

On the Multiplicity of the Second Eigenvalue of Laplacian in R^2

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We prove the multiplicity of the second eigenvalue is at most two for a bounded smooth domain in R^2 , which sharpened an earlier result of Cheng, S.Y. and generalized a result of Lin, C.S.

1. Introduction.

We are concerned here with the eigenfunction of a domain Ω which is the solution of the following problems

$$(1.1) \quad \begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \phi|_{\partial\Omega} = 0 \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian. Ω is a bounded smooth domain in R^2 , and λ is the corresponding eigenvalue. Here Ω is smooth domain means that $\partial\Omega$ can be parameterized as a smooth function. Let λ_2, ϕ_2 be the second eigenvalue and eigenfunction. It is well known that ϕ_2 changes sign in Ω . we know that the multiplicity of λ_2 is exactly two when the domain Ω is a disk and the corresponding eigenfunction can be written explicitly by Bessel functions. In [3], Cheng's arguments can be carried over to show that the multiplicity of the second eigenvalue of (1.1) is at most three. Later Lin in [5] sharpened his result by showing that the multiplicity is at most two provided Ω is convex. We will prove that the multiplicity of the second eigenvalue is at most two for any smooth domain.

Theorem 1.1. *For a bounded smooth domain Ω in R^2 , the multiplicity of second eigenvalues of the problem (1.1) is at most two.*

This problem is closely related to the nodal line conjecture. Let $N(\phi_2)$ be the closure of $\{(x, y) \in \Omega | \phi_2(x, y) = 0\}$ which is called the nodal line of ϕ_2 . It is easy to see that $N(\phi_2)$ is a regular curve at the point where $|\nabla\phi_2| \neq 0$. In fact, the singular set $S = \{(x, y) \in \Omega | \phi_2(x, y) = 0 \text{ and } \nabla\phi_2 = 0\}$ is

composed of finite points in our case.(see Cheng [3]) Therefore $N(\phi_2)$ is regular except at finite points. The nodal line conjecture states that $N(\phi_2)$ can not be a closed curve. There are many works on this problem in the former literatures. Recently, Melas [8] proved the conjecture provided Ω is a convex domain. In the following, we will see that theorem 1.1 is a corollary of the nodal line conjecture in some case.

In this paper, we use an idea of "domain variation" which the author believe to be new in this field. When Ω is convex in one direction, theorem 1.1 can be obtained easily, because we can regard domain translations as a special variation in this case. In general, we consider a local variation of the domain and compare the two second eigenfunctions. Finally we prove the theorem by contradictions.

Remark 1.2. It remains to be an interesting open problem of the similar results in high dimensions.

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2. Preliminary Lemmas.

In this section we present several basic lemmas. Although they have been proved before, we include their proof here for convenience of the reader. If ψ is a solution of (1.1), by Taylor expansion at origin we know that

$$(2.1) \quad \phi = C_0 + C_1 r \sin(\theta + \theta_1) + O(r^2).$$

where the first two leading terms are harmonic polynomials.

We call ϕ is first order vanishing at point p , if $\phi(p) = 0$ and second order vanishing at p , if $\phi(p) = 0, \nabla\phi(p) = 0$. The following lemma is essentially contained in [3].

Lemma 2.1. *The multiplicity of λ_2 of problem (1.1) in a bounded smooth domain is at most three.*

Proof. Suppose that $\phi_2, \phi_3, \phi_4, \phi_5$ are orthogonal eigenfunctions corresponding to λ_2 . Let $q \in \Omega$ be the origin of the coordinate. Choosing c_2, c_3, c_4, c_5 ,

such that

$$(2.2) \quad \phi = \sum_{i=2}^5 c_i \phi_i,$$

satisfies

$$(2.3) \quad \phi(q) = 0,$$

$$(2.4) \quad \nabla \phi(q) = 0.$$

We can always choose c_i , $i=2,3,4,5$, because there are only three equations.

ϕ is an eigenfunction corresponding to λ_2 with a second order vanishing at point q . Then we have

$$(2.5) \quad \phi(q) = c_2 r^2 \sin 2(\theta + \theta_2) + O(r^3),$$

near the point q . Then we can find a small circle centered at q on which ϕ changes sign at least four times. Thus at q , ϕ has at least two nodal lines transversely intersect each other. Moreover, they divided Ω into at least three parts. From Courant nodal domain theorem, we know that the nodal line of ϕ divided Ω into exactly two parts. Therefore ϕ can not have a second order vanishing point in Ω . Then we proved our lemma.

Remark 2.2. From the proof of lemma 2.1, we can see that the second eigenfunction can not have a second vanishing point in Ω . Therefore the nodal line $N(\phi_2)$ is a regular curve in Ω . But $N(\phi_2)$ may have a singular point on the boundary.

The following lemma gives us some nodal line behaviors on the boundary. It is essentially proved by Lin [5].

Lemma 2.3. *Assume the multiplicity of λ_2 is three, then for any two points $p, q \in \partial\Omega$, there exists a second eigenfunction ϕ such that $N(\phi)$ intersects the boundary at these two points. Moreover, p and q are the second order vanishing points of ϕ .*

Proof. Suppose ϕ_2, ϕ_3, ϕ_4 be three orthogonal second eigenfunctions. Again we let

$$(2.6) \quad \phi = c_2 \phi_2 + c_3 \phi_3 + c_4 \phi_4,$$

such that

$$(2.7) \quad \frac{\partial \phi}{\partial n}(p) = 0,$$

$$(2.8) \quad \frac{\partial \phi}{\partial n}(q) = 0,$$

where n is the normal direction. Such ϕ always exists, because there are two equations with three parameters. By Maximum principle we know that $N(\phi)$ intersects the boundary at p, q . Obviously p, q are the second order vanishing points of ϕ .

Under the assumptions of lemma 2.3, we have the following remarks.

Remark 2.4. If we choose p in lemma 2.3 be the origin of the coordinate and tangent direction be the x -direction. We have the following expansion near p

$$(2.9) \quad \phi = c_2 r^2 \sin 2\theta + O(r^3),$$

where $c_2 \neq 0$ if p, q are different points and connected in $\partial\Omega$. And similar results hold for point q . Moreover if q is in Ω , we can also find an eigenfunction ϕ , such that $N(\phi)$ passing through point p and q . In fact in this case (2.8) is replaced by $\phi(q) = 0$.

Remark 2.5. If we let

$$(2.10) \quad \frac{\partial^2 \phi}{\partial n \partial T}(p) = 0,$$

replace (2.8), where T is the tangent direction. Then p is a third order vanishing point of ϕ . And $N(\phi)$ touches the boundary at p . In the coordinate as given in remark 2.4, we have the following local expansion

$$(2.11) \quad \phi = c_3 r^3 \sin 3\theta + O(r^4),$$

where $c_3 \neq 0$. In fact, if $c_3 = 0$ we can use a similar discussion as in the proof of lemma 2.1 to obtain a contradiction. That is the vanishing order is at most three.

Remark 2.6. If Ω is a simply connected domain, then there exists a second eigenfunction ϕ , such that $N(\phi)$ is a closed curve. In fact, the function ϕ obtained in remark 2.5 is the desired eigenfunction.

Corollary 2.7. *If Ω is convex in one direction, then the multiplicity of λ_2 is at most two.*

As we mentioned before, we have a shorter proof of theorem 1.1 in this case. And the idea of "domain variation" is developed from this proof.

Proof. Suppose the multiplicity of λ_2 is three. And we may assume that Ω is convex in x-direction. From remark 2.6, we know that there exists a second eigenfunction ϕ such that the nodal line of ϕ is closed.

Now, we consider ϕ_x which is the derivative of ϕ in x-direction, it satisfies

$$(2.12) \quad \Delta\phi_x + \lambda_2\phi_x = 0.$$

Let p,q be the maximum and minimum points in y-direction on the boundary. Then ϕ_x is nonpositive on one part of \widehat{pq} and nonnegative on the other of \widehat{pq} . And ϕ_x is not identically zero on the boundary.

Let ψ be the second eigenfunction whose nodal line intersects the boundary at p,q. In the case of simply connected domain $N(\psi)$ intersects the boundary at exactly two points p,q. Then by compare (2.12) with (1.1), we have

$$(2.13) \quad \int_{\partial\Omega} \phi_x \frac{\partial\psi}{\partial n} = 0.$$

This is impossible, because ϕ_x and $\frac{\partial\psi}{\partial n}$ change sign simultaneously at points p,q.

Finally we should mention that theorem 1.1 is a corollary of nodal line conjecture provided Ω is simply connected domain. That is, if the nodal line conjecture is true, then our theorem is also true in a simply connected domain. This can be seen easily from remark 2.6.

3. Domain Variations.

In this section, we sometime denote Ω by Ω_0 . Let Ω_t be a variation of Ω_0 for some small t. That is, $\partial\Omega_t$ as t varying, is a smooth curve continuously depends on t and $\lim_{t \rightarrow 0} \partial\Omega_t = \partial\Omega_0$. Throughout this section we always assume the multiplicity of $\lambda_2(\Omega_0)$ is three. And sometimes $\lambda_2(\Omega_0)$ is written as λ_2 ; $\lambda_2(\Omega_t)$ as $\lambda_2(t)$.

In this paper we are only interested in the domain variation where the second eigenvalue remains unchanged. This is possible because of the monotonicity and continuity of eigenvalues.

Now, we consider a special kind of domain variations. Suppose the origin of the coordinate is at $p \in \partial\Omega$, x -direction is in the tangent direction and y -direction is in the interior normal direction. Let Γ be the part of boundary which can be written as

$$(3.1) \quad \Gamma = (x, l(x)) \quad -b \leq x \leq b.$$

And let $\Gamma^1 = (x, l(x)), -b \leq x \leq 0$, $\Gamma^2 = (x, l(x)), 0 \leq x \leq b$. Let $h(x) \in C_0^\infty(0, 1)$, $h(x) \geq 0$ and $h(x) \not\equiv 0$ in $(0, 1)$.

Put

$$(3.2) \quad \Gamma_t^1 = \left\{ \left(x, l(x) + \beta(t)h\left(-\frac{x}{t}\right) \right), \quad -bt \leq x \leq 0 \right\},$$

$$(3.3) \quad \Gamma_t^2 = \left\{ \left(x, l(x) - \alpha(t)h\left(\frac{x}{t}\right) \right), \quad 0 \leq x \leq bt \right\},$$

where $\beta(t) > 0, \alpha(t) > 0$ are to be determined for $0 < t < \delta$.

We consider the domain variation where Γ is replaced by Γ_t^1 and Γ_t^2 . And at present, we assume $\alpha(t), \beta(t)$ are chosen so that $\lambda_2(t) = \lambda_2$. Moreover $\alpha(t) \rightarrow 0$, as $t \rightarrow 0$; $\beta(t) \rightarrow 0$, as $t \rightarrow 0$.

Suppose $\lambda_1(t) < \lambda_2(t) \leq \lambda_3(t) \leq \lambda_4(t)$ be the first four eigenvalues of Ω_t . Let $\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)$ be the corresponding orthogonal eigenfunctions. That is

$$(3.4) \quad \int_{\Omega_t} \phi_i(t)\phi_j(t)dx dy = \delta_{ij}, \quad 1 \leq i, j \leq 4,$$

where $\phi_i(t)$ is a function of (x, y) and t is the parameter. Since we assumed that the multiplicity of Ω_0 is three, then

$$(3.5) \quad \lambda_3(t) \longrightarrow \lambda_2, \quad \text{as } t \rightarrow 0.$$

$$(3.6) \quad \lambda_4(t) \longrightarrow \lambda_2, \quad \text{as } t \rightarrow 0.$$

We may assume for convenience that as $t \rightarrow 0$

$$(3.7) \quad \phi_2(t) \longrightarrow \phi_2,$$

$$(3.8) \quad \phi_3(t) \longrightarrow \phi_3,$$

$$(3.9) \quad \phi_4(t) \longrightarrow \phi_4,$$

otherwise we choose a sequence of t , since our proof is still true for a sequence of $\{t_n\}$ with $t_n \rightarrow 0$.

Lemma 3.1. *The nodal line $N(\phi_2(t))$ intersects the boundary at Γ_t exactly twice with the second order intersection or once with the third order intersection for small t .*

Here the second order intersection means that the intersection point is a second order vanishing point.

Proof. We prove the lemma by contradictions.

First suppose the nodal line $N(\phi_2(t))$ does not intersect Γ_t at all. Let q be a point on the boundary which is not in Γ_t and connected in $\partial\Omega_t$ with the origin p . Now, consider domain Ω_0 , through points p, q by lemma 2.3, we know that there exists a second eigenfunction ϕ such that $N(\phi)$ intersects $\partial\Omega$ at points p, q . Moreover, by Courant nodal domain theorem, we know that $\frac{\partial\phi}{\partial n}$ is positive on one of \widehat{pq} and negative on the other \widehat{pq} . In particular, $\frac{\partial\phi}{\partial n}$ has different signs on Γ^1 and Γ^2 . And $\frac{\partial\phi}{\partial n}$ on Γ^2 and ϕ on Γ_t^1 have the same sign as t sufficiently small.

Since ϕ satisfies (1.1) and $\phi_2(t)$ satisfies

$$(3.10) \quad \begin{cases} \Delta\phi_2(t) + \lambda_2\phi_2(t) = 0 & \text{in } \Omega_t \\ \phi_2(t)|_{\partial\Omega_t} = 0. \end{cases}$$

We can deduce from (1.1) and (3.10)

$$(3.11) \quad \int_{\partial(\Omega_t \cap \Omega)} \left(\phi \frac{\partial\phi_2(t)}{\partial n} - \phi_2(t) \frac{\partial\phi}{\partial n} \right) = 0.$$

That is

$$(3.12) \quad \int_{\Gamma_t^1} \phi \frac{\partial\phi_2(t)}{\partial n} - \int_{\Gamma^2} \phi_2(t) \frac{\partial\phi}{\partial n} = 0.$$

But $\phi_2(t)$ on Γ^2 on and $\frac{\partial\phi_2(t)}{\partial n}$ on Γ_t^1 have different sign here. Then as t small (3.12) is impossible.

Now, suppose $N(\phi_2(t))$ intersects Γ_t only once with a second order intersection. We may assume this intersection point is p_t . If p_t is in Γ_t^2 , we assume as t sufficiently small, $N(\phi_2(t))$ intersects Γ at p_t^* and exactly once with the second order intersection. Otherwise ϕ_2 the limitation of $\phi_2(t)$ would have at least fourth order vanishing at p which is impossible.

Again we consider the domain Ω_0 . By lemma 2.3 and remark 2.4, we know that there exists an eigenfunction ϕ , such that $N(\phi)$ intersects the boundary of Ω at points p, p_t or p, p_t^* in the second case. Since $\frac{\partial\phi}{\partial n}$ has different signs on Γ^1 and Γ^2 , then $N(\phi)$ does not intersect Γ_t^1 as t sufficiently

small. Compare (1.1) with (3.10) we can also obtain (3.12). In this case, we can deduce also, by our choice of ϕ that $\phi \frac{\partial \phi_2(t)}{\partial n}$ on Γ_t^1 and $\phi_2(t) \frac{\partial \phi}{\partial n}$ on Γ^2 have different signs. We again reach a contradiction.

Corollary 3.2. *The multiplicity of $\lambda_2(t)$ on the domain Ω_t is one.*

Proof. Suppose the multiplicity of $\lambda_2(t)$ on Ω_t is not one. Let $\phi_2(t), \phi_3(t)$ be orthogonal eigenfunctions corresponding to $\lambda_2(t)$. Let q be the point as in the proof of lemma 3.1. We can find a linear combination of $\phi_2(t)$ and $\phi_3(t)$, such that the nodal line of it intersects the boundary at point q . Then by remark 2.2, we know that its nodal line intersects Γ_t at most once and with a second order intersection. This contradicts lemma 3.1.

Corollary 3.3. *The nodal line $N(\phi_2(t))$ can not intersect $\partial\Omega_t$ at point p .*

The proof is similar to that of lemma 3.1.

Remark 3.4. The second eigenfunction ϕ_2 which is the limitation of $\phi_2(t)$ has a third order vanishing at point p . And the ϕ_2 is actually unique. This can be seen easily from remark 2.5.

Now we turn to study the properties of ϕ_3 and ϕ_4 .

Lemma 3.5. *The nodal line $N(\phi_3)$ intersects the boundary at point p . Moreover it is a second order intersection.*

Proof. Suppose that $N(\phi_3)$ does not intersect the boundary at point p . For t sufficiently small neither does $N(\phi_3(t))$. And we may assume $\phi_3(t)$ near point p in Ω_t is positive, $\frac{\partial \phi_4}{\partial n}(t, p) \leq 0$.

We choose a constant $k = k(t) \geq 0$, such that

$$(3.13) \quad \frac{\partial}{\partial n}(\phi_4(t) - k(t)\phi_3(t))|_p = 0.$$

Now if $\frac{\partial}{\partial n}(\phi_4(t) - k(t)\phi_3(t)) \geq 0$ near p for a sequence of t 's going to zero, then for the limit $\phi_4 - k\phi_3$ we would have $\frac{\partial}{\partial n}(\phi_4 - k\phi_3) \geq 0$ near p and is 0 at p . So $\phi_4 - k\phi_3$ would have a third order vanishing at p , contradicting the fact that ϕ_2 is the only second eigenfunction with this property. Similarly one shows that $\frac{\partial}{\partial n}(\phi_4(t) - k(t)\phi_3(t))$ can not have more than one zeros near p . So (3.13) means that the nodal line $N(\phi_4(t) - k(t)\phi_3(t))$ intersects the

boundary $\partial\Omega_t$ at p . Moreover, $N(\phi_4(t) - k(t)\phi_3(t))$ intersects Γ_t only at point p for t sufficiently small. We can deduce

$$(3.14) \quad \begin{cases} \Delta(\phi_4(t) - k(t)\phi_3(t)) + \lambda_2(\phi_4(t) - k(t)\phi_3(t)) = w(t) & \text{in } \Omega_t, \\ (\phi_4(t) - k(t)\phi_3(t))|_{\partial\Omega_t} = 0, \end{cases}$$

where $w(t) = (\lambda_2 - \lambda_4(t))\phi_4(t) - (\lambda_2 - \lambda_3(t))k(t)\phi_3(t)$. As $t \rightarrow 0$, let

$$(3.15) \quad \frac{w(t)}{|w(t)|_{L^\infty(\Omega_t)}} \rightarrow w = \beta_2\phi_4 - \beta_1\phi_3,$$

where β_1, β_2 are constants. Since $\lambda_4(t) \geq \lambda_3(t) > \lambda_2$, then $\beta_1, \beta_2 \leq 0$ and at least one of them is not zero. In particular $\beta_2 \neq 0$.

We can check easily that $\phi_2, \beta_2\phi_4 - \beta_1\phi_3$ and $\beta_1\phi_4 + \beta_2\phi_3$ are orthogonal eigenfunctions corresponding to λ_2 in Ω . Moreover

$$(3.16) \quad \frac{\partial}{\partial n}(\beta_2\phi_3 + \beta_1\phi_4)|_p > 0,$$

from our assumption.

We claim that the nodal line $N(\phi_2 + \beta_2\phi_3 + \beta_1\phi_4)$ does not intersect $\partial\Omega$ near point p . In fact, ϕ_2 has a third vanishing order at point p . From the expansion (2.11), we can assume $\frac{\partial\phi_2}{\partial n} \geq 0$ on the boundary near p . Then $\phi_2 + \beta_2\phi_3 + \beta_1\phi_4$ have our desired property.

Notice that $\phi_2 + \beta_2\phi_3 + \beta_1\phi_4$ and w are orthogonal, then we can choose $c(t)$ such that

$$(3.17) \quad \int_{\Omega \cap \Omega_t} [\phi_2 + \beta_2\phi_3 + \beta_1\phi_4 + c(t)(\beta_2\phi_4 - \beta_1\phi_3)] \frac{w(t)}{|w(t)|_{L^\infty(\Omega_t)}} = 0,$$

and $c(t) \rightarrow 0$, as $t \rightarrow 0$. In fact, as $t \rightarrow 0$

$$(3.18) \quad \int_{\Omega \cap \Omega_t} (\beta_2\phi_4 - \beta_1\phi_3) \frac{w(t)}{|w(t)|_{L^\infty(\Omega_t)}} \rightarrow \int_{\Omega} w^2 > 0.$$

$$(3.19) \quad \int_{\Omega \cap \Omega_t} (\phi_2 + \beta_2\phi_3 + \beta_1\phi_4) \frac{w(t)}{|w(t)|_{L^\infty(\Omega_t)}} \rightarrow 0.$$

Set

$$(3.20) \quad \phi = \phi_2 + \beta_2\phi_3 + \beta_1\phi_4 + c(t)(\beta_2\phi_4 - \beta_1\phi_3).$$

We know ϕ is also an eigenfunction corresponding to λ_2 and $N(\phi)$ does not intersect $\partial\Omega$ near point p as t small. Now we compare ϕ with $\phi_4(t) - k(t)\phi_3(t)$, as in the proof of lemma 3.1, we have

$$(3.21) \quad \int_{\Gamma_t^1} \phi \frac{\partial}{\partial n} (\phi_4(t) - k(t)\phi_3(t)) - \int_{\Gamma^2} \frac{\partial \phi}{\partial n} (\phi_4(t) - k(t)\phi_3(t)) = 0,$$

which leads to a contradiction as t sufficiently small.

Remark 3.6. The ϕ_3 which is the limitation of $\phi_3(t)$ is unique. Since it has a second vanishing order at p and are orthogonal with ϕ_2 which has a third vanishing order at p . So ϕ_3 must be unique. And therefore ϕ_4 which is the limitation of $\phi_4(t)$ is also unique. The vanishing order of ϕ_4 at p is one.

Since ϕ_2, ϕ_3, ϕ_4 are unique, then

$$\begin{aligned} \lim_{t \rightarrow 0} \phi_2(t) &= \phi_2, \\ \lim_{t \rightarrow 0} \phi_3(t) &= \phi_3, \\ \lim_{t \rightarrow 0} \phi_4(t) &= \phi_4. \end{aligned}$$

We actually do not need to take a sequence in the above proof.

In the following, we let $\phi(t, p)$ be the value of $\phi(t)$ at point p .

Lemma 3.6. Assume $\alpha(t) = O(t^5)$ and $\beta(t) = O(t^5)$ as $t \rightarrow 0$ in the domain variation, then as $t \rightarrow 0$

$$(3.22) \quad \frac{\partial^2 \phi_2}{\partial y \partial x}(t, p) = O(t).$$

$$(3.23) \quad \frac{\partial \phi_2}{\partial y}(t, p) = O(t^2).$$

Proof. By lemma 3.1, we know that there exist $p_1, p_2 \in \Gamma_t$ and $p_1, p_2 \in N(\phi_2(t))$. Then

$$(3.24) \quad \frac{\partial \phi_2}{\partial n}(t, p_1) = \frac{\partial \phi_2}{\partial n}(t, p_2) = 0.$$

Moreover $\frac{\partial^2 \phi_2}{\partial n \partial T}(t, p_1)$ and $\frac{\partial^2 \phi_2}{\partial n \partial T}(t, p_2)$ have different sign, if p_1 and p_2 are different, where n is the exterior normal direction and T is the tangent direction

at the corresponding points. Then there always exists p_3 on $\Gamma_t \cap \widehat{p_1 p_2}$, such that

$$(3.25) \quad \frac{\partial^2 \phi_2}{\partial n \partial T}(t, p_3) = 0.$$

Note that on Γ_t^2 in the local chart

$$(3.26) \quad T = \left(\frac{1}{\sqrt{1 + (l'(x) + \frac{\alpha(t)}{t} h'(x/t))^2}}, \frac{l'(x) + \frac{\alpha(t)}{t} h'(x/t)}{\sqrt{1 + (l'(x) + \frac{\alpha(t)}{t} h'(x/t))^2}} \right) = (T_x, T_y)$$

$$(3.27) \quad n = \left(\frac{l'(x) + \frac{\alpha(t)}{t} h'(x/t)}{\sqrt{1 + (l'(x) + \frac{\alpha(t)}{t} h'(x/t))^2}}, -\frac{1}{\sqrt{1 + (l'(x) + \frac{\alpha(t)}{t} h'(x/t))^2}} \right) = (n_x, n_y)$$

Then

$$\frac{\partial \phi_2(t)}{\partial n} = \frac{\partial \phi_2(t)}{\partial x} n_x + \frac{\partial \phi_2(t)}{\partial y} n_y.$$

$$(3.28) \quad \frac{\partial^2 \phi_2(t)}{\partial T \partial n} = \frac{\partial}{\partial x} \left(\frac{\partial \phi_2(t)}{\partial x} n_x + \frac{\partial \phi_2(t)}{\partial y} n_y \right) T_x + \frac{\partial}{\partial y} \left(\frac{\partial \phi_2(t)}{\partial x} n_x + \frac{\partial \phi_2(t)}{\partial y} n_y \right) T_y.$$

For all $q \in \Gamma_t$ we have

$$(3.29) \quad \frac{\partial^2 \phi_2(t)}{\partial T \partial n} = -\frac{\partial^2 \phi_2(t)}{\partial_x \partial y} + O(t).$$

In particular at p_3 ,

$$(3.30) \quad \frac{\partial^2 \phi_2}{\partial_x \partial y}(t, p_3) = O(t).$$

From the regularity of elliptic equations, we know that $\phi_2 \in C^3$. Then

$$(3.31) \quad \frac{\partial^2 \phi_2}{\partial_x \partial y}(t, p) = O(t).$$

In fact, (3.31) is true for all points of Γ_t .

For the proof of (3.23), we observe that

$$(3.32) \quad \nabla \phi_2(t, p_1) = \nabla \phi_2(t, p_2) = 0,$$

together with (3.31) for some q on Γ_t we obtain the proof.

4. Proof of the Theorem.

In this section, we use the same notations as in section 3. We first specify the function $\alpha(t)$, $\beta(t)$ which given in (3.2) and (3.3).

Let $\beta_0(t) = t^m$ and $\alpha_0(t, m)$ be chosen so that $\lambda_2(t) = \lambda_2$. This is possible as we mentioned before. Notice that for each $1 > t > 0$, $\alpha_0(t, m)$ is a monotone decreasing function of m , and

$$(4.1) \quad \lim_{m \rightarrow \infty} \alpha_0(t, m) = 0.$$

Choose $m = m(t)$ be a smooth function for small t , and

$$\begin{aligned} 1^0 \quad & \beta(t) = t^{m(t)} = O(t^5) \quad \text{as } t \rightarrow 0, \\ 2^0 \quad & \alpha(t) = \alpha_0(t, m(t)) < t^5 \quad \text{for } t \text{ small.} \end{aligned}$$

It is easy to check that domain Ω_t is uniformly C^4 . And then as $t \rightarrow 0$

$$\phi_2(t) \rightarrow \phi_2 \quad \text{in } C^3(\Omega_t \cap \Omega).$$

Now we give the following lemmas.

Lemma 4.1. *Under the above notations and assumptions, the limit points of $\frac{\alpha(t)}{\beta(t)}$ are 1, k_0 and $\frac{1}{k_0}$ for some constant $1 > k_0 > 0$ as $t \rightarrow 0$, therefore $\alpha(t)$ and $\beta(t)$ have the same order as $t \rightarrow 0$.*

Proof. Compare $\phi_2(t)$ with ϕ_i , $i = 2, 3, 4$ as in the proof of lemma 3.1, we have

$$(4.2) \quad \int_{\Gamma_t^1} \frac{\partial \phi_2}{\partial n}(t) \phi_i = \int_{\Gamma^2} \frac{\partial \phi_i}{\partial n} \phi_2(t),$$

where $i = 2, 3, 4$.

Now we calculate the value of the two terms in (4.2).

$$(4.3) \quad \int_{\Gamma_t^1} \frac{\partial \phi_2}{\partial n}(t) \phi_i = \int_{-bt}^0 \left(\frac{\partial \phi_2}{\partial n}(t) \phi_i \right) |_{\Gamma_t^1} \sqrt{1 + (l'(x) + \frac{\beta(t)}{t} h'(-x/t))^2} dx.$$

Since

$$(4.4) \quad \phi_i(x, l(x) + \beta(t)h(-x/t)) = \frac{\partial \phi_i}{\partial y}(x, l(x))h(-x/t)\beta(t) + O(t^5),$$

then

$$(4.5) \quad \int_{\Gamma_t^1} \frac{\partial \phi_2}{\partial n}(t) \phi_i = \int_{-bt}^0 \frac{\partial \phi_2}{\partial n}(t, x, l(x) + \beta(t)h(-\frac{x}{t})) \left[\frac{\partial \phi_i}{\partial y}(x, l(x))h(-\frac{x}{t})\beta(t) + O(t^5) \right] L(x) dx,$$

where $L(x) = \sqrt{1 + (l'(x) + \beta(t)/th'(-\frac{x}{t}))^2}$. For convenience we assume $b = 1$ in the following.

By Taylor expansion at the origin in Γ_t , we have

$$(4.6) \quad \begin{aligned} \phi_2(t) &= a_0(t)y + a_1(t)xy + a_2(t)(x^2 - y^2) + a_3(t)yx^2 + \\ & a_4(t)x^3 + a_5(t)y^3 + a_6(t)y^2x + O(t^4), \end{aligned}$$

where we used the fact that for $t > 0$, the vanishing order of $\phi_2(t)$ is finite at p . (see Melas [8]). And the first two terms are harmonic polynomials. All $a_i(t)$ are continuous functions. If $\phi_2(t)$ has a third order vanishing at p , by remark 2.5 we know $a_3(t)$ is not zero.

Recall (3.27), we can deduce

$$(4.7) \quad \begin{aligned} \frac{\partial}{\partial n} \phi_2(t) |_{\Gamma_t^1} &= \\ &= [a_0(t) + a_1(t)x - 2a_2(t)y + a_3(t)x^2] n_y + [a_1(t)y + 2a_2(t)x] n_x + O(t^3), \end{aligned}$$

where $y = l(x) + \beta(t)h(-x/t)$, $n_x = O(t)$ and $n_y = 1 + O(t^2)$ in Γ_t^1 .

Put $x = ts$, then

$$(4.8) \quad \frac{\partial \phi_2}{\partial n}(t) |_{\Gamma_t^1} = a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2).$$

Similarly we can deduce for $0 \leq x \leq t$

$$(4.9) \quad \begin{aligned} \frac{\partial \phi_i}{\partial y}(x, l(x)) &= \frac{\partial \phi_i}{\partial y}(p) + \frac{\partial^2 \phi_i}{\partial x \partial y}(p)ts + \\ &+ \left[\frac{1}{2} \frac{\partial^3 \phi_i}{\partial x^2 \partial y}(p) + \frac{\partial^2 \phi_i}{\partial y^2}(p)l''(0) \right] t^2s^2 + O(t^3). \end{aligned}$$

Substitute (4.8),(4.9) into (4.5) we obtain

$$(4.10) \quad \int_{\Gamma_i^1} \frac{\partial \phi_2}{\partial n}(t) \phi_i = (1 + O(t^2)) \beta(t) t \int_{-1}^0 [a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2)] \\ \left[\frac{\partial \phi_i}{\partial y}(p) + \frac{\partial^2 \phi_i}{\partial x \partial y}(p)ts + \left(\frac{\partial^2 \phi_i}{\partial y^2}(p)l''(0) + \frac{1}{2} \frac{\partial^3 \phi_i}{\partial x^2 \partial y}(p) \right) t^2s^2 + O(t^3) \right] h(-s) ds.$$

Since ϕ_2 has a third vanishing order at p and ϕ_3 has a second order vanishing at p , we have for $i = 2, 3, 4$,

$$(4.11) \quad \int_{\Gamma_i^1} \frac{\partial \phi_2}{\partial n}(t) \phi_2 = \beta(t) t^3 \int_{-1}^0 [a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2)] \\ \left[1/2 \frac{\partial^3 \phi_2}{\partial x^2 \partial y}(p) s^2 + O(t) \right] h(-s) ds [1 + O(t^2)].$$

$$(4.12) \quad \int_{\Gamma_i^1} \frac{\partial \phi_2}{\partial n}(t) \phi_3 = \beta(t) t^2 \int_{-1}^0 [a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2)] \\ \left[\frac{\partial^2 \phi_3}{\partial x \partial y}(p) s + O(t) \right] h(-s) ds [1 + O(t^2)].$$

$$(4.13) \quad \int_{\Gamma_i^1} \frac{\partial \phi_2}{\partial n}(t) \phi_4 = \beta(t) t \int_{-1}^0 [a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2)] \\ \left[\frac{\partial \phi_4}{\partial y}(p) + O(t) \right] h(-s) ds [1 + O(t^2)].$$

where $a_0(t) = O(t^2)$, $a_1(t) = O(t)$ and $a_3(t) \rightarrow \frac{1}{2} \frac{\partial^3 \phi_2}{\partial x^2 \partial y}(p) \neq 0$, because of lemma 3.8.

Again on Γ^2 we can deduce

$$(4.14) \quad \int_{\Gamma^2} \frac{\partial \phi_4}{\partial n} \phi_2(t) = \alpha(t) t \int_0^1 [a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2)] \\ \left[\frac{\partial \phi_4}{\partial y}(p) + O(t) \right] h(s) ds [1 + O(t^2)].$$

$$(4.15) \quad \int_{\Gamma^2} \frac{\partial \phi_3}{\partial n} \phi_2(t) = \alpha(t)t^2 \int_0^1 [a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2)] \left[\frac{\partial^2 \phi_3}{\partial x \partial y}(p)s + O(t) \right] h(s)ds [1 + O(t^2)].$$

$$(4.16) \quad \int_{\Gamma^2} \frac{\partial \phi_2}{\partial n} \phi_2(t) = \alpha(t)t^3 \int_0^1 [a_0(t) + a_1(t)ts + a_3(t)t^2s^2 + o(t^2)] \left[\frac{\partial^3 \phi_2}{\partial x^2 \partial y}(p)s^2 + O(t) \right] h(-s)ds [1 + O(t^2)].$$

For some sequence $\{t_i\}$, $t_i \rightarrow 0$, suppose

$$(4.17) \quad \frac{a_0(t_i)}{t_i^2} \rightarrow x.$$

$$(4.18) \quad \frac{a_1(t_i)}{t_i} \rightarrow y.$$

$$(4.19) \quad a_3(t_i) \rightarrow z.$$

Then we have

$$(4.20) \quad \begin{aligned} \lim_{t_i \rightarrow 0} \frac{\beta(t_i)}{\alpha(t_i)} &= \frac{\int_0^1 (xs + ys^2 + zs^3)h(s)ds}{\int_{-1}^0 (xs + ys^2 + zs^3)h(-s)ds} \\ &= \frac{\int_0^1 (x + ys + zs^2)h(s)ds}{\int_{-1}^0 (x + ys + zs^2)h(-s)ds} \\ &= \frac{\int_0^1 (xs^2 + ys^3 + zs^4)h(s)ds}{\int_{-1}^0 (xs^2 + ys^3 + zs^4)h(-s)ds} \\ &= k \geq 0. \end{aligned}$$

Set $b_i = \int_0^1 s^i h(s)ds$, $i = 0, 1, 2, 3, 4$. We can deduce from (4.20)

$$(4.21) \quad \begin{cases} (k-1)b_0x - (k+1)b_1y + (k-1)b_2z = 0 \\ (k+1)b_1x - (k-1)b_2y + (k+1)b_3z = 0 \\ (k-1)b_2x - (k+1)b_3y + (k-1)b_4z = 0. \end{cases}$$

Since x, y, z are not all zero, then the determinant of the coefficient is zero. We have

$$(4.22) \quad D = \begin{vmatrix} (k-1)b_0 & -(k+1)b_1 & (k-1)b_2 \\ (k+1)b_1 & -(k-1)b_2 & (k+1)b_3 \\ (k-1)b_2 & -(k+1)b_3 & (k-1)b_4 \end{vmatrix} = 0.$$

A simple calculation shows that

$$(4.23) \quad D = \frac{k-1}{b_3} [(k-1)^2(b_2b_4(b_1b_2 - b_0b_3) - b_2^2(b_1b_4 - b_2b_3)) - (k+1)^2(b_3^2(b_1b_2 - b_0b_3) - b_1b_3(b_1b_4 - b_2b_3))].$$

Put

$$\begin{aligned} A &= b_2b_4(b_1b_2 - b_0b_3) - b_2^2(b_1b_4 - b_2b_3), \\ B &= b_3^2(b_1b_2 - b_0b_3) - b_1b_3(b_1b_4 - b_2b_3). \end{aligned}$$

From Holder inequality we can check $b_1b_2 < b_0b_3$ and $b_2b_3 < b_1b_4$. That is $A < 0$ and $B < 0$.

Then we know either $k = 1$ or

$$(4.24) \quad k^2 - 2\frac{A+B}{A-B}k + 1 = 0.$$

We can show that $A < B$. (see appendix) That is (4.24) always has two positive solutions. Then we finished the proof of lemma.

Lemma 4.2. *Under the assumptions of lemma 4.1, we have*

$$\lim_{t \rightarrow 0} \frac{\alpha(t)}{\beta(t)} = 1.$$

Proof. Suppose $\lim_{t \rightarrow 0} \frac{\alpha(t)}{\beta(t)} = k_0$, k_0 is a solution of (4.24). We will use the fact that $\lambda_2 \leq \lambda_3(t) \leq \lambda_4(t)$ to rule out this possibility.

Now compare $\phi_4(t)$ with ϕ_4 , we have

$$(4.25) \quad (\lambda_4(t) - \lambda_2) \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_4 = \int_{\Gamma^2} \frac{\partial \phi_4}{\partial n} \phi_4(t) - \int_{\Gamma^1_t} \frac{\partial \phi_4(t)}{\partial n} \phi_4.$$

We can calculate the right hand sides of (4.25) as in lemma 4.1, we deduce

$$\begin{aligned} (\lambda_2 - \lambda_4(t)) \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_4 &= \\ &= t(\alpha(t) - \beta(t)) \frac{\partial \phi_4}{\partial y}(p) \frac{\partial \phi_4}{\partial y}(t, p) \int_0^1 h(s) ds + O(\alpha(t)t^2). \end{aligned}$$

Since $\frac{\partial\phi_4}{\partial y}(t, p) \rightarrow \frac{\partial\phi_4}{\partial y}(p)$ and $\lambda_4(t) \geq \lambda_2$, we deduce

$$(4.26) \quad \alpha(t) \leq \beta(t),$$

for $t \geq 0$ and sufficiently small.

Now compare $\phi_3(t)$ with ϕ_3 we have

$$(4.27) \quad \begin{aligned} (\lambda_2 - \lambda_3(t)) \int_{\Omega \cap \Omega_t} \phi_3(t) \phi_3 &= \frac{\partial\phi_3}{\partial y}(t, p) \frac{\partial^2\phi_3}{\partial y\partial x}(p) \int sh(s)ds(\alpha(t) + \beta(t))t^2 \\ &+ \frac{\partial^2\phi_3}{\partial y\partial x}(t, p) \frac{\partial^2\phi_3}{\partial x\partial y}(p) \int s^2h(s)ds(\alpha(t) - \beta(t))t^3 + o(\alpha(t)t^3), \end{aligned}$$

here we used the fact that $\frac{\partial\phi_3}{\partial y}(p) = 0$.

First from (4.27) we know that

$$(4.28) \quad \lambda_2 - \lambda_3(t) = o(\alpha(t)t^2).$$

Compare $\phi_3(t)$ with ϕ_4 we have

$$(4.29) \quad \begin{aligned} (\lambda_2 - \lambda_3(t)) \int_{\Omega \cap \Omega_t} \phi_3(t) \phi_4 &= \frac{\partial\phi_3}{\partial y}(t, p) \frac{\partial\phi_4}{\partial y}(p) \int h(s)ds(\alpha(t) - \beta(t))t \\ &+ \frac{\partial^2\phi_3}{\partial y\partial x}(t, p) \frac{\partial\phi_4}{\partial y}(p) \int sh(s)ds(\alpha(t) + \beta(t))t^2 \\ &+ o(\alpha(t)t^2). \end{aligned}$$

From (4.29), (4.28) we have

$$(4.30) \quad \frac{\partial\phi_3}{\partial y}(t, p) = -\frac{\partial^2\phi_3}{\partial y\partial x}(t, p) \frac{\int sh(s)ds(\alpha(t) + \beta(t))t}{\int h(s)ds(\alpha(t) - \beta(t))} + o(t),$$

where $\frac{\partial^2\phi_3}{\partial y\partial x}(t, p) \rightarrow \frac{\partial^2\phi_3}{\partial y\partial x}(p)$.

Substitute (4.30) into (4.27) and notice $\lambda_2 \leq \lambda_3(t)$ we have

$$(4.31) \quad \begin{aligned} 0 \geq & -\frac{\partial^2\phi_3}{\partial y\partial x}(t, p) \frac{\partial^2\phi_3}{\partial y\partial x}(p) \frac{(\int sh(s)ds)^2(\alpha(t) + \beta(t))^2}{\int h(s)ds(\alpha(t) - \beta(t))} \\ & + \frac{\partial^2\phi_3}{\partial y\partial x}(t, p) \frac{\partial^2\phi_3}{\partial y\partial x}(p) \int s^2h(s)ds(\alpha(t) - \beta(t)) + o(\alpha(t)). \end{aligned}$$

Recall (4.26) and the notation of b_i , A and B , we have

$$(4.32) \quad 0 \leq -(1+k_0)^2 \frac{b_1^2}{b_0} + (1-k_0)^2 b_2.$$

Recall (4.24) then we have

$$b_0 b_2 B \geq A b_1^2.$$

That is

$$b_0 b_2 [b_3^2 (b_1 b_2 - b_0 b_3) - b_1 b_3 (b_1 b_4 - b_2 b_3)] \geq b_1^2 (b_2^3 b_3 - b_0 b_2 b_3 b_4)$$

$$(4.34) \quad (b_0 b_3 - b_1 b_2)^2 \leq 0.$$

This is a contradiction, since $b_0 b_3 > b_1 b_2$. Then we proved the lemma.

Now suppose the limitation of $\frac{\alpha(t)}{\beta(t)}$ is one as $t \rightarrow 0$, we distinct the following two cases

$$(i) \quad \underline{\lim}_{t \rightarrow 0} \frac{\alpha(t)t}{\alpha(t) - \beta(t)} = c \neq 0,$$

$$(ii) \quad \underline{\lim}_{t \rightarrow 0} \frac{\alpha(t)t}{\alpha(t) - \beta(t)} = 0,$$

where c could be infinity.

We first prove that case (i) cannot occur.

Lemma 4.3. *Under the assumptions of lemma 4.1, we have*

$$(4.35) \quad \lim_{t \rightarrow 0} \frac{\alpha(t)t}{\alpha(t) - \beta(t)} = 0.$$

Proof. Suppose case (i) occur. Notice (4.25), we have as in the proof of lemma 4.2,

$$(4.36) \quad (\lambda_2 - \lambda_4(t)) \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_4 = t(\alpha(t) - \beta(t)) \left(\frac{\partial \phi_4}{\partial y}(p) \frac{\partial \phi_4}{\partial y}(t, p) \right) \int_0^1 h(s) ds$$

$$+ (\alpha(t) + \beta(t)) t^2 \left[\frac{\partial^2 \phi_4}{\partial x \partial y}(t, p) \frac{\partial \phi_4}{\partial y}(p) + \frac{\partial \phi_4}{\partial y}(t, p) \frac{\partial^2 \phi_4}{\partial x \partial y}(p) \right] \int_0^1 h(s) s ds + o(\alpha(t)t^2).$$

where $\frac{\partial \phi_4}{\partial y}(p) \neq 0$, and $\frac{\partial \phi_4}{\partial y}(t, p) \rightarrow \frac{\partial \phi_4}{\partial y}(p)$, $\frac{\partial^2 \phi_4}{\partial x \partial y}(t, p) \rightarrow \frac{\partial^2 \phi_4}{\partial x \partial y}(p)$ and $\int_{\Omega \cap \Omega_t} \phi_4(t) \phi_4 \rightarrow 1$ as $t \rightarrow 0$.

Similarly, compare $\phi_4(t)$ with ϕ_3 , we have

$$(4.37) \quad (\lambda_2 - \lambda_4(t)) \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_3 = (\alpha(t) + \beta(t)) t^2 \frac{\partial \phi_4}{\partial y}(t, p) \frac{\partial^2 \phi_3}{\partial x \partial y}(p) \int h(s) ds + o(\alpha(t) t^2).$$

Here we again used the fact that $\frac{\partial \phi_3}{\partial y}(p) = 0$.

Then from (4.36) and (4.37) we obtain

$$(4.38) \quad \frac{\partial \phi_4}{\partial y}(t, p) \frac{\partial^2 \phi_3}{\partial x \partial y}(p) \int h(s) ds = \lim_{t \rightarrow 0} \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_3 \times \left[\frac{\partial \phi_4}{\partial y}(p) \frac{\partial^2 \phi_4}{\partial x \partial y}(p) \int sh(s) ds + \frac{1}{2c} \left(\frac{\partial \phi_4}{\partial y} \right)^2 \int h(s) ds \right],$$

where

$$c = \lim_{t \rightarrow 0} \frac{\alpha(t) t}{\alpha(t) - \beta(t)} \neq 0.$$

Notice that $\frac{\partial \phi_4}{\partial y}(p) \neq 0$ and $\frac{\partial^2 \phi_3}{\partial y \partial x}(p) \neq 0$, but $\lim_{t \rightarrow 0} \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_3 = 0$ since $\phi_4(t) \rightarrow \phi_4$. Then we reach a contradiction. This is the proof of the lemma.

Lemma 4.4. *Under the assumptions of lemma 4.1, as $t \rightarrow 0$ we have*

$$(4.39) \quad \phi_2(t) = c_2(t) \phi_2 + c_3(t) \phi_3 + c_4(t) \phi_4 + o(t^4) \quad \text{in } \Omega \cap \Omega_t$$

$$(4.40) \quad \phi_3(t) = d_2(t) \phi_2 + d_3(t) \phi_3 + d_4(t) \phi_4 + o(t^4) \quad \text{in } \Omega \cap \Omega_t$$

$$(4.41) \quad \phi_4(t) = e_2(t) \phi_2 + e_3(t) \phi_3 + e_4(t) \phi_4 + o(t^4) \quad \text{in } \Omega \cap \Omega_t,$$

where $c_i(t) = \int_{\Omega \cap \Omega_t} \phi_2(t) \phi_i$, $d_i(t) = \int_{\Omega \cap \Omega_t} \phi_3(t) \phi_i$ and $e_i(t) = \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_i$, $i = 2, 3, 4$.

Proof. We only need to show (4.39), the remains can be proved similarly. Put

$$(4.42) \quad w(t) = \phi_2(t) - c_2(t) \phi_2 - c_3(t) \phi_3 - c_4(t) \phi_4.$$

Then we know that $w(t)$ are orthogonal to ϕ_2, ϕ_3, ϕ_4 in the domain $\Omega \cap \Omega_t$.

Consider $\frac{w(t)}{t^4}$, it satisfies the equation of problem (1.1) in $\Omega \cap \Omega_t$. If there exists a sequence $\{t_i\}$ and $t_i \rightarrow 0$, such that

$$(4.43) \quad \left| \frac{w(t_i)}{t_i^4} \right|_{L^\infty} \rightarrow c \neq 0,$$

then $\left| \frac{w(t_i)}{t_i^4} \right|_{L^\infty}$ is archived in the interior of $\Omega \cap \Omega_t$, assume at p_i . We can suppose $p_i \rightarrow q \in \Omega$, because on the boundary $\partial(\Omega \cap \Omega_t)$, $\frac{w(t_i)}{t_i^4} \rightarrow 0$.

Consider $\left| \frac{w(t_i)}{t_i^4} \right|_{L^\infty}$, we know that w also satisfies (1.1) and $w \not\equiv 0$. Moreover, w is orthogonal to ϕ_2, ϕ_3, ϕ_4 , which contradicts the fact that the multiplicity of λ_2 is at most three. Therefore we proved (4.39).

Remark 4.5. Since ϕ_2 vanishes at p with a third order, ϕ_3 vanishes with a second order. From lemma 4.4 and the regularity argument of elliptic equations we know that

$$(4.44) \quad \frac{\partial \phi_2}{\partial y}(t, p) = c_4(t) \frac{\partial \phi_4}{\partial y}(p) + O(t^3).$$

$$(4.45) \quad \frac{\partial^2 \phi_2}{\partial x \partial y}(t, p) = c_3(t) \frac{\partial^2 \phi_3}{\partial x \partial y}(p) + O(t^2),$$

where $c_3(t), c_4(t)$ are given in (4.39) and $c_4(t) = O(t^2), c_3(t) = O(t)$.

In the following, we will rule out case (ii).

Lemma 4.6. *Under the assumptions of lemma 4.1, if case (ii) occurs we have*

$$(4.46) \quad e_2(t) = O(t^2).$$

$$(4.47) \quad e_3(t) = \frac{(\alpha(t) + \beta(t))t}{\alpha(t) - \beta(t)} e_3 + O\left(\frac{\alpha(t)t^2}{\alpha(t) - \beta(t)}\right),$$

where $e_3 \neq 0$.

Proof. First we compare $\phi_4(t)$ with ϕ_2 , as before we have

$$(4.48) \quad (\lambda_2 - \lambda_4(t)) \int_{\Omega \cap \Omega_t} \phi_4(t) \phi_2 \\ = \frac{1}{2} t^3 (\alpha(t) - \beta(t)) \frac{\partial \phi_4}{\partial y}(t, p) \frac{\partial^3 \phi_2}{\partial x^2 \partial y}(p) \int_0^1 s^2 h(s) ds + O(\alpha(t)t^4).$$

Then from (4.36) and (4.48) we obtain (4.46) easily.

Compare $\phi_4(t)$ with ϕ_3 , we have

$$(4.49) \quad (\lambda_2 - \lambda_4(t)) \int_{\Omega \cap \Omega_t} \phi_4(t)\phi_3 = t^2(\alpha(t) + \beta(t)) \frac{\partial \phi_4}{\partial y}(t, p) \frac{\partial^2 \phi_3}{\partial x \partial y}(p) \int_0^1 sh(s)ds + O(\alpha(t)t^3).$$

Then from (4.36) and (4.49) we can obtain (4.47) with

$$(4.50) \quad e_3 = \frac{b_1 \frac{\partial^2 \phi_3}{\partial x \partial y}(p)}{b_0 \frac{\partial \phi_4}{\partial y}(p)}.$$

Lemma 4.7. *Under the assumptions of lemma 4.1, if case (ii) occurs, we have*

$$(4.51) \quad \begin{vmatrix} b_0 & b_1 & b_2 \\ b_2 & b_3 & b_4 \\ b_1 & 0 & b_3 \end{vmatrix} = 0,$$

where $b_i = \int_0^1 s^i h(s)ds$.

Proof. From the proof of lemma 4.1, we have

$$(4.52) \quad (\alpha(t) - \beta(t)) \int_0^1 (a_0(t) + a_3(t)t^2s^2)h(s) + (\alpha(t) + \beta(t)) \int_0^1 a_1(t)tsh(s) = O(\alpha(t)t^3).$$

$$(4.53) \quad (\alpha(t) - \beta(t)) \int_0^1 (a_0(t)s^2 + a_3(t)t^2s^4)h(s) + (\alpha(t) + \beta(t)) \int_0^1 a_1(t)ts^3h(s) = O(\alpha(t)t^3).$$

$$(4.54) \quad (\alpha(t) + \beta(t)) \int_0^1 (a_0(t)s + a_3(t)t^2s^3)h(s) + (\alpha(t) - \beta(t)) \int_0^1 a_1(t)ts^2h(s) = O(\alpha(t)t^3).$$

where $a_0(t) = c_4(t) + O(t^3)$, $a_1(t) = c_3(t) + O(t^2)$.

Recall (4.7), where $a_2(t) = \frac{\partial^2 \phi}{\partial y^2}(t, p)$, now by lemma 4.4 we can deduce that $a_2(t) = O(t)$, then in (4.8) the term $o(t^2)$ can be replaced by $O(t^3)$. Therefore the right hand side of (4.52)-(4.54) become $O(\alpha(t)t^3)$.

Again we assume

$$\begin{aligned} \frac{a_0(t)}{t^2} &\longrightarrow x, \\ a_3(t) &\longrightarrow z, \\ \frac{(\alpha(t) + \beta(t))a_1(t)}{(\alpha(t) - \beta(t))t} &\longrightarrow y, \end{aligned}$$

where y is finite. In fact from the domain variation and

$$\int_{\Omega_t} \phi_2(t)\phi_4(t) = 0,$$

we know

$$(4.55) \quad \int_{\Omega \cap \Omega_t} \phi_2(t)\phi_4(t) = o(t^5).$$

Similarly for $i, j = 2, 3, 4$,

$$(4.56) \quad \int_{\Omega \cap \Omega_t} \phi_i \phi_j = \delta_{ij} + o(t^5).$$

Therefore

$$(4.57) \quad c_2(t)e_2(t) + c_3(t)e_3(t) + c_4(t)e_4(t) = o(t^5).$$

Then from remark 4.5, lemma 3.8 and lemma 4.6, we deduce

$$c_3(t)e_3(t) = O(t^2),$$

$$(4.58) \quad a_1(t) = O\left(\frac{(\alpha(t) - \beta(t))t}{\alpha(t)}\right).$$

Then we have

$$(4.59) \quad \begin{cases} b_0x + b_1y + b_2z = 0 \\ b_2x + b_3y + b_4z = 0 \\ b_1x + b_3z = 0. \end{cases}$$

Since $z \neq 0$, we know the determinant of the coefficient is zero. Then we proved the lemma.

Proof of Theorem 1.1. Suppose the multiplicity of the second eigenvalue is three, then from the above lemmas, we know (4.51) is true. That is

$$(4.60) \quad b_1(b_2b_3 - b_1b_4) - b_3(b_0b_3 - b_1b_2) = 0.$$

But by Holder inequality, we already know this is impossible because of $b_2b_3 < b_1b_4$ and $b_1b_2 < b_0b_3$. Then we proved the theorem 1.1.

Appendix.

We give an elementary proof of an inequality which we mentioned in section 4. We feel indebted to Prof. Liu Jiaqian for providing me the result.

Proposition A. Let $h(s) \geq 0$ in $[0, 1]$ and $h(s) \not\equiv 0$, set $b_i = \int_0^1 h(s)s^i ds$ then we have

$$b_2^3 + b_3^2b_0 + b_1^2b_4 < 2b_1b_2b_3 + b_0b_2b_4.$$

Proof. Consider

$$\int_0^1 \int_0^1 \int_0^1 h(s)h(r)h(t)(s-t)^2(s-r)^2(t-r)^2 dsdrdt > 0.$$

We can obtain the proof of the proposition by expansion of the above inequality.

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