Non-combability of Hilbert modular groups

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Introduction.

In the recent study of finitely generated infinite groups, important classes of groups have been defined and introduced by J.W. Cannon, M. Gromov, D.B.A. Epstein, D.F. Holt, S.V.F. Levy, M.S. Patterson, and W.P. Thurston: hyperbolic groups ([12]), automatic groups, combable groups ([5]), etc.. It has turned out that many well known classes of finitely generated groups belong (or do not belong) to them ([5, 8-12]). But so far not much seems to be known for arithmetic groups except the cocompact lattices in **R**-rank 1 semisimple Lie groups (which are Gromov hyperbolic). First we recall the known results:

(1) The group $SL(n, \mathbf{Z})$ is hyperbolic if $n = 2$ [11], and is not combable if $n \geq 3$ ([5]).

(2) For the ring $\mathcal{O}$ of algebraic integers of each imaginary quadratic number field, $SL(2, \mathcal{O})$ is automatic, because it is the fundamental group of a noncompact complete hyperbolic 3-manifold of finite volume, so is geometrically finite ([5]).

(3) Let $d \geq 2$ be a square-free integer and $\mathcal{O} = \mathcal{O}(\sqrt{d})$ the ring of algebraic integers in the real quadratic number field $\mathbf{Q}(\sqrt{d})$. Then the group $SL(2, \mathcal{O})$ is not biautomatic ([10]).

In this article, we show the following.

Theorem. *Let* **k** *($\neq \mathbf{Q}$) be a totally real algebraic number field and* $\mathcal{O}_{\mathbf{k}}$ *the ring of algebraic integers of* **k**. *Then the group* $SL(n, \mathcal{O}_{\mathbf{k}})$ *is not combable (therefore, not automatic) for* $n \geq 2$.

Remark. (1) There are several distinct notions of combing (see [1, 5, 8]). We claim in the above theorem that $SL(n, \mathcal{O}_{\mathbf{k}})$ is not combable in the sense of Epstein et al. ([5]).

(2) The case $n = [\mathbf{k} : \mathbf{Q}] = 2$ is one of the questions raised in [10]. And this gives the first example of arithmetic subgroups of **Q**-rank 1 algebraic

groups which are not combable. It seems that the description in [13, §3.H] is related to this case.

One of the central ideas in geometric group theory is that if a group acts isometrically, properly discontinuously, and cocompactly on a geodesic proper metric space, the geometry of the space affects the combinatorial nature of the group. The most natural space on which the arithmetic group $SL(n, \mathcal{O}_k)$ acts is probably the corresponding symmetric space X. Unfortunately the quotient space is not compact in this case. But we can consider the submanifold X' obtained by removing the portions corresponding to the ends of the quotient. In fact, Epstein and Thurston ([5]) investigated such a submanifold of the symmetric space $SO(n)\backslash SL(n, \mathbf{R})$ in the case $SL(n, \mathbf{Z})$ ($n \geq 3$). We clarify the geometric meaning of their argument and apply the method to wider classes of arithmetic groups.

They showed the following general result ([5, Theorem 10.3.5]): If a combable group acts on a k-connected Riemannian manifold, there exists a positive constant C with the following property. Let z be a lipschitz k-cycle in the manifold. Then there exists a lipschitz $(k + 1)$-chain u, such that $\partial u = z$ and that $mass_{k+1}(u) \leq C \cdot diam(z)mass_k(z)$, where $mass_i$ is the i-dimensional volume for $i = k, k + 1$ and $diam(z)$ is the diameter of z.

Thus the problem turns to finding a k-cycle z_m for each positive integer m such that the volume of z_m grows much faster than the infimum of the volumes of $(k + 1)$-chains filling z_m as m goes to infinity. And this is a matter of geometry of nonpositively curved manifolds.

In this part, the most important points of their argument are the following:

(1) The cycles z_m are constructed in a horosphere with respect to some suitable geodesic.

(2) In order to estimate the infimum of the volumes of chains filling z_m, they used a bounded closed differential form on the submanifold.

First we explain the former briefly. Horospheres in higher rank symmetric spaces have interesting metrical property while every horosphere (equipped with the induced Riemannian metric) in manifolds of constant negative sectional curvature is just the Euclidean space. For example, the typical horosphere in the Riemannian direct product $\mathcal{H} \times \mathcal{H}$ of the upper half planes, where each factor $\mathcal{H}$ has constant sectional curvature $-1/2$, is isometric to $\mathbf{R}^3$ equipped with the metric $ds^2 = e^z dx^2 + e^{-z} dy^2 + dz^2$. Let S_0 be a regular square in the xy-plane such that one pair of its opposite edges is parallel to the x-axis, the other pair is parallel to the y-axis, and all the edges have length $2\sqrt{2}e^{2m}$. If we push upward (resp. downward) S_0 along the z-axis till the z-coordinates change to $4m$ (resp. $-4m$), then the

length of the edges parallel to the y-axis (resp. the x-axis) of the resulting rectangle S_+ (resp. S_-) is equal to $2\sqrt{2}$ (see Figure 2 in §3). By using this property, we can construct 2-chain whose area is much larger than the length of its boundary. The situation is the same for $SO(n)\backslash SL(n, \mathbf{R})$. In case of other symmetric spaces, if we take a geodesic γ in some suitable direction, we can describe the metric of the horospheres with respect to γ by using the Langlands decomposition of the (standard) maximal parabolic $\mathbf{Q}$-subgroup (Lemma 2.1).

Concerning the latter, we use the pullback of the volume form on the unipotent part of the Langlands decomposition (see §4). Since this form Ω is not bounded on the whole space X, we need to know the structure of X' near its boundary to some extent. In the case $n \geq 3$, the structure of the end of the quotient is complicated, and we cannot know X' explicitly. But we can compute the norm of Ω explicitly on each horosphere (Lemma 4.2). So it suffices for our aim to prove that the submanifold X' is contained in the complement of a horoball with respect to γ. We show this by estimating the supremum of the injectivity radii on each horosphere (Lemma 5.7).

The organization of this paper is as follows. In §1, we take a finite index, normal subgroup Γ' of $\Gamma = SL(n, \mathcal{O}_{\mathbf{k}})$ and find a submanifold X' of the corresponding symmetric space X on which Γ acts cocompactly. In §2, we take a geodesic γ and describe the horospheres of X with respect to γ in terms of parabolic subgroups. We construct (preliminary) cycles z_m in §3 (, which we replace by new cycles $\mathscr{Z}_m$ in §5). In §4, we define a closed form Ω on X and compute its norm. In §5, we complete the proof of Theorem by finding a lower bound of the volumes of chains filling $\mathscr{Z}_m$.

Notation.

(1) We denote by $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{R}^+$, $\mathbf{C}^*$ the set of the complex numbers, the real numbers, the rational numbers, the (rational) integers, the positive real numbers, the nonzero complex numbers, respectively.

(2) For any algebraic group $\mathbf{H}$ defined over $\mathbf{Q}$, we denote by $\mathbf{H_R}$, $\mathbf{H_Q}$, $\mathbf{H_Z}$ the group of real points, rational points, integral points, respectively, of $\mathbf{H}$. And for any algebraic group $\mathbf{H}$ (resp. Lie group L), we denote by $\mathbf{H}^0$ (resp. L^0) the identity component of $\mathbf{H}$ (resp. L) in the Zariski topology (resp. in its topology as a Lie group).

1. Preliminaries.

Let $\mathbf{k}$ be a totally real algebraic number field and $\mathcal{O}_\mathbf{k}$ the ring of algebraic integers of $\mathbf{k}$. Let $l = [\mathbf{k} : \mathbf{Q}] \geq 2$ be the degree of $\mathbf{k}$ over $\mathbf{Q}$. Then the group $SL(n, \mathcal{O}_\mathbf{k})$ acts isometrically on the Riemannian product of l copies of the symmetric space $SO(n)\backslash SL(n, \mathbf{R})$. In this section, for the argument of the remainder sections, we regard $SL(n, \mathcal{O}_\mathbf{k})$ as the group $\Gamma = \mathbf{G}_\mathbf{Z}$ of integral points of a certain algebraic $\mathbf{Q}$-group $\mathbf{G}$. The general reference of this section is [18, Chap.6].

We recall that $\mathbf{k} = \mathbf{Q}(\theta)$ for some algebraic number θ. Let $\theta_1, \ldots, \theta_l$ ($\theta_1 = \theta$) be the l distinct roots of the minimum polynomial of θ over $\mathbf{Q}$. For each $i = 1, \ldots, l$, we denote by ϕ_i the field embedding $\mathbf{k} \longrightarrow \mathbf{C}$ such that $\phi_i(\theta) = \theta_i$. We remark that $\phi_1 = id.$ and that each ϕ_i can be extended to a field automorphism $\mathbf{C} \longrightarrow \mathbf{C}$. Let $\{\alpha_1, \ldots, \alpha_l\}$ be an integral basis of $\mathbf{k}$: i.e. $\mathcal{O}_\mathbf{k} = \mathbf{Z}\alpha_1 + \cdots + \mathbf{Z}\alpha_l$, $\mathbf{k} = \mathbf{Q}\alpha_1 + \cdots + \mathbf{Q}\alpha_l$. For each $i, j \in \{1, \ldots, l\}$, let $\alpha_i\alpha_j = \sum_{k=1}^l b_{ij}^k \alpha_k$; $b_{ij}^k \in \mathbf{Z}$. We denote by I_n the $n \times n$-matrix $diag(1, \ldots, 1)$ and put

$$\mathbf{S} = (\phi_j(\alpha_i)I_n) = \begin{pmatrix} \phi_1(\alpha_1)I_n & \phi_2(\alpha_1)I_n & \cdots & \phi_l(\alpha_1)I_n \\ \phi_1(\alpha_2)I_n & \phi_2(\alpha_2)I_n & \cdots & \phi_l(\alpha_2)I_n \\ \vdots & \vdots & & \vdots \\ \phi_1(\alpha_l)I_n & \phi_2(\alpha_l)I_n & \cdots & \phi_l(\alpha_l)I_n \end{pmatrix},$$

$$\mathbf{B}_k = \left(b_{ki}^j I_n\right) ; \ k = 1, \ldots l.$$

For each k and $h = (h_{ij}) \in SL(n, \mathbf{C})$, we denote by $\phi_k(h)$ the element $\left(\phi_k(h_{ij})\right)$ of $SL(n, \mathbf{C})$. We define a faithful representation $\iota : SL(n, \mathbf{C}) \longrightarrow GL(nl, \mathbf{C})$ by

$$\iota(h) = diag(\phi_1(h), \ldots, \phi_l(h)) \qquad \text{for } h \in SL(n, \mathbf{C}).$$

We also define another faithful representation $\rho : SL(nl, \mathbf{C}) \longrightarrow GL(nl, \mathbf{C})$ by $\rho(g) = \mathbf{S}^{-1}g\mathbf{S}$ for $g \in SL(nl, \mathbf{C})$. Let $\mathfrak{gl}(nl, \mathbf{C})$ be the set of $(nl) \times (nl)$ complex matrices. For

$$Z = (Z_{ij}) = \begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1l} \\ Z_{21} & Z_{22} & \cdots & Z_{2l} \\ \vdots & \vdots & & \vdots \\ Z_{l1} & Z_{l2} & \cdots & Z_{ll} \end{pmatrix} \in \mathfrak{gl}(nl, \mathbf{C}),$$

where each Z_{ij} is an $n \times n$-matrix, we consider the following system of linear equations:

$$(1.1) \quad \begin{pmatrix} Z_{i1} & Z_{i2} & \dots & Z_{il} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} & \dots & Z_{1l} \end{pmatrix} \mathbf{B}_1^{-1} \mathbf{B}_i \, ; \, i = 2, \dots, l.$$

We also consider the following equations:

$$(1.2) \qquad \det(p_r \circ \rho(Z)) = 1 \, ; \, r = 1, \dots, l,$$

where $p_r : \mathfrak{gl}(nl, \mathbf{C}) \longrightarrow \mathfrak{gl}(n, \mathbf{C})$ is the map defined by $p_r(Y) = Y_{rr}$ for $Y = (Y_{ij}) \in \mathfrak{gl}(nl, \mathbf{C})$. Let $\mathbf{G} \, (\subset SL(nl, \mathbf{C}))$ be the algebraic $\mathbf{Q}$-group defined by the equations (1.1) and (1.2). Then $\mathbf{G}$ is "the group obtained from $SL(n, \mathbf{C})$ by the restriction of scalars from $\mathbf{k}$ to $\mathbf{Q}$" in the sense of [19, Chap.I]. And we have $\mathbf{G} = (\rho^{-1} \circ \iota)(SL(n, \mathbf{C}))$, $\mathbf{G}_{\mathbf{Z}} = (\rho^{-1} \circ \iota)(SL(n, \mathcal{O}_{\mathbf{k}}))$. We put $\Gamma = \mathbf{G}_{\mathbf{Z}}$ and identify Γ with $SL(n, \mathcal{O}_{\mathbf{k}})$. We also have

$$\mathbf{G}_{\mathbf{R}} = \rho^{-1} \left(\{ diag(B_1, \dots, B_l) \, | \, B_1, \dots, B_l \in SL(n, \mathbf{R}) \} \right).$$

The group $\mathbf{G}_{\mathbf{R}}$ is isomorphic to the direct product of l copies of $SL(n, \mathbf{R})$. Let

$$K = \rho^{-1} \left(\{ diag(K_1, \dots, K_l) \, | \, K_1, \dots, K_l \in SO(n) \} \right).$$

Then K is a maximal compact subgroup of $\mathbf{G}_{\mathbf{R}}$ and is isomorphic to the direct product of l copies of $SO(n)$.

We put $X = K \backslash \mathbf{G}_{\mathbf{R}}$, and give it the standard metric induced from the Killing form of the Lie algebra of $\mathbf{G}_{\mathbf{R}}$. Then X is isometric to the Riemannian product of l copies of $SO(n) \backslash SL(n, \mathbf{R})$ equipped with the standard metric. (Notice that $SO(2) \backslash SL(2, \mathbf{R})$ has constant sectional curvature $-1/2$ when we give it the standard metric induced from the Killing form.) The discrete group Γ acts isometrically on the symmetric space X from the right. We take a torsion-free, normal subgroup Γ' of Γ of finite index.

By Raghunathan's construction ([17]), there exists a Γ-invariant smooth function $\tilde{f} : X \longrightarrow \mathbf{R}$ such that the induced smooth function f on the quotient manifold X/Γ' satisfies the following conditions: (1) f is proper, (2) f has no critical points outside some compact subset, (3) the image of f is $[t_0, \infty)$ for some positive number t_0.

Then, for sufficiently large $t > 0$, $f^{-1}([t_0, t])$ is a deformation retract of X/Γ'. If we put $X_t = \tilde{f}^{-1}([t_0, t])$, X_t is a submanifold of X such that $\pi_i(X_t) = 0$ for all $i \geqq 0$. We put $X' = X_{t_1}$ for a positive number t_1 which we determine in §5. The group Γ acts isometrically, properly discontinuously, and cocompactly on X'.

2. Horospheres.

We take a maximal $\mathbf{Q}$-split torus $\mathbf{T}$ of $\mathbf{G}$ defined by

$$\mathbf{T} = \rho^{-1}\left(\{diag(\beta_1,\ldots,\beta_n,\ldots\ldots,\beta_1,\ldots,\beta_n)\,|\,\beta_1\cdots\cdot\beta_n = 1\}\right).$$

For each $k = 1,\ldots,n-1$, we define the map $\widehat{\chi}_k : \mathbf{T} \longrightarrow \mathbf{C}^*$ by

$$\widehat{\chi}_k(a) = \beta_k \qquad \text{for } a = diag(\beta_1,\ldots,\beta_n,\ldots\ldots,\beta_1,\ldots,\beta_n) \in \mathbf{T},$$

and put $\chi_k = \dfrac{\widehat{\chi}_k}{\widehat{\chi}_{k+1}}$. The set of $\mathbf{Q}$-roots of $\mathbf{G}$ with respect to $\mathbf{T}$ is

$$\mathbf{Q}\Sigma = \left\{\left.\frac{\widehat{\chi}_i}{\widehat{\chi}_j}\,\right|\,i,j = 1,\ldots,n; i \neq j\right\}.$$

We can take $\mathbf{Q}\Sigma^+ = \left\{\left.\dfrac{\widehat{\chi}_i}{\widehat{\chi}_j}\,\right|\,i > j\right\}$ and $\mathbf{Q}\Delta = \{\chi_1,\ldots,\chi_{n-1}\}$ as a system of positive roots and simple roots respectively. Then the standard maximal parabolic $\mathbf{Q}$-subgroup $\mathbf{P}_\Theta$ corresponding to the subset $\Theta = \{\chi_1,\ldots,\chi_{n-2}\} \subset \mathbf{Q}\Delta$ is given by the following:

$$\mathbf{P}_\Theta = \mathbf{M}_\Theta\mathbf{T}_\Theta\mathbf{U}_\Theta,$$

$$\mathbf{T}_\Theta = \left(\bigcap_{\chi\in\Theta}\ker\chi\right)^0$$

$$= \rho^{-1}\left(\left\{diag(\underbrace{\beta,\ldots,\beta}_{n-1},\beta^{1-n},\ldots\right.\right.$$

$$\left.\left.\ldots,\underbrace{\beta,\ldots,\beta}_{n-1},\beta^{1-n})\,|\,\beta\in\mathbf{C}^*\right\}\right),$$

$$\mathbf{M}_\Theta =$$

$$\rho^{-1}\left(\left\{diag(B_1,b_1,\ldots,B_l,b_l)\,\left|\,\begin{array}{l}B_k \text{ is an } (n-1)\times(n-1) \text{ complex}\\ \text{matrix, } b_k \in \mathbf{C}^*, \text{ and } (\det B_k)b_k = 1\\ \text{for all } k;\,(\det B_1)\cdots\cdot(\det B_l) = \pm 1\end{array}\right.\right\}\right),$$

$$\mathbf{U}_\Theta = \rho^{-1}\left(\left\{\left(\begin{pmatrix} I_{n-1} & {}^t(\boldsymbol{\xi}^1) \\ & 1 \end{pmatrix} \\ \qquad\qquad \ddots \\ \qquad\qquad\qquad \begin{matrix} I_{n-1} & {}^t(\boldsymbol{\xi}^l) \\ & 1 \end{matrix} \end{pmatrix}\middle| \begin{array}{l} \boldsymbol{\xi}^k \in \mathbf{C}^{n-1} \text{ for} \\ k = 1, \dots, l \end{array}\right\}\right),$$

where I_{n-1} is the $(n-1) \times (n-1)$ unit matrix. The group $\mathbf{M}_\Theta\mathbf{T}_\Theta$ is the centralizer $Z(\mathbf{T}_\Theta)$ of $\mathbf{T}_\Theta$ in $\mathbf{G}$. We put

$$M_\Theta = ((\mathbf{M}_\Theta)_\mathbf{R})^0 =$$

$$\rho^{-1}\left(\left\{ diag(B_1, b_1, \dots, B_l, b_l) \middle| \begin{array}{l} B_k \text{ is an } (n-1) \times (n-1) \text{ real} \\ \text{matrix, } b_k > 0, \text{ and } (\det B_k)b_k = 1 \\ \text{for all } k;\, (\det B_1) \cdots (\det B_l) = 1 \end{array}\right\}\right),$$

$$A_\Theta = ((\mathbf{T}_\Theta)_\mathbf{R})^0$$

$$= \rho^{-1}\left(\left\{ diag(\underbrace{q, \dots, q}_{n-1}, q^{1-n}, \dots\dots, \underbrace{q, \dots, q}_{n-1}, q^{1-n}) \,|\, q > 0 \right\},\right)$$

$$N_\Theta = (\mathbf{U}_\Theta)_\mathbf{R}$$

$$= \rho^{-1}\left(\left\{\left(\begin{pmatrix} I_{n-1} & {}^t(\boldsymbol{\xi}^1) \\ & 1 \end{pmatrix} \\ \qquad\qquad \ddots \\ \qquad\qquad\qquad \begin{matrix} I_{n-1} & {}^t(\boldsymbol{\xi}^l) \\ & 1 \end{matrix} \end{pmatrix}\middle| \begin{array}{l} \boldsymbol{\xi}^k \in \mathbf{R}^{n-1} \text{ for} \\ k = 1, \dots, l \end{array}\right\}\right).$$

Then $(\mathbf{P}_\Theta)_\mathbf{R} = (\mathbf{M}_\Theta)_\mathbf{R} A_\Theta N_\Theta$ is a Langlands decomposition of $(\mathbf{P}_\Theta)_\mathbf{R}$.

Since it is well known that $\mathbf{G}_\mathbf{R} = K(\mathbf{P}_\Theta)_\mathbf{R}$, we have $\mathbf{G}_\mathbf{R} = K \cdot M_\Theta A_\Theta N_\Theta$. Let x_0 be the coset (in $X = K\backslash\mathbf{G}_\mathbf{R}$) of the identity element. Then we have $X = x_0 \cdot M_\Theta A_\Theta N_\Theta$.

We take a geodesic along the A_Θ-orbit of x_0. Let $\mathfrak{a}_\Theta$ be the Lie algebra of A_Θ. Let

$$\mathbf{V} =$$

$$(d\rho_e)^{-1}\left(\frac{1}{n\sqrt{2(n-1)l}} diag(-1, \dots, -1, n-1, \dots\dots, -1, \dots, -1, n-1)\right)$$

$$\in \mathfrak{a}_\Theta,$$

where $d\rho_e$ is the differential of ρ at the identity element e. We define a unit speed geodesic $\gamma : [0, \infty) \longrightarrow X$ by $\gamma(t) = x_0 \cdot (\exp t\mathrm{V})$ for $t \geqq 0$. Let h_γ be the Busemann function with respect to γ: i.e. $h_\gamma(x) = \lim_{t \to \infty} \{d_X(x, \gamma(t)) - t\}$ for $x \in X$, where d_X is the distance on X ([2], [7]).

Lemma 2.1. *The horosphere $h_\gamma^{-1}(t)$ is given by $h_\gamma^{-1}(t) = x_0 \cdot a_{-t} M_\Theta N_\Theta$, where*

$$a_{-t} = \exp(-t\mathrm{V}) = \rho^{-1} \left(diag \left(e^{\frac{t}{n\sqrt{2(n-1)l}}}, \dots, e^{\frac{t}{n\sqrt{2(n-1)l}}}, e^{-\frac{t}{n}\sqrt{\frac{n-1}{2l}}}, \dots \right. \right.$$
$$\left. \left. \dots, e^{\frac{t}{n\sqrt{2(n-1)l}}}, \dots, e^{\frac{t}{n\sqrt{2(n-1)l}}}, e^{-\frac{t}{n}\sqrt{\frac{n-1}{2l}}} \right) \right).$$

In particular, the horosphere through x_0 is $h_\gamma^{-1}(0) = x_0 \cdot M_\Theta N_\Theta$.

Proof. Let $P(nl, \mathbf{R})$ be the set of all positive definite, symmetric matrices contained in $SL(nl, \mathbf{R})$. We identify $P(nl, \mathbf{R})$ with the symmetric space $SO(nl) \backslash SL(nl, \mathbf{R})$ in the usual way : i.e. $SL(nl, \mathbf{R})$ operates transitively on $P(nl, \mathbf{R})$ by conjugation ($x^* \cdot h^* = {}^t h^* x^* h^*$ for $x^* \in P(nl, \mathbf{R})$, $h^* \in SL(nl, \mathbf{R})$), and the isotropy group of $x_0^* = I_{nl} = diag(1, \dots, 1) \in P(nl, \mathbf{R})$ is $SO(nl)$. We define a totally geodesic embedding $F : X = K \backslash \mathbf{G_R} \longrightarrow P(nl, \mathbf{R})$ by $F(Kh) = {}^t(\rho(h))(\rho(h))$ for $h \in \mathbf{G_R}$ ([6], [16]). Then the pull-back of the Riemannian metric of $P(nl, \mathbf{R})$ under the map F is identical with the original Riemannian metric of X multiplied by l. Hence the speed of the geodesic $F \circ \gamma : [0, \infty) \longrightarrow P(nl, \mathbf{R})$ is $\sqrt{l}$. We define a unit speed geodesic $\gamma^* : [0, \infty) \longrightarrow P(nl, \mathbf{R})$ by

$$\gamma^*(t) = F(\gamma(\frac{t}{\sqrt{l}})) = F(x_0 \cdot \exp \frac{t}{\sqrt{l}} \mathrm{V})$$
$$= diag \left(e^{-\frac{t}{nl}\sqrt{\frac{2}{n-1}}}, \dots, e^{-\frac{t}{nl}\sqrt{\frac{2}{n-1}}}, e^{\frac{\sqrt{2(n-1)}}{nl}t}, \dots \right.$$
$$\left. \dots, e^{-\frac{t}{nl}\sqrt{\frac{2}{n-1}}}, \dots, e^{-\frac{t}{nl}\sqrt{\frac{2}{n-1}}}, e^{\frac{\sqrt{2(n-1)}}{nl}t} \right)$$

for $t \geqq 0$. Let $h_{\gamma^*}^*$ be the Busemann function (on $P(nl, \mathbf{R})$) with respect to the geodesic γ^*. Then we have $h_\gamma(x) = \frac{1}{\sqrt{l}} h_{\gamma^*}^*(F(x))$ for $x \in X$. We take

the element V^* of the tangent space $T_{x^*}(P(nl, \mathbf{R}))$ defined by

$$V^* = F_*(\frac{1}{\sqrt{l}}V)$$

$$= \frac{1}{nl\sqrt{2(n-1)}} diag(\underbrace{-1, \dots, -1}_{n-1}, n-1, \dots$$

$$\dots, \underbrace{-1, \dots, -1}_{n-1}, n-1).$$

We also take the permutation τ of (nl) letters defined by the following :

$$\begin{cases} \tau(kn+b) = k(n-1)+b & \text{for } k = 0, 1, \dots, l-1 \\ & \text{and } b = 1, \dots, n-1 \\ \tau(kn) = l(n-1)+k & \text{for } k = 1, 2, \dots, l. \end{cases}$$

Let A^* be the group of diagonal matrices in $SL(nl, \mathbf{R})$ with positive entries. For each $k = 1, \dots, nl-1$, we define the map $\chi_k^* : A^* \longrightarrow \mathbf{R}^+$ as follows.

$$\chi_k^*(a^*) = \frac{a_k}{a_{k+1}} \quad \text{for } a^* = diag(a_1, \dots, a_{nl}) \in A^*.$$

For $x^* = (x_{ij}) \in P(nl, \mathbf{R})$, we denote by $\triangle_k(x^*)$, the $(k \times k)$-minor determinant in the top left corner. And for each $(nl) \times (nl)$-matrix $Z^* = (z_{ij})$, we denote by $Z^* \cdot \tau$ the matrix $(z_{\tau(i)\tau(j)})$.

Then, from Lemma 2.5 of [14], we obtain

$$h_{\gamma^*}^*(F(x_0 \cdot g)) = nl \cdot \log \prod_{k=1}^{l-1} \left\{ (\triangle_k(F(x_0 \cdot g) \cdot \sigma))^{-d\chi_k^*(V^* \cdot \tau)} \right\},$$

where $d\chi_k^*$ is the differential of χ_k^*. We have $d\chi_k^*(V^* \cdot \tau) \neq 0$ if and only if $k = (n-1)l$, and then, $d\chi_{(n-1)l}^*(V^* \cdot \tau) = -\frac{1}{l\sqrt{2(n-1)}}$. Therefore,

$$h_{\gamma}(x_0 \cdot g) = \frac{n}{\sqrt{2(n-1)l}} \log \triangle_{(n-1)l}(F(x_0 \cdot g) \cdot \tau).$$

We notice that $X = x_0 \cdot M_\Theta A_\Theta N_\Theta$. Let

$$g_1 = \rho^{-1}(diag(B_1, b_1, \dots B_l, b_l)) \in M_\Theta,$$

$$g_2 = \rho^{-1}(diag(q, \dots, q, q^{1-n} \dots \dots, q, \dots, q, q^{1-n})) \in A_\Theta,$$

$$g_3 = \rho^{-1} \left(\begin{pmatrix} I_{n-1} & \boldsymbol{\xi}^1 \\ & 1 \\ & & \ddots \\ & & & I_{n-1} & \boldsymbol{\xi}^l \\ & & & & 1 \end{pmatrix} \right) \in N_\Theta,$$

and $g = g_1 g_2 g_3$. Then we have

$$F(x_0 \cdot g) \cdot \tau = \left(\begin{array}{ccc|c} q^2(^t B_1 B_1) & & & \\ & \ddots & & * \\ & & q^2(^t B_l B_l) & \\ \hline & ** & & *** \end{array} \right).$$

So $\triangle_{(n-1)l}(F(x_0 \cdot g) \cdot \tau) = q^{2(n-1)l}$, and $h_\gamma(x_0 \cdot g) = n\sqrt{2(n-1)l}\log q$.
Therefore, we obtain $h_\gamma^{-1}(t) = x_0 \cdot a_{-t} M_\Theta N_\Theta$. $\qquad\square$

3. $\{(n-1)l - 1\}$-cycle z_m and $(n-1)l$-chain u_m.

The horosphere $\mathcal{L} = h_\gamma^{-1}(0) = x_0 \cdot M_\Theta N_\Theta$ has a fiber bundle structure : we define the projection map $\mathcal{P}_b : \mathcal{L} \longrightarrow x_0 \cdot M_\Theta$ by $\mathcal{P}_b(x_0 \cdot g_1 g_2) = x_0 \cdot g_1$ for $g_1 \in M_\Theta$ and $g_2 \in N_\Theta$. We regard N_Θ as $\mathbf{R}^{(n-1)l}$ by identifying $(y_1, \ldots, y_{(n-1)l}) \in \mathbf{R}^{(n-1)l}$ with

$$\rho^{-1} \left(\begin{pmatrix} I_{n-1} & {}^t(\boldsymbol{\xi}^1) \\ & 1 \\ & & \ddots \\ & & & I_{n-1} & {}^t(\boldsymbol{\xi}^l) \\ & & & & 1 \end{pmatrix} \right) \in N_\Theta,$$

where $\boldsymbol{\xi}^k = (y_{(n-1)(k-1)+1}, y_{(n-1)(k-1)+2}, \ldots, y_{(n-1)k})$ for all k. We define a map $\mathcal{P}_f : \mathcal{L} \longrightarrow N_\Theta = \mathbf{R}^{(n-1)l}$ in the direction of the fibers by $\mathcal{P}_f(x_0 \cdot g_1 g_2) = g_2$ for $g_1 \in M_\Theta$ and $g_2 \in N_\Theta$. In this section, by using this fiber bundle structure of $\mathcal{L}$, we construct a lipschitz $\{(n-1)l - 1\}$-cycle z_m for each positive integer m ; we also construct a $(n-1)l$-chain u_m whose boundary is z_m.

First, we decompose the group M_Θ. By identifying $diag(B_1, \ldots, B_l)$, where $B_1, \ldots, B_l \in SL(n-1, \mathbf{C})$, with $diag(B_1, 1, \ldots, B_l, 1) \in \rho(\mathbf{G})$, we can regard the direct product $(SL(n-1, \mathbf{C}))^l$ as a subgroup of $\rho(\mathbf{G})$. Let

$\overline{\mathbf{G}} = \rho^{-1}((SL(n-1,\mathbf{C}))^l)$. Then $\overline{\mathbf{G}}$ is an algebraic $\mathbf{Q}$-group. We put $\overline{K} = K \cap \overline{\mathbf{G}}$. Let $A^\perp$ be the center of M_Θ : i.e.

$$A^\perp = \rho^{-1}(\{\ diag(\underbrace{q_1,\ldots,q_1}_{n-1},(q_1)^{1-n},\ldots\ldots,\underbrace{q_l,\ldots,q_l}_{n-1},(q_l)^{1-n})$$

$$|q_k > 0 \text{ for all } k,\ q_1 \cdots \cdot q_l = 1\ \}).$$

Then we have $M_\Theta = (\mathbf{G_R}) \cdot A^\perp$ and $K \cap M_\Theta = \overline{K}$. The space $x_0 \cdot M_\Theta$ is isometric to the Riemannian direct product $(SO(n-1)\backslash SL(n-1,\mathbf{R}))^l \times \mathbf{R}^{l-1}$, where each $SO(n-1)\backslash SL(n-1,\mathbf{R})$ (resp. $\mathbf{R}^{l-1}$) is equipped with the standard metric multiplied by $n/(n-1)$ (resp. the standard Euclidean metric). Let $\overline{\mathbf{P}}$ be the standard minimal parabolic $\mathbf{Q}$-subgroup of $\overline{\mathbf{G}}$ with respect to the maximal $\mathbf{Q}$-split torus $\overline{\mathbf{T}} = \mathbf{T} \cap \overline{\mathbf{G}}$ of $\overline{\mathbf{G}}$. As in the case of $\mathbf{P}_\Theta$ in §2, we decompose $\overline{\mathbf{P}}$ as $\overline{\mathbf{P}} = \overline{\mathbf{M}}\ \overline{\mathbf{T}}\ \overline{\mathbf{U}}$, and put

$$\overline{M} = (\overline{\mathbf{M}}_\mathbf{R})^0 = \rho^{-1}(\left\{ diag(q_1^1,\ldots,q_{n-1}^1,1,\ldots\ldots,q_1^l,\ldots,q_{n-1}^l,1) \left| \begin{array}{l} \text{each } q_j^i \text{ is a positive real number },\\[2mm] \prod_{j=1}^{n-1} q_j^i = 1 \text{ for all } i, \prod_{i=1}^{l} q_j^i = 1 \text{ for all } j \end{array} \right. \right\}),$$

$$\overline{A} = (\overline{\mathbf{T}}_\mathbf{R})^0 = \rho^{-1}(\{ diag(q_1,\ldots,q_{n-1},1,\ldots\ldots,q_1,\ldots,q_{n-1},1) \left| \begin{array}{l} \text{each } q_j \text{ is a positive real number },\\[2mm] q_1 \cdots \cdot q_{n-1} = 1 \end{array} \right. \}),$$

$$\overline{N}_\Theta = \overline{\mathbf{U}}_\mathbf{R} =$$

$$\rho^{-1}\left(\left\{ diag(\mathfrak{m}_1,1,\ldots,\mathfrak{m}_l,1) \left| \begin{array}{l} \text{each } \mathfrak{m}_k = \begin{pmatrix} 1 & & \beta_{ij}^k \\ & \ddots & \\ & & 1 \end{pmatrix} \text{ is }\\ \text{an upper triangular, real,}\\ (n-1)\times(n-1) \text{ matrix} \end{array} \right. \right\}\right).$$

Then we have $M_\Theta = (\overline{\mathbf{G}}_\mathbf{R}) \cdot A^\perp = \overline{K}\ \overline{M}\ \overline{A}\ \overline{N}A^\perp$.

Let

$$H_\Theta = \overline{M}\ \overline{A}A^\perp =$$

$$= \rho^{-1}\left(\left\{ diag(q_1,\ldots,q_{nl}) \left| \begin{array}{c} q_i > 0 \text{ for all } i, q_{kn+1} \cdots \cdot q_{kn+n} = 1\\ \text{for } k = 0,1,\ldots,l-1,\\ q_n \cdot q_{2n} \cdots \cdot q_{ln} = 1 \end{array} \right. \right\}\right).$$

$$X \quad \Omega$$

$$\pi \downarrow$$

the image
of $z_{I,J,m}$ $\subset x_0 \cdot H_\Theta N_\Theta \subset \quad \mathcal{L} \xrightarrow{\mathcal{P}_f} N_\Theta = \mathbf{R}^{(n-1)l} \supset \square_{I,J,m}$

$$n^{(n-1)l/2} dy_1 \wedge \cdots \wedge dy_{(n-1)l}$$

$$\downarrow \qquad \downarrow \mathcal{P}_b$$

$$H \xrightarrow{\Phi} \mathfrak{h}_\Theta \xrightarrow{\exp} H_\Theta \xrightarrow{\cong} x_0 \cdot H_\Theta \subset x_0 \cdot M_\Theta$$
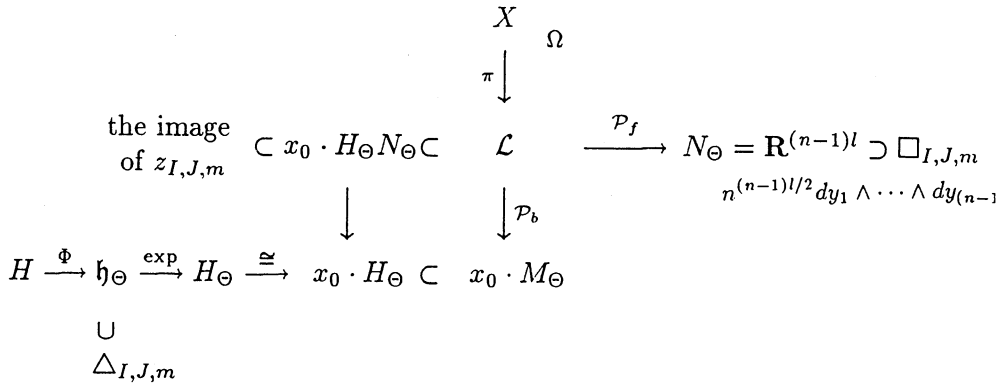
$$\cup$$

$$\triangle_{I,J,m}$$

Figure 1: The construction of the chain $z_{I,J,m}$ and the form Ω.

(Remark that $H_\Theta = M_\Theta$ if $n = 2$.) Then the restriction of $\mathcal{P}_b$ to $x_0 \cdot H_\Theta N_\Theta$ is also a fiber bundle.

From now on to Corollary 3.2 exclusive, we fix one positive integer m. We define z_m as a union $\sum z_{I,J,m}$ of $\{(n-1)l - 1\}$-chains, where the summation is taken over all the pair (I, J) of subsets of $\{1, \ldots, (n-1)l\}$ such that $I \cap J$ is the empty set ϕ and that $I \cup J \neq \phi$. We fix one such pair (I, J) and define the $\{(n-1)l - 1\}$-chain $z_{I,J,m}$ as follows (see Figure 1).

Let $\mathfrak{h}_\Theta$ be the Lie algebra of H_Θ and

$$H = \left\{ (h_1, \ldots, h_{(n-1)l}) \in \mathbf{R}^{(n-1)l} \,|\, h_1 + \cdots + h_{(n-1)l} = 0 \right\}$$

be a hyperplane in $\mathbf{R}^{(n-1)l}$. We define a map $\Phi : H \longrightarrow \mathfrak{h}_\Theta$ by

$$\Phi((h_1, \ldots, h_{(n-1)l})) =$$
$$\rho^{-1}(diag(h_1, \ldots, h_{n-1}, -(h_1 + \cdots + h_{n-1}), \ldots$$
$$\ldots, h_{(n-1)(k-1)+1}, \ldots, h_{(n-1)k}, -(h_{(n-1)(k-1)+1} + \cdots + h_{(n-1)k}), \ldots$$
$$\ldots, h_{(n-1)(l-1)+1}, \ldots, h_{(n-1)l}, -(h_{(n-1)(l-1)+1} + \cdots + h_{(n-1)l}))).$$

This map Φ is a lipschitz diffeomorphism when we give $\mathfrak{h}_\Theta$ the inner product induced from the Killing form of the Lie algebra of $\mathbf{G_R}$. We define $(n-1)l$ points $w_{1,m}, \ldots, w_{(n-1)l,m}$ of H as follows : for each $k = 0, \ldots, l-1$ and
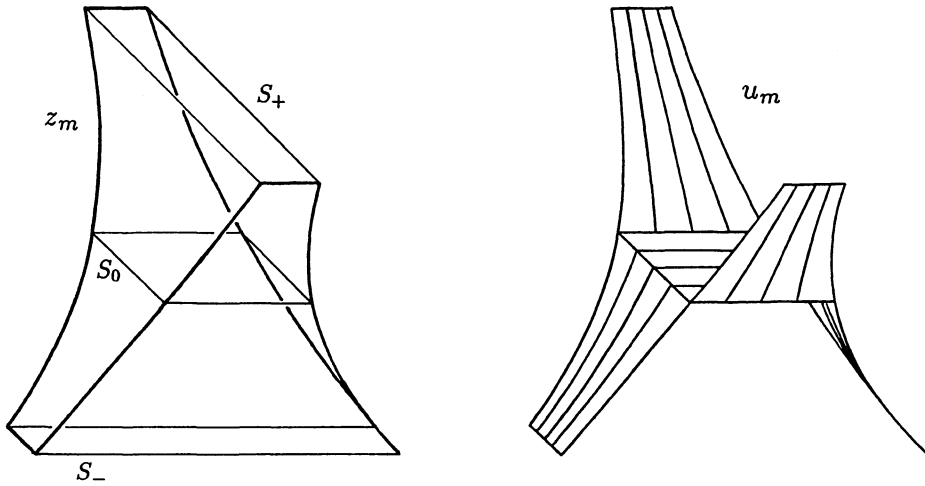
$j = 1, \ldots, n - 1$, let

$$
\begin{aligned}
w_{(n-1)k+j,m} \\
= (-m, \ldots\ldots\ldots, -m, &-\{(n-1)l+1\}\, m, \ldots, -\{(n-1)l+1\}\, m, \\
&\{(n-1)^2 l - 1\}\, m, -\{(n-1)l+1\}\, m, \ldots, -\{(n-1)l+1\}\, m, \\
&\qquad\qquad\qquad\qquad\qquad - m, \ldots\ldots\ldots - m),
\end{aligned}
$$

where the i-th entry for $(n-1)k+1 \leq i \leq (n-1)(k+1)$ with $i \neq (n-1)k+j$ is $-\{(n-1)l+1\}\, m$, the $\{(n-1)k+j\}$-th entry is $\{(n-1)^2 l - 1\}\, m$, and the rests are all equal to $-m$. We put $v_{i,m} = \Phi(w_{i,m})$ for $i = 1, \ldots, (n-1)l$.

Let $I \cup J = \{i_1, \ldots, i_s\} : i_1 < \ldots < i_s$. We denote by $\triangle_{I,J,m}$ the simplex whose vertices are $v_{i_1,m}, \ldots, v_{i_s,m}$: i.e. $\triangle_{I,J,m} = |v_{i_1,m} \cdots\cdots v_{i_s,m}|$. Let $\square_{I,J,m}$ be the $\{(n-1)l - s\}$-dimensional cube in $N_\Theta = \mathbf{R}^{(n-1)l}$ defined by

$$
\square_{I,J,m} =
$$
$$
\left\{ (y_1, \ldots, y_{(n-1)l}) \in \mathbf{R}^{(n-1)l} = N_\Theta \;\middle|\;
\begin{array}{l}
y_i = e^{nm} \text{ if } i \in I \\
y_j = -e^{nm} \text{ if } j \in J \\
-e^{nm} \leq y_k \leq e^{nm} \text{ if } k \notin I \cup J
\end{array}
\right\}.
$$

We define the map $z_{I,J,m} : \triangle_{I,J,m} \times \square_{I,J,m} \longrightarrow x_0 \cdot H_\Theta N_\Theta$ by $z_{I,J,m}(v,y) = x_0 \cdot (\exp v) y$ for $v \in \triangle_{I,J,m}$, $y \in \square_{I,J,m}$. By considering the standard simplicial decomposition of $\triangle_{I,J,m} \times \square_{I,J,m}$, we regard the map $z_{I,J,m}$ as an $\{(n-1)l - 1\}$-chain (see Figure 2).

Lemma 3.1. *The image of $z_{I,J,m}$ (equipped with the induced metric from X) is isometric to the direct product of the simplex $|v_{i_1,m} \cdots\cdots v_{i_s,m}|$ and $[-\sqrt{n}, \sqrt{n}]^{(n-1)l-s}$.*

Proof. Any C^∞-curve $c : (-\delta, \delta) \longrightarrow x_0 \cdot H_\Theta N_\Theta$ $(\delta > 0)$ in the image of $z_{I,J,m}$ can be written as $c(t) = x_0 \cdot \widehat{c}(t)$ by a curve $\widehat{c} : (-\delta, \delta) \longrightarrow H_\Theta N_\Theta$ in $H_\Theta N_\Theta$ such that

$$
\rho(\widehat{c}(t)) = diag(e^{b_1^1(t)}, \ldots, e^{b_n^1(t)}, \ldots\ldots, e^{b_1^l(t)}, \ldots, e^{b_n^l(t)})
$$
$$
\times
\begin{pmatrix}
I_{n-1} & {}^t(\xi^1(t)) & & & \\
& 1 & & & \\
& & \ddots & & \\
& & & I_{n-1} & {}^t(\xi^l(t)) \\
& & & & 1
\end{pmatrix}
$$

Figure 2: $(n = l = 2)$

for $t \in (-\delta, \delta)$, where all $\boldsymbol{\xi}^k(t) = (\xi_1^k(t), \ldots, \xi_{n-1}^k(t))$ (resp. $b_j^k(t)$) are C^∞-maps $(-\delta, \delta) \longrightarrow \mathbf{R}^{n-1}$ (resp. C^∞-functions on $(-\delta, \delta)$), and $\sum_{i=1}^n b_i^k \equiv 0$ for all k. Since the right translations are isometries, we take the following element g of $H_\Theta N_\Theta$ and put $\widetilde{c}(t) = c(t) \cdot g = x_0 \cdot \widehat{c}(t)g$.

$$
\rho(g) = diag(e^{-b_1^1(0)}, \ldots, e^{-b_n^1(0)}, \ldots\ldots, e^{-b_1^l(0)}, \ldots, e^{-b_n^l(0)})
$$

$$
\times \begin{pmatrix} I_{n-1} & {}^t\boldsymbol{\zeta}^1 & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & I_{n-1} & {}^t\boldsymbol{\zeta}^l \\ & & & & 1 \end{pmatrix},
$$

$$
\boldsymbol{\zeta}^k = (-e^{b_1^k(0)-b_n^k(0)}\xi_1^k(0), \ldots - e^{b_{n-1}^k(0)-b_n^k(0)}\xi_{n-1}^k(0)) \text{ for } k = 1, \ldots, l.
$$

We remark that $\widetilde{c} : (-\delta, \delta) \longrightarrow x_0 \cdot H_\Theta N_\Theta$ is a curve such that $\widetilde{c}(0) = x_0$. Since

$$
\widehat{c}(t)g = \rho^{-1}\left(\begin{pmatrix} \begin{pmatrix} B^1(t) & {}^t(\boldsymbol{\eta}^1(t)) \\ & e^{b_n^1(t)-b_n^1(0)} \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} B^l(t) & {}^t(\boldsymbol{\eta}^l(t)) \\ & e^{b_n^l(t)-b_n^l(0)} \end{pmatrix} \end{pmatrix}\right),
$$

where,

$$B^k(t) = diag(e^{b_1^k(t)-b_1^k(0)}, \dots, e^{b_{n-1}^k(t)-b_{n-1}^k(0)})$$

and

$$\boldsymbol{\eta}^k(t) = (e^{b_1^k(t)-b_n^k(0)}(\xi_1^k(t) - \xi_1^k(0)), \dots, e^{b_{n-1}^k(t)-b_n^k(0)}(\xi_{n-1}^k(t) - \xi_{n-1}^k(0)))$$

for $k = 1, \dots, l$, we have

$$\left.\frac{d}{dt}(\widehat{c}(t)g)\right|_{t=0} =$$

$$(d\rho_e)^{-1}\left(\left(\begin{pmatrix} (B^1)'(0) & {}^t\{(\boldsymbol{\eta}^1)'(0)\} \\ & (b_n^1)'(0) \end{pmatrix} \right.\right. \\ \left.\left. \ddots \right.\right. \\ \left.\left. \begin{pmatrix} (B_l)'(0) & {}^t\{(\boldsymbol{\eta}^l)'(0)\} \\ & (b_n^l)'(0) \end{pmatrix} \right)\right)$$

and

$$(\boldsymbol{\eta}^k)'(0) = (e^{b_1^k(0)-b_n^k(0)}(\xi_1^k)'(0), \dots, e^{b_{n-1}^k(0)-b_n^k(0)}(\xi_{n-1}^k)'(0))$$

for $k = 1, \dots, l$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of $\mathbf{G_R}$ (resp. K). We identify the tangent space $T_{x_0}(X)$ with the orthogonal complement $\mathfrak{p}$ of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Under this identification, we can express the tangent vector $\widetilde{c}(0) = \mathbf{v}$ as follows:

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2; \ \mathbf{v}_1 \in T_{x_0}(x_0 \cdot M_\Theta), \ \mathbf{v}_2 \in T_{x_0}(x_0 \cdot N_\Theta),$$

$$\mathbf{v}_1 = (d\rho_e)^{-1}(diag((b_1^1)'(0), \dots, (b_n^1)'(0), \dots\dots (b_1^l)'(0), \dots, (b_n^l)'(0))),$$

$$\mathbf{v}_2 = (d\rho_e)^{-1}\left(\left(\begin{pmatrix} 0 & \frac{{}^t((\boldsymbol{\eta}^1)'(0))}{2} \\ \frac{(\boldsymbol{\eta}^1)'(0)}{2} & 0 \end{pmatrix} \right.\right. \\ \left.\left. \ddots \right.\right. \\ \left.\left. \begin{pmatrix} 0 & \frac{{}^t((\boldsymbol{\eta}^l)'(0))}{2} \\ \frac{(\boldsymbol{\eta}^l)'(0)}{2} & 0 \end{pmatrix} \right)\right).$$

We remark that two subspaces $T_{x_0}(x_0 \cdot M_\Theta)$ and $T_{x_0}(x_0 \cdot N_\Theta)$ are orthogonal to each other in $T_{x_0}(X)$. Let $\varphi : (-\delta, \delta) \longrightarrow x_0 \cdot N_\Theta$ be the curve given by

$$\varphi(t) = x_0 \cdot \left(\rho^{-1}\left(\left(\begin{pmatrix} I_{n-1} & {}^t(\xi^1(t)) \\ & 1 \end{pmatrix} \right.\right.\right. \\ \left.\left.\left. \ddots \right.\right.\right. \\ \left.\left.\left. \begin{pmatrix} I_{n-1} & {}^t(\xi^1(t)) \\ & 1 \end{pmatrix} \right)\right)\right)$$

for $t \in (-\delta, \delta)$.

Assertion. *Let $i = (n-1)k + j : 0 \leqq k \leqq l - 1, 1 \leqq j \leqq n - 1$.*

(1) *If $i \notin I \cup J$, then we have $b_j^k(t) - b_n^k(t) = -nm$ for all $t \in (-\delta, \delta)$.*

(2) *If $i \in I \cup J$, then $(\xi_j^k)'(0) = 0$.*

Proof of Assertion. Since (2) is an immediate consequence of the definition of ξ_j^k, we prove (1).

(case 1) $(n-1)k + \nu \notin I \cup J$ for all $\nu \in \{1, \dots, n-1\} - \{j\}$

We have $b_\mu^k(t) = -m$ for all $\mu \in \{1, \dots, n-1\}$ and all t. So $b_n^k \equiv (n-1)m$, and $b_j^k - b_n^k \equiv -nm$.

(case 2) $(n-1)k + \nu \in I \cup J$ for some $\nu \in \{1, \dots, n-1\} - \{j\}$

We can take the smooth function $\lambda : (-\delta, \delta) \longrightarrow [0, 1]$ such that

$$b_j^k(t) = \lambda(t)[-\{(n-1)l + 1\}m] + (1 - \lambda(t))(-m) = -\{\lambda(t)(n-1)l + 1\}m.$$

For each $\mu \in \{1, \dots, n-1\} - \{j\}$ with $(n-1)k + \mu \in I \cup J$, the (nk)-(nk) entry of $v_{(n-1)k+\mu,m} = \Phi(w_{(n-1)k+\mu,m})$ is equal to

$$-(n-2)[-\{(n-1)l+1\}m] - \{(n-1)^2l - 1\}m = -(nl - l - n + 1)m.$$

And for each $k' \neq k$ and $\mu' \in \{1, \dots, n-1\}$, the (nk)-(nk) entry of $v_{(n-1)k'+\mu',m}$ is equal to $(n-1)m$. So we have

$$\begin{aligned}
b_n^k(t) = \lambda(t)\left\{-(nl - l - n + 1)m\right\} + \\
+ (1 - \lambda(t))(n-1)m = -\{\lambda(t)(n-1)l + 1 - n\}m.
\end{aligned}$$

Hence, $b_j^k(t) - b_n^k(t) = -nm$ for all t. $\square$

By this assertion, $\|\mathbf{v}_2\| = e^{-nm}\|\dot{\varphi}(0)\|$. When we identify $N_\Theta = \mathbf{R}^{(n-1)l}$ with $x_0 \cdot N_\Theta$ in the natural way, the metric of $\mathbf{R}^{(n-1)l}$ induced from X is given by $n(dy_1^2 + \cdots + dy_{(n-1)l}^2)$. Therefore, the image of $z_{I,J,m}$ is isometric to $|v_{i_1,m} \cdots v_{i_s,m}| \times [-\sqrt{n}, \sqrt{n}]^{(n-1)l-s}$. $\square$

From the above lemma, the $\{(n-1)l - 1\}$-chain $z_{I,J,m}$ is lipschitz.

Let $z_m = \sum z_{I,J,m}$, where the summation is taken over all the pair (I, J) of subsets of $\{1, \dots, (n-1)l\}$ such that $I \cup J \neq \phi$ and that $I \cap J = \phi$. As is shown in [5, pp. 239-240], $\sum(\triangle_{I,J,m} \times \square_{I,J,m})$ is topologically

the $\{(n-1)l-1\}$-dimensional sphere $S^{(n-1)l-1}$. Hence, z_m is a lipschitz $\{(n-1)l-1\}$-cycle. The following is an immediate consequence of Lemma 3.1.

Corollary 3.2. *The $\{(n-1)l-1\}$-dimensional volume $mass_{(n-1)l-1}(z_m)$ and the diameter $diam(z_m)$ of z_m are bounded above by some polynomials in m.*

We again fix one positive integer m, and define an $(n-1)l$-chain u_m which bounds z_m as follows. For each pair (I,J) of subsets of $\{1,\ldots,(n-1)l\}$ such that $I \cup J \neq \phi$, $I \cap J = \phi$, we define a map $u'_{I,J,m} : \triangle_{I,J,m} \times \square_{I,J,m} \times [0,1] \longrightarrow x_0 \cdot H_\Theta N_\Theta$ by $u'_{I,J,m}(v,y,t) = x_0 \cdot (\exp tv)y$ for $v \in \triangle_{I,J,m}$, $y \in \square_{I,J,m}$, and $t \in [0,1]$. As before, we regard the map $u'_{I,J,m}$ as a lipschitz $(n-1)l$-chain, and put $u'_m = \sum_{I,J} u'_{I,J,m}$. We put

$$\square_m = \left\{ (y_1, \ldots, y_{(n-1)l}) \in N_\Theta = \mathbf{R}^{(n-1)l} \ \middle| \ \begin{array}{c} -e^{nm} \leqq y_k \leqq e^{nm} \\ \text{for all } k \end{array} \right\},$$

and define a map $u''_m : \square_m \longrightarrow x_0 \cdot H_\Theta N_\Theta$ by $u''_m(y) = x_0 \cdot y$ for $y \in \square_m$. Let $u_m = u'_m + u''_m$. Then u_m is a lipschitz $(n-1)l$-chain which bounds z_m (see Figure 2).

4. Closed $(n-1)l$-form Ω and its norm.

We identify $\mathbf{R}^{(n-1)l}$ with N_Θ as in §3. We also identify N_Θ with $x_0 \cdot N_\Theta$ in the natural way. Then the metric of $\mathbf{R}^{(n-1)l}$ induced from X is $n(dy_1^2 + \cdots + dy_{(n-1)l}^2)$, and the volume form of $\mathbf{R}^{(n-1)l}$ is $n^{(n-1)l/2}dy_1 \wedge \cdots \wedge dy_{(n-1)l}$. Let $\pi : X \longrightarrow \mathcal{L} = x_0 \cdot M_\Theta N_\Theta$ be the map given by $\pi(\gamma(t) \cdot g_1 g_2) = x_0 \cdot g_1 g_2$ for $t \in \mathbf{R}$, $g_1 \in M_\Theta$, and $g_2 \in N_\Theta$. Let Ω be the pullback of the $(n-1)l$-form $n^{(n-1)l/2}dy_1 \wedge \cdots \wedge dy_{(n-1)l}$ on $\mathbf{R}^{(n-1)l}$ under the map $\mathcal{P}_f \circ \pi : X \longrightarrow N_\Theta = \mathbf{R}^{(n-1)l}$ (see Figure 1). (Notice that $(\mathcal{P}_f \circ \pi)(\gamma(t) \cdot g_1 g_2) = g_2$ for $t \in \mathbf{R}$, $g_1 \in M_\Theta$, and $g_2 \in N_\Theta$.)

For $x \in X$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l} \in T_x(X)$, we denote by $\mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l})$ the $(n-1)l$-dimensional volume of parallelotope spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l}$.

Definition 4.1. We define the norm $\|\Omega\|_x$ of Ω at $x \in X$ to be

$$\|\Omega\|_x = \sup \left\{ \frac{|\Omega(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l})|}{\mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l})} \ \middle| \ \begin{array}{c} \mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l} \in T_x(X) \\ \mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l}) \neq 0 \end{array} \right\}.$$

Lemma 4.2. $\quad \|\Omega\|_x = \exp(\sqrt{\frac{(n-1)l}{2}}\, t) \quad$ for all $x \in h_\gamma^{-1}(-t)$.

Proof. Let $M_\Theta = \overline{K}\,\overline{M}\,\overline{A}\,\overline{N}A^\perp$ as in §3. As in Lemma 2.1, we put

$$
a_t = \exp(t\mathrm{V}) = \rho^{-1}\left(diag\left(e^{-\frac{t}{n\sqrt{2(n-1)l}}},\ldots,e^{-\frac{t}{n\sqrt{2(n-1)l}}},e^{\frac{t}{n}\sqrt{\frac{n-1}{2l}}},\ldots\right.\right.
$$
$$
\left.\left.\ldots,e^{-\frac{t}{n\sqrt{2(n-1)l}}},\ldots,e^{-\frac{t}{n\sqrt{2(n-1)l}}},e^{\frac{t}{n}\sqrt{\frac{n-1}{2l}}}\right)\right)
$$

for $t \in \mathbf{R}$. For $\mathbf{y} = (y_1,\ldots,y_{(n-1)l}) \in \mathbf{R}^{(n-1)l}$, we put

$$
n_{\mathbf{y}} = \rho^{-1}\left(\begin{pmatrix} \begin{pmatrix} I_{n-1} & {}^t(\boldsymbol{\xi}^1) \\ & 1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} I_{n-1} & {}^t(\boldsymbol{\xi}^l) \\ & 1 \end{pmatrix} \end{pmatrix}\right) \in N_\Theta,
$$

where $\boldsymbol{\xi}^k = (y_{(n-1)(k-1)+1}, y_{(n-1)(k-1)+2},\ldots,y_{(n-1)k})$ for each k. Notice that $x_0 \cdot M_\Theta$ is diffeomorphic to $\overline{M}\,\overline{A}\,\overline{N}A^\perp$. So we can take a diffeomorphism $\Psi : \mathbf{R}^{(n^2-n)l/2-1} \times \mathbf{R} \times \mathbf{R}^{(n-1)l} \longrightarrow X$ such that $\Psi(\mathbf{r},t,\mathbf{y}) = x_0 \cdot \psi(\mathbf{r})a_t n_{\mathbf{y}}$ for $\mathbf{r} = (r_1,\ldots,r_{(n^2-n)l/2-1}) \in \mathbf{R}^{(n^2-n)l/2-1}$, $t \in \mathbf{R}$, and $\mathbf{y} = (y_1,\ldots,y_{(n-1)l}) \in \mathbf{R}^{(n-1)l}$, where $\psi : \mathbf{R}^{(n^2-n)l/2-1} \longrightarrow \overline{M}\,\overline{A}\,\overline{N}A^\perp$ is a diffeomorphism with $\psi(0) = I_{nl}$. Suppose that $x = x_0 \cdot \psi(\mathbf{r})a_t n_{\mathbf{y}} = x_0 \cdot n_{\mathbf{y}'}\psi(\mathbf{r})a_t$, $\mathbf{y}' = (y_1',\cdots,y_{(n-1)l}')$, and

$$
\psi(\mathbf{r}) = diag(B_1,b_1,\ldots,B_l,b_l) \in \overline{M}\,\overline{A}\,\overline{N}A^\perp,
$$

where each B_k is an $(n-1) \times (n-1)$ matrices. Then the relation between $\mathbf{y}$ and $\mathbf{y}'$ is given by the following :

$$
{}^t\mathbf{y} = e^{\frac{t}{\sqrt{2(n-1)l}}}\begin{pmatrix} b_1 B_1^{-1} & & \\ & \ddots & \\ & & b_l B_l^{-1} \end{pmatrix}{}^t(\mathbf{y}').
$$

Let

$$
\widetilde{e}_i = \frac{d}{d\kappa}\left\{ x_0 \cdot n_{(y_1',\ldots,y_{i-1}',y_i'+\kappa,y_{i+1}',\ldots,y_{(n-1)l}')}\psi(\mathbf{r})a_t \right\}\bigg|_{\kappa=0}
$$

and $e_i = \widetilde{e}_i/\|\widetilde{e}_i\|$ for $i = 1,\ldots,(n-1)l$. Then $\{e_1,\ldots,e_{(n-1)l}\}$ is an orthonormal basis of the tangent space $T_x(x_0 \cdot \psi(\mathbf{r})a_t N_\Theta)$. We put

$$
\widehat{e}_i = \left(\frac{\partial}{\partial y_i}\right)_{\mathbf{y}}\bigg/\left\|\left(\frac{\partial}{\partial y_i}\right)_{\mathbf{y}}\right\| \qquad \text{for} \quad i = 1,\ldots,(n-1)l.
$$

Under the identification $\mathbf{R}^{(n-1)l} = x_0 \cdot N_\Theta$, $\{\widehat{e}_1, \ldots, \widehat{e}_{(n-1)l}\}$ is an orthonormal basis of the tangent space $T_{x_0 \cdot n_\mathbf{y}}(x_0 \cdot N_\Theta)$. The restriction of the differential $(\mathcal{P}_f \circ \pi)_{*x}$ of the map $\mathcal{P}_f \circ \pi : X \longrightarrow N_\Theta = \mathbf{R}^{(n-1)l}$ to the tangent space $T_x(x_0 \cdot M_\Theta A_\Theta n_\mathbf{y})$ is the zero map. And the matrix of the restriction $(\mathcal{P}_f \circ \pi)_* |_{T_x(x_0 \cdot \psi(\mathbf{r}) a_t N_\Theta)} : T_x(x_0 \cdot \psi(\mathbf{r}) a_t N_\Theta) \longrightarrow T_{x_0 \cdot n_\mathbf{y}}(x_0 \cdot N_\Theta)$ of the linear map $(\mathcal{P}_f \circ \pi)_*$ to $T_x(x_0 \cdot \psi(\mathbf{r}) a_t N_\Theta)$ with respect to the basis $\{e_1, \ldots, e_{(n-1)l}\}$ and $\{\widehat{e}_1, \ldots, \widehat{e}_{(n-1)l}\}$ is

$$Q = (q_{ij}) = e^{\frac{t}{\sqrt{2(n-1)l}}} \begin{pmatrix} b_1 B_1^{-1} & & \\ & \ddots & \\ & & b_l B_l^{-1} \end{pmatrix}.$$

We suppose that $\mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l}) \neq 0$. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_{(n-1)l}\}$ be an orthonormal basis of the subspace $\mathbf{R}\mathbf{v}_1 + \cdots + \mathbf{R}\mathbf{v}_{(n-1)l}$ of $T(X)_x$. Each $\mathbf{v}_k$ is expressed as $\mathbf{v}_k = \sum_{j=1}^{(n-1)l} \xi_{kj} \mathbf{w}_j$, where every ξ_{kj} is a real number, and then, $\mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l}) = |\det(\xi_{kj})|$. For each k, we can write $\mathbf{w}_k = \mathbf{w}'_k + \mathbf{w}''_k$, $\mathbf{w}''_k = \sum_{i=1}^{(n-1)l} \eta_{ki} e_i$, where every η_{ki} is a real number and $\mathbf{w}'_k \perp e_i$ for all i. (Remark that $\|\mathbf{w}''_k\| \leqq \|\mathbf{w}_k\| = 1$.) Noticing that $T_x(x_0 \cdot M_\Theta A_\Theta n_\mathbf{y}) \perp T_x(x_0 \cdot \psi(\mathbf{r}) a_t N_\Theta)$, we put $\mathbf{v}_k = \mathbf{v}'_k + \mathbf{v}''_k$: $\mathbf{v}'_k \in T_x(x_0 \cdot M_\Theta A_\Theta n_\mathbf{y})$, $\mathbf{v}''_k \in T_x(x_0 \cdot \psi(\mathbf{r}) a_t N_\Theta)$. Then

$$\mathbf{v}''_k = \sum_{j=1}^{(n-1)l} \xi_{kj} \mathbf{w}''_j = \sum_{j=1}^{(n-1)l} \xi_{kj} \left(\sum_{\nu=1}^{(n-1)l} \eta_{j\nu} e_\nu \right).$$

Let $(\zeta_{ij}) = (\xi_{ij})(\eta_{ij})$. Then $\mathbf{v}''_k = \sum_{\nu=1}^{(n-1)l} \zeta_{k\nu} e_\nu$. Hence

$$(\mathcal{P}_f \circ \pi)_*(\mathbf{v}_k) = (\mathcal{P}_f \circ \pi)_*(\mathbf{v}''_k) = \sum_{i=1}^{(n-1)l} \zeta_{ki} (\mathcal{P}_f \circ \pi)_*(e_i)$$

$$= \sum_{i=1}^{(n-1)l} \zeta_{ki} \left(\sum_{\nu=1}^{(n-1)l} q_{\nu i} \widehat{e}_\nu \right) = \sum_{\nu=1}^{(n-1)l} \left(\sum_{i=1}^{(n-1)l} \zeta_{ki} q_{\nu i} \right) \widehat{e}_\nu,$$

and

$$\Omega(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l}) = \det((\zeta_{ij})\,{}^t Q) = e^{\sqrt{(n-1)l/2}\,t} \cdot \det(\xi_{ij}) \cdot \det(\eta_{ij}).$$

We have

$$\frac{|\Omega(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l})|}{\mathcal{V}(\mathbf{v}_1, \ldots, \mathbf{v}_{(n-1)l})} = e^{\sqrt{(n-1)l/2}\,t} \cdot |\det(\eta_{ij})|$$

$$\leqq e^{\sqrt{(n-1)l/2}\,t} \cdot \|\mathbf{w}''_1\| \cdot \cdots \cdot \|\mathbf{w}''_{(n-1)l}\| \leqq e^{\sqrt{(n-1)l/2}\,t}$$

Since the equality holds when $\mathbf{v}'_k = 0$ for all k, we obtain $\|\Omega\|_x = e^{\sqrt{(n-1)l/2}\,t}$.

$\square$

5. Isoperimetric inequality on the complement of a horoball.

Let $\Pi : X \longrightarrow X/\Gamma'$ be the projection. If $n = 2$, $H_\Theta = M_\Theta$. And in this case, it is known that $\Pi(x_0 \cdot H_\Theta N_\Theta)$ is compact (see [15, §1]). In the case $n \geq 3$, $\Pi(x_0 \cdot H_\Theta N_\Theta)$ is not compact. In fact, we can show that the geodesic $\Pi \circ \gamma' : [0, \infty) \longrightarrow X/\Gamma'$ is a ray (i.e. $d_{X/\Gamma'}(\Pi \circ \gamma'(t), \Pi \circ \gamma'(t')) = |t - t'|$ for all $t, t' \geq 0$) for the following geodesic $\gamma' : [0, \infty) \longrightarrow X$ by the same argument in §2 of [14].

$$(5.1) \quad \begin{cases} \gamma'(t) = x_0 \cdot (\exp t\mathbf{W}) \text{ for } t \geq 0 \\ \mathbf{W} = (d\rho_e)^{-1}(\dfrac{1}{2\sqrt{nl}}diag(-1,\underbrace{0,\ldots,0}_{n-3},1,0,\ldots \\ \qquad\qquad\qquad \ldots,-1,\underbrace{0,\ldots,0}_{n-3},1,0)). \end{cases}$$

So, from now on to the paragraph including the formula (5.4), we suppose $n \geq 3$. And we "change" the direction of $x_0 \cdot H_\Theta N_\Theta$ in the horosphere $\mathcal{L} = x_0 \cdot M_\Theta N_\Theta$ so that it projects onto a compact subset of X/Γ'.

First, we recall that $H_\Theta = \overline{M}\,\overline{A}A^\perp$ (§3).

Let $\mathbf{k}'$ be a totally real algebraic number field of degree $n-1$ over $\mathbf{Q}$ such that $\mathbf{k} \cap \mathbf{k}' = \mathbf{Q}$. We can take such an algebraic number field $\mathbf{k}'$ as follows. (The author learned it from Dr. C. Nakayama.) Let p be a prime such that $p \equiv 1 \mod 2(n-1)$ and that p is unramified in $\mathbf{k}|\mathbf{Q}$. Let ζ_p be a primitive pth root of unity. Then the field $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$ is a Galois extension of $\mathbf{Q}$ and totally real. Since the Galois group of $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$ over $\mathbf{Q}$ is the cyclic group of order $(p-1)/2$, we can take a normal subgroup of index $n-1$. Take the corresponding intermediate field. Then this field is the desired field $\mathbf{k}'$.

Let $\{\alpha'_1, \ldots, \alpha'_{n-1}\}$ be an integral basis of $\mathbf{k}'$ and

$$\alpha'_i\alpha'_j = \sum_{s=1}^{n-1} b'^s_{ij}\alpha'_s; b'^s_{ij} \in \mathbf{Z}$$

for $i, j \in \{1, \ldots, n-1\}$. For each $k = 1, \ldots, n-1$, let $\mathbf{B}'_k = \left(b'^{\,j}_{ki} \right)$, where the i-j element of $\mathbf{B}'_k$ is $b'^{\,j}_{ki}$. For $Z = (z_{ij}) \in \mathfrak{gl}(n-1, \mathbf{C})$, we consider the following system of linear equations:
(5.2)
$$\begin{pmatrix} z_{i1} & z_{i2} & \ldots & z_{i,n-1} \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & \ldots & z_{1,n-1} \end{pmatrix} \mathbf{B}'^{-1}_1 \mathbf{B}'_i \quad ; i = 2, \ldots n-1.$$

Let $\mathbf{H}(\subset \mathfrak{gl}(n-1, \mathbf{C}))$ be the algebraic group defined by the system of equations (5.2) and the condition $\det Z = 1$. Let $\phi'_1 = id., \phi'_2, \ldots, \phi'_{n-1} :$ $\mathbf{k}' \longrightarrow \mathbf{C}$ be the distinct field embeddings, and $\mathbf{S}' = \left(\phi'_j(\alpha'_i) \right)$ be the $(n-1) \times (n-1)$ matrix. We define a representation $\rho' : \mathbf{H} \longrightarrow \mathfrak{gl}(n-1, \mathbf{C})$ by $\rho'(Z) = \mathbf{S}'^{-1} Z \mathbf{S}'$ for $Z \in \mathbf{H}$. We also define an embedding $\iota' : \mathbf{k}' \longrightarrow$ $\mathfrak{gl}(n-1, \mathbf{C})$ by $\iota'(\alpha') = diag(\phi'_1(\alpha'), \phi'_2(\alpha'), \ldots, \phi'_{n-1}(\alpha'))$ for $\alpha' \in \mathbf{k}'$. Then the group $\mathbf{H}$ is defined over $\mathbf{Q}$, and $H_{\mathbf{Z}}$ is isomorphic to the group U' of all units in the ring $\mathcal{O}_{\mathbf{k}'}$ of algebraic integers of $\mathbf{k}'$ under the map $\iota'^{-1} \circ \rho'$. (The group $\mathbf{H}_{\mathbf{Z}}$ is essentially the same as the transpose of the group K constructed in the proof of Proposition 10.4.10 of [5, p.237].) We remark that $\rho'(\mathbf{H}_{\mathbf{Z}} \cap (\mathbf{H}_{\mathbf{R}})^0)$ is a subgroup of finite index of $\iota'(U')$. Therefore, by the Dirichlet units theorem, $\mathbf{H}_{\mathbf{R}}/\mathbf{H}_{\mathbf{Z}}$ is compact. Let

$$\widetilde{\mathbf{H}} = \left\{ \begin{pmatrix} Z & \\ & 1 \end{pmatrix} \,\middle|\, Z \in \mathbf{H} \right\} \subset SL(n, \mathbf{C}).$$

Then $\widetilde{\mathbf{H}}$ is an algebraic $\mathbf{Q}$-subgroup of $SL(n, \mathbf{C})$. And the quotient $\widetilde{\mathbf{H}}_{\mathbf{R}}/\widetilde{\mathbf{H}}_{\mathbf{Z}}$ is compact. By Borel-Harish-Chandra's theorem ([4, Theorem 12.3], we have $X_{\mathbf{Q}}(\widetilde{\mathbf{H}}) = 1$, where $X_{\mathbf{Q}}(\widetilde{\mathbf{H}})$ is the group of $\mathbf{Q}$-characters of $\widetilde{\mathbf{H}}$. By the construction of $\widetilde{\mathbf{H}}$, we have $X_{\mathbf{k}'}(\widetilde{\mathbf{H}}) = X(\widetilde{\mathbf{H}}) \cong \mathbf{Z}^{n-2}$, where $X_{\mathbf{k}'}(\widetilde{\mathbf{H}})$ (resp. $X(\widetilde{\mathbf{H}})$) is the group of $\mathbf{k}'$-characters (resp. characters) of $\widetilde{\mathbf{H}}$. Let $\hat{\mathbf{H}} = \rho^{-1} \circ \iota(\widetilde{\mathbf{H}})$ be the group obtained from $\widetilde{\mathbf{H}}$ by the restriction of scalars from $\mathbf{k}$ to $\mathbf{Q}$ as in §1. Since $\mathbf{k} \cap \mathbf{k}' = \mathbf{Q}$, the group $X_{\mathbf{k}}(\widetilde{\mathbf{H}})$ of $\mathbf{k}$-characters of $\hat{\mathbf{H}}$ is trivial. Therefore, the quotient $\hat{\mathbf{H}}_{\mathbf{R}}/\hat{\mathbf{H}}_{\mathbf{Z}}$ is compact by the Borel-Harish-Chandra's theorem.

We put $\mathfrak{d} = (\det \mathbf{S}')^{n-1} \mathbf{S}'^{-1}$ if $\det \mathbf{S}' > 0$,

$$\mathfrak{d} = (-\det \mathbf{S}')^{n-1} diag(-1, \underbrace{1, \ldots, 1}_{n-1}) \mathbf{S}'^{-1}$$

if $\det \mathbf{S}' < 0$. Then, $\mathfrak{d} \in SL(n-1, \mathbf{R})$. Let $R = \rho^{-1}(diag(\mathfrak{d}, 1, \ldots, \mathfrak{d}, 1)) \in M_\Theta$. We have $\overline{M}\,\overline{A} = R(\hat{\mathbf{H}}_{\mathbf{R}})^0 R^{-1}$, $R^{-1}M_\Theta R = M_\Theta$, and $R^{-1}N_\Theta R = N_\Theta$. Each element of $A^\perp$ commutes with R. We put $\Lambda = \hat{\mathbf{H}}_{\mathbf{Z}} \subset \Gamma = \mathbf{G}_{\mathbf{Z}}$. Then the quotient $x_0 \cdot R^{-1}\overline{M}\,\overline{A}R/\Lambda$ is compact. Since $[\Gamma : \Gamma'] < \infty$, we

can assume that $\Lambda \subset \Gamma'$ by replacing it with its subgroup of finite index if necessary.

According to the Dirichlet units theorem, the group U of all units in $\mathcal{O}_{\mathbf{k}}$ is isomorphic to the direct product $W \times \mathbf{Z}^{l-1}$, where W is the finite cyclic group of even order consisting of all roots of unity in U. We take a set $\{v_1, \ldots, v_{l-1}\} \subset U$ of generators of $\mathbf{Z}^{l-1}$ under the above isomorphism. For each $i \in \{1, \ldots, l-1\}$, let $\sigma_{1,i}$ be the element of $\Gamma \cap A^{\perp}$ given by

$$(5.3) \qquad \begin{cases} \sigma_{1,i} = \rho^{-1}(diag(\mathfrak{m}_i^1, \ldots, \mathfrak{m}_i^l)), \\ \mathfrak{m}_i^k = diag(\phi_k(v_i)^{-1}, \ldots, \phi_k(v_i)^{-1}, \phi_k(v_i)^{n-1}) \,;\, k = 1, \ldots l, \end{cases}$$

where each $\mathfrak{m}_i^k$ is an $n \times n$ matrix such that its n-n entry is $\phi_1(v_i)^{n-1}$, and other diagonal elements are all equal to $\phi_1(v_i)^{-1}$.

Recall that $\{\alpha_1, \ldots, \alpha_l\}$ is an integral basis of $\mathbf{k}$.
For each $i \in \{1, \ldots, n-1\}$ and $j \in \{1, \ldots, l\}$, let $\sigma_{2,i,j}$ be the element of $\Gamma \cap N_{\Theta}$ given by

$$(5.4) \qquad \begin{cases} \sigma_{2,i,j} = \rho^{-1}\left(\begin{pmatrix} \begin{pmatrix} I_{n-1} & {}^t(\xi_{i,j}^1) \\ & 1 \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} I_{n-1} & {}^t(\xi_{i,j}^l) \\ & 1 \end{pmatrix} \end{pmatrix}\right), \\ \xi_{i,j}^k = (0, \ldots, 0, \phi_k(\alpha_j), 0 \ldots, 0) \,:\, k = 1, \ldots, l, \end{cases}$$

where the i-th entry of $\xi_{i,j}^k$ is $\phi_k(\alpha_j)$ and other entries are all equal to zero. Since $[\Gamma : \Gamma'] < \infty$, we can assume that $\sigma_{1,i}, \sigma_{2,i,j} \in \Gamma'$ for all i, j by replacing them with their powers if necessary. We denote by Λ' the subgroup of Γ' generated by

$$\Lambda \cup \{\sigma_{1,i} \mid i = 1, \ldots, l-1\} \cup \left\{\sigma_{2,i,j} \,\middle|\, \begin{matrix} i = 1, \ldots, n-1 \\ j = 1, \ldots, l \end{matrix}\right\}.$$

Then $R^{-1}H_{\Theta}N_{\Theta}R/\Lambda'$ is compact, and hence $\Pi(x_0 \cdot R^{-1}H_{\Theta}N_{\Theta}R)$ is compact.

In order to simplify the succeeding description, we also put $R = I_{2l}$ in the case $n = 2$.

We define a diffeomorphism $\Xi : x_0 \cdot H_{\Theta}N_{\Theta} \longrightarrow x_0 \cdot R^{-1}H_{\Theta}N_{\Theta}R$ by $\Xi(x_0 \cdot g_1 g_2) = x_0 \cdot R^{-1}g_1 g_2 R$ for $g_1 \in H_{\Theta}$, $g_2 \in N_{\Theta}$. We remark that both $H_{\Theta}N_{\Theta}$ and $R^{-1}H_{\Theta}N_{\Theta}R$ have group structures. The map Ξ is a bilipschitz diffeomorphism between the Riemannian manifolds $x_0 \cdot H_{\Theta}N_{\Theta}$ and $x_0 \cdot R^{-1}H_{\Theta}N_{\Theta}R$.

Now we define a new $\{(n-1)l-1\}$-cycle $\mathcal{Z}_m$ (resp. $(n-1)l$-chain $\mathcal{U}_m$) by $\mathcal{Z}_m = \Xi \circ z_m$ (resp. $\mathcal{U}_m = \Xi \circ u_m$) for each positive integer m.

Lemma 5.5. *The pullback of the restriction of Ω to $x_0 \cdot R^{-1}H_\Theta N_\Theta R$ under the map Ξ is identical with the restriction of Ω to $x_0 \cdot H_\Theta N_\Theta$: i.e.* $\Xi^*(\Omega|_{x_0 \cdot R^{-1}H_\Theta N_\Theta R}) = \Omega|_{x_0 \cdot H_\Theta N_\Theta}$.

Proof. Let $n_\mathbf{y}$ for $\mathbf{y} = (y_1, \ldots, y_{(n-1)l}) \in \mathbf{R}^{(n-1)l}$ as in the proof of Lemma 4.2.

We define a linear map $\widehat{R} : \mathbf{R}^{(n-1)l} \longrightarrow \mathbf{R}^{(n-1)l}$ by

$$
{}^t(\widehat{R}\mathbf{y}) = \begin{pmatrix} \mathfrak{d}^{-1} & & \\ & \ddots & \\ & & \mathfrak{d}^{-1} \end{pmatrix} {}^t\mathbf{y} \qquad \text{for } \mathbf{y} \in \mathbf{R}^{(n-1)l}.
$$

Suppose that

$$
(5.5.1) \qquad \begin{cases} g = g_1 g_2 \in H_\Theta N_\Theta, \\ g_1 = \rho^{-1}(diag(B_1, b_1, \ldots, B_l, b_l)) \in H_\Theta, \\ g_2 = n_\mathbf{y} \in N_\Theta, \end{cases}
$$

where each B_k is an $(n-1) \times (n-1)$ diagonal matrix. Then we have

$$
R^{-1}gR = \rho^{-1}(diag(\mathfrak{d}^{-1}B_1\mathfrak{d}, b_1, \ldots, \mathfrak{d}^{-1}B_l\mathfrak{d}, b_l))n_{\widehat{R}\mathbf{y}}
$$

and $\mathcal{P}_f(x_0 \cdot R^{-1}gR) = \widehat{R}\mathbf{y}$. So, $\mathcal{P}_f \circ \Xi = \widehat{R} \circ \mathcal{P}_f$.

Since $\det \begin{pmatrix} \mathfrak{d}^{-1} & & \\ & \ddots & \\ & & \mathfrak{d}^{-1} \end{pmatrix} = 1$, we obtain

$$
\Xi^*(\Omega|_{x_0 \cdot R^{-1}H_\Theta N_\Theta R}) = \Xi^* \circ \mathcal{P}_f^*(\Omega|_{x_0 \cdot N_\Theta}) = \mathcal{P}_f^* \circ \widehat{R}^*(\Omega|_{x_0 \cdot N_\Theta})
$$
$$
= \mathcal{P}_f^*(\Omega|_{x_0 \cdot N_\Theta}) = \Omega|_{x_0 \cdot H_\Theta N_\Theta}.
$$

$\square$

Definition 5.6. We define the injectivity radius $Injrad_x$ at $x \in X$ to be the injectivity radius at $\Pi(x)$ on the Riemannian manifold X/Γ' : i.e.

$$
Injrad_x = \frac{1}{2} \min \left\{ d_X(x, x \cdot \sigma) \mid \sigma \in \Gamma', \sigma \neq id. \right\}.
$$

Lemma 5.7. *There exists a positive number C such that the following holds for each $t \in \mathbf{R}$:*

$$Injrad_x \leqq Ce^{\frac{-t}{\sqrt{2(n-1)l}}}$$

for any point x on the horosphere $h_\gamma^{-1}(-t)$.

Proof. Let $\overline{\mathbf{G}} = \rho^{-1}((SL(n-1,\mathbf{C}))^l)$, $\overline{\mathbf{P}}$ be as in §3. We put $\overline{\Gamma'} = \Gamma' \cap \overline{\mathbf{G}}$. Since $\overline{\Gamma'}$ is an arithmetic subgroup of $\overline{\mathbf{G}}$, the double coset classes $\overline{\mathbf{P}}_{\mathbf{Q}}\backslash\overline{\mathbf{G}}_{\mathbf{Q}}/\overline{\Gamma'}$ are known to be finite by Borel [3]. Let $\overline{\sigma}_1,\ldots,\overline{\sigma}_\nu$ be a complete representative system of $\overline{\mathbf{P}}_{\mathbf{Q}}\backslash\overline{\mathbf{G}}_{\mathbf{Q}}/\overline{\Gamma'}$. We assume the following.

(5.7.1)
$$\begin{cases} \rho(\overline{\sigma}_i) = diag(\phi_1(\mathfrak{c}_i),1,\ldots,\phi_l(\mathfrak{c}_i),1), \\ \mathfrak{c}_i \in SL(n-1,\mathbf{k}) \quad \text{for } i = 1,\ldots,\nu. \end{cases}$$

We remark that the horosphere

$$h_\gamma^{-1}(-t) = x_0 \cdot M_\Theta a_t N_\Theta = x_0 \cdot \overline{M}\,\overline{A}\,\overline{N}A^\perp a_t N_\Theta$$

is invariant under the right translation by each element of M_Θ.

By virtue of the Borel's result ([3]) on the fundamental open set for $\overline{\Gamma'}$ combined with the use of the action of the group generated by

$$\{\sigma_{1,i} \mid i = 1,\ldots,l-1\},$$

we can assume the following.

(5.7.2) There exist positive constants C_1, C_2, C_3, C_4 such that the following holds. For any $x \in h_\gamma^{-1}(-t)$, there exist $\mu \in \{1,\ldots,\nu\}$ and $x' \in h_\gamma^{-1}(-t)$ of the following form such that $\Pi(x) = \Pi(x')$:

(5.7.2.a) $x' = x_0 \cdot g_2 g_3 \overline{\sigma}_\mu g_4 a_t n_{\mathbf{y}},$

(5.7.2.b)
$$g_2 = \rho^{-1}(diag(\beta_1^1,\ldots,\beta_{n-1}^1,1,\ldots\ldots,\beta_1^l,\ldots,\beta_{n-1}^l,1))$$
$$\times \rho^{-1}(diag(\kappa_1,\ldots,\kappa_{n-1},1,\ldots\ldots,\kappa_1,\ldots,\kappa_{n-1},1)) \in \overline{M}\,\overline{A};$$
$$\beta_i^k > 0 \text{ for all } i,k\,;\, \prod_{i=1}^{n-1}\beta_i^k = 1 \text{ for all } k\,;\, \prod_{k=1}^{l}\beta_i^k = 1 \text{ for all } i\,;$$
$$\kappa_i > 0 \text{ for all } i\,;\, \kappa_1 \cdots \kappa_{n-1} = 1,$$

(5.7.2.c)
$$g_3 = \rho^{-1}(diag(\mathfrak{n}_1, 1, \ldots, \mathfrak{n}_l, 1)) \in \overline{N},$$

where each $\mathfrak{n}_k = \begin{pmatrix} 1 & & b_{i,j}^k \\ & \ddots & \\ & & 1 \end{pmatrix}$ is an upper triangular, real $(n-1)$

$\times (n-1)$ matrix such that its diagonal entries are all equal to 1,

(5.7.2.d)
$$g_4 = \rho^{-1}(diag(q_1, \ldots, q_1, (q_1)^{1-n}, \ldots\ldots, q_l, \ldots, q_l, (q_l)^{1-n})) \in A^{\perp},$$

(5.7.2.e)　　　　　　　　　$\dfrac{\kappa_i}{\kappa_{i+1}} \leqq C_1$　　for all i,

(5.7.2.f)　　　　　　　　$\dfrac{1}{C_2} \leqq \beta_i^k \leqq C_2$　　for all i, k,

(5.7.2.g)　　　　　　　　$|b_{i,j}^k| \leqq C_3$　　for all i, j, k,

(5.7.2.h)　　　　　　　　$\dfrac{1}{C_4} \leqq q_k \leqq C_4$　　for all k.

For each i, we can find an element $\mathfrak{v}_i$ of $\mathbf{k}^{n-1}$ such that $\mathfrak{c}_i({}^t\mathfrak{v}_i) = {}^t(1, 0, \ldots, 0)$, because $\mathfrak{c}_i \in SL(n-1, \mathbf{k})$. By replacing $\mathfrak{v}_1, \ldots, \mathfrak{v}_\nu$ with their constant multiples if necessary, we can assume the following.

(5.7.3)　　　There exist positive integer C_5 and elements $\mathfrak{v}_1, \ldots, \mathfrak{v}_\nu$ of $(\mathcal{O}_\mathbf{k})^{n-1}$ such that $\mathfrak{c}_i({}^t\mathfrak{v}_i) = {}^t(C_5, 0, \ldots, 0)$ for $i = 1, \ldots, \nu$.

Let $\mathbf{y}_j = (\phi_1(\mathfrak{v}_j), \ldots, \phi_l(\mathfrak{v}_j))$; $j = 1, \ldots, \nu$, where each $\phi_i : \mathbf{k}^{n-1} \longrightarrow \mathbf{C}^{n-1}$ is the natural linear map induced from ϕ_i. By replacing $n_{\mathbf{y}_1}, \ldots, n_{\mathbf{y}_\nu}$ with their powers if necessary, we can assume that

(5.7.4)　　　　　　　　　　　$n_{\mathbf{y}_1}, \ldots, n_{\mathbf{y}_\nu} \in \Gamma'$.

Let $g_2 g_3 = \rho^{-1}(diag(B_1, 1, \ldots, B_l, 1))$ and $x' = x_0 \cdot n_{\mathbf{y}'} g_2 g_3 \overline{\sigma}_\mu g_4 a_t$. Then we have

$${}^t\mathbf{y}' = e^{-\frac{t}{\sqrt{2(n-1)l}}} \begin{pmatrix} (q_1)^n B_1 & & \\ & \ddots & \\ & & (q_l)^n B_l \end{pmatrix} \begin{pmatrix} \phi_1(\mathfrak{c}_\mu) & & \\ & \ddots & \\ & & \phi_l(\mathfrak{c}_\mu) \end{pmatrix} {}^t\mathbf{y},$$

where each $\phi_i : SL(n-1, \mathbf{k}) \longrightarrow SL(n-1, \mathbf{R})$ is the natural homomorphism induced from ϕ_i. So, if $x' \cdot n_{\mathbf{y}_\mu} = x_0 \cdot n_{\mathbf{y}''} g_2 g_3 \overline{\sigma}_\mu g_4 a_t$, then we have

$$
{}^t(\mathbf{y}'' - \mathbf{y}')
$$

$$
= e^{-\frac{t}{\sqrt{2(n-1)l}}}
\begin{pmatrix} (q_1)^n B_1 & & \\ & \ddots & \\ & & (q_l)^n B_l \end{pmatrix}
\begin{pmatrix} \phi_1(\mathfrak{c}_\mu) & & \\ & \ddots & \\ & & \phi_l(\mathfrak{c}_\mu) \end{pmatrix} {}^t \mathbf{y}_\mu
$$

$$
= e^{-\frac{t}{\sqrt{2(n-1)l}}} diag((q_1)^n B_1, \ldots, (q_l)^n B_l)
$$
$$
\times {}^t(C_5, 0, \ldots, 0, \ldots \ldots, C_5, 0, \ldots, 0)
$$

$$
= C_5 e^{-\frac{t}{\sqrt{2(n-1)l}}} {}^t((q_1)^n \kappa_1 \beta_1^1, 0, \ldots, 0, \ldots \ldots, (q_l)^n \kappa_1 \beta_1^l, 0, \ldots, 0).
$$

Hence,

$$
Injrad_x = Injrad_{x'}
$$
$$
\leqq \frac{1}{2} d_X(x', x' \cdot n_{\mathbf{y}_\mu}) = \frac{1}{2} d_X(x_0 \cdot n_{\mathbf{y}'}, x_0 \cdot n_{\mathbf{y}''})
$$
$$
\leqq \frac{1}{2} C_5 e^{-\frac{t}{\sqrt{2(n-1)l}}} \kappa_1 \sqrt{n \times \{(q_1)^{2n}(\beta_1^1)^2 + \cdots + (q_l)^{2n}(\beta_1^l)^2\}}.
$$

From (5.7.2.e) and that $\kappa_1 \cdots \kappa_{n-1} = 1$, we have $\kappa_1 \leqq (C_1)^{n/2-1}$. Thus (5.7.2.f) and (5.7.2.h) imply that

$$
Injrad_x \leqq \frac{1}{2} C_5 e^{-\frac{t}{\sqrt{2(n-1)l}}} (C_1)^{n/2-1} \sqrt{n \times \{(C_4)^{2n}(C_2)^2 \times l\}}
$$
$$
= \frac{1}{2} (C_1)^{n/2-1} C_2 (C_4)^n C_5 \sqrt{nl} e^{-\frac{t}{\sqrt{2(n-1)l}}}.
$$

Therefore, it suffices to put $C = \frac{1}{2}(C_1)^{n/2-1} C_2 (C_4)^n C_5 \sqrt{nl}$. $\square$

Concerning the function $f : X/\Gamma' \longrightarrow \mathbf{R}$ in §1, we have

$$
\Pi(x_0 \cdot R^{-1} H_\Theta N_\Theta R) \subset f^{-1}([t_0, t_1])
$$

for a sufficiently large $t_1 > 0$, because $\Pi(x_0 \cdot R^{-1} H_\Theta N_\Theta R)$ is compact. Let $X' = \widetilde{f}^{-1}([t_0, t_1])$ and $\varepsilon_1 > 0$ be the minimum of the injectivity radii on X'. From Lemma 5.7, we have $X' \cap h_\gamma^{-1}((-\infty, -t_2)) = \phi$ for a sufficiently large $t_2 > 0$. Hence it suffices to consider the isoperimetric inequalities on the complement of the horoball $h_\gamma^{-1}((-\infty, -t_2))$.

Lemma 5.8. *For any lipschitz $(n-1)l$-chain $\mathcal{U}'_m$ in $X - h_\gamma^{-1}((-\infty, -t_2))$ which bounds $\mathcal{Z}_m$, we have*

$$
mass_{(n-1)l}(\mathcal{U}'_m) \geqq 2^{(n-1)l} n^{(n-1)l/2} e^{-\sqrt{(n-1)l/2}\, t_2} \cdot e^{n(n-1)l \cdot m}.
$$

Proof. Since $H^{(n-1)l}(X - h_\gamma^{-1}((-\infty, -t_2))) = 0$, we can take an $\{(n-1)l - 1\}$-form ω such that $\Omega = d\omega$. Lemma 5.5 implies that

$$\int_{\mathcal{U}'_m} \Omega = \int_{\mathcal{Z}_m} \omega = \int_{\mathcal{U}_m} \Omega = \int_{\Xi \circ u_m} \Omega = \int_{u_m} \Xi^* \Omega = \int_{u_m} \Omega$$
$$= n^{(n-1)l/2}(2e^{nm})^{(n-1)l} = 2^{(n-1)l} n^{(n-1)l/2} e^{n(n-1)l \cdot m}.$$

From Lemma 4.2, we have

$$\|\Omega\|_x \lessgtr e^{\sqrt{(n-1)l/2}\, t_2} \qquad \text{for all } x \in X - h_\gamma^{-1}((-\infty, -t_2)).$$

Hence,

$$mass_{(n-1)l}(\mathcal{U}'_m) \gtreqqless e^{-\sqrt{(n-1)l/2}\, t_2} \int_{\mathcal{U}'_m} \Omega$$
$$= 2^{(n-1)l} n^{(n-1)l/2} e^{-\sqrt{(n-1)l/2}\, t_2} \cdot e^{n(n-1)l \cdot m}.$$

$\square$

We are now ready to prove the theorem.

Proof of Theorem. We know by Corollary 3.2 that $mass_{(n-1)l}(\mathcal{Z}_m)$ and $diam(\mathcal{Z}_m)$ are bounded above by some polynomials in m, because Ξ is a bilipschitz diffeomorphism. This and Lemma 5.8 imply that the isoperimetric inequality does not hold on X'. Therefore, $\Gamma = SL(n, \mathcal{O}_\mathbf{k})$ is not combable. $\qquad$ Q.E.D.

Remark. The isoperimetric inequality for $\mathcal{Z}_m$'s does hold on the whole space X: Move $\mathcal{Z}_m$ along the geodesics orthogonal to the horosphere $\mathcal{L}$ and fill the resulting cycle $\mathcal{Z}'_m$. The locus of the cycle $\mathcal{Z}_m$ and the chain filling $\mathcal{Z}'_m$ give us the desired chain. In fact, the fundamental groups of compact manifolds of nonpositive curvature are known to be combable (see [8]). But in the above process, we must move $\mathcal{Z}_m$ a distance about a constant multiple of m along the orthogonal direction. If m is large, the chain $\mathcal{Z}_m$ crosses the horosphere $h_\gamma^{-1}(-t_2)$ on the way to $\mathcal{Z}'_m$. Therefore we cannot make this construction when we remove the horoball $h_\gamma^{-1}((-\infty, -t_2))$.

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