

Energy identity for a class of approximate harmonic maps from surfaces

WEIYUE DING AND GANG TIAN¹

1. Introduction.

Consider the heat equation for harmonic maps from a closed Riemannian surface (M, g) into a closed submanifold N in R^K :

$$(1.1) \quad \frac{\partial u}{\partial t} = \tau_g(u),$$

where

$$\tau_g(u) = \Delta_g u + g^{\alpha, \beta} A(u)(u_{, \alpha}, u_{, \beta}),$$

and $A(u)$ is the 2nd fundamental form of N in R^K . Let $u : M \times (0, \infty) \rightarrow N$ be a global weak solution to (1.1). (See [St].) Then u is smooth away from a finite number of singular points $\{(p_i, T_i)\} \subset M \times (0, \infty)$. Let (p, T) be a singular point. Then we may assume that for some $t_0 \in (0, T)$ and a neighborhood U of p , u is smooth on $\bar{U} \times [t_0, T)$ and $u(t) = u(\cdot, t) \rightarrow u(T) = u(\cdot, T)$ smoothly on $\bar{U} \setminus \{p\}$ as t tends to T from left. Moreover, by choosing a suitable sequence $t_i \rightarrow T$ and by rescaling $u(t_i)$ properly near p , one can show that there exist finitely many “bubbles” (i.e. harmonic maps $\varphi_j : S^2 \rightarrow N, 1 \leq j \leq m$,) associated with the sequence $\{u(t_i)\}$. These bubbles are responsible for the loss of energy and we have

$$\lim_{t_i \rightarrow T} E_U(u(t_i)) - E_U(u(T)) \geq \sum_{j=1}^m E(\varphi_j).$$

It is generally believed that the above inequality should be an identity, which we will call the “energy identity”. Since it can be easily shown that the limit

$$\lim_{t \rightarrow T-0} E_U(u(t))$$

¹This work is supported in part by the National Natural Science Fundation of China. The second author is also partailly supported by NSF grant.

exists, we may write the energy identity as

$$(1.2) \quad \lim_{t \rightarrow T-0} E_U(u(t)) = E_U(u(T)) + \sum_{j=1}^m E(\varphi_j).$$

(But note that we do not know if the set $\{\varphi_j\}$ of "bubbles" is unique, i.e. there is some possibility that it may depend on the specific time sequence t_i .) The identity (1.2) means that "blowing bubbles" is the only reason for the energy loss. J. Qing [Q] proved that (1.2) holds in the special case where N is the standard sphere S^n . In this note we will give a simple proof of (1.2) for the general targets N .

If we consider the solution $u(t)$ of (1.1) as maps which in some sense approximate harmonic maps, we must have certain control of the tension field $\tau(u(t))$. We will see that a good control is the boundedness of the L^2 norm, i.e.

$$(1.3) \quad \|\tau(u(t_i))\|_{L^2(M)} \leq C$$

for some sequence $t_i \rightarrow T$. However, from Eq. (1.1) we can only get

$$\int_{t_0}^T \|\tau(u(t))\|_{L^2(M)}^2 dt < \infty.$$

Since $T < \infty$, there is no guarantee for the existence of $t_i \rightarrow T$ such that (1.3) holds. On the other hand, if $T = \infty$ then such a sequence is easy to get. Therefore, our analysis in the following easily leads to the energy identity for blow-ups at time-infinity. More precisely, we have

Theorem 1. *Let u be a global weak solution to (1.1) and $t_i \rightarrow \infty$ be a sequence such that $u(t_i) \rightarrow u_\infty$ weakly in $W^{1,2}(M, R^K)$ and (1.3) holds. Suppose that the convergence $u(t_i) \rightarrow u_\infty$ is not $W^{1,2}$ -strong. Then there exists a subsequence, still denoted by $u(t_i)$, and a finite set of harmonic maps $\varphi_j : S^2 \rightarrow N$ ($j = 1, \dots, m$) such that*

$$\lim_{t_i \rightarrow \infty} E(u(t_i)) = E(u_\infty) + \sum_{j=1}^m E(\varphi_j).$$

Note that the limit on the left does not depend on the specific sequence $u(t_i)$, but u_∞ and the set $\{\varphi_j\}$ may depend on it. Theorem 1 follows easily from the following

Theorem 2. *Let u_i be a sequence of maps in $W^{2,2}(D, N)$ with bounded energy $E(u_i) \leq C$, where D is the unit disk. Assume that (i) u_i converge strongly to u_0 in $W_{loc}^{1,2}(D \setminus \{0\}, N)$, but the convergence is not $W^{1,2}(D, N)$ -strong; and (ii) the L^2 -norms of the tension fields $\tau(u_i)$ are uniformly bounded. Then there exists a subsequence, still denoted by u_i , and a finite set of harmonic maps $\varphi_j : S^2 \rightarrow N$ ($j = 1, \dots, m$) such that*

$$(1.4) \quad \lim_{i \rightarrow \infty} E(u_i) = E(u_0) + \sum_{j=1}^m E(\varphi_j).$$

The derivation of (1.2) from Theorem 2 needs more work. By an argument of Qing (see §2 in [Q]), it can be shown that Theorem 2 implies that the energy identity (1.2) holds true, and hence we have

Theorem 2. *Let u be a global weak solution to (1.1) and $T \in (0, \infty)$. Assume that $u(t) \rightarrow u(T)$ weakly in $W^{1,2}(M, R^K)$, but the convergence is not $W^{1,2}$ -strong. Then there exists a finite set of harmonic maps $\varphi_j : S^2 \rightarrow N$ ($j = 1, \dots, m$) such that*

$$\lim_{t \rightarrow T-0} E(u(t)) = E(u(T)) + \sum_{j=1}^m E(\varphi_j).$$

We should mention that when u_i is a sequence of harmonic maps (i.e. $\tau(u_i) = 0$), Theorem 2 was first proved by J. Jost in [J] (Lemma 4.3.1, p.127). More recently, T. Parker proved in [P] that if the u_i is a sequence of harmonic maps, the limit $u_0 \cup_j \{\varphi_j\}$ maps to a connected set.

Part of this work was done when the second author was at Peking University in the summer of 1994. After we finished the writing of this paper, we learned from F. H. Lin that L. B. Mou and C. Y. Wang also obtained results similar to ours, and their methods are different from ours.

2. Proof of Theorem 2.

Let $u_i \in W^{2,2}(D, N)$ be a sequence of maps which satisfies the conditions of Theorem 2. Before proving the energy identity (1.4) we need some preliminaries. The following estimate is similar to the main estimate in [S-U] and will be important for our analysis.

Lemma 2.1. *Let $u \in W^{2,p}(D, N)$, $1 < p \leq 2$. There exists $\epsilon_0 > 0$ such that if $E(u) < \epsilon_0$ then*

$$(2.1) \quad \|u - \bar{u}\|_{W^{2,p}(D_{1/2})} \leq C[\|\nabla u\|_{L^p(D)} + \|\tau(u)\|_{L^p(D)}],$$

where \bar{u} is the mean value of u over the unit disk D .

Proof. It will be convenient to assume that $\bar{u} = 0$. Then we have the inequality

$$\|u\|_{0,p} \leq C_p \|\nabla u\|_{0,p}.$$

Here and in the following, we use $\|\cdot\|_{k,p}$ to denote the $W^{k,p}(D)$ -norm. Consider u as a solution to the equation

$$\Delta u + A(u)(du, du) = \alpha,$$

where $\alpha = \tau(u) \in L^p(D)$. Assume first that $1 < p < 2$. Let ϕ be a cut-off function which equals 1 on $D_{3/4}$, 0 on ∂D , and $0 \leq \phi \leq 1$ otherwise. From the equation we get that

$$|\Delta(\phi u)| \leq C(|\nabla(\phi u)| |\nabla u| + |\nabla u| + |u| + |\alpha|).$$

By the standard L^p -estimate we have

$$\|\phi u\|_{2,p} \leq C(\|\nabla(\phi u)\|_{0,p} \|\nabla u\|_{0,p} + \|\nabla u\|_{0,p} + \|\alpha\|_{0,p}).$$

But

$$\|\nabla(\phi u)\|_{0,p} \leq \|\nabla(\phi u)\|_{0,q} \|\nabla u\|_{0,2},$$

where $q = 2p/(2-p)$. Since we have the Sobolev's imbedding $W^{2,p} \subset W^{1,q}$, so

$$\|\phi u\|_{2,p} \leq C(E(u))^{1/2} \|\phi u\|_{2,p} + \|\nabla u\|_{0,p} + \|\alpha\|_{0,p}.$$

Thus, if $E(u, D)$ is small, (2.1) follows.

Next, if $p = 2$, one can first derive the above estimate with $p = 4/3$. Such an estimate gives a $L^4(D_{3/4})$ -bound for $|\nabla u|$. Then one can apply the interior L^2 -estimate to the equation for u and get (2.1) with $p = 2$.

Remark. We will only need the case $p = 2$ in Lemma 2.1. But note that even for $1 < p < 2$, the lemma provides the necessary compactness in the blow-up analysis of Sacks-Uhlenbeck, since the imbedding $W^{2,p} \subset W^{1,2}$ is compact. For instance, if $u_i : D \rightarrow N$ is a sequence of $W^{2,p}$ maps with $E(u_i) < \epsilon_0$ and $\tau(u_i) \rightarrow 0$ in $L^p(D)$, then we see that u_i subconverges strongly in $W^{1,2}(D_{1/2})$ to some harmonic map.

Now we want to prove for certain subsequence of u_i , still denoted by u_i , the energy identity

$$(2.2) \quad \lim_{i \rightarrow \infty} E(u_i) = E(u_0) + \sum_{j=1}^m E(\phi_j)$$

where $\varphi_j : S^2 \rightarrow N$ are harmonic spheres obtained by suitable rescaling of the sequence u_i and taking the limits. Since u_i converge strongly in $W^{1,2}(D \setminus D_\delta, N)$ for any $\delta \in (0, 1)$, we see that (2.2) is equivalent to

$$(2.3) \quad \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} E(u_i, D_\delta) = \sum_{j=1}^m E(\varphi_j).$$

For simplicity we assume that the first bubble is obtained by the scaling as follows. For some $0 < \lambda_i \rightarrow 0$ and $x_i \rightarrow 0$, let

$$\tilde{u}_i(x) = u_i(x_i + \lambda_i(x - x_i)).$$

By choosing λ_i properly we may assume that \tilde{u}_i converges strongly in $W^{1,2}(D_R)$, for every $R \geq 1$, to a harmonic map $\psi_1 : R^2 \rightarrow N$ with finite energy. Then $\varphi_1 = \pi^* \psi_1$ is the first harmonic sphere with the same energy as ψ_1 , where $\pi : S^2 \setminus \{N\} \rightarrow R^2$ is the stereoprojection. If for given $\delta > 0$ small and $R > 0$ large, we set

$$A(\delta, R, i) = \{x \in R^2 : \lambda_i R \leq |x - x_i| \leq \delta\},$$

then it is easy to see that (2.3) is equivalent to

$$(2.4) \quad \lim_{R \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{i \rightarrow \infty} E(u_i, A(\delta, R, i)) = \sum_{j=2}^m E(\varphi_j)$$

It will be convenient to make a conformal change of the domain. Let (r_i, θ_i) be the polar coordinates centered at x_i . Let $f : R^1 \times S^1 \rightarrow R^2$ be the mapping given by $r_i = e^{-t}$, $\theta_i = \theta$ for $(t, \theta) \in R^1 \times S^1$. Let $R^1 \times S^1$ be given the product metric $g = dt^2 + d\theta^2$. Then g is conformal to the Euclidean metric ds^2 . In fact, we have

$$(f^{-1})^* g = \frac{1}{r_i^2} ds^2.$$

Fix a large number $R > 0$ and a small number $\delta > 0$, we will be concerned with the energy of u_i on $A(\delta, R, i)$. Let $v_i = f^* u_i$. Then

$$E(u_i, A(\delta, R, i)) = E(v_i, B_i),$$

where

$$B_i = [|\log \delta|, |\log(\lambda_i R)|] \times S^1.$$

We list some important properties of v_i inherited from u_i .

(1) We will consider $v_i : (|\log \delta|, \infty) \times S^1 \rightarrow N$ as a solution to the equation

$$(2.5) \quad \Delta v_i + A(v_i)(dv_i, dv_i) = \alpha_i,$$

where $\alpha_i = \tau(v_i)$. Note that

$$\|\tau(v_i)\|_{L^2}^2 = \int_{D_\delta(x_i)} r_i^2 |\tau(u_i)|^2 dx \leq 2\delta^2 C.$$

Thus, α_i has very small L^2 -norm if only δ is very small. On the other hand, since the L^1 -norm of the tension field is conformally invariant, the L^1 -norm of α_i is also very small.

(2) For any mapping $u \in W^{2,2}(D, N)$, we may define a “quadratic differential” $\Phi(u) = \phi(u)dz^2$ by

$$\phi(u) = |u_x|^2 - |u_y|^2 - 2\sqrt{-1} u_x \cdot u_y.$$

It is well known that if u is harmonic then $\Phi(u)$ is holomorphic, i.e. $\bar{\partial}\phi(u) = 0$. In fact, if $u \in W^{2,p}$ this can be checked directly by differentiation. One finds that

$$\bar{\partial}\phi(u) = (u_x - \sqrt{-1} u_y) \cdot \tau(u).$$

So for our maps u_i we have

$$|\bar{\partial}\phi(u_i)| \leq |\nabla u_i| |\tau(u_i)|.$$

It follows that

$$(2.6) \quad \|\bar{\partial}\phi(u_i)\|_{L^1} \leq E(u_i)^{1/2} \|\tau(u_i)\|_{L^2} \leq C.$$

Writing $\phi(u_i) = \phi_i$, we have

$$2\pi\rho\sqrt{-1}\phi_i(z) = \int_{\partial D_\rho} \frac{\phi_i(\theta)}{\theta - z} d\theta - \int_{D_\rho} \frac{\bar{\partial}\phi_i(\xi)}{\xi - z} d\xi,$$

where ρ can be any number in $(0, 1]$. We choose $\rho = 3/4$. Note that by assumption, $u_i \rightarrow u_0$ strongly in $W^{1,2}$. Hence the energy of u_i is locally small in a neighborhood U of ∂D_ρ , and we may apply Lemma 2.1 to conclude that u_i have uniformly bounded $W^{2,2}$ -norms on U . Thus, by the imbedding theorem for trace spaces we see that $|\phi_i|$ are uniformly bounded in $L^1(\partial D_\rho)$.

It is then easy to see that the first integral in the above identity is small in $L^1(D_\delta)$ if δ is small. For the $L^1(D_\delta)$ norm of the 2nd integral we derive that

$$\begin{aligned} \int_{D \times D_\delta} \frac{|\bar{\partial}\phi_i|}{|\xi - z|} d\xi dz &= \int_D d\xi |\bar{\partial}\phi_i| \int_{D_\delta} \frac{dz}{|\xi - z|} \\ &\leq C\delta \int_D |\bar{\partial}\phi_i| d\xi \leq C\delta, \end{aligned}$$

by (2.6). This shows that the $L^1(D_\delta)$ -norm of $|\phi(u_i)|$ is small if δ is small. Since the L^1 -norm of the quadratic differential is conformally invariant, we conclude that

$$\phi(v_i) = |(v_i)_t|^2 - |(v_i)_\theta|^2 - 2\sqrt{-1}(v_i)_t \cdot (v_i)_\theta$$

has small L^1 -norm provided δ is small.

For simplicity, we will first prove the energy identity in the case of $m = 1$, i.e. φ_1 is the only bubble. The general case can be treated similarly, as will be explained at the end of this section. Since $m = 1$, we need only to prove that $E(v_i, B_i)$ can be arbitrarily small if δ is sufficiently small and R and i is large enough. Given any small $\epsilon > 0$, we may assume by the above results, δ is so small that

$$(2.7) \quad \|\tau(v_i)\|_{L^1}, \|\tau(v_i)\|_{L^2} \text{ and } \|\phi(v_i)\|_{L^1} < \epsilon.$$

Let $T_0 = |\log \delta|$. Then since $v_i \rightarrow v_0 = f^*u_0$ $W^{1,2}$ -strongly on $P_M = [T_0, T_0 + M] \times S^1$ as $i \rightarrow \infty$ for any fixed $M > 0$, we have

$$E(v_i, P_M) \rightarrow E(u_0, D_\delta \setminus D_{\delta e^{-M}}), \quad i \rightarrow \infty.$$

We may assume that $E(u_0, D_\delta) < \epsilon/2$ for small δ . It follows that for any $M > 0$ there is $i(M)$ such that

$$(2.8) \quad E(v_i, P_M) \leq \epsilon/2 \text{ if } i \geq i(M).$$

Next, let $T_i = |\log(\lambda_i R)|$ and $Q_{M,i} = [T_i - M, T_i] \times S^1$. Similar arguments show that we may take R and $i(M)$ large so that

$$(2.9) \quad E(v_i, Q_{M,i}) \leq \epsilon/2 \text{ if } i \geq i(M).$$

Now we claim that there exists $I > 0$ such that if $i \geq I$ then

$$(2.10) \quad \int_{[t, t+1] \times S^1} |\nabla v_i|^2 < \epsilon \text{ for } t \in [T_0, T_i - 1].$$

Suppose that the claim is false, then we may assume that as $i \rightarrow \infty$, there exist t_i such that

$$\int_{[t_i, t_i+1] \times S^1} |\nabla v_i|^2 \geq \epsilon.$$

In view of (2.8)-(2.9), we must have $t_i - T_0 \rightarrow \infty$ and $T_i - t_i \rightarrow \infty$. Thus, by translation $t \rightarrow t - t_i$ we may consider v_i as a map on $[-M_i, M_i] \times S^1$ with $M_i \rightarrow \infty$ and

$$(2.11) \quad \int_{[0,1] \times S^1} |\nabla v_i|^2 \geq \epsilon.$$

We may assume that $v_i \rightarrow v_\infty$ weakly in $W_{loc}^{1,2}(R^1 \times S^1, N)$. Moreover, by the definition of v_i and the assumption that $\tau(u_i)$ has bounded L^2 -norm, it is easy to see that

$$(2.12) \quad |\tau(v_i)| \rightarrow 0 \text{ in } L_{loc}^2(R^1 \times S^1).$$

Now, if the convergence $v_i \rightarrow v_\infty$ is $W^{1,2}$ -strong on $[0, 1] \times S^1$, then (2.11) and (2.12) imply that v_∞ is a nonconstant harmonic map with finite energy. Since $R^1 \times S^1$ is conformal to $S^2 \setminus \{N, S\}$, v_∞ gives rise to a nonconstant harmonic sphere $\phi_2 : S^2 \rightarrow N$, a contradiction to the assumption that $m = 1$. Next, if the convergence is not strong, then by Lemma 2.1 there exists some $p \in [0, 1] \times S^1$ and energy concentration near the point p , namely along some subsequence we have

$$\lim_{i \rightarrow \infty} E(v_i, D_r(p)) \geq \epsilon_0 > 0,$$

for any $r > 0$. In such a case we can still obtain a second harmonic sphere φ_2 by the rescaling argument. (Note that such an argument can go through because we have Lemma 2.1.) This shows that the claimed (2.10) must be true.

Now we remark that (2.7), (2.10) and the estimate (2.1) in Lemma 2.1, together with the imbedding $W^{2,2} \subset C^\gamma$ ($\gamma \in (0, 1)$) implies that

$$(2.13) \quad \|v_i - \bar{v}_{i,t}\|_{C^\gamma([t,t+1] \times S^1, N)} \leq C\epsilon^{1/2},$$

for each $t \in [T_0, T_i - 1]$, at least for i large enough. Here $\bar{v}_{i,t}$ is the mean value of v_i over $[t, t + 1] \times S^1$, and C may depend on γ .

To estimate the energy of v_i on $[T_0, T_i] \times S^1$ we employ the technique used by Sacks and Uhlenbeck in their proof of removable singularity theorem. Let n be an integer such that $d = (T_i - T_0)/n \leq 1$, and let $t_k = T_0 + kd$

for $k = 0, 1, \dots, n$. We will simply write $v = v_i$, $P = [T_0, T_i] \times S^1$, $P_k = [t_k, t_{k+1}] \times S^1$, and $S_k = \{t_k\} \times S^1$. Let $h(t)$ be an R^k valued piecewise linear function $h(t)$ such that $h(t_i)$ equals the mean value of u over S_i . Consider h as a map from R into R^K . Then we have

$$\Delta(v - h) + A = \alpha,$$

where $A = A(v)(dv, dv)$. Taking the inner product of this equation with $v - h$ and integrating over R_k we get

$$\int_{P_k} |\nabla(v - h)|^2 = \int_{P_k} (v - h) \cdot (A - \alpha) + \int_{S_{k+1}} - \int_{S_k} (v - h) \cdot (v_t - h_t) d\theta.$$

Note that the boundary integrals of $(v - h) \cdot h_t$ vanish, also that $|v - h| \leq C\sqrt{\epsilon}$ by (2.13). So we have

$$\begin{aligned} \int_{P_k} |\nabla(v - h)|^2 &\leq C\epsilon^{1/2} \int_{P_k} |\nabla v|^2 \\ &\quad + C\epsilon^{1/2} \int_{P_k} |\alpha| + \int_{S_{k+1}} - \int_{S_k} (v - h) \cdot v_t d\theta. \end{aligned}$$

Summing the inequality over i we get

$$\begin{aligned} \int_P |\nabla(v - h)|^2 &\leq C\epsilon^{1/2} \left[\int_P |\nabla v|^2 + \int_P |\alpha| \right] \\ &\quad + \int_{S_0} (v - h) \cdot v_t - \int_{S_n} (v - h) \cdot v_t \\ &\leq C\epsilon^{1/2} (E(v) + C\epsilon). \end{aligned}$$

Here we have used (2.7) and the fact that the integrals of $|v_t|$ over S_0 and S_n are dominated by the $W^{2,2}$ -norm of v on a neighborhood of the two circles, and the later is small by Lemma 2.1 (noting that v has small local energy). Since h is independent of θ , the above estimate implies that

$$\int_P |v_\theta|^2 \leq C\epsilon^{1/2} E(v) + C\epsilon.$$

But it is clear that $|v_t|^2 \leq |\phi(v)| + |v_\theta|^2$, so we get

$$\int_P |v_t|^2 \leq \int_P |\phi(v)| + C\epsilon^{1/2} E(v) + C\epsilon \leq +C\epsilon^{1/2} E(v) + C\epsilon.$$

Combining the above two inequalities we finally obtain

$$E(v) \leq (1 - 2C\epsilon^{1/2})^{-1} C\epsilon \leq C\epsilon.$$

This proves the energy identity for the case of $m = 1$.

For the general case that $m > 1$, the essential part of the proof is completely same as the above proof for $m = 1$. One just has to distinguish more “bubble domains” and “neck domains”, and show that the energy on the neck domains is arbitrarily small. The proof can be completed by induction in m , the number of bubbles. For simplicity, we just indicate how to reduce the proof of $m = 2$ to that of $m = 1$.

If $m = 2$, the inequality (2.10) no longer holds, and after translations we must have (2.11). Then we have two possibilities: (i) the convergence $v_i \rightarrow v_\infty$ is $W^{1,2}$ -strong and v_∞ gives rise to a nonconstant harmonic map $\varphi_2 : S^2 \rightarrow N$; (ii) energy of v_i concentrates near some point $p \in [0, 1] \times S^1$, and after suitable rescalings we get the second bubble φ_2 . Letting $t_i - T_0 = A_i$ and $T_i - t_i = B_i$, we need to show that the energy of v_i on the cylinder

$$[-A_i, B_i] \times S^1$$

can be very close to $E(\varphi_2)$ in either case. Note that we have A_i and $B_i \rightarrow \infty$ as $i \rightarrow \infty$.

In case (i), we see that since $m = 2$ there is no local energy concentration, hence $v_i \rightarrow v_\infty$ in $W_{loc}^{1,2}(R^1 \times S^1, N)$. In other words, we have

$$\lim_{M \rightarrow \infty} \lim_{i \rightarrow \infty} E(v_i, [-M, M] \times S^1) = E(\varphi_2).$$

On the other hand, on the two cylinders $[-A_i, M] \times S^1$ and $[M, B_i] \times S^1$, one can show using the same method as in the proof for $m = 1$ that the energy of v_i is small provided we choose δ small and i, R , and M large. This shows the energy identity holds.

In case (ii), we may assume that $p = (0, 1)$. Then since $m = 2$, there is only one bubble in $(-1, 1) \times S^1$ for the sequence v_i . It follows from the result for $m = 1$ that

$$\lim_{i \rightarrow \infty} E(v_i, (-1, 1) \times S^1) = E(\varphi_2),$$

while the energy of v_i on the two cylinders $(-A_i, -1) \times S^1$ and $(1, B_i) \times S^1$ can be estimated as before. Thus, we obtain the energy identity in this case too. This completes our proof of Theorem 2.

3. Two Remarks.

It is likely that our proof of Theorem 2 can be generalized to certain singular targets, such as Alexandrov spaces with curvature bounded from above and algebraic varieties. We hope to treat this problem in the future.

In the case $\tau(u_i) \equiv 0$, i.e. u_i is harmonic, by an asymptotic analysis (cf. [S], [CT]), we can show that the neck connecting the bubbles and u_0 in Theorem 2 converges exponentially to a geodesic in N . The key idea of the proof is as follows: On the neck cylinder $[-A, B] \times S^1$, the gradient of u_i is pointwisely small. Therefore, one can rewrite the harmonicity equation as a perturbation of the Laplacian equation on $[-A, B] \times S^1$. The unperturbed linear system has two special kinds of solutions, namely constant and linear functions, both independent of the variable $\theta \in S^1$. These constant and linear solutions are respectively the infinitesimal form of constant maps and maps with geodesic image. Any solution perpendicular to those special ones either decays or grows exponentially. It is natural to expect that such behaviours are preserved for solutions of the perturbed equation. By an argument in [CT], one can then prove that connecting neck converges exponentially to a geodesic.

One can also apply the same arguments to harmonic maps from Riemann surfaces with varying conformal structure. Though energy may lose from necks connecting bubbles and the limiting map, one can show that those connecting necks converge exponentially to geodesics. Simple examples show that this is the best one can get in the general case of harmonic maps.

We expect that similar result holds true for solutions of the heat equation (1.1). Namely, if $u(t_i)$ are given as in Theorem 1, then the connecting neck converges to a geodesic in N .

References.

- [CT] Cheeger, J. and Tian, G. *The cone structure of complete Ricci-flat manifolds with quadratic curvature decay*, Invent. Math. 1994.
- [J] Jost, J. *Two-Dimensional Geometric Variational Problems*, John Wiley & Sons, Chichester, 1991.
- [P] Parker, T. *Bubble tree convergence for harmonic maps*, preprint.
- [Q] Qing, J. *On singularities of the heat flow for harmonic maps from surfaces into spheres*, Comm. in Analysis and Geometry, **3** (1994), 297-317.
- [S] Simon, L. *Asymptotics for a class of nonlinear evolution equations with applications to geometric problems*, Ann. of Math. **118** (1983), 525-571.
- [S-U] Sacks, J. and Uhlenbeck, K. *The existence of minimal immersions of two spheres*, Ann. Math. **113** (1981), 1-24.

- [St] Struwe, M. *On the evolution of harmonic maps of Riemannian surfaces*,
Comment. Math. Helv. **60** (1985), 558-581.

RECEIVED JUNE 12TH, 1995.

INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
BEIJING

AND

COURANT INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY