On Singularities of the Heat Flow for Harmonic maps from Surfaces into Spheres

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In this paper we prove that there is no unaccounted energy loss for blow-ups of solutions of the heat equation for harmonic maps from surfaces into spheres. Any energy drop corresponds precisely to the production of bubbles, that is, nontrivial harmonic maps from 2-sphere into the target

1. Introduction.

In this paper we shall study the local behavior of solutions of the heat equation for harmonic maps from surfaces into spheres near singular points. About ten years ago, Michael Struwe [17] proved that global solutions of the heat equation for harmonic maps from surfaces exist and are regular except possibly at finitely many isolated points. Recently K.C.Chang, W.Ding and R.Ye [6] showed that these singular points indeed exist in some cases. Naturally then, one would like to know what happens to the solutions near these singular points. In fact, to get a complete description of the set of harmonic maps by Morse theory it might be helpful to know the local behavior of the solutions near singular points, since the solutions of the heat equation for harmonic maps can be naturally interpreted as the trajectories of gradient-like vector fields with respect to the energy functional.

For simplicity, we assume that the domain is just R^2. The heat equation for harmonic maps is as follows:

$$(1.1) \qquad \begin{cases} \partial_t u = \triangle u + A(u)(\nabla u, \nabla u) & in \ R^2 \\ u(x,0) = u_0(x), \end{cases}$$

where $A(\cdot)(\cdot, \cdot)$ is the second fundamental form of the target manifold N in some R^k, and u_0 is a map in $W^{1,2}(R^2, N)$ with finite energy. By Struwe's results [17] we know, for any u_0 with finite energy, there exists a global solution

for (1.1) which is regular except at finitely many points $\{(x_j, T_j)\}_{j=1}^l = S$. Let T be any one of $\{T_j\}_{j=1}^l$, and let $u(x, T)$ be a weak limit of the solution $u(x, t)$ as $t \to T$. Then we have

Theorem 1.1. *Suppose that N is a sphere and $u(x, t)$ is a solution of (1). Furthermore, assume that $(x_0, T) \in S$. Then there exist finitely many, non-constant, harmonic maps $\{\omega_k\}_{k=1}^p$ from S^2 into N such that*

$$(1.2) \qquad \lim_{t \to T} \int_{B_R} |\nabla u|^2(\cdot, t) dx = \int_{B_R} |\nabla u(\cdot, T)|^2 dx + \sum_{k=1}^p E(\omega_k),$$

for any disk $B_R = B_R(x_0)$ with $\overline{B_{2R}(x_0)} \bigcap \{x \neq x_0 : (x, T) \in S\} = \Phi$. Moreover, for some sequence $\{t_i\}$ with $\lim_{i \to \infty} t_i = T$ there exist
(a) sequences $\{a_i^1\}, \{a_i^2\}, \cdots \{a_i^p\}$ in R^2 with $\lim_{i \to \infty} a_j^k = x_0$ for $1 \leq k \leq p$, and
(b) sequences $\{\lambda_i^1\}, \{\lambda_i^2\}, \cdots \{\lambda_i^p\}$, λ_i^k positive for $1 \leq k \leq p$ and any i, and $\lim_{i \to \infty} \lambda_i^k = 0$ for $1 \leq k \leq p$, satisfying, for $j \neq k$,

$$(1.3) \qquad \frac{\lambda_i^k}{\lambda_i^j} + \frac{\lambda_i^j}{\lambda_i^k} + \frac{|a_i^k - a_i^j|^2}{\lambda_i^k \lambda_i^j} \to \infty, \; as \; i \to \infty,$$

and

$$(1.4) \qquad u(x, t_i) - \sum_{k=1}^p (\omega_k(\frac{x - a_i^k}{\lambda_i^k}) - \omega_k(\infty)) \to u(x, T)$$

strongly in $W^{1,2}(B_R, R^k)$, where $\omega_k(\infty)$ is the image of ω_k at North pole of S^2.

Remark 1. (1.2) implies there is no unaccounted energy loss. In other words, the necks connecting bubbles hardly contribute any energy. In fact, this point is one of the main difficulties in the proof. Jost, in the book [11], has proved a similar result for certain min-max values.

Remark 2. (1.4) is only proved for certain sequence $\{t_i\}$, since we do not know whether this configuration of bubbles is unique or not when the sequence varies: This is a very interesting and challenging open problem.

The proof of Theorem 1.1 is indirect. Our strategy is to grab all bubbles in one time. Then by appropriate scaling, turn the problem into that of blow-ups of a Palais-Smale sequence. For this purpose we prove the following proposition, which may be considered as an improvement of Proposition 5.1 in [17].

Proposition 1.2. *Suppose that $u_n \in W^{1,2}(R^2, S^N)$ with $E(u_n) \leq C$, and that $E'(u_n) \to 0$ in $L^2_{loc}(R^2, R^{K+1})$. Then there may exist*
(a) sequences $\{a_n^1\}, \{a_n^2\}, \cdots \{a_n^p\}$ in R^2 with $\lim_{i \to \infty} a_n^k = x^k$ for $1 \leq k \leq p$, and
(b) sequences $\{\lambda_n^1\}, \{\lambda_n^2\}, \cdots \{\lambda_n^p\}$ with λ_n^k positive for $1 \leq k \leq p$ any n, and $\lim_{n \to \infty} \lambda_n^k = 0$ for any $1 \leq k \leq p$, satisfying, for $j \neq k$,

$$(1.5) \qquad \frac{\lambda_n^k}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^k} + \frac{|a_n^k - a_n^j|^2}{\lambda_n^k \lambda_n^j} \to \infty, \ as \ n \to \infty,$$

and
(c) a harmonic map ω_0 from S^2 into S^N and a sequence of non-constant harmonic maps $\{\omega_k\}_{k=1}^p$ from S^2 into S^N with the properties

$$(1.6) \qquad u_n(x) - \sum_{k=1}^p (\omega_k(\frac{x - a_n^k}{\lambda_n^k}) - \omega_k(\infty)) \to \omega_0(x),$$

strongly in $W^{1,2}_{loc}(R^2, R^{K+1})$ therefore

(1.7)

$$\lim_{n \to \infty} \int_{B_K} |\nabla u_n|^2 dx = \int_{B_K} |\nabla \omega_0|^2 dx + \sum_{k=1}^p E(\omega_k) \ \text{for any } K < \infty.$$

Remark 3. Possible bubbles drifting to infinity indeed could happen here as noticed by a referee. But Theorem 1.1 only concerns the local behavior of the solution of heat equation near a given singular point.

Note. In this paper, to simplify the exposition, we make no effort to distinguish sequences and subsequences: that is, we will often replace sequences with their subsequences and use the same indices.

Our proof of Proposition 1.2 is based on energy estimates over the necks connecting bubbles, together with a procedure for analyzing blow-ups which is similar to one used by Brezis and Coron in [4].

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2. Solutions of Heat Equations.

In this section we shall study the local behavior of solutions of the heat equation for harmonic maps from surfaces near their singular points. First we will use the local energy comparison lemma from Struwe's paper [17] to prove the following:

Proposition 2.1. *Suppose that u is a solution of (1.1), and that (x_0, T) and B_R are given as in the statement of Theorem 1.1.*
Then $\lim_{t \to T} \int_{B_R} |\nabla u|^2(\cdot, t) dx$ exists.

Proof. Suppose that

(2.1)
$$\limsup_{t \to T} \int_{B_R} |\nabla u|^2(\cdot, t) = \lim_{t_i \to T} \int_{B_R} |\nabla u|^2(\cdot, t_i) = M + \int_{B_R} |\nabla u|^2(\cdot, T),$$

and

(2.2)
$$\liminf_{t \to T} \int_{B_R} |\nabla u|^2(\cdot, t) = \lim_{s_i \to T} \int_{B_R} |\nabla u|^2(\cdot, s_i) = m + \int_{B_R} |\nabla u|^2(\cdot, T).$$

Then to prove this proposition we only need to prove $m = M$. Take $\lambda_j = \frac{1}{j} R$. Since $u(x, t) \to u(x, T)$ strongly in C^∞ on any compact subset of $B_R \backslash \{x_0\}$) as $t \to T$ (see Struwe's [17]), for each j there exist i_j such that

(2.3)
$$\int_{B_R \backslash B_{\lambda_j}(x_0)} ||\nabla u(\cdot, t_{i_j})|^2 - |\nabla u(\cdot, T)|^2|dx < \frac{1}{2} \int_{B_{\lambda_j}(x_0)} |\nabla u(\cdot, T)|^2 dx,$$

and by (2.1),

(2.4) $$\int_{B_R} |\nabla u(\cdot, t_{i_j})|^2 dx - \int_{B_R} |\nabla u(\cdot, T)|^2 dx$$
$$\geq M - \frac{1}{2} \int_{B_{\lambda_j}(x_0)} |\nabla u(\cdot, T)|^2 dx.$$

Therefore we have

$$(2.5) \qquad \int_{B_{\lambda_j}(x_0)} |\nabla u(\cdot, t_j)|^2 dx \geq M,$$

where $t_j = t_{i_j}$. Assume that,

$$s_j \leq t_j \leq s_{j+1} \leq t_{j+1},$$

by taking subsequences from $\{t_j\}$ and $\{s_j\}$ if necessary. For any $\delta > 0$, by Lemma 3.6 in [17], we get

$$(2.6)$$
$$\int_{B_\delta(x_0)} |\nabla u(\cdot, s_j)|^2 dx \geq \int_{B_{\lambda_j}(x_0)} |\nabla u(\cdot, t_j)|^2 dx - \frac{c|t_j - s_j|}{(\delta - \lambda_j)^2} E(u_0).$$

This implies

$$(2.7) \quad \lim_{s_j \to T} \int_{B_R} |\nabla u|^2 (\cdot, s_j) =$$

$$= \lim_{s_j \to T} \left\{ \int_{B_R \setminus B_\delta(x_0)} |\nabla u(\cdot, s_j)|^2 + \int_{B_\delta(x_0)} |\nabla u(\cdot, s_j)|^2 \right\}$$

$$\geq M + \int_{B_R \setminus B_\delta(x_0)} |\nabla u(\cdot, T)|^2 dx.$$

Taking $\delta \to 0$, we can conclude that $m = M$, which completes the proof. To prove Theorem 1.1, we will choose the scaling as λ_j defined in the above proof, and an appropriate time sequences to convert the problem into that of blow-ups of a Palais-Smale sequence. We then apply Proposition 1.2.

Proof of Theorem 1.1. By Proposition 2.1, we know there exists a number L such that

$$(2.8) \qquad \lim_{t \to T} \int_{B_R} |\nabla u|^2 (\cdot, t) = L + \int_{B_R} |\nabla u|^2 (\cdot, T),$$

for B_R given as in the statement of Theorem 1.1. Moreover, in the light of (2.5) there are sequences $\lambda_j \to 0, t_j \to T$ such that

$$(2.9) \qquad \int_{B_{\lambda_j}(x_0)} |\nabla u(\cdot, t_j)|^2 dx \geq L.$$

Now we do the scaling procedure by setting

$$v_j(x,t) = u(\lambda_j x + x_0, -\lambda_j^2 t + t_j).$$

First of all,

(2.10) $E(v_j(\cdot, t)) = E(u(\cdot, -\lambda_j^2 t + t_j)) \le E(u_0),$

for any $t \le \frac{t_j}{\lambda_j^2}$. Secondly,

(2.11) $\int_0^S \int_\Omega |\partial_t v_j|^2 dx dt = \int_{t_j - \lambda_j^2 S}^{t_j} \int_{\Omega_j} |\partial_t u|^2 dx dt \to 0,$

as $j \to \infty$ for any given positive number S, where Ω is any bounded domain in R^2 and $\Omega_j = \lambda_j \Omega + x_0$. Because one has

(2.12)
$$\begin{aligned}
&\int_{t_1}^{t_2} \int_{B_R} |\partial_t u|^2 \phi^2 dx dt + \int_{t_1}^{t_2} \frac{d}{dt} \int_{B_R} |\nabla u|^2 \phi^2 dx\, dt \\
&\le C \int_{t_1}^{t_2} \int_{B_r} |\nabla u||\partial_t u||\nabla \phi|\phi dx dt,
\end{aligned}$$

using $\partial_t u \phi^2$ as a testing function for equation (1.1), where $\phi \in C_0^\infty(B_R), \phi \equiv 1$ in $B_{\frac{R}{2}}$ (see the proof of Lemma 3.6 in Struwe's paper [17]). Therefore

(2.13) $\displaystyle\int_{t_1}^{t_2} \int_{B_R} |\partial_t u|^2 dx dt$

$$\le \int_{B_R} |\nabla u|^2(\cdot, t_1) dx - \int_{B_R} |\nabla u|^2(\cdot, t_2) dx + C \frac{t_2 - t_1}{R^2}$$

$$\le C \sup_{[t_1, t_2]} \int_{B_R \setminus B_{\frac{R}{2}}} |\nabla u|^2(\cdot, t) dx + \int_{B_{\frac{R}{2}}} |\nabla u|^2(\cdot, t_1) dx$$

$$- \int_{B_{\frac{R}{2}}} |\nabla u|^2(\cdot, t_2) dx + C \frac{t_2 - t_1}{R^2}$$

Thus (2.11) is proved. Then for some number $\tau_j \in (0, S)$ we have

(2.14) $E(v_j) \le C, \text{ and } \displaystyle\int_\Omega |\partial_t v_j|^2 dx \to 0, \text{ as } j \to \infty,$

where $v_j(x) = v_j(x, \tau_j)$. Now on one hand, for any positive number K,

(2.15) $\displaystyle\int_{B_K} |\nabla v_j|^2 dx = \int_{B_{\lambda_j K}} |\nabla u(\cdot, -\lambda_j^2 \tau_j + t_j)|^2 dx.$

Applying Lemma 3.6 in [17] again, we have,

$$(2.16) \quad \int_{B_{\lambda_j K}} |\nabla u(\cdot, -\lambda_j^2 \tau_j + t_j)|^2 dx$$

$$\geq \int_{B_{\lambda_j}} |\nabla u(\cdot, t_j)|^2 dx - \frac{c\tau_j}{(K-1)^2} E(u_0).$$

Therefore,

$$(2.17) \quad \int_{B_K} |\nabla v_j|^2 dx \geq L - \frac{cS}{(K-1)^2} E(u_0).$$

On the other hand, for any positive small number δ,

$$(2.18) \quad \int_{B_{\frac{\delta}{\lambda_j}}} |\nabla v_j|^2 dx = \int_{B_\delta} |\nabla u(\cdot, -\lambda_j^2 \tau_j + t_j)|^2 dx.$$

By the Proposition 2.1 we have,

(2.19)

$$\lim_{j\to\infty} \int_{B_\delta} |\nabla u(\cdot, -\lambda_j^2 \tau_j + t_j)|^2 dx = L + \int_{B_\delta} |\nabla u(\cdot, T)|^2 dx.$$

Thus

$$(2.20) \quad \lim_{K\to\infty} \lim_{j\to\infty} \int_{B_K} |\nabla v_j|^2 dx = L.$$

By the homogeneity of the scaling and (2.12), we can conclude that $\{v_j(x)\}$ is a Palais-Smale sequence to which we can apply Proposition 1.2. The only thing left is the convergence (1.4). In view of Proposition 1.2 we define

$$(2.21) \quad f_j(x) = \omega_0(x) + \sum_{k=1}^{p} (\omega_k(\frac{x - a_j^k}{\lambda_j^k}) - \omega_k(\infty)),$$

which has the property: $v_j(x) - f_j(x)$ strongly converges to zero in $W_{loc}^{1,2}(R^2, R^{K+1})$. So we have, for any given K large enough,

(2.22)

$$\int_{B_R} \left| \nabla u(x, \hat{t}_j) - \nabla f_j\left(\frac{x - x_0}{\lambda_j}\right) \right|^2 dx = \int_{B_{\frac{R}{\lambda_j}}} |\nabla v_j - \nabla f_j|^2 dx$$

$$= \int_{B_{\frac{R}{\lambda_j}} \setminus B_K} |\nabla v_j - \nabla f_j|^2 dx + \int_{B_K} |\nabla v_j - \nabla f_j|^2 dx$$

$$= \int_{B_{\frac{R}{\lambda_j}} \setminus B_K} |\nabla v_j - \nabla f_j|^2 dx + o(1),$$

where

$$(2.23) \quad \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} |\nabla v_j - \nabla f_j|^2 dx$$

$$= \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} |\nabla v_j|^2 dx + \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} |\nabla f_j|^2 dx - 2 \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} \nabla v_j \cdot \nabla f_j dx$$

$$= \int_{B_R \backslash B_{\lambda_j K}} |\nabla u|^2 (\cdot, -\lambda_j^2 \tau_j + t_j) dx$$

$$+ \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} |\nabla f_j|^2 dx - 2 \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} \nabla v_j \cdot \nabla f_j dx.$$

By the definition of λ_j in (16), it is easily seen that

$$(2.24)$$
$$\int_{B_R \backslash B_{\lambda_j K}} |\nabla u|^2 (\cdot, -\lambda_j^2 \tau_j + t_j) dx = o(1) + \int_{B_R} |\nabla u|^2 (\cdot, T) dx,$$

also,

$$(2.25) \quad \begin{array}{c} \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} |\nabla f_j|^2 dx = o(1) \text{ as } K \to \infty \\ \int_{B_{\frac{R}{\lambda_j}} \backslash B_K} \nabla v_j \cdot \nabla f_j dx = o(1), \text{ as } K \to \infty \end{array}$$

since f_j is a combination of the standard bubbles centered at given points $\{x^k\}$. Therefore

$$(2.26)$$
$$\lim_{j \to \infty} \int_{B_R} \left| \nabla u(x, -\lambda_j^2 \tau_j + t_j) - \nabla f_j \left(\frac{x - x_0}{\lambda_j} \right) \right|^2 dx = \int_{B_R} |\nabla u|^2 (\cdot, T) dx.$$

On the other hand, again by the special form of f_j, we have

$$(2.27) \quad u(x, \hat{t}_j) - \left(f_j \left(\frac{x - x_0}{\lambda_j} \right) - \omega_0(\infty) \right) \rightharpoonup u(x, T)$$

$$\text{as } j \to \infty \text{ in } W^{1,2}(B_R, R^{K+1}).$$

Therefore we have the convergence (1.4). This finishes the proof of Theorem 1.1.

3. Proof of Proposition 1.2.

In this section we shall discuss the behavior of Palais-Smale sequences and prove Proposition 1.2. We all know that the deep regularity theory of minimizing harmonic maps due to Rick Schoen and Karen Uhlenbeck [14, 15] has had profound influences not only in the field of geometric analysis but also in the field of nonlinear partial differential equations. It also has been noticed that the recent works of Frederic Helein [10], Fabric Bethuel [1] and C.L.Evans [9] on the regularity of stationary harmonic maps captured important features of certain nonlinear differential equations by introducing Hardy space and Fefferman-Stein's duality theorem of BMO spaces and Hardy spaces to the nonlinear analysis. Helein first observed the special form of the equation for harmonic maps into spheres, though Yunmei Chen [8] and J. Shatah [16] seem to have been the first to deduce useful analytic consequences from this observation for weakly harmonic maps. We shall use the special form of the equation for harmonic maps into spheres in the proof of our key lemma in this section. Now let us state a lemma about ϵ-compactness for a special kind of Palais-Smale sequences, whose proof is very standard in the light of Helein [10] and Qing [12].

Lemma 3.1. *Suppose that N is a compact manifold. And suppose that $u_n \in W^{1,2}(R^2, N)$ with*

$$(3.1) \qquad -\Delta u_n = A(u_n)(\nabla u_n, \nabla u_n) + h_n$$

where h_n goes to zero in $L^2_{loc}(R^2, R^k)$, and that $u_n \rightharpoonup u$ in $W^{1,2}(R^2, R^k)$, $E(u_n) \leq C$. Then u is a harmonic map. And, there exists an ϵ_0, u_n strongly converges to u in $W^{1,2}(B_R, R^k)$ for any $r < R$ provided that $\int_{B_R} |\nabla u_n|^2 dx \leq \epsilon_0$.

Remark 4. About the harmonicity of the weak limit, in [2] Bethuel recently proved that a weak limit of any Palais-Smale sequence is still harmonic when the target is a general compact manifold.

Proof. First let us assume that N has a global orthonormal frame on it without loss of any generality. Otherwise we extend the metric of N to the tubular neighborhood $\tilde{N}$ of N in R^K so that N is a totally geodesic submanifold of $\tilde{N}$ (see [10]), and we replace N by $\tilde{N}$ if it is necessary. Note that the equation (3.1) stays the same since the image of u_n lies in N. Then

using the same notation in [12] (see Lemma 3 in [12]), there exists an ϵ_0 such that

$$(3.2) \qquad \bar{\partial}\left(M^{ij}\left\langle\frac{\partial u}{\partial x}-\sqrt{-1}\frac{\partial u}{\partial y}, e_i\right\rangle\right) = \langle h, e_j\rangle \text{ in } \Omega \subset R^2,$$

where $\{e_i\}$ is a global orthonormal frame on N (we ignore the index n here for a moment without confusion), and, as stated in Lemma 3 in [12], the norms of the metrix (M_{ij}) and its inverse are bounded in Ω by a universal constant (independent on n) provided that $\int_\Omega |\nabla u|^2 dx \leq \epsilon_0$. So by standard elliptic estimates and Sobolev Embedding Theorem we have ϵ-compactness since $\|\langle h, e_j\rangle\|_{L^2} \leq \|h\|_{L^2}$. Then, this lemma follows from the standard argument due to Sacks and Uhlenbeck [13]. So the proof is complete.

Next, we will state and prove a lemma which will be used to estimate the energy over the necks connecting bubbles. Let $\Omega = B_R \backslash B_r$ with $0 < r < R \leq 1$.

Lemma 3.2. *Suppose that* $u \in W^{1,2}(B_R, S^K)$ *with* $\int_\Omega |\nabla u|^2 dx \leq \varepsilon$ *satisfies*

$$(3.3) \qquad \triangle u = u|\nabla u|^2 + h \text{ in } B_R,$$

where $\|h\|_{L^2(B_R)} \leq \varepsilon$. *Assume that* v *is given as follows*

$$(3.4) \qquad \begin{cases} \triangle v = 0 \text{ in } \Omega \\ v|_{\partial\Omega} = u|_{\partial\Omega} \end{cases}$$

Then there exists an ε_0 *such that*

$$(3.5) \qquad \int_\Omega |\nabla u|^2 dx \leq C \int_\Omega |\nabla v|^2 dx + C \|h\|_{L^2(B_R)},$$

provided that $\varepsilon \leq \varepsilon_0$, *where* C *and* ε_0 *are constants independent on* Ω.

Proof. By the assumptions we have

$$(3.6) \qquad \begin{aligned} -\Delta(u-v)^k &= \sum_{l,\alpha}\left\{(u^k u^l_\alpha - u^l u^k_\alpha)u^l_\alpha\right\} + h^k \\ &= \sum_{l,\alpha}\left\{(u^k u^l_\alpha - u^l u^k_\alpha)(u^l - v^l)_\alpha\right\} \\ &\quad + \sum_{l,\alpha}\left\{(u^k u^l_\alpha - u^l u^k_\alpha)v^l_\alpha\right\} + h^k. \end{aligned}$$

Using the special algebraic structure we let $A^{kl} = (u^k u^l_1 - u^l u^k_1, u^k u^l_2 - u^l u^k_2)$. Then $\text{div} A^{kl} = u^l h^k - u^k h^l$. Suppose we have ϕ^{kl} satisfying

$$(3.7) \qquad \begin{cases} -\Delta\phi^{kl} = u^k h^l - u^l h^k \text{ in } B_R \\ \phi^{kl}|_{B_R} = 0. \end{cases}$$

Then we have $\text{div}(A^{kl} - \nabla\phi^{kl}) = 0$ in B_R. Next we rewrite the equation (3.6) into

(3.8)
$$
\begin{aligned}
-\Delta(u-v)^k = & \sum_l (A^{kl} - \nabla\phi^{kl})\nabla(u-v)^l \\
& + \text{div}\left(\sum_l \nabla\phi^{kl}(u-v)^l\right) - \sum_l \Delta\phi^{kl}(u-v)^l \\
& + \sum_l A^{kl} \cdot \nabla v^l + h^k.
\end{aligned}
$$

Now let w_1 satisfy

(3.9)
$$
\begin{cases}
-\Delta w_1^k = \sum_l (A^{kl} - \nabla\phi^{kl})\nabla(u-v)^l & in\ \Omega \\
w_1^k|_{\partial\Omega} = 0,
\end{cases}
$$

and let w_2 satisfy

(3.10)
$$
\begin{cases}
-\Delta w_2^k = \text{div}\left(\sum_l \nabla\phi^{kl}(u-v)^l\right) & in\ \Omega \\
w_2^k|_{\partial\Omega} = 0.
\end{cases}
$$

To estimate w_1, we notice that $\text{div}(A^{kl} - \nabla\phi^{kl}) = 0$ in B_R. By Theorem 1.2 in [3] (see also [7] of S.Chanillo and Yanyan Li) we have

(3.11)
$$
\|\nabla w_1\|_{L^2(\Omega)} \le C \sum_{k,l} \|A^{kl} - \nabla\phi^{kl}\|_{L^2(\Omega)} \|\nabla(u-v)^l\|_{L^2(\Omega)},
$$

where C is independent of Ω. The key point here is that, by the fact that B_R is simply connected, we can rewrite the equation into a form to which Theorem 1.2 in [3] can be applied. And to estimate w_2, we simply multiply two sides of the equation (3.10) by w_2 and integrate over Ω

(3.12)
$$
\begin{aligned}
\|\nabla w_2\|_{L^2(\Omega)} & \le \sum_k \left\|\sum_l \nabla\phi^{kl}(u-v)^l\right\|_{L^2(\Omega)} \\
& \le \|\nabla\phi\|_{L^4(B_R)}\|u-v\|_{L^4(\Omega)} \\
& \le C\|h\|_{L^2(B_R)}\|\nabla(u-v)\|_{L^2(\Omega)},
\end{aligned}
$$

where again C is independent of Ω. Now we multiply the equation (3.8) by $u - v$ and integrate over Ω. We get

(3.13)

$$\int_\Omega |\nabla(u-v)|^2 dx = \sum_k \left\{ \int_\Omega (-\Delta w_1^k)(u-v)^k dx \int_\Omega (-\Delta w_2^k)(u-v)^k dx \right\}$$

$$+ \sum_{k,l} \left\{ \int_\Omega (u^k h^l - u^l h^k)(u-v)^k(u-v)^l dx + \int_\Omega A^{kl} \nabla v^l (u-v)^k dx \right\}$$

$$+ \sum_k \int_\Omega h^k (u-v)^k dx.$$

$$\leq C\epsilon \int_\Omega |\nabla(u-v)|^2 dx + C \int_\Omega |\nabla u| |\nabla v| dx + C\|h\|_{L^2(B_R)}.$$

Since all C's here are independent of Ω, when ε is small enough we have the energy control for $u - v$ as follows:

(3.14) $$\int_\Omega |\nabla(u-v)|^2 dx \leq c \int_\Omega |\nabla u||\nabla v| dx + c \|h\|_{L^2(B_R)}.$$

This implies this lemma in view of the following Schwarz inequality

(3.15) $$\int_\Omega |\nabla u|^2 dx \leq 2 \int_\Omega |\nabla(u-v)|^2 dx + 2 \int_\Omega |\nabla v|^2 dx.$$

Thus the proof is completed.

Remark 5. In the proof of the above lemma, the use of Theorem 1.2 in [3] seems very crucial. The independence of C on the domain is understood as a special feature of the equations studied in [2] and [7].

By Lemma 3.1, which is the so-called ε-compactness lemma, through a standard argument we know that for u_n and u as given in Lemma 3.1, there exist at most finitely many points $x^1, x^2, \cdots, x^l$ such that u_n converges to u strongly in $W_{loc}^{1,2}(R^2 \setminus \{x^1, x^2, \cdots, x^l\}, S^K)$. So what we are really concerned with is the local behavior of u_n around each of these special points. For simplicity, we will focus our attention on a single blow-up point $\bar{x} \in \{x^k\}_{k=1}^l$. Let

$$B_\delta(\bar{x}) \bigcap (\{x^k\}_{k=1}^l \setminus \bar{x}) = \phi.$$

Now we employ the quantity $Q_n(t)$ used by Brezis and Coron in [4] to detect bubbles. Namely,

(3.16) $$Q_n(t) = \sup_{x \in B_\delta(\bar{x})} \int_{x+tD} |\nabla u_n|^2 dx,$$

where D is the unit disk in R^2. Then for ε_0 given in Lemma 3.1 and Lemma 3.2 we can find λ_n and a_n such that

$$(3.17) \qquad \int_{a_n + \lambda_n D} |\nabla u_n|^2 dx = Q_n(\lambda_n) = \frac{\varepsilon_0}{2}.$$

It is clear that $\inf_{n \to \infty} \lambda_n = 0$ and $\lim_{n \to \infty} a_n = \bar{x}$, otherwise $\bar{x}$ would not be a blow-up point by Lemma 3.1.

Proof of Proposition 1.2. It is quite easy to take out the first bubble by scaling with λ_n and a_n, namely let $\widetilde{u}_n(x) = u_n(\lambda_n x + a_n)$. Then $\widetilde{u}$ satisfies

$$(3.18) \qquad \triangle \widetilde{u}_n = A(\widetilde{u}_n)(\nabla \widetilde{u}_n, \nabla \widetilde{u}_n) + \lambda_n^2 h_n(\lambda_n x + a_n).$$

It is easy to see that $\lambda_n^2 h_n(\lambda_n x + a_n) \to 0$ in L^2. Therefore by Lemma 3.1 we know $\widetilde{u}_n \to \omega$ strongly in $W_{loc}^{1,2}(R^2, R^{K+1})$, where ω is a non-constant harmonic map from S^2 into S^K since $\int_D |\nabla \omega|^2 dx = \frac{\varepsilon_0}{2} > 0$. Moreover, if we let

$$(3.19) \qquad \omega_0 = u, \ \lambda_n^1 = \lambda_n, \ a_n^1 = a_n \ and \ \omega_1 = \omega,$$

we have

$$(3.20) \qquad \int_{B_\delta} \left| \nabla \left(u_n(x) - \omega_1 \left(\frac{x - a_n^1}{\lambda_n^1} \right) \right) \right|^2 dx$$
$$= \int_{B_\delta} |\nabla u_n|^2 dx - \int_{R^2} |\nabla \omega_1|^2 dx + o(1),$$

and

$$(3.21) \qquad u_n(x) - \left[\omega_1 \left(\frac{x - a_n^1}{\lambda_n^1} \right) - \omega_1(\infty) \right] \rightharpoonup \omega_0(x),$$

in $W^{-1,2}(R^2, R^{K+1})$. Now we are going to detect the second bubble. As we said in Note 1, we may assume that $\lim_{n \to \infty} \int_{B_\delta} |\nabla u_n|^2 dx$ exists. By (3.21) we know that

$$(3.22) \qquad \int_{B_\delta} |\nabla \omega_0|^2 dx \leq \lim_{n \to \infty} \int_{B_\delta} |\nabla u_n|^2 dx - \int_{R^2} |\nabla \omega_1|^2 dx,$$

and equality holds if and only if

$$(3.23) \qquad u_n(x) - \left[\omega_1 \left(\frac{x - a_n^1}{\lambda_n^1} \right) - \omega_1(\infty) \right] \to \omega_0(x)$$

strongly in $W_{loc}^{1,2}(R^2, R^{K+1})$. If this happens, we can stop searching for bubbles, and Proposition 1.2 holds. Otherwise, we have

(3.24)
$$\int_{B_\delta} |\nabla \omega_0|^2 dx < \lim_{n\to\infty} \int_{B_\delta} \left| \nabla \left(u_n - \omega_1 \left(\frac{x - a_n^1}{\lambda_n^1} \right) \right) \right|^2 dx,$$

which implies

(3.25)
$$\limsup_{t\to 0} \limsup_{n\to\infty} \int_{B_t(\bar{x})} \left| \nabla \left(u_n - \omega_1 \left(\frac{x - a_n^1}{\lambda_n^1} \right) \right) \right|^2 dx = r > 0,$$

since $\bar{x}$ is an isolated blow-up point. Now let $v_n(x) = u_n(x) - (\omega_1(\frac{x-a_n^1}{\lambda_n^1}) - \omega_1(\infty))$ and redefine $Q_n(t)$ as follows:

(3.26)
$$Q_n(t) = \sup_{x\in B_\delta} \int_{x+tD} |\nabla v_n|^2 dx.$$

By (3.25) again we can find λ_n and a_n such that

(3.27)
$$\int_{a_n+\lambda_n D} |\nabla v_n|^2 dx = Q_n(\lambda_n) = \min\left(\frac{r}{2}, \frac{\varepsilon_0}{2}\right) > 0,$$

and

(3.28)
$$\lim_{n\to\infty} a_n = \bar{x}, \quad \lim_{n\to\infty} \lambda_n = 0.$$

Next we are going to prove that

(3.29)
$$\frac{\lambda_n}{\lambda_n^1} + \frac{|a_n - a_n^1|}{\lambda_n^1} \to \infty \ as \ n \to \infty$$

by contradiction. In fact, if $\frac{\lambda_n}{\lambda_n^1} < M$ and $\frac{|a_n - a_n^1|}{\lambda_n^1} < M$ for some big number M, then

(3.30)
$$\int_{a_n+\lambda_n D} |\nabla v_n|^2 dx$$
$$\leq \int_{a_n^1+3M\lambda_n^1 D} |\nabla v_n|^2 dx = \int_{3MD} |\nabla(\tilde{u}_n - \omega_1)|^2 dx = o(1),$$

because $\widetilde{u}_n \to \omega$ strongly in $W^{1,2}_{loc}(R^2, R^{K+1})$. This is obviously contradicts the definition of a_n and λ_n. Now we do the scaling procedure again for v_n with a_n and λ_n, which yields

(3.31)
$$\widetilde{v}_n(x) = u_n(\lambda_n x + a_n) - \left[\omega_1 \left(\frac{\lambda_n}{\lambda_n^1} x + \frac{a_n - a_n^1}{\lambda_n^1} \right) - \omega_1(\infty) \right] \rightharpoonup \omega(x),$$

for some ω. By (3.29) it is not hard to see that $\omega_1(\frac{\lambda_n}{\lambda_n^1} x + \frac{a_n - a_n^1}{\lambda_n^1}) - \omega_1(\infty) \to 0$. Therefore ω is a harmonic map from S^2 into S^K by Lemma 3.1. What we need now is to prove that $\widetilde{v}_n \to \omega$ strongly in $W^{1,2}_{loc}(R^2, R^{K+1})$, which will imply that ω is not a constant map by (3.27). To do so, we consider two different cases.
Case 1

(3.32)
$$\frac{\lambda_n}{\lambda_n^1} < M, \frac{|a_n - a_n^1|}{\lambda_n^1} \to \infty; \ or \ \frac{\lambda_n}{\lambda_n^1} \to \infty, \frac{|a_n - a_n^1|}{\lambda_n} \to \infty.$$

In fact this is the case of joint bubbles (see Figure 1), since they are essentially separated. In other words, given any big number L we have

(3.33)
$$(a_n + L\lambda_n D) \bigcap (a_n^1 + L\lambda_n^1 D) = \phi,$$

for n large enough. Therefore $\omega_1(\frac{\lambda_n}{\lambda_n^1} x + \frac{a_n - a_n^1}{\lambda_n^1}) \to \omega_1(\infty)$ in $W^{1,2}_{loc}(R^2, R^{K+1})$. Then by Lemma 3.1 with (3.27) (apply Lemma 3.1 to $u_n(\lambda_n x + a_n)$) and some covering lemma we can conclude that $\widetilde{v}_n \to \omega$ strongly in $W^{1,2}_{loc}(R^2, R^{K+1})$.

Before we go to the second case, we need to prove a key lemma derived from Lemma 3.2 as follows:

Lemma 3.3. *Suppose that u and Ω are given as in Lemma 3.2, and that*

(3.34)
$$|u - P|_{W^{\frac{1}{2},2}(\partial B_R, R^{K+1})} < \delta, |u - Q|_{W^{\frac{1}{2},2}(\partial B_R, R^{K+1})} < \delta,$$

for some points $P, Q \in S^K$. Then

(3.35)
$$\int_\Omega |\nabla u|^2 dx \le C\delta + C\frac{|P - Q|^2}{\log \frac{R}{r}} + C\|h\|_{L^2(B_R)},$$
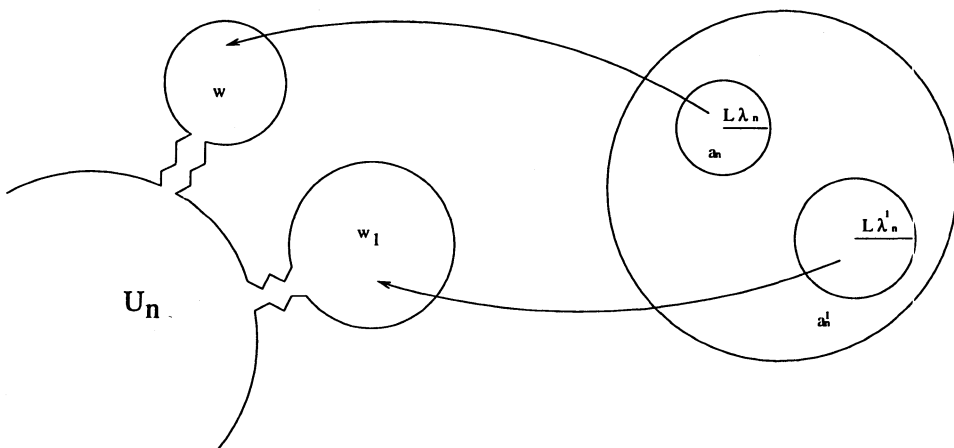
where $C's$ are independent on Ω.

Figure 1.

Proof. Since we can explicitly solve the equations:

$$(3.36) \qquad \begin{cases} \triangle w = 0 \\ w|_{\partial B_R} = P, w|_{\partial B_R} = Q, \end{cases}$$

and compute the energy as in the paper [12]. Namely,

$$(3.37) \qquad \int_\Omega |\nabla w|^2 dx = 2\pi \frac{|P - Q|^2}{\log \frac{R}{r}}.$$

It is easy to get the result from Lemma 3.2 since

$$(3.38) \qquad \int_\Omega |\nabla v|^2 \le C\delta + \int_\Omega |\nabla w|^2.$$

Now we proceed to the second case.
Case 2

$$(3.39) \qquad \frac{\lambda_n}{\lambda_n^1} \to \infty \ and \ \frac{|a_n - a_n^1|}{\lambda_n} < M.$$

This is the case of bubbles over bubbles (see Figure 2). We need to control the energy over the necks connecting bubbles. Using Lemma 3.1 as in the first case we notice that, first, $u_n(\lambda_n x + a_n)$ converges to $\omega(x)$ strongly in $W_{loc}^{1,2}(R^2 \backslash B_\epsilon(\frac{a_n^1 - a_n}{\lambda_n}))$ for any small given ϵ; second, $u_n(\lambda_n^1 x + a_n^1)$ converges
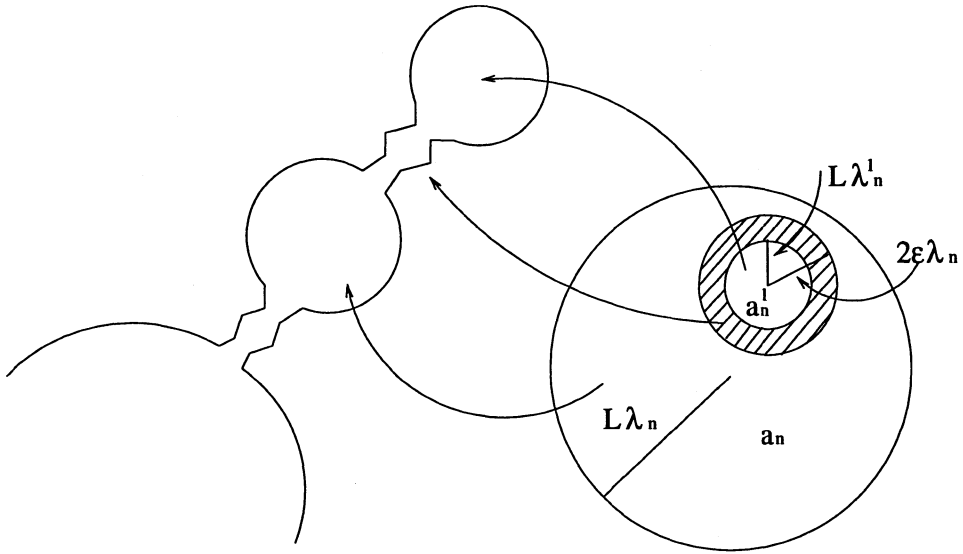
Figure 2.

to ω_1 strongly in $W^{1,2}(B_K, R^{K+1})$ for any given large K. Therefore, for any large number L

(3.40)
$$\int_{B_L(0)} |\nabla(\tilde{v}_n - \omega)|^2 dx$$
$$\leq \int_{B_{2\epsilon}(\frac{a_n^1 - a_n}{\lambda_n})} |\nabla(\tilde{v}_n - \omega)|^2 dx + o(1)$$
$$\leq \int_{B_{2\epsilon\lambda_n}(a_n^1)\backslash B_{K\lambda_n^1}(a_n^1)} |\nabla u_n|^2 dx + o(1).$$

Now we need to estimate the energy over the circular annulus $B_{2\epsilon\lambda_n}(a_n^1)\backslash B_{K\lambda_n^1}(a_n^1)$. It is not hard to see that

$$\left| u_n - \omega(P) \right|_{W^{\frac{1}{2},2}(\partial B_{2\epsilon\lambda_n}(a_n^1), R^{K+1})} < o(1),$$

where P is the limit of $\frac{1}{\lambda_n}(a_n - a_n^1)$ and

$$\left| u_n - \omega_1(\infty) \right|_{W^{\frac{1}{2},2}(\partial B_{K\lambda_n^1}(a_n^1), R^{K+1})} < o(1).$$

This follows from the strong $W^{1,2}$-convergence away from this circular annulus. Moreover by (3.27)

(3.41)
$$\int_{B_{2\epsilon\lambda_n}(a_n^1)\backslash B_{K\lambda_n^1}(a_n^1)} |\nabla u_n|^2 dx < \varepsilon_0,$$

when ϵ is small enough and K is large enough. So using Lemma 3.3 and the fact that $\frac{\lambda_n}{\lambda_n^1} \to \infty$ we can conclude that $\widetilde{v}_n \to \omega$ strongly in $W_{loc}^{1,2}$. To finish the proof of Proposition 1.2 we only need to do the above procedure of searching for bubbles over and over until

(3.42)
$$\int_{B_\delta} |\nabla \omega_0|^2 dx = \lim_{n \to \infty} \int_{B_\delta} |\nabla u_n|^2 dx - \sum_{k=1}^p \int_{R^2} |\nabla \omega_k|^2 dx.$$

This implies

(3.43)
$$u_n - \sum_{k=1}^p \left(\omega_k \left(\frac{x - a_n^k}{\lambda_n^k} \right) - \omega_k(\infty) \right) \to \omega_0,$$

strongly in $W_{loc}^{1,2}(R^2, R^{K+1})$. Thus the proof is complete.

Remark 6. From the works of Sacks and Uhlenbeck, Schoen and Uhlenbeck [13, 14, 15], it is not difficult to see that Proposition 1.2 holds if the domain is any compact surface equipped with a given metric with or without boundary. Let us be reminded that Theorem 1.2 in [3] (Theorem 0.2 in [7]) is valid not only for the laplacian of R^2 but also for a much larger class of elliptic operators including the laplacians of any surfaces with a given metric. For the case when the domain has a boundary we recall the argument in Chang's paper [5], which essentially says that all blow-ups occur in the interior in the following sense,

(3.44)
$$\frac{dist(a_k^j, \partial\Omega)}{\lambda_k^j} \to \infty \text{ as } j \to \infty.$$

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