One-harmonic maps on Riemann surfaces

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In this paper we study the properties of an "antiholomorphic energy functional" on a compact Riemann surface R. An interesting feature of this functional, not shared by the usual energy of harmonic maps, is its symmetry with respect to the inversion of the map. For surfaces of negative curvature we prove that, in any homotopy class containing an orientation-preserving diffeomorphisms, there exist a unique critical point. We also relate the functional to lagrangian area minimizing maps, as well as to Teichmüller theory and the abelian vortex equation. The functional is then seen to be related to the "distance" of an invariant Lorentzian metric. We will explicitly write the exponential coordinates of such metric. In these coordinates the Euler-Lagrange equation of the functional turns out to be linear, it also coincides with the equation of orthogonality to the orbits of the group $\mathcal{D}$ of diffeomorphism of the surface. The Lorentzian metric restricted to $\mathcal{M}_{-1}/\mathcal{D}$ is the Weil-Petersson metric, this allows us to express the equation of Weil-Petersson geodesics only in terms of the hyperbolic geometry of the surface.

1. Introduction.

Let R be a compact connected Riemann surface of genus bigger then one. Let ϕ be a smooth map of R into itself, let g, h be hermitian metrics on R, such that it is endowed with the conformal structures which they induce. Consider the functional

$$(1.1) \qquad T_\partial(g, h, \phi) = \int_R ||\partial\phi|| dVol_g.$$

Where $|| \; ||$ denotes the metric norm of $\partial\phi$. One might consider as well the analogous functional with the $\bar\partial$ and with the d operator. This is the L^1 version of the usual energy functionals

$$E_\partial(g, h, \phi) = \int_R ||\partial\phi||^2 dVol_g$$

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$$E_{\bar{\partial}}(g, h, \phi) = \int_R ||\bar{\partial}\phi||^2 dVol_g$$

$$E_d(g, h, \phi) = \int_R ||d\phi||^2 dVol_g.$$

which for harmonic diffeomorphisms are related by the simple equations

$$\frac{E_d}{2} = E_\partial + E_{\bar{\partial}}$$
$$E_\partial - E_{\bar{\partial}} = Vol_h(R).$$

The above energy functional have been studied by many people see for example [3], [10]. In particular it has been proved that, if h has strictly negative Gauss curvature, then in any homotopy class of smooth maps from R to itself, there exist a unique critical point which is also smooth. Moreover if the homotopy class contains a diffeomorphism, then the critical point is itself a diffeomorphism, see [10]. Such critical points are known as harmonic maps. Critical points of the functionals T_∂, $T_{\bar{\partial}}$, and T_d, seem to have different behaviour. However they share a symmetry property which is not enjoyed by harmonic maps, namely if ϕ is a diffeomorphism, then $T_\partial(g, h, \phi) = T_\partial(h, g, \phi^{-1})$. The same holds for the functional, $T_{\bar{\partial}}$, and T_d. In this paper we will study T_∂. We will show that, if the curvatures of g and h are both negative, then in any homotopy class, containing an "orientation preserving" diffeomorphism, there exists a unique critical map of T_∂. Such map preserves the curvature form. Let us denote with $\mathcal{M}$ the space of hermitian metrics on R, with $\mathcal{M}_{-1}$ the ones with constant curvature -1, with $\mathcal{D}$ the group of orientation preserving diffeomorphism and with $\mathcal{D}_0$ its identity component. In section 3 we introduce coordinates (P, Q) on $\mathcal{M}$ which linearize the Euler-Lagrange equation of T_∂. We also show that, for metrics of negative curvature, the orientation preserving critical diffeomorphisms of T_∂ have a graph which is a minimal lagrangian submanifold in $R \times R$ with a suitable Riemannian metric. Section 4 is devoted to the special case where both g and h belong to $\mathcal{M}_{-1}$. The non antiholomorphic critical points are then simplectic. In this case the existence and uniqueness Theorem for critical points of T_∂ follows from [10], and [12]. In light of this we can also relate the functional to the standard energy as a map from Teichmüller space into the reals, mainly studied by Tromba and Wolf, see [13], [14]. In section 5 we deal with an invariant generalized Lorentzian metric whose exponential coordinates coincide with (P, Q). The Lorentzian "distance" on $\mathcal{M}$ can be expressed in terms of T_∂, and the Euler-Lagrange equation of T_∂ coincides with the equation of orthogonality to the $\mathcal{D}_0$ orbits. We also show that the medium point m of the Lorentzian geodesics in

$\mathcal{M}/\mathcal{D}_0$ between two points g and h in $\mathcal{M}_{-1}/\mathcal{D}_0$, is such that the diagonal map from (R, m) into $(R, g) \times (R, h)$ is harmonic and conformal, moreover (R, m) satisfies the abelian vortex equation, see [6]. The Lorentzian metric restricted to $\mathcal{M}_{-1}$ is positive definite, and the induced metric on Teichmüller space $\mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0$ coincides with the Weil-Petersson metric. This allows us to determine the equation of the Weil-Petersson geodesics only in terms of the hyperbolic geometry of the surface. In the last section we prove the existence and uniqueness Theorem for critical points of T_∂ homotopic to the identity. To obtain this result we use a Wente-type estimate, the Mumford, and Cheger-Gromov compactness Theorems, (see [2], [9], and [8]) and apply the "continuity method". The authors would like to thank professor G. Anzellotti, L. Simon, R. Schoen and the referee for many discussions and helpful suggestions. This work was completed while the second author was visiting Brest and Stanford University, he would like to express his gratitude for a very warm hospitality.

2. The functional.

Let R be a compact connected Riemann surface of genus strictly bigger then 1. Let g, h be hermitian metrics on R, and $\phi : R \to R$ be a smooth map of R into itself; let us consider the functional

$$T_\partial(g, h, \phi) = \int_R ||\partial\phi|| dVol_g.$$

In an analogous way one can define the functional $T_{\bar\partial}$ and T_d. Let us give to R the Riemann surface structures given by g and h respectively, then in holomorphic local coordinate we can write

$$g = \rho^2 dz d\bar{z}, h = \sigma^2 dw d\bar{w}.$$

Lemma 2.1. *Let ψ be an orientation preserving diffeomorphism of R, and ϕ a smooth map of R into itself, then we have:*

(i) *If ϕ is a diffeomorphism then $T_\partial(g, h, \phi) = T_\partial(h, g, \phi^{-1})$*

(ii) $T_\partial(g, \psi^*(h), \phi) = T_\partial(g, h, \phi \circ \psi)$

(iii) $T_\partial(a^2 g, h, \phi) = a T_\partial(g, h, \phi)$ *for every positive constant a.*

(iv) *If $g = h$ then the identity map is a critical point of T_∂.*

The same relations hold for $T_{\bar\partial}$ and for T_d.

Proof. Let us prove (i) In holomorphic local coordinates we have

$$\|\partial\phi^{-1}\| = \frac{|\frac{\partial\phi^{-1}}{\partial w}|\rho(\phi^{-1}(w))}{\sigma(w)}.$$

Since, by Cramer role for linear systems we have

$$|(\frac{\partial\phi^{-1}}{\partial w})(\phi)| = \frac{|\frac{\partial\phi}{\partial z}|}{|det(Jac(\phi))|},$$

in the case of T_∂, we conclude by change of variable via the diffeomorphism ϕ. Here Jac is the jacobian matrix. The case of $T_{\bar\partial}$ is similar.

In the case of T_d we have in real coordinates

$$\|d(\phi^{-1})\| = \sqrt{\frac{\rho^2(\phi^{-1})}{\sigma^2}trace(Jac(\phi^{-1})^T Jac(\phi^{-1}))}.$$

Hence the claim follows from change of variable in the integral via the diffeomorphism ϕ, together with the identity

$$\sqrt{(trace(Jac(\phi)Jac(\phi)^T)^{-1}}|det(Jac(\phi))| = \sqrt{(trace(Jac(\phi)Jac(\phi)^T)},$$

which can be proved by using the eigenvalues of the positive definite symmetric two by two real matrix $Jac(\phi)Jac(\phi)^T$.

The properties (ii) and (iii) are clear.

For (iv) we may consider a variation ϕ_t of ϕ, differentiate the functional with respect to t, and compute at t=0. Since, denoting the identity map with I, we have $\|\bar\partial(I)\| \equiv 0, \|\partial(I)\| = \|d(I)\| \equiv 1$, and the identity is harmonic, we conclude. $\square$

In this paper we will only study the functional T_∂. The Euler-Lagrange operator $\mathcal{E}(\phi, g, h)$ for the functional T_∂ is given in local holomorphic coordinates by:

$$\mathcal{E}(\phi, g, h) = \rho\sigma(\phi)\left(\partial\log(\frac{\partial\phi}{\overline{\partial\phi}}) + (\partial_w\log\sigma^2)(\phi)(\partial\phi)\right)\frac{\overline{\partial\phi}}{|\partial\phi|}$$

$$-\rho\sigma(\phi))\left((\bar\partial_w\log\sigma^2)(\phi)\partial\bar\phi\right)\frac{\overline{\partial\phi}}{|\partial\phi|}$$

$$-\rho\sigma(\phi)\left(\partial_z\left(\log\rho^2\right)\right)\frac{\overline{\partial\phi}}{|\partial\phi|}.$$

Note that if ϕ is an orientation preserving diffeomorphism, then

$$|\partial\phi| \geq \sqrt{|\partial\phi)|^2 - |\bar\partial\phi|^2} > 0,$$

hence $\mathcal{E}(\phi, g, h)$ is well defined and smooth on the whose surface.

Let ϕ be a smooth map from R into itself, let $B = \{x \in R : \partial\phi(x) \neq 0\}$. We say that ϕ is a critical point of T_∂ if $\mathcal{E}(\phi, g, h)(x) = 0$ for every x in B.

Lemma 2.2. *If ϕ is a smooth critical point of T_∂ then it preserves the curvature forms in every point of B, i.e. we have:*

$$K_g dVol_g(x) = \phi^*(K_h dVol_h)(x)$$

for every $x \in B$. Where K_g and K_h are the Gauss curvatures of g, h respectively.

Proof. Consider the operator

$$\mathcal{E}'(\phi, g, h) = \frac{\mathcal{E}(\phi, g, h)|\partial\phi|}{\rho\sigma(\phi)\overline{\partial\phi}}.$$

Since ϕ is critical, then the real part of $\bar\partial\mathcal{E}'$ vanishes identically, we will denote with Re the real and with Im the imaginary part of a complex number. We have:

$$Re\bar\partial\mathcal{E}'(\phi, h, g) = \bar\partial_w\partial_w(\log\rho^2)(\phi)(|\bar\partial\phi|^2 - |\partial\phi|^2)$$
$$+\bar\partial\partial(\log\sigma^2).$$

But $\bar\partial\partial\left(\log\rho^2\right) = -K_g\frac{\rho^2}{2}$ and $\bar\partial\partial\left(\log\sigma^2\right) = -K_h\frac{\sigma^2}{2}$. The result follows.
$\square$

Remark 1. If g and h have negative curvature everywhere, then every smooth non antiholomorphic critical point of T_∂, must be an orientation preserving diffeomorphism. In fact, by the above Lemma, since ϕ is smooth, then the jacobian determinant of ϕ is positive on the closure of B, hence B is open and closed, so if it is non empty it must coincide with R. In particular if $K_g \equiv K_h \equiv -1$ then every non antiholomorphic critical point of T_∂ is a simplectic diffeomorphism. Similarly, if g has negative curvature everywhere and there exists a smooth critical point of T_∂ which is an orientation preserving diffeomorphism, then h has also negative curvature everywhere.

Lemma 2.3. *Let g and h be metrics on R with negative curvature, let ϕ : $(R, g) \to (R, h)$ be an orientation preserving critical diffeomorphism of T_∂. We have that the second variation of T_∂ at ϕ is positive definite.*

Proof. Recall that

$$\mathcal{E} = \frac{\mathcal{E}' \rho \sigma(\phi) \overline{\partial \phi}}{|\partial \phi|}.$$

Hence if ϕ is critical, then

$$\frac{\partial \mathcal{E}}{\partial t}_{|t=0} = \frac{\frac{\partial \mathcal{E}'}{\partial t}_{|t=0} \rho \sigma(\phi) \overline{\partial \phi}}{|\partial \phi|}.$$

If u is a non zero vector field on R and L the linearized of the operator $\mathcal{E}'$ we have

$$L(u) = \partial \left(\frac{\partial u}{\partial \phi} - \frac{\bar{\partial} \bar{u}}{\bar{\partial} \bar{\phi}} \right)$$

$$+ \partial_w{}^2 (\log \sigma^2)(\partial \phi) u - \bar{\partial}_w^2 (\log \sigma^2)(\partial \bar{\phi})(\bar{u})$$

$$+ \partial_w \bar{\partial}_w (\log \sigma^2)(\bar{u} \partial \phi - u \bar{\partial} \phi) + \partial_w (\log \sigma^2) \partial u - \bar{\partial}_w (\log \sigma^2) \partial \bar{u}.$$

In other words

$$L(u) = \partial \left(\left(\frac{\partial u}{\partial \phi} + \partial_w (\log \sigma^2) u \right) - \left(\frac{\bar{\partial} \bar{u}}{\bar{\partial} \bar{\phi}} + \bar{\partial}_w (\log \sigma^2) \bar{u} \right) \right)$$

$$+ 2 \partial_w \bar{\partial}_w (\log \sigma^2)(\bar{u} \partial \phi - u \bar{\partial} \phi).$$

If we set

$$v = \frac{\bar{u}}{\bar{\partial} \bar{\phi}}$$

and

$$\mathcal{F} = \rho \sigma(\phi) |\partial \phi|$$

the above equation becomes

$$L(u) = \partial \left(\partial \bar{v} + \bar{v} \left(\partial_w (\log \sigma^2) \partial \phi + \partial (\log \partial \phi) \right) \right)$$

$$- \partial \left(\bar{\partial} v + v \left(\bar{\partial}_w (\log \sigma^2) \bar{\partial} \bar{\phi} + \bar{\partial}(\log \bar{\partial} \bar{\phi}) \right) \right)$$

$$+ 2 \partial_w \bar{\partial}_w (\log \sigma^2) \left(v |\partial \phi|^2 - \bar{v} \partial \phi \partial \bar{\phi} \right).$$

Set

$$\gamma = \mathcal{F} v = \frac{\bar{u} \sigma(\phi) \rho |\partial \phi|}{\bar{\partial} \bar{\phi}}$$

and $\omega = \gamma dz + \bar{\gamma} d\bar{z}$, we may think of L as a function of ω, and using the Euler-Lagrange equation we find

(2.1)

$$L(\omega) = \partial \left(\frac{1}{\mathcal{F}} \left(\partial \bar{\gamma} - \bar{\partial} \gamma \right) \right) + 2 \partial_w \bar{\partial}_w (\log \sigma^2) \frac{\partial \phi}{\mathcal{F}} \left(\gamma \bar{\partial} \bar{\phi} - \bar{\gamma} \bar{\partial} \phi \right).$$

The second variation of T_∂ is given by

$$2 \int_R Re \left(L(u) u \frac{\sigma(\phi) \bar{\partial} \bar{\phi}}{\rho |\partial \phi|} \right) dVol_g = 2 \int_R Re \left(L(\omega) \bar{\gamma} \right) (i/2) dz \wedge d\bar{z}$$

Hence by adding formula (2.1) with its conjugate, and integrating by part we find

$$\frac{\partial^2 T_\partial}{\partial t^2}_{|t=0} = \int_R \frac{-1}{\mathcal{F}} \left(\partial \bar{\gamma} - \bar{\partial} \gamma \right)^2 (i/2) dz \wedge d\bar{z}$$

$$+ 2Re \left(\int_R \frac{-K_h \sigma^2(\phi)}{\mathcal{F}} \partial \phi \left(\gamma \bar{\partial} \bar{\phi} - \bar{\gamma} \partial \bar{\phi} \right) \bar{\gamma} (i/2) dz \wedge d\bar{z} \right).$$

We have

$$2Re \left(\partial \phi \left(\gamma \bar{\partial} \bar{\phi} - \bar{\gamma} \partial \bar{\phi} \right) \bar{\gamma} \right)$$

$$= 2 |\gamma|^2 |\partial \phi|^2 \left(1 - Re \left(\frac{\gamma^2}{|\gamma|^2} \frac{\bar{\partial} \phi}{\partial \phi} \right) \right) > 0,$$

since ϕ is an orientation preserving diffeomorphism. $\qquad \square$

Let us consider the function $S = -\phi^*(K_h dVol_h)/dVol_g + K_g$. By looking at the one-form $2Re\mathcal{E}' dz = \mathcal{E}' dz + \bar{\mathcal{E}}' d\bar{z}$, we find as in Lemma 2.2 that $d^*(2Re\mathcal{E}') = S$, where $d^* = *d*$ and $*$ is the Hodge $*$ operator with respect to g. Consider the operator $M = 2Re\mathcal{E}' dz + dS$ then we have the following

Lemma 2.4. *If g and h have negative curvature, and ϕ is an orientation preserving diffeomorphism, then the operator M is elliptic and $\mathcal{E}'(\phi) = 0$ if and only if $M(\phi) = 0$.*

Proof. We know from Lemma 2.2 that if $\mathcal{E}' = 0$ then $M(\phi) = 0$, viceversa if $M(\phi) = 0$ then $(I + dd^*)(2Re(\mathcal{E}'(\phi) dz)) = 0$ which implies that $\mathcal{E}'(\phi) = 0$ since $I + dd^*$ is a positive operator. Now for a fixed non zero complex number ξ, set $\alpha = \frac{1}{\partial \phi}$, $\beta = \frac{-2K_h \sigma^2 \bar{\partial} \bar{\phi}}{\rho^2}$, and $\gamma = \frac{-2K_h \sigma^2 \partial \bar{\phi}}{\rho^2}$, then the top order symbol of M is represented by the matrix

$$A = \begin{pmatrix} (\alpha - \beta) \xi^2 + \gamma |\xi|^2 & -(\bar{\alpha} + \bar{\beta}) |\xi|^2 + \bar{\gamma} \xi^2 \\ -(\alpha + \beta) |\xi|^2 + \gamma \bar{\xi}^2 & (\bar{\alpha} - \bar{\beta}) \bar{\xi}^2 + \bar{\gamma} |\xi|^2 \end{pmatrix}.$$

So

$$det(A) = -4|\xi|^4(Re(\alpha\bar{\beta} - \frac{\alpha\bar{\gamma}\xi^2}{|\xi|^2})) = 8K_h|\xi|^4\frac{\sigma^2}{\rho^2}\left(1 - Re\left(\frac{\xi^2\bar{\partial}\phi}{|\xi|^2\partial\phi}\right)\right)$$

which is negative since ϕ is an orientation preserving diffeomorphism. Note that $|det(A)| \geq \frac{-4K_h|\xi|^4\sigma^2 Jac(\phi)}{\rho^2|\partial\phi|^2}$. Hence when ϕ is T_∂ critical, Lemma 2.2, gives $|detA| \geq \frac{-4K_g|\xi|^4}{|\partial\phi|^2}$. $\square$

3. The coordinates P,Q.

Let us now fix a square root $k^{1/2}$ of the canonical bundle of R, let r be an integer, let $S(r)$ be the smooth sections of $k^{r/2} \times \bar{k}^{-r/2}$. For example the sections of $S(-2)$ are known as Beltrami differentials. Let $\mathcal{M}$ be the space of hermitian metrics on R. Let us fix an element $g \in \mathcal{M}$, and conformal coordinates on R where g is given as $g = \rho^2 dz d\bar{z}$. We have a natural map from $S(0) \times S(-2)$ into $\mathcal{M}$ given by $\alpha(B, A) = \bar{A}\rho^2 dz d\bar{z} + (2B+1)\rho^2 dz d\bar{z} + A\rho^2 dz d\bar{z}$. Then $\alpha(B, A)$ is a metric if and only if $B+1/2$ and $(B+1/2)^2 - |A|^2$ are positive. Consider the open set $\Omega \subseteq S(0) \times S(-2)$ given by

$$\Omega = \{(P,Q) : P + 1 > 0 \text{ and } (P+1)^2 - |Q|^2 > 0\}.$$

We have the following

Proposition 3.1. *The map Ψ from Ω into $\mathcal{M}$ given by*

$$\Psi(P,Q) = \left((P+1)\bar{Q} + ((P+1)^2 + |Q|^2) + (P+1)Q)\right) g$$

is an open embedding.

Proof. In coordinate (B, A) the map reads

$$\Psi(P,Q) \to (\frac{(P+1)^2 + |Q|^2) - 1}{2}, (P+1)Q),$$

whereas we have

$$\Psi^{-1}(B,A)$$
$$= \left(\sqrt{\frac{2B+1+\sqrt{(2B+1)^2 - 4|A|^2}}{2}} - 1, \frac{A}{\sqrt{\frac{2B+1+\sqrt{(2B+1)^2 - 4|A|^2}}{2}}}\right)$$

and these are both smooth maps. Note that, after having identified the tangent space to $\mathcal{M}$ at g with $S(0) \times S(-2)$ via the map α, we find that $\Psi(0,0) = g$ and $d\Psi(0,0) = I$. $\square$

Denote by P_h, Q_h, the coordinates of the metric h given by the above Proposition. Set for convenience $F_h = P_h + 1$. Let $\phi : R \to R$ be a smooth map, and h a smooth metric, set $\alpha = \frac{\partial \phi}{\partial z}$ and $\beta = \frac{\partial \phi}{\partial \bar{z}}$, $F_{\phi^*(h)} = F$ and $Q_{\phi^*(h)} = \tilde{Q} d\bar{z}/dz$. Then $F = \frac{\sigma |\alpha|}{\rho}$ and $\tilde{Q} = \frac{\sigma \bar{\alpha} \beta}{\rho |\alpha|}$. So

$$\partial(\log F^2) = \partial(\log \alpha) + \partial(\log \bar{\alpha}) + \partial_w(\log \sigma^2)\alpha + \partial_{\bar{w}}(\log \sigma^2)\bar{\beta} - \partial(\log \rho^2).$$

Hence the Euler-Lagrange equation $\mathcal{E}' = 0$ is equivalent to

$$(3.1) \qquad\qquad \partial(\log F) = \partial(\log \bar{\alpha}) + \partial_{\bar{w}}(\log \sigma^2)\bar{\beta}.$$

Observe that $\bar{\partial}\alpha = \partial\beta$. Define $M = \sigma^2 \bar{\alpha}\beta = \rho^2 F\tilde{Q}$. We have

$$\partial(\log(M)) = \partial(\log \sigma^2) + \partial(\log \bar{\alpha}) + \frac{\bar{\partial}\alpha}{\beta}.$$

However $\frac{\alpha}{\beta} = \frac{F}{\tilde{Q}}$ so

$$\partial(\log(M)) = \partial(\log \sigma^2) + \partial(\log \bar{\alpha}) + \bar{\partial}(\log \alpha)\frac{F}{\tilde{Q}}.$$

And by plugging (3.1) and its conjugate into the above equation we find

$$\partial \log M = \partial \log \sigma^2 + \partial \log F - \bar{\partial}_w \log \sigma^2 \bar{\beta} - \frac{F}{\tilde{Q}}\partial_w \log \sigma^2 \beta + \frac{F}{\tilde{Q}}\bar{\partial} \log F$$

i.e.

$$\partial \log F + \partial \log \tilde{Q} + \partial \log \rho^2 = \partial \log M = \partial \log F + \frac{F\bar{\partial} \log F}{\tilde{Q}}$$

that is

$$\frac{\partial \tilde{Q}}{\tilde{Q}} + \partial \log \rho^2 - \frac{\bar{\partial}F}{\tilde{Q}} = 0$$

or

$$\rho^{-3}\partial(\rho^2 \tilde{Q}) = \rho^{-1}\bar{\partial}F.$$

There are natural global differential operators on (R, g),

$$L_r : S(r) \to S(r-1)$$
$$K_r : S(r) \to S(r+1)$$

(essentially the covariant complex derivatives) defined as follows:

$$K_r(a) = \rho^{r-1}\partial(\rho^{-r}a)$$
$$L_r(a) = \rho^{-r-1}\bar{\partial}(\rho^r a)$$

The main properties of the operators L_r, K_r are the following

$K_r = \overline{L_{-r}}$
$\Delta_r = 4L_{r+1}K_r + r(r+1)I =$ where Δ_r is the ∂ laplacian on S_r.
$L_{r+1}K_r = K_{r-1}L_r + (r/2)K_g$
$\int_R K_r a(\bar{b})dVol_g = -\int_R a(\overline{L_{r+1}b})dVol_g$
$\Delta_{r+1}K_r = K_r\Delta_r$ and
$\Delta_r L_{r+1} = L_{r+1}\Delta_{r+1}$

whenever g has constant curvature -1.

See e.g. [16], [17]. Recall that a Beltrami differential μ is harmonic if and only if we can write $\mu = \frac{\bar{q}}{dVol_g}$, where q is a holomorphic quadratic differential, and this is the case if and only if $K_{-2}(\mu) = 0$.

We say that a metric $h \in \mathcal{M}$ is critical with respect to the fixed metric g if the identity map is critical for T_∂ between (R, g) and (R, h).

By the above computation we conclude

Theorem 3.2. *The metric h is critical with respect to g if and only if*

$$K_{-2}(Q_h) = L_0(P_h).$$

Let us denote with $\mathcal{C}_g$ the space of metrics which are critical with respect to g.

Remark 2. If ϕ is an orientation preserving diffeomorphism of R, then

$$T_\partial(\phi, g, h) = T_\partial(I, g, \phi^*(h)) = \int_R (P_{\phi^*(h)} + 1)dVol_g.$$

When the metrics have negative curvature, we can interpret the critical point of T_∂ as area minimizing maps in suitable Riemannian 4-manifolds. Let g and h be two metrics on R, both with strictly negative curvature, consider the product $R \times R$ with the metric

$$s(x,y) = \sqrt{\frac{K_g(x)}{K_h(y)}}g(x) + \sqrt{\frac{K_h(y)}{K_g(x)}}h(y).$$

The above Riemannian manifold will henceforth be denoted by $R * R$. We have the following

Lemma 3.3. *Let g and h be two metrics with strictly negative curvature, let $\phi : R \to R$ be a non antiholomorphic smooth map, and $\tilde{\phi} : R \to R * R$ be the graph map given by $\tilde{\phi}(x) = (x, \phi(x))$. Then ϕ is critical with respect to $T_{\partial}(g, h)$, if and only if $\tilde{\phi}$ is minimal and with lagrangian image with respect to the simplectic form $K_g dVol_g - K_h dVol_h$. In this case $Vol(\tilde{\phi}(R)) = 2T(g, h, \phi)$.*

Proof. Set $p = \tilde{\phi}^*(s)$, $a = \sqrt{-K_g}$, $b = \sqrt{-K_h}$. We have $p = \frac{a}{b(\phi)} g + \frac{b(\phi)}{a} \phi^*(h)$. If we write $g = \rho^2 dz d\bar{z}$, $h = \sigma^2 dw d\bar{w}$, $p = \delta^2 d\zeta d\bar{\zeta}$ and

$$.\phi^*(h) = \rho^2 \left((P + 1) Q d\bar{z}^2 + ((P + 1)^2 + |Q|^2) dz d\bar{z} + (P + 1) \bar{Q} dz^2 \right),$$

then

$$(\frac{dVol_p}{dVol_g})^2 = \left(\left(\frac{a}{b(\phi)} + \frac{b(\phi)}{a} ((P + 1)^2 + |Q|^2) \right) \right)^2 - 4 \frac{b(\phi)}{a} (P + 1) Q|^2 \right)$$

$$= \left(\frac{a^2}{b^2(\phi)} + \frac{b^2(\phi)}{a^2} ((P + 1)^2 + |Q|^2)^2 + 2((P + 1)^2 + |Q|^2)) \right.$$

$$\left. -4 |\frac{b(\phi)}{a} (P + 1) Q|^2 \right)$$

$$= \left(\frac{a^2}{b^2(\phi)} + \frac{b^2(\phi)}{a^2} ((P + 1)^2 - |Q|^2) + 4(P + 1)^2 - 2((P + 1)^2 - |Q|^2) \right)$$

$$= 4(P + 1)^2 + \left(\frac{a}{b(\phi)} - \frac{b(\phi)}{a} \frac{dVol_{\phi^*(h)}}{dVol_g} \right)^2 .$$

Set $q(\phi) = \left(\frac{a}{b(\phi)} - \frac{b(\phi)}{a} \frac{dVol_{\phi^*(h)}}{dVol_g} \right)$. Note that the map ϕ preserves curvature if and only if $\tilde{\phi}$ has lagrangian image if and only if $q(\phi) = 0$. Let $\tilde{\phi}_t$ be a variation of $\tilde{\phi}$ which for small t may be assumed to be given by the graphs of a variation ϕ_t of ϕ. Then

$$\frac{\partial(Area(\tilde{\phi}_t(R)))}{\partial t}\bigg|_{t=0} = \int_R \frac{4(P + 1) \frac{\partial P}{\partial t}|_{t=0} + q(\phi) \frac{\partial q(\phi_t)}{\partial t}|_{t=0}}{\sqrt{4(P + 1)^2 + q(\phi)^2}} dVol_g.$$

Now the lemma follows from Lemma 2.2 and Remark 2.   □

Remark 3. Let ϕ be a critical and $\Psi = (\Psi_1, \Psi_2)$ be its graph map, then we define

$$\Phi_1(\Psi) = -K_g(\Psi_1)\rho^2 \partial_z \Psi_1 \partial_z \bar{\Psi}_1$$
$$\Phi_2(\Psi) = -K_h(\Psi_2)\sigma^2 \partial \Psi_2 \partial_w \bar{\Psi}_2$$
$$A_1(\Psi) = \log(\frac{K_h(\Psi_2)}{K_g(\Psi_1)})$$
$$A_2(\Psi) = \log(\frac{K_g(\Psi_1)}{K_h(\Psi_2)})$$
$$\mathcal{H}_1(\Psi) = -K_g(\Psi_1)\frac{\rho^2|\partial\Psi_1|^2}{\delta^2}$$
$$\mathcal{H}_2(\Psi) = -K_h(\Psi_2)\frac{\sigma^2|\partial\Psi_2|^2}{\delta^2}$$
$$\mathcal{L}_1(\Psi) = -K_g(\Psi_1)\frac{\rho^2|\bar{\partial}\Psi_1|^2}{\delta^2}$$
$$\mathcal{L}_2(\Psi) = -K_h(\Psi_2)\frac{\sigma^2|\bar{\partial}\Psi_2|^2}{\delta^2}.$$

Let us collect some formulas for future reference. Since Φ is harmonic, conformal, and with lagrangian image, it satisfies the following equations (in the ζ coordinates.)

(3.2)
$$\begin{cases} \bar{\partial}\partial\Psi_1 + \partial_z(\log\rho^2)\bar{\partial}\Psi_1\partial\Psi_1 = 1/2 \left(\bar{\partial}\Psi_1\partial_\xi A_1 + \partial\Psi_1\bar{\partial}_\xi A_1\right) \\ \bar{\partial}\partial\Psi_2 + \partial_w(\log\sigma^2)\bar{\partial}\Psi_2\partial\Psi_2 = 1/2 \left(\bar{\partial}\Psi_2\partial_\xi A_2 + \partial\Psi_2\bar{\partial}_\xi A_2\right). \end{cases}$$

Moreover from above we derive the Bochner type formula

(3.3)
$$\partial_\zeta\bar{\partial}_\zeta \left(\log(\frac{|\partial\Psi_1|^2}{|\bar{\partial}\Psi_1|^2}\right) =$$

$$2\partial_z\bar{\partial}_z(\log(\rho^2)\left(|\partial\Psi_1|^2 - |\bar{\partial}\Psi_1|^2\right)$$

$$-(1/2)\partial\left(\frac{(|\partial\Psi_1|^2 - |\bar{\partial}\Psi_1|^2)\partial_\zeta A_1}{\partial\Psi_1\partial\bar{\Psi}_1}\right))$$

$$-(1/2)\bar{\partial}\left(\frac{(|\partial\Psi_1|^2 - |\bar{\partial}\Psi_1|^2)\bar{\partial}_\zeta A_1}{\bar{\partial}\Psi_1\bar{\partial}\bar{\Psi}_1}\right)$$

(see Appendix)

Remark 4. If both metric g and h have constant negative curvature, it is observed in [12] that every minimal immersion has lagrangian image.

4. The constant curvature case.

Throughout this section we will assume that both g and h have constant Gauss curvature -1.

Lemma 4.1. *There is a unique critical point of T_∂ in the homotopy class of an orientation preserving diffeomorphism.*

Proof. Let ϕ_0 an orientation preserving diffeomorphism, let $\tilde{\phi}_0$ be its graph map; It follows from Schoen-Yau [11], that there exists a metric $m(g, h, \phi_0)$ of constant curvature -1 on R, and a map $\Psi = (\Psi_1, \Psi_2) : (R, m(g, h, \phi_0)) \to R * R$ which is harmonic, conformal, and homotopic to $\tilde{\phi}_0$. Since $K_g = K_h \equiv -1$, then $R * R = (R, g) \times (R, h)$ and both Ψ_1, Ψ_2 are harmonic. Then, by [10], they are orientation preserving diffeomorphism, and by Remark 4, the map $\Psi_2 \circ \Psi_1^{-1}$ is a critical point for T_∂. It follows from Schoen [12], that there exists only one such critical point. Therefore also the metric $m(g, h, \phi_0)$ is uniquely determined by g h and ϕ_0. $\qquad\square$

Remark 5. Note that, in the case of constant curvature, the lagrangian condition implies that the maps Ψ_1 and Ψ_2 have the same energy density, so they have the same energy. Moreover we have

$$2T_\partial = Vol(\tilde{\phi}) = E(\Psi_1) + E(\Psi_2) = 2E(\Psi_1) = 2E(\Psi_2).$$

Theorem 4.2. *Let g and h be hermitian metrics on R, both with constant curvature -1, then*

  (i) *Suppose an homotopy class of maps of R into itself contains an antiholomorphic non constant map, then there is only one such map and it is the unique critical point of T_∂ in the class.*

  (ii) *Suppose an homotopy class contains the constants, then these are the only critical points of T_∂ in the class.*

  (iii) *Suppose an homotopy class contains an orientation preserving diffeomorphism, then there exists a unique critical point of T_∂ in the class. Such point is a simplectic diffeomorphism.*

  (iv) *In any other homotopy class there is no critical points of T_∂.*

Proof. Two non constant homotopic antiholomorphic maps coincide, an antiholomorphic map homotopic to a constant is itself constant. The Theorem then follows from Remark 1 and Lemma 4.1 $\qquad\square$

If the starting diffeomorphism ϕ_0 is chosen to be the identity, and if for fixed g we set $m_g(h)$ to be the Teichmüller class of $m(g, h, I)$, then $m_g(h)$ defines a smooth map from $\mathcal{T}$ into $\mathcal{T}$. Where $\mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0$ is Teichmüller space.

Given two metrics g and h with constant curvature -1, let $\Phi(g, h)$ be the $(2, 0)$ part of h with respect to g, and let $E(g, h)$ be the energy of the harmonic map from (R, g) to (R, h) homotopic to the identity.

Then the map m_g described above is defined by the following equation

$$(4.1) \qquad \Phi(m_g(h), g)) + \Phi(m_g(h), h) = 0.$$

Also, if ϕ denotes the corresponding critical point of T_∂, then, by Remark 5 we have

$$(4.2) \qquad T_\partial(\phi, g, h) = E(m_g(h), g) = E(m_g(h), h).$$

Equation (4.1), implies that the Teichmüller class of g is the only fixed point of m_g. Moreover, since, according to Wolf, [14], the holomorphic quadratic differentials give coordinates in Teichmüller space, it follows that m_g is bijective, and by the implicit function Theorem, m_g is a diffeomorphism of Teichmüller space. Again from equation (4.1) we derive that $m_g(h) = m_h(g)$.

Lemma 4.3. *We have that the differential of m_g at g equals 1/2 Identity.*

Proof. Consider the energy of harmonic maps as a function $E : \mathcal{T} \times \mathcal{T} \to \mathbf{R}$. Recall that $-\Phi(h, g)$ is the Weil Peterson gradient of the energy with respect to the first variable, see [13], hence equation (4.1) says

$$\frac{\partial E}{\partial x_i}(m_g(h), g) = -\frac{\partial E}{\partial x_i}(m_g(h), h)$$

Differentiating with respect to h and evaluating in g we find

$$2 \sum_j \frac{\partial^2 E}{\partial x_i \partial x_j}(g, g) \frac{\partial m_g^j}{\partial x_k} = -\frac{\partial^2 E}{\partial x_i \partial y_k}(g, g).$$

Since E takes its minimum on the diagonal of $\mathcal{T} \times \mathcal{T}$, by differentiating the equality $\frac{\partial E}{\partial x_i}(h, h) = 0$, and evaluating at g, we find $\frac{\partial^2 E}{\partial x_i \partial x_j}(g, g) +$

$\frac{\partial^2 E}{\partial x_i \, \partial y_j}(g,g) = 0$. Since the Hessian of $E(\,\cdot\,,g)$ at g is invertible, the Lemma follows. $\square$

We will describe later a metric on the space $\mathcal{M}$ of metrics on R, such that $m_g(h)$ is the Teichmüller class of the medium point of a geodesic joining suitable representatives of g and h. Let us define $T_\partial[g] : \mathcal{T} \to \mathbf{R}$ by $T_\partial[g](h) = T_\partial(\phi, g, h)$ where ϕ is the unique critical point of T_∂ homotopic to the identity.

Theorem 4.4. *The function $T_\partial[g]$ is smooth and proper, its absolute minimum is taken at g, which is also its only critical point. Moreover the hessian of $T_\partial[g]$ at g is two times the the Weil-Petersson metric.*

Proof. The Theorem follows from the analogous results for the energy function, see [13], together with formula (4.2) and Lemma 4.3. $\square$

Remark 6. It is easy to see from the above, that in the case of the sphere with constant curvature $+1$, the only critical points of T_∂ are the antiholomorphic maps and the isometries. In the case of the flat torus T the situation is more complicated, in fact, for example, the map $z \to z + \bar{z}$, from the complex plane into itself, induces a critical point ϕ of T_∂, such that $\partial\phi$ is nowhere zero, however ϕ is not an isomorphism.

5. The generalized Lorentzian metric.

Let $g \in \mathcal{M}$, given by $g = \rho^2 dz d\bar{z}$. Let k be a tangent vectors to $\mathcal{M}$ at g, so k is a bilinear symmetric form. Let $\tilde{k}$ be the contraction of k with the metric g. Define the quadratic form on $T_g(\mathcal{M})$ given by

$$\|k\|_g^2 = -1/4 \int_R det(\tilde{k}) dVol_g.$$

If $k_1 = \bar{A'}_1 \rho^2 dz d\bar{z} + 2B'_1 \rho^2 dz d\bar{z} + +A'_1 \rho^2 dz d\bar{z}$ and $k_2 = \bar{A'}_2 \rho^2 dz d\bar{z} + 2B'_2 \rho^2 dz d\bar{z} + +A'_2 \rho^2 dz d\bar{z}$ then the Lorentzian metric resulting from polarization is

$$<< k_1, k_2 >>_g = \int_R \left(\bar{A'}_1 A'_2/2 + \bar{A'}_2 A'_1/2 - B'_1 B'_2 \right) dVol_g.$$

The above metric is "generalized Lorentzian" since the spaces $\{P' = 0\}$ and $\{Q' = 0\}$ decompose the tangent space to $\mathcal{M}$ at g in orthogonal direct

sum; moreover the metric is positive definite on the space $\{P' = 0\}$ locally parameterized by a pair of real functions, whereas it is negative definite on $\{Q' = 0\}$, which is locally parameterized by a single real function. Note that the group $\mathcal{D}$ of orientation preserving diffeomorphisms of R acts by pull-back on $\mathcal{M}$. Then we have the following

Lemma 5.1. *The above scalar product is $\mathcal{D}$ invariant.*

Proof. It is a straightforward calculation from the definitions. $\square$

Proposition 5.2. *The orthogonal space with respect to $<<>>$ to the $\mathcal{D}_0$ orbit is the tangent space to $\mathcal{C}_g$.*

Proof. Let $\phi_t : (-\varepsilon, \varepsilon) \to \mathcal{D}$, be a one parameter group of diffeomorphisms with $\phi_0 = I$ and $\phi_0' = u$. Then the Lie derivative $\mathcal{L}_u(g)$ is given by

$$\mathcal{L}_u(g) = \frac{\partial}{\partial t}(\phi_t^*(g))_{t=0} = \rho^2(\partial \bar{u})dz^2 + \left(\partial(\rho^2 u) + \bar{\partial}(\rho^2 \bar{u})\right) dzd\bar{z} + \rho^2(\bar{\partial}u)d\bar{z}^2.$$

If we set $\theta = (\rho^2 dzd\bar{z})^{1/2}u$, then θ is in $S(-1)$ and

$$\mathcal{L}_u(g) = \left(L_{-1}(\theta) + 2Re(K_{-1}(\theta)) + \overline{L_{-1}(\theta)}\right) g.$$

In other words a vector $\begin{pmatrix} P' \\ Q' \end{pmatrix}$ is tangent to the g orbit, if and only if there exists an element $\theta \in S(-1)$ such that $P' = ReK_{-1}(\theta), Q' = L_{-1}(\theta)$. If we write the metric v in the form $v = \left(\bar{Q}' + 2P' + Q'\right) g$ then we obtain

$$<< v, \mathcal{L}_u(g) >>= Re \int_R (L_{-1}(\theta)\bar{Q}' - K_{-1}(\theta)P')dVol_g$$

$$= -Re \int_R (\theta(\overline{K_{-2}(Q') - L_0(P')})dVol_g.$$

Since θ is arbitrary, the Lemma follows from Theorem 3.2. $\square$

Lemma 5.3. *If g is a metric with negative curvature everywhere, then the Lorentzian metric is positive definite on the $\mathcal{D}_0$ orbit of g.*

Proof. Let $\mathcal{L}_u g$ be a vector in $T_g(\mathcal{D}_0 g)$, we have

$$
\begin{aligned}
||\mathcal{L}_u g||^2 &= \int_R (|L_{-1}(\theta)|^2 - (Re(K_{-1}(\theta))^2) dVol_g \\
&= \int_R (|L_{-1}(\theta)|^2 - |K_{-1}(\theta)|^2 + (ImK_{-1}(\theta))^2) dVol_g \\
&= \int_R (|L_{-1}(\theta)|^2 + (L_0 K_{-1}(\theta)\bar{\theta} + (ImK_{-1}(\theta))^2) dVol_g \\
&= \int_R (|L_{-1}(\theta)|^2 + K_{-2}L_{-1}(\theta)\bar{\theta} dVol_g \\
&\quad + \int_R (-\frac{K_g}{2}|\theta|^2 + (ImK_{-1}(\theta))^2) dVol_g \\
&= \int_R ((ImK_{-1}(\theta))^2 - \frac{K_g}{2}|\theta|^2) dVol_g.
\end{aligned}
$$

$\square$

Proposition 5.4. *If g is a metric of constant curvature -1, then we have the equality:*

(*) $\quad \left((T_g(\mathcal{D}_0 g))^\perp \bigcap T_g(\mathcal{M}_{-1}) \right)$

$$
= \left\{ \begin{pmatrix} 0 \\ Q' \end{pmatrix} \text{ with } Q' \text{ harmonic Beltrami differential} \right\}.
$$

It follows that the restriction to $\mathcal{M}_{-1}/\mathcal{D}_0$ of the Lorentzian metric induced on $\mathcal{M}/\mathcal{D}_0$ coincides with the Weil-Petersson metric. Moreover the Lorentzian metric is positive definite on $\mathcal{M}_{-1}$.

Proof. By the proof of Lemma 4.1 we see that the projection map from $\mathcal{C}_g \bigcap \mathcal{M}_{-1}$ into $\mathcal{M}_{-1}/\mathcal{D}_0$ is an isomorphism. In particular we have $T_g(\mathcal{M}_{-1}) = T_g(\mathcal{D}_0 g) \oplus T_g(\mathcal{C}_g)$, which, by Proposition 5.2, coincides with $T_g(\mathcal{D}_0 g) \oplus \left((T_g(\mathcal{D}_0 g))^\perp \cap T_g(\mathcal{M}_{-1}) \right)$. Hence the space

$$
\left((T_g(\mathcal{D}_0 g))^\perp \cap T_g(\mathcal{M}_{-1}) \right)
$$

has complex dimension $3 genus(R) - 3$. However, by Theorem 3.2 and by Theorem 2.4.1 of [13], it contains all the vectors of the form $\begin{pmatrix} 0 \\ Q' \end{pmatrix}$ with Q' harmonic, and equality (*) follows. The Lorentzian metric is positive definite on $T_g(\mathcal{M}_{-1})$ because of (*) together with Lemma 5.3. $\square$

Theorem 5.5. *If we identify the tangent space to $\mathcal{M}$ at g with $S(0) \times S(-2)$ via the map $d\alpha$, then the map Ψ given in Proposition 3.1 becomes the exponential map of the Lorentzian metric.*

Proof. We already observed that $\Psi(0,0) = g$ and $d\Psi(0,0) = I$. Let h be a metric with coordinates P_h, Q_h with respect to g, then the square norm of a tangent vector to h is given by

$$\|(P',Q')\|^2 = \int_R (|Q'|^2 - P'^2) - 1/4 \int_R \frac{(\bar{Q}Q' - Q\bar{Q}')^2}{(P+1)^2 - |Q|^2} dVol_g.$$

The energy of a path $\gamma(t)$ is

$$E(\gamma) = \int_0^1 \int_R (|Q'(t)|^2 - P'^2(t)) dVol_g dt$$

$$- 1/4 \int_0^1 \int_R \frac{(\bar{Q(t)}Q'(t) - Q(t)\bar{Q'}(t))^2}{(P(t)+1)^2 - |Q(t)|^2} dVol_g dt.$$

Take a variation $\gamma(t,s)$ with $\gamma(0,s) = \gamma(0)$ and $\gamma(1,s) = \gamma(1)$. Let us indicate with ' the differentiation with respect to t and with $\cdot$ the one with respect to s. Differentiating the energy with respect to s and evaluating at $(0,0)$ we obtain

$$\frac{\partial E(t,s)}{\partial s} = \int_0^1 \int_R \left(Q'^\cdot \bar{Q}' + Q'\bar{Q}^\cdot - 2P'P'^\cdot \right) dVol_g dt.$$

(Recall that $Q(t,0) = tQ'$). Integrating by parts and observing that $P''(t,0) = Q''(t,0) = 0$, we find that $\frac{\partial E(t,s)}{\partial s}\big|_{s=0} \equiv 0$. $\qquad\square$

Remark 7. It follows from the above Proposition that the geodesics are the paths of the form $P(t) = tP', Q(t) = tQ'$. In particular every two points in $\mathcal{M}$ can be joined by a unique geodesic. We deduce from the formula for the square norm of a vector tangent to $\mathcal{M}$, and from Remark 2 that the energy of the geodesic joining g and h is given by $E(\gamma) = 2T(I,g,h) - Vol(g) - Vol(h)$.

Remark 8. Let us consider the geodesics $\gamma(t)$, $t \geq 0$ with $\gamma'(0) = \begin{pmatrix} P' \\ Q' \end{pmatrix}$. If $Max_R(|Q'(0)| - P'(0)) \leq 0$, then γ is defined in $[0, +\infty[$. Otherwise its maximum interval of definition is $[0, \frac{1}{max_R(|Q'(0)| - P'(0))}[$.

Theorem 5.6. *Let* $q : \mathcal{M} \to \mathcal{M}/\mathcal{D}_0$, *be the projection map, let* $\gamma(t) = \begin{pmatrix} tP' \\ tQ' \end{pmatrix}, t \in [0,1]$ *be a Lorentzian geodesic in* $\mathcal{M}$ *with* $\gamma(0) = g$ *and* $\gamma'(0) = \begin{pmatrix} P' \\ Q' \end{pmatrix}$. *Then the following facts are equivalent*

(i) $\gamma'(t_0)$ *is orthogonal to the* $\mathcal{D}_0$ *orbit of* $\gamma(t_0)$ *for some* $t_0 \in [0,1]$

(ii) $\gamma'(t)$ *is orthogonal to the* $\mathcal{D}_0$ *orbit of* $\gamma(t)$ *for every* $t \in [0,1]$

(iii) $\gamma(t_0)$ *is critical with respect to g for some* $t_0 \in]0,1]$

(iv) $\gamma(t)$ *is critical with respect to g for every* $t \in [0,1]$.

If the above conditions are satisfied, then $q(\gamma)$ *is a geodesic in* $\mathcal{M}/\mathcal{D}_0$ *with the induced Lorentzian metric.*

Proof. From Proposition 5.2 and Theorem 3.2 it follows that $\gamma'(0)$ is orthogonal to the $\mathcal{D}_0$ orbit of g, if and only if (iii) is satisfied, if and only if (iv) is satisfied. Assume that $t_0 \in [0,1]$ and $\gamma'(t_0)$ is orthogonal to the $\mathcal{D}_0$ orbit of $\gamma(t_0)$. Let us consider the complex structure on R that makes $h_0 = \gamma(t_0)$ conformal. We find as above that for every $t \in [0,1]$ the metric $\gamma(t)$ is critical with respect to $\gamma(t_0)$, hence $\gamma(t_0)$ is critical with respect to $\gamma(t)$. It follows that $\gamma'(t)$ is orthogonal to the $\mathcal{D}_0$ orbit of $\gamma(t)$. If γ' is orthogonal to the $\mathcal{D}_0$ orbit at every point of γ, then $q(\gamma)$ is a geodesic in $\mathcal{M}/\mathcal{D}_0$. $\square$

Let us now consider two metrics g and h, both with constant curvature -1. Let $\gamma(t)$ be the Lorentzian geodesic joining g and h, assume that h is critical with respect to g. Let m be the metric corresponding to the medium point of the geodesic. Let us consider on R the complex structure that makes m conformal, set $m = \rho^2 dz d\bar{z}$. Denote the corresponding exponential coordinates with P, Q. We must have $P_g = -P_h$, $Q_g = -Q_h$. We have

$$dVol_g = ((P_g + 1)^2 - |Q_g|^2)dVol_m$$
$$dVol_h = ((P_h + 1)^2 - |Q_h|^2)dVol_m$$

Moreover by Proposition 2.2 we have $dVol_g = dVol_h$, hence $P_g = P_h \equiv 0$. Since $K_{-2}(Q_g) = L_0(P_g)$, it follows that Q_g is harmonic, so $\Phi_g = \bar{Q}_g \rho^2 dz d\bar{z}$ and $\Phi_h = -\Phi_g$ are holomorphic quadratic differentials. Note that $||\Phi_g|| = ||\Phi_h|| < 1$. Since orientation preserving diffeomorphisms with holomorphic Hopf differential are harmonic, we conclude that the diagonal map from $(R, m) \to (R, g) \times (R, h)$ is both harmonic and conformal. In other words if $\pi : \mathcal{M} \to \mathcal{M}_{-1}$ denotes the maps that associates to every metric the unique metric, conformal to it, of constant curvature -1, then we have $m_g(h) = q(\pi(m))$. Moreover

$$K_m dVol_m = -(1 - |Q|^2)dVol_m = -(1 - ||\Phi||^2)dVol_m$$

i.e.

$$K_m = -(1 - ||\Phi||^2).$$

So we have the following

Proposition 5.7. *There is a one-to-one correspondence*

$$(g, h) \rightarrow (m, \Phi)$$

from the set

$$X = \{(g, h) : \quad g \text{ and } h \text{ both have constant curvature } -1,$$
$$\text{and they are critical with respect to each other}\}$$

and the set

$$Y = \{(m, \Phi), \text{ where } m \text{ is a metric}$$
$$\text{and } \Phi \text{ is a holomorphic quadratic differential with respect to } m,$$
$$\text{such that } ||\Phi|| < 1, \text{ and } K_m = -(1 - ||\Phi||^2)\}$$

Here m is the medium point of the Lorentzian geodesic between g and h, whereas Φ is the $(2, 0)$ part of g with respect to m, Moreover $m_g(h) = q(\pi(m))$.

Proof. We already described the map from X to Y, let us describe its inverse. Set $g = \Phi + \bar{\Phi} + (1 + ||\Phi||^2)m$, and $h = -\Phi - \bar{\Phi} + (1 + ||\Phi||^2)m$, then g and h are critical with respect to m, hence by Proposition 2.2 they have constant curvature -1, moreover the diagonal map from (R, m) into $(R, g) \times (R, h)$ is harmonic and conformal, in particular g and h are critical with respect to each other. Let m' be the medium point of the geodesics between g and h. From above we see that the diagonal map between (R, m') and $(R, g) \times (R, h)$ is also harmonic and conformal. It follows that m and m' are conformal. Moreover the P coordinates of g with respect to m and m' are both identically zero, so $m = m'$. Moreover, by construction, the quadratic differential Φ is the $(2, 0)$ part of g with respect to $m = m'$. The rest of the Proposition has already been proved.   □

Remark 9. It can by proved as above that given $(m, \Phi) \in Y$ and $\theta \in [0, 2\pi]$ the metric $g_\theta = e^{i\theta}\Phi + e^{-i\theta}\bar{\Phi} + (1 + |\Phi|^2)m$, has constant curvature -1 and it is critical with respect to m

The set Y was already considered by Hitchin [6], (in fact, among other things, he proved the above Remark). The system of equations

$$\begin{cases} K_m = -(1 - ||\Phi||^2) \\ \bar{\partial}\Phi = 0 \end{cases}$$

is the abelian vortex equation.

The orthogonal space to $\mathcal{M}_{-1}$.

Let g be a metric with constant curvature -1. It follows from Proposition 5.2, Theorem 3.2 and Corollary 5.4 that a vector $\begin{pmatrix} P' \\ Q' \end{pmatrix} \in T_g(\mathcal{M})$ is in $T_g(\mathcal{M}_{-1})^\perp$ if and only if Q' is L^2 orthogonal to the space of harmonic Beltrami differentials and $K_{-2}(Q') = L_0(P')$.

Set $D = (-1/2(\Delta - 2I))^{-1}$, where Δ is the Laplace-Beltrami operator on R with negative spectrum. The operator D is known to be compact, L^2 self adjoint and positive, see for example [15]. As in [15] we can prove the following

Proposition 5.8. *A vector* $\begin{pmatrix} P' \\ Q' \end{pmatrix} \in T_g(\mathcal{M})$ *is orthogonal to* $\mathcal{M}_{-1}$ *if and only if*

$$Q' = -2L_{-1}L_0 D(P').$$

Proof. Assume that $Q' = -2L_{-1}L_0 D(P')$, and let μ be an harmonic Beltrami differential, then $\int_R Q'\bar{\mu}dVol_g = 2\int_R L_0(D(P')\overline{K_{-2}(\mu)}dVol_g = 0$. On the other hand we have

$$
\begin{aligned}
4L_{-1}K_{-2}(-2L_{-1}L_0(D(P')) &= (\Delta_{-2} - 2I)(-2L_{-1}L_0(D(P')) \\
&= -2\Delta_{-2}L_{-1}L_0 D(P') + 4L_{-1}L_0 D(P') \\
&= -2(L_{-1}\Delta_{-1}L_0 D(P') + 4L_{-1}L_0 D(P') \\
&= -2L_{-1}L_0\Delta(D(P')) + 4L_{-1}L_0 D(P') \\
&= 4L_{-1}L_0(P')
\end{aligned}
$$

i.e. $4L_{-1}K_{-2}(Q') = 4L_{-1}L_0(P')$. However, since R does not have non trivial holomorphic vector fields, it follows that L_{-1} has trivial kernel. Viceversa, if $\begin{pmatrix} P' \\ Q' \end{pmatrix}$ is orthogonal to $\mathcal{M}_{-1}$, then the differentials Q' and

$-2L_{-1}L_0(D(P'))$ have the same K_{-2} image and they are both L^2 orthogonal to the harmonic differentials, hence they coincide. $\square$

Set $H = -2L_{-1}L_0D$.

Proposition 5.9. *We have*

$$T_g(\mathcal{M}) = T_g(\mathcal{M}_{-1}) \oplus T_g(\mathcal{M}_{-1})^\perp.$$

Moreover the Lorentzian metric is positive definite on $T_g(\mathcal{M}_{-1})$ and negative definite on its orthogonal space.

Proof. We already know that the Lorentzian metric is positive definite on $\mathcal{M}_{-1}$, in particular $\mathcal{M}_{-1}$ has trivial intersection with its orthogonal. Let $\begin{pmatrix} P' \\ Q' \end{pmatrix} \in T_g(\mathcal{M})$, and consider the decomposition $\begin{pmatrix} P' \\ Q' \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} Z \\ W \end{pmatrix}$, where $\begin{pmatrix} X \\ Y \end{pmatrix} \in T_g(\mathcal{D}_0g)$ and $\begin{pmatrix} Z \\ W \end{pmatrix} \in (T_g(\mathcal{D}_0g))^\perp$. Decompose farther $\begin{pmatrix} Z \\ W \end{pmatrix}$ as $\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} Z \\ Q_1 \end{pmatrix} + \begin{pmatrix} 0 \\ Q_2 \end{pmatrix}$, where Q_2 is harmonic and Q_1 is L^2 orthogonal to the harmonic differentials. So $\begin{pmatrix} X \\ Y + Q_2 \end{pmatrix}$ is tangent to $\mathcal{M}_{-1}$ and $\begin{pmatrix} Z \\ Q_1 \end{pmatrix}$ is orthogonal to it. By Proposition 5.8, together with the properties of the complex covariant differentials K_r, L_r it follows, as in [15] pag. 135, that if $\begin{pmatrix} P' \\ Q' \end{pmatrix} \in (T_g(\mathcal{M}_{-1}))^\perp$ then

$$\int_R |Q'|^2 dVol_g = \int_R |P'|^2 dVol_g - \int_R D(P')P' dVol_g$$

hence $<< \begin{pmatrix} P' \\ Q' \end{pmatrix}, \begin{pmatrix} P' \\ Q' \end{pmatrix} >> = -\int_R D(P')P' dVol_g$. Since D is a positive operator and $Q' = H(P')$, we conclude that $<<>>$ is negative definite on $T_g(\mathcal{M}_{-1})^\perp$. $\square$

From the above Proposition we see that the orthogonal projection from $T_g(\mathcal{M})$ to $T_g(\mathcal{M}_{-1})$ is well defined and it is easy to show that the Levi-Civita connection on $\mathcal{M}_{-1}$ coincides with the orthogonal projection of the connection on $\mathcal{M}$.

The Weil-Petersson geodesics.

Let $g \in \mathcal{M}_{-1}$ and $\gamma(t)$ for $t \in [0,1]$ a Weil-Petersson geodesics in Teichmüller space $\mathcal{T} = \mathcal{M}_{-1}/\mathcal{D}_0$, with $\gamma(0) = q(g)$. Let $\tilde{\gamma} = \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} \in \mathcal{M}_{-1}$ be the lift of γ starting at g which is, at every point, Lorentz orthogonal to the $\mathcal{D}_0$ orbit. So $\tilde{\gamma}$ is a Lorentzian geodesics. Note that, since the Lorentzian metric and the usual L^2 metric on $\mathcal{M}_{-1}$, (see e.g. [13]) have the same orthogonal space to the $\mathcal{D}_0$ orbits, then $\tilde{\gamma}$ can also be regarded as the L^2 lift of γ.

Lemma 5.10. *We have $dVol_{\tilde{\gamma}(t)} = dVol_g$ for every $t \in [0,1]$.*

Proof. Given $t_0 \in [0,1]$ let $h_0 = \tilde{\gamma}(t_0)$ and let P_{h_0}, Q_{h_0} be the exponential coordinates with respect to h_0. For positive t close to 0 we have

$$dVol_{\gamma(t+t_0)} = \left(\left((P(t+t_0)+1)^2 - |Q(t+t_0)|^2\right) dVol_g \right.$$
$$= \left((P_{h_0}(t)+1)^2 - |Q_{h_0}(t)|^2|\right) dVol_{h_0}.$$

Since the tangent vector to the curve at h_0 is tangent to $\mathcal{M}_{-1}$ and orthogonal to the $\mathcal{D}_0$ orbit, it follows from Proposition 5.4 that

$$\frac{\partial \left((P_{h_0}(t)+1)^2 - |Q_{h_0}(t)|^2|\right)}{\partial t}\bigg|_{t=0} = 2\frac{\partial P_{h_0}}{\partial t}\bigg|_{t=0} \equiv 0.$$

So

$$\frac{\partial \left((P+1)(t)^2 - |Q(t)|^2|\right)}{\partial t}\bigg|_{t=t_0} \equiv 0.$$

Hence

(5.1) $$(P+1)^2 - |Q|^2 \equiv 1.$$

$\square$

Lemma 5.11. *We have the equation $\frac{\partial^2 Q}{\partial t^2}(0) = H(|\frac{\partial Q}{\partial t}(0)|^2)$.*

Proof. Since $\begin{pmatrix} P \\ Q \end{pmatrix}$ gives the exponential coordinates for the Lorentzian metric and $\tilde{\gamma}$ is a geodesic in $\mathcal{M}_{-1}$, then the vector $\begin{pmatrix} P''(0) \\ Q''(0) \end{pmatrix}$ is orthogonal

to $\mathcal{M}_{-1}$. Moreover by differentiating twice equation (5.1) and evaluating at 0 we find $P''(0) = |Q'(0)|^2$. Hence the Lemma follows from Proposition 5.8
$\square$

Let g be a metric let z be local holomorphic coordinates such that $g = \rho^2 dz d\bar{z}$. Let h_0 be another metric with coordinates P^{h_0}, Q^{h_0} with respect to g. Let us consider on R holomorphic local coordinates such that $h_0 = \sigma^2 dw d\bar{w}$. Let $f = w^{-1} z$. So we have $f^*(\sigma^2 dw d\bar{w})$ is conformal to h_0, hence

$$\frac{\rho^2 f \overline{\frac{\partial f}{\partial z}} \frac{\partial f}{\partial \bar{z}} d\bar{z}}{\sigma^2 dz} = r(P+1)Q$$

$$\frac{\rho^2 f |\frac{\partial f}{\partial z}|^2}{\sigma^2} = r(P+1)^2.$$

for some positive smooth function r. It follows that

(5.2) $$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}$$

where

$$\mu = \frac{Q_{h_0}}{P_{h_0} + 1}$$

Since h_0 is a metric, then $|\mu| < 1$ and the Beltrami equation (5.2) admits a global solution f^μ which is a diffeomorphism of R. Since any pair of local solution of (5.2) differ by a holomorphic map, see [1], it follows that $f^\mu : (R, g) \to (R, h)$ is holomorphic, by changing the holomorphic coordinates, if needed, we might assume that $w = f^\mu(z)$, hence that the function r is identically 1, i. e.

(5.3) $$\frac{\rho^2 f^\mu \overline{\frac{\partial f^\mu}{\partial z}} \frac{\partial f^\mu}{\partial \bar{z}} d\bar{z}}{\sigma^2 dz} = (P+1)Q$$

$$\frac{\rho^2 f^\mu |\frac{\partial f^\mu}{\partial z}|^2}{\sigma^2} = (P+1)^2.$$

We are going to compute the geodesics equation by pulling back equation in Lemma 5.11 on a given point t_0 of $\tilde{\gamma}$, via a global solution f^μ of the Beltrami equation. So let us fix $t_0 \in [0, 1]$ and t positive close to 0. Let z be holomorphic local coordinates for the complex structure induced by g and w holomorphic local coordinates for the one induced by h_0, with these notations we have

(5.4) $$(P(t + t_0) + 1)\bar{Q}(t + t_0)\rho^2 dz \, d\bar{z}$$

$$+ \left((P(t+t_0)+1)^2 + |Q(t+t_0)|^2\right)\rho^2 dz\,d\bar{z}$$
$$+(P(t+t_0)+1)Q(t+t_0)\rho^2 dz\,d\bar{z}$$
$$= f^{\mu *}\left((P_{h_0}(t)+1)\overline{Q_{h_0}(t)}\sigma^2 dwd\bar{w}\right)$$
$$+f^{\mu *}\left(((P_{h_0}(t)+1)^2 + |Q_{h_0}|(t)|^2)\sigma^2 dwd\bar{w}\right)$$
$$+f^{\mu *}\left((P_{h_0}(t)+1)Q_{h_0}(t)\sigma^2 dwd\bar{w}\right).$$

If $B = \tilde{B}(w)\frac{d\bar{w}}{dw}$ is a Beltrami differential, then $f^{\mu *}(B)$ is given, in holomorphic local coordinates z, by the equation

$$f^{\mu *}(B) = \tilde{B}(f^\mu)\frac{\overline{\frac{\partial f^\mu}{\partial z}}}{\frac{\partial f^\mu}{\partial z}}\frac{d\bar{z}}{dz}.$$

and for every smooth test function θ on R, we have, by definition (Here H_{h_0} is the operator H computed with respect to the metric h_0.)

$$\left(f^{\mu *}(H_{h_0})\right)\left(\theta(f^\mu)\right) = f^{\mu *}\left(H_{h_0}(\theta)\right).$$

Moreover observe that $\mu_{h_0}{}'(0) = Q_{h_0}{}'(0)$. From general Teichmüller theory (see for example [16]), we know that

$$(5.5) \qquad f^{\mu *}(Q_{h_0}{}'(0)) = \frac{\partial\mu}{\partial t}(1 - |\mu|^2)^{-1}.$$

If we keep in mind the equations (5.2), (5.3) and equation (5.1) relative to the exponential coordinates based on h_0, then by differentiating equation (5.4) twice, evaluating at 0 taking the $(0,2)$ and $(1,1)$ part divided by $\rho^2 dzd\bar{z}$, we find the equations

$$(5.6) \qquad \frac{\partial^2((P+1)Q)}{\partial t^2}\bigg|_{t=t_0} = f^{\mu *}(H_{h_0})(|Q_{h_0}{}'(0)|^2(f^\mu))(P+1)^2$$

$$+4|Q_{h_0}{}'(0)|^2(f^\mu)(P+1)Q + \left(\overline{f^{\mu *}(H_{h_0})}(|Q_{h_0}{}'(0)|(f^\mu)Q\right)Q.$$

and

$$(5.7)$$
$$\frac{\partial^2((P+1)^2 + |Q|^2)}{\partial t^2}\bigg|_{t=t_0} = 2(f^{\mu *}(H_{h_0}))(|Q_{h_0}{}'(0)|^2(f^\mu))(P+1)\bar{Q}$$

$$+2(\overline{(f^{\mu *}(H_{h_0})}(|Q_{h_0}{}'(0)|^2(f^\mu))(P+1)Q$$
$$+4|Q_{h_0}{}'(0)|^2(f^\mu))((P+1)^2 + |Q|^2).$$

By equation (5.1) we find $Q = (1-|\mu|^2)^{-1/2}\mu$ and $P = (1-|\mu|^2)^{-1/2}-1$. So we can write equations (5.6) and (5.7) in terms of μ. Set

$$\delta(\mu) = f^{\mu*}\left(Q_{h_0}{}'(0)\right) = \frac{\partial\mu}{\partial t}(1-|\mu|^2)^{-1}$$

and

$$H(\mu) = f^{\mu*}(H_{h_0}).$$

(5.8)
$$\frac{\partial^2 (P+1)^2}{\partial t^2} = (P+1)^2 \left((2Re(H(\mu)(|\delta(\mu)|^2)\bar\mu) + 4|\delta(\mu)|^2\right) - 2|\delta(\mu)|^2.$$

Moreover since f^μ preserves the volume form, from [18] Lemma 4.1, we find

(5.9) $$f^{\mu*}(K_r) = (1-|\mu|^2)^{-1/2}\left(K_r - \bar\mu L_r\right)$$

$$+r/2(1-|\mu|^2)^{-1/2}\left(K_0 - \bar\mu L_0\right)(\log(1-|\mu|^2))\, I$$

$$+r(1-|\mu|^2)^{-1/2}(\rho^2 dz d\bar z)^{-1/2}\left(\overline{\partial\mu}\right) I.$$

Since $L_r = \overline{K_{-r}}$, and $\Delta = 4L_1 K_0$, it follows that the operator $H(\mu)$ depends only on μ and its derivatives on the surface R; it does not explicitly depend on the chosen solution to the Beltrami equation.

Here $H(\mu)$ is the operator H_{h_0}, with h_0 the metric with the same volume form as g, and with Beltrami differential μ.

So, by using equations (5.7) and (5.8) we prove the following

Theorem 5.12. *Consider the equation*

(5.10) $$\frac{\partial^2\mu}{\partial t^2} = (1-|\mu|^2)H(\mu)(|\delta(\mu)|^2)$$

$$+2\frac{\partial\log(1-|\mu|^2)}{\partial t}\frac{\partial\mu}{\partial t}$$

$$+2|\frac{\partial\mu}{\partial t}|^2(1-|\mu|^2)^{-1}\mu$$

with the initial conditions $\mu(0) = 0$ and $\frac{\partial\mu}{\partial t}(0)$ a fixed harmonic Beltrami differential in (R,g). Then if $\mu(t)$ is the solution of the above equation, and $R(t)$ is the Riemann surface structure on R such that $f^{\mu(t)} : (R,g) \to R((t)$ is holomorphic, then the curve given by $t \to$ Teichmüller classes of $R(t)$, is the Weil-Petersson geodesics starting at the class of g with initial tangent vector $\frac{\partial\mu}{\partial t}(0)$.

Note that equation (5.10) is equivalent to the system of first order equations:

$$\begin{cases} \frac{\partial \delta}{\partial t} = H(\mu)(|\delta|^2) - 2i(Im(\delta\bar{\mu}))\delta \\ \frac{\partial \mu}{\partial t} = \delta(1 - |\mu|^2) \end{cases}$$

with $\mu(0) = 0$ and $\delta(0) = \frac{\partial \mu}{\partial t}(0)$. Moreover, since f^μ preserves the volume form, then equation (5.5) implies that $\int_R |\delta(\mu_t)|^2 dVol_g$ is independent on t, and it is the energy density of γ.

Remark 10. In other words, if μ_t is the solution of equation (5.10), then the (Lorentzian or L^2) lifting of the Weil-Petersson geodesics starting at g with initial tangent vector

$$\frac{\partial \mu}{\partial t}(0)g + \overline{\frac{\partial \mu}{\partial t}(0)}g,$$

is given by

$$\frac{\mu_t}{1 - |\mu_t|^2}g + \left(\frac{1 + |\mu_t|^2}{1 - |\mu_t|^2}\right)g + \frac{\bar{\mu}_t}{1 - |\mu_t|^2}g.$$

6. The general existence Theorem.

In this last section we are going to show that, whenever g and h have strictly negative curvature, there exists a unique critical point of T_∂ in any homotopy class containing an orientation preserving diffeomorphism. By using Lemma (2.1) ii) we may reduce to the case of maps homotopic to the identity. Let τ be the harmonic map from $(R, h) \to (R, g)$ homotopic to the identity, so the identity is harmonic from (R, h) to $(R, \tau^*(g))$. Let P, Q the Lorentzian exponential coordinates of $\tau^*(g)$ with respect to h. Set $\delta = 1/2 \log(P + 1)$, then the identity is harmonic from $(R, e^{2\delta}h)$ to $(R, \tau^*(g))$. Since the P coordinate of $\tau^*(g)$ with respect to $e^{2\delta}h$ is identically zero, it follows from Theorem 3.2 and from Lemma 2.1, that τ is a critical point of $T_\partial(e^{2\delta}h, g)$ and $\phi_0 = \tau^{-1}$ is critical with respect to $T_\partial(g, e^{2\delta}h)$. Consider the product $\mathcal{D}_0 \times \mathbf{R}$, and the subset $M \subseteq \mathcal{D}_0 \times \mathbf{R}$, given by $M = \{(\phi, t) \text{ such that } \mathcal{E}(g, e^{2(1-t)\delta}h, \phi) \equiv 0\}$. Note that, by Lemma 2.2, if (ϕ, t) is in M, then $e^{2(1-t)\delta}h$ has negative curvature. Let $\pi : M \to \mathbf{R}$ be the restriction of the natural projection. It follows from Lemma 2.3 and Lemma 2.4, together with the implicit function Theorem, that M is a smooth submanifold of $\mathcal{D}_0 \times \mathbf{R}$, and π is a local diffeomorphism. The connected component containing $(\phi_0, 0)$ of the preimage through π of the interval $[0, 1]$, is a parameterized path γ in M. We must show that γ is

defined in the whole interval $[0, 1]$. If this were not the case, then γ would be defined in an interval of the form $[0, t_0[$, it is then sufficient to show the existence of a sequence t_n converging to t_0, such that ϕ_{t_n} converges to a critical diffeomorphism ϕ_{t_0}. Set $h_t = e^{2(1-t)\delta} h$ and $h'_t = \phi_t^*(h_t)$.

Lemma 6.1. *We have $T_\partial(g, h_t, \phi_t), t \in [0, t_0[$ is uniformly bounded.*

Proof. Since each ϕ_t is critical, it follows that

$$\frac{\partial T_\partial(g, h_t, \phi_t)}{\partial t} \le Max_R |\delta| T_\partial(g, h_t, \phi_t),$$

so

$$T_\partial(g, h_t, \phi_t) \le T_\partial(g, h, \phi_0) e^{Max_R |\delta|}.$$

$\square$

We have that the metrics g and h'_t are critical with respect to each other, let m_t the medium point of the Lorentzian geodesics joining g and h'_t. In the Lorentzian exponential coordinates centered at m_t, the geodesics is given by $P_s = sP, Q_s = sQ$, where g has coordinates $\begin{pmatrix} P_{-1} \\ Q_{-1} \end{pmatrix}$ and h'_t has coordinates $\begin{pmatrix} P_1 \\ Q_1 \end{pmatrix}$. (here $\begin{pmatrix} P \\ Q \end{pmatrix}$ depends on t but, for simplicity, we will drop the suffix). From above we find

(6.1) $1 - P \ge |Q|$

$$1 + P \ge |Q|, \text{ so}$$

$$-1 < P < 1, \text{ and } 0 \le |Q| < 1.$$

(6.2) $K_{m_t} = K_g((1-P)^2 - |Q|^2) = K_{h'_t}((1+P)^2 - |Q|^2)$

which gives

(6.3) $4K_g < K_{m_t} < 0.$

On the other hand, by using exponential coordinates P^*, Q^* centered in g, it can be easily seen that

(6.4) $dVol(m_t) = 1/4(dVol(g) + dVol(h'_t) + 2(P^* + 1)dVol_g$

Let now t_n be a sequence in $[0, t_0[$ converging to t_0. Set $\phi_n = \phi_{t_n}, h_n = h_{t_n}$, $h'_n = h'_{t_n}$, and $m_n = m_{t_n}$.

Lemma 6.2. *The energy of the map* $\tilde{\phi}_n : (R, m_n) \to (R \times R, g \times h_n)$ *is uniformly bounded.*

Proof. Let us denote with $Diag$ the diagonal map; we have

$$(6.5) \qquad E(m_n, g \times h_n, \tilde{\phi}_n) = E(m_n, g \times h_n', Diag)$$

$$= \int_R ((1+P)^2 + (1-P)^2 + 2|Q|^2) dVol_{m_n}$$

which, because of estimates (6.1) is bounded by $10 Vol_{m_{t_n}}$. This in turns is uniformly bounded because of formula (6.4), Remark 2 and Lemma 6.1. $\square$

From this we immediately find

Lemma 6.3. *The sequence of the Teichmüller classes of* m_n *has a convergent subsequence.*

Proof. By using the above Lemma we can argue as in [11]. $\square$

If we act on the metrics m_n, and in the same time on the map $\tilde{\phi}_n$ with the group $\mathcal{D}_0$, we still find a sequence of minimal immersions with lagrangiam image, let us denote such immersions with $\Psi_n = (\Psi(1)_n, \Psi(2)_n)$. Note that $\Psi(1)_n$ and $\Psi(2)_n$ are T_∂ critical with respect to the pairs of metrics m_n, g and and m_n, h_n respectively.

Remark 11. From formula (6.1) we can derive an L^∞ bound of the differentials of $\Psi(1)_n$ and $\Psi(2)_n$ in terms of L^∞ bounds of m_n.

We need to show that a subsequence of m_n converges up to elements in $\mathcal{D}_0$. To do this we will use an easy version of Cheeger-Gromov compactness Theorem, see [9]. So it is necessary to have a bound on the curvatures, on the total volumes (which are immediate from equations (6.3), (6.4) and Lemma 6.1), and on the diameters. Let m'_n be the metric conformal to m_n with constant curvature -1. We can assume that m'_n converges in C^∞ norm, therefore to bound the diameter of m_n it is sufficient to bound the volume forms of m_n.

Let us write in conformal coordinates

$$m_n = \nu_n^2(\xi_n) d\xi_n d\bar{\xi}_n$$

and

$$m'_n = \mu_n{}^2(\xi_n)d\xi_n d\bar{\xi}_n.$$

Lemma 6.4. *There exists a uniform L^1 bound on $\log(\frac{\nu_n{}^2}{\mu_n{}^2})$. Where L^1 is defined with respect to the measure of m'_n. (Hence with respect to any fixed metric on R).*

Proof. We have $|\log(\frac{\nu_n{}^2}{\mu_n{}^2})| \leq \frac{\nu_n{}^2}{\mu_n{}^2} + \frac{\mu_n{}^2}{\nu_n{}^2}$. Formula (6.4) gives an L^1 bound on $\frac{\nu_n{}^2}{\mu_n{}^2}$. By Lemma 2.2 we have the equations

(6.6)
$$\partial_n\bar{\partial}_n \log \nu_n{}^2 = -(1/2)K_g(\Psi(1)_n)det(Jac(\Psi(1)_n))(i/2)d\xi_n \wedge d\bar{\xi}_n.$$

$$\partial_n\bar{\partial}_n \log \mu_n{}^2 d\xi_n \wedge d\bar{\xi}_n = (\frac{\mu_n{}^2}{2})(i/2)d\xi_n \wedge d\bar{\xi}_n.$$

For any positive constant C, we define the set $S_C = \{z \in R : \log(\frac{\nu_n{}^2}{\mu_n{}^2}) < -C\}$. Since the curvature of g is uniformly bounded, by using formulas (6.1), and (6.6), we see that, for large enough C (independent on n) the function $\Delta_n(\log(\frac{\nu_n{}^2}{\mu_n{}^2}))$ is negative on S_C, on the other hand, whenever S_C is nonempty, the minimum of $\log(\frac{\nu_n{}^2}{\mu_n{}^2})$ on $\overline{S_C}$ is in S_C, this yields a contradiction. Note that, we not only found an L^1 bound on $|\log(\frac{\nu_n{}^2}{\mu_n{}^2})|$, but also an L^∞ lower bound on $\log(\frac{\nu_n{}^2}{\mu_n{}^2})$.   □

To obtain an L^∞ upper bound on $|\log(\frac{\nu_n{}^2}{\mu_n{}^2})|$ we will use a so-called Wente estimate, see [2], applied to a relatively compact neighborhood Ω of a fundamental domain of R in its universal covering space D. For simplicity any function on R and its lift to D will be denoted with the same symbol.

Lemma 6.5. *There is an L^∞ uniform bound on $|\log(\frac{\nu_n}{\mu_n})|$.*

Proof. Since μ_n converges and (by the proof of Lemma 6.4) the function $\log(\frac{\nu_n}{\mu_n})$ has an L^∞ lower bound, we only need to bound $\log(\nu_n{}^2)$. Let us write $\log(\nu_n) = \tau_n + \eta_n$, where τ_n is harmonic on Ω and coincides with ν_n on $\partial\Omega$, whereas

(6.7) $$\partial_n\bar{\partial}_n \log \eta_n{}^2 = -1/2K_g(\Psi(1)_n)Jac(\Psi(1)_n).$$

Since the right hand side of (6.7) has a determinant form and the conformal structure of m_n converges, we can apply Theorem 1.2 in [2], to obtain the estimate

$$|\eta_n|_{L^\infty} \le C|K_g|_{L^\infty}|d\Psi(1)_n|_{L^2(\Omega)}$$

where C is a positive constant independent on n, and the L^2 norm is computed with respect to the volume form of m_n'. Lemma 6.2 will give a uniform L^∞ bound on η_n. Since τ_n is harmonic, then we can estimate its L^∞ norm in terms of its L^1 norm on a domain Ω' slightly smaller then Ω. Now the Lemma follows from Lemma 6.4. $\square$

Similar Lemmas can be proved for the sequence $\Psi(2)_n$. By Remark (11) and Lemma 6.5 we derive an L^∞ uniform bound on $d\Psi_n$. (All the bounds have to be considered with respect to the convergent sequence of hyperbolic metric, or, which is the same, with respect to a fixed metric). By using equation (6.6) we find an L^∞ bound on the second derivative of $dVol_{m_n}$. By Sobolev inequality in dimension 2, we obtain that (a subsequence of) m_n converges to a metric m in $C^{1,\alpha}$ (whereas $m'_n \to m'$ in norm C^∞) and $\Psi_n \to \Psi$ in $C^{0,\alpha}$. Let us show that Ψ is an immersion. To do this we will compare the metric m_n with the metric p_n which is the pull-back via Ψ_n of the metric on $(R * R)_n$, we have

$$p_n = \frac{K_{m_n}\nu_n{}^2 d\xi_n d\bar{\xi}_n}{\sqrt{K_g K_h}} \left\{ \frac{1}{(P+1)^2 - |Q|^2} \left((P+1)Q + ((P+1)^2 + |Q|^2)\right) \right\}$$

$$+ \frac{K_{m_n}\nu_n{}^2 d\xi_n d\bar{\xi}_n}{\sqrt{K_g K_h}} \frac{1}{(P+1)^2 - |Q|^2}(P+1)\bar{Q}$$

$$+ \frac{K_{m_n}\nu_n{}^2 d\xi_n d\bar{\xi}_n}{\sqrt{K_g K_h}} \left\{ \frac{1}{(1-P)^2 - |Q|^2} \left(-(1-P)Q + ((1-P)^2 + |Q|^2)\right) \right\}$$

$$+ \frac{K_{m_n}\nu_n{}^2 d\xi_n d\bar{\xi}_n}{\sqrt{K_g K_h}} \left\{ \frac{-1}{(1-P)^2 - |Q|^2)}(1-P)\bar{Q} \right\}.$$

By a tedious but elementary computation using (6.1) we find

$$p_n = \frac{\nu_n{}^2 d\xi_n d\bar{\xi}_n}{\sqrt{K_g K_h}} \left\{ (K_h(1+P) - K_g(1-P))\bar{Q} \right\}$$

$$+ \frac{\nu_n{}^2 d\xi_n d\bar{\xi}_n}{\sqrt{K_g K_h}} 2((1+P)^2 K_g + |Q|^2 K_h)$$

$$+\left\{\frac{\nu_n{}^2 d\xi_n d\bar{\xi_n}}{\sqrt{K_g K_h}}(K_h(1+P)-K_g(1-P))Q.\right\}$$

Therefore we get

$$dVol_{p_n} = 2((1-P^2)+|Q|^2)dVol_{m_n}.$$

Since the curvature of h_n is bounded away from zero, from the above formulas we derive the following

Lemma 6.6. *The metrics m_n and p_n are uniformly comparable i.e. there exist positive constants C_1 and C_2, such that*

$$C_1 m_n \leq p_n \leq C_2 m_n.$$

It follows that Ψ is an immersion.

Since Ψ is an immersion then (by using notations of Remark 3) $\mathcal{H}_1(\Psi_n)$ and $\mathcal{L}_1(\Psi_n)$ as well as $\mathcal{H}_2(\Psi_n)$ and $\mathcal{L}_2(\Psi_n)$ do not have common zeros. Moreover since the maps Ψ_n have lagrangian image, we have $\mathcal{H}_1(\Psi_n) = \mathcal{L}_1(\Psi_n)$ and $\mathcal{H}_2(\Psi_n) = \mathcal{L}_2(\Psi_n)$. Let us now prove that Ψ_n converges to Ψ in C^∞ norm. The maps $\Psi_n : (R, p_n) \to (R * R)_n$ are minimal immersions with lagrangian image, then they satisfy equation (3.2), which, because of Lemma 6.6, is a uniformly elliptic equation in divergence form. By standard estimates, see for example [4], and by formulas (6.1), we find a uniform $C^{1,\alpha}$ estimate on Ψ_n. Now by bootstrapping the equations (3.2) and (6.6) we conclude that Ψ_n converges C^∞ to Ψ. So Ψ is a minimal immersion with lagrangian image To conclude we need to show that the components $\Psi(1)$ and $\Psi(2)$ of Ψ are diffeomorphisms. Since they are limits of orientation preserving diffeomorphisms, then they have non negative jacobian determinants, moreover, $\mathcal{H}_1 = \mathcal{H}_2$ and $\mathcal{L}_1 = \mathcal{L}_2$, and since Ψ is an immersion $\mathcal{H}_1$ and $\mathcal{L}_1$ as well as $\mathcal{H}_2$ and $\mathcal{L}_2$ do not have common zeros. If p is a zero of the jacobian of, say $\Psi(1)$, then $\mathcal{H}_1$ and $\mathcal{L}_1$ do not vanish at p. If we let $l = \log(\frac{\mathcal{H}_1}{\mathcal{L}_1})$ and $J^* = \mathcal{H}_1 - \mathcal{L}_1$ we have that, near the point p, $J^* = fl$ where f is a smooth function. Moreover, we already observed that $J^* \geq 0$, that is $l \geq 0$.

$$|\Delta(l)| \leq C(|l|+|d(l)|) \text{ near } p$$

where C is a positive constant. From above, by using a standard Harnack inequality see [5], we would conclude that l vanishes in a neighborhood of p. Since $\Psi(1)$ and $\Psi(2)$ are non constant, it follows that they are orientation preserving diffeomorphism. Since the jacobians of $\Psi(1)$ and $\Psi(2)$ are

bounded away from 0, it follows that $\phi_n = \Psi(2)_n \circ \Psi(1)_n{}^{-1}$ converges C^∞ to $\phi_{t_0} = \Psi(2) \circ \Psi(1)^{-1}$. Hence ϕ is a T_∂ critical map with respect to the metrics g and h_{t_0}. We are left to prove uniqueness. Let λ and μ be smooth functions such that $e^\lambda g$ and $e^\mu h$ have constant curvature -1. For a small positive ε and for $t \in (-\varepsilon, 1 + \varepsilon)$ we have that $e^{t\lambda}g$ and $e^{t\mu}h$ have negative curvature. Consider the set

$$S = \{(t, \phi) : \quad \phi \text{ is critical with respect to } T_\partial(e^{t\lambda}g, e^{t\mu}h)\}.$$

It can be proved as above that the projection map $S \to (-\varepsilon, 1 + \varepsilon)$ is a covering. Since the fiber over $t = 1$ is a single point, we conclude. We have then proved that Theorem 4.2 is valid for metrics with negative variable curvature as well, that is

Theorem 6.7. *Let g and h be hermitian metrics on R, both with strictly negative curvature, then*

(i) *Suppose an homotopy class of maps of R into itself contains an anti-holomorphic non constant map, then there is only one such map and it is the unique critical point of T_∂ in the class.*

(ii) *Suppose an homotopy class contains the constants, then these are the only critical points of T_∂ in the class.*

(iii) *Suppose an homotopy class contains an orientation preserving diffeo-morphism, then there exists a unique critical point of T_∂ in the class. Such point is a diffeomorphism preserving the curvature forms.*

(iv) *In any other homotopy class there is no critical points of T_∂.*

If we denote by $\mathcal{M}_-$ the subset of $\mathcal{M}$ given by metrics of negative curvature, we can state the above Theorem in the following geometric form

Theorem 6.8. *Any two points in $\mathcal{M}_-/\mathcal{D}_0$ can be joined by a unique Lorentzian geodesics.*

7. Appendix.

In this appendix we want to derive formula (3.2) and the Bochner formula (3.3). Set $a = \sqrt{-K_g}$ and $b = \sqrt{-K_h}$, then the Dirichlet energy of the map Ψ is given

$$E(\Psi) = \int_R \left(\frac{a}{b}\rho^2 |d\Psi_1|^2 + \frac{b}{a}\sigma^2 |d\Psi_2|^2 \right) (i/2)d\zeta \wedge \bar{d}\zeta.$$

Let Ψ_t be a variation of Ψ, then the first derivative evaluated at $t = 0$ is given by twice the real part of

$$\int_R \left(\frac{a}{b}\rho^2 |d\Psi_1|^2 \left(\partial_z(\log(a)) + \partial_z(\log(\rho^2)) \right) \Psi_1' \right) (i/2)d\zeta \wedge d\bar{\zeta}$$

$$- \int_R \left(\frac{b}{a}\sigma^2 |d\Psi_2|^2 \left(\partial_z(\log(a)) \right) \Psi_1' \right) (i/2)d\zeta \wedge d\bar{\zeta}$$

(7.1) $$+ \int_R \frac{a}{b}\rho^2 \left(\bar{\partial}\Psi_1'\partial\bar{\Psi}_1 + \partial\Psi_1'\bar{\partial}\bar{\Psi}_1 \right) (i/2)d\zeta \wedge d\bar{\zeta}$$

plus the analogue term in Ψ_2'.

By integration by parts, formula (7.1) equals to

$$\int_R \left(\frac{a}{b}\rho^2 |d\Psi_1|^2 \left(\partial_z(\log(a)) + \partial_z(\log(\rho^2)) \right) \Psi_1' \right) (i/2)d\zeta \wedge d\bar{\zeta}$$

$$- \int_R \left(\frac{b}{a}\sigma^2 |d\Psi_2|^2 \left(\partial_z(\log(a)) \right) \Psi_1' \right) (i/2)d\zeta \wedge d\bar{\zeta}$$

(7.2) $$- \int_R \left(\bar{\partial}_\zeta \left(\frac{a}{b}\rho^2\partial\bar{\Psi}_1 \right) + \partial_\zeta \left(\frac{a}{b}\rho^2\bar{\partial}\bar{\Psi}_1 \right) \right) (i/2)d\zeta \wedge d\bar{\zeta}.$$

Performing the differentiation in the above formula and collecting the term $\frac{a}{b}\rho^2$ we find the first equation of the Euler-Lagrange system, i.e.

(7.3)
$$\begin{cases} \bar{\partial}\partial\Psi_1 + \partial_z(\log\rho^2)\bar{\partial}\Psi_1\partial\Psi_1 - 1/2 \left(\partial\Psi_1\partial_\zeta A_1 + \partial\Psi_1\bar{\partial}_\zeta A_1 \right) \\ = 1/2 \left(\bar{\partial}_z(\log(a)) \left(|\partial\Psi_1|^2 - \frac{b^2}{a^2}\frac{\sigma^2}{\rho^2}|\partial\Psi_2|^2 \right) \right). \end{cases}$$

The same equation holds for Ψ_2. Notice that if Ψ is minimal with lagrangian image, then the right hand side of the above equation vanishes, and we find equation (3.2).

In this case we derive a Bochner formula.

$$\partial_\zeta\bar{\partial}_\zeta(\log(\rho^2|\partial\Psi_1|^2)) = \partial_\zeta\bar{\partial}_\zeta(\log(\rho^2)) + \partial_\zeta\bar{\partial}_\zeta\log(\partial\Psi_1) + \partial_\zeta\bar{\partial}_\zeta\log(\bar{\partial}\bar{\Psi}_1)) =$$

$$\bar{\partial}_\zeta \left(\partial_z(\log(\rho^2))\partial\Psi_1 + \partial_z(\log(\rho^2))\partial\bar{\Psi}_1 \right) + \partial_\zeta \left(\frac{\partial\bar{\partial}\Psi_1}{\partial\Psi_1} \right) + \bar{\partial}_\zeta \left(\frac{\partial\bar{\partial}\bar{\Psi}_1}{\bar{\partial}\bar{\Psi}_1} \right)$$

By using (3.2) we find

$$\partial_\zeta \bar{\partial}_\zeta (\log(\rho^2 |\partial \Psi_1|^2)) = \bar{\partial}_\zeta \left(\partial_z (\log(\rho^2) \partial \Psi_1) - \partial_\zeta (\partial_z (\log(\rho^2) \bar{\partial} \Psi_1)) \right)$$

$$+ 1/2 \partial_\zeta (\bar{\partial}_\zeta + \frac{\bar{\partial} \Psi_1}{\partial \Psi_1} \partial_\zeta) A_1$$

$$+ 1/2 \bar{\partial}_\zeta \left((\partial_\zeta + \frac{\partial \bar{\Psi}_1}{\bar{\partial} \bar{\Psi}_1} \bar{\partial}_\zeta) A_1 \right).$$

By standard computations we finally get

$$\partial_\zeta \bar{\partial}_\zeta (\log(\rho^2 |\partial \Psi_1|^2)) = \partial_z \bar{\partial}_z \log(\rho^2) \left(|\partial \Psi_1|^2 - |\bar{\partial} \Psi_1|^2 \right)$$

$$+ 1/2 \partial_\zeta \left(\bar{\partial} + \frac{\bar{\partial} \Psi_1}{\partial \Psi_1} \partial \right) A_1$$

$$+ 1/2 \bar{\partial}_\zeta \left(\partial + \frac{\partial \bar{\Psi}_1}{\bar{\partial} \bar{\Psi}_1} \bar{\partial} \right) A_1.$$

Similarly

$$\partial_\zeta \bar{\partial}_\zeta (\log(\rho^2 |\bar{\partial} \Psi_1|^2)) = -\partial_z \bar{\partial}_z \log(\rho^2) \left(|\partial \Psi_1|^2 - |\bar{\partial} \Psi_1|^2 \right)$$

$$+ 1/2 \partial_\zeta \left(\bar{\partial} + \frac{\bar{\partial} \Psi_1}{\partial \bar{\Psi}_1} \partial_\zeta \right) A_1 + 1/2 \bar{\partial}_\zeta \left(\partial + \frac{\partial_\zeta \Psi_1}{\bar{\partial} \Psi_1} \bar{\partial}_\zeta \right) A_1.$$

By subtraction we obtain formula (3.3).

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