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Local entropy rigidity for hyperbolic manifolds

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We study deformations of compact hyperbolic manifolds of a given total volume. We show that along any non-trivial deformation the topological entropy and the difference between topological entropy and Liouville entropy are locally strictly convex functions of the deformation parameter, thus providing a partial positive answer to a conjecture of A. Katok.

§1. Introduction.

1.1. Statement of the problem.

1.1.1. Notation. Throughout this article, M denotes a compact smooth manifold of dimension n. Given a C^2 Riemannian metric g on M, the geodesic flow determined by g on the unit tangent bundle S_gM is denoted by T_g or T_g^t, or simply by g^t when this abuse of notation will not be confusing.

1.1.2. Geodesic flows on the unit tangent bundles S_gM of compact Riemannian manifolds (M, g) of negative curvature are the chief examples of transitive Anosov flows.

We recall that a continuous flow $T : (v,t) \in N \times \mathbb{R} \mapsto T^t v \in N$ is *transitive* if it has a dense orbit. A flow T on a compact manifold N is an *Anosov flow* [Ano67] if it is C^1 and the tangent bundle TN of N splits continuously in T-invariant subbundles $TN = E^0 \oplus E^u \oplus E^s$ satisfying the conditions:

1. E^0 is the tangent space to the orbits of flow.
2. There exists positive constants λ and C such that $\|dT^t|E^s\| < Ce^{-\lambda t}$ and $\|dT^{-t}|E^u\| < Ce^{\lambda t}$ for all $t > 0$.

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1.1.3. Anosov flows have many invariant measures. Given a T^t-invariant probability measure m, we denote by $h(T, m)$ or $h(T^t, m)$ the measure theoretic entropy of the time 1 transformation T^1 with respect to the measure m [Pet83]. The supremum

$$h_{\text{top}}(T) = \sup\{h(T, m) \mid T^t\text{-invariant probability measure } m\}$$

of all measure theoretic entropies is attained, for Anosov flows, by a unique measure, the Margulis measure m_0, also called the measure of maximal entropy [Mar70, Bow72a, Bow74, BR75].

In the case of a geodesic flow T_g^t, we also have a smooth T_g^t-invariant measure on S_gM, the Liouville measure Liouv(g), which arises from the contact structure on S_gM.

1.1.4. For locally symmetric manifolds of negative curvature, it is not difficult to see that the Liouville measure and the Margulis measure coincide. In other words, for geodesic flows on locally symmetric manifolds of negative curvature, the measure theoretical entropy of Liouville measure coincides with the topological entropy. The converse of this statement is the content of the following conjecture of A. Katok.

1.1.5. Conjecture [Kat82, BK85]. *The topological entropy and the entropy of the Liouville measure for a geodesic flow on a negatively curved manifold coincide, (if and) only if the manifold is locally symmetric.*

A. Katok himself showed in [Kat82] that the conjecture is true if one considers metrics conformally equivalent to a locally symmetric metric. In particular, the conjecture is true for surfaces of genus greater than 1.

The topological entropy has also another geometric interpretation [Man79]. Denote by $\widetilde{M}$ the universal cover of M. For a negatively curved metric g on M, let $B_{g,r}(p)$ be the ball of radius r centred at $p \in \widetilde{M}$ for the lift to $\widetilde{M}$ of metric g. Then

$$h_{\text{top}}(T_g) = \lim_{r\to\infty} \frac{1}{r} \log \mathrm{Vol}_g\big(B_{g,r}(p)\big).$$

In words, the topological entropy of the geodesic flow coincides with the volume growth of balls in the universal cover of M.

A related conjecture of Gromov, formulated at about the same time as Katok's conjecture, states:

1.1.6. Conjecture ([Gro83]). *Among all metrics of volume equal to the volume of a locally symmetric metric g_0 on a manifold M, the volume growth of balls in the universal cover $\widetilde{M}$ of M is minimized at the metrics isometric to g_0.*

Besson, Courtois and Gallot ([BCG94a] and [BCG94b]) have recently announced a proof of Gromov's conjecture for rank-one neg. curved locally symmetric manifolds which builds up on their 1992 theorem:

1.1.7. Theorem ([BCG]). *Let g_0 be a metric of constant negative curvature on a compact manifold M of dimension n. Then, in the space of H^s metrics with volume equal to the volume of g_0, there exists a neighbourhood $\mathcal{U}$ of g_0 in the H^s topology ($s > n/2 + 2$), such that the minimum of the topological entropy in $\mathcal{U}$ is attained only by metrics isometric to g_0.*

1.1.8. In consideration of the fact that the method of Besson, Courtois and Gallot applies so far only to topological entropy, in order to shade some light on Katok's entropy conjecture 1.1.5, it is very interesting to study the functions

$$\mathrm{Ent_{top}} : g \in \mathcal{M}^r(M) \mapsto h_{\mathrm{top}}(T_g)$$

$$\mathrm{Ent_{Liouv}} : g \in \mathcal{M}^r(M) \mapsto h(T_g, \mathrm{Liouv}(g))$$

in a neighbourhood of the locally symmetric metric g_0. Here we have denoted by (M, g_0) a compact locally symmetric manifold of negative curvature and set

$$\mathcal{M}^r(M) = \{C^r \text{ metrics } g \text{ on } M \mid \mathrm{Vol}_g(M) = \mathrm{Vol}_{g_0}(M)\}.$$

1.1.9. We remark that the above functions $\mathrm{Ent_{top}}$ and $\mathrm{Ent_{Liouv}}$ are known to be smooth by previous works respectively of Katok, Knieper, Pollicott and Weiss [KKPW89] (for the topological entropy) and of Contreras (for the Liouville entropy) [Con92], provided we restrict ourselves to negatively curved metrics. It is also known [KKW91] that a locally symmetric metric g_0 is a critical point of $\mathrm{Ent_{top}}$ and $\mathrm{Ent_{Liouv}}$.

1.2. Statement of the theorems.

In this paper we obtain estimates for the second derivative of the functions $\mathrm{Ent_{top}}$ and $\mathrm{Ent_{Liouv}}$ at a metric of constant negative curvature which allow us to establish that along a path through a metric of constant negative curvature g_0, locally, the only metric for which one has $\mathrm{Ent}_{\mathrm{top}}(g) = \mathrm{Ent}_{\mathrm{Liouv}}(g)$ is g_0. More exactly:

Theorem A. *Let g_0 be a metric of constant negative curvature on a compact manifold M of dimension n and let g_ε be a C^2 curve of C^5 metrics of constant volume. Then, if g_ε is not tangent to the orbit of g_0 under the diffeomorphism group, the function*

$$\varepsilon \mapsto \mathrm{Ent}_{\mathrm{top}}(g_\varepsilon) - \mathrm{Ent}_{\mathrm{Liouv}}(g_\varepsilon)$$

is strictly convex at $\varepsilon = 0$. In particular, it follows that along the path g_ε and for small ϵ's, the equality $\mathrm{Ent}_{\mathrm{top}}(g_\varepsilon) = \mathrm{Ent}_{\mathrm{Liouv}}(g_\varepsilon)$ occurs only at g_0.

In [Pol94], Pollicott proved that at a locally symmetric metric the topological entropy is convex for volume preserving conformal deformations. Along the way to the proof of Theorem A we find the following theorem.

Theorem B. *Under the same hypothesis as Theorem A, the function*

$$\varepsilon \mapsto \mathrm{Ent}_{\mathrm{top}}(g_\varepsilon),$$

is locally strictly convex at g_0.

For the measure theoretical entropy the volume normalization does not yield any useful convexity or concavity property. Surprisingly, we have:

Theorem C. *There exists an hyperbolic 3-manifold (M, g_0) for which the function $\mathrm{Ent}_{\mathrm{Liouv}} : \mathcal{M}^5(M) \to R^+$ has second derivative at g_0 with mixed signature. In particular the entropy $\mathrm{Ent}_{\mathrm{Liouv}}$ of the Liouville measure does not have either a minimum nor a maximum at g_0, when restricted to $\mathcal{M}^5(M)$.*

1.3. Outline of the proofs.

The scheme for the estimate can be divided into three parts. In the first part, since geodesic flows on a negatively curved manifold can be represented as a flow built under a function on a topologically mixing subshift of finite type, we investigate these flows.

We recall some basic definitions about cocycles. Let T be a Borel $\mathbb{R}$-flow on a Borel space X. An *($\mathbb{R}$-valued) cocycle for the flow T* is a Borel function $c : X \times \mathbb{R} \to \mathbb{R}$ satisfying the relation

$$c(x, t + s) = c(x, t) + c(T^t x, s).$$

Cocycles for the flow T form a group under addition and a cocycle c for T is called a coboundary if $c(x,t) = b(T^t x) - b(x)$ for some Borel function b. Coboundaries form a subgroup and the elements of the quotient group are the Borel cohomology classes of the flow. If a cocycle c is "differentiable along the orbits", i.e. if $c(x,t) = \int_0^t A(T^s x)\, ds$, for some Borel function A on X, we say that A *generates* c. In a cohomology class we can always find a representative which is smooth along the orbits. If X has a C^α structure, we say that a cohomology class is C^α if it has a representative generated by a C^α function on X.

If (X,T) is a symbolic flow, i.e. a flow built over a topologically mixing subshift of finite type with a Hölder ceiling function λ, then to each C^α cohomology class $[c]$ is attached a T-invariant measure $m_{[c]}$ called *the Gibbs state* for $[c]$. If A is a Hölder function generating a representative of $[c]$, it is costumery to refer to A as *the potential for the Gibbs state* $m_{[c]}$. The function identically equal to 1, generates a cocycle, called the *length cocycle* and its corresponding Gibbs state is the measure of maximal entropy for T.

We prove the following Proposition about derivatives of entropy of Gibbs states for symbolic flows in terms of the variation of the generating cocycles. The $\mathrm{Cov}_u^T(v,w)$ below denotes the "covariance" of Hölder continuous functions v, w for the flow T and with respect to the Gibbs state for the potential u (see [Rue78] and §2) and $\mathrm{Var}_u^T(v) = \mathrm{Cov}_u^T(v,v)$.

1.3.1. Proposition (A). *Let (Σ,σ) be a topologically mixing subshift of finite type and let λ_ε be a C^2 curve positive C^α functions on Σ. Let $(X_\varepsilon, T_\varepsilon^t)$ be the special flow built over (Σ,σ) with ceiling function λ_ε. Let δl_ε be a C^α function on $X_\varepsilon \approx \{(p,t)\,|\,p\in\Sigma, 0\le t<\lambda_\varepsilon(p)\}$ with the property that*

$$\int_0^{\lambda_\varepsilon(p)} \delta l_\varepsilon(p,t)\, dt = d/d\varepsilon\, \lambda_\varepsilon(p).$$

Then, denoting with primes differentiations with respect to ε, we have :

$$(A1)\qquad h'_{\mathrm{top}}(T_0) \equiv \frac{d}{d\varepsilon} h_{\mathrm{top}}(T_\varepsilon)_{\varepsilon=0} = -h_{\mathrm{top}}(T_0) m_0(\delta l_0)$$

where m_0 is the measure of maximal entropy for the flow T_0. If $h'_{\mathrm{top}}(T_0) = 0$ and $\delta^2 l_\varepsilon$ denotes a C^α function on X_ε with the property that

$$\int_0^{\lambda_\varepsilon(p)} \delta^2 l_\varepsilon(p,t)\, dt = d^2/d\varepsilon^2 \lambda_\varepsilon(p),$$

then

$$(A2)\qquad h''_{\mathrm{top}}(T_0) = -h_{\mathrm{top}}(T_0)\, M_0(\delta^2 l_0) + h_{\mathrm{top}}(T_0)^2\ \mathrm{Var}_0^{T_0}(\delta l_0).$$

(B). *Let $u_\varepsilon = u_\varepsilon(p,t)$, $(p \in \Sigma,\ t \in \mathbb{R})$, be a C^2 family of Hölder continuous functions of pressure zero for the flow T_ε. Let m_{u_ε} denote the Gibbs state for the flow T_ε and potential u_ε. We have:*
$(B1)$

$$h'(T_0, m_{u_0}) \equiv \frac{d}{d\varepsilon} h(T_\varepsilon, m_{u_\varepsilon})_{\varepsilon=0} = \mathrm{Cov}_{u_0}^{T_0}(u_0, \delta u_0) - h(T_0, m_{u_0})\, m_{u_0}(\delta l_0),$$

where δu_ε is the (generator of) first variation of the cocycle generated by u_ε and is defined by $\int_0^{\lambda_\varepsilon} \delta u_\varepsilon(p,t)\, dt = d/d\varepsilon \int_0^{\lambda_\varepsilon} u_\varepsilon(p,t)\, dt$. If u_0 is cohomologous to a constant and $m_{u_0}(\delta l_0) = 0$, then $h'(T, m_u) = 0$ and

$$\text{(B2)} \qquad \begin{aligned} h''(T_0, m_{u_0}) = &- \mathrm{Var}_{u_0}^{T_0}(\delta u_0) - 2h(T_0, m_{u_0})\, \mathrm{Cov}_{u_0}^{T_0}(\delta l_0, \delta u_0) \\ &- h(T_0, m_{u_0})\, m_{u_0}(\delta^2 l_0). \end{aligned}$$

(C). *Finally, if in addition to the hypothesis in* (B), *we have* (a) $m_{u_\varepsilon}(\delta l_\varepsilon) = 0$ *for all ε and* (b) $m_0 = m_{u_0}$, *then*

$$\text{(C1)} \qquad h''_{\mathrm{top}}(T_0) = \mathrm{Cov}_{m_0}^{T_0}\left(h_{\mathrm{top}}(T_0)\, \delta l_0\, , \delta u_0 + h_{\mathrm{top}}(T_0)\, \delta l_0\right).$$

$$\text{(C2)} \qquad h''(T_0, m_{u_0}) = -\, \mathrm{Cov}_{m_0}^{T_0}\left(\delta u_0\, , \delta u_0 + h_{\mathrm{top}}(T_0)\, \delta l_0\right)$$

(C3)

$$h''_{\mathrm{top}}(T_0) - h''(T_0, m_{u_0}) = \mathrm{Var}_{m_0}^{T_0}\left(\delta u_0 + h_{\mathrm{top}}(T_0)\delta l_0\right).$$

The formula (A1) was essentially a step of [KKW91] and (A2) has also been proved independently by Pollicott [Pol94].

The above proposition yields formulas for the second derivative of $\mathrm{Ent_{top}}$ and $\mathrm{Ent_{Liouv}}$ along paths g_ε of metrics in $\mathcal{M}(M)$ once one knows how to determine the variations of the cocycles generating the Margulis and the Liouville measures—i.e. the length cocycle and the Liapunov cocycle—in terms of the variation $S = \frac{d}{d\varepsilon} g_\varepsilon$ of the Riemannian metric. In fact, since Gibbs states only depend on the cohomology class of the generating cocycles it is sufficient to determine the variation of the cohomology class of these cocycles. This is the second step of the proof.

1.3.2. Notation. For a symmetric covariant tensor field S of rank 2 on a Riemannian manifold (M, g_0), denote by $S^\vee$ the quadratic form field $S^\vee :$ $v \mapsto S(v,v)$ defined on the sphere bundle $S_{g_0}M$.

Then for the length cocycle we have $\delta l_0 = \frac{1}{2} S^\vee |_{\varepsilon=0}$. For the Liapunov cocycle we have the following Proposition:

1.3.3. Proposition. *Let g_ε be a C^1 curve of C^4 metrics on a manifold M, and assume that g_0 has constant negative curvature. Let $S = \frac{d}{d\varepsilon} g_\varepsilon$ at $\varepsilon = 0$.*

Then the generator δu of the first variation of the Liapunov cocycle at $\varepsilon = 0$ is cohomologous to

$$\delta u \approx T^{\vee} + \frac{1-n}{2} S^{\vee}$$

where T is the symmetric tensor field defined by $T = -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S + \frac{1}{2}(\mathrm{Tr}_g\, S)g - \frac{1}{2}\delta^*\delta S$.

In the Proposition above, $\nabla^*\nabla$ is the rough Laplacian for the metric g_0, δ^* denotes the symmetrization of the covariant derivative for g_0 and δ denotes its formal adjoint, the divergence.

The final step consists in estimating the covariance $\mathrm{Cov}_{m_0}(\delta l_0, \delta u_0) = \mathrm{Cov}_{m_0}(\frac{1}{2}S^{\vee}, T^{\vee} + \frac{1-n}{2}S^{\vee})$ which appears in the formulas (C1-3). We prove:

1.3.4. Proposition. *Let g_0 be a metric on M of constant negative curvature -1. Let S be a C^3 a symmetric covariant tensor field of rank 2 on M and let $T = -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S + \frac{1}{2}(\mathrm{Tr}_g\, S)g - \frac{1}{2}\delta^*\delta S$. Then, we have:*

$$\mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(S^{\vee}, T^{\vee}) \geq \frac{n-2}{4}\, \mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(S^{\vee}, S^{\vee})$$

$$\mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(T^{\vee}, T^{\vee}) \geq \left(\frac{n-2}{4}\right)^2 \mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(S^{\vee}, S^{\vee})$$

The above estimate is the heart of the proof. It follows from observing that since (M, g_0) is locally symmetric of constant negative curvature, the functions $S^{\vee}, T^{\vee}$ on the unit tangent bundle $S_{g_0}M$ lift to the orthonormal frame bundle FM of M. The group $G \approx SO_0(1,n)$ acts transitively on FM, and the bilinear form $\mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}$ is diagonal with respect to this action of G , i.e. it respects the decomposition of $L^2(FM)$ into irreducible subspaces. But the linear map $S \mapsto T$ also respects this decomposition and, on each irreducible subspace, T is a multiple of S. In the end, the estimate reduces to an estimate of the smallest eigenvalue of the operator $S \mapsto T$ on the space orthogonal to the orbit of g_0 under Diff(M). This is achieved via a suitable Weitzenböck formula.

1.3.5. Corollary. *Under the hypothesis of Theorem A and setting $S = \frac{d}{d\varepsilon}g_\varepsilon$ at $\varepsilon = 0$, the second derivatives of the topological entropy and of the Liouville entropy satisfy:*

$$\mathrm{Ent}''_{\mathrm{top}}(g_0) \geq \frac{1}{4}\binom{n-1}{2}\, \mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(S^{\vee}, S^{\vee}),$$

$$\mathrm{Ent}''_{\mathrm{top}}(g_0) - \mathrm{Ent}''_{\mathrm{Liouv}}(g_0) \geq \frac{(n-2)^2}{4}\, \mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(S^{\vee}, S^{\vee}),$$

Proof of Theorems A and B. The equality $\mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(S^\vee, S^\vee) = 0$, is equivalent to saying that $S^\vee$ is cohomologous to a constant, in our case to zero. Applying a theorem of Guillemin and Kazhdan [GK79], we obtain that this implies that the curve g_ε is tangent to the orbit of the diffeomorphism group at g_0, in contradiction to the hypothesis. Thus $\mathrm{Cov}^{g_0^t}_{\mathrm{Liouv}(g_0)}(S^\vee, S^\vee) > 0$ and by Corollary 1.3.5 we obtain $\mathrm{Ent}''_{\mathrm{top}}(g_0) > 0$ and $\mathrm{Ent}''_{\mathrm{top}}(g_0) - \mathrm{Ent}''_{\mathrm{Liouv}}(g_0) > 0$. □

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§ 2. Derivatives of entropy for symbolic flows.

2.1.

For generalities on topologically mixing subshifts of finite type, pressure and Gibbs states we refer to [Bow75] and [Rue78].

Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing subshift of finite type. For any Hölder continuous potential ϕ denote by μ_ϕ the Gibbs state for ϕ, and by $P^\sigma(\phi)$ or $P(\sigma, \phi)$ the pressure of the potential ϕ. We recall that pressure $P^\sigma(\phi)$ and the entropy $h(\sigma, \mu_\phi)$ of σ with respect to the Gibbs state μ_ϕ are related by the variational principle:

$$P^\sigma(\phi) = h(\sigma, \mu_\phi) + \mu_\phi(\phi) = \sup_\mu (h(\sigma, \mu) + \mu(\phi)),$$

where the supremum is taken as μ ranges on the set of probability σ-invariant measures on Σ. It is also known that, for $\phi \in C^\alpha(\Sigma)$, the equality $P^\sigma(\phi) = h(\sigma, \mu_\phi) + \mu_\phi(\phi)$ completely characterizes the Gibbs state μ_ϕ among the σ-invariant measures μ on Σ (see [Bow75]).

We define the *covariance* of Hölder continuous functions ψ and ζ with respect to the Gibbs state μ_ϕ by

$$\mathrm{Cov}^\sigma_\phi(\psi, \zeta) = \sum_{i=-\infty}^{\infty} \mu_\phi \left(\psi \cdot \zeta \circ \sigma^i - \mu_\phi(\psi)\mu_\phi(\zeta)\right).$$

We also set

$$\mathrm{Var}^\sigma_\phi(\psi) = \mathrm{Cov}^\sigma_\phi(\psi, \psi)$$

and we call the above limit is called the *variance* of ψ w.r.t. the Gibbs state μ_ϕ. We have $\mathrm{Cov}^\sigma_\phi(\psi,\zeta) = 0$ for all ψ if and only if ζ is cohomologous to a constant, i.e. $\zeta = \chi \circ \sigma - \chi + \mathrm{Const}$ for some Hölder continuous function $\chi : \Sigma \to \mathbb{R}$.

The convergence of the above series follows from the exponential rate of mixing of Hölder continuous functions.

2.2.

From [Con92] the map

$$P^\sigma : \phi \in C^\alpha(\Sigma) \mapsto P^\sigma(\phi) \in \mathbb{R}$$

is real analytic. Setting

$$D^n P^\sigma_{\phi_0}(\phi_1, \dots, \phi_n) = \frac{d}{dt_n} \cdots \frac{d}{dt_1} P^\sigma(\phi_0 + t_1\phi_1 + \cdots + t_n\phi_n)\Big|_{t_1=t_2=\cdots=t_n=0}.$$

we have, (cf. [Con92] and [Rue78, Ch.5, Exerc. 5])

(2.1)
$$D^1 P^\sigma_{\phi_0}(\phi_1) = \mu_{\phi_0}(\phi_1) \quad \text{and} \quad D^2 P^\sigma_{\phi_0}(\phi_1, \phi_2) = \mathrm{Cov}^\sigma_{\phi_0}(\phi_1, \phi_2).$$

Thus the maps

$$\phi \in C^\alpha(\Sigma) \mapsto \mu_\phi \in C^\alpha(\Sigma)^*$$

and

$$\mathrm{Cov}^\sigma : \phi \in C^\alpha(\Sigma) \mapsto \mathrm{Cov}^\sigma_\phi \in L(C^\alpha(\Sigma), C^\alpha(\Sigma); \mathbb{R})$$

are real analytic.

2.2.1. Observation. If ϕ_ε is a C^2 curve of potentials in $C^\alpha(\Sigma)$, writing $\phi_\varepsilon = \phi_0 + \varepsilon\phi_1 + \frac{\varepsilon^2}{2}\phi_2 + o(\varepsilon^2)$, we obtain

(2.2)
$$P^\sigma(\phi_\varepsilon) = P^\sigma(\phi_0) + \varepsilon\mu_{\phi_0}(\phi_1) + \frac{\varepsilon^2}{2}\left(\mathrm{Cov}^\sigma_{\phi_0}(\phi_1, \phi_1) + \mu_{\phi_0}(\phi_2)\right) + o(\varepsilon^2).$$

2.3.

Let $\sigma : \Sigma \to \Sigma$ be a topologically mixing subshift of finite type and λ be a positive C^α function on Σ.

We recall the definition of special flow T built under λ. Let $\tilde{T}$ be the flow on $\Sigma \times \mathbb{R}$ defined by $\tilde{T}^t(p,s) = (p, s+t)$ and consider on $\Sigma \times \mathbb{R}$ the equivalence relation $\sim$ generated by $(p,t) \sim (\sigma p, t - \lambda(p))$. Then, setting $X = \Sigma \times \mathbb{R}/\sim$, the flow $\tilde{T}$ descends to a flow T on X: the flow (X,T) is called *the special flow built under* λ. Sometimes we shall simply write $(\Sigma, \sigma, \lambda, T)$ or $(\Sigma, \sigma, \lambda)$ to refer to (X,T). Clearly a fundamental domain for X is the set $\{(p,t) \in \Sigma \times \mathbb{R} : 0 \leq t < \lambda(p)\}$ and X can be easily turned in a metric space (see [BW72]).

2.3.1. Notation. Let $A : X \to \mathbb{R}$ be a C^α continuous potential on X and let $P^T(A) = P(T,A)$, m_A and $h(T, m_A)$ denote respectively the pressure of A for the flow T, the Gibbs state for the potential A and the entropy of the flow T with respect to the measure m_A.

In analogy to the case of shifts, the pressure $P^T(A)$ and the entropy $h(T, m_A)$ of T with respect to the Gibbs state m_A are related by the variational principle:

$$P^T(A) = h(T, m_A) + m_A(A) = \sup_m (h(T,m) + m(A)),$$

where the supremum is taken as m ranges on the set of probability T-invariant measures on X. Also, m_A is the unique T-invariant measure on X for which the equality $P^T(A) = h(T, m_A) + m_A(A)$ is achieved.

2.3.2. Definition. Let c be the cocycle for the special flow $(X,T) = (\Sigma, \sigma, \lambda)$ generated by $A : X \to \mathbb{R}$. Then the *induced cocycle on* Σ is the cocycle for σ generated by $\mathrm{I}[A] : \Sigma \to \mathbb{R}$ where $\mathrm{I}[A]$ is defined by

$$\mathrm{I}[A](p) = \int_0^{\lambda(p)} A(T^t p)\, dt.$$

2.3.3. Notation. Given a function $A : X \to \mathbb{R}$ let $\Phi[A] : \Sigma \to \mathbb{R}$ be the function defined by

$$\Phi[A] = \mathrm{I}[A] - P^T(A)\lambda = \mathrm{I}[A - P^T(A)]$$

If $A : X \to \mathbb{R}$ is a Hölder continuous on X, then $\mathrm{I}[A]$ and $\Phi[A]$ are also Hölder continuous on Σ.

We can reduce the study of flows to the case of shifts by the following theorem that clarifies the relation between Gibbs states for (Σ, σ) and (X,T).

2.3.4. Theorem (Bowen, Franco-Sanchez). *The Gibbs measure m_A for the flow T and the potential A is given by*

$$m_A = \frac{\mu_{\Phi[A]} \otimes dt}{(\mu_{\Phi[A]} \otimes dt)(X)} = \frac{\mu_{\Phi[A]} \otimes dt}{\mu_{\Phi[A]}(\lambda)} \tag{2.3}$$

where $\mu_{\Phi[A]}$ denotes the Gibbs state of the potential $\Phi[A]$ on Σ. Furthermore the pressure $P^\sigma(\Phi[A])$ of $\Phi[A]$ is zero.

2.3.5. Remark. From the above Theorem, Contreras' result mentioned in 2.2, the formula 2.1 for the first derivative for $P^T(A)$ and the implicit function theorem, it follows that for $\lambda \in C^\beta(\Sigma)$ and $\alpha \leq \beta$, the map

$$P^T : A \in C^\alpha(X) \mapsto P^T(A) \in \mathbb{R}$$

is real analytic. The maps $m : A \in C^\alpha(X) \mapsto m_A \in C^\alpha(X)^*$ and $A \in C^\alpha(X) \mapsto h(T, m_A) \in \mathbb{R}$ are real analytic as well.

2.3.6. Remark. The variational characterization of Gibbs states implies that the measure of maximal entropy for the flow T is the Gibbs state for the potential zero (or, more generally, for a potential cohomologous to a constant).

2.3.7. Remark. Again by the variational principle, the potential A has pressure zero if and only if

$$h(T, m_A) = -m_A(A)$$

2.4.

We recall now the definition of covariance for special flows. Since an exponential mixing rate for Hölder functions is not guaranteed, the definition is not as immediate as in the case of shifts.

2.4.1. Notation. Retaining the previous notations let $B : X \to \mathbb{R}$ be another Hölder continuous function on X. Define $\Psi_A[B] : \Sigma \to \mathbb{R}$ by setting

$$\Psi_A[B] = \mathrm{I}\,[B] - m_A(B)\,\lambda = \mathrm{I}\,[B - m_A(B)].$$

The definition of covariance was given by Marina Ratner who proved in [Rat73a] the following theorem:

2.4.2. Theorem (Ratner). *For a Hölder continuous function $B : X \to \mathbb{R}$ and any Gibbs measure m_A on X the limit*

$$\mathrm{Var}_A^T(B) = \lim_{T\to\infty} \frac{1}{T} \int_X \left(\int_0^T (B \circ T^t - m_A(B))\, dt \right)^2 dm_A$$

exists, equals $\frac{1}{\mu_{\Phi[A]}(\lambda)} \mathrm{Var}^{\sigma}_{\Phi[A]}(\Psi_A[B])$ *and it is called the* variance of B *with respect to the Gibbs state m_A.*

By polarization, given any three Hölder continuous functions $A, B, C : X \to \mathbb{R}$, the limit

$$\mathrm{Cov}_A^T(B,C) = \lim_{T\to\infty} \frac{1}{T} \int_X \left(\int_0^T (B \circ T^t - m_A(B))\, dt \right) \cdot \\ \cdot \left(\int_0^T (C \circ T^{t'} - m_A(C))\, dt' \right) dm_A$$

exists and we have

$$\mathrm{Cov}_A^T(B,C) = \frac{1}{\mu_{\Phi[A]}(\lambda)} \mathrm{Cov}^{\sigma}_{\Phi[A]}(\Psi_A[B], \Psi_A[C]). \tag{2.4}$$

We call $\mathrm{Cov}_A^T(B,C)$ the *covariance* of B and C with respect to the Gibbs state m_A. We have $\mathrm{Cov}_A^T(B,C) = 0$ for all B if and only if C is cohomologous to a constant, i.e. if there exists a Hölder function D differentiable along the flow T^t such that $C = \left.\frac{dD \circ T^t}{dt}\right|_{t=0} + \mathrm{Const}$ (see [Rat73a]).

2.4.3. Remark. Notice that if the integral $\int_{-\infty}^{\infty} |m_A(B \cdot C \circ T^t) - m_A(B)m_A(C)|\, dt$ exists, then

$$\mathrm{Cov}_A^T(B,C) = \int_{-\infty}^{\infty} \left(m_A(B \cdot C \circ T^t) - m_A(B)m_A(C) \right) dt.$$

2.5.

Let λ_ε be a C^r curve of positive C^α functions on Σ and let $(X_\varepsilon, T_\varepsilon)$ be the family of special flows built under λ_ε.

Let u_ε be the a C^r family of C^α potentials on X_ε, i.e. a curve admitting a C^r lift to $C^\alpha(\Sigma \times \mathbb{R})$.

2.5.1. Definition. For λ_ε and u_ε as above we define *the i-th variation of the cocycle (generated by)* u_ε as (the cocycle generated by) any C^α function $\delta^i u_\varepsilon$ be a on X_ε with the property that

$$\mathrm{I}_\varepsilon[\delta^i u_\varepsilon](p) = \int_0^{\lambda_\varepsilon(p)} \delta^i u_\varepsilon(p,t)\,dt = \frac{d^i}{d\varepsilon^i}\int_0^{\lambda_\varepsilon(p)} u_\varepsilon(p,t)\,dt. \tag{2.5}$$

(for simplicity, $\delta^1 = \delta$ and $\delta^0 u_\varepsilon = u_\varepsilon$). In other words the i-th variation $\delta^i u_\varepsilon$ induces on Σ the i-th derivative of the induced cocycle $\mathrm{I}_\varepsilon[u_\varepsilon]$. Notice that the i-th variation $\delta^i u_\varepsilon$ of the cocycle u_ε is only defined up to a coboundary for the flow T_ε.

We denote by $\delta^i l_\varepsilon$, the i-th variation of the length cocycle:

$$\mathrm{I}_\varepsilon[\delta^i l_\varepsilon](p) = \int_0^{\lambda_\varepsilon(p)} \delta^i l_\varepsilon(p,t)\,dt = \frac{d^i}{d\varepsilon^i}\lambda_\varepsilon(p). \tag{2.6}$$

Having stated the set up the proof of Proposition 1.3.1 is rather elementary. To simplify notation, we denote with primes differentiation with respect to ε and suppress the dependence on ε, e.g. $P'(T,u) = d/d\varepsilon P(T_\varepsilon, u_\varepsilon)$ and $P'(T_0,u_0) = d/d\varepsilon P(T_\varepsilon,u_\varepsilon)|_\varepsilon = 0$.

2.5.2. Proposition (Derivatives of the Pressure and Entropy). *Let λ_ε be a C^2 curve of positive C^α functions on Σ and let $(X_\varepsilon, T_\varepsilon)$ be the family of special flows built under λ_ε. Let u_ε be the a C^2 family of C^α potentials on X_ε. Then, retaining the previous notation for the first and second variation for length cocycle and the cocycle generated by u_ε and setting*

$$v_\varepsilon = \delta u_\varepsilon - P(T_\varepsilon,u_\varepsilon)\,\delta l_\varepsilon \qquad \textit{and} \qquad w_\varepsilon = \delta^2 u_\varepsilon - P(T_\varepsilon,u_\varepsilon)\,\delta^2 l_\varepsilon$$

we have:

$$P'(T,u) = m_u(v)$$
$$P''(T,u) = \mathrm{Var}_u^T(v) + m_u(w) - 2m_u(v)\,m_u(\delta l)$$

and

$$\begin{aligned} h'(T,m_u) &= -\operatorname{Cov}_u(v,u) + h(T,m_u)\,m_u(\delta l)\\ h''(T,m_u) &= -D^3P_u^T(u,v,v) - 2h(T,m_u)\operatorname{Cov}_u^T(\delta l, v)\\ &\qquad + 2\big(\operatorname{Cov}_u^T(v,u) + h(T,m_u)\,m_u(\delta l)\big)\,m_u(\delta l)\\ &\qquad - \operatorname{Cov}_u^T(u,w) + 2m_u(v)\operatorname{Cov}_u^T(u,\delta l)\\ &\qquad - h(T,m_u)\,m_u(\delta^2 l) - \operatorname{Cov}_u^T(v,v). \end{aligned}$$

2.6.

We shall not give the proof of the proposition above. However, for completeness, we shall give independent proofs of the following corollaries which we use in this paper.

2.6.1. Corollary. *Let λ_ε be a C^2 curve of positive C^α functions on Σ and let $(X_\varepsilon, T_\varepsilon)$ be the family of special flows built under λ_ε. We retain here the previous notation and denote by $h_{\text{top}}(T_\varepsilon)$ the topological entropy of the flow T_ε and by M_ε the measure of maximal entropy for the flow T_ε. Also denote by $\text{Var}_0^{T_\varepsilon}$ the variance of the flow T_ε with respect to the measure of maximal entropy m_ε. Then, suppressing the dependence on ε, we have*

$$h'_{\text{top}}(T) = -h_{\text{top}}(T)\, M(\delta l). \tag{2.7}$$

If $h'_{\text{top}}(T_0) = 0$ we have

$$h''_{\text{top}}(T_0) = -h_{\text{top}}(T_0)\, M_0(\delta^2 l_0) + h_{\text{top}}(T_0)^2\ \text{Var}_0^{T_0}(\delta l_0). \tag{2.8}$$

Proof. Observe that since $\varepsilon \mapsto \lambda_\varepsilon \in C^\alpha(\Sigma)$ is C^2, the first and second variation δl_ε and $\delta^2 l_\varepsilon$ of the length cocycle exist (as Hölder functions). By the Remark 2.3.6 the topological entropy equals the pressure for the potential 0 and from Theorem 2.3.4 and 2.3 we obtain that

$$m_\varepsilon = \frac{\mu_{\phi(\varepsilon)} \otimes dt}{\mu_{\phi(\varepsilon)}(\lambda_\varepsilon)},$$

where $\phi(\varepsilon) = -h_{\text{top}}(T_\varepsilon)\lambda_\varepsilon$. Again by Theorem 2.3.4, we have that $P^\sigma(\phi(\varepsilon)) = 0$ for all ε. By Remark 2.3.5, the curve $\phi(\varepsilon)$ is a C^2 curve of C^α potentials on Σ. Thus from (2.2) we obtain that

$$\mu_\phi(\phi') = 0 \quad \text{and} \quad \text{Var}_\phi^\sigma(\phi') + \mu_\phi(\phi'') = 0. \tag{2.9}$$

Since $\phi' = -h'_{\text{top}}(T)\lambda - h_{\text{top}}(T)\lambda'$ from the first of (2.9) we conclude that

$$h'_{\text{top}}(T) = -\frac{h_{\text{top}}(T)\ \mu_\phi(\lambda')}{\mu_\phi(\lambda)} = -\frac{h_{\text{top}}(T)\ \mu_\phi \otimes dt(\delta l)}{\mu_\phi(\lambda)} = -h_{\text{top}}(T)\, m_0(\delta l)$$

proving (2.7).

The hypothesis $h'_{\text{top}}(T_0) = 0$ implies that $\phi'(0) = -h'_{\text{top}}(T_0)\lambda_0$ and $\phi''(0) = -h''_{\text{top}}(T_0)\lambda_0 - h_{\text{top}}(T_0)\lambda''_0$. Thus, using the second of the formulas (2.9), we obtain

$$\text{Var}_{\phi(0)}^\sigma(-h_{\text{top}}(T_0)\lambda'_0) + \mu_{\phi(0)}(-h''_{\text{top}}(T_0)\lambda_0 - h_{\text{top}}(T_0)\lambda''_0) = 0.$$

We conclude that

$$\begin{aligned} h''_{\text{top}}(T_0) &= -h_{\text{top}}(T_0)\frac{\mu_{\phi(0)}(\lambda''_0)}{\mu_{\phi(0)}(\lambda_0)} + h_{\text{top}}(T_0)^2\frac{\operatorname{Var}^\sigma_{\phi_0}(\lambda'_0)}{\mu_{\phi(0)}(\lambda_0)} \\ &= -h_{\text{top}}(T_0)M_0(\delta l_0) + h_{\text{top}}(T_0)^2\frac{\operatorname{Var}^\sigma_{\phi_0}(\mathrm{I}\,[\delta\lambda_0])}{\mu_{\phi(0)}(\lambda_0)} \end{aligned}$$

From $h'_{\text{top}}(T_0) = 0$ we have $M_0(\delta l_0) = 0$ and therefore $\mathrm{I}\,[\delta\lambda_0] = \Psi_0[l'_0]$. Using (2.4) we conclude

$$\frac{\operatorname{Var}^\sigma_{\phi_0}(\mathrm{I}\,[\delta\lambda_0])}{\mu_{\phi(0)}(\lambda_0)} = \operatorname{Var}^{T^t}_{m_0}(\delta l_0)$$

proving (2.8) and Part A of Proposition 1.3.1. □

2.6.2. Corollary. *Let λ_ε be a C^2 curve of positive C^α functions on Σ and let $(X_\varepsilon, T_\varepsilon)$ be the family of special flows built under λ_ε. Let u_ε be the a C^2 family of C^α potentials on X_ε. In the hypothesis that $P(T^t_\varepsilon, u_\varepsilon) = 0$ for all ε we have:*

$$h'(T, m_u) = -\operatorname{Cov}^T_u(u, \delta u) - h(T, m_u)\, m_u(\delta l). \tag{2.10}$$

If u_0 is cohomologous to a constant and $m_{u_0}(\delta l_0) = 0$, then $h'(T, m_u) = 0$ and

$$\begin{aligned} h''(T_0, m_{u_0}) = &-\operatorname{Var}^{T_0}_{u_0}(\delta u_0) - 2h(T_0, m_{u_0})\operatorname{Cov}^{T_0}_{u_0}(\delta l_0, \delta u_0) \\ &- h(T_0, m_{u_0})\, m_{u_0}(\delta^2 l_0), \end{aligned} \tag{2.11}$$

Proof. Since by hypothesis $P(T^t_\varepsilon, u_\varepsilon) = 0$ for all ε, we have that $\Phi[u_\varepsilon](p) = \int_0^{\lambda_\varepsilon(p)} u_\varepsilon(T^t_\varepsilon p)\, dt$. For simplicity set $\psi_\varepsilon = \Phi[u_\varepsilon]$. The Gibbs state m_{u_ε} is thus given by

$$m_{u_\varepsilon} = \frac{\mu_{\psi_\varepsilon} \otimes dt}{\mu_{\psi_\varepsilon}(\lambda_\varepsilon)},$$

where μ_{ψ_ε} denotes as before the Gibbs measure for the shift σ and the potential ψ_ε. From equation 2.2 and the fact that $P(\sigma, \psi_\varepsilon) = 0$ for all ε (see 2.2), we obtain

$$\mu_\psi(\psi') = 0. \tag{2.12}$$

By the variational characterization of Gibbs states and $P(\sigma, \psi_\varepsilon) = 0$ we have $h(\sigma, \mu_\psi) = -\mu_\psi(\psi)$ and, using (2.12), we obtain

(2.13)
$$h'(\sigma, \mu_\psi) = -\operatorname{Cov}^\sigma_\psi(\psi, \psi') - \mu_\psi(\psi') = -\operatorname{Cov}^\sigma_\psi(\psi, \psi').$$

Differentiating w.r.t. ε Abramov's formula

$$h(\sigma, \mu_{\psi_\varepsilon}) = \mu_{\psi_\varepsilon}(\lambda_\varepsilon)\, h(T_\varepsilon, m_{u_\varepsilon}),$$

and using (2.13), we have

$$-\operatorname{Cov}^\sigma_\psi(\psi, \psi') = h(T, m_u)\left(\operatorname{Cov}^\sigma_\psi(\lambda, \psi') + \mu_{\psi_\varepsilon}(\lambda')\right) + \mu_{\psi_\varepsilon}(\lambda)\, h'(T, m_u)$$

or

(2.14)
$$h'(\sigma, m_u) = -\frac{\operatorname{Cov}^\sigma_\psi(\psi + h(T, m_u)\,\lambda, \psi')}{\mu_{\psi_\varepsilon}(\lambda)} - h(T, m_u)\frac{\mu_{\psi_\varepsilon}(\lambda')}{\mu_{\psi_\varepsilon}(\lambda)}.$$

Observe that we have $\Psi_u[u] = \psi - m_u(u)\,\lambda = \psi + h(T, m_u)\,\lambda$ and $\Psi_u[\delta u] = \psi' - m_u(\delta u)\,\lambda = \psi' - \mu_\psi(\psi')\,\lambda = \psi'$. Thus, using (2.4) we have $\operatorname{Cov}^\sigma_\psi(\psi + h(T, m_u)\,\lambda, \psi') = \operatorname{Cov}^T_u(u, \delta u)\ \mu_{\psi_\varepsilon}(\lambda)$, which proves (2.10).

Assume now that the u_0 is cohomologous to a constant. Then $P(T, u) = 0$ implies that the function $u_0 + h(T_0, m_{u_0})$ is cohomologous to zero and therefore also $\psi_0 + h(m_{u_0})\,\lambda_0$ is cohomologous to a zero. The further assumption $m_{u_0}(\delta l_0) = 0$ entails $\mu_{\psi_0}(\lambda_0') = 0$ and therefore $h'(T_0, m_{u_0}) = 0$. In this case one further differentiation of (2.14) yields

$$\begin{aligned} h''(m_{u_0}) = &-\frac{\operatorname{Cov}^\sigma_{\psi_0}(\psi_0' + h(m_{u_0})\lambda_0', \psi_0')}{\mu_{\psi_0}(\lambda_0)} - \\ &- h(m_{u_0})\left[\frac{\operatorname{Cov}^\sigma_{\psi_0}(\psi_0', \lambda_0')}{\mu_{\psi_0}(\lambda_0)} + \frac{\mu_{\psi_0}(\lambda_0'')}{\mu_{\psi_0}(\lambda_0)}\right] \\ = &-\frac{\operatorname{Cov}^\sigma_{\psi_0}(\psi_0', \psi_0')}{\mu_{\psi_0}(\lambda_0)} - h(m_{u_0})\left[2\frac{\operatorname{Cov}^\sigma_{\psi_0}(\psi_0', \lambda_0')}{\mu_{\psi_0}(\lambda_0)} + \frac{\mu_{\psi_0}(\lambda_0'')}{\mu_{\psi_0}(\lambda_0)}\right]. \end{aligned}$$

and (2.11) is proved. Part B of Proposition 1.3.1 is now proved.

□

2.6.3. Corollary. *Let λ_ε be a C^2 curve of positive C^α functions on Σ and let $(X_\varepsilon, T_\varepsilon)$ be the family of special flows built under λ_ε. Let u_ε be the a C^2 family of C^α potentials on X_ε. Assume that*

1. *$P(T^t_\varepsilon, u_\varepsilon) = 0$ for all ε;*
2. *$m_{u_\varepsilon}(\delta l_\varepsilon) = 0$ for all ε;*
3. *u_0 is cohomologous to a constant.*

Then $m_0 = m_{u_0}$, *then we obtain* $m_0(\delta^2 l_0) = -\operatorname{Cov}^{T_0}_{m_0}(\delta l_0, \delta u_0)$. *Therefore we obtain* $h'_{\text{top}}(T_0) = h'(T_0, m_{u_0}) = 0$ *and*

$$\begin{aligned} h''_{\text{top}}(T_0) &= \operatorname{Cov}_{m_0}(h_{\text{top}}(T_0)\,\delta l_0\,, u'_0 + h_{\text{top}}(T_0)\,\delta l_0). \\ h''(T_0, m_{u_0}) &= -\operatorname{Cov}_{m_0}(u'_0\,, u'_0 + h_{\text{top}}(T_0)\,\delta l_0) \\ h''_{\text{top}}(T_0) - h''(T_0, m_{u_0}) &= \operatorname{Cov}_{m_0}(u'_0 + h_{\text{top}}(T_0)\delta l_0, u'_0 + h_{\text{top}}(T_0)\delta l_0) \end{aligned}$$

Proof. The claim $m_0 = m_{u_0}$ is equivalent to u_0 being cohomologous to a constant. Retain the notation of the proof of the previous Corollary. Since $m_{u_\varepsilon}(\delta l_\varepsilon) = 0$ implies $\mu_{\psi_\varepsilon}(\lambda'_\varepsilon) = 0$ for all ε, differentiating we obtain

$$\operatorname{Cov}^{\sigma}_{\psi_\varepsilon}(\lambda'_\varepsilon, \psi'_\varepsilon) + \mu_{\psi_\varepsilon}(\lambda''_\varepsilon) = 0.$$

From $m_{u_\varepsilon}(\lambda') = 0$ we obtain $\Psi_{u_\varepsilon}[\delta l] = \lambda'$ and since $\Psi_{u_\varepsilon}[\delta u] = \psi'$, using 2.4, we obtain that

$$m_u(\delta^2 \lambda) = -\operatorname{Cov}^{T}_{u}(\delta l, \delta u).$$

The rest is now mere rephrasing. This also concludes the proof of Proposition 1.3.1. □

§ 3. The first variation of the length and Liapunov cocycles.

In this section we collect some well known facts that we allow us to connect the results of §2 to the study of our original problem. Some of the theorems stated are valid in a more general setting of Anosov or Axiom-A flows. We state them for the case of geodesic flows for metrics of negative curvature.

3.1. Symbolic Coding.

The geodesic flow T_g of a metric of negative curvature g on the unit tangent bundle S_gM of a manifold M is isomorphic to a symbolic flow [Rat69], [Rat73b], [Bow72b]. More exactly, there is symbolic flow (X, T) built over a topologically mixing subshift of finite type (Σ, σ) with a positive Hölder continuous ceiling function λ and a finite-to-one Hölder continuous surjection $\pi : X \to S_gM$ such that π intertwines the flow T on X with the flow T_g on S_gM. Furthermore, denoting by m_u and $n_{u\circ\pi}$ respectively the Gibbs state for $u \in C^\alpha(S_gM)$ with respect to the flow T_g and Gibbs state for $u \circ \pi \in C^\beta(X)$ with respect to the flow T, we have that the surjection

π is a measure theoretical isomorphism of $(X, T, n_{u\circ\pi})$ onto (S_g, T_g, m_u). In particular $u \in C^\alpha(S_gM)$ is cohomologous to zero if and only if $u \circ \pi$ is cohomologous to zero.

The Margulis measure and the Liouville measure on S_gM are the Gibbs states respectively for the length cocycle and for the Liapunov cocycle. We recall the definition of Liapunov cocycle.

3.1.1. The Liapunov cocycle. For $v \in S_gM$ let $W^{ss}(v)$ denote the strong stable manifold passing through v, i.e. the set of points $w \in S_gM$ with $d(\phi^t v, \phi^t w) \to 0$ as $t \to +\infty$. The Liapunov cocycle is given by the growth of the volume of strong stable manifold: more exactly, if μ^{ss} is the volume form induced on strong stable manifolds by some Riemannian metric on S_gM, the Liapunov cocycle is defined by $\mathcal{L}(v,t) = \frac{d(\phi^t)^*\mu^{ss}}{d\mu^{ss}}(v)$. Changing the Riemannian metric on S_gM does not affect the cohomology class of the Liapunov cocycle.

The Liapunov cocycle has an interpretation in terms of Jacobi fields. The natural projection $p : S_gM \to M$ maps bijectively the strong stable manifold $W_g^{ss}(v)$ onto the stable horosphere $\mathcal{H}_g(v)$ of v. The second fundamental form $U_g(v)$ of the stable horosphere $\mathcal{H}(v)$ at $p(v)$ considered as a tensor of type $\binom{1}{1}$ satisfies [2] along the orbit $T_g^t v$ the Riccati equation

$$\frac{\nabla}{dt} U_g(T_g^t v) + U_g^2(T_g^t v) = R(p_* T_g^t v, \cdot\,) p_* T_g^t v. \tag{3.1}$$

The Liapunov cocycle can be expressed as

$$\mathcal{L}_g(v,t) = \int_0^t \operatorname{Tr} U_g(T_g^\tau v)\, d\tau. \tag{3.2}$$

It is not difficult to see that $\operatorname{Tr} U_g$ is a Hölder continuous function on S_gM.

3.1.2. The construction yielding $(\Sigma, \sigma, \lambda)$ and the semi-conjugacy π, proceeds by choosing in S_gM finitely many disjoint smooth disks D_i transversal to the flow T_g. Inside each D_i, we choose closed connected sets R_i so that each R_i equals the closure of its interior R_i° and the union R of all R_i's forms a cross-section to the flow T_g. Let P be the Poincaré map of R. Then Σ is obtained as the collection of sequences $\omega = (\omega_i)_{i\in Z}$ such that

[2]Technically the map $v \mapsto U_g(v)$ is a section of the pull-back to S_gM of the bundle $T^{(1,1)}M \to M$ via the natural projection $p : SM \to M$. The Riccati equation can then be interpreted in these terms, ∇ being the pull-back to S_gM of the Levi-Civita connection on M.

$\cap_{i\in Z} P^{-i} R_{\omega_i} \neq \varnothing$. By the expansiveness of the flow T_g if $\omega \in \Sigma$ then the set $\cap_{i\in Z} P^{-i} R_{\omega_i}$ contains exactly one point $x = x(\omega)$ of R, thus defining a surjection $\pi_\Sigma : \omega \mapsto x(\omega)$. For each $p \in R$ the first return time $\overline{\lambda}(p)$ to P under the Poincaré map of P is well defined and positive. Let $\lambda = \overline{\lambda} \circ \pi_\Sigma$ and let σ be the shift on Σ and define (X, T) as the symbolic flow (X, T) built over (Σ, σ) with ceiling function λ. It is plain that p_Σ extents to a a surjection $\pi : X \to S_g M$ intertwining the flow T on X with the flow T_g on $S_g M$.

3.2. Structural Stability.

With M a compact manifold, denote by SM the bundle of oriented directions on M, i.e. $SM = \{v \in TM \,|\, v \neq 0\}/\sim$, where $v \sim v'$ and if and only if $\exists c > 0, v = cv'$. For each Riemannian metric g there is a natural identification between the bundle SM and the unit tangent bundle $S_g M$. By means of this identification, we regard the geodesic flow T_g for the metric g as a flow on SM, which we also denote by T_g with abuse of notation.

Denote by $\mathcal{U}$ the space of C^2-metrics of negative curvature on M, and more generally by $\mathcal{U}^s$ the space of C^s-metrics of negative curvature on M.

It is well known, (cf. [Mor24], [Ano67] and [Gro]), that the various flows T_g for $g \in \mathcal{U}$ are all orbit-equivalent. This means that if g_0 and $g \in \mathcal{U}$, there is a homeomorphism $H_g : SM \to SM$ sending the orbits of the flow T_{g_0} to the orbits of the flow T_g. In fact the homeomorphism H_g is homotopic to the identity and can always be chosen smooth along the orbits of T_{g_0} and Hölder continuous on SM. We call such an homeomorphism a *(g_0, g)-Morse correspondence*. The Morse correspondence is not unique: if H_1 and H_2 are two (g_0, g)-Morse correspondences then there exists a real valued function $t(v)$ on SM such that $\forall v \in SM, H_1^{-1} \circ H_2(v) = T_{g_0}^{t(v)}(v)$. Thus we have a sort of transversal uniqueness:

3.2.1. Definition. For g in a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$, we will call *the family of Morse correspondences normal at g_0*, the family $g \mapsto H_g$ of (g_0, g)-Morse correspondences H_g which is uniquely defined by the property

> "for each $v \in SM$ the footpoint of $H_g(v)$ belongs to the hypersurface obtained by exponentiating in the metric g_0 a small ball in the subspace g_0-perpendicular to v"

We have described some of the generalities of the construction of the symbolic flow in order to understand what happens if we perturb the flow

T_{g_0} by perturbing a metric $g_0 \in \mathcal{U}$. Let π_{g_0} be a semi-conjugacy between a symbolic flow $(\Sigma, \sigma, \lambda_{g_0})$ and (SM, T_{g_0}) mentioned in Section 3.1.

For g in some subset of $\mathcal{U}$, denoting by H_g the family of Morse correspondences normal at g_0, from the fact that H_g sends the orbits of the flow T_{g_0} to the orbits of the flow T_g, we obtain that the set $H_g \circ \pi_{g_0}(\Sigma)$ provides a global cross-section for the flow T_g with a Hölder continuous return function $\lambda_g : \Sigma \to \mathbb{R}$.

We obtain a finite-to-one Hölder continuous surjection $\pi_g : X_g \to SM$ intertwining the symbolic flow on $X_g = \Sigma \times \mathbb{R} / [(p, \lambda_g(p) + t) \sim (\sigma p, t)]$ and the flows T_g on SM. [Equivalently we can look at the composition $H_g \circ \pi$ as giving an orbit equivalence between $(\Sigma, \sigma, \lambda_{g_0})$ and (SM, T_g).] Thus we have the following diagram

$$\begin{array}{ccc} (\Sigma, \sigma, \lambda_{g_0}) & \xrightarrow[\pi_{g_0}]{} & (SM, T_{g_0}) \\ \updownarrow \overline{H}_g & & \updownarrow H_g \\ (\Sigma, \sigma, \lambda_g) & \xrightarrow[\pi_g]{} & (SM, T_g) \end{array}$$

with dashed arrows representing orbit equivalences. Observe that, identifying the space X_g supporting the flow $(\Sigma, \sigma, \lambda_g)$ with $X_g = \{(p, t) \in \Sigma \times \mathbb{R} | 0 \leq t < \lambda_g\}$ and Σ with $\Sigma \times \{0\}$, the orbit equivalences $\overline{H}_g$ are the identity on Σ.

3.3. Variation of cocycles.

Now we are in the situation in which we can apply the results of §2. To this purpose we need to pull-back the length and the Liapunov cocycles to the flow $(\Sigma, \sigma, \lambda_g)$ or $(\Sigma, \sigma, \lambda_{g_0})$, and determine in geometric terms their first variations δl, δu which entered in Proposition 1.3.1. Finally we need to resolve the smoothness issues related to this construction.

Assume that for each $g \in \mathcal{U}$ we are given a cocycle c_g for the flow T_g and orbit equivalences $H_g : SM \to SM$ sending the orbits of the flow T_{g_0} to the orbits of the flow T_g.

3.3.1. Definition. The *pull-back of the family of cocycles c_g along the orbit equivalences H_g* is the family of cocycles $H_g^* c_g$ for the flow T_{g_0} given by

$$H_g^* c_g(v, t) = c_g(H_g v, t'),$$

where t' is defined by $T_g^{t'} H_g v = H_g T_{g_0}^t v$.

3.3.2. Example. The length cocycles $\ell_g(v,t) = t$ for the flow T_g is generated by $A_g = 1$. If the orbit equivalences are differentiable along the flow T_{g_0}, then, denoting by Y_g the generator of the flow T_g, we have

$$(H_g)_* X_{g_0} := \frac{d}{dt} H_g \circ T^t_{g_0} = l_{H_g} Y_g \tag{3.3}$$

or equivalently

$$H_g T^t_{g_0} v = T_g^{\int_0^t l_{H_g}(T^s_{g_0} v)\, ds} H_g v.$$

Therefore the pull-back of the length cocycle ℓ_g along H_ϕ is given by

$$H_g^* \ell_g(v,t) = \int_0^t l_{H_g}(T^s_{g_0} v)\, ds$$

and it has as generator the function l_{H_g} defined in 3.3. Denoting by p the natural projection $SM \to M$, and by $\|\cdot\|_g$ the norm induced by g on TM, from $\|p_* Y_g\|_g = 1$ we have

$$l_{H_g}(v) = \left\| p_*(H_g)_* X_{g_0}(v) \right\|_g. \tag{3.4}$$

The following Proposition is elementary.

3.3.3. Proposition. *If the cocycles c_g have generators A_g, then their pull-back along a family H_g of T_{g_0}-differentiable orbit equivalences is differentiable along the flow T_{g_0} and it is generated by $(A_g \circ H_g) l_{H_g}$.*

3.3.4. Definition. For a curve of metrics g_ε we define the *the i-th variation of the cocycle c_{g_ε}* as the cocycle for the flow T_{g_0} defined by $d^i/d\varepsilon^i H^*_{g_\varepsilon} c_{g_\varepsilon}$ at $\varepsilon = 0$.

3.3.5. Remark. Let u_g be the generator of a cocycle c_g for the flow $(\Sigma, \sigma, \lambda_g)$. Let $\overline{H}_g$ be orbit equivalences sending the orbits of the flow $(\Sigma, \sigma, \lambda_{g_0})$ to the orbits of the flow $(\Sigma, \sigma, \lambda_g)$. Assume further that $\overline{H}_g$ is the identity on Σ (identifying the space X_g supporting the flow $(\Sigma, \sigma, \lambda_g)$ with $X_g = \{(p,t) \in \Sigma \times \mathbb{R} | 0 \le t < \lambda_g\}$ and Σ with $\Sigma \times \{0\}$). Then the pull-back of the length cocycle for $(\Sigma, \sigma, \lambda_g)$ along the orbit equivalences $\overline{H}_g$ has a generator l_g satisfying

$$\int_0^{\lambda_{g_0}} l_g(p,t)\, dt = \lambda_g(p)$$

and the pull-back $\overline{H}_g^* c_g$ is the the cocycle generated by a function $\delta^0 u_g$ such that

$$\int_0^{\lambda_{g_0}} \delta^0 u_g(p,t)\,dt = \int_0^{\lambda_g} u_g(p,t)\,dt.$$

We conclude that the two definitions 2.5.1 and 3.3.4 of the i-th variation of a cocycle agree.

Now we are ready to tackle the question of the smoothness of the dependence on g.

Improving on previous results of [dlLMM86], Katok et al. proved:

3.3.6. Theorem ([KKPW89]). [3] *For sufficiently small $\beta > 0$, in a neighbourhood $\mathcal{V}^s$ of g_0 in $\mathcal{U}^s$ the family of Morse correspondences normal at g_0, $g \mapsto H_g$, exists and has the following properties:*

1. *The homeomorphisms H_g are T_{g_0}-differentiable, that is they are differentiable along the orbits of the flow T_{g_0}.*
2. *The map $g \to H_g$ is of class C^{s-2} as a map from the Banach manifold $\mathcal{V}^s$ to the Banach manifold of T_{g_0}-differentiable C^β-maps of SM into itself.* [4]
3. *The map $g \mapsto l_{H_g}$ is of class C^{s-2} as a map with values in $C^\beta(SM, \mathbb{R}^+)$.*
4. *The topological entropy $g \mapsto \mathrm{Ent}_{top}(g)$ is of class C^{s-1}.*

Furthermore Contreras showed:

3.3.7. Theorem ([Con92]). *Under the same hypotheses, denoting by E_g^s the stable bundle for the flow T_g and by u_g the generator of the Liapunov cocycle for T_g we have:*

5. *the maps $g \in \mathcal{V}^s \to u_g \circ H_g$ and $g \in \mathcal{V}^s \to E_g^s \circ H_g$is of class C^{s-3} as a map with values respectively in $C^\beta(SM)$ and in the space of C^β distributions on SM.*

[3] The theorem proved in [KKPW89] was stated for Anosov flows and it is valid more generally for hyperbolic attractors: see also [Con92]. We quote its application to geodesic flows. The loss of one further derivative is due to the fact that the vector field defining the geodesic flow depends on the first derivatives of the metric.

[4] More exactly the target space is the space $C^\beta_{g_0}(SM, SM)$ of C^β continuous maps $f : SM \to SM$ whose derivatives $X_{g_0} f$ along the orbits of the flow T_{g_0} are C^β continuous maps $SM \to T(SM)$ endowed with the norm $\|f\|_\beta + \|X_{g_0} f\|_\beta$.

6. *The entropy of Liouville measure* $g \mapsto \mathrm{Ent}_{Liouv}(g)$ *as a function of* $g \in \mathcal{V}^s$ *is of class* C^{s-3}.

3.3.8. In particular, from (2) above, if $s \geq 3$ the derivative of the (g_0, g)-Morse correspondence H_g normal at g_0, is a linear map from the C^3 sections S of the symmetric tensor bundle S^2M to Hölder continuous vector fields $\Xi_g(S)$ along H_g.

3.3.9. Definition. We call the vector field $\Xi_{g_0}(S)$ the *infinitesimal Morse correspondence at* g_0 *in the direction of* S

3.3.10. The the infinitesimal Morse correspondence $\Xi_{g_0}(S)$ is differentiable along the g_0-geodesics and it satisfies a differential equation given in [FF93]. Furthermore, by the definition of normal Morse correspondence, the vector field $\Xi_{g_0}(S)$ is everywhere perpendicular to X_{g_0} in the natural metric on $S_{g_0}M$, i.e. the projection of $\Xi_{g_0}(S)$ at $v \in S_{g_0}M$ on M is g_0-perpendicular to v:

$$g_0\big(v, p_*\Xi_{g_0}(S)(v)\big) = 0. \tag{3.5}$$

3.3.11. Lemma. *Let* g_ε *is a* C^1*-curve of* C^5 *Riemannian negatively curved metrics on a compact manifold* M, *and let* $S_\varepsilon = \frac{d}{d\varepsilon} g_\varepsilon \big|_{\varepsilon=0}$.

Then the generator of the first variation of the length cocycle at g_ε *at* $\varepsilon = 0$ *is is cohomologous to the function:*

$$v \in SM \mapsto \delta l_0(v) = \frac{1}{2} S_0^{\vee}(v) = \frac{1}{2} S_0(v, v),$$

and the generator of the first variation of the Liapunov cocycle u_ε *at* $\varepsilon = 0$ *is cohomologous to the Hölder function:*

$$v \in SM \mapsto \delta u_0(v) = \frac{d}{d\varepsilon} u_\varepsilon \circ H_{g_\varepsilon}(v) \big|_{\varepsilon=0} + \frac{1}{2} S_0(v, v)\, u_0(v).$$

Proof. By Proposition 3.3.3, the pull-back of the T_{g_ε}-cocycle generated by u_ε has generator $u_\varepsilon(H_{g_\varepsilon}) l_\varepsilon$ and by (1)-(6) above the derivatives $d\, u_\varepsilon \circ H_{g_\varepsilon}/d\varepsilon$ and $dl_\varepsilon/d\varepsilon$ both exist as limits in $C^\beta(SM)$. Thus for each g_0-geodesic closed orbit γ_0, denoting by γ_ε the geodesic g_ε-geodesic closed orbit homotopic to

γ_0 we have

$$\begin{aligned}&\frac{d}{d\varepsilon}\int_{\gamma_\varepsilon} u_\varepsilon(T^s_{g_\varepsilon}v)\,ds\\ &= \frac{d}{d\varepsilon}\int_{\gamma_0} u_\varepsilon(H_{g_\varepsilon}T^s_{g_0}v)l_{H_{g_\varepsilon}}(T^s_{g_0}v)\,ds\\ &= \int_{\gamma_0}\left[\frac{d}{d\varepsilon}\Big(u_\varepsilon(H_{g_\varepsilon}T^s_{g_0}v)\Big)l_{H_{g_\varepsilon}}(T^s_{g_0}v) + u_\varepsilon(H_{g_\varepsilon}T^s_{g_0}v)\frac{d}{d\varepsilon}l_{H_{g_\varepsilon}}(T^s_{g_0}v)\right]ds\end{aligned}$$

At $\varepsilon = 0$, we have $l_{H_{g_\varepsilon}}(T^s_{g_0}v) = 1$ and, using 3.4 and 3.5, we see from the equation of the geodesics (cf. [FF93] that

$$\frac{d}{d\varepsilon}\,l_{H_{g_\varepsilon}}(w)\big|_{\varepsilon=0} = \frac{1}{2}S(w,w).$$

Since, by a well known theorem of Livšic [Liv71], the collection of integrals along periodic orbits determine the cohomology class of Hölder cocycles, our claim is proved. □

§4. Relation of the Liapunov cocycle to the variation of metric.

In this section we find a formula for the first variation of the Liapunov cocycle in terms of the first order variation of the Riemannian metric.

4.1.

Throughout this section, M denotes a compact connected manifold with no boundary. We denote by SM the bundle of oriented directions on M. As usual given a Riemannian metric g of class C^s on M we identify SM with the unit tangent bundle $S_gM = \{v \in TM \,|\, g(v,v) = 1\}$, via the obvious C^s-diffeomorphism. The symbol g^t denotes both the geodesic flow on S_gM determined by g can and the flow induced on SM, via the above identification.

We shall consider a C^1 path $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto g_\varepsilon$ of C^s-metrics, with $s \geq 4$ and set $S_\varepsilon = \frac{dg_\varepsilon}{d\varepsilon}$. We denote by $H_\varepsilon : SM \to SM$ the (g_0, g_ε)-Morse correspondence normal at g_0 and denote by $\Xi_{g_0}(S_{g_0})$ the infinitesimal Morse correspondence.

4.2.

As usual let $p : SM \to M$ be the canonical projection. Let $\gamma_0(t)$ be a unit-speed geodesic of initial velocity $\gamma_0'(0) = v \in SM$. Clearly we have $\gamma_0(t) = p(g_0^t v)$. Define

$$\overline{\gamma} : D = \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \mapsto \overline{\gamma}(t, \varepsilon) = p(h_\varepsilon(g_0^t v)) \in M.$$

We set, for all t, $\xi = \frac{\partial \overline{\gamma}}{\partial \varepsilon}\Big|_\varepsilon = 0$, $Y = \frac{\partial \overline{\gamma}}{\partial t}\Big|_\varepsilon = 0$. Of course $Y(t) = g_0^t v$ and $\xi(t) = p_*(\Xi_{g_0}(S_{g_0})(g_0^t v))$.

The map $\overline{\gamma}$ satisfies the following conditions:

1. The map $\overline{\gamma}$ is differentiable, by Theorem 3.3.6 and our smoothness hypotheses.
2. The curves $\gamma_\varepsilon : t \mapsto \overline{\gamma}(t, \varepsilon)$ are geodesics for the metric g_ε on M, by definition of Morse correspondence.
3. We have $g_0(\xi, Y) = 0$, since we are considering the Morse correspondence normal at g_0.
4. We have also $g_0(Y, Y) = 1$, since $\gamma_0(t)$ is a unit-speed geodesic.

Let E be the pullback to D of the tangent bundle of M via the map $\overline{\gamma} : D \to M$. Then the bundle E is endowed with Riemannian metrics g^ε on E, Levi-Civita connections ∇^ε for the metrics g^ε, and we let R^ε denote curvature of ∇^ε.

Let U be the section of $E^* \otimes E$, i.e. the $\binom{1}{1}$-tensor field on M along $\overline{\gamma}$, defined as $U(t, \varepsilon) = U_{g_\varepsilon}(h_\varepsilon(g_0^t v))$, where, as usual $U_g(v)$ is the second fundamental form of the stable horosphere of the metric g at $p(v)$ considered as tensor of type $\binom{1}{1}$. By Theorem 3.3.7, the section U is C^1.

Finally to lighten the notation we write g, g^t, S, γ, ∇, R, etc. for g_0, g_0^t, S_0, γ_0, ∇^0, R^0, etc.

4.2.1. Lemma. *Denoting by u_ε the generator of the Liapunov cocycle for that flow g_ε^t, along the geodesic $\gamma(t) = \gamma_0(t)$, we have*

$$\frac{d}{d\varepsilon} u_\varepsilon \circ H_{g_\varepsilon}(g^t v)\big|_{\varepsilon=0} = L_\xi \operatorname{Tr} U(\gamma(t)) = \operatorname{Tr} \nabla_\xi U(\gamma(t))$$

Proof. The first equality follows from (3.2) and the definition of ξ. The second follows from the fact that taking traces of endomorphisms commutes with covariant differentiation.

□

4.2.2. Notation. Set for simplicity

$$B(v) = \frac{d}{d\varepsilon} u_\varepsilon \circ H_{g_\varepsilon}(g^t v)\big|_{\varepsilon=0}.$$

Recall that the first variation of the Liapunov cocycle along the path g_ε at $\varepsilon = 0$ is given by

$$\delta u(v) = B(v) + \frac{1}{2} S(v,v) \operatorname{Tr} U.$$

4.2.3. Lemma. *Along the geodesic $\gamma(t)$, the field of endomorphisms $\nabla_\xi U(\gamma(t))$ satisfies the following differential equation:*

$$\begin{aligned} &\nabla_Y(\nabla_\xi U) + U \circ \nabla_\xi U + \nabla_\xi U \circ U = \\ &-\frac{1}{2} S(Y,Y)\nabla_Y U - [\Gamma(Y), U] - [R(\xi, Y), U] - S(Y,Y)U^2 + \\ &\left.\frac{\partial R^\varepsilon_{\gamma(t)}}{\partial \varepsilon}\right|_{\varepsilon=0} (Y, \cdot\,) Y + \nabla_\xi R(Y, \cdot\,)Y + R(\nabla_Y \xi, \cdot)Y + R(Y, \cdot)\nabla_Y \xi. \end{aligned}$$

where $[\cdot,\cdot]$ denotes the ordinary commutator of endomorphisms and Γ denotes the $\binom{1}{2}$-tensor field along γ given by $\Gamma = \frac{d\nabla^\varepsilon}{d\varepsilon}\big|_{\varepsilon=0}$. For simplicity we have written $\Gamma(Y)$ for the contraction $\left.\frac{d\nabla^\varepsilon_Y}{d\varepsilon}\right|_{\varepsilon=0}$.

Proof. The field of endomorphisms U satisfies, along each geodesic γ_ε, the Riccati equation 3.1 suitably corrected to take into account the fact that $t \mapsto \gamma_\varepsilon(t)$ is not a g_ε-unit speed geodesic. Thus:

$$\left\|\frac{\partial \overline{\gamma}}{\partial t}\right\|_\varepsilon \frac{\nabla^\varepsilon}{\partial t} U + \left\|\frac{\partial \overline{\gamma}}{\partial t}\right\|_\varepsilon^2 U^2 = R^\varepsilon\left(\frac{\partial \overline{\gamma}}{\partial t}, \cdot\right)\frac{\partial \overline{\gamma}}{\partial t} \tag{4.1}$$

where R^ε denotes the Riemann curvature tensor of the metric g^ε.

Observe that we have:

$$\begin{aligned} L_{\frac{\partial}{\partial\varepsilon}} \left(\left\|\frac{\partial \overline{\gamma}}{\partial t}\right\|_\varepsilon^2\right)\Bigg|_{\varepsilon=0} &= \\ \frac{d}{d\varepsilon}\left(g^\varepsilon_{\overline{\gamma}(t,\varepsilon)}\left(\frac{\partial \overline{\gamma}}{\partial t}_{\overline{\gamma}(t,\varepsilon)}, \frac{\partial \overline{\gamma}}{\partial t}_{\overline{\gamma}(t,\varepsilon)}\right)\right)\Bigg|_{\varepsilon=0} &= \\ S(Y,Y) + 2g(\nabla_\xi Y, Y) &= \\ S(Y,Y) + 2g(\nabla_Y \xi, Y) &= S(Y,Y). \end{aligned} \tag{4.2}$$

where in the last equality we have used the fact that $\nabla_Y \xi$ and Y are g-perpendicular.

Similarly one has

$$L_{\frac{\partial}{\partial\varepsilon}} \left(\left\| \frac{\partial \overline{\gamma}}{\partial t} \right\|_{\varepsilon} \right) \Big|_{\varepsilon=0} = \frac{1}{2} S(Y,Y) \tag{4.3}$$

since $\|Y\|_0 = 1$.

Furthermore:

(4.4)

$$\nabla_{\frac{\partial}{\partial\varepsilon}} (\nabla^{\varepsilon}_{\frac{\partial}{\partial t}} U_{\overline{\gamma}(t,\varepsilon)}) \Big|_{\varepsilon=0} = \Gamma(Y) U_{\overline{\gamma}(t,0)} - U_{\overline{\gamma}(t,0)} \Gamma(Y) + \nabla_{\frac{\partial}{\partial\varepsilon}} (\nabla_{\frac{\partial}{\partial t}} U_{\overline{\gamma}(t,\varepsilon)}) \Big|_{\varepsilon=0} =$$
$$\Gamma(Y) U_{\overline{\gamma}(t,0)} - U_{\overline{\gamma}(t,0)} \Gamma(Y) + \nabla_Y (\nabla_\xi U_{\overline{\gamma}(t,0)}) + R(\xi, Y) U_{\overline{\gamma}(t,0)} - U_{\overline{\gamma}(t,0)} R(\xi, Y)$$

where, in the last equality, we have used the fact that for commuting vector fields X, Z and $\binom{1}{1}$ tensor fields W one has:

$$[\nabla_X, \nabla_Z] W = R(X,Z) \circ W - W \circ R(X,Z).$$

Finally observe that

$$\nabla_{\frac{\partial}{\partial\varepsilon}} \left(R^{\varepsilon}(\frac{\partial \overline{\gamma}}{\partial t}, \cdot) \frac{\partial \overline{\gamma}}{\partial t} \right) \Big|_{\varepsilon=0} = \tag{4.5}$$
$$\frac{\partial R^{\varepsilon}_{\gamma(t,0)}}{\partial \varepsilon} \Big|_{\varepsilon=0} (Y, \cdot) Y + \nabla_{\frac{\partial}{\partial\varepsilon}} \left(R(\frac{\partial \overline{\gamma}}{\partial t}, \cdot) \frac{\partial \overline{\gamma}}{\partial t} \right) \Big|_{\varepsilon=0} =$$
$$\frac{\partial R^{\varepsilon}_{\gamma(t,0)}}{\partial \varepsilon} \Big|_{\varepsilon=0} (Y, \cdot) Y + \nabla_\xi R(Y, \cdot) Y + R(\nabla_\xi Y, \cdot) Y + R(Y, \cdot) \nabla_\xi Y =$$
$$\frac{\partial R^{\varepsilon}_{\gamma(t,0)}}{\partial \varepsilon} \Big|_{\varepsilon=0} (Y, \cdot) Y + \nabla_\xi R(Y, \cdot) Y + R(\nabla_Y \xi, \cdot) Y + R(Y, \cdot) \nabla_Y \xi.$$

Taking the covariant derivative of equation (4.1) along ξ and using the above observations (4.2)–(4.5) we obtain that $\nabla_\xi U$ satisfies the given differential equation along the geodesic γ. □

4.3.

Assume that (M, g) is a rank-1 locally symmetric space of dimension n. Then, we can find parallel orthonormal vector fields $Y_1 = Y$, Y_2, ... ,Y_n

along the geodesic γ satisfying $R(Y,Y_i)Y = \lambda_i^2 Y_i$, with $\lambda_1 = 0$, $\lambda_j = 2$ for $1 < j \leq r$, and $\lambda_j = 1$ for $r < j \leq n$. Here $r = 1, 2, 4$ or 8 depending on the type of symmetric space.

Furthermore, from the definition of U, along the geodesic γ, we have $U(\cdot) = -\sqrt{R(Y,\cdot)Y}$, i.e. $UY_j = -\lambda_j Y_j$.

4.3.1. Lemma. *Retaining the above notation, define* $B_j = \langle (\nabla_\xi U) Y_j, Y_j \rangle$, $j = 1, \ldots, n$, *where* $\langle \cdot, \cdot \rangle$ *denotes the* g*-inner-product. Then, along the geodesic* γ, *we have that the* B_j *satisfy the differential equation*

$$L_Y B_j - 2\lambda_j B_j = -S(Y,Y)\lambda_j^2 + \left\langle \frac{\partial R^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} (Y, Y_j)Y, Y_j \right\rangle$$

Proof. Since U is symmetric, we have:

$$\begin{gathered} \langle (U \circ \nabla_\xi U + \nabla_\xi U \circ U) Y_j, Y_j \rangle = \\ \langle (\nabla_\xi U) Y_j, U Y_j \rangle + \langle (\nabla_\xi U) U Y_j, Y_j \rangle = -2\lambda_j B_j. \end{gathered} \tag{4.6}$$

Also:

$$\begin{gathered} \langle [\Gamma(Y), U] Y_j, Y_j \rangle = \\ \langle \Gamma(Y) U Y_j, Y_j \rangle - \langle U \Gamma(Y) Y_j, Y_j \rangle = \langle \Gamma(Y) U Y_j, Y_j \rangle - \langle \Gamma(Y) Y_j, U Y_j \rangle = 0. \end{gathered} \tag{4.7}$$

Similarly,

$$\langle [R(\xi, Y), U] Y_j, Y_j \rangle = 0. \tag{4.8}$$

From $\langle \nabla_Y \xi, Y \rangle = 0$, and since we can also assume $R(Y_j, Y)Y_j = \lambda Y$, we have:

$$\begin{gathered} \langle R(\nabla_Y \xi, Y_j) Y, Y_j \rangle = \\ \langle R(Y, Y_j)(\nabla_Y \xi), Y_j \rangle = \langle \nabla_Y \xi, R(Y_j, Y) Y_j \rangle = \lambda \langle \nabla_Y \xi, Y \rangle = 0. \end{gathered} \tag{4.9}$$

Similarly

$$\langle R(Y, Y_j) \nabla_Y \xi, Y_j \rangle = 0. \tag{4.10}$$

Evaluating on Y_j the right hand-side of the equation of Lemma 4.2.3 and taking the inner product of with Y_j, after using 4.6–4.10 and noticing that

$$\nabla_Y U = 0, \qquad \nabla R = 0, \qquad \text{and} \qquad \langle U^2 Y_j, Y_j \rangle = \lambda_j^2,$$

we obtain the given equation for B_j. □

4.3.2. Corollary. *If (M, g) has constant negative curvature -1 and dimension n, $\nabla_\xi \operatorname{Tr} U$, along γ , satisfies the following equation:*

$$L_Y(\nabla_\xi \operatorname{Tr} U) - 2(\nabla_\xi \operatorname{Tr} U) = (1-n)S(Y,Y) - \left.\frac{\partial \operatorname{Ric}^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} (Y,Y).$$

Proof.
Clearly $\sum_j B_j = \sum_j \langle (\nabla_\xi U) Y_j, Y_j\rangle = \nabla_\xi \operatorname{Tr} U$. From

$$\operatorname{Ric}(X,Z) = \operatorname{Tr}(W \mapsto R(X,W)Z),$$

we have $\sum_j \langle \frac{\partial R^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} (Y, Y_j)Y, Y_j\rangle = \frac{\partial \operatorname{Ric}^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}$. Summing over j the equation of Lemma 4.3.1, since $\lambda_j = 1$ we obtain our claim. □

4.4.

Let $\nabla^*\nabla$ denote the rough Laplacian for the metric $g = g_0$ on M. Recall that the Lichnerowicz Laplacian for a metric g is defined as [Bes87, 1.180b]:

$$\Delta_L S = \nabla^*\nabla S + \operatorname{Ric} \circ S + S \circ \operatorname{Ric} - 2R^\circ(S), \tag{4.11}$$

where $\circ$ denotes the contraction symmetric tensors of rank 2 identified with $\binom{1}{1}$ tensor via g and $R^\circ(S)$ is defined by [5]

$$R^\circ(S)(Y,Z) = \operatorname{Tr}_g S(R(\cdot\,, X)Y, \cdot). \tag{4.12}$$

Then [Bes87, 1.174]:

$$\left.\frac{\partial \operatorname{Ric}^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} = \frac{1}{2}\Delta_L S - \delta^*\delta S - \nabla d(\operatorname{Tr} S) \tag{4.13}$$

where δ^* is the symmetrization of covariant derivative and δ is its formal adjoint, the divergence.

Proof of Proposition 1.3.3. Form Corollary 4.3.2 we obtain that along the geodesic γ we have

$$B(g^t v)\nabla_\xi = \operatorname{Tr} U(\gamma(t)) \approx \frac{n-1}{2} S(Y(t), Y(t)) + \frac{1}{2} \left.\frac{\partial \operatorname{Ric}^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} (Y(t), Y(t)),$$

[5] Our definition agrees with [Bes87, 1.131], in spite of the opposite convention on the sign of the Riemann curvature R.

with $\approx$ meaning that the difference is a coboundary. In fact, the difference at $\gamma(t) = p(g^t v)$ is $\frac{1}{2} L_Y(\nabla_\xi \operatorname{Tr} U)(\gamma(t)) = \frac{1}{2}\frac{\partial}{\partial t} B(g^t v)$

We claim that since g has constant curvature -1 we have:

$$\Delta_L S(Y,Y) = \nabla^*\nabla S(Y,Y) - 2nS(Y,Y) + 2\operatorname{Tr}_g S.$$

In fact, $\operatorname{Ric} = -(n-1)g$, and therefore

$$\operatorname{Ric}\circ S + S\circ\operatorname{Ric} = -2(n-1)S.$$

Furthermore, since $R(Y,X)Z = -\langle X,Z\rangle Y + \langle Y,Z\rangle X$,

$$\begin{aligned}(4.14)\quad R^\circ(S)(Y,Y) &= \sum_i S(R(Y_i,Y)Y,Y_i) = \\ &= \sum_i S(-Y_i + \langle Y,Y_i\rangle Y, Y_i) = -\operatorname{Tr}_g(S) + S(Y,Y).\end{aligned}$$

proving our claim. We obtain that

$$\begin{aligned}\nabla_\xi \operatorname{Tr} U \approx{}& \frac{n-1}{2} S(Y,Y) + \frac{1}{4}\nabla^*\nabla S(Y,Y) - \frac{n}{2} S(Y,Y) + \\ &+ \frac{1}{2}\operatorname{Tr}_g S - \frac{1}{2}\delta^*\delta S(Y,Y) - \frac{1}{2}\nabla d(\operatorname{Tr} S)(Y,Y) \\ \approx{}& -\frac{1}{2} S(Y,Y) + \frac{1}{4}\nabla^*\nabla S(Y,Y) + \frac{1}{2}\operatorname{Tr}_g S - \\ &- \frac{1}{2}\delta^*\delta S(Y,Y) - \frac{1}{2}\nabla d(\operatorname{Tr} S)(Y,Y).\end{aligned}$$

and setting

$$T = -\frac{1}{2} S + \frac{1}{4}\nabla^*\nabla S + \frac{1}{2}(\operatorname{Tr}_g S) g - \frac{1}{2}\delta^*\delta S$$

and noticing that the term $\nabla d(\operatorname{Tr} S)(Y,Y) = Y^2(\operatorname{Tr} S)$ is cohomologous to zero, we have concluded that

$$B(g^t v) = \nabla_\xi \operatorname{Tr} U(\gamma(t)) \approx T^\vee(g^t v)$$

or $B \approx T^\vee$. Thus the first variation of the Liapunov cocycle along the path g_ε at $\varepsilon = 0$ is

$$\delta u = B + \frac{1}{2} S^\vee \operatorname{Tr} U \approx T^\vee - \frac{n-1}{2} S^\vee.$$

□

§ 5. Proof of Proposition 1.3.4 and Theorem C.

5.1.

Let $g_\varepsilon \in \mathcal{U}$ be a curve of metrics of constant volume on M. Then if we set $S_\varepsilon = \frac{\partial g_\varepsilon}{\partial \varepsilon}$ and denoting by s_n the volume of the n-sphere, we have,

$$\frac{1}{2}\int_{Sg_\varepsilon M} S^\vee(v)\, d\,\mathrm{Liouv}(g_\varepsilon)(v) =$$
$$= \frac{s_{n-1}}{2n}\int_M \mathrm{Tr}_{g_\varepsilon} S_\varepsilon\, d\,\mathrm{Vol}\, g_\varepsilon = -\frac{s_{n-1}}{n}\frac{\partial\, \mathrm{Vol}_{g_\varepsilon}(M)}{\partial \varepsilon} = 0.$$

Since $\frac{1}{2}S^\vee$ is the first variation δl_ε of the length cocycle l_ε, and $\mathrm{Liouv}(g_\varepsilon)$ the Gibbs state for the Liapunov cocycle u_ε of the flow T_{g_ε}, we obtain that $m_{u_\varepsilon}(\delta l_\varepsilon) = 0$ for all ε. If g_0 is locally symmetric, the measure of maximal entropy M_0 and the Liouville measure $\mathrm{Liouv}(g_0)$ coincide and thus all the conditions of Proposition 1.3.1 (C) are verified. Denoting for simplicity g_0 by g, the maximal measure entropy M_0 by m and the covariance of the flow T_g with respect to the measure of m by Cov, we can summarize the results so far achieved (cf. Propositions 1.3.1, 1.3.3) in the following Proposition:

5.1.1. Proposition. *Let g_ε be a C^2-curve of C^5-metrics on a manifold M of constant volume, and assume that that $g = g_0$ has constant negative curvature -1. Denoting with δl and δu the first variation of the length cocycle and of Liapunov cocycle at $\varepsilon = 0$ we have:*

$$\mathrm{Ent}''_{\mathrm{top}}(g) = \mathrm{Cov}\,\big((n-1)\delta l + \delta u, (n-1)\delta l\big)$$

and

$$\mathrm{Ent}''_{\mathrm{Liouv}}(g) = -\,\mathrm{Cov}\,\big((n-1)\delta l + \delta u, \delta u\big).$$

Setting $T = \mathcal{T}(S) := -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S + \frac{1}{2}(\mathrm{Tr}_g S)g - \frac{1}{2}\delta^*\delta S$, since $\delta l \approx \frac{1}{2}S^\vee$ and $\delta u \approx T^\vee - \frac{n-1}{2}S^\vee$, we have

$$\mathrm{Ent}''_{\mathrm{top}}(g) = \mathrm{Cov}\,\big(T^\vee, \frac{n-1}{2}S^\vee\big),$$

$$\mathrm{Ent}''_{\mathrm{Liouv}}(g) = -\,\mathrm{Cov}\,\big(T^\vee, T^\vee - \frac{n-1}{2}S^\vee\big)$$

and

$$\mathrm{Ent}''_{\mathrm{top}}(g) - \mathrm{Ent}''_{\mathrm{Liouv}}(g) = \mathrm{Cov}\,\big(T^\vee, T^\vee\big).$$

5.2.

If g has constant sectional curvature, then the group G of isometries of $\widetilde{M}$, the universal cover of M, is isomorphic to $SO_0(1,n)$, where $n = \dim(M)$. We have an identification of M with the space $\Gamma\backslash G/K$, where $\Gamma \approx \pi_1(M)$ is a discrete group of isometries acting without fixed points on $\widetilde{M}$, and $K \approx SO(n)$ is the stabilizer of a point $p_0 \in \widetilde{M}$. Furthermore, the unit tangent unit bundle S_gM and the orthonormal frame bundle FM of M are identified with $\Gamma\backslash G/K_1$ and $\Gamma\backslash G$, where $K_1 \approx SO(n-1)$ is the stabilizer of a vector $v_0 \in T_{p_0}\widetilde{M}$. The parallel transport of an orthonormal frame $(v_1, v_2, \dots, v_n) \in FM$ along the geodesic determined by v_1 is identified with the action on $\Gamma\backslash G$ given by multiplication on the right by the split-Cartan $A_1 \approx \mathbb{R}$ commuting with K_1. It is plain that this parallel transport projects via the natural projection $(v_1, v_2, \dots, v_n) \in FM \mapsto v_1 \in S_gM$ to the geodesic flow on S_gM.

5.2.1. Notation. Denote by $\mathcal{C}^r(S^2M)$ the space of C^r-sections of the bundle S^2M of symmetric covariant tensors of rank 2. Similarly $\mathcal{L}^2(S^2M)$ denotes the L^2 sections of this bundle. Clearly the map ${}^\vee : S \to S^\vee$ defined, for $v \in S_gM$, by $S^\vee(v) = S(v,v)$ maps $\mathcal{C}^r(S^2M)$ to $C^r(S_gM)$ and $\mathcal{L}^2(S^2M)$ to $L^2(S_gM)$.

Observe also that we have an injection $L^2(S_gM) \hookrightarrow L^2(FM)$, regarding $L^2(S_gM)$ as the subset of $L^2(FM)$ of K_1-invariant vectors. Similarly, $C^r(S_gM) \hookrightarrow C^r(FM)$. In the sequel these identifications will be implicit.

5.2.2. Left invariant differential operators on G commuting with K_1 act on $C^\infty(S_gM)$. This is in particular true of the the Casimir operator $\mathbf{Cas}_{SO(1,n)}$ of G, which we normalize so that on $C^\infty(M)$ it coincides with the rough Laplacian $\nabla^*\nabla$.

Observe that the differential operators $\nabla^*\nabla$, δ, δ^* etc. acting on symmetric tensor field can be viewed by means of the above maps as left invariant differential operators on $C^\infty(FM)$ commuting with K_1. In fact, denote by P_i the parallel transport of an orthonormal frame $(v_1, v_2, \dots, v_n) \in FM$ along the geodesic determined by v_i and let Y_i be vector field generating the flow P_i. For $i < j$, set $R_{ij} = [Y_i, Y_j]$. Then it is plain that the vector fields Y_i, R_{ij}, $(i < j)$, form a basis for the Lie algebra $\mathfrak{g} \approx \mathfrak{so}(1,n)$ of G and that R_{ij} exponentiate to the flow $\exp(\theta R_{ij})$ which rotates an orthonormal frame $(v_1, \dots, v_i, \dots, v_j, \dots, v_n) \in FM$ to $(v_1, \dots, \cos(\theta)v_i - \sin(\theta)v_j, \dots, \sin(\theta)v_i + \cos(\theta)v_j, \dots, v_n)$. In particular given $S \in \mathcal{C}^{\in}(\mathcal{S}^{\in}\mathcal{M})$, setting, for $(v_1, \dots, v_n) \in FM$,

$$S_{ij}(v_1, \dots, v_n) = S(v_i, v_j)$$

and identifying S_{11} with the lift to FM of $S^\vee$ we have

$$S_{1i} = -\frac{1}{2} L_{R_{1i}} S^\vee.$$

Thus, if $(v_1, \dots, v_i, \dots, v_j, \dots, v_n) \in FM$ and $S \in C^2(S^2M)$, we have:

$$\begin{aligned}
(\delta S)^\vee(v_1) = (\delta S)(v_1) &= -\sum_i \nabla_{v_i} S(v_i, v_1) = -\sum_i \frac{\partial}{\partial t} S(P_i^t v_i, P_i^t v_1)\Big|_{t=0} \\
&= -\sum_i S_{1i}(P_i^t v_1, \dots, P_i^t v_n)\big|_{t=0} = -\sum_i L_{Y_i} S_{1i}(v_1, \dots, v_n) \\
&= \frac{1}{2} \sum_i L_{Y_i} L_{R_{1i}} S^\vee(v_1, \dots, v_n)
\end{aligned}$$

Similarly, we have

$$\nabla^*\nabla = -\sum_1 L_{Y_i}^2 = (-\sum_i L_{Y_i}^2 + \sum_{i<j} R_{ij}^2) - \sum_{i<j} R_{ij}^2 = \mathbf{Cas}_{SO(1,n)} - \mathbf{Cas}_{SO(n)},$$

where $\mathbf{Cas}_G$ denotes the Casimir operator of the group G. Since, for $S \in \mathcal{C}^{\in}(\mathcal{S}^{\in}\mathcal{M})$, we have $\mathbf{Cas}_{SO(n)} S^\vee = 2\operatorname{Tr} S - 2nS^\vee$, we can express the operator $\mathcal{T}(\mathcal{S})$ defined in 5.1.1 as

(5.1)

$$\begin{aligned}
\mathcal{T}(\mathcal{S}) &= -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S + \frac{1}{2}(\operatorname{Tr}_g S) \otimes g - \frac{1}{2}\delta^*\delta S \\
&= -\frac{1}{2}S + \frac{1}{4}(\mathbf{Cas}_{SO(1,n)} - \mathbf{Cas}_{SO(n)})S + \frac{1}{2}(\operatorname{Tr}_g S) \otimes g - \frac{1}{2}\delta^*\delta S \\
&= \frac{n-1}{2}S + \frac{1}{4}\mathbf{Cas}_{SO(1,n)} S - \frac{1}{2}\delta^*\delta S.
\end{aligned}$$

5.3.

By a well known theorem on the unitary representation of $SO_0(1, n)$, the Hilbert space $L^2(FM) \approx \Gamma\backslash G$ splits as a direct sum $L^2(FM) = \sum H_i$ of topologically irreducible components H_i of the right action of G.

5.3.1. Lemma. *Each irreducible representation space H_i splits as a direct sum of irreducible representation $H_{j,\tau} \approx \tau$ for the action of the maximal compact subgroup $K \approx SO(n)$ of G. Let τ be an irreducible representation of $SO(n)$. Then the multiplicity of τ in H_i is at most one.*

Proof. The first statement is trivial. Denote by $P = MAN$ a minimal parabolic of $G = SO_0(1,n)$. By a Theorem of Harish-Chandra [War72, 5.5.1.7], there is a irreducible representation σ of $M \approx SO(n-1)$, such that the multiplicity of τ in H_i is at most the multiplicity of σ in the restriction of τ to M. This concludes the proof, since the multiplicity of any irreducible representations of $SO(n-1)$ in a given irreducible representation of $SO(n)$ is at most one (cf. [Žel73, §129] and [GC50]).

□

5.3.2. Notation. For each equivalence class of irreducible representations τ of K, let H_τ be the subspace of H of K-type τ i.e. the set of vectors transforming under K according to the irreducible representation τ of K.

The following lemma is an easy consequence of the ellipticity of the rough Laplacian $\nabla^*\nabla$ acting on symmetric tensors. For $S \in \mathcal{C}^\infty(S^pM)$, we set $S^\vee(v) = S(v, v, \dots, v)$.

5.3.3. Lemma. *Let $S \in \mathcal{C}^\infty(S^pM)$ be a symmetric tensor field of rank p and K-type τ. Let $S^\vee = \sum s_i$, $s_i \in L^2(FM)$, be its decomposition in irreducible components. Then we have*

1. *For each s_i there exists a symmetric tensor field $S_i \in \mathcal{C}^\infty(S^pM)$ of rank p and K-type τ with $s_i = S_i^\vee$.*
2. *Each S_i is an eigenfunction of the rough Laplacian $\nabla^*\nabla$ and, in particular, S_i is C^∞.*
3. *The series $S = \sum S_i$, or equivalently $S^\vee = \sum s_i$, converges in the C^∞ topology.*

We will only be interested with symmetric tensor field of rank 2 and therefore with only two K-types: the trivial K-type and the K-type of traceless symmetric tensor of rank 2.

5.3.4. Notation. Set

$$\mathrm{Conf}^r(g) = \{S : S = f \otimes g, f \in C^r(M)\}$$

and

$$\mathrm{Teich}^r(g) := \{S \in C^r(S^2M) : \delta S = 0, \mathrm{Tr}_g\, S = 0\}.$$

Then $\mathrm{Conf}^r(g)$ is the space tangent to C^r-conformal deformation of g and it is the space of vectors of the trivial K-type in $C^r(FM)$. The space $\mathrm{Teich}^r(g)$ is the formal L^2-orthogonal space in $C^r(S^2M)$ to the conformal fibers and to the local orbit of g under the diffeomorphism group. Furthermore, each $S \in \mathrm{Teich}^r(g)$ has the K-type of traceless symmetric tensor of rank 2.

5.3.5. Notation. Let $S \in L^2(S^2M)$. Let $H(S)$ be the cyclic subspace of $L^2(FM)$ generated by G acting on $S^\vee$.

5.3.6. Corollary. *Let $R \in \mathrm{Conf}^\infty(g)$ and write $R^\vee = \sum_i r_i$, with $r_i \in H_i$. Then, for all i, we have $r_i = R_i^\vee$ with $R_i \in \mathrm{Conf}^\infty(g)$.*

Similarly, let $S \in \mathrm{Teich}^\infty(g)$ and write $S^\vee = \sum_i s_i$, with $s_i \in H_i$. Then, for all i, we have $s_i = S_i^\vee$ with $S_i \in \mathrm{Teich}^\infty(g)$.

Proof. The first claim is a mere restatement of Lemma 5.3.3. If $S \in \mathrm{Teich}^\infty(g)$ and $S^\vee = \sum_i s_i$, by Lemma 5.3.3 we have $s_i = S_i^\vee$ with S_i smooth traceless symmetric tensor field of rank 2. The discussion in 5.2.2 implies that the divergence operator on smooth traceless symmetric tensor field of rank 2 coincides with an element of the enveloping algebra of $\mathfrak{g} \approx \mathfrak{so}(1,n)$. Thus we have $\delta s_i \in H_i$ for all i. We conclude that $\delta S = 0$ implies $\delta S_i = 0$ for all i, showing that $S_i \in \mathrm{Teich}^\infty(g)$ for all i.

□

5.3.7. Lemma. *Let $R \in \mathrm{Conf}^\infty(g)$ and $S \in \mathrm{Teich}^\infty(g)$ then $H(R) \perp H(S)$.*

Proof. By Corollary 5.3.6, is sufficient to consider the case in which R and S belong to the same irreducible subspace H_i. Arguing by contradiction, we may assume that $H_i = H(R) = H(S)$. Then $R = r \otimes g$ with $r \in C^\infty(M)$. By the ergodicity of the geodesic flow, we have that $\delta^*\delta^* r = \nabla dr$ is not zero. In fact, for $v \in S_gM$ we have $(\delta^*\delta^* r)^\vee(v) = \nabla^2_{v,v} r = d^2/dt^2 r(g^t v)|_{t=0}$, thus $\delta^*\delta^* r = 0$ would imply that r is constant, in which case there is nothing to prove.

It is not difficult to see from the compactness of M that the traceless part $(\nabla dr)_0 := \nabla dr - \frac{\Delta r}{n} \otimes g$ of ∇dr is not zero. Since S is traceless, the K orbits of $(\nabla dr)_0^\vee$ and $S^\vee$ generate K-representations both isomorphic to the representation of $SO(n)$ on the space of harmonic homogeneous polynomial of degree two on $\mathbb{R}^n$. By Lemma 5.3.1 the multiplicity of this representation in H_i is 1, and therefore $(\nabla dr)_0^\vee$ and $S^\vee$ have the same K-orbit. However, since $(\nabla dr)_0^\vee$ and $S^\vee$ are both K_1-invariant we obtain that, for some constant $C \neq 0$, we have $S^\vee = C(\nabla dr)_0^\vee$ and equivalently $S = C(\nabla dr)_0$. From $\langle S, (\nabla dr)_0\rangle = \langle S, (\delta^*\delta^* r)_0\rangle = \langle S, \delta^*\delta^* r\rangle$, we obtain $\langle S, \delta^*\delta^* r\rangle \neq 0$. This is impossible because we have $\langle S, \delta^*\delta^* r\rangle = \langle \delta S, \delta^* r\rangle$, and the latter term is zero since $S \in \mathrm{Teich}(g)$ and therefore S is divergence free.

□

5.3.8. Remark. A simple modification of the proof above shows that if $S \in \mathrm{Teich}^\infty(g)$, then $H(S)$ is orthogonal to all one forms. This means that if $S \in \mathrm{Teich}^\infty(g)$, then the minimal K-type occurring in $H(S)$ is the K-type of S.

5.3.9. Lemma. *Let S_1, $S_2 \in C^\infty(S^pM)$ and assume that $H(S_1)$ and $H(S_2)$ are orthogonal. Then*

$$\mathrm{Cov}\left(S_1^\vee, S_2^\vee\right) = \mathrm{Cov}\left(\mathcal{T}(S_1)^\vee, S_2^\vee\right) = \mathrm{Cov}\left(\mathcal{T}(S_1)^\vee, \mathcal{T}(S_1)^\vee\right) = 0.$$

Proof. By formula 5.1 the operator $\mathcal{T}$ belongs to the enveloping algebra of G and therefore $\mathcal{T}(S_i) \in H(S_i)$.

Since the flow g^t (lifted to FM) is given by right translation of by some split-Cartan in G, by the exponential decay of matrix coefficients, we have

$$\mathrm{Cov}(F, G) = \int_{-\infty}^{\infty} \langle F, G \circ g^t\rangle\, dt$$

for any pair of smooth functions F and G on FM, provided that either F or G integrates to zero on FM. Thus if F and G belong to orthogonal G-invariant subspaces we have that at least one of them has average zero on FM and $\mathrm{Cov}(F, G) = 0$. The lemma follows.

□

Proof of Proposition 1.3.4. Recall that for simplicity we write g instead of g_0. It is sufficient to prove the claim for a dense set of $S \in C^3(S^2M)$, e.g. for $S \in C^\infty(S^2M)$, since the bilinear form Cov is continuous in the C^α topology. For S in $C^\infty(S^2M)$ we can write

$$S = S_T + S_C + S_0,$$

with $S_T \in \mathrm{Teich}^\infty(g)$, $S_C \in \mathrm{Conf}^{\,\infty}(g)$ and with S_0 belonging to the image under δ^* of the space $\Lambda^1(M)$ of C^∞ one-forms. This decomposition of S is unique, since the sum

$$C^\infty(S^2M) = \mathrm{Teich}^\infty(g) + \mathrm{Conf}^{\,\infty}(g) + \delta^*\big(\Lambda^1(M)\big)$$

is direct (Cf. [Bes87, Lemma 4.57]). By the linearity of the operator $\mathcal{T}$, we can decompose T as $T = \mathcal{T}(S_T) + \mathcal{T}(S_C) + \mathcal{T}(S_0)$. Now, we observe that, in computing $\mathrm{Cov}\,\big(T^\vee, S^\vee\big)$ and $\mathrm{Cov}\,\big(T^\vee, T^\vee\big)$, the terms S_0 and $\mathcal{T}(S_0)$ are irrelevant: in fact the terms $S_0^\vee$ and $\mathcal{T}(S_0)^\vee$ are cohomologous to zero, since we have $S_0 = L_X g$ for some C^∞ vector field X on M. Thus the proof of Proposition 1.3.4 will be complete if we prove it simply for $S \in \mathrm{Teich}^\infty(g) + \mathrm{Conf}^{\,\infty}(g)$.

By Lemma 5.3.9, we have

$$\mathrm{Cov}\,\big(\mathcal{T}(S_C)^\vee, S_T^\vee\big) = \mathrm{Cov}\,\big(\mathcal{T}(S_T)^\vee, S_C^\vee\big) = \mathrm{Cov}\,\big(\mathcal{T}(S_C)^\vee, \mathcal{T}(S_T)^\vee\big) = 0$$

and thus it suffices to prove the claim of Proposition 1.3.4 separately for $S \in \mathrm{Teich}^\infty(g)$ and for $S \in \mathrm{Conf}^\infty(g)$.

Let $S \in \mathrm{Teich}^\infty(g)$. From $\mathcal{T}(S) = -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S + \frac{1}{2}(\mathrm{Tr}_g\, S)g - \frac{1}{2}\delta^*\delta S$ and the definition of $\mathrm{Teich}^\infty(g)$ we obtain

$$\mathcal{T}(S) = -\frac{1}{2}S + \frac{1}{4}\nabla^*\nabla S, \tag{5.2}$$

Regard S as a one-form with values in T^*M. Let d^∇ be the differential induced on $\Lambda^r(M) \otimes T^*(M)$ by the Levi-Civita connection and let d^{∇^*} be its formal adjoint. We have the following Weitzenbock formula (cf. [Bes87, p.335])

$$(d^{\nabla^*}d^\nabla + d^\nabla d^{\nabla^*})S = \nabla^*\nabla S - R^\circ(S) + S \circ \mathrm{Ric}\,.$$

where $R^\circ(S)$ was defined in (4.12). Using the formula (4.14) and the fact that $\mathrm{Ric} = -(n-1)g$, we obtain

$$(d^{\nabla^*}d^\nabla + d^\nabla d^{\nabla^*})S = \nabla^*\nabla S - S - (n-1)S = \nabla^*\nabla S - nS.$$

Since the left hand side above is positive, we conclude that the spectrum of the restriction of the $\nabla^*\nabla$ to $\mathrm{Teich}^\infty(g)$ lies in $[n, \infty)$ and (5.2) implies that the spectrum of the restriction of $\mathcal{T}(S)$ to $\mathrm{Teich}^\infty(g)$ lies in the interval $[(n-2)/4, \infty)$.

By Lemma 5.3.3, we can decompose S as a C^∞ convergent series $S = \sum S_i$ with $S_i^\vee$ belonging to the irreducible component H_i and S_i eigenfunction of $\nabla^*\nabla$ of eigenvalue $\lambda_i \geq (n-2)/4$. Then for $i \neq j$ the spaces $H(S_i)$ and $H(S_j)$ are orthogonal.

Lemma 5.3.9 implies that

$$\mathrm{Cov}\left(S_i^\vee, S_j^\vee\right) = \mathrm{Cov}\left(\mathcal{T}(S_i)^\vee, S_j^\vee\right) = \mathrm{Cov}\left(\mathcal{T}(S_i)^\vee, \mathcal{T}(S_j)^\vee\right) = 0.$$

We conclude that

(5.3)

$$\begin{aligned}\mathrm{Cov}\left(\mathcal{T}(S)^\vee, S^\vee\right) &= \sum_i \mathrm{Cov}\left(\mathcal{T}(S_i)^\vee, S_i^\vee\right) = \sum_i \lambda_i \,\mathrm{Cov}\left(S_i^\vee, S_i^\vee\right) \\ &\geq \sum_i \frac{n-2}{4}\mathrm{Cov}\left(S_i^\vee, S_i^\vee\right) = \frac{n-2}{4}\mathrm{Cov}\left(S^\vee, S^\vee\right)\end{aligned}$$

and, similarly,

(5.4)

$$\begin{aligned}\mathrm{Cov}\left(\mathcal{T}(S)^\vee, \mathcal{T}(S)^\vee\right) &= \sum_i \mathrm{Cov}\left(\mathcal{T}(S_i)^\vee, \mathcal{T}(S_i)^\vee\right) = \sum_i \lambda_i^2 \,\mathrm{Cov}\left(S_i^\vee, S_i^\vee\right) \\ &\geq \sum_i \left(\frac{n-2}{4}\right)^2 \mathrm{Cov}\left(S_i^\vee, S_i^\vee\right) = \left(\frac{n-2}{4}\right)^2 \mathrm{Cov}\left(S^\vee, S^\vee\right),\end{aligned}$$

proving the desired estimate for the case $S \in \mathrm{Teich}^\infty(g)$.

Consider now the case $S \in \mathrm{Conf}^\infty(g)$. We have $\mathrm{Tr}_g\, S = nS$ and therefore $\mathcal{T}(S) = \frac{n-1}{2}S + \frac{1}{4}\nabla^*\nabla S - \frac{1}{2}\delta^*\delta S$. Writing $S = F \otimes g$, with $F \in C^\infty(M)$, we see that the term $\delta^*\delta S$ appearing in $\mathcal{T}(S)$ can be rewritten as δdF and it gives rise in $\mathcal{T}(S)^\vee$ to the term $(\delta^*\delta S)^\vee = (\delta dF)^\vee$, which is cohomologous to zero. Thus, in computing $\mathrm{Cov}\left(\mathcal{T}(S)^\vee, S^\vee\right)$ and $\mathrm{Cov}\left(\mathcal{T}(S)^\vee, T^\vee\right)$ we can replace $\mathcal{T}(S)$ by

$$\mathcal{T}_0(S) := \frac{n-1}{2}S + \frac{1}{4}\nabla^*\nabla S.$$

The positivity of the rough Laplacian $\nabla^*\nabla$ implies that the elliptic operator $\mathcal{T}_0$ has spectrum in $[(n-1)/2, \infty)$. Reasoning as before we obtain , for $S \in \mathrm{Conf}^\infty(g)$, estimates similar to (5.3) and (5.4), but with the constant $(n-1)/2$ replacing the smaller constant $(n-2)/4$. Thus the estimates (5.3) and (5.4) are also valid for $S \in \mathrm{Conf}^\infty(g)$. This concludes our proof □

Proof of Theorem C. Let $n = \dim M = 3$. By Proposition 5.1.1 and formulas (5.1) and (5.2), we see that, for $S \in \mathrm{Teich}^\infty(g)$,

$$\begin{aligned}\mathrm{Ent}''_{\mathrm{Liouv}}(g) &= -\,\mathrm{Cov}\left(T^\vee, T^\vee - S^\vee\right) = \\ &= -\,\mathrm{Cov}\left(S^\vee + \frac{1}{4}\,\mathbf{Cas}_{SO(1,3)}\, S^\vee, \frac{1}{4}\,\mathbf{Cas}_{SO(1,3)}\, S^\vee\right).\end{aligned}$$

Thus assuming that $\mathbf{Cas}_{SO(1,3)}\, S^\vee = \lambda S^\vee$, we obtain $\mathrm{Ent}''_{\mathrm{Liouv}}(g) = -\frac{1}{4}\lambda(1+\frac{1}{4}\lambda)\,\mathrm{Cov}(S^\vee, S^\vee)$. Thus $\mathrm{Ent}''_{\mathrm{Liouv}}(g) < 0$ is equivalent to $\lambda > 0$ or $\lambda < -4$. The latter eventuality is impossible, since we have shown that, in dimension 3, the spectrum of the Laplacian on $\mathrm{Teich}(g)$ is contained in the interval $[3, \infty)$ and $\mathbf{Cas}_{SO(1,3)} = \nabla^*\nabla + \mathbf{Cas}_{SO(3)} = \nabla^*\nabla - 6$. In any case, since the spectrum of $\nabla^*\nabla$ is discrete, we find that on a infinite dimensional subspace of $\mathrm{Teich}(g)$ the derivative $\mathrm{Ent}''_{\mathrm{Liouv}}(g)$ in a curve of metrics in a direction in this subspace is negative.

However if the operator $\mathbf{Cas}_{SO(1,3)}$ on $\mathrm{Teich}^\infty(g)$ has an eigenvector S of eigenvalue $\lambda < 0$, it follows that the Liouville entropy has positive second derivative in the direction of S. Now recall that such an S would generate an irreducible representation of $SO(1,3)$ with minimal K-type given by the K-orbit of S, i.e. with minimal K-type τ_2 (here we have denoted by τ_2 the representation of $K \approx SO(3)$ on the space of traceless symmetric tensors of rank 2). The unitary irreducible representations of $SO(1,3)$ with minimal K-type τ_2 belong to the unitary principal series of $SO(1,3)$ and determined up to unitary equivalence by the value of the Casimir operator on them. From [Tay86] we obtain that the values of the eigenvalues of the Casimir operator, with our normalization, are given by $\mu^2 - 3$, $\mu \in \mathbb{R}$. Thus, since the Plancherel measure has support on all the interval $[-3, \infty)$, by Theorem 5.4 of [DW79], we find a cocompact lattice Γ in $S0(1,3)$, with spectrum of the Casimir on $\mathrm{Teich}^\infty(g)$ in $(-3, 0)$. Then $\Gamma\backslash SO(1,3)/SO(3)$ provides us with the desired counterexample. □

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