1)_{E_2^{-1}}\ A_1^{-1} = h_1^{-1}$ from (1). $E_1^{-2}(E_1^2)_{E_2^{-1}} = \eta_2$ by Lemma IV.5.3. $(A_1)_{E_2^{-1}} = (h_1)_{\tilde{T}_2^{-1}} = h_2^{-1}h_1$ since $\tilde{T}_1$ and $\tilde{T}_2$ are not orderly adjacent (by Lemma IV.6.3).

Thus, $A_2 = h_1^{-1}\eta_2 h_2^{-1}h_1$. By Lemma IV.6.2, η_2 commutes with h_2^{-1} and $\eta_2 h_1 = h_1\eta_2\nu$ ($\nu \in \text{Center}(G)$). Thus, $A_2 = \nu h_1^{-1}h_2^{-1}h_1\eta_2$. By the same Lemma, $h_1^{-1}h_2^{-1}h_1 = \nu h_2^{-1}$. Thus, $A_2 = \nu^2 h_2^{-1}\eta_2 = h_2^{-1}\eta_2$.

(3) By Corollary II.7, $A_3 = E_1^{-2}A_1^{-1}(A_1)_{E_3^{-1}}(E_1^2)_{E_3^{-1}}$. Like in (2) we can write, $A_3 = A_1^{-1}E_1^{-2}(E_1^2)_{E_3^{-1}}(A_1)_{E_3^{-1}}$.

$A_1^{-1} = h_1^{-1}$ from (1). $E_1^{-2}(E_1^2)_{E_3^{-1}} = \eta_3^{-1}\ \mu$. (Lemmas IV.5.3 and IV.5.1(d)). $(A_1)_{E_3^{-1}} = (h_1)_{\tilde{T}_3^{-1}} = h_1h_3$. (Lemma IV.6.3). Thus, $A_3 = h_1^{-1}\mu\eta_3^{-1}h_1h_3$. Since $[h_1^{-1}, \eta_3^{-1}] = \nu$, and $\mu \in \text{Center}(G)$, $A_3 = \mu\eta_3^{-1}h_1^{-1}\nu h_1h_3$.

Since $\mu, \nu \in \text{Center}(G)$, $A_3 = \mu\nu\eta_3^{-1}h_3^{-1}$ Since η_3 commutes with h_3, $A_3 = h_3\eta_3^{-1}\mu\nu$.

(4) By Corollary II.7,

$$(A_4)_{E_2^{-1}E_4^{-1}} = E_4^2A_3E_3^2A_2(E_3^{-2})_{E_2^{-1}}(A_3^{-1})_{E_2^{-1}}(E_4^{-2})_{E_2^{-1}}.$$

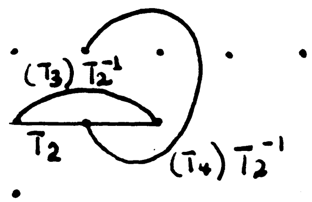


Figure IV.8.

Since A_3 is a product of prime elements with s.h.t. $\tilde{T}_3$, A_3 commutes with E_3^2 $(= \alpha(\tilde{T}_3^2))$ (Lemma IV.8.0). Thus,

$$(A_4)_{E_2^{-1}E_4^{-1}} = E_4^2 E_3^2 A_3 A_2 (E_3^{-2})_{E_2^{-1}} (A_3^{-1})_{E_2^{-1}} (E_4^{-2})_{E_2^{-1}}.$$

Now, $E_4 = \alpha(\tilde{T}_4)$, $\tilde{T}_4$ is a half-twist which is transversal to $\tilde{T}_3$. Thus, by Corollary IV.8.0(ii), E_4^{-2} commutes with A_3^{-1}. Thus, $(A_4)_{E_2^{-1},E_4^{-1}} = E_1^2 E_3^2 A_3 A_2 (E_3^{-2})_{E_2^{-1}} (E_4^{-2})_{E_2^{-1}} (A_3^{-1})_{E_2^{-1}}$. $Z_1 = (T_3)_{T_2^{-1}}$ and $Z_2 = (T_4)_{T_2^{-1}}$ are 2 half-twists which are adjacent to T_3 and to T_2 (Fig. IV.8). Thus, by Lemma IV.6.1, $[\alpha(Z_i^2), f] = \nu$ where f is a prime element with s.h.t. T_3 or T_2 and central element ν and $[\alpha(Z_i^2), \eta] = \mu$ where η is a prime element with s.h.t. T_3 or T_2 and central element μ $(i = 1, 2)$.

$A_3 A_2$ is a product of 4 prime elements with s.h.t. $\tilde{T}_2$ or $\tilde{T}_3$. Two of them have a central element μ, and 2 of them have a central element ν. Thus, $A_3 A_2 \alpha(Z_1^{-2})\alpha(Z_2^{-2}) = (\nu\mu)^2 \alpha(Z_1^{-2})\alpha(Z_2^{-2}) A_3 A_2$. Since $\alpha(Z_1) = (E_3)_{E_2^{-1}}$, $\alpha(Z_2) = (E_4)_{E_2^{-1}}$ and $\mu^2 = \nu^2 = 1$, we have $A_3 A_2 (E_3^{-2})_{E_2^{-1}} (E_4^{-2})_{E_2^{-1}} = (E_3^{-2})_{E_2^{-1}} (E_4^{-2})_{E_2^{-1}} \cdot A_3 A_2$. Thus, $(A_4)_{E_2^{-1}E_4^{-1}} = E_4^2 E_3^2 (E_3^{-2})_{E_2^{-1}} (E_4^{-2})_{E_2^{-1}} A_3 A_2 (A_3^{-1})_{E_2^{-1}}$.

From Corollary IV.4.2, $\tilde{T}_4^2 (\tilde{T}_4^2)^{-2}_{\tilde{T}_2^{-1}} = \tilde{T}_3^{-2} (\tilde{T}_3^2)_{\tilde{T}_2^{-1}}$. We use Lemma IV.4.1(4) and $c^2 = 1$ to exchange factors and rewrite this equation as $\tilde{T}_4^2 \tilde{T}_3^2 (\tilde{T}_3^{-2})_{\tilde{T}_2^{-1}} (\tilde{T}_4^{-2})_{\tilde{T}^{-1}} = 1$. We apply α to it to get: $E_4^2 E_3^2 (E_3^{-2})_{E_2^{-1}} (E_4^{-2})_{E_2^{-1}} = 1$. Thus, $(A_4)_{E_2^{-1}E_4^{-1}} = A_3 A_2 (A_3^{-1})_{E_2^{-1}}$. By (2) (3),

$$\begin{aligned}(A_4)_{E_2^{-1}E_4^{-1}} &= h_3 \eta_3^{-1} \mu\nu h_2^{-1} \eta_2 (\eta_3 h_3^{-1})_{\tilde{T}_2^{-1}} \mu\nu \\ &= h_3 \eta_3^{-1} h_2^{-1} \eta_2 \eta_3 \eta_2 h_2^{-1} h_3^{-1}. \text{ (By Lemma IV.6.3)}\end{aligned}$$

Since $[h_2^{-1}, \eta_3^{-1}] = \nu$ and $[\eta_2, \eta_3^{-1}] = \mu$ (Lemma IV.6.2), we get $(A_4)_{E_2^{-1}E_4^{-1}} = \mu\nu h_3 h_2^{-1} \eta_2^2 h_2^{-1} h_3^{-1}$. Since h_2 commutes with η_2 and

$[h_3, h_2^{-1}] = [h_3, \eta_2] = \nu$, we get $(A_4)_{E_2^{-1}E_4^{-1}} = \nu^5\mu\eta_2^2 h_2^{-2} = \nu\mu\eta_2^2 h_2^{-2}$. Thus,

$$A_4 = (\nu\mu\eta_2^2 h_2^{-2})_{E_4E_2}.$$

Since

$$\begin{aligned}(\eta_2)_{E_2} &= \eta_2^{-1}\ \mu\\ (h_2)_{E_2} &= h_2^{-1}\nu\\ (h_4)_{E_2} &= h_4 h_2^{-1}\\ (\eta_4)_{E_2} &= \eta_4\eta_2^{-1},\end{aligned}$$

So,

$$\begin{aligned}A_4 &= \nu\mu\left(\eta_2^{-1}\mu\eta_2\eta_4^{-1}\right)^2 \circ \left(h_2^{-1}\nu h_2 h_4^{-1}\right)^{-2}\\ &= \nu\mu\eta_4^{-2}h_4^{+2}.\end{aligned}$$

(5) By Corollay II.7.

$$\begin{aligned}A_5 &= (A_4)_{E_2^{-1}E_3E_7^{-1}E_8} = \qquad \text{(from (4))}\\ &= \nu\mu(\eta_4)^{-2}_{E_2^{-1}E_3E_7^{-1}\ E_8}(h_4)^2_{E_2^{-1}E_3E_7^{-1}E_8}.\end{aligned}$$

By Lemma IV.2.3 , $(h_4, \tilde{T}_4)$ is coherent with $\left((h_4)_{E_2^{-1}E_3E_7^{-1}E_8}, \tilde{T}_5\right)$. But, $(h_5, \tilde{T}_5)$ is coherent with $(h_1, \tilde{T}_1)$. Thus, $\left((h_4)_{E_2^{-1}E_3E_7^{-1}E_8}, \tilde{T}_5\right)$ is coherent with $(h_1, \tilde{T}_1)$. From uniqueness $(h_4)_{E_2^{-1}E_3E_7^{-1}E_8} = h_5$. Similarly, or from Claim IV.5.3, $(\eta_4)_{E_2^{-1}E_3E_7^{-1}E_8} = \eta_5$. Thus,

$$A_5 = \nu\mu\eta_5^{-2}h_5^{+2}.$$

(6) By Corollary II.7,

$$A_6 = E_4^{-2}A_4^{-1}(A_4)_{E_6^{-1}}(E_4^2)_{E_6^{-1}}.$$

By (4) above, A_4 is a product of prime elements, with s.h.t. $\tilde{T}_4$. Thus, A_4 commutes with E_1^2. Thus:

$$A_6 = A_4^{-1}E_4^{-2}(E_4^2)_{E_6^{-1}}(A_4)_{E_6^{-1}}.$$

By Lemma IV.5.3, $E_4^{-2}(E_4^2)_{E_6^{-1}} = \eta_6$. Thus,

$$A_6 = \nu\mu\ h_4^{+2}\eta_4^{-2}\eta_6\nu\mu(\eta_4^{-2})_{E_6^{-1}}(h_4^{+2})_{E_6^{-1}}.$$

Since $\tilde{T}_4$ and $\tilde{T}_6$ are not orderly adjacent,

$$\begin{aligned}(\eta_4)_{E_6^{-1}} = (\eta_4)_{\tilde{T}_6^{-1}} &= \eta_6^{-1}\eta_4\\ (h_4)_{E_6^{-1}} &= h_6^{-1}h_4.\end{aligned}$$

Thus,

$$\begin{aligned}A_6 &= h_4^{-2}\eta_4^{+2}\eta_6(\eta_6^{-1}\eta_4)^{-2}(h_6^{-1}h_4)^2\\ &= h_4^{-2}\eta_4^{+2}\eta_6\eta_4^{-1}\eta_6\eta_4^{-1}\eta_6h_6^{-1}h_4h_6^{-1}h_4.\end{aligned}$$

Since T_4 and T_6^{-1} are adjacent, $[h_4, \eta_6] = [h_4, h_6] = \nu$ (Lemma IV.6.2), and thus, $h_4h_6^{-1} = h_6^{-1}h_4\nu$ and $\eta_6h_4 = h_4\eta_6\nu$. Moreover, h_4 and η_4 commute. Thus,

$$A_6 = \nu^9\eta_4^{+2}\eta_6\eta_4^{-1}\eta_6\eta_4^{-1}\eta_6h_6^{-2}.$$

Since $[\eta_6, \eta_4] = \mu$, $\eta_6\eta_4^{-1} = \eta_4^{-1}\eta_6\mu$, and thus, $A_6 = \nu^9\mu^3\eta_6^3h_6^{-2} = \eta_5^3h_6^{-2}\nu\mu$.

(7) and (8): Similar to the proof of (5).

(9) By Corollary II.7,

$$A_9 = E_5^{-2}A_5^{-1}(A_5)_{E_9^{-1}}(E_5^2)_{E_9^{-1}}.$$

Since A_5 is a product of prime elements in G with s.h.t. $\tilde{T}_5$, it commutes with E_5^2. Thus,

$$A_9 = A_5^{-1}E_5^{-2}(E_5^2)_{E_9^{-1}}(A_5)_{E_9^{-1}}.$$

By Lemma IV.5.3, $E_5^{-2}(E_5^{+2})_{E_9^{-1}} = \mu\eta_9^{-1}$. Thus,

$$A_9 = \nu\mu h_5^{-2}\eta_5^2 \cdot \mu\eta_9^{-1} \cdot \left(\nu\mu\eta_5^{-2}h_5^{+2}\right)_{E_9^{-1}}.$$

Since T_5 and T_9 are orderly adjacent,

$$\begin{aligned}(\eta_5)_{E_9^{-1}} = (\eta_5)_{\tilde{T}_9^{-1}} &= \eta_5\eta_9\\ (h_5)_{E_9^{-1}} &= h_5h_9.\end{aligned}$$

Thus,

$$A_9 = h_5^{-2}\eta_5^2\mu\eta_9^{-1}(\eta_5\eta_9)^{-2}(h_5h_9)^2.$$

Since $[\eta_5, \eta_9] = \mu$, then $\eta_9^{-1}\eta_5^{-1} = \eta_5^{-1}\eta_9^{-1}\mu$. Since $[h_5, h_9] = [h_5, \eta_9] = \nu$, then $h_9h_5 = h_5h_9^{-1}\nu$ and $\eta_9^{-1}h_5 = h_5\eta_9^{-1}\nu$. Thus we can collect all h_5 and η_5 at the left to get

$$A_9 = h_5^{-2}\eta_5^2\eta_5^{-2}h_5^2\eta_9^{-1}\eta_9^{-2}h_9^2 \cdot \mu^4\ \nu^3.$$

Thus, $A_9 = \eta_9^{-3}h_9^2\nu$.

□

Lemma IV.8.2. (a) $h_i^3 = \eta_i^3 \quad \forall i = 1 \ldots 9$.
(b) $\nu\mu = 1$.

Proof. For T_i and T_j orderly adjacent

$$\begin{aligned}(h_i)_{E_j^{-1}E_i^{-1}} &= (h_ih_j)_{E_i^{-1}}\\ &= \nu h_i^{-1}h_jh_i \quad \text{(By Lemma IV.6.3)}\\ &= \nu h_i^{-1}\nu h_ih_j \quad \text{(By Lemma IV.6.2)}\\ &= h_j \qquad\quad \text{(Since } \nu^2 = 1, \nu \in \text{Center}(G)).\end{aligned}$$

Similarly,

$$(\eta_i)_{E_j^{-1}E_i^{-1}} = \eta_j$$

Thus

$$\begin{aligned}(h_j)_{E_iE_j} &= h_i\\ (\eta_j)_{E_iE_j} &= \eta_i.\end{aligned}$$

(a) By Corollary II.7,

$$(A_9)_{E_8^{-1}E_9^{-1}} = E_9^2\ A_8(E_9^{-2})_{E_8^{-1}} = E_9^2\ h_8^{-1}\ \eta_8(E_9^{-2})_{E_8^{-1}}.$$

Since T_9 is a half-twist adjacent to T_8 and h_8, η_8 are prime elements with s.h.t. $\tilde{T}_8$, we get by Lemma IV.6.1 that $E_9^2\ h_8^{-1} = \nu h_8^{-1}\ E_9^2$ and $E_9^2\ \eta_8 = \mu\eta_8\ E_9^2$. Thus,

$$(A_9)_{E_8^{-1}E_9^{-1}} = E_9^2\ A_8(E_9^{-2})_{E_8^{-1}} = \nu\ \mu\ h_8^{-1}\ \eta_8\ E_9^2(E_9^{-2})_{E_8^{-1}}.$$

By Lemmas IV.5.3 and IV.8.0 $E_9^2(E_9^{-2})_{E_8^{-1}} = \mu\eta_8^{-1}$. Thus,

$$E_9^2 A_8 (E_9^{-2})_{E_8^{-1}} = \nu h_8^{-1}.$$

Thus, $(A_9)_{E_8^{-1}E_9^{-1}} = \nu h_8^{-1}$ and $A_9 = \nu(h_8^{-1})_{E_9 E_8} = \nu h_9^{-1}$. We compare this with the previous Lemma to get $\eta_9^{-3}h_9^2\nu = h_9^{-1}\nu$. Thus, $\eta_9^{-3}h_9^3 = 1$. Since $(\eta_9, \tilde{T}_9)$ is coherent with $(\eta_i, \tilde{T}_i)$ and $(h_9, \tilde{T}_9)$ is coherent with $(h_i, \tilde{T}_i)$, we can use Corollary III.3.5 to conclude that $\forall i\ \exists \tilde{B}_i \in \tilde{B}_9$ s.t. $\eta_i = (\eta_9)_{\tilde{B}_i}$ and $h_i = (h_9)_{\tilde{B}_i}$. Thus, $\eta_i^{-3}h_i^3 = 1 \ \ \forall i$.

(b) By Corollary II.7, $(A_9)_{E_8^{-1}E_9^{-1}} = E_9^2 A_8 (E_9^{-2})_{E_8^{-1}}$.

Thus,

$$\begin{aligned}
A_8 &= E_9^{-2}(A_9)_{E_8^{-1}E_9^{-1}}(E_9^2)_{E_8^{-1}} = && \text{(See IV.8.1)}\\
&= E_9^{-2}(h_9^{-2}\eta_9^{-3}\nu)_{E_8^{-1}E_9^{-1}}(E_9^{-2})_{E_8^{-1}} = && \text{(See above)}\\
&= E_9^{-2}h_8^2\eta_8^{-3}\nu(E_9^2)_{E_8^{-1}} = && \text{(See IV.6.1)}\\
&= E_9^{-2}(E_9^2)_{E_8^{-1}}h_8^2\eta_8^{-3}\nu^3\mu^3 = && \text{(See IV.5.3)}\\
&= \eta_8 h_8^2\eta_8^{-3}\nu\mu = && \text{(See IV.6.2)}\\
&= \eta_8^{-2}h_8^2\nu\mu.
\end{aligned}$$

We compare with the previous result to get $\nu\mu = 1$.

□

Proposition IV.8.3. *Let $A_i = E_{i'}E_i^{-1}$. Let $h_i, \eta_i\mu, \nu$ be as before. Then:*

(1) $A_1 = h_1$.

(2) $A_2 = h_2^{-1}\eta_2$.

(3) $A_3 = h_3\eta_3^{-1}$.

(4) $A_4 = h_4^{-1}\eta_4$.

(5) $A_5 = h_5^{-1}\eta_5$.

(6) $A_6 = h_6$.

(7) $A_7 = h_7\eta_7^{-1}$.

(8) $A_8 = h_8^{-1}\eta_8$.

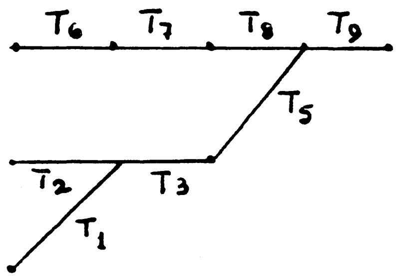


Figure V.1.

(9) $A_9 = h_9^{-1}\nu$.

Proof. Immediately from the previous 2 lemmas. □

Lemma IV.8.4. *$\{E_j, \eta_j, h_j\}_{j=1}^{9}$ generates G.*

Proof. By Lemma IV.7.1, $\{A_j, E_j\}$ generates G. By the previous lemma, A_j is a product of ν, h_j, η_j. However, ν is the central element of $(h_1, \tilde{T}_1)$. Thus, $\nu = (h_1)(h_1)_{\tilde{T}_1}$. □

V. Construction of $\mathbf{G_9}$, $\hat{\alpha} : \mathbf{G_9} \tilde{\to} \mathbf{G}$.

Construction of $\mathbf{G_0(9)}$.

Let $G_0(9)$ be the group generated by g_i $i = 1 \dots 9$ $i \neq 4$ with the following list of relations:

$[g_1, g_2]^2 = 1.$

$[g_1, g_2] \in \text{Center}(G_0(9).)$

$$[g_i, g_j] = \begin{cases} [g_1, g_2] & T_i, T_j \text{ are adjacent} \\ 1 & T_i, T_j \text{ are disjoint.} \end{cases}$$

where T_i are described as follows:

Denote $\nu = [g_1, g_2]$.

Let us reformulate the relations of $G_0(9)$ as follows:

$$G_0(9) = \Big\langle g_1,\dots,\check{g}_4,\dots,g_9 \;\Big|\; [g_i,g_j] = \begin{cases} \tau & T_i,T_j \text{ are adjacent,} \\ 1 & T_i,T_j \text{ are disjoint.} \end{cases} \quad \tau^2 = 1 \qquad \tau_{g_0} = \tau \Big\rangle$$

Remark. $G_0(9)$ or $G_0(n)$ in general can be described in a different way: Take A_{n-1}, a free abelian group on $n-1$ generators $A_{n-1} = \langle w_1,\dots,w_{n-1}\rangle$. Define a skew-symmetric form on A_{n-1} as follows:

$$w_i \cdot w_j = \begin{cases} 1 & |i-j| = 1 \\ 0 & |i-j| \neq 1. \end{cases}$$

Let $G_0(n)$ be the unique central extension that satisfies

$$1 \to \mathbb{Z}/2 \xrightarrow{b} G_0(n) \xrightarrow{a} A_{n-1} \to 1$$

where $G_0(n)$ is generated by $u_1 \dots u_{n-1}$, $a(u_i) = w_i$ and $[u_i,u_j] = \begin{cases} 1 & |i-j| \neq 1 \\ \tau & |i-j| = 1 \end{cases}$ and $b(1) = \tau$. We have: $\mathrm{Ab}(G_0(n)) = A_{n-1}$, $G_0(n)' = \{\tau, 1\} \simeq \mathbb{Z}/2$.

$\mathbf{G_0(9)}$ as a $\mathbf{\tilde{B}_9}$-group.

$$(g_i)_{\tilde{T}_k} = \begin{cases} g_i^{-1}\tau & i = k \\ g_i & T_i \text{ and } T_k \text{ are disjoint} \\ g_k g_i & T_i \text{ and } T_k \text{ are orderly adjacent} \\ g_i g_k^{-1} & T_i \text{ and } T_k \text{ are not orderly adjacent.} \end{cases}$$

Remark V.0. Let $G_0(9)$ and g_i be as above. Then g_i is a prime element of $G_0(9)$ with s.h.t. $\tilde{T}_i$ and central element τ.

Proof. By the actions of $\tilde{B}_9$ on g_i and the axioms of prime element. □

Consider the semidirect product: $\tilde{B}_9 \ltimes G_0(9)$.

Construction of $\overset{\rtimes}{\alpha} : \tilde{\mathbf{B}}_9 \rtimes \mathbf{G}_0(\mathbf{9}) \to \mathbf{G}$.

$\overset{\rtimes}{\alpha} \mid_{\tilde{B}_9} = \alpha$

$\overset{\rtimes}{\alpha}(\tilde{T}_i) = E_i$

$\overset{\rtimes}{\alpha}(\xi_i) = \alpha(\xi_i) = \eta_i$ (Lemma IV.5.3)

$\overset{\rtimes}{\alpha}(c) = \alpha(c) = \mu$

$\overset{\rtimes}{\alpha} \mid_{G_0(9)}$ defined by $\overset{\rtimes}{\alpha}(g_i) = h_i$.

$\overset{\rtimes}{\alpha}(\tau) = \overset{\rtimes}{\alpha}[g_1, g_2] = [h_1, h_2] = \nu$.

Since ν is the central element of h_i it belong to Center(G) and is of order 2. Thus, by Proposition IV.4.2 all relations that g_i satisfy are also satisfied by h_i, and thus $\overset{\rtimes}{\alpha} \mid_{G_0(9)}$ is well-defined.

By Lemma IV.4.3, $\hat{\alpha} \mid_{G_0(9)}$ is compatible with the action of $\tilde{B}_9$ on $G_0(9)$ and thus we have $\overset{\rtimes}{\alpha} : \tilde{B}_9 \rtimes G_0(9) \to G$.

Construction of $\mathbf{G_9}$.

Let g_i be the generators of $G_0(9)$ as above.

Let $\tau = [g_1, g_2]$.

Let ξ_i be the prime elements in $\tilde{B}_9$ defined in Chapter IV.4 (ξ-situation).

Let c be the central element of ξ_i. $c = [\tilde{T}_1^2, \tilde{T}_2^2]$.

Let $N_9 \subseteq \tilde{B}_a \rtimes G_0(9)$ be the normal subgroup generated by τc and $(g_i\xi_i)^3$:

$N_9 = \langle (\tau c^{-1}, \quad (g_i\xi_i^{-1})^3 \ i = 1 \dots 9 \ i \neq 4 \rangle$.

$G_9 = \tilde{B}_9 \rtimes G_0(9)/N_9$.

Construction of $\hat{\alpha}$, $\hat{\alpha} : \mathbf{G_9} \to \mathbf{G}$.

By Lemma IV.4.5, $\nu\mu = 1$, $(\eta_i\eta_i^{-1})^3 = 1$, so $\overset{\rtimes}{\alpha}(N_9) = 1$. Thus $\overset{\rtimes}{\alpha}$ induces a map on G_9 denoted by $\hat{\alpha}$. $\hat{\alpha} : G_9 \to G$.

Construction of $\hat{\beta} : \mathbf{G} \to \mathbf{G_9}$.

We start by using a set of generators for G, $\{\Gamma_i, \Gamma_{i'}\}_{i=1}^9$.

We then choose as a set of generators $\{E_i, E_{i'}\}_{i=1}^9$ where $E_i = \begin{cases} \Gamma_i & i \neq 2,7 \\ \Gamma_{i'} & i = 2,7 \end{cases}$ and $E_{i'} = (E_i)\varphi_i$.

The third set of generators was $\{E_i, A_i\}_{i=1}^9$ where $A_i = E_{i'}E_i^{-1}$.
Finally, we take the following set of generators for $G: E_i, h_i, \eta_i$.
We define $\hat{\beta}: G \to G_9$ on the third set of generators as follows:
$\hat{\beta}(E_i) = \tilde{T}_i \quad i = 1 \dots 9$.
$\hat{\beta}(A_1) = g_1$.
$\hat{\beta}(A_2) = g_2^{-1}\xi_2$.
$\hat{\beta}(A_3) = g_3\xi_3^{-1}$.
$\hat{\beta}(A_4) = g_4^{-1}\xi_4$.
$\hat{\beta}(A_5) = g_5^{-1}\xi_5$.
$\hat{\beta}(A_6) = g_6$.
$\hat{\beta}(A_7) = g_7\xi_7^{-1}$.
$\hat{\beta}(A_8) = g_8^{-1}\xi_8$.
$\hat{\beta}(A_9) = \tau g_9^{-1}$.

Remark. By definition of $\hat{\alpha}$ and by the formula for expressing A_i in terms of h_i and η_i (Lemma IV.8.3), if $\hat{\beta}$ is well-defined, then $\hat{\alpha}\hat{\beta} = Id$.

Theorem V.1. *$\hat{\beta}$ is well-defined.*

Proof. We recall that $G = F_{18}/G(\varepsilon(18))$ where F_{18} is the free group generated on $\{\Gamma_i, \Gamma_{i'}\}_{i=1}^9$ and $G(\varepsilon(18))$ is the subgroup generated by the relations induced from the factors in the braid monodromy factorization $\varepsilon(18)$ (see Theorem I.1) by the Van Kampen method. To prove that $\hat{\beta}$ is well-defined, we have to prove that all relations induced by the Van Kampen Theorem are valid when each generator in a relation is replaced by its image under $\hat{\beta}$. In what follows we shall take every braid in the braid monodromy factorization $\varepsilon(18)$ and use the Van Kampen method to deduce from it a relation on $\pi_1(\mathbb{CP}^2 - S)$ in terms of Γ_i and $\Gamma_{i'}$. Then we shall present the relation in terms of E_i and A_i. The next step is to substitute $\tilde{T}_i$ instead of E_i and $\hat{\beta}(A_i)$ instead of A_i and confirm that the relation holds.

Denote:
$t_i = \hat{\beta}(E_i) = \tilde{T}_i$.
$t_{i'} = \hat{\beta}(E_i')$.
$t_{i''} = \hat{\beta}(E_iE_{i'}E_i^{-1})$.
$a_i = \hat{\beta}(A_i) = \beta(E_{i'}E_i^{-1}) = t_{i'}t_i^{-1}$.
We have expressions for a_i in terms of g_i, ξ_i, τ.
$a_1 = g_1$.
$a_2 = g_2^{-1}\xi_2$.
$a_3 = g_3\xi_3^{-1}$.

$a_4 = g_4^{-1}\xi_4$.
$a_5 = g_5^{-1}\xi_5$.
$a_6 = g_6$.
$a_7 = g_7\xi_7^{-1}$.
$a_8 = g_8^{-1}\xi_8$.
$a_9 = \tau g_9^{-1}$.
Then:
$t_i = \tilde{T}_i$.
$t_{i'} = a_i t_i$.
$t_{i''} = (a_i t_i)_{t_i} = (a_i)_{t_i} \cdot t_i = t_i (a_i)_{t_i^2} = t_i a_i$ (Lemma II.3).

$\varepsilon(18)$ implies relations of type $\langle b_1, b_2\rangle = 1$ relations of type $[b_1, b_2] = 1$ and relations of type $b_1 = b_2$. We shall treat all relations of type $\langle b_1, b_2\rangle$ together. To do this we prove the following two lemmas:

Lemma 1. $(a_i)_{t_i} = (a_i)_{t_i^{-1}} = \begin{cases} \tau a_i^{-1} & i = 1,6,9 \\ a_i^{-1} & \textit{otherwise.} \end{cases}$

Proof of Lemma 1. The elements g_i and ξ_i are prime elements with s.h.t. t_i. By Lemma II.5, for a prime element g with s.h.t. t, we have:
$(g)_t = (g)_{t^{-1}} = \tau g^{-1}$.

For such g we then also have:
$(g^{-1})_t = (g^{-1})_{t^{-1}} = \tau^{-1} g$.

Since $\tau^{-1} = \tau$ we have for every f a prime element with s.h.t. t or an inverse of a prime element with s.h.t. t:
$(f)_t = (f)_{t^{-1}} = \tau f$.

For $i = 1,6,9$, one can see from the above list that a_i is a prime element, or τ multiplied by an inverse of a prime element, and since $\tau \in \mathrm{Center}(G)$, we get the Lemma for $i = 1,6,9$.

For $i \neq 1,6,9$, a_i is a product of a prime element with an inverse of a prime element. Write $a_i = f_i \cdot f_i'$, thus, $(a_i)_{t_i} = (a_i)_{t_i^{-1}}$ will be the product of the inverse of the 2 factors times τ^2 ($\tau \in \mathrm{Center}(G)$). By Lemma IV.6.2 such two factors commute. Thus,

$$\begin{aligned}(a_i)_{t_i} &= (f_i)_{t_i}(f_i')_{t_i} \\ &= \tau f_i^{-1}\tau(f_i')^{-1} = \tau^2 f_i^{-1} f_i'^{-1} = f_i^{-1} f_i'^{-1} = f_i'^{-1} f_i^{-1} = a_i^{-1}\end{aligned}$$

The same is true for $(a_i)_{t_i^{-1}}$. □ Lemma 1

Lemma 2. *If T_i and T_j are adjacent and $d_i = a_i$ or $(a_i)_{t_i}$ then* $(d_i)_{t_{\underline{j}}^{-1} t_i^{-1}} = d_i^{-1}(d_i)_{t_{\underline{j}}^{-1}}$ *for* $t_{\underline{j}} = t_j$ *or* $t_{j'}$ *or* $t_{j''}$.

Proof of Lemma 2. By the above list:
$a_i = g_i$ or $g_i^{-1}\xi_i$ or $g_i\xi_i^{-1}$ or τg_i^{-1}.
By the above Lemma:
$(a_i)_{t_i} = \tau g_i^{-1}$ or $\xi_i^{-1} g_i$ or $\xi_i g_i^{-1}$ or g_i.
Thus, $d_i = g_i$ or τg_i^{-1} or $(g_i\xi_i^{-1})^{\pm 1}(g_i^{-1}\xi_i)^{\pm 1}$. □

If d_i is of the form g_i or τg_i^{-1} it satisfies the Lemma by the definition of prime element (axiom 2). For symmetry reasons we shall only treat the case $d_i = g_i^{-1}\xi_i$.

Case 1. $t_{\underline{j}} = t_j$

$$(d_i)_{t_j^{-1} t_i^{-1}} = (g_i^{-1}\xi_i)_{t_j^{-1} t_i^{-1}} = (g_i)^{-1}_{t_j^{-1} t_i^{-1}} \cdot (\xi_i)_{t_j^{-1} t_i^{-1}}$$
$$\overset{\text{By axiom 2 of the prime element}}{=} (g_i^{-1} g_{i t_j^{-1}})^{-1} \cdot \xi_i^{-1} \cdot (\xi_i)_{t_j^{-1}}$$
$$= (g_i^{-1})_{t_j^{-1}}\ g_i\ \xi_i^{-1}\ (\xi_i)_{t_j^{-1}}$$
$$\overset{\text{By Proposition IV.6.2}}{=} \xi_i^{-1} g_i (g_i^{-1})_{t_j^{-1}} \cdot (\xi_i)_{t_j^{-1}}$$
$$= (g_i^{-1}\xi_i)^{-1}(g_i^{-1}\xi)_{t_j^{-1}} = d_i^{-1}(d_i)_{t_j^{-1}}.$$

Case 2. $t_{\underline{j}} = t_{j'}$
$(d_i)_{t'_j{}^{-1}} = (d_i)_{(t_j a_j)^{-1}} = ([a_j, d_i] d_i)_{t_j^{-1}} = [a_j d_i]_{t_j^{-1}} \cdot (d_i)_{t_j^{-1}}$.
By Proposition IV.6.2, Claim II.4 and by the formulas for a_i, $[a_j, d_i]$ is a product of τ. Since τ is of order 2, $[a_j, d_i] = \tau^{\varepsilon}$ $\varepsilon = 0, 1$. Thus, $(d_i)_{t_{j'}^{-1}} = \tau^{\varepsilon}(d_i)_{t_j^{-1}}$. So we get the claim from case 1 when multiplying each side of the equation there by τ^{ε} to get the equation for $t_{j'}$.

□ Case 3

$t_{\underline{j}} = t_{j''}$
As in Case 2 we get $(d_i)_{t''_j} = (d_i)_{t_j}\tau^{\varepsilon}$ and we use Case 1 to get Case 3.
□ Lemma 2

Since T_i and T_j are adjacent, $\langle t_i, t_j \rangle = 1$. We use the above Lemma to deduce from Lemma IV.3.1 that $\langle d_i t_i, t_{\underline{j}} \rangle = 1 \quad \underline{j} = j, j', j''$. Since $d_i = a_i$ or $(a_i)_{t_i}$ we get $\langle t_{\underline{i}}, t_{\underline{j}} \rangle = 1$. This covers all the triple relations which are induced from $\varepsilon(18)$.

Lemma 3. *For i, j s.t. T_i and T_j are disjoint or transversal, we have $[a_i, t_j] = 1$ and $[t_{\underline{i}}, t_{\underline{j}}] = 1$ where $\underline{i} = i$ or i', $\underline{j} = j$ or j'.*

Proof of Lemma 3. It is enough to prove $[a_i, t_j] = [t_i, t_j] = [t_{i'}, t_j] = [t_{i'}, t_{j'}] = 1$.

If T_i and T_j are disjoint, then $[T_i, T_j] = 1$. If T_i and T_j are transversal, then $[\tilde{T}_i, \tilde{T}_j] = 1$. In any case, $[\tilde{T}_i, \tilde{T}_j] = 1$ and thus, $[t_i, t_j] = 1$.

Now, a_i is a prime element with s.h.t. t_i or a product of 2 prime elements. If t_j is disjoint from t_i we know that each of the prime factors in a_i commutes with t_j by the definition of prime element. If t_j is transversal to t_i then each of the prime factors in a_i commutes with t_j by Lemma III.3.5. In any case, $[a_i, t_j] = 1$. Now, $t_{i'} = a_i t_i, \quad [t_{i'}, t_j] = [a_i t_i, t_j]$. Since t_i commutes with t_j and a_i, then $[t_{i'}, t_j] = [a_i, t_j]$, which equal 1. Now, $[t_{i'}, t_{j'}] = [a_i t_i, a_j t_j]$. Since t_j commutes with $a_i t_i$ and a_j commutes with t_i to prove that $t_{i'}$ commutes with $t_{j'}$ it is enough to prove $[a_i, a_j] = 1$. This follows from Proposition IV.6.2. □

We use the following Lemma to show that all commutation relations in $\varepsilon(18)$ are satisfied when t_i is replacing E_i and a_i is replacing A_i.

Lemma 4. *Let $\tilde{Z}_{ij}^2$ be a braid in B_{18} s.t. $\tilde{Z}_{ij}$ connects q_i or $q_{i'}$ with q_j or $q_{j'}$ where outside of 2 small discs centered at $q_i, q_{i'}$ and $q_j, q_{j'}$ respectively, the path goes below the real line, except when it goes above some of the pairs $q_k, q_{k'}$ (for $i < k < j$ $k \in K$). Assume T_i and T_j are disjoint or transversal. Then the relation induced by $\tilde{Z}_{ij}^2$ via the Van Kampen-Zariski method is mapped to 1 under β.*

Proof. We cut $\tilde{Z}_{ij}$ into 2 pieces, one connects u with the disc around q_i and $q_{i'}$ from below and the other one connects u with the disc around q_j and $q_{j'}$ above the pairs $q_k q_{k'}$ $k \in K$. Thus the relation induced from $\tilde{Z}_{ij}^2$ is

$$\left[\Gamma_{\underline{i}}, \left(\prod_{k \in K} \Gamma_k^{-1} \Gamma_{k'}^{-1}\right) \Gamma_{\underline{j}} \left(\prod_{k \in K} \Gamma_{k'} \Gamma_k\right)\right] = 1$$

where $\Gamma_{\underline{i}} = (\Gamma_i)\rho_i^m$ for some m. Since $(\Gamma_i)\rho_i = \Gamma_{i'}$, $(\Gamma_{i'})\rho_i = \Gamma_{i'}\Gamma_i\Gamma_{i'}$, we know that $\Gamma_{\underline{i}} \in \langle(\Gamma_i, \Gamma_{i'}\rangle$.

It is enough to consider $\Gamma_{\underline{i}} = \Gamma_i$ or $\Gamma_{i'}$ and $\Gamma_{\underline{j}} = \Gamma_j$ or $\Gamma_{j'}$, since by proving that Γ_i and $\Gamma_{i'}$ commutes with $X\Gamma_j X^{-1}$, we can conclude that $X\Gamma_j X^{-1}$ commutes with every element g from $\langle \Gamma_i, \Gamma_{i'}\rangle$; in particular, $\Gamma_{\underline{i}}$ commutes with $X\Gamma_j X^{-1}$. Similarly, $\Gamma_{\underline{i}}$ commutes with $X\Gamma_{j'}X^{-1}$. Thus, $[X^{-1}\Gamma_{\underline{i}}X, \Gamma_j] = [X^{-1}\Gamma_{\underline{i}}X, \Gamma_{j'}] = 1$. So, $[X^{-1}\Gamma_{\underline{i}}X, \Gamma_{\underline{j}}] = 1$.

Now, $\beta(\Gamma_{\underline{i}}) = t_i$ or $t_{i'}$, $\beta(\Gamma_{\underline{j}}) = t_j$ or $t_{j'}$, $\beta(\Gamma_{k'}\Gamma_k) = t_{k'} \cdot t_k = a_k t_k^2$. So

$$\beta\left(\left(\prod_{k\in K} \Gamma_k^{-1}\Gamma_{k'}^{-1}\right)\Gamma_{\underline{j}}\left(\prod_{k\in K}\Gamma_{k'}\Gamma_k\right)\right) = \left(\prod_{k\in K} t_k^{-2}a_k^{-1}\right)a_j^{\delta}t_j\left(\prod_{k\in K} a_k t_k^2\right),$$

which is a product of squares of half-twists and prime elements. Thus, we can use Propositions IV.6.1 and IV.6.2 to rearrange the factors in the above product while multiplying the product with the appropriate ν^{ε} which is a central element to get

$$\tau^{\varepsilon}a_j^{\delta}\prod_{k\in K} a_k^{-1}(a_k)_{t_j^{-1}}\prod_{k\in K} t_k^{-2}(t_k^2)_{t_j^{-1}} \cdot t_j.$$

Now,

$$a_k^{-1}(a_k)_{t_j^{-1}} = \begin{cases} (a_k)_{t_j^{-1}t_k^{-1}} & t_j, t_k \text{ adjacent} \qquad\qquad \text{(Lemma 2)} \\ 1 & t_j, t_k \text{ disjoint or transversal (Lemma 3).}\end{cases}$$

Since a_k is a product of prime elements with τ by Lemma III.2.2, $(a_k)_{t_j^{-1}t_h^{-1}}$ is also such a product. Thus, $a_k^{-1}(a_k)_{t_j^{-1}}$ is 1 or a prime element supported on t_j.

By Lemma IV.4.0 when t_k and t_j are adjacent, $(t_k^{-2})(t_k^2)_{t_j^{-1}}$ is a product of c with an inverse of a prime element supported on t_j. By Remark III.1.1, if t_k and t_j are disjoint or transveral then $t_k^{-2}(t_k^2)_{t_j^{-1}} = 1$.

Thus,

$$\begin{aligned}\beta&\left(\left(\prod_{k\in K}\Gamma_k^{-1}\Gamma_{k'}^{-1}\right)\Gamma_{\underline{j}}\left(\prod_{k\in K}\Gamma_{k'}\Gamma_{k'}^{-1})\right)\right)\\ &= \tau^{\varepsilon}a_j^{\sigma}c^t\left(\prod\begin{pmatrix}\text{prime elements supported}\\ \text{on } t_j \text{ or their inverse}\end{pmatrix}\right)\cdot t_j\\ &= c^t\tau^{\varepsilon}(t_{j'}t_j^{-1})^{\delta}\left(\prod\begin{pmatrix}\text{prime elements supported}\\ \text{on } t_j \text{ or their inverse}\end{pmatrix}\right)\cdot t_j.\end{aligned}$$

We have to check if this product commutes with $\beta(\Gamma_{\underline{i}})$. Since $\Gamma_{\underline{i}}$ belongs to the subgroup generated by Γ_i and $\Gamma_{i'}$ which is the same as the subgroup generated by E_i and $E_{i'}$, $\beta(\Gamma_{\underline{i}})$ is a product of $\beta(E_i)$ $(= t_i)$ and of $\beta(E_{i'})$ $(= t_i')$. Thus it is enough to check that the above product commutes with t_i and $t_{i'}$.

Now, $t_i = \tilde{T}_i$ and $t_j = \tilde{T}_j$ are transversal or disjoint, thus by Remark III.1.1(d) $t_i, t_{i'}$ commutes with $t_j, t_{j'}$. By the definition of prime element (Axiom 3) and Lemma III.4.2, a prime element supported on t_j commutes with t_i and t_i'. Finally, $c, \tau \in \text{Center}(G_9)$, and thus all the factors in the above product commute with t_i and $t_{i'}$ and, therefore, the product commutes. □ Lemma 4

Lemma 4 covers all degree 2 elements that appear in $\varepsilon(18)$.

We still have to check the degree 1 elements that appear in $\varepsilon(18)$.

We recall from Lemma II.2 that for $E_i = \begin{cases} \Gamma_i & i \neq 2,7 \\ \Gamma_{i'} & i = 2,7 \end{cases}$ and $A_i = E_{i'}E_i^{-1}$ we have

(1) $\Gamma_{i'}\Gamma_i = E_{i'}E_i = A_iE_i^2$.

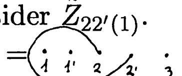
(2) $\Gamma_i = \begin{cases} E_i & i \neq 2,7 \\ E_i^{-1}E_{i'}E_i = E_i^{-1}A_iE_i^2 & i = 2,7. \end{cases}$

(3) $\Gamma_{i'} = \begin{cases} E_i = A_iE_i & i \neq 2,7 \\ E_i & i = 2,7 \end{cases}$

We have to consider each of the degree 1 elements in $\varepsilon(18)$ and to apply β on both sides of the induced relation in order to prove that we get the same element of G_9.

We want to rewrite each side of the induced relation as a product of the following form: $\tau^\varepsilon \cdot$ prime elements $\cdot$ half-twists, and then use formulas for the action of $\tilde{B}_9$ on G_9 and Lemma 1.

We shall only consider here one braid which is a degree 1 element. We shall consider $\tilde{Z}_{22'(1)}$.

$\tilde{z}_{22'(1)} =$ 

$\tilde{Z}_{22'(1)}$ implies by the RMS method the relation

$\Gamma_{2'} = \Gamma_{1^{-1}}\Gamma_{1'}^{-1}\Gamma_2\Gamma_{1'}\Gamma_1$.

From (2) above, this relation is actually:

$E_2 = E_1^{-2}A_1^{-1}E_2^{-1}A_2E_2^2A_1E_1^2$.

Recall that $t_i = \beta(E_i)$. $a_i = \beta_i(A_i)$. Thus,

$$
\begin{aligned}
\beta(\ell s) &= t_2 \\
\beta(rs) &= t_1^{-2} a_1^{-1} t_2^{-1} a_2 t_2^2 a_1 t_1^2 = \\
&= (a_1)^{-1}_{t_1^{+2}} (a_2)_{t_2 t_1^{+2}} (a_1)_{t_1^{-2} t_2 t_1^{+2}} \cdot t_1^{-2} t_2 t_1^2 = \text{ (using Lemma 1)} \\
&= a_1^{-1} \cdot (a_2^{-1})_{t_1^2} (a_1)_{t_2 t_1^2} t_1^{-2} t_2 \ t_1^2 = \\
&\qquad \text{(using Lemma III.2.1(2) applied on } a_2 \text{ and on } (a_1)_{t_2}) \\
&= g_1^{-1} (g_2^{-1} \xi_2)^{-1} (g_1)_{t_2} \cdot t_1^{-2} t_2 t_1^2 = \\
&= g_1^{-1} \xi_2^{-1} g_2 g_1 g_2^{-1} \cdot t_1^{-2} t_2 t_1^2 = \text{ (using } [g_1^{-1}, g_2] = \tau \text{ and } [g_1^{-1}, \xi_2^{-1}] = \tau) \\
&= \tau^2 \cdot \xi_2 g_2 g_1^{-1} g_1 g_2^{-1} \cdot t_1^{-2} t_2 t_1^2 \\
&= \xi_2^{-1} t_1^{-2} t_2 t_1^2 t_2^{-1} \cdot t_2 \\
&= \xi_2^{-1} \cdot t_1^{-2} (t_1^2)_{t_2^{-1}} \cdot t_2 \\
&= \xi_2^{-1} \cdot \quad \xi_2 \qquad \cdot t_2 = t_2 \ \text{ (Lemma IV.4.2)}
\end{aligned}
$$

□ Theorem V.1

Lemma V.2. $\hat{\alpha}\hat{\beta} = I_d \qquad \hat{\beta}\tilde{\alpha} = I_d$

Proof. By definition of $\hat{\alpha}$ and by the formula for expressing A_i in terms of h_i and η_i (Lemma IV.8.3), $\hat{\alpha}\hat{\beta} = Id$.

For $\hat{\beta}\hat{\alpha}$ recall that G_9 is generated by $\tilde{T}_i$ and g_i, i.e., by t_i and g_i, i.e., by $\tilde{T}_i, a_i$ and ξ_i.

$\hat{\beta}\hat{\alpha}(t_i) = \hat{\beta}\hat{\alpha}(\tilde{T}_i) = \hat{\beta}(\tilde{E}_i) = \tilde{T}_i = t_i$.

By Proposition IV.8.3 and by $\hat{\alpha}(\xi_i) = \eta_i$, $\quad \hat{\alpha}(g_i) = h_i$, we get $\hat{\alpha}(a_i) = A_i$. Thus, $\hat{\beta}\hat{\alpha}(a_i) = \hat{\beta}(A_i) \overset{\text{by def.}}{=} a_i$

In Lemma IV.5.3 we have expressions for η_i, in terms of E_i. If we apply $\hat{\beta}$ on these expressions, we get the same expressions where E_i is replaced by $\tilde{T}_i$. These expressions are exactly the expressions for ξ_i as products of $\tilde{T}_i$'s from Lemma IV.4.2. Thus, $\hat{\beta}(\eta_i) = \xi_i$.

For η_4 we could also use the expression $\eta_4 = (\eta_5)_{\tilde{T}_8^{-1}\tilde{T}_7\tilde{T}_3^{-1}\tilde{T}_2}$. Apply on it the $\tilde{B}_9$ homomorphism $\hat{\beta}$ to get $\hat{\beta}(\eta_4) = (\hat{\beta}(\eta_5))_{\tilde{T}_8^{-1}\ldots\tilde{T}_2} \overset{\text{from above}}{=} (\xi_5)_{\tilde{T}_8^{-1}\ldots\tilde{T}_2} \overset{\text{Lemma IV.4.2}}{=} \xi_4$.

So $\hat{\beta}\hat{\alpha}(\xi_i) = \hat{\beta}(\eta_i) = \xi_i$ by IV.4.2 and IV.5.3.

Thus, $\hat{\beta}\hat{\alpha} = Id$. □

Corollary V.3. $G \simeq \dfrac{\tilde{B}_9 \ltimes G_0(9)}{N_9} = G_9$

Proof. $\hat{\beta} : G \to G_9$ is an isomorphism. □

VI. Main Results and Formulations of Additional Results.

In this chapter we shall state the main results concerning the fundamental group of the complement of a branch curve of a Veronese surface of order 3 proven in the previous chapters. The theorem is formulated in Theorem VI.1. We shall also formulate additional results on fundamental groups that were proven in earlier works as well as future results.

VI.1. Main results and "forthcoming" results.

In order to phrase the main results we recall a few definitions.

Definition. *Braid group* $B_n = B_n[D, K]$

Let D be a closed disc in $\mathbb{R}^2$, $K \subset D$, K finite. Let B be the group of all diffeomorphisms β of D such that $\beta(K) = K$, $\beta|_{\partial D} = \mathrm{Id}_{\partial D}$. For $\beta_1, \beta_2 \in B$, we say that β_1 is equivalent to β_2 if β_1 and β_2 induce the same automorphism of $\pi_1(D - K, u)$. The quotient of B by this equivalence relation is called the braid group $B_n[D, K]$ $(n = \#K)$. We sometimes denote by $\overline{\beta}$ the braid represented by β. The elements of $B_n[D, K]$ are called braids.

Definition. $H(\sigma)$, *half-twist defined by* σ

Let D, K be as above. Let $a, b \in K$, $K_{a,b} = K - a - b$ and σ be a simple path in $D - \partial D$ connecting a with b s.t. $\sigma \cap K = \{a, b\}$. Choose a small regular neighborhood U of σ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{C}^1$ ($\mathbb{C}^1$ is taken with usual "complex" orientation) such that $f(\sigma) = [-1, 1]$, $f(U) = \{z \in \mathbb{C}^1 \mid |z| < 2\}$. Let $\alpha(r), r \geq 0$, be a real smooth monotone function such that $\alpha(r) = 1$ for $r \in [0, \frac{3}{2}]$ and $\alpha(r) = 0$ for $r \geq 2$.

Define a diffeomorphism $h : \mathbb{C}^1 \to \mathbb{C}^1$ as follows. For $z \in \mathbb{C}^1$, $z = re^{i\varphi}$ let $h(z) = re^{i(\varphi + \alpha(r))}$. It is clear that on $\{z \in \mathbb{C}^1 \mid |z| \leq \frac{3}{2}\}, h(z)$ is the positive rotation on 180° and that $h(z) =$ Identity on $\{z \in \mathbb{C}^1 \mid |z| \geq 2\}$, in particular on $\mathbb{C}^1 - f(U)$. Considering $(f \circ h \circ f^{-1})|_D$ (we always take composition from left to right) we get a diffeomorphism of D which switches

a and b and is the identity on $D-U$. Thus it defines an element of $B_n[D,K]$, called the half-twist defined by σ and denoted $H(\sigma)$.

Definition. *Frame of* $B_n[D,K]$

Let D be a disc in $\mathbb{R}^2$. Let $K = \{a_1, \dots, a_n\}$ $K \subset D$. Let $\sigma_1, \dots, \sigma_{n-1}$ be a system of simple paths in $D - \partial D$ such that each σ_i connects a_i with a_{i+1} and for

$$i,j \in \{1,\dots,n-1\}\ ,\ i<j \quad , \quad \sigma_i \cap \sigma_j = \begin{cases} \emptyset & \text{if } |i-j| \geq 2 \\ a_{i+1} & \text{if } j = i+1\,. \end{cases}$$

Let $H_i = H(\sigma_i)$. We call the ordered system of (positive) half-twists $(H_1, \dots, H_{n-1})$ a frame of $B_n[D,K]$ defined by $(\sigma_1, \dots, \sigma_{n-1})$, or a frame of $B_n[D,K]$ for short.

Definition. *Transversal half-twists*

The half-twists $H(\sigma_1)$ and $H(\sigma_2)$ will be called *transversal* if C_1 and C_2 intersect transversally in one point which is not an end point of either of the σ_i's.

Definition. $\tilde{B}_n$

Let T_n be the subgroup of B_n normally generated by $[X,Y]$ for X,Y transversal half-twists. $\tilde{B}_n$ is the quotient of B_n modulo T_n. We choose a frame X_i of B_n. We denote their images in $\tilde{B}_n$ by $\tilde{X}_i$.

Proposition-Definition. $G_0(n), \tau, u_1$

Let A_{n-1} be the free abelian group on $w_1, \dots, w_{n-1}d$. Let us define a $\mathbb{Z}/2$ skew-symmetric form on A_{n-1} as follows:

$$w_i \cdot w_j = \begin{cases} 1 & (i-j) = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

There exists a unique central extension $G_0(n)$, of $\mathbb{Z}/2$ by A_{n-1}, with generators $u_1 \dots u_{n-1}$ that satisfies

$$1 \to \mathbb{Z}/2 \xrightarrow{b} G_0(n) \xrightarrow{a} A_{n-1} \to 1$$

$$a(u_i) = w_i$$

$$[u_i, u_j] = b(w_i \cdot w_j) = \begin{cases} \tau & |i-j| = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

We always consider $G_0(n)$ with the standard $\tilde{B}_n$-action as follows:

$$(u_i)_{\tilde{X}_k} = \begin{cases} u_i^{-1}\tau & k = i \\ u_k u_i & |i-k| = 1 \\ u_i & |i-k| \geq 2 \end{cases} .$$

Claim. $\mathrm{Ab}(G_0(n)) = A_{n-1}$ *(free abelian group on $n-1$ generators),* $G_0(n)' = \{\tau, 1\} (\simeq \mathbb{Z}/2)$.

Proof. Claim III.6.4. □

Let ψ_n be the standard homomorphisms $B_n \overset{\psi_n}{\to} S_n$(= symmetric group).

Let Ab be the standard homomorphism $B_n \overset{\mathrm{Ab}}{\to} \mathbb{Z}$.

Since $\psi_n([X,Y]) = 1$, and $\mathrm{Ab}([X,Y]) = 1$, ψ_n and Ab induce homomorphisms on $\tilde{B}_n$.

Definition. $\tilde{\psi}_n, \tilde{P}_n, \tilde{P}_{n,0}, c$

$\tilde{\psi}_n : \tilde{B}_n \to S_n$, the induced homomorphism from ψ_n.

$\widetilde{\mathrm{Ab}} : \tilde{B}_n \overset{\mathrm{Ab}}{\to} \mathbb{Z}$, the induced homomorphism from $B_n \overset{\mathrm{Ab}}{\to} \mathbb{Z}$.

$\tilde{P}_n = \ker \tilde{\psi}_n$.

$\tilde{P}_{n,0} = \ker \tilde{\psi}_n \cap \ker \widetilde{\mathrm{Ab}} = \ker \tilde{P}_n \to \mathrm{Ab}(\tilde{B}_n) = \mathbb{Z}$.

$c = [\tilde{X}_1^2, \tilde{X}_2^2]$ for 2 consecutive half-twists.

Consider $\tilde{B}_9 \ltimes G_0(9)$ with respect to the standard $\tilde{B}_9$ action on $G_0(9)$.

Definition. $v_1, N_9, G_9, \tilde{\psi}_9 : G_9 \to S_9$

$v_1 = (\tilde{X}_2 \tilde{X}_1 \tilde{X}_2^{-1})^2 \tilde{X}_2^{-2}$ for a frame $X_1, \ldots, X_8$ of B_9.

$N_9 =$ normal subgroup generated by $c\tau^{-1}$, $(u_1 v_1^{-1})^3$.

$G_9 = \tilde{B}_9 \rtimes G_0(9)/N_9$.

$\tilde{\psi}_9 : G_9 \to S_9 \quad \tilde{\psi}_9(\alpha, \beta) = \tilde{\psi}_9(\alpha)$.

Definition. $\mathrm{Ab}_9, H_9, H_{9,0}$

$\mathrm{Ab}_9 : G_9 \to \mathbb{Z} \quad \mathrm{Ab}_9(\alpha, \beta) = \widetilde{\mathrm{Ab}}(\alpha)$.

$H_9 = \ker \tilde{\psi}_9$.

$H_{9,0} = \ker \tilde{\psi}_9 \cap \ker \mathrm{Ab}_9$.

Theorem VI.1. *Let V_3 be the Veronese surface of order 3. Let S_3 be the branch curve of a generic projection $V_3 \to \mathbb{CP}^2$. Let $\mathbb{C}^2$ be an "affine piece" of $\mathbb{CP}^2$. Let $S = S_3 \cap \mathbb{C}^2$. Let $G = \pi_1(\mathbb{C}^2 - S)$. Then $G \cong G_9$ s.t. $\psi : G \to S_9$ is compatible with $\tilde{\psi}_9 : G_9 \to S_9$.*

Proof. Corollary V.3. □

Proposition VI.2. *Let V_3 be the Veronese surface of order 3. Let S_3 be the branch curve of a generic projection $V_3 \to \mathbb{CP}^2$. Let $\bar{G} = \pi_1(\mathbb{CP}^2 - S_3)$. Consider v_1 as an element of G_9. Then there exist $w_0 \in H_{9,0}$ s.t. $\overline{G} \simeq \overline{G}_9$ where $\overline{G}_9 = G_9/\langle v_1^{18} w_0\rangle$.*

Proof. To appear in [MoTe9]. □

Proposition VI.3. *Let X_i be a frame in B_n. Let $c = [\tilde{X}_1^2, \tilde{X}_2^2]$. Then*
$c = [\tilde{X}_1^2, \tilde{X}_2^2] = [\tilde{X}_i^2, \tilde{X}_{i+1}^2] = \cdots = [\tilde{X}_{n-2}^2, \tilde{X}_{n-1}^2]$.
Moreover, $(\tilde{P}_n') = (\tilde{P}_{n,0})' = \{1, c\} \simeq \mathbb{Z}_2$.
$\mathrm{Ab}(\tilde{P}_n) =$ *free abelian group on n generators.*
$\tilde{B}_n$ acts on $\tilde{P}_{n,0}$ by conjugation.
$\tilde{P}_{n,0}$ with this action is isomorphic to $G_0(n)$ with the standard $\tilde{B}_n$-action as defined previously.
There exists a series: $1 \subseteq (\tilde{P}_{n,0})' \subseteq \tilde{P}_{n,0} \subseteq \tilde{P}_n \subset \tilde{B}_n$ *s.t.* $\tilde{B}_n/\tilde{P}_n = S_n$, $\tilde{P}_n/\tilde{P}_{n,0} \simeq \mathbb{Z}$, $\tilde{P}_{n,0}/(\tilde{P}_{n,0})' \simeq A_{n-1} = \mathrm{Ab}(G_0(n))$, $(\tilde{P}_{n,0})' \simeq \mathbb{Z}_2$.
$\mathrm{Ab}(B_n) = \mathbb{Z}$.

Proof. See in [MoTe4], Chapters 4, 5, definitions of $\tilde{P}_n$ and $\tilde{P}_{n,0}$ and Theorem III.6.4. □

Proposition VI.4. *There exists a series* $1 \subseteq H_{9,0}' \subset H_{9,0} \subset H_9 \subset G_9$, *where* $G_9/H_9 \simeq S_9$, $H_9/H_{9,0} \simeq \mathbb{Z}$, $H_{9,0}/H_{9,0}' \simeq (\mathbb{Z} + \mathbb{Z}/3)^8$, $H_{9,0}' = H_9' = \{1, c\} \cong \mathbb{Z}/2$.

Proof. To appear in [MoTe9]. □

Proposition VI.5. *Let $\overline{H}_9$ and $\overline{H}_{9,0}$ be the images of $H_{9,0}$ and H_q in $\overline{G}_9$. Then $\overline{H}_{9,0}' = \overline{H}_q'$ and*

$$1 \subseteq \overline{H}_{9,0}' \subseteq \overline{H}_{9,0} \subseteq \overline{H}_g \subset \overline{G}_g$$

where $\overline{G}_9/\overline{H}_9 \simeq S_9$, $\overline{H}_9/\overline{H}_{9,0} \simeq \mathbb{Z}_9$, $\overline{H}_{9,0}/\overline{H}_{9,0}' \simeq (\mathbb{Z} + \mathbb{Z}/3)^8$, $\overline{H}_{9,0}' \cong \mathbb{Z}/2$.

Proof. To appear in [MoTe9]. □

VI.2. The Galois cover of $\mathbf{V_n}$.

We shall quote here other results on fundamental groups related to Veronese surfaces proven in earlier works.

Theorem. *Let V_n be a Veronese surface of order n. Let $(V_n)_{\mathrm{Gal}}$ be its Galois cover with respect to f, a generic projection. Then $\pi_1((V_n)^{\mathrm{Aff}}_{\mathrm{Gal}})$ the fundamental group of the part of $(V_n)_{\mathrm{Gal}}$ that lies over a generic affine part of $\mathbb{CP}^2$ is a direct sum of n^2-1 cyclic groups of order n.*

Proof. See [MoTe3]. □

Theorem. *Let V_n be a Veronese surface of order n. Let $(V_n)_{\mathrm{Gal}}$ be its Galois cover with respect to f, a generic projection to $\mathbb{CP}^2$. Then $\pi_1(V_n)_{\mathrm{Gal}}$ is a direct sum of n^2-2 cyclic groups of order n.*

Proof. See [MoTe3]. □

The above results concern the computation of ker ψ for

$$\psi : \frac{\pi_1(\mathbb{C}^2 - S)}{\langle \Gamma_j^2 \rangle} \to S_n$$

for S, the branch curve of a generic projection to $\mathbb{CP}^2$ of V_n, and $\{\Gamma_j\}$ a g-base for $\pi_1(\mathbb{C} - S, *)$. To carry out the computation we used the relations induced from the Van Kampen method (Chapter II), $\Gamma_j^2 = \Gamma_{j'}^2 = 1$ and the RMS method without using the computations of Chapter IV. These results are easier since we assume there that all generators of fundamental groups are of order 2.

VI.3. The Galois cover of $\mathbf{X_{ab}}$.

We shall also quote here a few results concerning the fundamental group of the Galois cover of X_{ab}.

Definition. X_{ab}

Let $X = \mathbb{CP}^1 \times \mathbb{CP}^1$. Let ℓ_1 be the first $\mathbb{CP}^1$ and ℓ_2 the second one. Let $a, b \in N^+$. Look at $E = a\ell_1 + b\ell_2$. Embed X in some $\mathbb{P}^N$ with respect to the linear system $|E|$. Denote the image of the embedding by X_{ab}.

Theorem. *The fundamental group $\pi_1((X_{ab})^{\mathrm{Aff}}_{\mathrm{Gal}})$ is a finite commutative group on $n-1$ generators ($n = 2ab$), each of order C ($c = g.c.d.(a, b)$) and there are no further relations.*

Proof. Theorem 10.1 from [MoTe5]. □

Theorem. *The fundamental group $\pi_1((X_{ab})_{\mathrm{Gal}})$ is a finite abelian group with $n-2$ generators, each of order c ($c = g.c.d.(a, b)$) and there are no further relations.*

Proof. Theorem 10.2 from [MoTe5]. □

Corollary. *If a, b are relatively prime, then $(X_{ab})_{\mathrm{Gal}}$ is simply connected.*

These results give us very interesting examples of surfaces of general type. The Galois covers are minimal surfaces of general type. Their index is zero for $a = b = 5$ or for $a = 4$, and $b = 7$ and positive for $a \geq 5$, $b \geq 6$. By the above Corollary, they are simply connected for a, b relatively prime. Thus we get a series of simply connected surfaces of general type with positive index unlike the Bogomolov-watershed Conjecture (see [FH]). Moreover, X_{55} is an example of a surface of general type with zero index and even type with finite commutative fundamental group whose universal cover is homomorphic to a connected sum of $S^2 \times S^2$. X_{55} gives also an exotic differential structure on a connected sum of several copies of $S^2 \times S^2$. There are only a few other such examples (one of them is $X_{4,7}$). The other 3 examples will appear in [MoTe10].

In this work we have computed fundamental groups of complements of branch curves as part of our research on algebraic surfaces. This work also has implications to the topology of complements of curves in general. For general singular curves see, for example, [L1] and [L2].

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