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On Compactness of Isospectral Conformal Metrics of 4-Sphere

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Let (S^4, g_0) be 4-sphere with standard metric g_0. Let $g = u^2 g_0$ with $u > 0$ be a conformal metric. We show that if the scalar curvature function of g is near constant in the L^2 sence, then the set of conformal metrics isospectral to g, when suitably centered, is compact in the C^∞ topology.

1. Introduction.

This paper is the study of the compactness of conformal isospectral metrics on the standard 4-sphere.

Two Riemannian metrics g, g' on a compact manifold are said to be *isospectral* if the associated Laplace operators, on functions, have the same spectrum. If, in addition, g and g' are also pointwise conformal to each other, they will be said to be *isospectral conformal metrics.*

It is a well known problem to study the extent to which the spectrum determine the metric. Compactness property is one of them we should study. Much evidence indicates that this is a complicated question except for a compact Riemann surface. When the underlying manifold is a compact Riemann surface without boundary, Osgood, Phillips and Sarnak ([OPS]) have been able to prove that the set of isospectral metrics on a compact Riemann surface is a compact set in the C^∞ topology.

They used Wolpert's criterion ([W]) for compactness of the conformal structures in the Teichmüller space in terms of the determinant of the Laplace operator in the following way. It reduces the problem to studying the isospectral conformal metrics on a fixed Riemann surface.

For manifolds of dimension greater than two, we do not have such a nice criterion available. But let us assume we already have one. We would like to see if the subset which consists of isospectral conformal metrics is C^∞ compact which seems get us done on the ground level.

Naturally we would like to know whether there are any such nontrivial pairs which make our question meaningful. There exist many pairs, even many continuous families of metrics, such that they are isospectral and conformal to each other but not isometric. See, e.g. ([BPY],[BG]).

Along this line, when $n = 3$, the problem has been answered completely by Brooks-Perry-Yang ([BPY]) and Chang-Yang ([CY1, CY2]). We should mention that in dimension three if one assumes that the length of the shortest closed geodesic for each metric in the set of isospectral metrics has a uniformly lower bound, then the set will be compact in C^∞ topology. This has been proved by Anderson ([An]).

We will consider dimension four. Since the conformal Laplace and Dirac operators have certain invariant property among conformal class, the log determinant of those operators is available on certain manifolds of dimension four. Thus we can employ the same argument as in Riemann surface case to get H_1 compactness instead of C^∞ compactness on those manifolds. This has been done by Branson, Chang and Yang ([BCY]). As long as consider H_1 compactness, Gursky has done some H_1 compactness for higher dimensional manifolds with an extra assumption that the L^p norm of the full curvature tensor is bounded with $p > \frac{n}{2}$ ([Gu]). We also get some related results ([X1, X2]).

We are concerned about C^∞ compactness. In this note, we will mainly consider the standard 4-sphere.

As far as the standard sphere is concerned, we have to do something with the conformal transformation group since it is noncompact. In order to deal with this noncompactness, we first give the following

Definition. For a positive function u on S^4 and φ a conformal transformation, define

$$u_\varphi = (u \circ \varphi)|d\varphi|^{1/2}$$

where $|d\varphi|$ is the linear stretching factor of $d\varphi$ measured with respect to the standard metric g_0.

It is clear from the definition that $u_\varphi^2 g_0 = \varphi^*(u^2 g_0)$. For ease of presentation of our results, we denote all functions related in this way to a given function u by $[u]$, i.e.,

$$[u] = \left\{u_\varphi | \varphi \in G, \text{the conformal group of } S^4\right\}.$$

As can be easily seen , the noncompactness of conformal group of S^4 implies that the class $[u]$ is noncompact in $H_2\left(S^4\right)$, although the metrics $\{g = v^2 g_0 | v \in [u]\}$ are all isometric. If $u \equiv 1$, they all have constant scalar curvature.

Thus when we talk about compactness of metrics on the sphere, we have to modulo its conformal group.

First of all, we observe that if one of metrics in a sequence $\{g_i = u_i^2 g_0\}$ has constant scalar curvature, then the rest of them all have constant scalar curvature with same constant. This implies that, up to conformal group, the isospectral set of conformal metrics on S^4 is a point if one of those metrics has constant scalar curvature. Of course, it is compact in the C^∞ topology. Details about this observation will be given in next section.

Whenever you have such an observation, it is natural to ask yourself that what will happen if the metric is near the constant scalar curvature in certain sense. This motivates us to reach the following

Definition. A conformal metric $g = u^2 g_0$ is said to be an $\mu-$ constant scalar curvature metric in the L^2 sense with $\mu \geq 0$ if

$$\int_{S^4} \left\{ s_g - \left(\int_{S^4} u^4 dv_0 \right)^{-1} \int_{S^4} s_g u^4 dv_0 \right\}^2 u^4 dv_0 = \mu^2 vol\left(S^4\right).$$

Note that $0-$ constant scalar curvature metrics in the L^2 sense are constant scalar metrics. Thus we expect that the set of conformal metrics isospectral to an $\mu-$ constant scalar curvature metrics should be compact in C^∞ topology if μ is small enough. That turns out to be the main result we are going to prove in this note. We also give a precise upper bound for μ.

Theorem (Main Theorem). *Let $g_1 = u^2 g_0$ be an $\mu-$ constant scalar curvature metric on (S^4, g_0). If $\{g_i = u_i^2 g_0\}$ is a sequence of conformal metrics isospectral to g_1 and $\mu \leq \frac{11}{26}$, there exist a subsequence g_j and a conformal transformation φ_j such that $\{\varphi^*(g_j)\}$ converges in the C^∞ topology to a metric g which is also isospectral to g_1.*

The first thought about this theorem is if the conformal metrics with the small $\mu-$ constant scalar curvature are near the standard metric g_0 in the C^0 sense up to some action by the conformal group. If this is the case, then our result will follow easily. But we will show that this is not the case. We can construct a large set of conformal metrics on S^4 with small $\mu-$ constant scalar curvature which are not C^0 close to the standard metric. We give the details in the appendix.

The plan for the proof of our main result in this paper is the following: first we show that the conformal metrics which are isospectral to an $\mu-$ constant scalar curvature metric are all $\mu-$ constant scalar curvature metrics.

In other words, this condition is a spectral invariant among conformal class. This can be done by using the third heat invariant and topological invariant. Details will be given in section two together with some preliminaries.

In section 3, we combine the above with the fourth heat invariant and prove that the L^4 norm of the curvature tensor is bounded.

Once we have the L^4 bound on the full curvature tensor, specially, on scalar curvature, we are in position to get pointwise control on the conformal factors. Here, of course, we need the lower bound of the first nonzero eigenvalues of metrics as well as the full conformal group of S^4. This is the C^0 version of the main theorem. It involves the main estimates we need to deal with and is also a place where we have to handle the conformal group. The details appear in section four.

The leading coefficients of the higher order heat invariants can be applied to get the bounds

$$\int_{S^4} \left|\nabla^k R\right|^2 dv_g \leq C(k)$$

for some constants $C(k)$ which depend on k, the heat invariants a_i and the geometric data of the standard metric g_0. The argument is similar to the one used in the three dimensional case, but we do need extra care. Once we have these bounds, it is not hard to get the bounds we want. We will carry out the details in section five.

2. Preliminaries.

For the materials in the first part of this section, one can consult reference [BPP].

The basic information we can get from the spectrum data is the heat kernel $H_t(x,y)$. The heat kernel is the fundamental solution of the heat equation

$$\left(\frac{\partial}{\partial t} + \Delta\right) u = 0.$$

It is well-known that $H_t(x,y)$ has an eigenfunction expansion:

$$H_t(x,y) = \sum_i e^{-\lambda_i t} \varphi_i(x)\varphi_i(y).$$

where the $\{\varphi_i(x)\}$ are an orthonormal basis of the eigenfunctions of Δ with eigenvalues λ_i.

It then follows that the trace of H_t is given by

$$tr(H_t) = \int_M H_t(x,x)dv_g = \sum_i e^{-\lambda_i t}.$$

As one might expect on physical grounds, the dominant factor in $H_t(x,x)$ for small time t is the local geometry of M at x. Precisely we have

Theorem 2.1([BGM]). *As $t \to 0^+$, $tr(H_t)$ has the following asymptotic expansion:*

$$tr(H_t) \sim \frac{1}{(4\pi t)^{n/2}}\left\{a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots\right\},$$

where $a_0, a_1, a_2, \cdots$ are integrals over M of local invariants of the geometry of M.

The point here is that $tr(H_t)$ is visibly a spectral invariant of M. Hence so are the a_i's.

For low values of i, one may compute the a_i's as follows:

$$a_0 = \int_M dv_g = vol(M);$$

$$a_1 = \frac{1}{6}\int_M s_g dv_g;$$

$$a_2 = \frac{1}{360}\int_M \left[5s_g^2 - 2\,|Ric|^2 + 2\,|R|^2\right] dv_g;$$

$$\begin{aligned} a_3 = \frac{1}{6!}\int_M \Big\{ & -\frac{142}{63}|\nabla s_g|^2 - \frac{26}{63}|\nabla Ric|^2 \\ & -\frac{1}{9}|\nabla R|^2 + \frac{5}{9}s_g^3 - \frac{2}{3}s_g|Ric|^2 + \frac{2}{3}|R|^2 s_g \\ & -\frac{4}{7}Ric_{ij}Ric_{jk}Ric_{ki} + \frac{20}{63}Ric_{ij}Ric_{kl}R_{ikjl} \\ & -\frac{8}{63}Ric_{ij}R_{ikln}R_{jkln} + \frac{8}{21}R_{ijkl}R_{ijmn}R_{mnkl}\Big\} dv_g \end{aligned}$$

where s_g, Ric, R and dv_g are the scalar curvature, Ricci curvature tensor, full Riemannian curvature tensor and the volume element associated to given metric g respectively. The formula for a_3 was computed by Sakai ([S]) and Gilkey ([G1]) verified it. The name, Sakai-Gilkey's formula, will be used in this paper.

Since we are going to work with conformal classes, it is convenient for us to reduce the Riemann curvature tensor R_{ijkl} into its component parts,

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(B_{ik}g_{jl} - B_{il}g_{jk} + B_{jl}g_{ik} - B_{jk}g_{il}) + \frac{s_g}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

where W_{ijkl} and B_{ij} are the Weyl tensor and the traceless Ricci tensor respectively.

As usual,

$$R_{ij} = g^{lk}R_{iljk};$$

$$B_{ij} = R_{ij} - \frac{s_g}{n}g_{ij};$$

$$s_g = g^{ij}R_{ij}.$$

In terms of W, B and s_g, the low heat invariants have expansions:

Theorem 2.2. *If $n = 4$, then*

$$a_2 = \frac{1}{180}\int_M \left[|W|^2 + |B|^2 + \frac{29}{12}s_g^2\right] dv_g;$$

$$\begin{aligned} a_3 = \frac{1}{7!}\int_M \Big\{ & -\frac{7}{3}|\nabla W|^2 - \frac{4}{3}|\nabla B|^2 - \frac{152}{9}|\nabla s_g|^2 \\ & + 5s_g|W|^2 + \frac{50}{9}s_g|B|^2 + \frac{555}{162}s_g^3 \\ & + \frac{38}{9}W_{ijkl}W_{klmn}W_{mnij} + 12B_{ij}W_{iklm}W_{jklm} \\ & + \frac{40}{3}B_{ij}B_{kl}W_{ikjl} + \frac{56}{9}W_{ijkl}W_{iukv}W_{julv}\Big\} dv_g. \end{aligned}$$

Proof. The formula for a_2 is well-known. The expression for a_3 seems also known to everyone in this community. But we cannot find a reference for it. The argument to get this is quite simple. We can start with Sakai-Gilkey's formula, using the computation done by Parker and Rosenberg [PR] for the relations between curvature tensors. We will easily reach our claim once we correct the minor misprints in their paper. Namely, on page 220 of [PR], the coefficient of B_{13} in the formula for A_{16} should read as $\frac{8(n-4)}{(n-2)^3}$ instead of

$\frac{8(n-4)}{(n-2)^2}$. Also on the bottom of page 219 of [PR], the expressions for C_6 and C_7 should read:

$$C_6 = -\frac{(n-2)^2}{4n^2}B_2 + B_4 + \frac{1}{n-1}B_{11} + \frac{n}{n-2}B_{13} - B_{14},$$

$$C_7 = -\frac{(n-2)(n-3)}{4n^2(n-1)}B_2 + \frac{n-3}{n-2}(B_3 - B_4)$$
$$- \frac{1}{4}B_5 - \frac{1}{2n}B_{12} - \frac{1}{2}B_{15} + \frac{1}{4}B_{16} + B_{17}.$$

We are interested in conformal geometry. When the underlying manifold is S^4, the standard unit sphere in R^5, we know that the Weyl tensor vanishes identically. In this simple case, we then have

Theorem 2.3. *If $g = u^2 g_0$ is a conformal metric on S^4, then*

$$a_0 = \int_{S^4} u^4 dv_0;$$

$$a_1 = \int_{S^4} s_g u64 dv_0;$$

$$a_2 = \frac{1}{180}\int_{S^4}\left[|B|_g^2 + \frac{29}{12}s_g^2\right]u^4 dv_0;$$

$$7!a_3 = \int_{S^4}\left[-\frac{4}{3}|\nabla B|_g^2 - \frac{152}{9}|\nabla s_g|_g^2 + \frac{50}{9}s_g|B|_g^2 + \frac{185}{54}s_g^3\right]u^4 dv_0.$$

Proof. When $g = u^2 g_0$, we have

$$W_{ijkl}(g) = u^2 W_{ijkl}(g_0) \equiv 0;$$

$$B_{ij}(g) = -2\left[\frac{u_{ij}}{u} - 2\frac{u_i u_j}{u^2} - \frac{1}{4}\left(\frac{\Delta u}{u} - 2\frac{|\nabla u|^2}{u^2}\right)g_{0ij})\right];$$

$$s_g = u^{-3}(s_0 u - 6\Delta u);$$

$$dv_g = u^4 dv_0.$$

Here we have used Δ to indicate the Laplace operator associated to the standard metric on S^4. These formulas can be found in Besse's book ([B]). From those relations, it is not hard to see that Theorem 2.3 is true.

Theorem 2.3 has a useful application.

Corollary 2.4. *On (S^4, g_0), if $g_1 = u^2 g_0, g_2 = v^2 g_0$ and g_1 is isospectral to g_2, then*

$$\int_{S^4} s_{g_1}^2 u^4 dv_0 = \int_{s^4} s_{g_2}^2 v^4 dv_0.$$

Proof. Since g_1 is isospectral to g_2, we have $a_2(g_1) = a_2(g_2)$. From Theorem 2.3, namely, we have

$$\int_{S^4} \left[|B_{g_1}|_{g_1}^2 + \frac{29}{12} s_{g_1}^2 \right] u^4 dv_0 = \int_{S^4} \left[|B_{g_2}|_{g_2}^2 + \frac{29}{12} s_{g_2}^2 \right] v^4 dv_0.$$

But the Gauss-Bonnet formula ([B],[PR]) for four dimensional closed manifolds tells us:

$$\int_{S^4} \left[s_{g_1}^2 - 12 |B_{g_1}|_{g_1}^2 \right] u^4 dv_0 = \int_{S^4} \left[s_{g_2}^2 - 12 |B_{g_2}|_{g_2}^2 \right] v^4 dv_0.$$

Combining these two identities, it is not hard to see that our claim holds.

A easy consequence of this corollary is this: if one of the metrics has constant scalar curvature and the metrics are conformal and isospectral to each other, then the other has constant scalar curvature too. By well-known theorem of Obata, up to conformal transformation on S^4, those two metrics are isometric. It says that the set of conformal metrics isospectral to a constant scalar curvature metric on S^4, modulo the action of the conformal group, is one point. This is also one motivation for our main theorem.

Another direct consequence of Theorem 2.3 and Corollary 2.4 is the following:

Theorem 2.5. *On S^4, if we define*

$$a_2'(g) = \int_{S^4} s_g^2 u^4 dv_0 - \left(\int_{S^4} u^4 dv_0 \right)^{-1} \left[\int_{S^4} s_g u^4 dv_0 \right]^2,$$

then a_2' is a spectral invariant.

Before we end this section, we would like to recall the Best Sobolev inequality on S^4 which will be used repeatly in what follows.

Theorem 2.6 ([Au],[Bec]). *On (S^4, g_0), for any function $f \in H_1^2(S^4)$, we have*

$$vol^{1/2}(S^4) \left(\int_{S^4} |f|^4 dv_0 \right)^{1/2} \leq \frac{1}{2} \int_{S^4} |\nabla f|^2 dv_0 + \int_{S^4} f^2 dv_0.$$

3. L^4 bounds of curvature tensor.

The main aim of this section is to get the L^4 norm bound of the curvature tensor. It will be done in several Lemmas. Let us start with

Lemma 3.1.

$$vol^{1/2}\left(S^4\right)\left(\int_{S^4} s_g^4 u^4 dv_0\right)^{1/2} \leq \frac{1}{2}\int_{S^4} |\nabla s_g|_g^2 u^4 dv_0 + \frac{1}{12}\int_{S^4} s_g^3 u^4 dv_0$$

Proof. Applying Theorem 2.6 to the function $f = s_g u$, we have

$$\begin{aligned} vol^{1/2}\left(S^4\right)\left(\int_{S^4} (s_g u)^4 dv_0\right)^{1/2} &\leq \frac{1}{2}\int_{S^4} |\nabla (s_g u)|_{g_0}^2 dv_0 + \int_{S^4} (s_g u)^2 dv_0 \\ &= \frac{1}{2}\int_{S^4} |\nabla s_g|_{g_0}^2 u^2 dv_0 + \int_{S^4} s_g u \nabla_{g_0} s_g \cdot \nabla_{g_0} u dv_0 \\ &\quad + \frac{1}{2}\int_{S^4} s_g^2 |\nabla u|_{g_0}^2 dv_0 + \int_{S^4} s_g^2 u^2 dv_0 \\ &= \frac{1}{2}\int_{S^4} |\nabla s_g|_g^2 u^4 dv_0 + \frac{1}{2}\int_{S^4} s_g^2 \frac{s_g u^3 - s_0 u}{6} u dv_0 + \int_{S^4} s_g^2 u^2 dv_0. \end{aligned}$$

Since $s_0 = 12$ for our case, we can simplify above inequality to get the desired conclusion.

Lemma 3.2.

$$vol^{1/2}\left(S^4\right)\left(\int_{S^4} |B|_g^4 u^4 dv_0\right)^{1/2} \leq \frac{1}{2}\int_{S^4} |\nabla B|_g^2 u^4 dv_0 + \frac{1}{12}\int_{S^4} s_g |B|_g^2 u^4 dv_0.$$

Proof. As before, the Best Sobolev inequality (Theorem 2.6) can be applied to function $|B|_g u$ to get

$$\begin{aligned} vol^{1/2}\left(S^4\right)\left(\int_{S^4} |B|_g^4 u^4 dv_0\right)^{1/2} &\leq \frac{1}{2}\int_{S^4} \left|\nabla\left(|B|_g u\right)\right|^2 dv_0 + \int_{S^4} |B|_g^2 u^2 dv_0 \\ &= \frac{1}{2}\int_{S^4} |\nabla |B|_g|^2 u^2 dv_0 - \frac{1}{4}\int_{S^4} |B|_g^2 \Delta u^2 dv_0 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{S^4}|B|_g^2|\nabla u|^2dv_0+\int_{S^4}|B|_g^2u^2dv_0\\
=&\frac{1}{2}\int_{S^4}|\nabla|B|_g|_g^2u^4dv_0+\frac{1}{2}\int_{S^4}|B|_g^2\frac{s_gu^3-s_0u}{6}udv_0\\
&+\int_{S^4}|B|_g^2u^2dv_0\\
\le&\frac{1}{2}\int_{S^4}|\nabla B|_g^2u^4dv_0+\frac{1}{12}\int_{S^4}|B|_g^2s_gu^4dv_0.
\end{aligned}
$$

Lemma 3.3. $\left|\int_{S^4}s_g|B|_g^2u^4dv_0\right|\le\frac{11}{26}vol^{1/2}\left(S^4\right)\left(\int_{S^4}|B|_g^4u^4dv_0\right)^{1/2}+C_1$ *for some positive constant C_1 which depends only on the heat invariants a_i.*

Proof. The following estimates give the proof of our statement:

$$
\begin{aligned}
\left|\int_{S^4}s_g|B|_g^2u^4dv_0\right|\le&\left|\int_{S^4}\left[s_gu^2-\left(\int_{S^4}u^4dv_0\right)^{-1}\int_{S^4}s_gu^4dv_0u^2\right]|B|_g^2u^2dv_0\right|\\
&+\left(\int_{S^4}u^4dv_0\right)^{-1}\left|\int_{S^4}s_gu^4dv_0\right|\int_{S^4}|B|_g^2u^4dv_0\\
\le&\left(a_2'(g)\right)^{1/2}\left(\int_{S^4}\left(|B|_g^2u^2\right)^2dv_0\right)^{1/2}+a_0^{-1}|a_1|a_2\\
\le&\frac{11}{26}vol^{1/2}\left(S^4\right)\left(\int_{S^4}|B|_g^4u^4dv_0\right)^{1/2}+C_1
\end{aligned}
$$

with $C_1=a_0|a_1|a_2$. Here we have used the assumption that $a_2'(g)$ is less than $(\frac{11}{26})^2vol\left(S^4\right)$.

Lemma 3.4. *With C_1 as in Lemma 3.3, we have*

$$
\left|\int_{S^4}s_g^3u^4dv_0\right|\le\frac{11}{26}vol^{1/2}\left(S^4\right)\left(\int_{S^4}s_g^4u^4dv_0\right)^{1/2}+C_1.
$$

Proof. As in the proof of lemma 3.3, we have

$$
\begin{aligned}
\left|\int_{S^4}s_g^3u^4dv_0\right|\le&\left|\int_{S^4}\left[s_gu^2-\left(\int_{S^4}u^4dv_0\right)^{-1}\int_{S^4}s_gu^4dv_0u^2\right]s_g^2u^2dv_0\right|+C_1\\
\le&\frac{11}{26}vol^{1/2}\left(S^4\right)\left(\int_{S^4}s_g^4u^4dv_0\right)^{1/2}+C_1.
\end{aligned}
$$

Lemma 3.5.

$$\frac{4}{3}\int_{S^4}|\nabla B|_g^2u^4dv_0+\frac{152}{9}\int_{S^4}|\nabla s_g|_g^2u^4dv_0$$
$$\leq\frac{50}{9}\int_{S^4}s_g|B|_g^2u^4dv_0+\frac{185}{54}\int_{S^4}s_g^3u^4dv_0+|a_3|.$$

Proof. This is the formula a_3 with the obvious estimate $-a_3\leq|a_3|$.

Lemma 3.6. *There exists a positive constant C_2 such that*

$$\left(\int_{S^4}|B|_g^4u^4dv_0\right)^{1/2}+\left(\int_{S^4}s_g^4u^4dv_0\right)^{1/2}\leq C_2.$$

Proof. Applying Lemmas 3.1 to 3.5, we have

$$vol^{1/2}\left(S^4\right)\left[\frac{4}{3}\left(\int_{S^4}|B|_g^4u^4dv_0\right)^{1/2}+\frac{152}{9}\left(\int_{S^4}s_g^4u^4dv_0\right)^{1/2}\right]$$
$$\leq\frac{2}{3}\int_{S^4}|\nabla B|_g^2u^4dv_0+\frac{1}{9}\int_{S^4}s_g|B|_g^2u^4dv_0$$
$$+\frac{152}{9}\left\{\frac{1}{2}\int_{S^4}|\nabla s_g|_g^2u^4dv_0+\frac{1}{12}\int_{S^4}s_g^3u^4dv_0\right\}$$
$$=\frac{1}{2}\left\{\frac{4}{3}\int_{S^4}|\nabla B|_g^2u^4dv_0+\frac{152}{9}\int_{S^4}|\nabla s_g|_g^2u^4dv_0\right\}$$
$$+\frac{1}{9}\int_{S^4}s_g|B|_g^2u^4dv_0+\frac{152}{108}\int_{S^4}s_g^3u^4dv_0$$
$$\leq\frac{1}{2}\left\{\frac{50}{9}\int_{S^4}s_g|B|_g^2u^4dv_0+\frac{185}{54}\int_{S^4}s_g^3u^4dv_0\right\}+\frac{1}{2}|a_3|$$
$$+\frac{1}{9}\int_{S^4}s_g|B|_g^2u^4dv_0+\frac{152}{108}\int_{S^4}s_g^3u^4dv_0$$
$$=\frac{26}{9}\int_{S^4}s_g|B|_g^2u^4dv_0+\frac{337}{108}\int_{S^4}s_g^3u^4dv_0+\frac{1}{2}|a_3|$$
$$\leq\frac{26}{9}\cdot\frac{11}{26}vol^{1/2}\left(S^4\right)\left[\int_{S^4}|B|_g^4u^4dv_0\right]^{1/2}$$
$$+\frac{337}{108}\cdot\frac{11}{26}vol^{1/2}\left(S^4\right)\left[\int_{S^4}s_g^4u^4dv_0\right]^{1/2}+\left(\frac{26}{9}+\frac{337}{108}\right)C_1+\frac{1}{2}|a_3|$$

$$
\begin{aligned}
&= \frac{11}{9} vol^{1/2}\left(S^4\right)\left(\int_{S^4} |B|_g^4 u^4 dv_0\right)^{1/2} \\
&+ \frac{337 \times 11}{108 \times 26} vol^{1/2}\left(S^4\right)\left(\int_{S^4} s_g^4 u^4 dv_0\right)^{1/2} + \left(\frac{26}{9} + \frac{337}{108}\right) C_1 + \frac{1}{2}|a_3|.
\end{aligned}
$$

Namely shifting the integral terms of the right to the left, we obtain

$$
\begin{aligned}
&\left(\frac{4}{3} - \frac{11}{9}\right)\left(\int_{S^4} |B|_g^4 u^4 dv_0\right)^{1/2} + \left(\frac{152}{9} - \frac{337 \times 11}{108 \times 26}\right)\left(\int_{S^4} s_g^4 u^4 dv_0\right)^{1/2} \\
&\leq \left[\left(\frac{26}{9} + \frac{1011}{324}\right) C_1 + \frac{1}{2}|a_3|\right] vol^{-1/2}\left(S^4\right).
\end{aligned}
$$

Now it is clear that

$$
\frac{4}{3} - \frac{11}{9} = \frac{1}{9} > 0,
$$

and

$$
\frac{152}{9} - \frac{337 \times 11}{108 \times 26} = \frac{43717}{108 \times 26} > \frac{1}{9} > 0.
$$

Therefore, we have

$$
\begin{aligned}
&\left(\int_{S^4} |B|_g^4 u^4 dv_0\right)^{1/2} + \left(\int_{S^4} s_g^4 u^4 dv_0\right)^{1/2} \\
&\qquad \leq 9\left\{\left(\frac{26}{9} + \frac{337}{108}\right) C_1 + \frac{1}{2}|a_3|\right\} vol^{-1/2}\left(S^4\right) \equiv C_2.
\end{aligned}
$$

Remark. By tracing through the above proofs,one can verify that the result remains valid for $a_2'(g) \leq \delta^2 vol\left(S^4\right)$ with $\delta < \frac{6}{13}$.

4. Proof of the C^0 version of main theorem.

From now on, we will reserve the letters a_0, a_1, a_2, a_2', a_3 for the spectral invariants. We would like to have estimates on the conformal factors in terms of these invariants and the first nonzero eigenvalues of the metrics $g = u^2 g_0$. Of course we will be free to use data about metric g_0, the background metric. The main aim of this section is to show the following theorem which is the C^0 version of the main theorem stated in introduction:

Theorem 4.1. *On (S^4, g_0), if $g = u^2 g_0$ is a conformal metric, then there exist a $v \in [u]$ and a constant $C_3 > 0$, depending only on $a_0(g)$, $a_1(g)$,*

$a_2(g)$, $a_2'(g)$, $\lambda_1(g)$ and the data of the metric g_0 such that the metric g is quasi-isometric to g_0 in the sense that

$$0 < \frac{1}{C_3} g_0 \leq g \leq C_3 g_0.$$

Proof. The proof will consist of several steps.
Step 1. Applying Lemma 1 of [CY2], we have a $v \in [u]$ such that

$$\int_{S^4} v^4 x_j dv_0 = 0$$

for $j = 1, 2, \cdots, 5$ where x_j are the ambient coordinates of S^4. The key point is that $u^2 g_0$ is isometric to $v^2 g_0$. Thus they have same geometry, for example, the same volume and the same first eigenvalues. Thus if we denote $v^2 g_0$ by g again, we have

$$\begin{aligned} \int_{S^4} v^4 x_j^2 dv_0 &\leq [\lambda_1(g)]^{-1} \int_{S^4} |\nabla x_j|_g^2 v^4 dv_0 \\ &= [\lambda_1(g)]^{-1} \int_{S^4} v^2 |\nabla x_j|_{g_0}^2 dv_0. \end{aligned} \tag{4.1}$$

Remember $x_1^2 + x_2^2 + \cdots + x_5^2 = 1$ and $|\nabla x_1|^2 + \cdots + |\nabla x_5|^2 = 4$. By summing inequality (4.1) from $j = 1$ to $j = 5$, we obtain

$$a_0 = \int_{S^4} u^4 dv_0 = \int_{S^4} v^4 dv_0 \leq (\lambda_1(g))^{-1} \int_{S^4} 4v^2 dv_0.$$

That is,

$$\int_{S^4} v^2 dv_0 \geq 4^{-1} a_0 \lambda_1(g) = C_4 > 0. \tag{4.2}$$

Let $\eta = \left[C_4/(2vol\,(S^4))\right]^{1/2}$ and $\Omega = \{x \in S^4 | v(x) \geq \eta\}$. Then we have $\eta > 0$ and the estimate

$$\begin{aligned} 0 < C_4 &\leq \int_{S^4} v^2 dv_0 = \int_\Omega v^2 dv_0 + \int_{S^4 \backslash \Omega} v^2 dv_0 \\ &\leq \left(\int_\Omega v^4 dv_0\right)^{1/2} (vol(\Omega))^{1/2} + \eta^2 vol\,(S^4 \backslash \Omega) \\ &\leq a_0{}^{1/2} vol(\Omega)^{1/2} + \frac{C_4}{2}. \end{aligned}$$

Therefore

$$\text{(4.3)} \qquad vol(\Omega) \geq \frac{C_4^2}{(4a_0)} > 0.$$

Step 2. From the information in Section two, we know that the scalar curvatures of the metrics g and g_0 satisfy the relation:

$$\text{(4.4)} \qquad 6\Delta v + s_g v^3 = s_0 v$$

Suppose $\delta > 0$, the precise value will be chosen later. Multiply equation (4.4) by $v^{-1-2\delta}$ and integrate by parts to get

$$6\int_{S^4} |\nabla v^{-\delta}|^2 dv_0 = -\frac{\delta^2}{(1+2\delta)}\int_{S^4} s_g v^{2-2\delta} dv_0 + \frac{(\delta^2 s_0)}{(1+2\delta)}\int_{S^4} v^{-2\delta} dv_0.$$

Now let $\lambda_1(g_0) = \lambda_1$ denote the first eigenvalue of Δ acting on functions, i.e., $\lambda_1 = 4$, then by the Rayleigh-Ritz characterization for λ_1, we get (4.5)

$$\begin{aligned}\int_{S^4} v^{-2\delta} dv_0 &\leq \left(vol\,(S^4)\right)^{-1}\left(\int_{S^4} v^{-\delta} dv_0\right)^2 + (1/\lambda_1)\int_{S^4} |\nabla v^{-\delta}|^2 dv_0 \\ &\leq vol\,(S^4)^{-1}\left(\int_{S^4} v^{-\delta} dv_0\right)^2 + \frac{\delta^2 s_0}{(6\lambda_1\,(1+2\delta))}\int_{S^4} v^{-2\delta} dv_0 \\ &\quad - \frac{\delta^2}{6\lambda_1(1+2\delta)}\int_{S^4} s_g v^{2-2\delta} dv_0.\end{aligned}$$

From Equation (4.3), we have

$$\begin{aligned}\int_{S^4} v^{-\delta} dv_0 &= \int_{\Omega} v^{-\delta} dv_0 + \int_{S^4\backslash\Omega} v^{-\delta} dv_0 \\ &\leq \eta^{-\delta} vol(\Omega) + \left(\int_{S^4\backslash\Omega} v^{-2\delta} dv_0\right)^{1/2} \left(vol\,(S^4\backslash\Omega)\right)^{1/2}.\end{aligned}$$

In above equation, square, divide by the volume and apply Hölder's inequality to obtain

$$\begin{aligned}\text{(4.6)} \quad & vol\,(S^4)^{-1}\left(\int_{S^4} v^{-\delta} dv_0\right)^2 \\ &\leq (1+1/\gamma)\left[\eta^{-2\delta} vol(\Omega)^2\right] + (1+\gamma)\frac{vol\,(S^4\backslash\Omega)}{vol\,(S^4)}\int_{S^4} v^{-2\delta} dv_0.\end{aligned}$$

for all positive γ. Now as $vol(\Omega) \geq \frac{C_4^2}{4a_0} > 0, vol\left(S^4\backslash\Omega\right) = (1-2\theta)vol\left(S^4\right)$ for some $\theta = \theta(vol(\Omega)) > 0$. Therefore we can take γ small enough such that $(1+\gamma)(1-2\theta) \leq (1-\theta)$. Now combining Equations (4.5) and (4.6), we obtain

$$
\begin{aligned}
\int_{S^4} v^{-2\delta} dv_0 &\leq C_5 + (1-\theta)\int_{S^4} v^{-2\delta} dv_0 + \frac{\delta^2 s_0}{(6\lambda_1(1+2\delta))}\int_{S^4} v^{-2\delta} dv_0 \\
&\quad - \frac{\delta^2}{6\lambda_1(1+2\delta)}\int_{S^4} s_g v^{2-2\delta} dv_0 \\
&\leq C_5 + (1-\theta)\int_{S^4} v^{-2\delta} dv_0 + \frac{\delta^2}{2(1+2\delta)}\int_{S^4} v^{-2\delta} dv_0 \\
&\quad - \frac{\delta^2}{6\lambda_1(1+2\delta)}\left[\int_{S^4} s_g v^{2-2\delta} dv_0\right].
\end{aligned} \tag{4.7}
$$

where $C_5 = (1+\frac{1}{\gamma})\eta^{-2\delta} vol^2\left(S^4\right) > 0$ and we have also used the facts that $s_0 = 12$, and $\lambda_1 = 4$. Rewrite this inequality as

$$
(\theta - \frac{\delta^2}{2(1+2\delta)})\int_{S^4} v^{-2\delta} dv_0 \leq C_5 - \frac{\delta^2}{6\lambda_1(1+2\delta)}\int_{S^4} s_g v^{2-2\delta} dv_0. \tag{4.8}
$$

Now we can estimate the second term on the right hand side of (4.8) by Hölder's inequality and Lemma 3.6 as follows:

$$
\begin{aligned}
&\left|\int_{S^4} s_g v^{2-2\delta} dv_0\right| \\
&\leq \left(\int_{S^4} s_g^4 v^4 dv_0\right)^{1/4}\left(\int_{S^4} v^{4(1-2\delta)/3} dv_0\right)^{3/4} \\
&\leq C_2^{1/4}\left(\int_{S^4} v^4 dv_0\right)^{(1-2\delta)/4}\left(vol\left(S^4\right)\right)^{(2+2\delta)/4} \\
&\equiv C_6
\end{aligned}
$$

if we choose δ smaller than one-half.

Now if we further choose δ such that

$$
\frac{\delta^2}{1+2\delta} < \min\{\theta, 1/2\},
$$

then we easily verify that

$$
\begin{aligned}
\int_{S^4} v^{-2\delta} dv_0 &\leq \frac{2C_5}{\theta} + \frac{2}{\theta}\frac{\delta^2}{6\lambda_1(1+2\delta)} C_6 \\
&\leq \frac{2C_5}{\theta} + \frac{1}{12} C_6 \\
&\equiv C_7.
\end{aligned} \tag{4.9}
$$

Step 3. Let $G(p,q)$ denote the Green's function for Δ with singularity at p. We may add a constant and assume $G(p,q)$ is positive. Then we have

$$
\begin{aligned}
v^{-\alpha}(p) &= \frac{1}{vol\,(S^4)}\int_{S^4} v^{-\alpha}dv_0 - \int_{S^4} G(p,q)\Delta v^{-\alpha}dv_0(q) \\
&= \frac{1}{vol\,(S^4)}\int_{S^4} v^{-\alpha}dv_0 - \frac{1}{6}\int_{S^4} G(p,q)\left[\alpha s_g v^{2-\alpha} - s_0\alpha v^{-\alpha}\right. \\
&\left. +6\alpha(1+\alpha)v^{-\alpha-2}|\nabla v|^2\right] dv_0(q) \\
&\leq \frac{1}{vol\,(S^4)}\int_{S^4} v^{-\alpha}dv_0 + \frac{(\alpha s_0)}{6}\int_{S^4} v^{-\alpha}G(p,q)dv_0 \\
&- \frac{\alpha}{6}\int_{S^4} G(p,q)s_g v^{2-\alpha}dv_0
\end{aligned}
\tag{4.10}
$$

Now by Hölder's inequality, we have
(4.11)

$$
\int_{S^4} v^{-\alpha}G(p,q)dv_0 \leq \left[\int_{S^4} v^{-\alpha r/(r-1)}dv_0\right]^{(r-1)/r}\left[\int_{S^4} G^r(p,q)dv_0\right]^{1/r}.
$$

If we choose $1 < r < 2$ and $\alpha < \frac{(2\delta)(r-1)}{r}$ where $\delta > 0$ has been defined above, then equation (4.11) says that the second term on the right hand side of equation (4.10) is bounded. Also, using Hölder's inequality, we get

$$
\begin{aligned}
&\left|\int_{S^4} G(p,q)s_g v^{2-\alpha}dv_0\right| \\
&= \left|\int_{S^4} G(p,q)s_g v v^{1-\alpha}dv_0\right| \\
&\leq \left[\int_{S^4} G^r(p,q)dv_0\right]^{1/r}\left[\int_{S^4} |s_g|^4 v^4 dv_0\right]^{1/4} \\
&\cdot \left[\int_{S^4} v^{\frac{(4r(2-\alpha)-4)}{(3r-4)}}dv_0\right]^{\frac{(3r-4)}{4r}}.
\end{aligned}
\tag{4.12}
$$

Now we choose r such that

$$
2 > r > \frac{4}{3}.
$$

Then choose α sufficiently small for a fixed r so that

$$
0 < \alpha < \min\left\{2\delta, \frac{[2\delta(3r-4)+4r]}{(4r)}, \frac{[2\delta(r-1)]}{r}\right\}.
\tag{4.13}
$$

Finally by Hölder's inequality, the first and third terms on the right hand side of equation (4.10) are bounded in terms of something which does not depend on the point p. Thus there is a constant $C_8 > 0$ such that $v \geq C_8$.

Step 4. Let $w = v^{1+\epsilon}$ with ϵ to be determined later. From the Sobolev inequality for w (theorem 2.6), we have

$$vol^{1/2}(S^4)\left(\int_{S^4} w^4 dv_0\right)^{1/2} \leq \frac{1}{2}\int_{S^4} |\nabla w|^2 dv_0 + \int_{S^4} w^2 dv_0. \tag{4.14}$$

On the other hand, multiplying (4.4) by $v^{1+2\epsilon}$, we obtain

$$6\Delta v v^{1+2\epsilon} + s_g v^{4+2\epsilon} = s_0 v^{2+2\epsilon}. \tag{4.15}$$

And now integrate (4.15) and use integration by parts to get

$$6\frac{(1+2\epsilon)}{(1+\epsilon)^2}\int_{S^4} |\nabla w|^2 dv_0 = \int_{S^4} s_g v^2 w^2 dv_0 - s_0 \int_{S^4} w^2 dv_0. \tag{4.16}$$

Notice that for $\epsilon < 1$, $\int_{S^4} w^2 dv_0$ is bounded by some constant depending on a_0 and the volume of S^4. From (4.14), we conclude that

$$vol^{1/2}(S^4)\left(\int_{S^4} w^4 dv_0\right)^{1/2} \leq \frac{(1+\epsilon)^2}{(12(1+2\epsilon))}\int_{S^4} s_g v^2 w^2 dv_0 + C_9(\epsilon). \tag{4.17}$$

For any $\eta > 0$, let $E = \{x \in M \| s_g| \geq (C_2\eta^{-1})^{1/2}\}$. Then, apply Lemma 3.6 to obtain

$$\begin{aligned} C_2 &\geq \int_{S^4} |s_g|^4 v^4 dv_0 \\ &\geq \int_E |s_g|^4 v^4 dv_0 \\ &\geq C_2\eta^{-1}\int_E s_g^2 v^4 dv_0. \end{aligned}$$

Therefore we have

$$\int_E s_g^2 v^4 dv_0 \leq \eta, \tag{4.18}$$

and

$$|s_g| \leq C_2^{1/2}\eta^{-1/2} \quad \text{on} \quad M\backslash E. \tag{4.19}$$

This implies that

$$\begin{aligned} &\left|\int_{S^4} s_g v^2 w^2 dv_0\right| \\ &\leq \int_{S^4} |s_g| w^2 v^2 dv_0 \\ &\leq C_2^{1/2}\eta^{-1/2}\int_{S^4} v^2 w^2 dv_0 + \eta^{1/2}\left[\int_{S^4} w^4 dv_0\right]^{1/2}. \end{aligned} \tag{4.20}$$

To estimate $\int_{S^4} v^2 w^2 dv_0$, we apply the Rayleigh-Ritz characterization of $\lambda_1(\Delta_g) \equiv \Lambda$

$$\lambda_1(\Delta_g) \leq \frac{\int_{S^4} |\nabla\psi|_g^2 v^4 dv_0}{\int_{S^4} \left[\psi - (vol\,(S^4))^{-1} \int_{S^4} \psi v^4 dv_0\right]^2 v^4 dv_0},$$

or equivalently,

$$\int_{S^4} \psi^2 v^4 dv_0 \leq \left(vol\,(S^4)\right)^{-1} \left(\int_{S^4} \psi v^4 dv_0\right)^2 + \Lambda^{-1} \int_{S^4} v^2 |\nabla\psi|^2 dv_0 \tag{4.21}$$

to $\psi = v^\epsilon$ to obtain

(4.22)

$$\begin{aligned}\int_{S^4} v^2 w^2 &\leq \left[\int_{S^4} v^4 dv_0\right]^{-1} \left[\int_{S^4} v^{4+\epsilon} dv_0\right]^2 + \Lambda^{-1} \int_{S^4} v^2 |\nabla v^\epsilon|^2 dv_0 \\ &= \left[\int_{S^4} v^4 dv_0\right]^{-1} \left[\int_{S^4} v^{4+\epsilon} dv_0\right]^2 + \frac{\epsilon^2}{(\Lambda(1+\epsilon))} \int_{S^4} |\nabla w|^2 dv_0 \\ &= \left[\int_{S^4} v^4 dv_0\right]^{-1} \left[\int_{S^4} v^{4+\epsilon} dv_0\right]^2 + \frac{\epsilon^2}{(6\Lambda(1+2\epsilon))} \left[\int_{S^4} s_g v^2 w^2 dv_0 \right. \\ &\left. - s_0 \int_{S^4} w^2 dv_0\right],\end{aligned}$$

where we have used Equation (4.16) to obtain the second equality.

To estimate $\int_M v^{4+\epsilon} dv_0$, first of all, from Step 3 we have $C_8 > 0$ with $v - C_8 \geq 0$. Therefore

$$\begin{aligned}\int_{S^4} v^{4+\epsilon} dv_0 &= \int_{S^4} \left(v^4 - C_8^4\right) v^\epsilon dv_0 + C_8^4 \int_{S^4} v^\epsilon dv_0 \\ &\leq \left[\int_{S^4} \left(v^4 - C_8^4\right) v^{2\epsilon} dv_0\right]^{1/2} \left[\int_{S^4} \left(v^4 - C_8^4\right) dv_0\right]^{1/2} + C_8^4 \int_{S^4} v^\epsilon dv_0,\end{aligned}$$

where the inequality comes from Cauchy-Schwartz and the positivity of $v^4 - C_8^4$. Thus we have

$$\begin{aligned}\left[\int_{S^4} v^{4+\epsilon} dv_0\right]^2 &\leq (1+\gamma) \left[\int_{S^4} (v^4 - C_8^4) v^{2\epsilon} dv_0\right] \left[\int_{S^4} (v^4 - C_8^4) dv_0\right] \\ &+ \left(1 + \frac{1}{\gamma}\right) C_8^8 \left[\int_{S^4} v^\epsilon dv_0\right]^2,\end{aligned} \tag{4.23}$$

where γ will be chosen later. But

$$\int_{S^4} \left(v^4 - C_8^4\right) dv_0 = \alpha \int_{S^4} v^4 dv_0$$

where $\alpha = 1 - \frac{(C_8^4 vol(S^4))}{\int_{S^4} v^4 dv_0}$ is a positive constant less than 1 and we conclude that

$$(4.24)\quad \left[\int_{S^4} v^4 dv_0\right]^{-1}\left[\int_{S^4} v^{4+\epsilon} dv_0\right]^2$$
$$\le (1+\gamma)\alpha\left[\int_{S^4}\left(v^4 - C_8^4\right)v^{2\epsilon}dv_0\right] + \left(1+\frac{1}{\gamma}\right)C_8^8\frac{\left[\int_{S^4} v^\epsilon dv_0\right]^2}{\left[\int_{S^4} v^4 dv_0\right]}$$
$$= (1+\gamma)\alpha\int_{S^4} v^{4+2\epsilon}dv_0$$
$$+\left[\left(1+\frac{1}{\gamma}\right)C_8^8\frac{\left[\int_{S^4} v^\epsilon dv_0\right]^2}{\left[\int_{S^4} v^4 dv_0\right]} - C_8^4(1+\gamma)\alpha\int_{S^4} v^{2\epsilon}dv_0\right].$$

Since we assume $\epsilon < 1$ and $\alpha_0 = \int_{S^4} v^4 dv_0 > 0$, we get the conclusion that the second term on the right hand side of Inequality (4.24) is bounded by some constant. Choose γ so that $(1+\gamma)\alpha = (1-\beta) < 1$. From (4.22) we then have

$$\int_{S^4} v^2 w^2 dv_0 \le (1-\beta)\int_{S^4} v^2 w^2 dv_0 + C_{10} + \frac{\epsilon^2}{(6\Lambda(1+2\epsilon))}\int_{S^4} s_g v^2 w^2 dv_0.$$

It is equivalent to

$$(4.25)\qquad \int_{S^4} v^2 w^2 dv_0 \le \frac{1}{\beta}\frac{\epsilon^2}{(6\Lambda(1+2\epsilon))}\int_{S^4} s_g v^2 w^2 dv_0 + \frac{1}{\beta}C_{10}.$$

Combine (4.25) and (4.20) to obtain

$$\int_{S^4} s_g v^2 w^2 dv_0$$
$$\le C_2^{\frac{1}{2}}\eta^{\frac{-1}{2}}\int_{S^4} v^2 w^2 dv_0 + \eta\left[\int_{S^4} w^4 dv_0\right]^{1/2}$$
$$\le \frac{(C_2^{1/2}\eta^{-1/2}\epsilon^2)}{[6\beta\Lambda(1+2\epsilon)]}\int_{S^4} s_g v^2 w^2 dv_0 + \frac{(C_2^{1/2}\eta^{-1/2}C_{10})}{\beta} + \eta\left[\int_{S^4} w^4 dv_0\right]^{1/2}.$$

Therefore

$$\left(1 - \frac{\left(C_2^{1/2}\eta^{-1/2}\epsilon^2\right)}{(6\beta\Lambda(1+2\epsilon))}\right)\int_{S^4} s_g v^2 w^2 dv_0 \le \eta\left[\int_{S^4} w^4 dv_0\right]^{1/2} + C_{11}$$

where $C_{11} = \frac{\left(C_2^{1/2}\eta^{-1/2}C_{10}\right)}{\beta}$. If we set $\theta = vol^{1/2}\left(S^4\right)\frac{(C_2^{1/2}\eta^{-1/2}\epsilon^2)}{(6\beta\Lambda(1+2\epsilon))}$, from Equation (4.17), we obtain

$$
\begin{aligned}
(4.26)\qquad & vol^{1/2}\left(S^4\right)\frac{(12(1+2\epsilon))}{(1+\epsilon)^2}(1-\theta)\left[\int_{S^4} w^4 dv_0\right]^{1/2} \\
& \leq \eta\left[\int_{S^4} w^4 dv_0\right]^{1/2} + (1-\theta)C_9(\epsilon)\frac{12(1+2\epsilon)}{(1+\epsilon)^2} + C_{11}.
\end{aligned}
$$

Now choose $\eta = 2$. Then choose $\epsilon = \epsilon_0 > 0$ small enough such that

$$\theta = \frac{1}{2}.$$

Finally we have reached the following

$$
\begin{aligned}
& \frac{(12(1+2\epsilon))}{(1+\epsilon)^2}(1-\theta) - 2 \\
&= \frac{(12(1+2\epsilon))}{(1+\epsilon)^2}\frac{1}{2} - 2 \\
&= 2\frac{\left[(2-\epsilon^2)+4\epsilon\right]}{(1+\epsilon)^2} > 0
\end{aligned}
$$

because $0 < \epsilon < 1$. Hence from equation (4.26),

$$
\begin{aligned}
& \left[\int_{S^4} w^4 dv_0\right]^{1/2} \\
& \leq \frac{2\left[((1-\theta)C_9(\epsilon) + C_{11})((2-\epsilon^2)+4\epsilon)\right]}{(1+\epsilon)^2} vol^{-1/2}\left(S^4\right) \\
& \equiv (C_{12})^{\frac{1}{2}}.
\end{aligned}
$$

Step 5. Apply Green's function to equation (4.4). Then

$$
\begin{aligned}
(4.27)\qquad v(p) - vol(M)^{-1}\int_{S^4} v dv_0 &= \int_{S^4}(-\Delta v)(q)G(p,q)dv_0(q) \\
&= \frac{1}{6}\int_{S^4}(s_g v^3 - s_0 v)G dv_0.
\end{aligned}
$$

Since $vol\left(S^4\right)^{-1}\int_{S^4} v dv_0$ and $\int_{S^4} vG dv_0$ are a priori bounded, to bound $v(p)$, it suffices to bound $\int_{S^4} s_g v^3 G dv_0$.

It is well known that $|G(p,q)| \le \frac{K}{d^2(p,q)}$ for some constant K[Au, p.108]. Recall the following estimate [Au, p. 37]: for $h(y) = \int_{R^4} \frac{f(x)}{\|x-y\|^2} dx$, we have

$$\|h\|_r \le C(r')\|f\|_{r'} \tag{4.28}$$

where $\frac{1}{r} = \frac{1}{2} + \frac{1}{r'} - 1 = \frac{1}{r'} - \frac{1}{2}$ with $r > 1$.

We will iterate this estimate with a sequence of suitable choice of r_j and r_j'. Start with $r_0' = \frac{4r_0}{8+r_0}, r_0 = 4 + 4\epsilon$ for $4\epsilon \le 4\epsilon_0$ where ϵ_0 has been determined in Step 4. We then have

$$\begin{aligned}\int_{S^4} |s_g v^3|^{r_0'} dv_0 &\le \left[\int_{S^4} |s_g|^4 v^4 dv_0\right]^{\frac{r_0'}{4}} \left[\int_{S^4} v^{\frac{8r_0}{4-r_0'}} dv_0\right]^{\frac{3r_0'}{4}} \\ &= \left\{\int_{S^4} |s_g|^4 v^4 dv_0\right\}^{\frac{r_0'}{4}} \left\{\int_{S^4} v^{r_0} dv_0\right\}^{\frac{3}{4}r_0'}.\end{aligned} \tag{4.29}$$

Apply (4.28) to get

$$\int_{S^4} v^{r_1} dv_0^{\frac{1}{r_1}} \le C(r_0') \left\{\int_{S^4} |s_g v^3|^{r_0'} dv_0\right\}^{\frac{1}{r_0'}} + C_{13}$$

where C_{13} is a constant and $\frac{1}{r_1} = \frac{1}{r_0'} - \frac{1}{2}$, i.e.,

$$\begin{aligned} r_1 \frac{(2r_0')}{2 - r_0'} &= \frac{4r_0}{8 - r_0} \\ &= \frac{r_0}{1-\epsilon} > r_0. \end{aligned}$$

Note that if we can choose ϵ such that $1-\epsilon < 0$, then $4+4\epsilon > 6+2 = 8$. Thus $r_0' > 2$ and we are done by virtue of Estimate (4.29) and Hölder's inequality. If $1-\epsilon = 0$, then we can replace ϵ by $\epsilon' < \epsilon$. So we have $1-\epsilon' = (\epsilon - \epsilon') > 0$. Thus we can assume that $1-\epsilon > 0$.

Continue this process with

$$r_2 = \frac{r_0}{1-\epsilon}; \quad r_1' = \frac{4r_1}{8+r_1};$$

$$\cdots\cdots$$

$$r_k = \frac{2r_{k-1}'}{2 - r_{k-1}'} = \frac{r_{k-1}}{1-\epsilon};$$

$$r_{k-1}' = \frac{4r_{k-1}}{8 + r_{k-1}}.$$

Notice that

$$r_{k+1} - r_k = \frac{\epsilon}{1-\epsilon} r_k > 0.$$

Thus there will be a k_0 with $r_{k_0} > 6+2 = 8$ and $r_0 < r_1 < \cdots < r_{k_0-1} < 8 < r_{k_0}$ with

$$r'_{k_0} = \frac{4r_{k_0}}{8+r_{k_0}} > 2.$$

So at the end of the iteration, we can find a bound for $\|s_g v^3\|_{r_{k_0}}, 2 < r'_{k_0} < 2+\delta_0$. This, together with Hölder's inequality, implies that $v \in L^\infty$,

$$\begin{aligned} \|v\|_\infty &\leq \|v\|_1 + \|s_g v^3\|_{r'_{k_0}} \|G\|_{q'} \\ &\leq C_{14} \end{aligned}$$

for some constant C_{14}. Here $\frac{1}{r'_{k_0}} + \frac{1}{q'} = 1$ with $q' < 2$.

Step 6. Now we can choose $C_3 = \max\{C_8^{-2}, C_{14}^2\}$ to finish the proof of Theorem 3.1 easily.

5. Proof of Main Theorem.

In this section we will prove the theorems stated in §1. The main tool one can use is the heat invariants, i.e., the coefficients a_i in the asymptotic expansion of the trace of the heat kernel

$$Z(t) = \sum e^{-\lambda_i t} \simeq \frac{1}{(4\pi t)^{\frac{n}{2}}} \sum a_i t^i$$

as $t \to 0$. What we can get from this information is the following:

Theorem 5.1. *Suppose (M, g_0) is a compact 4–dimensional Riemannian manifold without boundary. If $\int_M |R|^4 dv_g \leq C_{21}$ and*

$$|a_k| \leq b_k, \quad k = 3, 4, \cdots$$

and there is a constant $\lambda > 0$ such that $0 < \lambda^{-1} g_0 \leq g \leq \lambda g_0$. Then

$$\int_M |\nabla^k R|^2 dv \leq C(k)$$

for some constant $C(k)$ depending on k, b_k, C_{21}, λ and the geometry of g_0.

Proof. The higher coefficients $a_i(g)$ become rapid increasingly complex and difficult to compute. However, the exact forms of the a_i are not so important for our purpose. What is important is that they have the general leading coefficients [G2]

$$(5.1)\quad a_k(g) = (-1)^k \int_M \left(c_k \left|\nabla^{k-2} R\right|^2 + d_k \left|\nabla^{k-3} s_g\right|^2 \right) dv_g + \int_M Q_k dv_g$$

where c_k and d_k are positive constants and Q_k is a lower order term involving covariant derivatives of R and its contractions of order at most $k-3$. More precisely, Q_k is a polynomial of weight $2k$ in contractions of $R_{ijkl,I}$ with $|I| \leq k-3$, with coefficients depending only on the metric g. Each monomial in Q_k is a product of contractions of $R_{ijkl,I}$ of weight $2k$, where the weight of $R_{ijkl,I}$ is defined to be $|I|+2$ and the weight of the monomial is the sum of the weights of the factors.

First of all, since the coefficients c_k, d_k in (5.1) are positive, the bound on a_3 gives a bound

$$(5.2)\quad \int_M |\nabla R|^2 dv \leq h_{33} + h_{23} \int_M |R|^3 dv.$$

By Hölder's inequality, one sees that $\int_M |R|^3 dv_g \leq (\int_M |R|^4 dv_g)^{\frac{3}{4}} vol(M)^{\frac{1}{4}}$, i.e.,

$$\int_M |\nabla R|^2 dv_g \leq C(3)$$

where $C(3) = h_{33} + h_{23}\left(C_{21}^{\frac{3}{4}}\right) vol(M)^{\frac{1}{4}}$.

Next, the bound on a_4 gives a bound

$$(5.3)\quad \int_M \left|\nabla^2 R\right|^2 dv_g \leq h_{24} \int_M |R|^4 dv_g + h_{34} \int_M |\nabla R|^2 |R| dv_g + h_{44}.$$

By assumption, the first term on the right hand side of (5.3) is bounded. To bound the second term, choose $\eta = 2(C_{21})^{\frac{1}{2}} C_s h_{34} > 0$ where C_s is the Sobolev constant with respect to the metric g which can be chosen to depend only on the metric g_0 and λ since g is equivalent to g_0. Now let

$\Omega = \{x \in M, |R|(x) \geq \eta\}$ and β_s a constant. Then we have

$$
\begin{aligned}
h_{34}\int_M |\nabla R|^2|R|dv_g &= h_{34}\int_\Omega |\nabla R|^2|R|dv_g + h_{34}\int_{M\setminus\Omega} |\nabla R|^2|R|dv_g \\
&\leq h_{34}\left(\int_\Omega |R|^2 dv_g\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla R|^4 dv_g\right)^{\frac{1}{2}} \\
&\qquad + h_{34}\eta\int_M |\nabla R|^2 dv_g \\
&\leq h_{34}\left(\int_\Omega |R|^2 dv_g\right)^{\frac{1}{2}} \\
&\qquad \times\left[C_s\int_M \left|\nabla^2 R\right|^2 dv_g + \beta_s\int_M |\nabla R|^2 dv_g\right] \\
&\quad + h_{34}\eta\int_M |\nabla R|^2 dv_g \\
&\leq \frac{h_{34}C_s}{\eta}\left(\int_\Omega |R|^4 dv_g\right)^{\frac{1}{2}}\int_M \left|\nabla^2 R\right|^2 dv_g \\
&\quad + h_{34}\beta_s\left(\int_M |R|^2 dv_g\right)^{\frac{1}{2}}\left(\int_M |\nabla R|^2 dv_g\right) \\
&\quad + h_{34}\eta\int_M |\nabla R|^2 dv_g \\
&\leq \frac{h_{34}C_sC_{21}^{\frac{1}{2}}}{\eta}\int_M \left|\nabla^2 R\right|^2 dv_g + h_{45}
\end{aligned}
$$

which implies, together with (5.3), that

$$
\int_M \left|\nabla^2 R\right|^2 dv_g \leq C(4). \tag{5.4}
$$

Now apply (5.4) to get

$$
\begin{aligned}
\left(\int_M |R|^8 dv_g\right)^{\frac{1}{2}} &\leq C_s\int_M |\nabla R^2|^2 dv_g + \beta_s\int_M |R|^4 dv_g \\
&\leq 4C_s\int_M |R|^2|\nabla R|^2 dv_g + \beta_s\int_M |R|^4 dv_g \\
&\leq 4C_s\left(\int_M |R|^4 dv_g\right)^{\frac{1}{2}}\left(\int_M |\nabla R|^4 dv_g\right)^{\frac{1}{2}} + \beta_s\int_M |R|^4 dv_g
\end{aligned}
$$

$$\leq 4C_s \left(\int_M |R|^4 dv_g\right)^{\frac{1}{2}}$$
$$\times \left(C_s \int_M \left|\nabla^2 R\right|^2 dv_g + \beta_s \int_M |\nabla R|^2\right) + \beta_s \int_M |R|^4 dv_g$$
$$\equiv C_{22}.$$

Next we bound $\int_M \left|\nabla^3 R\right|^2 dv_g$ and $\int_M |\nabla R|^8 dv_g$ in essentially the same way. Namely, as above, the bound on a_5 gives a bound

$$\begin{aligned}\int_M \left|\nabla^3 R\right|^2 dv_g &\leq h_{15} \\ &+ h_{25} \int_M \left|\nabla^2 R\right|^2 |R| dv_g + h_{35} \int_M \left|\nabla^2 R\right| |\nabla R|^2 dv_g \\ &+ h_{45} \int_M |\nabla R|^2 |R|^2 dv_g + h_{55} \int_M |R|^5 dv_g.\end{aligned} \tag{5.5}$$

The last three terms on the right side of (5.5) are bounded by the above estimate, Hölder's inequality and the Sobolev inequality. The second term can be bounded as above. Repeating the above argument, we have

$$\int_M \left|\nabla^2 R\right|^4 dv_g \leq C_{23} \tag{5.6}$$

and

$$\int_M |\nabla R|^8 dv_g \leq C_{24}. \tag{5.7}$$

The proof is now completed by induction in a similar fashion. Thus suppose we have bounded

$$\int_M \left|\nabla^l R\right|^2 dv_g \leq C(l+2) \quad l \leq k-1 \tag{5.8}$$

with $k \geq 4$. We claim that

$$\int_M \left|\nabla^k R\right|^2 dv_g \leq C(k+2). \tag{5.9}$$

To see this first note, by Sobolev's embedding, that (5.8) implies the bounds:

$$\int_M \left|\nabla^{k-3} R\right|^8 dv_g \leq C_{25}, \tag{5.10}$$

$$|\nabla^m R|_{C^0} \le C_{26} \quad \text{for} \quad m \le k-4, \tag{5.11}$$

and

$$\int_M \left|\nabla^{k-2} R\right|^4 dv_g \le C_{27}. \tag{5.12}$$

Since the heat invariant a_{k+2} is bounded, the bounds in (5.10) follow from Equation (5.1) and a bound on the terms containing Q_{k+2} in terms of (5.11), (5.12), (5.10) and (5.8). Now recall that Q_{k+2} is a polynomial of weight $2k+4$, each monomial being a product of terms which are contractions of $R_{ijkl,I}$ with $|I| \le k-1$, the weight of $R_{ijkl,I}$ being $|I|+2$. Thus, modulo terms of the form $R_{ijkl,I}$ with $|I| \le k-4$ which are bounded by (5.11), Q_{k+2} at most contains terms of the form:

(*i*) $$\left|\nabla^{k-1} R\right|^2 |R|$$

(*ii*) $$\left|\nabla^{k-1} R\right| \left|\nabla^{k-2}\right| |\nabla R|$$

(*iii*) $$\left|\nabla^{k-1} R\right| \left|\nabla^{k-3} R\right| |R|^2$$

(*iv*) $$\left|\nabla^{k-2} R\right| \left|\nabla^{k-3} R\right| (|\nabla R|\,|R| + |\nabla^3 R|) \quad \text{or} \quad \left|\nabla^{k-2} R\right|^2 |R|^2$$

(*v*) $$\left|\nabla^{k-3} R\right|^4 \quad \text{if } k=4$$

(*vi*)
$\left|\nabla^{k-3} R\right|^p \cdot$ (some terms with derivatives of order $\le k-4$) with $p \le 3$.

Then terms (iii) - (vi) may be bounded in terms of (5.8), (5.10), (5.11) and (5.12). For (ii), we have

$$\begin{aligned} &\int_M \left|\nabla^{k-1} R\right| \left|\nabla^{k-2} R\right| |\nabla R| dv_g \\ &\le \int_M \left|\nabla^{k-1} R\right|^2 dv_g + \int_M \left|\nabla^{k-2} R\right|^4 + \int_M |\nabla R|^4 dv_g \end{aligned}$$

which is bounded by (5.8), (5.10) and (5.12). Now for term (i), it follows from (5.8), (5.10) and (5.11) since $|R|_{C^0} \le C_{26}$.

Now we are in position to show our main theorem.

After combining Theorem 4.1 and 5.1 with $M = S^4$, we have a $v \in [u]$ such that the metric $g = v^2 g_0$ satisfies

i) $\lambda^{-1}g_0 \le g \le \lambda g_0$ for a constant $\lambda > 0$;

ii) $\int_{S^4} \left|\nabla^k R_g\right|^2 dv_g \le C(k)$ for all $k \ge 0$.

Then we want to use the mathematical induction to show that

$$\int_{S^4} \left|\nabla^k v\right|^2 dv_0 \le D(k) \tag{5.13}$$

for some constants $D(k)$ depending only on $k, C(k), C(k-1), \cdots, C(0), \lambda$ and the geometry of g_0. To this end, it is clear that (5.13) is true for $k \le 2$. Now since v satisfies the equation

$$\Delta v + \frac{s_g}{6}v^3 = \frac{s_0}{6}v,$$

we have

$$\begin{aligned}
&\int_{S^4} \left|\nabla^k(\Delta v)\right|^2 dv_0 \\
&\le \frac{|s_0|^2}{18}\int_{S^4} \left|\nabla^k v\right|^2 dv_0 + \frac{1}{18}\int_{S^4} \left|\nabla^k(s_g v^3)\right|^2 dv_0 \\
&\le \frac{|s_0|^2}{18}\int_{S^4} \left|\nabla^k v\right|^2 dv_0 + \frac{1}{18}E(C(k), C(k-1), \cdots, C(0), \lambda)\int_{S^4} \left|\nabla^k v\right|^2 dv_0 \\
&\le \left[\frac{|s_0|^2}{18} + \frac{1}{18}E(C(k), C(k-1), \cdots, C(0), \lambda)\right]\int_{S^4} \left|\nabla^k v\right|^2 dv_0 \\
&\le D(k)\left[\frac{|s_0|^2}{18} + \frac{1}{18}E(C(k), C(k-1), \cdots, C(0), \lambda)\right]
\end{aligned}$$

where E is a constant depending on $C(k), C(k-1), \cdots, C(0), \lambda$.

However, the method of the integration by parts can be applied to get

$$\begin{aligned}
\int_{S^4} \left|\nabla^{k+2} v\right|^2 dv_0 &\le \int_{S^4} \left|\nabla^k(\Delta v)\right|^2 dv_0 + F(k, D(k), \cdots, D(0), \lambda, g_0) \\
&\le D(k)\left[\frac{|s_0|^2}{18} + \frac{1}{18}E(C(k), C(k-1), \cdots, C(0), \lambda)\right] \\
&\quad + F(k, D(k), \cdots, D(0), \lambda, g_0) \\
&\equiv D(k+2)
\end{aligned}$$

where F is another constant depending on the constants as indicated.

Thus (5.13) is true for all $k \ge 0$. Clearly (5.13) implies our main theorem.

Remark. In fact, (5.13) says that the set of the conformal factors $\{v_j\}$ when suitably centered form a compact set in the C^∞ topology which clearly is stronger than what we have stated in main theorem. In the preprint version, similar to 3-dimensional case, we have used Cheeger-Gromov's compactness theorem to get the result without going to prove (5.13). It is referee who suggested to prove the stronger statement as above. I would like to thank referee for this improvement.

Appendix.

The main aim of this appendix is to construct a family of conformal metrics g_λ on S^4 such that $\int_{S^4}\left[R_\lambda - (\int_{S^4} dv_{g_\lambda})^{-1}\int_{S^4} R_\lambda dv_{g_\lambda}\right]^2 dv_{g_\lambda}$ is as small as we want but g_λ is unbounded in the C^0 topology even though the possibility of the non-compactness from the conformal group of S^4 already has been ruled out.

Let P be the south pole and we stereo-graphically project to the equatorial plane of S^4. Let $x = (x_1, x_2, \cdots, x_5) \in S^4$ and let $y = (y_1, y_2, y_3, y_4) \in \mathfrak{R}^4$ denote the stereographic projection coordinates of x. It is standard that

$$x_i = \frac{2y_i}{1+|y|^2} \qquad i = 1,2,3,4;$$

$$x_5 = \frac{|y|^2-1}{1+|y|^2}.$$

Its inverse can be written as

$$y_i = \frac{x_i}{1-x_5} \qquad i = 1,2,3,4.$$

It follows that in the stereographic projection coordinates, the standard metric on S^4 can be written as

$$ds_0^2 = \sum_{i=1}^{5} dx_i^2 = \left[\frac{2}{1+|y|^2}\right]^2 \sum_{j=1}^{4} dy_j^2.$$

Recall that on a Riemannian manifold $(M, g), dimM = n$, the operator

$$L_g\psi = \Delta_g\psi - c(n)R_g\psi$$

is called the conformal Laplacian, where $c(n) = \frac{n-2}{4(n-1)}$ and R_g is the scalar curvature of g. It is well-known that under conformal change of metrics, the conformal Laplacian obeys the following rule:

For $g' = w^{\frac{4}{n-2}} g, w > 0$, we have

$$(A.1) \qquad L_{g'}\psi = w^{-\frac{n+2}{n-2}} L_g(\psi w) \quad \text{for all } \psi \in C^\infty(M).$$

Now suppose v is a positive C^2 solution of the equation

$$(A.2) \qquad \Delta_{g_0} v + \frac{1}{6} R v^3 = \frac{R_0}{6} v$$

on S^4 with $R_0 = 12$. Then if we set

$$u(y) = \frac{2}{1+|y|^2} v(y).$$

It follows from Equation (A.1), with $w = \frac{2}{1+|y|^2}, \psi = v, g' = g_0$ and $g =$ Euclidean metric on $\Re^4$, that u satisfies

$$(A.3) \qquad \Delta u + \frac{R}{6} u^3 = 0, \quad y \in \Re^4$$

where Δ is the usual Laplace operator on $\Re^4$. We also have

$$(A.4) \qquad \int_{S^4} v^4 dv_{g_0} = \int_{\Re}^4 u^4 dy;$$

$$(A.5) \qquad \int_{S^4} R(y) v^4 dv_{g_0} = \int_{\Re}^4 R(y) u^4 dy;$$

$$(A.6) \qquad \int_{S^4} R^2(y) v^4 dv_{g_0} = \int_{\Re}^4 R^2(y) u^4 dy.$$

In this appendix, we will show the following

Proposition. *There exists a family of conformal metrics $g_\lambda = v_\lambda^2 g_0$ on S^4 such that*

a. *$\{g_\lambda\}$ is not C^0 bounded;*

b. *$\int_{S^4} x dv_{g_\lambda} = 0$ for all λ;*

c.

$$\mu(g_\lambda) = \int_{S^4} \left[R_{g_\lambda} - \left(\int_{S^4} dv_{g_\lambda} \right)^{-1} \int_{S^4} R_\lambda dv_{g_\lambda} \right]^2 dv_{g_\lambda}$$

can be as small as we like when λ is small or large enough;

d. $\int_{S^4} dv_{g_\lambda}$ *is uniformly bounded;*

e. R_λ *is also uniformly bounded;*

f. $\int_{S^4}(\Delta_{g_0}(v_\lambda))^2 dv_{g_0} \to \infty$ *as* $\lambda \to 0$ *or as* $\lambda \to \infty$.

Proof. As noted previously, stereographic projection allows us to work in $\mathfrak{R}^4$. Let us start with the scalar curvature functions R_λ. We choose R_λ to be the function

$$R_\lambda = 48\frac{(\lambda^2+r^2)^3+(1+\lambda^2r^2)^3}{(1+\lambda^2)^3(1+r^2)^3}$$

where $r^2 = y_1^2+y_2^2+y_3^2+y_4^2$.

It is easy computation and also well known that

$$\Delta\left(\frac{\lambda}{1+\lambda^2r^2}\right) = -8\left(\frac{\lambda}{1+\lambda^2r^2}\right)^3; \tag{A.7}$$

and

$$\Delta\left(\frac{\lambda}{\lambda^2+r^2}\right) = -8\left(\frac{\lambda}{\lambda^2+r^2}\right)^3. \tag{A.8}$$

Therefore if we set

$$u_\lambda = \frac{\lambda}{1+\lambda^2r^2}+\frac{\lambda}{\lambda^2+r^2},$$

then we obtain, from (A.7) and (A.8),

$$\begin{aligned}\Delta(u_\lambda) &= \Delta\left(\frac{\lambda}{1+\lambda^2r^2}\right)+\Delta\left(\frac{\lambda}{\lambda^2+r^2}\right)\\ &= -8\left[\left(\frac{\lambda}{1+\lambda^2r^2}\right)^3+\left(\frac{\lambda}{\lambda^2+r^2}\right)^3\right]\\ &= -8\frac{(\lambda^2+r^2)^3+(1+\lambda^2r^2)^3}{(1+\lambda^2)^3(1+r^2)^3}u_\lambda^3.\end{aligned} \tag{A.9}$$

That is,

$$\Delta(u_\lambda)+\frac{R_\lambda}{6}u_\lambda^3 = 0.$$

Now elementary integrations show that
$(A.10)$

$$\begin{aligned}\int_{\Re}^{4} u_\lambda^4 dy &= vol\left(S^3\right)\int_0^\infty u_\lambda^4 r^3 dr\\ &= vol\left(S^3\right)\left\{\frac{1}{6}+\frac{2\lambda^2\left(\lambda^4-3\lambda^2+1\right)}{\left(1-\lambda^4\right)^2}+\frac{2\lambda^4\left(4\lambda^2-3-3\lambda^4\right)\ln\lambda^2}{\left(1-\lambda^4\right)^3}\right\}\\ &\equiv vol\left(S^3\right)\left\{\frac{1}{6}+f\left(\lambda\right)\right\}.\end{aligned}$$

$(A.11)$

$$\begin{aligned}&\int_{\Re}^{4} R_\lambda u_\lambda^4 dy\\ &= vol\left(S^3\right)\int_0^\infty R_\lambda u_\lambda^4 r^3 dr\\ &= vol\left(S^3\right)\int_0^\infty \left[\left(\frac{\lambda}{\lambda^2+r^2}\right)^3+\left(\frac{\lambda}{1+\lambda^2r^2}\right)^3\right]\left(\frac{\lambda}{\lambda^2+r^2}+\frac{\lambda}{1+\lambda^2r^2}\right)r^3dr\\ &= vol\left(S^3\right)\left\{\frac{1}{6}+\frac{\lambda^2}{2\left(1-\lambda^4\right)^2}+\frac{\lambda^6\ln\lambda^4}{\left(1-\lambda^4\right)^3}+\frac{\lambda^6}{\left(1-\lambda^4\right)^2}\right\}\\ &\equiv 48 vol\left(S^3\right)\left\{\frac{1}{6}+g\left(\lambda\right)\right\}.\end{aligned}$$

$(A.12)$

$$\begin{aligned}&\int_{\Re}^{4} R_\lambda^2 u_\lambda^4 dy\\ &= vol\left(S^3\right)\int_0^\infty R_\lambda^2 u_\lambda^4 r^3 dr\\ &= (48)^2 vol\left(S^3\right)\int_0^\infty \frac{\left[\left(\frac{\lambda}{\lambda^2+r^2}\right)^3+\left(\frac{\lambda}{1+\lambda^2r^2}\right)^3\right]^2}{\left(\frac{\lambda}{\lambda^2+r^2}+\frac{\lambda}{1+\lambda^2r^2}\right)^2}r^3dr\\ &= (48)^2 vol\left(S^3\right)\left\{\frac{1}{6}-\frac{2\lambda^2\left(4\lambda^4+5\lambda^2+2\right)}{3\left(1-\lambda^2\right)^2}-\frac{\lambda^4\left(4\lambda^2+3+3\lambda^4\right)\ln\lambda^2}{\left(1-\lambda^4\right)^3}\right\}\\ &\equiv (48)^2 vol\left(S^3\right)\left\{\frac{1}{6}+h\left(\lambda\right)\right\}.\end{aligned}$$

From the definitions of $h(\lambda), f(\lambda), g(\lambda)$, we can clearly see that as $\lambda \to 0$ or as $\lambda \to \infty$, $h(\lambda), f(\lambda), g(\lambda) \to 0$. Thus we have

$$
\begin{aligned}
&\int_{\mathfrak{R}}^4 \left[R_\lambda - \left(\int_{\mathfrak{R}}^4 u_\lambda^4 dy\right)^{-1} \int_{\mathfrak{R}}^4 R_\lambda u_\lambda^4 dy\right]^2 u_\lambda^4 dy \\
&= \left[\int_{\mathfrak{R}}^4 u_\lambda^4 dy\right]^{-1} \left\{\left[\int_{\mathfrak{R}}^4 R_\lambda^2 u_\lambda^4 dy\right]\left[\int_{\mathfrak{R}}^4 u_\lambda^4 dy\right] - \left[\int_{\mathfrak{R}}^4 R_\lambda u_\lambda^4 dy\right]^2\right\} \\
&= \left[\frac{1}{6} + f(\lambda)\right]^{-1} 48^2 vol\left(S^3\right) \\
&\quad \times \left\{\left(\frac{1}{6} + h(\lambda)\right)\left(\frac{1}{6} + f(\lambda)\right) - \left(\frac{1}{6} + g(\lambda)\right)^2\right\} \\
&= \left[\frac{1}{6} + f(\lambda)\right]^{-1} 48^2 vol\left(S^3\right) \\
&\quad \times \left\{\frac{1}{6}\left(f(\lambda) + h(\lambda) - 2g(\lambda)\right) + h(\lambda) f(\lambda) - g(\lambda)^2\right\} \\
&\to 0 \quad \text{as} \quad \lambda \to 0 \quad \text{or as} \quad \lambda \to \infty.
\end{aligned} \tag{A.13}
$$

Now for S^4, we set $v_\lambda = \frac{1+r^2}{2} u_\lambda$. From Equation (A.1), we get

$$6\Delta_{g_0} v_\lambda + R_\lambda v_\lambda^3 = 12 v_\lambda. \tag{A.14}$$

Our desired conformal metrics on S^4 will be the metrics $g_\lambda = v_\lambda^2 g_0$. Now we only need to check Parts a, b, c, d and e of the proposition.

To see part a, from the definition of the function v_λ, we have

$$v_\lambda = \frac{1+r^2}{2}\left(\frac{\lambda}{(\lambda^2 + r^2)} + \frac{\lambda}{(1 + \lambda^2 r^2)}\right).$$

Thus when we value them at $r = 0$ or $r = \infty$, we have

$$
\begin{aligned}
v_\lambda(0) = v_\lambda(\infty) &= \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) \\
&\to \infty \quad \text{as} \quad \lambda \to 0 \quad \text{or as} \quad \lambda \to \infty.
\end{aligned}
$$

To see part b, we observe that, for $i = 1, 2, 3$ and 4,

$$x_i = \frac{2y_i}{1 + |y|^2}.$$

Hence

$$\int_{S^4} x_i dv_{g_\lambda} = \int_{\Re}^4 \frac{2y_i}{1+|y|^2}\left[\frac{\lambda}{1+\lambda^2 r^2}+\frac{\lambda}{\lambda^2+r^2}\right]^4 dy.$$

Then, using spherical coordinates, we can easily see that this integral is zero.

For x_5, we have

$$x_5 = \frac{|y|^2-1}{|y|^2+1}.$$

Therefore,

$$\begin{aligned}\int_{S^4} x_5 dv_{g_\lambda} &= \int_{\Re}^4 \frac{|y|^2-1}{1+|y|^2}\left[\frac{\lambda}{1+\lambda^2 r^2}+\frac{\lambda}{\lambda^2+r^2}\right]^4 dy\\ &= vol\left(S^3\right)\int_0^\infty \frac{r^2-1}{1+r^2}\left[\frac{\lambda}{1+\lambda^2 r^2}+\frac{\lambda}{\lambda^2+r^2}\right]^4 r^3 dr\\ &= vol\left(S^3\right)\left\{\int_0^1 \frac{r^2-1}{1+r^2}\left[\frac{\lambda}{1+\lambda^2 r^2}+\frac{\lambda}{\lambda^2+r^2}\right]^4 r^3 dr\right.\\ &\left.+\int_1^\infty \frac{r^2-1}{1+r^2}\left[\frac{\lambda}{1+\lambda^2 r^2}+\frac{\lambda}{\lambda^2+r^2}\right]^4 r^3 dr\right\}.\end{aligned}$$

By using a variable change in one of these integrals, we can see that they cancel each other.

To see part c, we have shown the relations (A.4), (A.5), (A.6) and (A.13). Part c follows easily from these relations.

Part d follows from (A.4) and (A.10).

To see part e, we note that Equation (A.14) shows that the functions R_λ which have been given at the beginning of the proof of the proposition are exactly the scalar curvature functions of the conformal metrics g_λ. Thus, from that expression, we can conclude that R_λ is uniformly bounded. That means that there are upper and lower bounds on R_λ which do not depend on the parameter λ.

To see part f, we note from Equation (A. 14) and part e that

$$\int_{S^4} \left(\Delta_{g_0}\left(v_\lambda\right)\right)^2 dv_{g_0}$$

is equivalent to $\int_{S^4} \left(v_\lambda\right)^6 dv_{g_0}$ while the latter can be estimated as follows:

$$\begin{aligned}\int_{S^4} \left(v_\lambda\right)^6 dv_{g_0} &= \int_{\Re}^4 \left(\frac{1+r^2}{2}\right)^6 u_\lambda^6 \left(\frac{2}{1+r^2}\right)^4 dy\\ &= vol\left(S^3\right)\int_0^\infty \left[\frac{1+r^2}{2}\right]^2 \left(\frac{\lambda}{\lambda^2+r^2}+\frac{\lambda}{1+\lambda^2 r^2}\right)^6 r^3 dr\end{aligned}$$

$$\begin{aligned}
&\geq vol\left(S^3\right)\int_0^\infty \left[\frac{1+r^2}{2}\right]^2\left(\frac{\lambda}{\lambda^2+r^2}\right)^6 r^3dr\\
&=\frac{vol\left(S^3\right)}{8}(\lambda)^6\int_0^\infty \frac{(1+s)^2 s}{(\lambda^2+s)^6}ds\\
&=\frac{vol\left(S^3\right)}{8}(\lambda)^6\left\{\frac{1}{6\lambda^6}+\frac{\left(1-\lambda^2\right)^2}{20\lambda^8}\right\}\\
&=\frac{vol\left(S^3\right)}{480}\frac{3+4\lambda^2+3\lambda^4}{\lambda^2}\\
&\to\infty \text{ as } \lambda\to 0 \text{ or as } \lambda\to\infty.
\end{aligned}$$

We can even get a larger set of conformal metrics which satisfy the proposition by considering the different points on the Sphere as south pole.

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