

# Flips of moduli spaces and transition formulas for Donaldson polynomial invariants of rational surfaces

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## 1. Introduction.

In [7], Donaldson has defined polynomial invariants for smooth simply connected 4-manifolds with  $b_2^+ \geq 3$ . These invariants have also been defined for 4-manifolds with  $b_2^+ = 1$  in [25, 17, 18], along lines suggested by the work of Donaldson in [5]. In this case, however, they depend on an additional piece of information, namely a chamber defined on the positive cone of  $H^2(X; \mathbb{R})$  by a certain locally finite set of walls. Explicitly, let  $X$  be a simply connected, oriented, and closed smooth 4-manifold with  $b_2^+ = 1$  where  $b_2^+$  is the number of positive eigenvalues of the quadratic form  $q_X$  when diagonalized over  $\mathbb{R}$ . Let

$$\Omega_X = \{x \in H^2(X, \mathbb{R}) \mid x^2 > 0\}$$

be the positive cone. Fix a class  $\Delta$  in  $H^2(X, \mathbb{Z})$  and an integer  $c$  such that  $d = 4c - \Delta^2 - 3$  is nonnegative. A *wall of type*  $(\Delta, c)$  is a nonempty hyperplane:

$$W^\zeta = \{x \in \Omega_X \mid x \cdot \zeta = 0\}$$

in  $\Omega_X$  for some class  $\zeta \in H^2(X, \mathbb{Z})$  with  $\zeta \equiv \Delta \pmod{2}$  and  $\Delta^2 - 4c \leq \zeta^2 < 0$ . The connected components of the complement in  $\Omega_X$  of the walls of type  $(\Delta, c)$  are the *chambers of type*  $(\Delta, c)$ . Then the Donaldson polynomial invariants of  $X$  associated to  $\Delta$  and  $c$  are defined with respect to chambers of type  $(\Delta, c)$ . The invariants only depend on the class  $w = \Delta \pmod{2} \in H^2(X; \mathbb{Z}/2\mathbb{Z})$  and the integer  $p = \Delta^2 - 4c = -d - 3$ , and we shall often refer to walls and chambers of type  $(w, p)$  as well. We shall write  $D_{w,p}^X(\mathcal{C})$  for the Donaldson polynomial corresponding to the  $SO(3)$  bundle  $P$  with invariants  $w_2(P) = w$  and  $p_1(P) = p$ , depending on the chamber  $\mathcal{C}$ .

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A basic question is then the following: Suppose that  $\mathcal{C}_+$  and  $\mathcal{C}_-$  are separated by a single wall  $W^\zeta$ . Here there may be more than one class  $\zeta$  of type  $(\Delta, c)$  defining  $W^\zeta$ . Then find a formula for the difference

$$\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-) = D_{w,p}^X(\mathcal{C}_+) - D_{w,p}^X(\mathcal{C}_-).$$

We shall refer to such a difference as a *transition formula*.

There has been considerable interest in the above problem. The first result in this direction is due to Donaldson in [5], who gave a formula in case  $\Delta = 0$  and  $c = 1$ . Kotschick [17] showed that, on the part of the symmetric algebra generated by 2-dimensional classes,  $\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-) = \pm\zeta^d$  for  $\zeta^2 = -(4c - \Delta^2) = p$ , and that  $\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-)$  is in fact always divisible by  $\zeta$ , except when  $p = -5$  and  $\zeta^2 = -1$  (cf. also Mong [25] for some partial results along these lines). For a rational ruled surface  $X$ , all the transition formulas for  $\Delta = 0$  and  $2 \leq c \leq 4$  have been determined in [25, 35, 22]. Using a gauge-theoretic approach, Yang [37] settled the problem for  $\Delta = 0$  and  $c = 2$ , and computed the degree 5 Donaldson polynomials for rational surfaces. The known examples and the work of Kotschick and Morgan [18] raise the following rather natural conjecture:

**Conjecture.** *The transition formula  $\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-)$  is a homotopy invariant of the pair  $(X, \zeta)$ ; more precisely, if  $\phi$  is an oriented homotopy equivalence from  $X'$  to  $X$ , then*

$$\delta_{\phi^*w,p}^{X'}(\phi^*(\mathcal{C}_+), \phi^*(\mathcal{C}_-)) = \phi^* \delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-).$$

We remark that this conjecture is essentially equivalent to the following statement: the transition formula  $\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-)$  is a polynomial in  $\zeta$  and the quadratic form  $q_X$  with coefficients involving only  $\zeta^2$ , homotopy invariants of  $X$  (i.e.  $b_2^-(X)$ ), and universal constants.

Our goal in this paper is to study the corresponding problem in algebraic geometry. More precisely, let  $X$  be an algebraic surface (not necessarily with  $b_2^+(X) = 1$ ) and let  $L$  be an ample line bundle on  $X$ . We can then identify the moduli space of  $L$ -stable rank two bundles  $V$  on  $X$  with  $c_1(V) = \Delta$  and  $c_2(V) = c$  with the moduli space of equivalence classes of ASD connections on  $X$  with respect to a Hodge metric on  $X$  corresponding to  $L$ . Let  $\mathfrak{M}_L(\Delta, c)$  be the Gieseker compactification of this moduli space. It is known that  $\mathfrak{M}_L(\Delta, c)$  changes as we change  $L$ , and that  $\mathfrak{M}_L(\Delta, c)$  is constant on a set of chambers for the ample cone of  $X$  which are defined in a way analogous to the definition of chambers for  $\Omega_X$  given above. Using the recent result of Morgan [26] and Li [21] that the Donaldson polynomial of an algebraic

surface can be evaluated using the Gieseker compactification  $\mathfrak{M}_L(\Delta, c)$  of the moduli space of stable bundles, we shall work on  $\mathfrak{M}_L(\Delta, c)$  for suitable choices of  $L$  and in particular analyze the change in  $\mathfrak{M}_L(\Delta, c)$  for  $L \in \mathcal{C}_+$  or  $L \in \mathcal{C}_-$ , where  $\mathcal{C}_\pm$  are two adjacent chambers. It turns out that we can obtain  $\mathfrak{M}_{L_+}(\Delta, c)$  from  $\mathfrak{M}_{L_-}(\Delta, c)$  by a series of blowups and blowdowns (flips). Our results are thus very similar to those of Thaddeus in [33]. Thaddeus [34] and also Dolgachev-Hu [3] have developed a general picture for the variation of GIT quotients after a change of polarization, and although our methods are somewhat different it seems quite possible that they fit into their general framework. We have also found it convenient to borrow some of Thaddeus' notation.

Next we shall apply our results on the change in the moduli spaces to determine the transition formula for Donaldson polynomials in case  $X$  is a rational surface with  $-K_X$  effective. We shall give explicit formulas for  $\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-)$  in case the nonnegative integer  $\ell_\zeta = (\zeta^2 - p)/4 \leq 2$ . These formulas are in agreement with the above conjecture, in the sense that the transition formula is indeed a polynomial in  $\zeta$  and  $q_X$  with coefficients involving only universal constants,  $\zeta^2$ , and  $K_X^2 = 9 - b_2^-(X) = 12 - \chi(X)$  where  $\chi(X)$  is the topological Euler-Poincaré characteristic of  $X$ . We shall also give a formula in principle for  $\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-)$  in general (see Theorem 5.4), but to make this formula explicit involves more knowledge of the enumerative geometry of  $\text{Hilb}^n X$  than seems to be available at present. In case  $-K_X$  is effective, the moduli spaces are (essentially) smooth and the centers of the blowup are smooth as well; in fact they are  $\mathbb{P}^k$ -bundles over  $\text{Hilb}^{n_1} X \times \text{Hilb}^{n_2} X$  for appropriate  $k, n_1$  and  $n_2$ . In this way, we obtain general formulas which can be made explicit for low values of  $n$ . For instance, adopting Kotschick's notation [17]

$$\varepsilon(\zeta, \Delta) = (-1)^{\frac{(\zeta-\Delta) \cdot K_X}{2}} = (-1)^{\left(\frac{\zeta-\Delta}{2}\right)^2},$$

we show the following (see Theorem 6.4 for details):

**Theorem.** *Assume that the wall  $W^\zeta$  is defined only by  $\pm\zeta$  with  $\ell_\zeta = 1$  and that  $\mathcal{C}_\pm$  lies on the  $\pm$ -side of  $W^\zeta$ . Then, on the subspace of the symmetric algebra generated by  $H_2(X)$ ,  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)$  is equal to*

$$(-1)^{d+1} \cdot \varepsilon(\zeta, \Delta) \cdot \left\{ d(d-1) \cdot \left(\frac{\zeta}{2}\right)^{d-2} \cdot q_X + (2K_X^2 + 2d + 6) \cdot \left(\frac{\zeta}{2}\right)^d \right\}.$$

Along the direction of the work of Kronheimer and Mrowka [19, 20], we also consider the difference of Donaldson polynomial invariants involving the

natural generator  $x \in H_0(X; \mathbb{Z})$ . More precisely, let  $\nu$  be the corresponding 4-dimensional class in the instanton moduli space. For  $\alpha \in H_2(X; \mathbb{Z})$ , we give a formula for the difference  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)(\alpha^{d-2}, \nu)$  in Theorem 5.5. It is worthwhile to point out that the similarity between Theorem 5.4 and Theorem 5.5 may indicate that there exists a deep relation between  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)(\alpha^d)$  and  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)(\alpha^{d-2}, \nu)$ , and suggest a way to generalize the notion of *simple type* in [19, 20] from the case of  $b_2^+ > 1$  to the case of  $b_2^+ = 1$ . For instance, modulo some lower degree terms,  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)(\alpha^{d-2}, \nu)$  can be obtained from  $(-1/4) \cdot \delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)(\alpha^d)$  by replacing  $d$  by  $(d-2)$  (see Theorem 5.13 and Theorem 5.14). In fact, based on some heuristic arguments, it seems reasonable to conjecture that  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)(\alpha^{d-2}, \nu)$  is a combination of  $\delta_{w,p'}^X(\mathcal{C}_-, \mathcal{C}_+)(\alpha^{d-4k})$  for various nonnegative integers  $k$  if the degrees are properly arranged.

Our paper is organized as follows. In section 2, we study rank two torsion free sheaves which are semistable with respect to ample divisors in  $\mathcal{C}_-$  but not semistable with respect to ample divisors in  $\mathcal{C}_+$ . When the surface  $X$  is rational with  $-K_X$  effective, these sheaves are parametrized by an open subset of a union of projective bundles over the product of two Hilbert schemes of points in  $X$ . More precisely, if  $\zeta$  defines the wall separating  $\mathcal{C}_-$  from  $\mathcal{C}_+$ , define  $E_\zeta^{n_1, n_2}$  be the set of all isomorphism classes of nonsplit extensions of the form

$$0 \rightarrow \mathcal{O}_X(F) \otimes I_{Z_1} \rightarrow V \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \rightarrow 0,$$

where  $F$  is a divisor class such that  $2F - \Delta \equiv \zeta$  and  $Z_1$  and  $Z_2$  are two zero-dimensional subschemes of  $X$  with  $\ell(Z_i) = n_i$  such that  $n_1 + n_2 = \ell_\zeta$ . In case  $X$  is rational,  $E_\zeta^{n_1, n_2}$  is a  $\mathbb{P}^N$  bundle over  $\text{Hilb}^{n_1} X \times \text{Hilb}^{n_2} X$ , and the set of points of  $E_\zeta^{n_1, n_2}$  lying in  $\mathfrak{M}_{L_-}(\Delta, c)$  but not in  $\mathfrak{M}_{L_+}(\Delta, c)$  is a Zariski open subset of  $E_\zeta^{n_1, n_2}$ . The main technical difficulty is that it is hard to control the rational map from  $E_\zeta^{n_1, n_2}$  to  $\mathfrak{M}_{L_-}(\Delta, c)$ , and in particular this map is not a morphism. The general picture that we establish is the following: first, the map  $E_\zeta^{0, \ell_\zeta} \dashrightarrow \mathfrak{M}_{L_-}(\Delta, c)$  is a morphism, and it is possible to make an elementary transformation, or *flip*, along its image. The result is a new space for which the rational map  $E_\zeta^{1, \ell_\zeta - 1} \dashrightarrow \mathfrak{M}_{L_-}(\Delta, c)$  becomes a morphism, and it is possible to make a flip along *its* image. We continue in this way until we reach  $\mathfrak{M}_{L_+}(\Delta, c)$ .

It seems rather difficult to see that the above picture holds directly. Instead we shall proceed as follows. We define abstractly a sequence of moduli spaces, indexed by an integer  $k$  with  $0 \leq k \leq \ell_\zeta + 1$ , such that the moduli space for  $k = 0$  is  $\mathfrak{M}_{L_-}(\Delta, c)$ , the moduli space for  $k = \ell_\zeta + 1$  is

$\mathfrak{M}_{L_+}(\Delta, c)$ , and moreover the  $k^{\text{th}}$  moduli space contains an embedded copy of  $E_{\zeta}^{k, \ell_{\zeta} - k}$  such that the flip along this copy yields the  $(k + 1)^{\text{st}}$  moduli space. Thus the picture is very similar to that developed independently by Thaddeus in [33]. To define our sequence of moduli spaces, we define  $(L_0, \zeta, \mathbf{k})$ -semistability in section 3 for rank two torsion free sheaves, where  $L_0$  is any ample divisor contained in the common face of  $\mathcal{C}_+$  and  $\mathcal{C}_-$ ,  $\zeta$  is the set of classes of type  $(\Delta, c)$  defining the common wall of  $\mathcal{C}_+$  and  $\mathcal{C}_-$ , and  $\mathbf{k}$  is a set of integers. We show that  $\mathfrak{M}_{L_-}(\Delta, c)$  and  $\mathfrak{M}_{L_+}(\Delta, c)$  are linked by the moduli spaces  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  where the data  $\mathbf{k}$  is allowed to vary. When the surface  $X$  is rational with  $-K_X$  effective, we can obtain  $\mathfrak{M}_{L_+}(\Delta, c)$  from  $\mathfrak{M}_{L_-}(\Delta, c)$  by a series of flips. The fact that all  $(L_0, \zeta, \mathbf{k})$ -semistable rank two torsion free sheaves do form a moduli space  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  in the usual sense is proved in section 4 where we introduce an equivalent notion of stability called *mixed stability*. Our method follows Gieseker's GIT argument in [13]. Roughly speaking, the goal of mixed stability is to define stability for a sheaf of the form  $V \otimes \Xi$ , where  $V$  is a torsion free sheaf but  $\Xi$  is just a  $\mathbb{Q}$ -divisor. To make this idea precise, given actual divisors  $H_1$  and  $H_2$  and positive weights  $a_1$  and  $a_2$ , we shall define a notion of stability which "mixes" stability for  $V \otimes H_1$  with stability for  $V \otimes H_2$ , together with weightings of the stability condition for  $V \otimes H_i$ . The effect of this definition will be formally the same as if we had defined stability of  $V \otimes \Xi$ , where  $\Xi$  is the  $\mathbb{Q}$ -divisor

$$\frac{a_1}{a_1 + a_2} H_1 + \frac{a_2}{a_1 + a_2} H_2.$$

After the first draft of this paper was completed, we learned from the preprint of Ellingsrud and Göttsche [8] that the existence of the relevant mixed moduli spaces follows from more general results of Maruyama and Yokogawa [23, 38]. They constructed coarse moduli spaces for semistable torsion free parabolic sheaves over general smooth projective varieties, using results of Simpson [32]. However, the proof given here for the special case that we need is considerably shorter than the general proof and is based on different ideas, and so will be given in full.

In section 5, using our results on flips of moduli spaces, we give a formula for the transition formula of Donaldson polynomials when  $X$  is rational with  $-K_X$  effective, and compute the leading term in the transition formula. In section 6, we obtain explicit transition formulas when  $\ell_{\zeta} \leq 2$ .

Some of the material in our section 2 has been worked out independently by Hu and Li [16] and Göttsche [14]. Moreover Ellingsrud and Göttsche [8] have recently studied the change in the moduli space by similar methods and

have obtained results very similar to ours. Using very different methods, the results in Section 4 have also been obtained by Matsuki and Wentworth [24], who also consider the case of higher rank. They use branched covers of the surface  $X$  to study the change in the moduli space. We expect that a minor modification of the arguments in Section 4 of this paper will also handle the case of higher rank.

### Conventions and notations.

We fix some conventions and notations for the rest of this paper. Let  $X$  be a smooth algebraic surface. We shall be primarily interested in the case where  $X$  is simply connected and  $-K_X$  is effective and nonzero. Thus necessarily  $X$  is a rational surface. However much of the discussion in sections 1–4 will also apply to the general case. Stability and semistability with respect to an ample line bundle  $L$  will always be understood to mean Gieseker stability or semistability unless otherwise noted. We shall not mention the choice of  $L$  explicitly if it is clear from the context. Recall that a torsion free sheaf  $V$  of rank two is Gieseker  $L$ -stable if and only if, for every rank one subsheaf  $W$  of  $V$ , either  $\mu_L(W) < \mu_L(V)$  or  $\mu_L(W) = \mu_L(V)$  and  $2\chi(W) < \chi(V)$ , where  $\mu_L$  is the normalized degree with respect to  $L$ . Semistability is similarly defined, where the second inequality is also allowed to be an equality. For a torsion free sheaf  $V$ , we use  $V^{\vee\vee}$  to stand for its double dual. For two divisors  $D_1$  and  $D_2$  on  $X$ , the notation  $D_1 \equiv D_2$  means that  $D_1$  and  $D_2$  are numerically equivalent, that is,  $D_1 \cdot D = D_2 \cdot D$  for any divisor  $D$ . For a locally free sheaf (or equivalently a vector bundle)  $\mathcal{E}$  over a smooth variety  $Y$ , we use  $\mathbb{P}(\mathcal{E})$  to denote the associated projective space bundle, that is,  $\mathbb{P}(\mathcal{E})$  is the **Proj** of  $\bigoplus_{d \geq 0} S^d(\mathcal{E})$ .

Fix a divisor  $\Delta$  and an integer  $c$ . Let  $\mathcal{C}_-$  and  $\mathcal{C}_+$  be two adjacent chambers of type  $(\Delta, c)$  separated by the wall  $W^\zeta$ . We assume that  $\zeta \cdot \mathcal{C}_- < 0 < \zeta \cdot \mathcal{C}_+$ . Let  $L_\pm \in \mathcal{C}_\pm$  be an ample line bundle, so that  $L_- \cdot \zeta < 0 < L_+ \cdot \zeta$ , and denote by  $\mathfrak{M}_\pm$  the moduli space  $\mathfrak{M}_{L_\pm}(\Delta, c)$  of rank two Gieseker semistable torsion free sheaves  $V$  with  $c_1(V) = \Delta$  and  $c_2(V) = c$ . Let  $L_0$  be any ample divisor contained in the interior of the intersection of  $W^\zeta$  and the closures of  $\mathcal{C}_\pm$ . Let  $\zeta = \zeta_1, \dots, \zeta_n$  be all the positive rational multiples of  $\zeta$  such that  $\zeta_i$  is an integral class of type  $(w, p)$  which also defines the wall  $W^\zeta$ . In sections 5–6, we will assume that  $n = 1$  for notational simplicity.

Finally, we point out that our  $\mu$ -map is half of the  $\mu$ -map used in [17, 18] (see (viii) and (ix) in Notation 5.1). Thus our transition formula differs from the one defined in [18] by a universal constant.

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## 2. Preliminaries on the moduli space.

In this section, we study rank two torsion free sheaves which are related to walls. These sheaves arise naturally from the comparison of  $L_-$ -semistability and  $L_+$ -semistability. We will show that when the surface  $X$  is rational with  $-K_X$  effective, the moduli spaces  $\mathfrak{M}_\pm$  are smooth at the points corresponding to these sheaves. We start with the following lemma, which for simplicity is just stated for  $L_-$ -stability.

**Lemma 2.1.** *Let  $V$  be a rank two torsion free sheaf on  $X$  with  $c_1(V) = \Delta$  and  $c_2(V) = c$ . If  $V$  is  $L_-$ -semistable, then exactly one of the following holds:*

- (i) *Both  $V$  and  $V^{\vee\vee}$  are  $L_-$ -stable and Mumford  $L_-$ -stable.*
- (ii)  *$V$  sits in an exact sequence*

$$0 \rightarrow \mathcal{O}_X(F_1) \otimes I_{Z_1} \rightarrow V \rightarrow \mathcal{O}_X(F_2) \otimes I_{Z_2} \rightarrow 0$$

*where  $2F_1 \equiv \Delta \equiv 2F_2$ , and  $Z_1$  and  $Z_2$  are zero-dimensional subschemes of  $X$  such that  $\ell(Z_1) \geq \ell(Z_2)$ . Moreover in this case  $V$  is  $L_-$ -semistable for every choice of an ample line bundle  $L$  and  $V$  is strictly  $L_\pm$ -semistable if and only if  $\ell(Z_1) = \ell(Z_2)$ .*

*Proof.* Suppose that  $V$  is (Gieseker)  $L_-$ -semistable. The vector bundle  $V^{\vee\vee}$  satisfies  $c_1(V^{\vee\vee}) = \Delta$  and  $c_2(V^{\vee\vee}) \leq c$ . Standard arguments [10] show that  $V^{\vee\vee}$  is Mumford  $L_-$ -semistable. If  $V^{\vee\vee}$  is strictly Mumford  $L_-$ -semistable, then by [10, 31], either  $L_-$  must lie on a wall of type  $(\Delta, c)$  or if  $\mathcal{O}_X(F_1)$  is a destabilizing sub-line bundle then  $\Delta \equiv 2F_1$ . Since by assumption  $L_-$  does not lie on a wall of type  $(\Delta, c)$ , either  $V^{\vee\vee}$  is Mumford  $L_-$ -stable or there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(F_1) \rightarrow V^{\vee\vee} \rightarrow \mathcal{O}_X(F_2) \otimes I_Z \rightarrow 0,$$

where  $F_2 = \Delta - F_1 \equiv F_1$  and  $Z$  is a zero-dimensional subscheme of  $X$ . If  $V^{\vee\vee}$  is Mumford  $L_-$ -stable, then  $V$  is Mumford  $L_-$ -stable and therefore  $L_-$ -stable. Thus case (i) holds. Otherwise  $\mathcal{O}_X(F_1) \cap V$  is of the form  $\mathcal{O}_X(F_1) \otimes I_{Z_1}$  for some  $Z_1$  and  $V/\mathcal{O}_X(F_1) \otimes I_{Z_1}$  is a subsheaf of  $\mathcal{O}_X(F_2) \otimes I_Z$  and thus of the form  $\mathcal{O}_X(F_2) \otimes I_{Z_2}$  for some  $Z_2$ . Thus we are in case (ii) of the lemma. Since  $\mu(\mathcal{O}_X(F_1) \otimes I_{Z_1}) = \mu(V)$  and  $V$  is semistable, we have

$$2\chi(\mathcal{O}_X(F_1) \otimes I_{Z_1}) \leq \chi(V) = \chi(\mathcal{O}_X(F_1) \otimes I_{Z_1}) + \chi(\mathcal{O}_X(F_2) \otimes I_{Z_2}).$$

Hence  $\chi(\mathcal{O}_X(F_2) \otimes I_{Z_2}) - \chi(\mathcal{O}_X(F_1) \otimes I_{Z_1}) \geq 0$ . As  $F_1 \equiv F_2$  and  $\chi(\mathcal{O}_X(F_i) \otimes I_{Z_i}) = \chi(\mathcal{O}_X(F_i)) - \ell(Z_i)$ , we must then have  $\ell(Z_1) - \ell(Z_2) \geq 0$ . The last sentence of (ii) is a straightforward argument left to the reader.  $\square$

If  $V$  satisfies the conclusions of (2.1)(ii), we shall call  $V$  *universally semistable*.

Next we shall compare stability for  $L_-$  and  $L_+$ .

**Lemma 2.2.** *Let  $V$  be a torsion free rank two sheaf with  $c_1(V) = \Delta$  and  $c_2(V) = c$ .*

(i) *If  $V$  is  $L_-$ -stable but  $L_+$ -unstable, then there exist a divisor class  $F$  and two zero-dimensional subschemes  $Z_-$  and  $Z_+$  of  $X$  and an exact sequence*

$$0 \rightarrow \mathcal{O}_X(F) \otimes I_{Z_-} \rightarrow V \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_+} \rightarrow 0,$$

where  $L_- \cdot (2F - \Delta) < 0 < L_+ \cdot (2F - \Delta)$ . Moreover the divisor  $F$ , the schemes  $Z_-$  and  $Z_+$ , and the map  $F \otimes I_{Z_-} \rightarrow V$  are unique mod scalars, and  $\zeta = 2F - \Delta$  defines a wall of type  $(\Delta, c)$ .

(ii) *Conversely, suppose that there is a nonsplit exact sequence as above. Then  $V$  is simple. Moreover,  $V$  is not  $L_-$ -stable if and only if it is  $L_-$ -unstable if and only if there exist subschemes  $Z'$  and  $Z''$  and an exact sequence*

$$0 \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z'} \rightarrow V \rightarrow \mathcal{O}_X(F) \otimes I_{Z''} \rightarrow 0,$$

if and only if  $V^{\vee\vee}$  is a direct sum  $\mathcal{O}_X(F) \oplus \mathcal{O}_X(\Delta - F)$ . In this case the scheme  $Z'$  strictly contains the scheme  $Z_+$ ,  $\ell(Z') > \ell(Z_+)$  and  $\ell(Z') + \ell(Z'') = \ell(Z_-) + \ell(Z_+)$ . Finally, if  $Z_- = \emptyset$  then  $V$  is always  $L_-$ -stable.



*Proof.* We first show (i). Suppose that  $V$  is  $L_-$ -stable but  $L_+$ -unstable. Then by (2.1)  $V^{\vee\vee}$  is also  $L_-$ -stable and  $L_+$ -unstable. By [31], there is a uniquely determined line bundle  $\mathcal{O}_X(F)$  and a map  $\mathcal{O}_X(F) \rightarrow V^{\vee\vee}$  with torsion free quotient such that  $L_- \cdot (2F - \Delta) < 0 < L_+ \cdot (2F - \Delta)$ . Moreover  $\zeta = 2F - \Delta$  defines a wall of type  $(\Delta, c)$ . The subsheaf  $\mathcal{O}_X(F) \cap V$  of  $V^{\vee\vee}$  is a subsheaf of  $\mathcal{O}_X(F)$  and agrees with it away from finitely many points. Thus  $\mathcal{O}_X(F) \cap V = \mathcal{O}_X(F) \otimes I_{Z_-}$  for some well-defined subscheme  $Z_-$ . Moreover the quotient  $V / [\mathcal{O}_X(F) \otimes I_{Z_-}]$  is a subsheaf of  $\mathcal{O}_X(\Delta - F) \otimes I_Z$  for some zero-dimensional subscheme  $Z$ , and agrees with  $\mathcal{O}_X(\Delta - F)$  away from finitely many points. Thus the quotient is of the form  $\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}$  for some zero-dimensional subscheme  $Z_+$ . The uniqueness is clear.

To see (ii), suppose that  $V$  is given as a nonsplit exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \otimes I_{Z_-} \rightarrow V \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_+} \rightarrow 0$$

as above, where  $L_- \cdot (2F - \Delta) < 0 < L_+ \cdot (2F - \Delta)$ . Again by (2.1),  $V$  is  $L_-$ -semistable if and only if it is  $L_-$ -stable if and only if  $V^{\vee\vee}$  is  $L_-$ -stable. Now taking double duals of the above exact sequence, there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow V^{\vee\vee} \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_Z \rightarrow 0$$

for some zero-dimensional scheme  $Z$ . Moreover, by [31],  $V^{\vee\vee}$  is  $L_-$ -unstable if and only if the above exact sequence splits, and in particular if and only if  $Z = \emptyset$  and  $V^{\vee\vee} = \mathcal{O}_X(F) \oplus \mathcal{O}_X(\Delta - F)$ . In this case, the map  $\mathcal{O}_X(\Delta - F) \rightarrow V^{\vee\vee}$  induces a map  $\mathcal{O}_X(\Delta - F) \otimes I_{Z'} \rightarrow V$  for some ideal sheaf  $I_{Z'}$ . We may clearly assume that the quotient is torsion free, in which case it is necessarily of the form  $\mathcal{O}_X(F) \otimes I_{Z''}$  with  $\ell(Z') + \ell(Z'') = \ell(Z_-) + \ell(Z_+)$ . Using the nonzero map  $\mathcal{O}_X(\Delta - F) \otimes I_{Z'} \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_+}$ , we see that there is an inclusion  $I_{Z'} \subseteq I_{Z_+}$ ; moreover this inclusion must be strict since the defining exact sequence for  $V$  is nonsplit. Thus  $Z'$  strictly contains  $Z_+$  and in particular  $\ell(Z') > \ell(Z_+)$ . Conversely, if there exists a nonzero map  $\mathcal{O}_X(\Delta - F) \otimes I_{Z'} \rightarrow V$ , then there is a nonzero map  $\mathcal{O}_X(\Delta - F) \rightarrow V^{\vee\vee}$  and thus  $V^{\vee\vee}$  is the split extension.

We next show that  $V$  is simple. If  $V$  is stable then it is simple. If  $V$  is not stable, then  $V^{\vee\vee} = \mathcal{O}_X(F) \oplus \mathcal{O}_X(\Delta - F)$ . There is an inclusion  $\text{Hom}(V, V) \subseteq \text{Hom}(V^{\vee\vee}, V^{\vee\vee})$ . If  $V^{\vee\vee}$  is split, then  $\text{Hom}(V^{\vee\vee}, V^{\vee\vee}) = \mathbb{C} \oplus \mathbb{C}$ . In this case, using a nonscalar endomorphism of  $V$ , it is easy to see that we can split the exact sequence defining  $V$ .

Finally suppose that  $Z_- = \emptyset$  in the notation of (2.2). If  $V$  is  $L_-$ -unstable, then we can find  $Z'$  with  $\ell(Z') > \ell(Z_+)$  and a subscheme  $Z''$  such

that  $\ell(Z') + \ell(Z'') = \ell(Z_+)$ . Thus  $\ell(Z') \leq \ell(Z_+)$ , a contradiction. It follows that  $V$  is  $L_-$ -stable.  $\square$

For the rest of this section, we shall assume that  $-K_X$  is effective and nonzero and that  $q(X) = 0$ . Thus  $X$  is a rational surface.

**Lemma 2.3.** *Suppose that  $\mathfrak{M}_\pm$  is nonempty. Suppose that  $(w, p) \neq (0, 0)$ , or equivalently that  $\mathfrak{M}_\pm$  does not consist of a single point corresponding to a twist of the trivial vector bundle. Then the open subset of  $\mathfrak{M}_\pm$  corresponding to Mumford stable rank two vector bundles is nonempty and dense. Every component of  $\mathfrak{M}_\pm$  has dimension  $4c - \Delta^2 - 3 = -p - 3$ . The points of  $\mathfrak{M}_\pm$  corresponding to  $L_\pm$ -stable sheaves  $V$  are smooth points.*

*Proof.* Suppose that  $\mathfrak{M}_\pm$  is nonempty, and let  $V$  correspond to a point of  $\mathfrak{M}_\pm$ . Then by general theory (e.g. Chapter 7 of [10]),  $\mathfrak{M}_\pm$  is smooth of dimension  $4c - \Delta^2 - 3 = -p - 3$  at  $V$  if  $V$  is stable and  $\text{Ext}^2(V, V) = 0$ , since  $h^2(X; \mathcal{O}_X) = 0$ . Moreover, setting  $W = V^{\vee\vee}$ , there is a surjection from  $H^2(X; \text{Hom}(W, W))$  to  $\text{Ext}^2(V, V)$ . Thus to show that  $\text{Ext}^2(V, V) = 0$  it suffices to show that  $H^2(X; \text{Hom}(W, W)) = 0$ . Now  $H^2(X; \text{Hom}(W, W))$  is dual to  $H^0(X; \text{Hom}(W, W) \otimes K_X)$ . Since  $-K_X$  is effective, there is an inclusion of  $H^0(X; \text{Hom}(W, W) \otimes K_X)$  in  $H^0(X; \text{Hom}(W, W))$ . If  $W$  is stable, then  $H^0(X; \text{Hom}(W, W)) \cong \mathbb{C}$  and  $H^0(X; \text{Hom}(W, W) \otimes K_X) = 0$ . Thus  $\mathfrak{M}_\pm$  is smooth at  $V$ . Standard theory [1, 10] also shows that every torsion free sheaf  $V$  for which  $V^{\vee\vee}$  is stable is smoothable. Thus the set of locally free sheaves is nonempty and dense in the component containing  $V$  in this case.

Now consider a  $V$  such that  $W = V^{\vee\vee}$  is not stable. Using the exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow W \rightarrow \mathcal{O}_X(F) \otimes I_Z \rightarrow 0$$

for  $W$  which was given in the course of the proof of (2.1), it is easy to check that there is an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(I_Z, W \otimes \mathcal{O}_X(-F) \otimes K_X) &\rightarrow \text{Hom}(W, W \otimes K_X) \rightarrow \\ &\rightarrow H^0(W \otimes \mathcal{O}_X(-F) \otimes K_X). \end{aligned}$$

Since  $-K_X$  is effective and nonzero,  $H^0(W \otimes \mathcal{O}_X(-F) \otimes K_X) = \text{Hom}(I_Z, W \otimes \mathcal{O}_X(-F) \otimes K_X) = 0$ . Thus  $\text{Hom}(W, W \otimes K_X) = 0$  as well. Once again  $V$  is smoothable.

Now we claim that a general smoothing  $V'$  of  $V$  is Mumford stable. For otherwise by the proof of (2.1) there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \rightarrow V' \rightarrow \mathcal{O}_X(F) \otimes I_Z \rightarrow 0$$

as above, with  $\ell(Z) \leq \ell(\emptyset) = 0$ . In this case  $V'$  is an extension of  $\mathcal{O}_X(F)$  by  $\mathcal{O}_X(F)$ , forcing  $w = p = 0$  and (since  $h^1(\mathcal{O}_X) = 0$ )  $V' = \mathcal{O}_X(F) \oplus \mathcal{O}_X(F)$ .  $\square$

It is natural to make the following conjecture, which is true for geometrically ruled  $X$  by [30] and is verified in certain other cases by [36].

**Conjecture 2.4.** *If  $X$  is a rational surface with  $-K_X$  effective, then for every choice of  $L$ ,  $\Delta$  and  $c$ ,  $\mathfrak{M}_L(\Delta, c)$  is either empty or irreducible.*

Let us fix some notations for the rest of this paper.

**Definition 2.5.** Let  $X$  be an algebraic surface (not necessarily rational), and let  $\zeta$  be a fixed numerical equivalence class defining a wall of type  $(\Delta, c)$ . Set  $\ell_\zeta = (4c - \Delta^2 + \zeta^2)/4 = (\zeta^2 - p)/4$ . Choose two nonnegative integers  $n_-$  and  $n_+$  with  $n_- + n_+ = \ell_\zeta$ , and let  $E_\zeta^{n_-, n_+}$  be the set of all isomorphism classes of nonsplit extensions of the form

$$0 \rightarrow \mathcal{O}_X(F) \otimes I_{Z_-} \rightarrow V \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_+} \rightarrow 0$$

with  $\zeta \equiv 2F - \Delta$  and  $\ell(Z_\pm) = n_\pm$ .

We remark that since  $\zeta \equiv \Delta \pmod{2}$  and  $\Delta^2 - 4c \leq \zeta^2 < 0$ ,  $\ell_\zeta$  is a nonnegative integer. If  $V$  corresponds to a point of  $E_\zeta^{n_-, n_+}$ , then  $V$  is  $L_+$ -unstable since  $L_+ \cdot \zeta > 0$ . By (2.2)(ii),  $V$  is simple, and if it is  $L_-$ -semistable then it is actually stable. By (2.3), if  $X$  is a rational surface with  $-K_X$  effective, then  $\mathfrak{M}_-$  is smooth in a neighborhood of a point corresponding to a sheaf  $V$  lying in  $E_\zeta^{n_-, n_+}$  for some  $\zeta, n_-, n_+$ . We shall now study  $E_\zeta^{n_-, n_+}$  in more detail for rational surfaces.

**Lemma 2.6.** *Suppose that  $-K_X$  is effective and that  $q(X) = 0$ . For  $Z_-$  and  $Z_+$  two fixed zero-dimensional subschemes of  $X$  of lengths  $n_-$  and  $n_+$  respectively,*

$$\dim \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-}) = n_- + n_+ + h(\zeta) = \ell_\zeta + h(\zeta),$$

where

$$h(\zeta) = h^1(X; \mathcal{O}_X(2F - \Delta)) = \frac{(\zeta \cdot K_X)}{2} - \frac{\zeta^2}{2} - 1.$$

*Proof.* Note that  $\text{Hom}(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-}) \subseteq H^0(\mathcal{O}_X(2F - \Delta)) = 0$ , since  $L_- \cdot (2F - \Delta) < 0$ . Likewise  $\text{Ext}^2(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-})$  is Serre dual to  $\text{Hom}(\mathcal{O}_X(F) \otimes I_{Z_-}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_+} \otimes K_X) \subseteq H^0(\mathcal{O}_X(\Delta - 2F) \otimes K_X) \subseteq H^0(\mathcal{O}_X(\Delta - 2F))$ , since  $-K_X$  is effective. Thus as  $L_+ \cdot (\Delta - 2F) < 0$ ,  $\text{Ext}^2(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-}) = 0$  as well. If we set  $\chi(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-}) = \sum_i (-1)^i \dim \text{Ext}^i(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-})$ , then

$$\begin{aligned} \chi(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-}) &= \\ &= -\dim \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-}). \end{aligned}$$

Now a standard argument [28] shows that

$$\begin{aligned} &\chi(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-}) \\ &= \int_X \text{ch}(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+})^\vee \cdot \text{ch}(\mathcal{O}_X(F) \otimes I_{Z_-}) \cdot \text{Todd}_X. \end{aligned}$$

Here given a class  $a = \sum a_i \in \bigoplus_i A^i(X)$ , we denote by  $a^\vee$  the class  $\sum_i (-1)^i a_i$ . An easy computation gives

$$\begin{aligned} &\int_X \text{ch}(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+})^\vee \cdot \text{ch}(\mathcal{O}_X(F) \otimes I_{Z_-}) \cdot \text{Todd}_X \\ &= \int_X \text{ch}(\mathcal{O}_X(\Delta - F))^\vee \cdot \text{ch}(\mathcal{O}_X(F)) \cdot \text{Todd}_X - \ell(Z_-) - \ell(Z_+). \end{aligned}$$

Reversing the above argument, we see that

$$\begin{aligned} \int_X \text{ch}(\mathcal{O}_X(\Delta - F))^\vee \cdot \text{ch}(\mathcal{O}_X(F)) \cdot \text{Todd}_X &= \chi(\mathcal{O}_X(2F - \Delta)) \\ &= -h^1(X; \mathcal{O}_X(2F - \Delta)) = \frac{\zeta^2}{2} - \frac{(\zeta \cdot K_X)}{2} + 1 = -h(\zeta). \end{aligned}$$

Putting these together we see that  $\dim \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-})$  is equal to  $n_- + n_+ + h(\zeta)$ .  $\square$

Let us describe the scheme structure on  $E_\zeta^{n_-, n_+}$  more carefully. For  $Z_-$  and  $Z_+$  fixed, the set of extensions in  $E_\zeta^{n_-, n_+}$  corresponding to  $Z_-, Z_+$ , is equal to  $\mathbb{P} \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_+}, \mathcal{O}_X(F) \otimes I_{Z_-})$ . To make a universal construction, let  $H_{n_\pm} = \text{Hilb}^{n_\pm} X$ . Let  $Z_{n_\pm}$  be the universal codimension two subscheme of  $X \times H_{n_\pm}$ . Let  $\pi_1, \pi_2$  be the projections of  $X \times H_{n_-} \times H_{n_+}$

to  $X$ ,  $H_{n_-} \times H_{n_+}$  respectively, and let  $\pi_{1,2}$ ,  $\pi_{1,3}$  be the projections of  $X \times H_{n_-} \times H_{n_+}$  to  $X \times H_{n_-}$ ,  $X \times H_{n_+}$  respectively. Define

$$\mathcal{E}_\zeta^{n_-, n_+} = \text{Ext}_{\pi_2}^1(\pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_{n_+}}, \pi_1^* \mathcal{O}_X(F) \otimes \pi_{1,2}^* I_{Z_{n_-}}).$$

The previous lemma and standard base change results show that  $\mathcal{E}_\zeta^{n_-, n_+}$  is locally free of rank  $h(\zeta) + \ell_\zeta$  over  $H_{n_-} \times H_{n_+}$ . We set  $E_\zeta^{n_-, n_+} = \mathbb{P}((\mathcal{E}_\zeta^{n_-, n_+})^\vee)$ , if  $h(\zeta) + \ell_\zeta > 0$ . Moreover by standard facts about relative Ext sheaves there is an exact sequence

$$\begin{aligned} 0 \rightarrow R^1 \pi_{2*} \text{Hom}\left(\pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_{n_+}}, \pi_1^* \mathcal{O}_X(F) \otimes \pi_{1,2}^* I_{Z_{n_-}}\right) \rightarrow \\ \rightarrow \mathcal{E}_\zeta^{n_-, n_+} \rightarrow \\ \rightarrow \pi_{2*} \text{Ext}^1\left(\pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_{n_+}}, \pi_1^* \mathcal{O}_X(F) \otimes \pi_{1,2}^* I_{Z_{n_-}}\right) \rightarrow 0. \end{aligned}$$

**Corollary 2.7.** *With  $X$  as in (2.6), if  $h(\zeta) + \ell_\zeta = h^1(X; \mathcal{O}_X(2F - \Delta) + \ell_\zeta) \neq 0$ ,  $E_\zeta^{n_-, n_+}$  is a  $\mathbb{P}^{N_\zeta}$ -bundle over  $H_{n_-} \times H_{n_+}$ , where  $N_\zeta = \dim \text{Ext}^1 - 1 = h(\zeta) + \ell_\zeta - 1$ . Thus if  $h(\zeta) + \ell_\zeta \neq 0$ , then  $\dim E_\zeta^{n_-, n_+} = 3\ell_\zeta + h(\zeta) - 1$ . Moreover in this case  $E_{-\zeta}^{n_+, n_-}$  is a  $\mathbb{P}^{N_{-\zeta}}$ -bundle over  $H_{n_+} \times H_{n_-}$ , and  $N_\zeta + N_{-\zeta} + 2\ell_\zeta = -p - 4$ . If  $h(\zeta) + \ell_\zeta = 0$ , then  $E_\zeta^{0,0} = \emptyset$  and  $E_{-\zeta}^{0,0} = \mathbb{P}^{-p-3}$  is a component of  $\mathfrak{M}_+$ . Finally this last case arises if and only if  $\zeta^2 = p$  and  $\zeta \cdot K_X = \zeta^2 + 2 = p + 2$ .*

*Proof.* Note that  $N_\zeta \geq 0$  unless  $h(\zeta) + \ell_\zeta = 0$ . Under this assumption, we have

$$N_\zeta + N_{-\zeta} + 2\ell_\zeta = 4\ell_\zeta - \zeta^2 - 4 = -p - 4.$$

The case where  $h(\zeta) + \ell_\zeta = 0$  is similar. Moreover if  $h(\zeta) + \ell_\zeta = 0$ , then it follows from (2.2)(ii) that all of the sheaves  $V$  corresponding to points of  $E_{-\zeta}^{0,0}$  are  $L_+$ -stable. By (2.2)(i) the map  $E_{-\zeta}^{0,0} \rightarrow \mathfrak{M}_+$  is one-to-one. Since  $\mathfrak{M}_+$  is of dimension  $-p - 3$  and smooth at points corresponding to the sheaves in  $E_{-\zeta}^{0,0} \rightarrow \mathfrak{M}_+$ , the map  $E_{-\zeta}^{0,0} \rightarrow \mathfrak{M}_+$  must be an embedding onto a component of  $\mathfrak{M}_+$ . The final statement follows from the formulas  $\zeta^2 = 4\ell_\zeta + p$  and

$$h(\zeta) = \frac{(\zeta \cdot K_X)}{2} - \frac{\zeta^2}{2} - 1. \quad \square$$

If  $h(\zeta) + \ell_\zeta \neq 0$ , then by Lemma 2.2 there is a rational map from  $E_\zeta^{n_-, n_+}$  to the moduli space  $\mathfrak{M}_-$  which is birational onto its image. However this map will not in general be a morphism if  $n_- > 0$  (see [16]). We shall study this more carefully in the next sections.

Let us also remark that standard theory gives a universal sheaf  $\mathcal{V}$  over  $E_\zeta^{n_-, n_+}$ :

**Proposition 2.8.** *Let  $\rho : X \times E_\zeta^{n_-, n_+} \rightarrow X \times H_{n_-} \times H_{n_+}$  be the natural projection, and let  $\pi_2 : X \times E_\zeta^{n_-, n_+} \rightarrow E_\zeta^{n_-, n_+}$  be the projection. Then there is a coherent sheaf  $\mathcal{V}$  over  $X \times E_\zeta^{n_-, n_+}$  and an exact sequence*

$$\begin{aligned} 0 \rightarrow \rho^* \left( \pi_1^* \mathcal{O}_X(F) \otimes \pi_{1,2}^* I_{Z_{n_-}} \right) \otimes \pi_2^* \mathcal{O}_{E_\zeta^{n_-, n_+}}(1) \\ \rightarrow \mathcal{V} \rightarrow \rho^* \left( \pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_{n_+}} \right) \rightarrow 0. \end{aligned}$$

**Remark 2.9.** Very similar results hold in the case where  $-K_X$  is effective and nonzero (corresponding to certain elliptic ruled surfaces) or  $K_X = 0$  (corresponding to  $K3$  or abelian surfaces). For example, in the case of a  $K3$  surface  $X$ , the moduli space is smooth of dimension  $-p - 6$  away from the sheaves which are strictly semistable for every ample divisor (although there exist components consisting entirely of non-locally free sheaves for small values of  $-p$ ). In this case however  $h(\zeta) = -\zeta^2/2 - 2$  and  $N_\zeta + N_{-\zeta} + 2\ell_\zeta = -p - 6$ , which is equal to the dimension  $d$  of the moduli space instead of to  $d - 1$ . For example, if  $\ell_\zeta = 0$ , then  $N_\zeta = N_{-\zeta} = d/2$ . In this case  $E_\zeta^{0,0} \cong \mathbb{P}^{d/2}$  is a maximal isotropic submanifold of the symplectic manifold  $\mathfrak{M}_-$ . In other words, the natural holomorphic 2-form  $\omega$  on  $\mathfrak{M}_-$  vanishes on  $E_\zeta^{0,0}$  and identifies the normal bundle of  $E_\zeta^{0,0}$  in  $\mathfrak{M}_-$  with the cotangent bundle of  $E_\zeta^{0,0}$ .

### 3. Flips of moduli spaces.

In this section, we begin by assuming again that  $X$  is an arbitrary algebraic surface. Let  $\zeta = \zeta_1, \dots, \zeta_n$  be the positive rational multiples of  $\zeta$  such that  $\zeta_i$  is an integral class also defining the wall  $W^\zeta$ . Our goal in this section is to deal with the problem that there is only a rational map in general from  $E_{\zeta_i}^{n_-, n_+}$  to  $\mathfrak{M}_-$ . We shall do so by finding a sequence of spaces between  $\mathfrak{M}_-$  and  $\mathfrak{M}_+$ , each one given by blowing up and down the previous one, such that for an appropriate member of the sequence the rational map  $E_{\zeta_i}^{n_-, n_+} \dashrightarrow \mathfrak{M}_-$  becomes a morphism (and a smooth embedding in the case of rational surfaces). Throughout the rest of this paper,  $L_0$  shall denote any ample divisor contained in the interior of the intersection of  $W^\zeta$  and the closures of  $\mathcal{C}_\pm$ . Recall that we have defined universal semistability after the proof of (2.1).

**Definition 3.1.** Let  $k$  be an integer. A rank two torsion free sheaf  $V$  with  $c_1(V) = \Delta$  and  $\Delta^2 - 4c_2(V) = p$  is  $(L_0, \zeta, k)$ -semistable if  $V$  is Mumford  $L_0$ -semistable and if it is strictly Mumford semistable, then either it is universally semistable or, for all divisors  $F$  such that  $2F - \Delta \equiv \zeta$ , we have the following:

(i) If there exists an exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \otimes I_{Z_1} \rightarrow V \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \rightarrow 0,$$

then  $\ell(Z_2) \leq k$  and thus  $\ell(Z_1) \geq \ell_\zeta - k$ .

(ii) If there exists an exact sequence

$$0 \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_1} \rightarrow V \rightarrow \mathcal{O}_X(F) \otimes I_{Z_2} \rightarrow 0,$$

then  $\ell(Z_1) \geq k + 1$  and thus  $\ell(Z_2) \leq \ell_\zeta - k - 1$ .

Likewise, setting  $\zeta = (\zeta_1, \dots, \zeta_n)$  and  $\mathbf{k} = (k_1, \dots, k_n)$ , we say that  $V$  is  $(L_0, \zeta, \mathbf{k})$ -semistable if  $V$  is  $(L_0, \zeta_i, k_i)$ -semistable for every  $i$ . Let  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  denote the set of isomorphism classes of  $(L_0, \zeta, \mathbf{k})$ -semistable rank two sheaves  $V$  with  $c_1(V) = \Delta$  and  $\Delta^2 - 4c_2(V) = p$ .

Next we give some easy properties of  $(L_0, \zeta, \mathbf{k})$ -semistability.

**Lemma 3.2.** (i) If  $k_i \geq \ell_{\zeta_i}$  for all  $i$ , and  $V$  is not universally semistable, then  $V$  is  $(L_0, \zeta, \mathbf{k})$ -semistable if and only if it is  $L_-$ -stable. Likewise if  $k_i \leq -1$  for all  $i$  and  $V$  is not universally semistable, then  $V$  is  $(L_0, \zeta, \mathbf{k})$ -semistable if and only if it is  $L_+$ -stable.

(ii) If  $k_i \geq \ell_{\zeta_i}$  for all  $i$ , then  $\mathfrak{M}_0^{(\zeta, \mathbf{k})} = \mathfrak{M}_-$ . Likewise if  $k_i \leq -1$  for all  $i$ , then  $\mathfrak{M}_0^{(\zeta, \mathbf{k})} = \mathfrak{M}_+$ .

(iii) For  $n_2 > k_i$ ,  $\mathfrak{M}_0^{(\zeta, \mathbf{k})} \cap E_{\zeta_i}^{n_1, n_2} = \emptyset$ .

(iv) There is an injection  $E_{\zeta_i}^{\ell_{\zeta_i} - k_i, k_i} \rightarrow \mathfrak{M}_0^{(\zeta, \mathbf{k})}$ . Likewise there is an injection  $E_{-\zeta_i}^{k_i+1, \ell_{\zeta_i} - k_i - 1} \rightarrow \mathfrak{M}_0^{(\zeta, \mathbf{k})}$ . Finally, the images of  $E_{\zeta_i}^{\ell_{\zeta_i} - k_i, k_i}$  and  $E_{\zeta_j}^{\ell_{\zeta_j} - k_j, k_j}$  are disjoint if  $i \neq j$ .

*Proof.* If  $k_i \geq \ell_{\zeta_i}$  for all  $i$ , then the condition that  $\ell(Z_2) \leq \ell_{\zeta_i}$  and  $\ell(Z_1) \geq 0$  are trivially always satisfied and the conditions  $\ell(Z_2) \leq -1$  and  $\ell(Z_1) \geq 0$

$\ell_{\zeta_i} + 1$  are vacuous. A similar argument handles the case  $k_i \leq -1$  for all  $i$ . It is easy to see that this implies (i). Statement (ii) follows from (i), and (iii) follows from the definitions. As for (iv), let  $V \in E_{\zeta_i}^{\ell_{\zeta_i} - k_i, k_i}$ . To decide if  $V$  is in  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$ , we look for potentially destabilizing subsheaves with torsion free quotient. Similar arguments as in [31] show that the only potentially destabilizing subsheaves with torsion free quotient must be either  $\mathcal{O}_X(F) \otimes I_{Z_1}$  or  $\mathcal{O}_X(\Delta - F) \otimes I_Z$ . By hypothesis, there is a unique subsheaf of  $V$  of the form  $\mathcal{O}_X(F) \otimes I_{Z_1}$ , and it is not destabilizing. If there is a subsheaf of the form  $\mathcal{O}_X(\Delta - F) \otimes I_Z$  with torsion free quotient, then by Lemma 2.2 we have  $\ell(Z) > \ell(Z_2) = k_i$  and so  $\ell(Z) \geq k_i + 1$ . Hence such a subsheaf is also not destabilizing. Thus by Definition 3.1  $V$  is  $(L_0, \zeta, \mathbf{k})$ -semistable. The fact that the map  $E_{\zeta_i}^{\ell_{\zeta_i} - k_i, k_i} \rightarrow \mathfrak{M}_0^{(\zeta, \mathbf{k})}$  is one-to-one and that  $E_{\zeta_i}^{\ell_{\zeta_i} - k_i, k_i}$  and  $E_{\zeta_j}^{\ell_{\zeta_j} - k_j, k_j}$  are disjoint if  $i \neq j$  also follow from similar arguments in [31]. The statement about  $E_{-\zeta_i}^{k_i + 1, \ell_{\zeta_i} - k_i - 1}$  is similar.  $\square$

Next suppose that we are given two integral vectors  $\mathbf{k}$  and  $\mathbf{k}'$  and a subset  $I$  of  $\{1 \dots, n\}$  such that  $k'_i = k_i$  if  $i \notin I$  and  $k'_i = k_i - 1$  if  $i \in I$ . We investigate the change as we pass from  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  to  $\mathfrak{M}_0^{(\zeta, \mathbf{k}')}$ .

**Lemma 3.3.** *The set of sheaves  $V$  in  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  which are not  $(L_0, \zeta, \mathbf{k}')$ -semistable is exactly the image of  $\bigcup_{i \in I} E_{\zeta_i}^{\ell_{\zeta_i} - k_i, k_i}$ . Likewise the set of  $V \in \mathfrak{M}_0^{(\zeta, \mathbf{k}')$  which are not  $(L_0, \zeta, \mathbf{k})$ -semistable is exactly the image of  $\bigcup_{i \in I} E_{-\zeta_i}^{k_i, \ell_{\zeta_i} - k_i}$ .*

*Proof.* If  $V$  is  $(L_0, \zeta, \mathbf{k})$ -semistable but not  $(L_0, \zeta, \mathbf{k}')$ -semistable, then  $V$  must be Mumford strictly  $L_0$ -semistable. Suppose that the  $(L_0, \zeta, \mathbf{k}')$ -destabilizing subsheaf is of the form  $\mathcal{O}_X(F) \otimes I_{Z_1}$ , where  $F$  corresponds to  $\zeta_i$  for some  $i \in I$ . Then  $\ell(Z_2) \leq k_i$  (since  $V \in \mathfrak{M}_0^{(\zeta, \mathbf{k})}$ ) but  $\ell(Z_2) \geq k_i$  (since the subsheaf is  $(L_0, \zeta, \mathbf{k}')$ -destabilizing, for  $k'_i = k_i - 1$ ) so that  $\ell(Z_2) = k_i$ . Thus  $V \in E_{\zeta_i}^{\ell_{\zeta_i} - k_i, k_i}$ . The other possibility is that the destabilizing subsheaf is of the form  $\mathcal{O}_X(\Delta - F) \otimes I_{Z_1}$ . Here we need  $\ell(Z_1) \geq k_i + 1$  but  $\ell(Z_1) < k_i$  and there are no such sheaves. The statement about  $\mathfrak{M}_0^{(\zeta, \mathbf{k}')$  follows by symmetry.  $\square$

We shall now describe a sequence of actual moduli spaces  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  for which the integral vector  $\mathbf{k}$  change in the way described before the statement



of (3.3).

**Definition 3.4.** Suppose that  $\zeta_i = r_i \zeta_1$ , where  $r_i$  is a positive rational number. Given  $t \in \mathbb{Q}$ , let  $t_i = r_i t$ , so that  $t_1 = t$ . Suppose that  $\frac{\ell_{\zeta_i} + t_i}{2}$  is not an integer for any  $i$ . In this case, define

$$k_i(t) = \left\lfloor \frac{\ell_{\zeta_i} + t_i}{2} \right\rfloor,$$

where  $[x]$  is the greatest integer function, and define  $\mathbf{k}(t)$  to be the vector formed by the  $k_i(t)$ . A rational number  $t$  is  $\zeta_i$ -critical if  $\frac{\ell_{\zeta_i} + t_i}{2} \in \mathbb{Z}$  and  $-1 \leq \frac{\ell_{\zeta_i} + t_i}{2} \leq \ell_{\zeta_i}$ . We shall also say that  $t_i$  is  $\zeta_i$ -critical. Finally  $t$  is  $\zeta$ -critical if it is  $\zeta_i$ -critical for some  $i$ . Note that there are only finitely many such  $t$ .

Given  $t \in \mathbb{Q}$ , let  $I(t) = \{i : t \text{ is } \zeta_i\text{-critical}\}$ . Suppose that  $\varepsilon$  is chosen so that, for every  $i$ , either there is no  $\zeta_i$ -critical rational number in  $[t_i - r_i \varepsilon, t_i + r_i \varepsilon]$  or  $t_i$  is the unique  $\zeta_i$ -critical rational number in  $[t_i - r_i \varepsilon, t_i + r_i \varepsilon]$ . Equivalently either there is no  $\zeta$ -critical number in  $[t - \varepsilon, t + \varepsilon]$  or  $t$  is the unique  $\zeta$ -critical number in  $[t - \varepsilon, t + \varepsilon]$ . Then we clearly have:

$$k_i(t - \varepsilon) = \begin{cases} k_i(t + \varepsilon), & \text{if } i \notin I(t) \\ k_i(t + \varepsilon) - 1, & \text{if } i \in I(t). \end{cases}$$

In particular if there is no  $\zeta$ -critical number in  $[t - \varepsilon, t + \varepsilon]$ , so that  $I(t) = \emptyset$ , then  $k_i(t - \varepsilon) = k_i(t + \varepsilon)$  for every  $i$ . Further note that if  $t \gg 0$ , then  $k_i(t) > \ell_{\zeta_i}$  for every  $i$ , and if  $t \ll 0$ , then  $k_i(t) < -1$  for every  $i$ .

We then have the following theorem, whose proof will be given in the next section:

**Theorem 3.5.** *For all  $t \in \mathbb{Q}$  which are not  $\zeta$ -critical, there exists a natural structure of a projective scheme on  $\mathfrak{M}_0^{(\zeta, \mathbf{k}(t))}$  for which it is a coarse moduli space.*

The proof of (3.5) will also show that  $\mathfrak{M}_0^{(\zeta, \mathbf{k}(t))}$  has the usual properties of a coarse moduli space: all sheaves corresponding to points of  $\mathfrak{M}_0^{(\zeta, \mathbf{k}(t))}$  will turn out to be simple (as they will turn out to be stable for an appropriate notion of stability), a classical or formal neighborhood of a point of  $\mathfrak{M}_0^{(\zeta, \mathbf{k}(t))}$  may be identified with the universal deformation space of the corresponding

sheaf, and there exists a universal sheaf locally in the classical or étale topology around every point of  $\mathfrak{M}_0^{(\zeta, \mathbf{k}(t))}$ .

For the rest of this section, we shall again restrict to the case where  $X$  is a rational surface with  $-K_X$  effective, unless otherwise noted. Let  $\zeta = \zeta_i$  for some  $i$  and let  $\mathbf{k} = \mathbf{k}(t)$  for some  $t$  which is not  $\zeta$ -critical. The first step is to make some infinitesimal calculations concerning the differential of the map  $E_\zeta^{\ell_\zeta - k, k} \rightarrow \mathfrak{M}_0^{(\zeta, \mathbf{k})}$  and the normal bundle to its image.

**Proposition 3.6.** *The map  $E_\zeta^{\ell_\zeta - k, k} \rightarrow \mathfrak{M}_0^{(\zeta, \mathbf{k})}$  is an immersion. The normal bundle  $\mathcal{N}_\zeta^{\ell_\zeta - k, k}$  to  $E_\zeta^{\ell_\zeta - k, k}$  in  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  is exactly  $\rho^* \mathcal{E}_{-\zeta}^{k, \ell_\zeta - k} \otimes \mathcal{O}_{E_\zeta^{\ell_\zeta - k, k}}(-1)$ , in the notation of the previous section.*

*Proof.* Since every sheaf in  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  is actually stable and therefore simple (which was also proved in (2.2)) we may identify an analytic neighborhood of  $V \in \mathfrak{M}_0^{(\zeta, \mathbf{k})}$  with the germ of the universal deformation space for  $V$ , i.e. with  $\text{Ext}^1(V, V)$ . Let us now calculate the tangent space to  $E_\zeta^{\ell_\zeta - k, k}$  at  $V$ : suppose that  $\xi \in \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}, \mathcal{O}_X(F) \otimes I_{Z_1}) = \text{Ext}^1$  is a nonzero extension class corresponding to  $V$ , where  $\ell(Z_1) = \ell_\zeta - k$  and  $\ell(Z_2) = k$ . Let  $H_{\ell_\zeta - k} = \text{Hilb}^{\ell_\zeta - k} X$  and  $H_k = \text{Hilb}^k X$ . Then there is the following exact sequence for the tangent space to  $E_\zeta^{\ell_\zeta - k, k}$  at  $\xi$ :

$$0 \rightarrow \text{Ext}^1 / \mathbb{C} \cdot \xi \rightarrow T_\xi E_\zeta^{\ell_\zeta - k, k} \rightarrow T_{Z_1} H_{\ell_\zeta - k} \oplus T_{Z_2} H_k \rightarrow 0.$$

Note further that the tangent space to  $\text{Hilb}^n X$  at  $Z$  is equal to  $\text{Hom}(I_Z, \mathcal{O}_Z)$ , which we may further canonically identify with  $\text{Ext}^1(I_Z, I_Z)$  since  $X$  is rational and by a local calculation. We then have the following:

**Proposition 3.7.** *For all nonzero  $\xi \in \text{Ext}^1$ , the natural map from a neighborhood of  $\xi$  in  $E_\zeta^{\ell_\zeta - k, k}$  to  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  is an immersion at  $\xi$ . The image of  $T_\xi E_\zeta^{\ell_\zeta - k, k}$  in  $\text{Ext}^1(V, V)$  is exactly the kernel of the natural map*

$$\text{Ext}^1(V, V) \rightarrow \text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}),$$

and the normal space to  $E_\zeta^{\ell_\zeta - k, k}$  at  $\xi$  in  $\mathfrak{M}_0^{(\zeta, \mathbf{k})}$  may be canonically identified with  $\text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2})$ .

*Proof.* Consider the natural map from  $\text{Ext}^1(V, V)$  to

$$\text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}).$$

We claim that this map is surjective and will describe its kernel in more detail. The map factors into two maps:

$$\begin{aligned} \text{Ext}^1(V, V) &\rightarrow \text{Ext}^1(V, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}) \\ \text{Ext}^1(V, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}) &\rightarrow \text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}). \end{aligned}$$

The cokernel of the first map is contained in  $\text{Ext}^2(V, \mathcal{O}_X(F) \otimes I_{Z_1})$ . To see that this group is zero, apply Serre duality: it suffices to show that  $\text{Hom}(\mathcal{O}_X(F) \otimes I_{Z_1}, V \otimes K_X) = 0$ . From the defining exact sequence for  $V$ , we have an exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Hom}(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(F) \otimes I_{Z_1} \otimes K_X) \\ &\rightarrow \text{Hom}(\mathcal{O}_X(F) \otimes I_{Z_1}, V \otimes K_X) \rightarrow \text{Hom}(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}). \end{aligned}$$

The first term is just  $H^0(K_X) = 0$  and the third is contained in  $H^0(\mathcal{O}_X(\Delta - 2F) \otimes K_X) = 0$ . Thus  $\text{Hom}(\mathcal{O}_X(F) \otimes I_{Z_1}, V \otimes K_X) = 0$ . The vanishing of the cokernel of the second map, namely  $\text{Ext}^2(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2})$ , is similar. Thus  $\text{Ext}^1(V, V) \rightarrow \text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2})$  is onto. If  $K$  is the kernel, then arguments as above show that there is an exact sequence

$$\begin{aligned} 0 &\rightarrow \text{Ext}^1(V, \mathcal{O}_X(F) \otimes I_{Z_1}) \rightarrow K \rightarrow \\ &\rightarrow \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}) \rightarrow 0. \end{aligned}$$

Here  $\text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}) = \text{Ext}^1(I_{Z_2}, I_{Z_2})$  is the tangent space to  $H_k$ . Moreover, there is an exact sequence

$$\begin{aligned} &\text{Hom}(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(F) \otimes I_{Z_1}) \rightarrow \\ &\rightarrow \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}, \mathcal{O}_X(F) \otimes I_{Z_1}) \rightarrow \\ &\rightarrow \text{Ext}^1(V, \mathcal{O}_X(F) \otimes I_{Z_1}) \rightarrow \\ &\rightarrow \text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(F) \otimes I_{Z_1}) \rightarrow 0. \end{aligned}$$

The last term is  $\text{Ext}^1(I_{Z_1}, I_{Z_1})$  which is the tangent space to  $H_{\ell_\zeta - k}$  at  $Z_1$ , and the first two terms combine to give  $\text{Ext}^1/\mathbb{C} \cdot \xi$ . Thus the kernel  $K$  looks very much like the tangent space to  $E_\zeta^{\ell_\zeta - k, k}$  at  $\xi$  and both spaces have the same dimension.

Let us describe the tangent space to  $E_\zeta^{\ell_\zeta-k,k}$  at  $\xi$  and the differential of the map  $E_\zeta^{\ell_\zeta-k,k}$  to  $\mathfrak{M}_0^{(\zeta,k)}$  in more intrinsic terms. It is easy to see that a  $\text{Spec } \mathbb{C}[\epsilon]$ -valued point of  $E_\zeta^{\ell_\zeta-k,k}$  which restricts to  $\xi$  defines two codimension two subschemes  $Z_1 \subseteq X \times \text{Spec } \mathbb{C}[\epsilon]$ ,  $Z_2 \subseteq X \times \text{Spec } \mathbb{C}[\epsilon]$ , flat over  $\text{Spec } \mathbb{C}[\epsilon]$ , restricting to  $Z_i$  over  $X$ , and an extension  $\mathcal{V}$  over  $X \times \text{Spec } \mathbb{C}[\epsilon]$  of the form

$$0 \rightarrow \pi_1^* \mathcal{O}_X(F) \otimes I_{Z_1} \otimes \rightarrow \mathcal{V} \rightarrow \pi_1^* \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \rightarrow 0.$$

Conversely such a choice of  $Z_1$ ,  $Z_2$  and  $\mathcal{V}$  define a  $\text{Spec } \mathbb{C}[\epsilon]$ -valued point of  $E_\zeta^{\ell_\zeta-k,k}$ . Thus there is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X(F) \otimes I_{Z_1} & \longrightarrow & V & \longrightarrow & \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_1^* \mathcal{O}_X(F) \otimes I_{Z_1} & \longrightarrow & \mathcal{V} & \longrightarrow & \pi_1^* \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_X(F) \otimes I_{Z_1} & \longrightarrow & V & \longrightarrow & \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Here the extension  $\mathcal{V}$  of  $V$  by  $V$ , viewed as a point of  $\text{Ext}^1(V, V)$ , corresponds to the Kodaira-Spencer map of the deformation  $\mathcal{V}$  of  $V$ . Likewise the left and right hand columns give classes in  $\text{Ext}^1(I_{Z_1}, I_{Z_1})$  and  $\text{Ext}^1(I_{Z_2}, I_{Z_2})$  corresponding to  $Z_1$  and  $Z_2$ . A straightforward diagram chase shows that if  $\mathcal{V}$  fits into this commutative diagram then the image of the extension class  $\xi \in \text{Ext}^1(V, V)$  corresponding to  $\mathcal{V}$  in  $\text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2})$  is zero. To see the converse, that every element in the kernel  $K$  of the map  $\text{Ext}^1(V, V) \rightarrow \text{Ext}^1(\mathcal{O}_X(F) \otimes I_{Z_1}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2})$  is the image of a tangent vector to  $E_\zeta^{\ell_\zeta-k,k}$  at  $\xi$ , use the arguments above which show that there is a surjection from  $K$  to

$$\text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}, \mathcal{O}_X(\Delta - F) \otimes I_{Z_2}) = \text{Ext}^1(I_{Z_2}, I_{Z_2}).$$

Thus there is an induced extension of  $\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}$  by  $\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}$ , necessarily of the form  $\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}$ , and a map from  $\mathcal{V}$  to  $\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}$

$I_{Z_2}$ , necessarily a surjection. The kernel of this surjection then defines an extension  $\mathcal{O}_X(F) \otimes I_{Z_1}$  of  $\mathcal{O}_X(F) \otimes I_{Z_1}$  by  $\mathcal{O}_X(F) \otimes I_{Z_1}$ . It follows that  $K$  is in the image of the tangent space to  $E_\zeta^{\ell_\zeta - k, k}$  at  $\xi$ . By counting dimensions the map on tangent spaces from  $T_\xi E_\zeta^{\ell_\zeta - k, k}$  to  $\text{Ext}^1(V, V)$  is injective, showing that the map from  $E_\zeta^{\ell_\zeta - k, k}$  to  $\mathfrak{M}_0^{(\zeta, k)}$  is an immersion and identifying the normal space at  $\xi$ .  $\square$

Let us continue the proof of Proposition 3.6. To give a global description of the normal bundle to  $E_\zeta^{\ell_\zeta - k, k}$  in  $\mathfrak{M}_0^{(\zeta, k)}$ , recall by standard deformation theory [10] that the pullback of the tangent bundle of  $\mathfrak{M}_0^{(\zeta, k)}$  to  $E_\zeta^{\ell_\zeta - k, k}$  is just  $\text{Ext}_{\pi_2}^1(\mathcal{V}, \mathcal{V})$ , where  $\mathcal{V}$  is the universal sheaf over  $X \times E_\zeta^{\ell_\zeta - k, k}$  described in (2.8) and  $\pi_2 : X \times E_\zeta^{\ell_\zeta - k, k} \rightarrow E_\zeta^{\ell_\zeta - k, k}$  is the second projection. Moreover the calculations above globalize to show that the normal bundle is exactly

$$\text{Ext}_{\pi_2}^1(\rho^*(\pi_1^* \mathcal{O}_X(F) \otimes \pi_{1,2}^* I_{Z_1}) \otimes \pi_2^* \mathcal{O}_{E_\zeta^{\ell_\zeta - k, k}}(1), \rho^*(\pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_2})),$$

where  $\rho : X \times E_\zeta^{\ell_\zeta - k, k} \rightarrow X \times H_{\ell_\zeta - k} \times H_k$  is the natural projection. Using standard base change results and the projection formula, we see that this sheaf is equal to

$$\rho^* \text{Ext}_{\pi_2}^1(\pi_1^* \mathcal{O}_X(F) \otimes \pi_{1,2}^* I_{Z_1}, \pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_2}) \otimes \mathcal{O}_{E_\zeta^{\ell_\zeta - k, k}}(-1),$$

which is the same as  $\rho^* \mathcal{E}_{-\zeta}^{k, \ell_\zeta - k} \otimes \mathcal{O}_{E_\zeta^{\ell_\zeta - k, k}}(-1)$ .  $\square$

Finally, to compare the moduli space  $\mathfrak{M}_0^{(\zeta, k(t+\varepsilon))}$  with  $\mathfrak{M}_0^{(\zeta, k(t-\varepsilon))}$ , where  $t$  is the unique  $\zeta$ -critical point in  $[t - \varepsilon, t + \varepsilon]$ , we shall need the following result which is a straightforward generalization of (A.2) of [11].

**Proposition 3.8.** *Let  $X$  be a smooth projective scheme or compact complex manifold, and let  $T$  be smooth. Suppose that  $\mathcal{V}$  is a rank two reflexive sheaf over  $X \times T$ , flat over  $T$ . Let  $D$  be a reduced divisor on  $T$ , not necessarily smooth and let  $i : D \rightarrow T$  be the inclusion. Suppose that  $L$  is a line bundle on  $X$  and that  $Z$  is a codimension two subscheme of  $X \times D$ , flat over  $D$ . Suppose further that  $\mathcal{V} \rightarrow i_* \pi_1^* L \otimes I_Z$  is a surjection, and let  $\mathcal{V}'$  be its kernel:*

$$0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow i_* \pi_1^* L \otimes I_Z \rightarrow 0.$$

Then there is a line bundle  $M$  on  $X$  and a subscheme  $Z'$  of  $X \times D$  codimension at least two, flat over  $D$ , with the following properties:

- (i)  $\mathcal{V}'$  is reflexive and flat over  $T$ .
- (ii) There are exact sequences

$$\begin{aligned} 0 &\rightarrow \pi_1^* M \otimes I_{Z'} \rightarrow \mathcal{V}|_D \rightarrow \pi_1^* L \otimes I_Z \rightarrow 0; \\ 0 &\rightarrow \pi_1^* L \otimes I_Z \otimes \mathcal{O}_D(-D) \rightarrow \mathcal{V}'|_D \rightarrow \pi_1^* M \otimes I_{Z'} \rightarrow 0, \end{aligned}$$

which restrict for each  $t \in D$  to give exact sequences

$$\begin{aligned} 0 &\rightarrow M \otimes I_{Z'} \rightarrow V_t \rightarrow L \otimes I_Z \rightarrow 0; \\ 0 &\rightarrow L \otimes I_Z \rightarrow (V_t)' \rightarrow M \otimes I_{Z'} \rightarrow 0. \end{aligned}$$

Here  $Z$  is the subscheme of  $X$  defined by  $\mathcal{Z}$  for the slice  $X \times \{t\}$  and  $Z'$  is likewise defined by  $\mathcal{Z}'$ .

- (iii) If  $D$  is smooth, then the extension class corresponding to  $(V_t)'$  in  $\text{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$  is defined by the image of the normal vector to  $D$  at  $t$  under the composition of the Kodaira-Spencer map from the tangent space of  $T$  at  $t$  to  $\text{Ext}^1(V_t, V_t)$ , followed by the natural map  $\text{Ext}^1(V_t, V_t) \rightarrow \text{Ext}^1(M \otimes I_{Z'}, L \otimes I_Z)$ .

Here  $\mathcal{V}'$  is called the *elementary modification* of  $\mathcal{V}$  along  $D$ . This construction has the following symmetry: if we make the elementary modification of  $\mathcal{V}'$  along  $D$  corresponding to the surjection  $\mathcal{V}' \rightarrow i_*(\pi_1^* M \otimes I_{Z'})$ , then the result is  $\mathcal{V} \otimes \mathcal{O}_{X \times T}(-(X \times D))$ .

Here is the typical way that we will apply the above: given  $X$ , let  $M$  be a smooth manifold and  $Y$  a submanifold of  $M$ . Let  $T$  be the blowup of  $M$  along  $Y$  and let  $D$  be the exceptional divisor. Let  $\pi : T \rightarrow M$  be the natural map. Then, given  $\xi \in D$ , the image in the normal space to  $\pi(\xi)$  of the normal direction at  $\xi$  to  $D$  under  $\pi_*$  may be identified with the line in the normal space corresponding to  $\xi$ .

We can now state the main result as follows:

**Theorem 3.9.** *Suppose that  $t$  is the unique  $\zeta$ -critical point in  $[t - \varepsilon, t + \varepsilon]$ . If  $h(\pm\zeta_i) + \ell_{\pm\zeta_i} \neq 0$  for every  $i$ , then the rational map  $\mathfrak{M}_0^{(\zeta, \mathbf{k}(t+\varepsilon))} \dashrightarrow \mathfrak{M}_0^{(\zeta, \mathbf{k}(t-\varepsilon))}$  is obtained as follows. For every  $i$ , fixing  $\zeta_i = \zeta$  and  $k_i(t + \varepsilon) = k$ , blow up  $E_\zeta^{\ell_\zeta - k, k}$  in  $\mathfrak{M}_0^{(\zeta, \mathbf{k}(t+\varepsilon))}$ . Then the exceptional divisor  $D$  is a  $\mathbb{P}^{N_\zeta} \times \mathbb{P}^{N-\zeta}$ -bundle over  $\text{Hilb}^{\ell_\zeta - k} X \times \text{Hilb}^k X$ . Moreover this divisor can*

be contracted in two different ways. Contracting the  $\mathbb{P}^{N-\zeta}$  fibers for all possible  $\zeta$  gives  $\mathfrak{M}_0^{(\zeta, k(t+\varepsilon))}$ . Contracting the  $\mathbb{P}^{N\zeta}$  fibers for all possible  $\zeta$  gives  $\mathfrak{M}_0^{(\zeta, k(t-\varepsilon))}$ . Moreover the morphism from the blowup to  $\mathfrak{M}_0^{(\zeta, k(t-\varepsilon))}$  is induced by an elementary modification as in (3.8), and the image of the the component of the exceptional divisor which is the blowup of  $E_\zeta^{\ell_\zeta-k, k}$  is  $E_{-\zeta}^{k, \ell_\zeta-k}$ . Finally the construction is symmetric.

Similar statements hold if  $h(\pm\zeta_i) + \ell_{\pm\zeta_i} = 0$  for some  $i$ , where we must also add in or delete an extra component coming from  $\pm\zeta_i$ .

*Proof.* Begin by blowing up  $E_\zeta^{\ell_\zeta-k, k}$  in  $\mathfrak{M}_0^{(\zeta, k(t+\varepsilon))}$  for all possible  $\zeta$ . For simplicity we shall just write down the argument in case there is only one  $\zeta$ ; the general case is just additional notation. Let  $\widetilde{\mathfrak{M}}_0^{(\zeta, k(t+\varepsilon))}$  denote the blowup and  $D$  the exceptional divisor. Note that the normal bundle  $\mathcal{N}_\zeta^{\ell_\zeta-k, k}$  to  $E_\zeta^{\ell_\zeta-k, k}$  in  $\mathfrak{M}_0^{(\zeta, k(t+\varepsilon))}$  is  $\rho^* \mathcal{E}_{-\zeta}^{k, \ell_\zeta-k} \otimes \mathcal{O}_{E_\zeta^{\ell_\zeta-k, k}}(-1)$ , where  $\rho : E_\zeta^{\ell_\zeta-k, k} \rightarrow \text{Hilb}^{\ell_\zeta-k} X \times \text{Hilb}^k X$  is the projection. In particular  $\mathcal{N}_\zeta^{\ell_\zeta-k, k}$  restricts to each fiber  $\mathbb{P}^{N\zeta}$  to a bundle of the form  $\left[ \mathcal{O}_{\mathbb{P}^{N\zeta}}^N \right] \otimes \mathcal{O}_{\mathbb{P}^{N\zeta}}(-1)$ , and an easy calculation using (2.7) shows that  $N = N_{-\zeta} + 1$ . It follows that the fibers of the induced map from  $D$  to  $\text{Hilb}^{\ell_\zeta-k} X \times \text{Hilb}^k X$  are naturally  $\mathbb{P}^{N\zeta} \times \mathbb{P}^{N-\zeta}$ . Moreover it is easy to see that  $\mathcal{O}(D)|_{\mathbb{P}^{N\zeta}} = \mathcal{O}_{\mathbb{P}^{N\zeta}}(-1)$ , using for example the fact that  $\mathcal{O}(D)|_{\mathbb{P}^{N\zeta} \times \mathbb{P}^{N-\zeta}} = \mathcal{O}(a, -1)$  for some  $a$  and the fact that

$$\mathcal{N}_\zeta^{\ell_\zeta-k, k}|_{\mathbb{P}^{N\zeta}} = R^0 \pi_{1*} [\mathcal{O}(-D)|_{\mathbb{P}^{N\zeta} \times \mathbb{P}^{N-\zeta}}] = \left[ \mathcal{O}_{\mathbb{P}^{N\zeta}}^{N-\zeta+1} \right] \otimes \mathcal{O}_{\mathbb{P}^{N\zeta}}(-a).$$

For the rest of the argument, we assume that there exists a universal family on  $X \times \mathfrak{M}_0^{(\zeta, k(t+\varepsilon))}$ . Of course, such a family will only exist locally in the classical or étale topology, but this will suffice for the argument. Let  $\mathcal{U}$  be the pullback of the universal family to  $X \times \widetilde{\mathfrak{M}}_0^{(\zeta, k(t+\varepsilon))}$ . Locally again we may assume that the restriction of  $\mathcal{U}$  to  $X \times D$  is the pullback of the universal extension  $\mathcal{V}$  of (2.8):

$$\begin{aligned} 0 &\rightarrow \rho^* \left( \pi_1^* \mathcal{O}_X(F) \otimes \pi_{1,2}^* I_{Z_{n-}} \right) \otimes \pi_2^* \mathcal{O}_{E_\zeta^{n-, n+}}(1) \\ &\rightarrow \mathcal{V} \rightarrow \rho^* \left( \pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_{n+}} \right) \rightarrow 0. \end{aligned}$$

Now consider the effect of making an elementary transformation of  $\mathcal{U}$  on  $X \times \widetilde{\mathfrak{M}}_0^{(\zeta, k(t+\varepsilon))}$  along the divisor  $D$ , using the morphism from  $\mathcal{U}$  to the pullback of  $\rho^* \left( \pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{Z_k} \right)$  given by considering the pullback

of the universal extension. Applying (3.8) to the elementary transformation  $\mathcal{U}'$ , we see that the fiber of  $\mathcal{U}'$  at a point of the fiber  $\mathbb{P}^{N_\zeta} \times \mathbb{P}^{N-\zeta}$  lying over a point  $(Z_1, Z_2) \in \text{Hilb}^{\ell_\zeta-k} X \times \text{Hilb}^k X$  is given by a nonsplit extension of the form

$$0 \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \rightarrow U \rightarrow \mathcal{O}_X(F) \otimes I_{Z_1} \rightarrow 0.$$

Moreover the extension class corresponding to  $U$  is given by the projectivized normal vector in  $\mathbb{P}^{N-\zeta}$ . Thus it is independent of the first factor  $\mathbb{P}^{N_\zeta}$  and the set of all possible such classes is parametrized by the second factor  $\mathbb{P}^{N-\zeta}$ . There is then an induced morphism from  $\widetilde{\mathfrak{M}}_0^{(\zeta, k(t+\varepsilon))}$  to  $\mathfrak{M}_0^{(\zeta, k(t-\varepsilon))}$  and clearly it has the effect of contracting  $D$  along its first ruling and has the property that the image of  $D$  is exactly  $E_{-\zeta}^{k, \ell_\zeta-k}$ . We leave the symmetry of the construction to the reader. This concludes the proof of (3.9).  $\square$

**Remark 3.10.** In the  $K3$  or abelian case, the arguments of this section show that the rational map  $\mathfrak{M}_0^{(\zeta, k(t+\varepsilon))} \dashrightarrow \mathfrak{M}_0^{(\zeta, k(t-\varepsilon))}$  is a Mukai elementary transformation [27, 29].

We can also use (3.8) to analyze the rational map from  $E_\zeta^{n-, n+}$  to  $\mathfrak{M}_-$ , in the case where it is not a morphism. For simplicity we shall only consider the case of  $E_\zeta^{1,0}$ , i.e.  $\ell_\zeta = 1$ . In this case  $Z_- = p \in X$  and  $I_{Z_-} = \mathfrak{m}_p$  is the maximal ideal sheaf of  $p$ . Moreover  $\text{Ext}^1(\mathcal{O}_X(\Delta - F), \mathcal{O}_X(F) \otimes \mathfrak{m}_p) = H^1(\mathcal{O}_X(2F - \Delta) \otimes \mathfrak{m}_p)$ . There is an exact sequence

$$0 \rightarrow H^0(\mathbb{C}_p) \rightarrow H^1(\mathcal{O}_X(2F - \Delta) \otimes \mathfrak{m}_p) \rightarrow H^1(\mathcal{O}_X(2F - \Delta)) \rightarrow 0.$$

Moreover, for  $p$  fixed, the extensions  $V$  corresponding to a split extension for  $V^{\vee\vee}$  are exactly the kernel of the map from  $H^1(\mathcal{O}_X(2F - \Delta) \otimes \mathfrak{m}_p)$  to  $H^1(\mathcal{O}_X(2F - \Delta))$ , i.e. the image of  $H^0(\mathbb{C}_p)$ . The normal space is thus identified with  $H^1(\mathcal{O}_X(2F - \Delta))$ . Now if the extension for  $V^{\vee\vee}$  is split, then there is a map  $\mathcal{O}_X(\Delta - F) \otimes \mathfrak{m}_p \rightarrow V$  with quotient  $\mathcal{O}_X(F)$ . This way of realizing  $V$  as an extension gives a surjection  $\text{Ext}^1(V, V) \rightarrow \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes \mathfrak{m}_p, \mathcal{O}_X(F))$ , and we must look at the image of the normal space  $H^1(\mathcal{O}_X(2F - \Delta))$  in this extension group. On the other hand, we have an exact sequence

$$0 \rightarrow H^1(\mathcal{O}_X(2F - \Delta)) \rightarrow \text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes \mathfrak{m}_p, \mathcal{O}_X(F)) \rightarrow H^0(\mathbb{C}_p) \rightarrow 0$$

coming from the long exact Ext sequence, and it is an easy diagram chase to see that the induced map  $\text{Ext}^1(\mathcal{O}_X(\Delta - F), \mathcal{O}_X(F) \otimes \mathfrak{m}_p) \rightarrow \text{Ext}^1(\mathcal{O}_X(\Delta -$



$F) \otimes \mathfrak{m}_p, \mathcal{O}_X(F)$ ) factors through the map  $\text{Ext}^1(\mathcal{O}_X(\Delta - F), \mathcal{O}_X(F) \otimes \mathfrak{m}_p) \rightarrow H^1(\mathcal{O}_X(2F - \Delta))$  and that the image is exactly the natural subgroup  $H^1(\mathcal{O}_X(2F - \Delta))$  of  $\text{Ext}^1(\mathcal{O}_X(\Delta - F) \otimes \mathfrak{m}_p, \mathcal{O}_X(F))$ .

The above has the following geometric interpretation: the locus  $U$  in  $E_\zeta^{1,0}$  of  $L_-$ -unstable sheaves is in fact a section of  $E_\zeta^{1,0}$ . If we blow up this section and then make the elementary transformation, the result is exactly the set of elements of  $E_\zeta^{0,1}$  corresponding to nonlocally free sheaves. This set is already a divisor in  $E_\zeta^{0,1}$ . There is thus a morphism from the blowup of  $E_\zeta^{1,0}$  along  $U$  to  $\mathfrak{M}_-$  which is an embedding into  $\mathfrak{M}_-$ . Its image  $(E_\zeta^{1,0})'$  in  $\mathfrak{M}_-$  meets  $E_\zeta^{0,1}$  exactly along the divisor in  $E_\zeta^{0,1}$  of nonlocally free sheaves.

We can now give a picture of the birational map from  $\mathfrak{M}_-$  to  $\mathfrak{M}_+$  in this case. Begin with the subvariety  $E_\zeta^{0,1}$  in  $\mathfrak{M}_-$  and blow it up. Let  $D^{0,1}$  be the exceptional divisor, ruled in two different ways. As  $E_\zeta^{0,1}$  meets  $(E_\zeta^{1,0})'$  along a divisor, the proper transform of  $(E_\zeta^{1,0})'$  in the blowup is again  $(E_\zeta^{1,0})'$ . Making the elementary modification along  $D^{0,1}$ , we then blow down  $D^{0,1}$  to get a new moduli space. This moduli space then contains  $E_\zeta^{1,0}$ . At this point we can then blow up  $E_\zeta^{1,0}$  and contract the new exceptional divisor  $D^{1,0}$  to obtain  $\mathfrak{M}_+$  (a few extra details need to be checked here concerning the Kodaira-Spencer class). Note again the symmetry of the situation. In principle we could hope to carry through this analysis to the case where  $\ell_\zeta > 1$  as well, but we run into trouble with the birational geometry of  $\text{Hilb}^n X$ . Somehow the construction of our auxiliary sequence of moduli spaces has eliminated the necessity for understanding this birational geometry in detail.

#### 4. Mixed stability and mixed moduli spaces.

Our goal in this section is to give a proof of Theorem 3.5 (for an arbitrary algebraic surface  $X$ ). By way of motivation for our construction, let us analyze Gieseker semistability more closely. In the notation of the last section, we suppose that  $L_0$  is an ample line bundle lying on a unique wall  $W$  of type  $(w, p)$ , and let  $\zeta_1, \dots, \zeta_n$  be the integral classes of type  $(w, p)$  defining  $W$ . Let  $V$  be an  $L_0$ -semistable rank two sheaf. Thus either  $V$  is Mumford  $L_0$ -stable or it is Mumford strictly semistable. In the second case, let  $\mathcal{O}_X(F) \otimes I_{Z_1}$  be a destabilizing subsheaf and suppose that there is an exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \otimes I_{Z_1} \rightarrow V \rightarrow \mathcal{O}_X(\Delta - F) \otimes I_{Z_2} \rightarrow 0.$$

Let  $\zeta = 2F - \Delta$ . We shall assume that  $\zeta = \zeta_i$  for some  $i$ , or equivalently that  $\zeta$  is not numerically equivalent to zero (i.e.,  $V$  is not universally semistable). By assumption  $\mu_{L_0}(V) \geq \mu_{L_0}(\mathcal{O}_X(F) \otimes I_{Z_1})$ , and so  $\chi(V) \geq 2\chi(\mathcal{O}_X(F) \otimes I_{Z_1})$ . Since  $\chi(V) = \chi(\mathcal{O}_X(F) \otimes I_{Z_1}) + \chi(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2})$ , we may rewrite this last condition as

$$\chi(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}) - \chi(\mathcal{O}_X(F) \otimes I_{Z_1}) \geq 0.$$

Now from the exact sequence

$$0 \rightarrow \mathcal{O}_X(F) \otimes I_{Z_1} \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_{Z_1} \rightarrow 0,$$

we see that  $\chi(\mathcal{O}_X(F) \otimes I_{Z_1}) = \chi(\mathcal{O}_X(F)) - \ell(Z_1)$ , and similarly  $\chi(\mathcal{O}_X(\Delta - F) \otimes I_{Z_2}) = \chi(\mathcal{O}_X(\Delta - F)) - \ell(Z_2)$ . By Riemann-Roch,

$$\begin{aligned} \chi(\mathcal{O}_X(\Delta - F)) - \chi(\mathcal{O}_X(F)) &= \frac{1}{2}((\Delta - F)^2 - (\Delta - F) \cdot K_X - F^2 + F \cdot K_X) \\ &= \frac{1}{2}(\Delta^2 - 2\Delta \cdot F + \zeta \cdot K_X) \\ &= \frac{1}{2}\zeta \cdot (K_X - \Delta) = t. \end{aligned}$$

Thus we have the following conditions on  $Z_1$  and  $Z_2$ :

$$\begin{aligned} \ell(Z_2) - \ell(Z_1) &\leq t; \\ \ell(Z_2) + \ell(Z_1) &= \ell_\zeta, \end{aligned}$$

and so  $2\ell(Z_2) \leq \ell_\zeta + t$ . Setting  $k = \left\lfloor \frac{\ell_\zeta + t}{2} \right\rfloor$ , we have  $\ell(Z_2) \leq k$ . Applying a similar analysis to a subsheaf of the form  $\mathcal{O}_X(\Delta - F) \otimes I_{Z_1}$  shows that, if there is such a subsheaf, with a torsion free quotient  $\mathcal{O}_X(F) \otimes I_{Z_2}$ , then

$$\ell(Z_2) \leq \frac{\ell_\zeta - t}{2} = \ell_\zeta - \frac{\ell_\zeta + t}{2}.$$

In particular, if  $\frac{\ell_\zeta + t}{2}$  is not an integer, then this condition becomes  $\ell(Z_2) \leq \ell_\zeta - k - 1$ . Thus, provided  $\frac{\ell_\zeta + t}{2}$  is not an integer for every  $\zeta$  defining the wall  $W$  (i.e.  $t$  is not  $\zeta$ -critical for every  $\zeta$ ),  $V$  is  $(L_0, \zeta, k)$ -semistable for  $k = \left\lfloor \frac{\ell_\zeta + t}{2} \right\rfloor$  and indeed  $V$  is  $(L_0, \zeta, \mathbf{k})$ -semistable, where  $\mathbf{k}$  is defined in the obvious way. Conversely, assuming that  $t$  is not  $\zeta$ -critical for every  $\zeta$ ,

$V$  is Gieseker  $L_0$ -semistable, indeed Gieseker  $L_0$ -stable, if it is  $(L_0, \zeta, \mathbf{k})$ -semistable for  $\mathbf{k}$  as above.

We would like to produce a similar condition where  $t$  is allowed to be any rational number which is not  $\zeta$ -critical. One way to think of this problem is to consider the analogous problem where we replace  $\Delta$  by  $\Delta + 2\Xi$  and make the corresponding change in  $c$ , so that  $\Delta$  and  $p$  remain the same. This corresponds to twisting  $V$  by  $\mathcal{O}_X(\Xi)$ ; and  $t$  is replaced by  $t - \zeta \cdot \Xi$ . In particular, we see that the notion of Gieseker stability is rather sensitive to twisting by a line bundle. Moreover if  $W$  is defined by exactly one  $\zeta$  such that there exists a divisor  $\Xi$  with  $\zeta \cdot \Xi = 1$ , for example if  $\zeta$  is primitive and  $p_g(X) = 0$ , it is easy to see that we can construct the appropriate moduli spaces as Gieseker moduli spaces corresponding to twists of  $V$  by various multiples of  $\Xi$ . In general however we will need to consider a problem which is roughly analogous to allowing twists of  $V$  by a  $\mathbb{Q}$ -divisor  $\Xi$ . This is the goal of the following definition of mixed stability:

**Definition 4.1.** Let  $X$  be an algebraic surface and let  $L_0$  be an ample line bundle on  $X$ . Fix line bundles  $H_1$  and  $H_2$  on  $X$  and positive integers  $a_1$  and  $a_2$ . For every torsion free sheaf  $V$  on  $X$  of rank  $r$ , define

$$p_{V;H_1,H_2,a_1,a_2}(n) = \frac{a_1}{r} \chi(V \otimes H_1 \otimes L_0^n) + \frac{a_2}{r} \chi(V \otimes H_2 \otimes L_0^n).$$

A torsion free sheaf  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable if, for all subsheaves  $W$  of  $V$  with  $0 < \text{rank } W < \text{rank } V$  and for all  $n \gg 0$ ,

$$p_{V;H_1,H_2,a_1,a_2}(n) > p_{W;H_1,H_2,a_1,a_2}(n).$$

$(H_1, H_2, a_1, a_2)$   $L_0$ -semistable and unstable are defined similarly.

The usual arguments show the following:

**Lemma 4.2.** *If  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable, then it is simple.*

In the case of rank two on a surface  $X$  (which is the only case which shall concern us),  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable if and only if, for all rank one subsheaves  $W$ , and for all  $n \gg 0$ , we have

$$a_1(\chi(V \otimes H_1 \otimes L_0^n) - 2\chi(W \otimes H_1 \otimes L_0^n)) + a_2(\chi(V \otimes H_2 \otimes L_0^n) - 2\chi(W \otimes H_2 \otimes L_0^n)) > 0.$$

In particular, if  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable then either  $V \otimes H_1$  or  $V \otimes H_2$  is stable, and a similar statement holds for semistability. A short calculation

shows that the coefficient of  $n$  in the above expression (which is a degree two polynomial in  $n$ ) is  $(a_1 + a_2)(L_0 \cdot (c_1(V) - 2F))$  where  $F = c_1(W)$  and that the constant term is

$$(a_1 + a_2)(\chi(V) - 2\chi(W)) + a_1 H_1 \cdot (c_1(V) - 2F) + a_2 H_2 \cdot (c_1(V) - 2F).$$

Thus  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable (resp. semistable) if and only if it is either Mumford  $L_0$ -stable or Mumford strictly semistable and the above constant term is positive (resp. nonnegative). It is easy to see, comparing this with the discussion at the beginning of this section, that formally this is the same as requiring that  $V \otimes \Xi$  is (Gieseker)  $L_0$ -stable or semistable, where  $\Xi$  is the  $\mathbb{Q}$ -divisor

$$\frac{a_1}{a_1 + a_2} H_1 + \frac{a_2}{a_1 + a_2} H_2.$$

Thus for example taking  $H_2 = 0$  and replacing  $H_1$  by a positive integer multiple we see that we can take for  $\Xi$  an arbitrary  $\mathbb{Q}$ -divisor.

Let us explicitly relate mixed stability to our previous notion of  $(L_0, \zeta, \mathbf{k})$ -semistability:

**Lemma 4.3.** *Given  $\Delta$  and  $c$  and the corresponding  $w$  and  $p$ , let  $L_0$  be an ample divisor lying on a unique wall of type  $(w, p)$  and let  $V$  be a rank two torsion free sheaf with  $c_1(V) = \Delta$  and  $c_2(V) = c$ . Let  $\Xi$  be the  $\mathbb{Q}$ -divisor  $\frac{a_1}{a_1 + a_2} H_1 + \frac{a_2}{a_1 + a_2} H_2$  and suppose that the rational number  $t_i = \frac{1}{2} \zeta_i \cdot (K_X - \Delta) - \zeta_i \cdot \Xi$  is not  $\zeta_i$ -critical for every  $\zeta_i$  of type  $(w, p)$  defining  $W$ . Then, with  $t = t_1$ ,  $V$  is  $(L_0, \zeta, \mathbf{k}(t))$ -semistable if and only if it is  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable if and only if it is  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable.*

*Proof.* Using the additivity of the polynomials  $p_{V; H_1, H_2, a_1, a_2}$  over exact sequences, it is easy to check that  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable if and only if it is Mumford  $L_0$ -semistable, and for every Mumford destabilizing subsheaf of the form  $\mathcal{O}_X(F) \otimes I_{Z_1}$ , either  $V$  is universally semistable or we have

$$\chi(V) - 2\chi(\mathcal{O}_X(F) \otimes I_{Z_1}) - \zeta_i \cdot \Xi > 0,$$

where  $\zeta_i = 2F - \Delta$ . Using our calculations above, this works out to

$$\ell(Z_2) - \ell(Z_1) \leq \frac{1}{2} \zeta_i \cdot (K_X - \Delta) - \zeta_i \cdot \Xi = t_i.$$

Equivalently since  $\ell(Z_1) + \ell(Z_2) = \ell_{\zeta_i}$ , this becomes  $\ell(Z_2) \leq \left\lfloor \frac{\ell_{\zeta_i} + t_i}{2} \right\rfloor$ . Thus  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable if and only if it is  $(L_0, \zeta, \mathbf{k}(t))$ -semistable. Moreover, since  $t$  is not  $\zeta_i$ -critical, the inequalities are automatically strict, so that  $V$  is also  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable.  $\square$

Now choosing a  $\Xi_0$  such that  $\zeta_1 \cdot \Xi_0 \neq 0$ , every rational number  $t$  is of the form  $\frac{1}{2}\zeta_1 \cdot (K_X - \Delta) - \zeta_1 \cdot r\Xi_0$  for some rational number  $r$ . Thus Theorem 3.5 will follow from Lemma 4.3 and from the more general result below:

**Theorem 4.4.** *Let  $X$  be an algebraic surface  $X$  and let  $L_0$  be an ample line bundle on  $X$ . Given a divisor  $\Delta$  and an integer  $c$ , line bundles  $H_1$  and  $H_2$  on  $X$  and positive integers  $a_1$  and  $a_2$ , suppose that every rank two torsion free sheaf  $V$  with  $c_1(V) = \Delta$ ,  $c_2(V) = c$  which is  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable is actually  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable. Then there exists a projective coarse moduli space  $\mathfrak{M}_{L_0}(\Delta, c; H_1, H_2, a_1, a_2)$  of isomorphism classes of rank two torsion free sheaves  $V$  with  $c_1(V) = \Delta$ ,  $c_2(V) = c$ , which are  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable.*

*Proof.* The argument will follow the arguments in [13] as closely as possible, and we shall assume a familiarity with that paper.

Suppose that  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable. Then either  $V \otimes H_1$  or  $V \otimes H_2$  is  $L_0$ -semistable, and thus by [13], Lemma 1.3 the set of all such  $V$  is bounded. We may thus choose an  $n$  such that, for all  $V$  which are  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable,  $V \otimes H_i \otimes L_0^n$  is generated by its global sections and has no higher cohomology, for  $i = 1, 2$ . Fix such an  $n$  for the moment, and let  $d_i = h^0(V \otimes H_i \otimes L_0^n)$ . Then  $d_i$  is independent of  $V$  and  $V$  is a quotient of  $(H_i^{-1} \otimes L_0^{-n})^{\oplus d_i}$ . Let  $Q_i$  be the open subset of the corresponding Quot scheme associated to  $(H_i^{-1} \otimes L_0^{-n})^{\oplus d_i}$  consisting of quotients which are rank two torsion free sheaves  $V_i$  with  $c_1(V_i) = \Delta$  and  $c_2(V_i) = c$ , and such that  $V_i \otimes H_i \otimes L_0^n$  is generated by its global sections and has no higher cohomology. We will write a point of  $Q_i$  as  $V_i$ , suppressing the surjection  $(H_i^{-1} \otimes L_0^{-n})^{\oplus d_i} \rightarrow V_i$ . Inside  $Q_1 \times Q_2$ , we have the closed subscheme  $I_0$  consisting of quotients  $V_1$  and  $V_2$  such that  $\dim \text{Hom}(V_1, V_2) \geq 1$ . There is also the open subvariety  $I'_0$  of  $I_0$  consisting of  $(V_1, V_2)$  with  $\dim \text{Hom}(V_1, V_2) = 1$ . Using the universal sheaves  $\mathcal{U}_i$  over  $X \times Q_i$ , we can construct a  $\mathbb{C}^*$  bundle  $I$  over  $I'_0$  whose points are  $(V_1, V_2, \varphi)$ , where  $\varphi : V_1 \rightarrow V_2$  is a nonzero homomorphism, unique up to scalars.

For  $i = 1, 2$ , let  $E_i$  be a fixed vector space of dimension equal to  $d_i = h^0(V \otimes H_i \otimes L_0^n)$ . Fix once and for all an isomorphism  $(H_i^{-1} \otimes L_0^{-n})^{\oplus d_i} \cong$

$(H_i^{-1} \otimes L_0^{-n}) \otimes E_i$ . A surjection  $(H_i^{-1} \otimes L_0^{-n})^{\oplus d_i} \rightarrow V_i$  then gives a map  $E_i \rightarrow H^0(V_i \otimes H_i \otimes L_0^n)$  and via such a surjection a basis  $v_1, \dots, v_{d_1}$  of  $E_1$  gives  $d_1$  sections of  $V_1 \otimes H_1 \otimes L_0^n$  and similarly for a basis  $w_1, \dots, w_{d_2}$  of  $E_2$ . Moreover  $GL(d_i)$  acts on  $(H_i^{-1} \otimes L_0^{-n})^{\oplus d_i}$  and on  $Q_i$ . By the universal property of the Quot scheme, this action extends to a  $GL(d_i)$ -linearization of the universal sheaf  $\mathcal{U}_i$  over  $X \times Q_i$ . Thus there is a right action of  $GL(d_1) \times GL(d_2)$  on  $I$ , and it is easy to see that the elements  $(\lambda \text{Id}, \lambda \text{Id})$  act trivially. Let  $F_i$  be the fixed vector space  $H^0(\Delta \otimes H_i^2 \otimes L_0^{2n})$ , and  $F$  the fixed vector space  $H^0(\Delta \otimes H_1 \otimes H_2 \otimes L_0^{2n})$ . Let

$$U = \text{Hom}\left(\bigwedge^2 E_1, F_1\right) \oplus \text{Hom}\left(\bigwedge^2 E_2, F_2\right) \oplus \text{Hom}(E_1 \otimes E_2, F).$$

(The factor  $\text{Hom}(E_1 \otimes E_2, F)$  is there to make sure that the destabilizing subsheaves for  $V \otimes H_1$  and  $V \otimes H_2$  are in fact the same.) Note that  $GL(d_1) \times GL(d_2)$  operates on the right on  $U$  and  $\mathbb{P}U$ . For example, the pair  $(\lambda \text{Id}, \mu \text{Id})$  acts on the triple  $(T_1, T_2, T) \in U$  via  $(T_1, T_2, T) \mapsto (\lambda^2 T_1, \mu^2 T_2, \lambda \mu T)$ . Thus  $(A_1, A_2)$  acts trivially on  $\mathbb{P}U$  if and only if  $(A_1, A_2) = (\lambda \text{Id}, \lambda \text{Id})$ . Given a quintuple  $\underline{V} = (V_1, V_2, \psi_1, \psi_2, \varphi)$ , where  $V_i \in Q_i$ ,  $\psi_i : E_i \rightarrow H^0(V_i \otimes H_i \otimes L_0^n)$  is an isomorphism, and  $\varphi : V_1 \rightarrow V_2$  is a nonzero map, we will define a point  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V})) \in \mathbb{P}U$ . To do so, fix an isomorphism  $\alpha_2 : \det V_2 \rightarrow \mathcal{O}_X(\Delta)$ , and set  $\alpha_1 = \alpha_2 \circ \det \varphi$ . (Thus  $\alpha_1 = 0$  if  $\varphi$  is not an isomorphism.) Given  $v, v' \in E_1$  and  $w, w' \in E_2$ , identify  $v, v'$  with their images in  $H^0(V_i \otimes H_1 \otimes L_0^n)$  and similarly for  $w, w'$ , and let

$$T_1(\underline{V})(v \wedge v') = \alpha_1(v \wedge v') = \alpha_2 \circ \det \varphi(v \wedge v') \in H^0(\Delta \otimes H_1^2 \otimes L_0^{2n});$$

$$T_2(\underline{V})(w \wedge w') = \alpha_2(w \wedge w') \in H^0(\Delta \otimes H_2^2 \otimes L_0^{2n});$$

$$T(\underline{V})(v \otimes w) = \alpha_2(\varphi(v) \wedge w) \in H^0(\Delta \otimes H_1 \otimes H_2 \otimes L_0^{2n}).$$

Changing  $\alpha_2$  by a nonzero scalar  $\lambda$  multiplies  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  by  $\lambda$ , so that the induced element of  $\mathbb{P}U$  is well defined. Similarly, if we replace  $\varphi$  by  $\lambda \varphi$ , then  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is replaced by  $(\lambda^2 T_1(\underline{V}), T_2(\underline{V}), \lambda T(\underline{V}))$ . It is easy to check that the map  $\underline{V} \mapsto T(\underline{V})$  induces a morphism from  $I$  to  $\mathbb{P}U$  which is  $GL(d_1) \times GL(d_2)$ -equivariant. Further note that we can define  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  more generally if we are given the data  $\underline{V}$  of two rank two torsion free sheaves  $V_1$  and  $V_2$  with  $\det V_i = \Delta$ , a morphism  $\varphi : V_1 \rightarrow V_2$ , and linear maps  $\psi_i : E_i \rightarrow H^0(V_i \otimes H_i \otimes L_0^n)$ , not necessarily isomorphisms, although it is possible for  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  to be zero in this case.

We have not yet introduced the extra parameters  $a_1$  and  $a_2$ . To do so, define  $G(a_1, a_2) \subset GL(d_1) \times GL(d_2)$  as follows:

$$G(a_1, a_2) = \{ (A_1, A_2) \mid \det A_1^{a_1} \det A_2^{a_2} = \text{Id} \}.$$

Thus unlike Thaddeus we don't change the polarization or the linearization but the actual group which we use to determine stability; still our construction could probably be interpreted in his general framework. Fixing  $a_1$  and  $a_2$  for the rest of the discussion, we shall denote  $G(a_1, a_2)$  by  $G$ . Since  $a_1$  and  $a_2$  are positive, the matrix  $(\lambda \text{Id}, \lambda \text{Id})$  lies in  $G$  if and only if  $\lambda$  is an  $m^{\text{th}}$  root of unity, where  $m = a_1 d_1 + a_2 d_2$ . Thus a quotient of  $G$  by a finite group acts faithfully on  $\mathbb{P}U$ . Moreover, the problem of finding a good quotient of  $\mathbb{P}U$  (for an appropriate open subset of  $\mathbb{P}U$ ) for  $G$  is the same as that of finding a good quotient of  $\mathbb{P}U$  for  $GL(d_1) \times GL(d_2)$ , since

$$G \cdot \mathbb{C}^*(\text{Id}, \text{Id}) = GL(d_1) \times GL(d_2).$$

This last statement follows since  $G$  clearly contains  $SL(d_1) \times SL(d_2)$  and since  $\mathbb{C}^* \times \mathbb{C}^*$  is generated by its diagonal subgroup and by the subgroup

$$\{(\lambda, \mu) : \lambda^{a_1 d_1} \mu^{a_2 d_2} = 1\}.$$

We may thus apply the general machinery of GIT to the group  $G$  acting on  $\mathbb{P}U$ . A one parameter subgroup of  $G$  is given by a basis  $\{v_i\}$  of  $E_1$ , a basis  $\{w_k\}$  of  $E_2$  and weights  $n_i, m_k \in \mathbb{Z}$ , such that  $v_i^\lambda = \lambda^{n_i} v_i$ ,  $w_k^\lambda = \lambda^{m_k} w_k$ , and

$$a_1 \sum_i n_i + a_2 \sum_k m_k = 0.$$

We shall always arrange our choice of basis so that  $n_1 \leq n_2 \leq \dots \leq n_{d_1}$  and  $m_1 \leq m_2 \leq \dots \leq m_{d_2}$ . Given  $(T_1, T_2, T) \in U$  and a one parameter subgroup of  $G$  as above, we see that  $\lim_{\lambda \rightarrow 0} (T_1, T_2, T)^\lambda = 0$  if and only if  $T_1(v_i \wedge v_j) = 0$  for every pair of indices  $i, j$  such that  $n_i + n_j \leq 0$ ,  $T_2(w_k \wedge w_\ell) = 0$  for every pair of indices  $k, \ell$  such that  $m_k + m_\ell \leq 0$ , and  $T(v_i \otimes w_j) = 0$  for every pair  $i, k$  such that  $n_i + m_k \leq 0$ . Likewise the condition that  $\lim_{\lambda \rightarrow 0} (T_1, T_2, T)^\lambda$  exists is similar, replacing the  $\leq$  by strict inequality. Finally note that if  $n_i + n_j \leq 0$ , then  $n_1 + n_j \leq 0$ , if  $m_k + m_\ell \leq 0$  then  $m_1 + m_\ell \leq 0$ , and if  $n_i + m_k \leq 0$  then  $n_1 + m_k \leq 0$  and  $n_i + m_1 \leq 0$ .

We then have the following:

**Lemma 4.5.** (i) *Suppose that we are given the data  $\underline{V}$  of two rank two torsion free sheaves  $V_1$  and  $V_2$  with  $\det V_i = \Delta$ , a morphism  $\varphi : V_1 \rightarrow V_2$ , and a linear map  $E_i \rightarrow H^0(V_i \otimes H_i \otimes L_0^n)$ , not necessarily an isomorphism. If  $E_i \rightarrow H^0(V_i \otimes H_i \otimes L_0^n)$  is not injective for some  $i$  or if  $\varphi$  is not an isomorphism, then  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is either zero or  $G$ -unstable.*

- (ii) For  $n$  sufficiently large depending only on  $\Delta$  and  $c$  and for  $V$  a rank two torsion free sheaf with  $\det V = \Delta$  and  $c_2(V) = c$ ,  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -unstable if and only if  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable for all choices of data  $\underline{V}$  such that  $E_i \rightarrow H^0(V_i \otimes H_i \otimes L_0^n)$  is injective and  $\varphi : V_1 \rightarrow V_2 \cong V$  is an isomorphism, and  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -strictly semistable if and only if  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -strictly semistable for all such  $\underline{V}$ . Thus  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -stable if and only if  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -stable for all such  $\underline{V}$ .

*Proof.* First let us prove (i). We may assume that  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V})) \neq 0$ . Suppose for example that  $v_1 \in E_1 \mapsto 0 \in H^0(V_1 \otimes H_1 \otimes L_0^n)$ . Complete  $v_1$  to a basis of  $E_1$  and choose a basis  $\{w_k\}$  for  $E_2$ . Then  $T_1(\underline{V})(v_1 \wedge v_i) = 0$  for all  $i$  and  $T(\underline{V})(v_1 \otimes w_k) = 0$  for all  $k$ . Define a one parameter subgroup of  $G$  as follows: let  $v_1^\lambda = \lambda^{-N}v_1$ ,  $v_i^\lambda = \lambda^a v_i$  for  $i > 1$ , and  $w_k^\lambda = \lambda^b w_k$  for all  $k$ . Clearly  $\lim_{\lambda \rightarrow 0} (T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))^\lambda = 0$  provided that  $a$  and  $b$  are positive, so that  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable provided that the one parameter subgroup so constructed lies in  $G$ , or on other words provided that

$$a_1(-N + a(d_1 - 1)) + a_2 b d_2 = 0.$$

It thus suffices to take  $a$  an arbitrary positive integer,  $b = a_1$ , and  $N = a(d_1 - 1) + a_2 d_2$ .

The argument in case  $\varphi$  has a kernel is similar: in this case let  $v_1 \in \text{Ker } \varphi$ . Then  $T_1(\underline{V}) = 0$  and  $T(\underline{V})(v_1 \otimes w_k) = 0$  for all  $k$ , so that the previous argument handles this case also.

Next we show (ii). Let  $p_{V \otimes H_i}$  be the usual normalized Hilbert polynomial of  $V \otimes H_i$ , and similarly for  $p_{W \otimes H_i}$ , where  $W$  is a rank one subsheaf of  $V$ . Thus  $p_{V \otimes H_i}$  and  $p_{W \otimes H_i}$  have the same leading term. Given a polynomial  $p$ , let  $\Delta p$  denote the difference polynomial. In our case, all of the polynomials  $p$  that occur are quadratic polynomials with the same fixed degree two term. Thus if  $p_1$  and  $p_2$  are two such polynomials, then  $p_1(n) > p_2(n)$  for all  $n \gg 0$  if and only if the linear term of  $p_1$  is greater than or equal to the linear term of  $p_2$ , and if the linear terms are equal then the constant term of  $p_1$  is greater than the constant term of  $p_2$ . In this last case, where the linear terms are also equal, we see that  $p_1(n) > p_2(n)$  for all  $n \gg 0$  if and only if  $p_1(n) > p_2(n)$  for some  $n$ . Finally the linear term of  $p_1$  is greater than or equal to the linear term of  $p_2$  if and only if  $\Delta p_1(n) \geq \Delta p_2(n)$  for all  $n$ , which we shall write as  $\Delta p_1 \geq \Delta p_2$ . Thus if  $\Delta p_1 \geq \Delta p_2$  and  $p_1(n) > p_2(n)$  for some  $n$ , then  $p_1(n) > p_2(n)$  for all  $n \gg 0$ . If  $\Delta p_1 = \Delta p_2$ , then  $p_1(n) > p_2(n)$  for some  $n$  if and only if  $p_1(n) > p_2(n)$  for all  $n$ .



We shall show that, for sufficiently large  $n$ , if  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable and  $\underline{V}$  corresponds to data where  $E_i \rightarrow H^0(V \otimes H_i \otimes L_0^n)$  is injective and  $\varphi$  is an isomorphism, then  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -semistable. Note that  $V$  is Mumford semistable. First we may choose  $n$  so that  $V \otimes H_i$  is generated by its global sections and has no higher cohomology, and so  $\chi(V \otimes H_i \otimes L_0^n) = h^0(V \otimes H_i \otimes L_0^n) = d_i$ . Hence, since  $E_i \rightarrow H^0(V \otimes H_i \otimes L_0^n)$  is injective, it is an isomorphism. Let  $W$  be a rank one subsheaf of  $V$ . Since  $V \otimes H_i$  is Mumford semistable,  $\Delta p_{W \otimes H_i} \leq \Delta p_{V \otimes H_i}$ . Now the proof of (3) of Lemma 1.2 in [13] shows that there exists an  $N$  so that, for all  $n \geq N$ , with  $d_i$  as above, if  $W$  is a rank one subsheaf of  $V$  and such that  $h^0(W \otimes H_i \otimes L_0^n) \geq d_i/2$  for at least one  $i$  ( $i = 1, 2$ ), then in fact  $\Delta p_{V \otimes H_i} = \Delta p_{W \otimes H_i}$  for all such  $W$ , and thus  $\mu_{L_0}(V) = \mu_{L_0}(W)$ . It is then easy to see that there is a twist  $W \otimes H_i \otimes L_0^{-k}$ , depending only on  $L_0$  and  $\Delta$ , such that  $h^0((V/W) \otimes H_i \otimes L_0^{-k}) = 0$ . The proof of Proposition 3.1 in [13] shows that in this case  $h^1(W \otimes H_i \otimes L_0^{-k})$  is bounded by  $Q$ , where  $Q$  is some universal bound for the numbers  $h^1(V \otimes H_i \otimes L_0^{-k})$  as  $V \otimes H_i$  ranges over the appropriate set of  $L_0$ -semistable sheaves. Thus by (4) of Lemma 1.2 in [13], the  $W$  satisfying the condition that  $h^0(W \otimes H_i \otimes L_0^n) \geq d_i/2$  for at least one  $i$  form a bounded family, and we may choose  $n$  so large, depending only on  $L_0, \Delta, c$ , such that  $h^j(W \otimes H_i \otimes L_0^n) = 0$  for  $j \geq 1$  and  $i = 1, 2$ .

Now suppose that  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable. Then there exists a one parameter subgroup of  $G$  as above such that

$$\lim_{\lambda \rightarrow 0} (T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))^\lambda = 0.$$

Let

$$\begin{aligned} s_1 &= \#\{j : T_1(\underline{V})(v_1 \wedge v_j) = 0\} \geq \max\{j : n_1 + n_j \leq 0\}; \\ s_2 &= \#\{j : T_2(\underline{V})(w_1 \wedge w_\ell) = 0\} \geq \max\{\ell : m_1 + m_\ell \leq 0\}. \end{aligned}$$

Since  $a_1 \sum_i n_i + a_2 \sum_k m_k = 0$ , at least one of  $n_1, m_1$  is negative. By symmetry we may assume that  $n_1$  is negative, and that  $n_1 \leq m_1$ . Since for  $j \leq s_1$ ,  $v_1 \wedge v_j$  is zero as a section of  $\det(V \otimes H_1 \otimes L_0^n)$ , the sections corresponding to  $v_j$ ,  $1 \leq j \leq s_1$ , are all sections of a rank one subsheaf  $W_1$  of  $V$ . Likewise the sections  $w_\ell$ ,  $1 \leq \ell \leq s_2$ , if there are any such, are all sections of a rank one subsheaf  $W_2$  of  $V$ . The condition that  $T(\underline{V})(v_1 \otimes w_1) = 0$  insures that  $W_1$  and  $W_2$  are contained in a saturated rank one subsheaf  $W$ , if  $s_2 \neq 0$ , otherwise we shall just take for  $W$  the saturated rank one subsheaf containing  $W_1$ . Moreover  $h^0(W \otimes H_1 \otimes L_0^n) \geq s_1$  and  $h^0(W \otimes H_2 \otimes L_0^n) \geq s_2$ . Suppose that we show that

$$a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) < 0.$$

Thus in particular  $s_i \geq d_i/2$  for at least one  $i$ . By our choice of  $n$  and the previous paragraph, if  $s_i \geq d_i/2$  for at least one  $i$ , then  $h^0(W \otimes H_i \otimes L_0^n) = \chi(W \otimes H_i \otimes L_0^n)$  and furthermore  $\mu_{L_0}(V) = \mu_{L_0}(W)$ . Thus

$$h^0(W \otimes H_i \otimes L_0^n) = \chi(W \otimes H_i \otimes L_0^n) \geq s_i$$

for  $i = 1, 2$  and so  $p_{V;H_1,H_2,a_1,a_2}(n) < p_{W;H_1,H_2,a_1,a_2}(n)$ . On the other hand,  $p_{V;H_1,H_2,a_1,a_2}$  and  $p_{W;H_1,H_2,a_1,a_2}$  are two quadratic polynomials with the same linear and quadratic terms (since  $\mu_{L_0}(V) = \mu_{L_0}(W)$ ), and

$$p_{V;H_1,H_2,a_1,a_2}(n) < p_{W;H_1,H_2,a_1,a_2}(n)$$

for one value of  $n$ . Thus the constant term of  $p_{W;H_1,H_2,a_1,a_2}$  must be larger than that of  $p_{V;H_1,H_2,a_1,a_2}$ . This contradicts the  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistability of  $V$ .

To see that  $a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) < 0$ , let

$$t_1 = \#\{j : n_j + m_1 \leq 0\} \leq s_1.$$

Here  $t_1 \leq s_1$  since  $T(\underline{V})(v_j \otimes w_1) = 0$  implies that  $v_j$  and  $w_1$  are contained in a rank one subsheaf of  $V$ , necessarily  $W$ , and thus that  $v_1 \wedge v_j = 0$ . Let

$$t_2 = \#\{\ell : n_1 + m_\ell \leq 0\} \leq s_2.$$

We have assumed that  $n_1 \leq m_1$ . Then consider the expression

$$a_1 \sum_j (n_1 + n_j) + a_2 \sum_\ell (n_1 + m_\ell).$$

On the one hand from the definition of the one parameter subgroup we have

$$a_1 \sum_j (n_1 + n_j) + a_2 \sum_\ell (n_1 + m_\ell) = a_1 d_1 n_1 + a_2 d_2 n_1.$$

On the other hand, to estimate  $\sum_j (n_1 + n_j)$ , we can ignore the positive terms where  $n_1 + n_j \geq 0$  and each of the  $s_1$  negative terms are at least  $n_1 + n_1 \geq 2n_1$ . Thus  $\sum_j (n_1 + n_j) \geq 2s_1 n_1$ . Since  $n_1 < 0$ , this term is  $\geq 2s_1 n_1$ . Also this inequality is strict or  $n_1 + n_i \leq 0$  for every  $i$ , which would say that every section of  $V \otimes H_1 \otimes L_0^n$  is really a section of  $W \otimes H_1 \otimes L_0^n$  contradicting the fact that  $V \otimes H_1 \otimes L_0^n$  is generated by global sections. So  $\sum_j (n_1 + n_j) < 2s_1 n_1$ . Likewise we claim that  $\sum_\ell (n_1 + m_\ell) \geq 2s_2 n_1$ . Here, to estimate  $\sum_\ell (n_1 + m_\ell)$ , we may ignore the terms with  $n_1 + m_\ell$  positive, leaving  $t_2$  terms  $n_1 + m_\ell$  which are  $\leq 0$ , and moreover each such term is at

least  $n_1 + m_1 \geq 2n_1$ . Thus  $\sum_{\ell}(n_1 + m_{\ell}) \geq 2t_2n_1$ , and since  $t_2 \leq s_2$  and  $n_1 < 0$ , we have  $2t_2n_1 \geq 2s_2n_1$ .

Putting this together we have

$$\begin{aligned} a_1d_1n_1 + a_2d_2n_1 &= a_1 \sum_j (n_1 + n_j) + a_2 \sum_{\ell} (n_1 + m_{\ell}) \\ &> a_1(2s_1n_1) + a_2(2s_2n_1), \end{aligned}$$

so that

$$a_1(d_1 - 2s_1)n_1 + a_2(d_2 - 2s_2)n_1 > 0.$$

As  $n_1 < 0$ , we must have  $a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) < 0$ , as desired.

We have thus shown that, if  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable, then  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -unstable. A very similar argument handles the  $G$ -strictly semistable case.

Now we turn to the converse statement, that if  $V$  is  $(H_1, H_2, a_1, a_2)$   $L_0$ -unstable then  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable. Suppose instead that

$$(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$$

is  $G$ -semistable. Let  $W$  be a rank one subsheaf of  $V$  such that

$$p_{W;H_1,H_2,a_1,a_2}(m) > p_{V;H_1,H_2,a_1,a_2}(m)$$

for all  $m \gg 0$ . We may assume that the quotient  $W' = V/W$  is torsion free. Thus  $p_{W';H_1,H_2,a_1,a_2}(m) < p_{V;H_1,H_2,a_1,a_2}(m)$  for all  $m \gg 0$ , and so  $\Delta p_{W' \otimes H_i} \leq \Delta p_{V \otimes H_i}$ . Now we have the map  $E_i \rightarrow H^0(V \otimes H_i \otimes L_0^n)$ . Consider  $E_i \cap H^0(W \otimes H_i \otimes L_0^n) \subseteq E_i$ . Let  $\dim E_i \cap H^0(W \otimes H_i \otimes L_0^n) = s_i$ . Suppose first that  $a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) < 0$ . We claim that in this case  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable, a contradiction. To see this, choose a basis  $v_1, \dots, v_{d_1}$  for  $E_1$  such that

$$v_i \in E_1 \cap H^0(W \otimes H_1 \otimes L_0^n)$$

for  $i \leq s_1$ , and similarly choose a basis  $w_1, \dots, w_{d_2}$  for  $E_2$  such that  $w_k \in E_2 \cap H^0(W \otimes H_2 \otimes L_0^n)$  for  $k \leq s_2$ . Thus, if  $i, j \leq s_1$  then  $T_1(\underline{V})(v_i \wedge v_j) = 0$ ; if  $k, \ell \leq s_2$  then  $T_2(\underline{V})(w_k \wedge w_{\ell}) = 0$ ; if  $i \leq s_1$  and  $k \leq s_2$  then  $T(\underline{V})(v_i \otimes w_k) = 0$ .

We will try to find a one parameter subgroup of  $G$  of the form

$$v_i^{\lambda} = \begin{cases} \lambda^{-m} v_i, & \text{for } i \leq s_1; \\ \lambda^n v_i, & \text{for } i > s_1, \end{cases}$$

and similarly

$$w_k^\lambda = \begin{cases} \lambda^{-m} w_k, & \text{for } i \leq s_2; \\ \lambda^n w_k, & \text{for } i > s_2. \end{cases}$$

It is easy to check that  $\lim_{\lambda \rightarrow 0} (T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))^\lambda = 0$  if and only if  $n > m$ . What we must arrange is the condition

$$a_1(-ms_1 + n(d_1 - s_1)) + a_2(-ms_2 + n(d_2 - s_2)) = 0.$$

Now consider the linear function with rational coefficients

$$f(t) = a_1(-s_1 + t(d_1 - s_1)) + a_2(-s_2 + t(d_2 - s_2)).$$

Since the coefficient of  $t$  is strictly positive  $f(t)$  is increasing, and

$$\begin{aligned} f(1) &= a_1(-s_1 + (d_1 - s_1)) + a_2(-s_2 + (d_2 - s_2)) \\ &= a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) < 0. \end{aligned}$$

Thus there is a rational number  $t = n/m > 1$  such that  $f(t) = 0$ , and this gives the desired choice of  $n$  and  $m$ . Thus if  $a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) < 0$ , then  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable, contradicting our hypothesis.

The other possibility is that  $a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) \geq 0$ . In this case  $d_i \geq 2s_i$  for at least one  $i$ . Recalling that we have the quotient  $W'$  of  $V$  by  $W$ , it then follows that for such an  $i$  the image of  $E_i$  in  $H^0(W' \otimes H_i \otimes L_0^n)$  must have dimension at least  $d_i/2$ . Arguing as in Proposition 3.2 of [13], it then follows from Lemma 1.2 of [13] that  $\Delta p_{W' \otimes H_i} = \Delta p_{W \otimes H_i}$  and so that  $V$  is Mumford  $L_0$ -semistable and  $\mu_{L_0}(V) = \mu_{L_0}(W)$ . Moreover, after enlarging  $n$  if necessary (independently of  $V$ ) we may assume that  $h^j(V \otimes H_i \otimes L_0^n) = 0$  for  $j > 0$ . In particular,  $d_i = \dim H^0(V \otimes H_i \otimes L_0^n)$  for  $i = 1, 2$ , and  $E_i \rightarrow H^0(V \otimes H_i \otimes L_0^n)$  is an isomorphism; so  $s_i = h^0(W \otimes H_i \otimes L_0^n)$ . As  $\mu_{L_0}(V) = \mu_{L_0}(W)$ , the polynomials  $p_{W;H_1,H_2,a_1,a_2}$  and  $p_{V;H_1,H_2,a_1,a_2}$  have the same terms in degree one and two, and thus since  $p_{W;H_1,H_2,a_1,a_2}(m) > p_{V;H_1,H_2,a_1,a_2}(m)$  for some  $m$  the same is true for all  $m$ , in particular for  $m = n$ . Moreover, for a general choice of a smooth curve  $C$  in the linear system corresponding to  $L_0$ , there is a fixed bound on the line bundle  $W \otimes H_i|_C$ . A standard argument as in the proof of (2) of Lemma 1.2 of [13] shows that, for  $n$  sufficiently large but independent of  $V$ , we have  $H^2(W \otimes H_i \otimes L_0^n) = 0$ . Thus  $s_i = h^0(W \otimes H_i \otimes L_0^n) \geq p_{W \otimes H_i}(n)$ . It follows that

$$\begin{aligned} a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) &\leq a_1(d_1 - 2p_{W \otimes H_1}(n)) + a_2(d_2 - 2p_{W \otimes H_2}(n)) \\ &= 2(p_{V;H_1,H_2,a_1,a_2}(n) - p_{W;H_1,H_2,a_1,a_2}(n)) < 0. \end{aligned}$$

This contradicts the assumption that  $a_1(d_1 - 2s_1) + a_2(d_2 - 2s_2) \geq 0$ . It then follows that  $(T_1(\underline{V}), T_2(\underline{V}), T(\underline{V}))$  is  $G$ -unstable.

The strictly semistable case is similar.  $\square$

We may now finish the proof of Theorem 4.4. Let  $\mathbb{P}U_{\text{ss}}$  be the set of  $G$ -semistable points of  $\mathbb{P}U$ . Let  $I_{\text{ss}}$  be the inverse image of  $\mathbb{P}U_{\text{ss}}$  under the morphism  $I \rightarrow \mathbb{P}U$ . Since every semistable sheaf is stable,  $I_{\text{ss}}$  is a  $\mathbb{C}^*$ -bundle over its image in  $Q_1 \times Q_2$ . Moreover the representable functor corresponding to  $I_{\text{ss}}$  is easily seen to be formally smooth over the moduli functor. Arguments very similar to those for Lemma 4.3 and 4.5 of [13] show that the morphism  $I_{\text{ss}} \rightarrow \mathbb{P}U_{\text{ss}}$  is one-to-one and proper, and thus in particular finite. Thus we may construct a quotient  $\mathfrak{M}_{L_0}(\Delta, c; H_1, H_2, a_1, a_2)$  of  $I_{\text{ss}}$  by  $G$ . This quotient maps in a one-to-one and proper way to the GIT quotient of  $\mathbb{P}U_{\text{ss}}$  and is therefore projective. By the discussion at the beginning of the proof of Theorem 4.4 the points of  $\mathfrak{M}_{L_0}(\Delta, c; H_1, H_2, a_1, a_2)$  may be identified with isomorphism classes of  $(H_1, H_2, a_1, a_2)$   $L_0$ -semistable rank two sheaves. Standard arguments then show that  $\mathfrak{M}_{L_0}(\Delta, c; H_1, H_2, a_1, a_2)$  has the usual properties of a coarse moduli space.  $\square$

## 5. The transition formula for Donaldson polynomial invariants.

From now on, we will assume that the surface  $X$  is rational with  $-K_X$  effective, and will study the transition formula of Donaldson polynomial invariants:

$$\delta_{w,p}^X(\mathcal{C}_+, \mathcal{C}_-) = D_{w,p}^X(\mathcal{C}_+) - D_{w,p}^X(\mathcal{C}_-)$$

where  $\mathcal{C}_-$  and  $\mathcal{C}_+$  are two adjacent chambers separated by a single wall  $W^\zeta$  of type  $(w, p)$  or equivalently of type  $(\Delta, c)$ . For simplicity, we assume that the wall  $W^\zeta$  is only represented by  $\pm\zeta$  since the general case just involves additional notation. We use  $\mathfrak{M}_0^{(k)}$  to stand for the moduli space  $\mathfrak{M}_0^{(\zeta, k)}$ . When  $\ell_\zeta = 0$ , we also assume that

$$h(\zeta) = h^1(X; \mathcal{O}_X(2F - \Delta)) \neq 0$$

(see Corollary 2.7). The special case when  $\ell_\zeta = h(\zeta) = 0$  will be treated in Theorem 6.1. By Theorem 3.9 and Lemma 3.2 (ii), we have the following

diagram:

$$\begin{array}{ccccccc}
 & & \widetilde{\mathfrak{M}}_0^{(\ell_\zeta)} & & \dots & & \widetilde{\mathfrak{M}}_0^{(0)} & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
 \mathfrak{M}_0^{(\ell_\zeta)} & & & & \mathfrak{M}_0^{(\ell_\zeta-1)} & & & & \mathfrak{M}_0^{(0)} & & & & & & \mathfrak{M}_0^{(-1)} \\
 \parallel & & & & & & & & & & & & & & \parallel \\
 \mathfrak{M}_- & & & & & & & & & & & & & & \mathfrak{M}_+
 \end{array}$$

where the morphism  $\widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k)}$  is the blowup of  $\mathfrak{M}_0^{(k)}$  at  $E_\zeta^{\ell_\zeta-k,k}$ , and the morphism  $\widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k-1)}$  is the blowup of  $\mathfrak{M}_0^{(k-1)}$  at  $E_{-\zeta}^{k,\ell_\zeta-k}$ .

Next, we collect and establish some notations. Recall that in section 2 we have constructed the bundle  $\mathcal{E}_\zeta^{\ell_\zeta-k,k}$  over  $H_{\ell_\zeta-k} \times H_k$ , where  $H_k = \text{Hilb}^k X$ .

**Notation 5.1.** Let  $\zeta$  define a wall of type  $(w, p)$ .

- (i)  $\lambda_k$  is the tautological line bundle over  $E_\zeta^{\ell_\zeta-k,k} = \mathbb{P}((\mathcal{E}_\zeta^{\ell_\zeta-k,k})^\vee)$ ; for simplicity, we also use  $\lambda_k$  to denote its first Chern class;
- (ii)  $\rho_k : X \times E_\zeta^{\ell_\zeta-k,k} \rightarrow X \times H_{\ell_\zeta-k} \times H_k$  is the natural projection;
- (iii)  $p_k : \widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k)}$  is the blowup of  $\mathfrak{M}_0^{(k)}$  at  $E_\zeta^{\ell_\zeta-k,k}$ ;
- (iv)  $q_{k-1} : \widetilde{\mathfrak{M}}_0^{(k)} \rightarrow \mathfrak{M}_0^{(k-1)}$  is the contraction of  $\widetilde{\mathfrak{M}}_0^{(k)}$  to  $\mathfrak{M}_0^{(k-1)}$ ;
- (v)  $\mathcal{N}_k$  is the normal bundle of  $E_\zeta^{\ell_\zeta-k,k}$  in  $\mathfrak{M}_0^{(k)}$ ; by Proposition 3.7, we have
 
$$\mathcal{N}_k = \rho_k^* \mathcal{E}_{-\zeta}^{k,\ell_\zeta-k} \otimes \lambda_k^{-1};$$
- (vi)  $D_k = \mathbb{P}(\mathcal{N}_k^\vee)$  is the exceptional divisor in  $\widetilde{\mathfrak{M}}_0^{(k)}$ ;
- (vii)  $\xi_k = \mathcal{O}_{\widetilde{\mathfrak{M}}_0^{(k)}}(-D_k)|_{D_k}$  is the tautological line bundle on  $D_k$ ; again, for simplicity, we also use  $\xi_k$  to denote its first Chern class;
- (viii)  $\mu^{(k)}(\alpha) = -\frac{1}{4}p_1(\mathcal{U}^{(k)})/\alpha$  where  $\alpha \in H_2(X; \mathbb{Z})$  and  $\mathcal{U}^{(k)}$  is a universal sheaf over  $X \times \mathfrak{M}_0^{(k)}$ . Let  $\mu^{(\ell_\zeta)}(\alpha) = \mu_-(\alpha)$  and that  $\mu^{(-1)}(\alpha) = \mu_+(\alpha)$ .
- (ix)  $\nu^{(k)} = -\frac{1}{4}p_1(\mathcal{U}^{(k)})/x$  where  $x \in H_0(X; \mathbb{Z})$  is the natural generator. Let  $\nu^{(\ell_\zeta)} = \nu_-$  and that  $\nu^{(-1)} = \nu_+$ .

Note that, in (viii) and (ix) above, the sheaf  $\mathcal{U}^{(k)}$  is only defined locally in the classical topology. However, since it is defined on the level of the Quot scheme a straightforward argument shows that  $p_1(\mathcal{U}^{(k)})$  is a well-defined element in the rational cohomology of  $X \times \mathfrak{M}_0^{(k)}$ , at least in the complement of the universally semistable sheaves. In case there are universally semistable sheaves, then the work of Li [21] extends the  $\mu$ -map to  $\mathfrak{M}_0^{(k)}$ , at least for the two-dimensional algebraic classes. We can then extend the  $\mu$ -map to the 4-dimensional class via a blowup formula due to O'Grady (unpublished). Moreover, there is a universal sheaf  $\mathcal{V}_k$  over  $X \times E_\zeta^{\ell_\zeta - k, k}$ . In what follows, we shall work as if there were a universal sheaf  $\mathcal{U}^{(k)}$ , and leave it to the reader to check that our final Chern class calculations can be verified directly even when no universal sheaf exists.

In the following lemma, we study the restrictions of  $p_k^* \mu^{(k)}(\alpha)$  and  $p_k^* \nu^{(k)}$  to  $D_k$ .

**Lemma 5.2.** *Let  $\alpha \in H_2(X; \mathbb{Z})$  and  $a = (\zeta \cdot \alpha)/2$ . Let  $\tau_1$  and  $\tau_2$  be the projections of  $E_\zeta^{\ell_\zeta - k, k}$  to  $H_{\ell_\zeta - k}$  and  $H_k$  respectively. Then,*

$$\begin{aligned} (\text{Id} \times p_k)^* c_1(\mathcal{U}^{(k)})|(X \times D_k) &= \pi_1^* \Delta + (p_k|D_k)^* \lambda_k \\ p_k^* \mu^{(k)}(\alpha)|D_k &= (p_k|D_k)^* [\tau_1^*([\mathcal{Z}_{\ell_\zeta - k}]/\alpha) + \tau_2^*([\mathcal{Z}_k]/\alpha) - a\lambda_k] \\ p_k^* \nu^{(k)}|D_k &= \frac{1}{4}(p_k|D_k)^* [4\tau_1^*([\mathcal{Z}_{\ell_\zeta - k}]/x) + 4\tau_2^*([\mathcal{Z}_k]/x) - \lambda_k^2]. \end{aligned}$$

*Proof.* Note that  $\mathcal{U}^{(k)}|X \times E_\zeta^{\ell_\zeta - k, k} = \mathcal{V}_k$ , where the sheaf  $\mathcal{V}_k$  is constructed by Proposition 2.8 and sits in the exact sequence:

$$0 \rightarrow \pi_1^* \mathcal{O}_X(F) \otimes \rho_k^* \pi_{1,2}^* I_{\mathcal{Z}_{\ell_\zeta - k}} \otimes \pi_2^* \lambda_k \rightarrow \mathcal{V}_k \rightarrow \pi_1^* \mathcal{O}_X(\Delta - F) \otimes \rho_k^* \pi_{1,3}^* I_{\mathcal{Z}_k} \rightarrow 0.$$

Thus,  $c_1(\mathcal{V}_k) = \pi_1^* \Delta + \pi_2^* \lambda_k$  and  $(\text{Id} \times p_k)^* c_1(\mathcal{U}^{(k)})|(X \times D_k) = \pi_1^* \Delta + (p_k|D_k)^* \lambda_k$ . Moreover,  $c_2(\mathcal{V}_k) = \rho_k^* \pi_{1,2}^* [\mathcal{Z}_{\ell_\zeta - k}] + \rho_k^* \pi_{1,3}^* [\mathcal{Z}_k] + (\pi_1^* F + \pi_2^* \lambda_k) \cdot \pi_1^*(\Delta - F)$ . Since

$$p_k^* \mu^{(k)}(\alpha)|D_k = (p_k|D_k)^* [\mu^{(k)}(\alpha)|E_\zeta^{\ell_\zeta - k, k}] = (p_k|D_k)^* [-\frac{1}{4} p_1(\mathcal{V}_k)/\alpha],$$

we have

$$p_k^* \mu^{(k)}(\alpha)|D_k = (p_k|D_k)^* [\tau_1^*([\mathcal{Z}_{\ell_\zeta - k}]/\alpha) + \tau_2^*([\mathcal{Z}_k]/\alpha) - a\lambda_k].$$

Similarly,  $p_k^* \nu^{(k)}|D_k = \frac{1}{4}(p_k|D_k)^* [4\tau_1^*([\mathcal{Z}_{\ell_\zeta-k}]/x) + 4\tau_2^*([\mathcal{Z}_k]/x) - \lambda_k^2]$ .  $\square$

It follows from the work of Morgan [26] and Li [21], together with unpublished work of Morgan, that  $D_{w,p}^X(\mathcal{C}_\pm)(\alpha^d) = \delta(\Delta) \cdot \mu_\pm(\alpha)^d$  and

$$D_{w,p}^X(\mathcal{C}_\pm)(\alpha^{d-2}, x) = \delta(\Delta) \cdot \mu_\pm(\alpha)^{d-2} \cdot \nu_\pm$$

where  $d = -p - 3$ ,  $\delta(\Delta) = (-1)^{(\Delta^2 + \Delta \cdot K_X)/2}$  is the difference between the complex orientation and the standard orientation on the instanton moduli space (see [6]), and  $x \in H_0(X; \mathbb{Z})$  is the natural generator. Strictly speaking, their methods only handle the case of  $D_{w,p}^X(\mathcal{C}_\pm)(\alpha^d)$ . To handle the case of  $D_{w,p}^X(\mathcal{C}_\pm)(\alpha^{d-2}, x)$ , one needs a blowup formula in algebraic geometry, which has been established by O'Grady (unpublished). To compute the differences

$$\mu_+(\alpha)^d - \mu_-(\alpha)^d \quad \text{and} \quad \mu_+(\alpha)^{d-2} \cdot \nu_+ - \mu_-(\alpha)^{d-2} \cdot \nu_-,$$

we need to know how  $\mu^{(k)}(\alpha)$  and  $\mu^{(k-1)}(\alpha)$  are related, and also how  $\nu^{(k)}$  and  $\nu^{(k-1)}$  are related. The following lemma handles this problem.

**Lemma 5.3.** *For  $\alpha \in H_2(X; \mathbb{Z})$  and the natural generator  $x \in H_0(X; \mathbb{Z})$ , we have*

$$\begin{aligned} q_{k-1}^* \mu^{(k-1)}(\alpha) &= p_k^* \mu^{(k)}(\alpha) - aD_k \\ q_{k-1}^* \nu^{(k-1)} &= p_k^* \nu^{(k)} - \frac{1}{4}[D_k^2 + 2(p_k|D_k)^* \lambda_k]. \end{aligned}$$

*Proof.* From the construction, the sheaf  $(\text{Id} \times q_{k-1})^* \mathcal{U}^{(k-1)}$  on  $X \times \widetilde{\mathfrak{M}}_0^{(k)}$  is the elementary modification of  $(\text{Id} \times p_k)^* \mathcal{U}^{(k)}$  along the divisor  $X \times D_k$ , using the surjection from  $(\text{Id} \times p_k)^* \mathcal{U}^{(k)}$  to the pullback of  $\rho_k^*(\pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{\mathcal{Z}_k})$ :

$$\begin{aligned} 0 &\rightarrow (\text{Id} \times q_{k-1})^* \mathcal{U}^{(k-1)} \rightarrow (\text{Id} \times p_k)^* \mathcal{U}^{(k)} \\ &\rightarrow (\text{Id} \times p_k|D_k)^* \rho_k^*(\pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{\mathcal{Z}_k}) \rightarrow 0 \end{aligned}$$

where  $(2F - \Delta) = \zeta$  and  $\pi_1$  is the natural projection  $X \times H_{\ell_\zeta-k} \times H_k \rightarrow X$ . Note that  $(\text{Id} \times p_k|D_k)^* \rho_k^*(\pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_{1,3}^* I_{\mathcal{Z}_k})$  is a sheaf supported on  $X \times D_k$ , and that its first and second Chern classes are equal to  $(X \times D_k)$  and  $(X \times D_k^2) - \pi_1^*(\Delta - F) \cdot (X \times D_k)$  respectively. It follows that

$$\begin{aligned} (\text{Id} \times q_{k-1})^* c_1(\mathcal{U}^{(k-1)}) &= (\text{Id} \times p_k)^* c_1(\mathcal{U}^{(k)}) - (X \times D_k) \\ (\text{Id} \times q_{k-1})^* c_2(\mathcal{U}^{(k-1)}) &= (\text{Id} \times p_k)^* c_2(\mathcal{U}^{(k)}) - (\text{Id} \times p_k)^* c_1(\mathcal{U}^{(k)}) \cdot (X \times D_k) \\ &\quad + \pi_1^*(\Delta - F) \cdot (X \times D_k). \end{aligned}$$



By Lemma 5.2,  $(\text{Id} \times p_k)^* c_1(\mathcal{U}^{(k)}) \cdot (X \times D_k) = (\Delta \times D_k) + (X \times (p_k|D_k)^* \lambda_k)$ . Thus,

$$\begin{aligned} (\text{Id} \times q_{k-1})^* p_1(\mathcal{U}^{(k-1)}) &= (\text{Id} \times p_k)^* p_1(\mathcal{U}^{(k)}) + (X \times D_k^2) - 4(\Delta - F) \times D_k \\ &\quad + 2(\text{Id} \times p_k)^* c_1(\mathcal{U}^{(k)}) \cdot (X \times D_k) \\ &= (\text{Id} \times p_k)^* p_1(\mathcal{U}^{(k)}) + 2(2F - \Delta) \times D_k \\ &\quad + X \times [D_k^2 + 2(p_k|D_k)^* \lambda_k]. \end{aligned}$$

Now the conclusions follow from some straightforward calculations.  $\square$

In the next two theorems, we will give formulas for the differences  $[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d$  and  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  in terms of the intersections in  $H_{\ell_\zeta - k} \times H_k$  and the Segre classes of the vector bundles  $\mathcal{E}_\zeta^{\ell_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee$  on  $H_{\ell_\zeta - k} \times H_k$ , where  $k = 0, 1, \dots, \ell_\zeta$ . The arguments are a little complicated, but the idea is that we are trying to get rid of the exceptional divisors  $D_k$  as well as the Chern classes of the tautological line bundles  $\xi_k$  and  $\lambda_k$ .

**Theorem 5.4.** *Let  $\zeta$  define a wall of type  $(w, p)$ , and  $d = (-p - 3)$ . For  $\alpha \in H_2(X; \mathbb{Z})$ , put  $a = (\zeta \cdot \alpha)/2$ . Then,  $[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d$  is equal to*

$$\begin{aligned} \sum_{j=0}^{2\ell_\zeta} \binom{d}{j} \cdot (-1)^{h(\zeta) + \ell_\zeta + j} \cdot a^{d-j} \cdot \\ \cdot \sum_{k=0}^{\ell_\zeta} ([Z_{\ell_\zeta - k}]/\alpha + [Z_k]/\alpha)^j \cdot s_{2\ell_\zeta - j}(\mathcal{E}_\zeta^{\ell_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee). \end{aligned}$$

*Proof.* By Lemma 5.3, we have  $q_{k-1}^* \mu^{(k-1)}(\alpha) = p_k^* \mu^{(k)}(\alpha) - aD_k$ . Since  $p_k$  and  $q_{k-1}$  are birational morphisms,

$$\begin{aligned} [p_k^* \mu^{(k)}(\alpha)]^d &= [\mu^{(k)}(\alpha)]^d \text{ and} \\ [q_{k-1}^* \mu^{(k-1)}(\alpha)]^d &= [\mu^{(k-1)}(\alpha)]^d. \end{aligned}$$

Thus,  $[\mu^{(k-1)}(\alpha)]^d - [\mu^{(k)}(\alpha)]^d$  is equal to

$$\begin{aligned} \sum_{i=1}^d \binom{d}{i} \cdot [p_k^* \mu^{(k)}(\alpha)|D_k]^{d-i} \cdot (-D_k|D_k)^{i-1} \cdot (-a^i) \\ = \sum_{i=1}^d \binom{d}{i} \cdot [p_k^* \mu^{(k)}(\alpha)|D_k]^{d-i} \cdot \xi_k^{i-1} \cdot (-a^i). \end{aligned}$$

By Lemma 5.2,  $p_k^* \mu^{(k)}(\alpha) | D_k = (p_k | D_k)^* ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha - a \lambda_k)$ . So we have

$$\begin{aligned}
& [\mu^{(k-1)}(\alpha)]^d - [\mu^{(k)}(\alpha)]^d \\
&= \sum_{i=1}^d \binom{d}{i} \cdot \\
&\quad \cdot \sum_{j=0}^{2\ell_\zeta} \binom{d-i}{j} \cdot ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha)^j \cdot (-a \lambda_k)^{d-i-j} \cdot \xi_k^{i-1} \cdot (-a^i) \\
&= \sum_{j=0}^{2\ell_\zeta} \sum_{i=1}^{d-j} \binom{d}{j} \cdot \binom{d-j}{i} \cdot \\
&\quad \cdot (-a^{d-j}) \cdot ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha)^j \cdot \xi_k^{i-1} \cdot (-\lambda_k)^{d-i-j} \\
&= \sum_{j=0}^{2\ell_\zeta} \binom{d}{j} \cdot (-a^{d-j}) \cdot ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha)^j \cdot \\
&\quad \cdot \sum_{i=1}^{d-j} \binom{d-j}{i} \cdot \xi_k^{i-1} \cdot (-\lambda_k)^{d-i-j} \\
&= \sum_{j=0}^{2\ell_\zeta} \binom{d}{j} \cdot (-a^{d-j}) \cdot ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha)^j \cdot \\
&\quad \cdot \sum_{i=0}^{d-1-j} \binom{d-j}{i+1} \cdot \xi_k^i \cdot (-\lambda_k)^{d-1-j-i}
\end{aligned}$$

Now, our formula follows from the following claim by summing  $k$  from 0 to  $\ell_\zeta$ .

**Claim.**

$$\begin{aligned}
& ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha)^j \cdot \sum_{i=0}^{d-1-j} \binom{d-j}{i+1} \cdot \xi_k^i \cdot (-\lambda_k)^{d-1-j-i} \\
&= ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha)^j \cdot (-1)^{h(\zeta) + \ell_\zeta + j - 1} \cdot s_{2\ell_\zeta - j} (\mathcal{E}_\zeta^{\ell_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee).
\end{aligned}$$

*Proof.* For simplicity, on the exceptional divisor  $D_k$ , we put

$$\sigma_s = ([Z_{\ell_\zeta - k}] / \alpha + [Z_k] / \alpha)^j \cdot \sum_{i=0}^s \binom{s+1}{i+1} \cdot \xi_k^i \cdot (-\lambda_k)^{s-i}.$$

So we must compute  $\sigma_{d-1-j}$ . Notice the relation

$$\sigma_s + \lambda_k \cdot \sigma_{s-1} = ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot (\xi_k - \lambda_k)^s.$$

Thus for  $0 \leq t \leq s$ , we have

$$\sigma_s = (-\lambda_k)^t \cdot \sigma_{s-t} + ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \sum_{i=0}^{t-1} (\xi_k - \lambda_k)^{s-i} \cdot (-\lambda_k)^i.$$

Put  $s = d - 1 - j$  and  $t = s - N_{-\zeta} = d - 1 - j - N_{-\zeta}$ , where  $N_{-\zeta} = \ell_{-\zeta} + h(-\zeta) - 1 = \ell_\zeta + h(-\zeta) - 1$  as defined in Corollary 2.7. Then,  $\sigma_{d-1-j}$  is equal to

$$\begin{aligned} & (-\lambda_k)^{d-1-j-N_{-\zeta}} \cdot \sigma_{N_{-\zeta}} + ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \\ & \quad \cdot \sum_{i=0}^{d-2-j-N_{-\zeta}} (\xi_k - \lambda_k)^{(d-1-j)-i} \cdot (-\lambda_k)^i. \end{aligned}$$

Since  $\dim E_\zeta^{\ell_\zeta-k,k} = d - 1 - N_{-\zeta}$ , we see that  $(-\lambda_k)^{d-1-j-N_{-\zeta}} \cdot \sigma_{N_{-\zeta}}$  is equal to

$$\begin{aligned} & (-\lambda_k)^{d-1-j-N_{-\zeta}} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \sum_{i=0}^{N_{-\zeta}} \binom{N_{-\zeta}+1}{i+1} \cdot \xi_k^i \cdot (-\lambda_k)^{N_{-\zeta}-i} \\ & = (-\lambda_k)^{d-1-j-N_{-\zeta}} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \xi_k^{N_{-\zeta}} \\ & = ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot (-\lambda_k)^{d-1-j-N_{-\zeta}} \cdot (\xi_k - \lambda_k)^{N_{-\zeta}} \end{aligned}$$

since the restriction of  $\xi_k$  to a fiber of  $D_k \rightarrow E_\zeta^{\ell_\zeta-k,k}$  is a hyperplane. Therefore,

$$\sigma_{d-1-j} = ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \sum_{i=0}^{d-1-j-N_{-\zeta}} (\xi_k - \lambda_k)^{(d-1-j)-i} \cdot (-\lambda_k)^i.$$

Now, we shall simplify  $(\xi_k - \lambda_k)^{(d-1-j)-i}$ . Since  $\xi_k$  is the tautological line bundle on  $D_k = \mathbb{P}(\mathcal{N}_k^\vee)$ , the line bundle  $(\xi_k \otimes \lambda_k^{-1})$  is the tautological line bundle on

$$\mathbb{P}(\mathcal{N}_k^\vee \otimes \lambda_k^{-1}) = \mathbb{P}[(\rho_k|_{E_\zeta^{k,\ell_\zeta-k}})^\vee \mathcal{E}_{-\zeta}^{k,\ell_\zeta-k}].$$

Since  $N_{-\zeta} + 1$  is the rank of  $\mathcal{E}_{-\zeta}^{k,\ell_\zeta-k}$ , it follows that

$$(\xi_k - \lambda_k)^{1+N_{-\zeta}} = - \sum_{j=1}^{1+N_{-\zeta}} c_j(\mathcal{E}_{-\zeta}^{k,\ell_\zeta-k}) \cdot (\xi_k - \lambda_k)^{1+N_{-\zeta}-j}.$$

One verifies that in general, for  $u' \geq N_{-\zeta}$ , one has

$$(\xi_k - \lambda_k)^{u'} = s_{u'-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}) \cdot (\xi_k - \lambda_k)^{N_{-\zeta}} + O((\xi_k - \lambda_k)^{N_{-\zeta}-1})$$

where  $s_i(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})$  is the  $i$ th Segre class of  $\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}$ . Therefore, since  $(d-1-j) - i \geq N_{-\zeta}$ , we see that  $(\xi_k - \lambda_k)^{(d-1-j)-i}$  is equal to

$$s_{d-1-j-i-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}) \cdot (\xi_k - \lambda_k)^{N_{-\zeta}} + O((\xi_k - \lambda_k)^{N_{-\zeta}-1})$$

and that  $([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot (\xi_k - \lambda_k)^{(d-1-j)-i} \cdot (-\lambda_k)^i$  is equal to

$$\begin{aligned} & ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \left[ s_{d-1-j-i-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}) \cdot (\xi_k - \lambda_k)^{N_{-\zeta}} \right] \cdot (-\lambda_k)^i \\ &= ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot s_{d-1-j-i-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}) \cdot (-\lambda_k)^i. \end{aligned}$$

Next, we note that  $([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot s_{d-1-j-i-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})$  is a cycle on  $E_\zeta^{\ell_\zeta - k, k}$  pulled-back from  $H_{\ell_\zeta - k} \times H_k$ . So this term is zero unless  $d-1-i-N_{-\zeta} \leq 2\ell_\zeta$ , that is,  $i \geq d-1-N_{-\zeta}-2\ell_\zeta$ . Note that by Corollary 2.7,  $d-1-N_{-\zeta}-2\ell_\zeta = N_\zeta$  and  $N_\zeta + 1 = h(\zeta) + \ell_\zeta$  is the rank of  $\mathcal{E}_\zeta^{\ell_\zeta - k, k}$ . Since  $\lambda_k$  is the tautological line bundle on  $E_\zeta^{\ell_\zeta - k, k} = \mathbb{P}((\mathcal{E}_\zeta^{\ell_\zeta - k, k})^\vee)$ , we see as before that

$$\lambda_k^i = s_{i-N_\zeta}(\mathcal{E}_\zeta^{\ell_\zeta - k, k}) \cdot \lambda_k^{N_\zeta} + O(\lambda_k^{N_\zeta - 1}).$$

Putting all these together, we conclude that  $\sigma_{d-1-j}$  is equal to

$$\begin{aligned} & ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \\ & \cdot \sum_{i=N_\zeta}^{d-1-j-N_{-\zeta}} s_{d-1-j-i-N_{-\zeta}}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}) \cdot (-1)^i \cdot s_{i-N_\zeta}(\mathcal{E}_\zeta^{\ell_\zeta - k, k}) \\ &= ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \sum_{i=0}^{2\ell_\zeta - j} (-1)^{i+N_\zeta} \cdot s_{(2\ell_\zeta - j) - i}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}) \cdot s_i(\mathcal{E}_\zeta^{\ell_\zeta - k, k}) \\ &= ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot (-1)^{j+N_\zeta} \cdot \\ & \cdot \sum_{i=0}^{2\ell_\zeta - j} s_{(2\ell_\zeta - j) - i}((\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee) \cdot s_i(\mathcal{E}_\zeta^{\ell_\zeta - k, k}) \\ &= ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot (-1)^{j+N_\zeta} \cdot s_{2\ell_\zeta - j}(\mathcal{E}_\zeta^{\ell_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee) \end{aligned}$$

This completes the proof of the Theorem.  $\square$

For the difference  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$ , we have the following.

**Theorem 5.5.** *Let  $\zeta$  define a wall of type  $(w, p)$ , and  $d = -p - 3$ . For  $\alpha \in H_2(X; \mathbb{Z})$ , put  $a = (\zeta \cdot \alpha)/2$ . Then,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to*

$$\frac{1}{4} \cdot \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot (-1)^{h(\zeta) + \ell_\zeta - 1 + j} \cdot a^{d-2-j} \cdot \sum_{k=0}^{\ell_\zeta} ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot [s_{2\ell_\zeta - j} - 4([\mathcal{Z}_{\ell_\zeta - k}] + [\mathcal{Z}_k])/x \cdot s_{2\ell_\zeta - 2 - j}]$$

where  $s_i$  stands for the  $i^{\text{th}}$  Segre class of  $\mathcal{E}_\zeta^{\ell_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee$ .

*Proof.* By Lemma 5.3, we have  $q_{k-1}^* \mu^{(k-1)}(\alpha) = p_k^* \mu^{(k)}(\alpha) - aD_k$  and

$$q_{k-1}^* \nu^{(k-1)} = p_k^* \nu^{(k)} - \frac{1}{4}[D_k^2 + 2(p_k | D_k)^* \lambda_k].$$

It follows that  $[\mu^{(k-1)}(\alpha)]^{d-2} \cdot \nu^{(k-1)} - [\mu^{(k)}(\alpha)]^{d-2} \cdot \nu^{(k)} = I_1 + I_2$  where

$$\begin{aligned} I_1 &= [\mu^{(k)}(\alpha) - aD_k]^{d-2} \cdot \frac{1}{4}[-D_k^2 - 2(p_k | D_k)^* \lambda_k] \\ &= [\mu^{(k)}(\alpha) | D_k + a\xi_k]^{d-2} \cdot \frac{1}{4}(\xi_k - 2\lambda_k) \\ I_2 &= \sum_{i=1}^{d-2} \binom{d-2}{i} \cdot \mu^{(k)}(\alpha)^{d-2-i} \cdot (-aD_k)^i \cdot \nu^{(k)} \\ &= \sum_{i=1}^{d-2} \binom{d-2}{i} \cdot [\mu^{(k)}(\alpha) | D_k]^{d-2-i} \cdot \xi_k^{i-1} \cdot (-a^i) \cdot (\nu^{(k)} | D_k). \end{aligned}$$

First of all, since  $\mu^{(k)}(\alpha) | D_k = ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha - a\lambda_k)$ , we see that

$$\begin{aligned} I_1 &= [([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha) + a(\xi_k - \lambda_k)]^{d-2} \cdot \frac{1}{4}(\xi_k - 2\lambda_k) \\ &= \frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot a^{d-2-j} \cdot ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot (\xi_k - \lambda_k)^{d-2-j} \cdot (\xi_k - 2\lambda_k) \\ &= \frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot a^{d-2-j} \cdot ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \end{aligned}$$

$$\begin{aligned}
& \cdot [(\xi_k - \lambda_k)^{d-1-j} - \lambda_k \cdot (\xi_k - \lambda_k)^{d-2-j}] \\
& = \frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot a^{d-2-j} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \\
& \quad \cdot [s_{d-1-j-N_\zeta}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta-k}) - \lambda_k \cdot s_{d-2-j-N_\zeta}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta-k})].
\end{aligned}$$

Next, by Lemma 5.2, we have  $\nu^{(k)}|D_k = \frac{1}{4} [4[\mathcal{Z}_{\ell_\zeta-k}]/x + 4[\mathcal{Z}_k]/x - \lambda_k^2]$ . Thus, as in the proof of Theorem 5.4, we can verify that  $I_2$  is equal to

$$\begin{aligned}
& \frac{1}{4} [4[\mathcal{Z}_{\ell_\zeta-k}]/x + 4[\mathcal{Z}_k]/x - \lambda_k^2] \cdot \\
& \quad \cdot \sum_{i=1}^{d-2} \binom{d-2}{i} \cdot [\mu^{(k)}(\alpha)|D_k]^{d-2-i} \cdot \xi_k^{i-1} \cdot (-a^i) \\
& = \frac{1}{4} [4[\mathcal{Z}_{\ell_\zeta-k}]/x + 4[\mathcal{Z}_k]/x - \lambda_k^2] \cdot \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot (-a^{d-2-j}) \cdot \\
& \quad \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \sum_{i=0}^{2\ell_\zeta+N_\zeta-2-j} s_{d-3-j-i-N_\zeta}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta-k}) \cdot (-\lambda_k)^i \\
& = \frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot (-a^{d-2-j}) \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \\
& \quad \cdot [4([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)/x \cdot (-1)^{j+N_\zeta} \cdot s' - \\
& \quad \quad - \sum_{i=0}^{2\ell_\zeta+N_\zeta-2-j} s_{d-3-j-i-N_\zeta}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta-k}) \cdot (-\lambda_k)^{i+2}] \\
& = \frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot a^{d-2-j} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \\
& \quad \cdot \left[ \sum_{i=0}^{2\ell_\zeta+N_\zeta-2-j} s_{d-3-j-i-N_\zeta}(\mathcal{E}_{-\zeta}^{k, \ell_\zeta-k}) \cdot (-\lambda_k)^{i+2} - \right. \\
& \quad \quad \left. -4([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)/x \cdot (-1)^{j+N_\zeta} \cdot s' \right]
\end{aligned}$$

where  $s'$  stands for  $s_{2\ell_\zeta-2-j}(\mathcal{E}_{-\zeta}^{\ell_\zeta-k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta-k})^\vee)$ . Thus,  $I_1 + I_2$  is equal to

$$\frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot a^{d-2-j} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot$$

$$\begin{aligned}
 & \cdot \left[ \sum_{i=-2}^{2\ell_\zeta + N_\zeta - 2 - j} s_{d-3-j-i-N_\zeta} (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}) \cdot (-\lambda_k)^{i+2} - \right. \\
 & \quad \left. -4([\mathcal{Z}_{\ell_\zeta - k}] + [\mathcal{Z}_k])/x \cdot (-1)^{j+N_\zeta} \cdot s' \right] \\
 &= \frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot a^{d-2-j} \cdot ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \\
 & \quad \cdot [(-1)^{j+N_\zeta} \cdot s'' - 4([\mathcal{Z}_{\ell_\zeta - k}] + [\mathcal{Z}_k])/x \cdot (-1)^{j+N_\zeta} \cdot s'] \\
 &= \frac{1}{4} \sum_{j=0}^{2\ell_\zeta} \binom{d-2}{j} \cdot (-1)^{h(\zeta) + \ell_\zeta - 1 + j} \cdot a^{d-2-j} \cdot ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot \\
 & \quad \cdot [s'' - 4([\mathcal{Z}_{\ell_\zeta - k}] + [\mathcal{Z}_k])/x \cdot s']
 \end{aligned}$$

since  $N_\zeta = h(\zeta) + \ell_\zeta - 1$ , where  $s''$  stands for  $s_{2\ell_\zeta - j}(\mathcal{E}_\zeta^{\ell_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee)$ . Letting  $k$  run from 0 to  $\ell_\zeta$ , we obtain the desired formula.  $\square$

**Remark 5.6.** For the sake of convenience, we record here the following relation among the Chern classes and the Segre classes of a vector bundle:

$$s_n = -c_1 \cdot s_{n-1} - c_2 \cdot s_{n-2} - \dots - c_n$$

with the convention that  $s_0 = 1$ . We refer to [12] for details.

In the next section, using Theorem 5.4 and Theorem 5.5, we shall compute  $[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d$  and  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  explicitly when  $0 \leq \ell_\zeta \leq 2$ . In principle, Theorem 5.4 and Theorem 5.5 give formulas for these differences in terms of certain intersections in  $H_{\ell_\zeta - k} \times H_k$ . However, it is difficult to evaluate these intersection numbers in general. In the following, we shall compute the term

$$(5.7) \quad S_j = \sum_{k=0}^{\ell_\zeta} ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot s_{2\ell_\zeta - j}(\mathcal{E}_\zeta^{\ell_\zeta - k, k} \oplus (\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k})^\vee)$$

in the special cases when  $j = 2\ell_\zeta$  and  $2\ell_\zeta - 1$ . We start with a simple lemma.

**Lemma 5.8.** *Let  $\alpha, \beta \in H_2(X; \mathbb{Z})$ . Then*

$$([\mathcal{Z}_k]/\alpha)^{2k} = \frac{(2k)!}{2^k \cdot k!} \cdot (\alpha^2)^k$$

$$([\mathcal{Z}_k]/\alpha)^{2k-1} \cdot ([\mathcal{Z}_k]/\beta) = \frac{(2k)!}{2^k \cdot k!} \cdot (\alpha^2)^{k-1} \cdot (\alpha \cdot \beta)$$

$$\begin{aligned} ([\mathcal{Z}_k]/\alpha)^{2k-2} \cdot ([\mathcal{Z}_k]/\beta)^2 &= \\ \frac{(2k-2)!}{2^{k-1} \cdot (k-1)!} \cdot (\alpha^2)^{k-1} \cdot \beta^2 &+ \frac{(2k-2)!}{2^{k-2} \cdot (k-2)!} \cdot (\alpha^2)^{k-2} \cdot (\alpha \cdot \beta)^2. \end{aligned}$$

*Proof.* The first equality is well-known (see [29] for instance). The other statements follow from the first one by considering

$$([\mathcal{Z}_k]/\alpha + [\mathcal{Z}_k]/\beta)^{2k} = \frac{(2k)!}{2^k \cdot k!} \cdot ((\alpha + \beta)^2)^k,$$

and formally equating the terms involving  $(2k-1)$  copies of  $\alpha$  and one  $\beta$  or  $(2k-2)$  copies of  $\alpha$  and two copies of  $\beta$ .  $\square$

The next result computes the term (5.7) when  $j = 2\ell_\zeta$ .

**Proposition 5.9.** *Let  $\zeta$  define a wall of type  $(w, p)$ , and  $\alpha \in H_2(X; \mathbb{Z})$ . Then,*

$$S_{2\ell_\zeta} = \sum_{k=0}^{\ell_\zeta} ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^{2\ell_\zeta} = \frac{(2\ell_\zeta)!}{\ell_\zeta!} \cdot (\alpha^2)^{\ell_\zeta}.$$

*Proof.* This follows in a straightforward way from Lemma 5.8 (i):

$$\begin{aligned} &\sum_{k=0}^{\ell_\zeta} ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^{2\ell_\zeta} \\ &= \sum_{k=0}^{\ell_\zeta} \binom{2\ell_\zeta}{2k} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha)^{2(\ell_\zeta-k)} \cdot ([\mathcal{Z}_k]/\alpha)^{2k} \\ &= \sum_{k=0}^{\ell_\zeta} \binom{2\ell_\zeta}{2k} \cdot \left[ \frac{(2\ell_\zeta - 2k)!}{2^{\ell_\zeta-k} \cdot (\ell_\zeta - k)!} \cdot (\alpha^2)^{\ell_\zeta-k} \right] \cdot \left[ \frac{(2k)!}{2^k \cdot k!} \cdot (\alpha^2)^k \right] \\ &= \sum_{k=0}^{\ell_\zeta} \binom{\ell_\zeta}{k} \cdot \frac{(2\ell_\zeta)!}{2^{\ell_\zeta} \cdot \ell_\zeta!} \cdot (\alpha^2)^{\ell_\zeta} \\ &= \frac{(2\ell_\zeta)!}{\ell_\zeta!} \cdot (\alpha^2)^{\ell_\zeta} \end{aligned}$$



□

To compute the term (5.7) when  $j = (2\ell_\zeta - 1)$ , we study  $\mathcal{E}_{-\zeta}^{k, \ell_\zeta - k}$  and  $\mathcal{E}_{\zeta}^{\ell_\zeta - k, k}$ , and evaluate their first Chern classes. We begin with a general lemma.

**Lemma 5.10.** *Let  $Z, W$  be codimension 2 cycles in a smooth variety  $Y$ .*

- (i) *If  $Z \subseteq W$ , then  $\text{Hom}(I_W, I_Z) = \mathcal{O}_Y$ ;*
- (ii) *If  $(Z - Z \cap W)$  is open and dense in  $Z$ , then  $\text{Hom}(I_W, I_Z) = I_Z$ ;*
- (iii) *If  $Z$  and  $W$  are local complete intersections meeting properly, then there is an exact sequence:*

$$0 \rightarrow \text{Ext}^1(I_W, I_Z) \rightarrow \mathcal{O}_W \otimes \det N_W \rightarrow \mathcal{O}_{W \cap Z} \otimes \det N_W \rightarrow 0$$

where  $N_W$  is the normal bundle of  $W$  in  $Y$ ;

- (iv) *Assume that  $Z \cap W$  is nowhere dense in  $W$  and that  $W$  is smooth at a generic point. Then, as a sheaf on  $W$ ,  $\text{Ext}^1(I_W, I_Z)$  is of rank 1; thus,  $c_0(\text{Ext}^1(I_W, I_Z)) = c_1(\text{Ext}^1(I_W, I_Z)) = 0$ ,  $c_2(\text{Ext}^1(I_W, I_Z)) = -[W]$ .*

*Proof.* (i) Applying the functor  $\text{Hom}(I_W, \cdot)$  to the exact sequence

$$0 \rightarrow I_Z \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Z \rightarrow 0,$$

we obtain  $0 \rightarrow \text{Hom}(I_W, I_Z) \rightarrow \text{Hom}(I_W, \mathcal{O}_Y) = \mathcal{O}_Y$ . Thus,  $\text{Hom}(I_W, I_Z) = I_U$  for some closed subscheme  $U$  of  $Y$ . On the other hand, since  $Z \subseteq W$ ,

$$H^0(Y; \text{Hom}(I_W, I_Z)) = \text{Hom}(I_W, I_Z) \neq 0.$$

Thus,  $U$  must be empty, and  $\text{Hom}(I_W, I_Z) = \mathcal{O}_Y$ .

(ii) As in the proof of (i),  $\text{Hom}(I_W, I_Z) = I_U$  for some closed subscheme  $U$  of  $Y$ . Applying the functor  $\text{Hom}(\cdot, I_Z)$  to the exact sequence

$$0 \rightarrow I_W \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_W \rightarrow 0,$$

we get  $0 \rightarrow I_Z \rightarrow \text{Hom}(I_W, I_Z) = I_U \rightarrow \text{Ext}^1(\mathcal{O}_W, I_Z)$ . Thus,  $U \subseteq Z$ ; moreover, since  $\text{Ext}^1(\mathcal{O}_W, I_Z) = 0$  on  $(X - W)$ , we have  $(Z - Z \cap W) = (U - U \cap W)$ . So

$$(Z - Z \cap W) \subseteq U \subseteq Z.$$

Since  $(Z - Z \cap W)$  is open and dense in  $Z$ , it follows that  $U = Z$ .

(iii) We begin with the local identification: let  $R$  be a regular local ring, and let  $Z$  and  $W$  be two codimension 2 local complete intersection subschemes of  $R$  meeting properly. Applying the functor  $\text{Hom}_R(\cdot, I_Z)$  to the Koszul resolution of  $W$

$$0 \rightarrow R \rightarrow R \oplus R \rightarrow I_W \rightarrow 0$$

gives  $I_Z \oplus I_Z \rightarrow I_Z \rightarrow \text{Ext}_R^1(I_W, I_Z) \rightarrow 0$ . It follows that  $\text{Ext}_R^1(I_W, I_Z) = I_Z/(I_Z \cdot I_W)$ . Since  $Z$  and  $W$  are codimension 2 local complete intersections meeting properly, we have  $I_Z \cdot I_W = I_Z \cap I_W$ . Thus,  $\text{Ext}_R^1(I_W, I_Z) \cong I_Z/(I_Z \cap I_W)$ , and we can fit it into an exact sequence

$$0 \rightarrow \text{Ext}_R^1(I_W, I_Z) \rightarrow R/I_W \rightarrow R/(I_W + I_Z) \rightarrow 0.$$

Here  $(I_W + I_Z)$  corresponds to the intersection  $W \cap Z$ . The identification of  $\text{Ext}_R^1(I_W, I_Z)$  and  $I_Z/(I_Z \cap I_W)$  is not canonical. Globally we must correct by  $\det N_W$ . Thus globally we have an exact sequence:

$$0 \rightarrow \text{Ext}^1(I_W, I_Z) \rightarrow \mathcal{O}_W \otimes \det N_W \rightarrow \mathcal{O}_{W \cap Z} \otimes \det N_W \rightarrow 0.$$

(iv) It is clear that  $\text{Ext}^1(I_W, I_Z)$  is a sheaf supported on  $W$ . To show that it has rank 1 as a sheaf on  $W$ , it suffices to verify that it has rank 1 at a generic point  $w$  of  $W$ . Since  $Z \cap W$  is nowhere dense in  $W$  and  $W$  is smooth at a generic point, we may assume that  $w \notin Z$  and that  $w$  is a smooth point of  $W$ . Then it follows from (iii) that  $\text{Ext}^1(I_W, I_Z)$  is of rank 1 at  $w$ .  $\square$

**Lemma 5.11.** *Let  $\text{Hom} = \text{Hom}(I_{Z_k}, I_{Z_{\ell_\zeta - k}})$ ,  $\text{Ext}^1 = \text{Ext}^1(I_{Z_k}, I_{Z_{\ell_\zeta - k}})$ ,  $\pi_1$  and  $\pi_2$  be the projections from  $X \times (H_{\ell_\zeta - k} \times H_k)$  to  $X$  and  $(H_{\ell_\zeta - k} \times H_k)$  respectively.*

(i) *There exist a row exact sequence and a column exact sequence:*

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & \pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes \mathcal{O}_{Z_{\ell_\zeta - k}}) & & & \\ & & & \downarrow & & & \\ 0 \rightarrow & R^1 \pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes \text{Hom}) & \rightarrow & \mathcal{E}_\zeta^{\ell_\zeta - k, k} & \rightarrow & \pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes \text{Ext}^1) & \rightarrow 0; \\ & & & \downarrow & & & \\ & & & [\mathcal{O}_{H_{\ell_\zeta - k} \times H_k}]^{\oplus h(\zeta)} & & & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

$$(ii) \quad c_1(R^1\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Hom)) = [Z_{\ell_\zeta-k}]/(\zeta - K_X/2) + \pi_{2*}[c_3(\mathcal{O}_{Z_{\ell_\zeta-k}})]/2;$$

$$(iii) \quad c_1(\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Ext^1)) = [Z_k]/(\zeta - K_X/2) + \pi_{2*}[c_3(Ext^1)]/2.$$

*Proof.* (i) Note that the bundle  $\mathcal{E}_\zeta^{\ell_\zeta-k,k}$  is defined as

$$\begin{aligned} Ext_{\pi_2}^1(\pi_1^*\mathcal{O}_X(\Delta - F) \otimes I_{Z_k}, \pi_1^*\mathcal{O}_X(F) \otimes I_{Z_{\ell_\zeta-k}}) &= \\ &= Ext_{\pi_2}^1(I_{Z_k}, \pi_1^*\mathcal{O}_X(\zeta) \otimes I_{Z_{\ell_\zeta-k}}). \end{aligned}$$

Since  $R^2\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Hom) = 0$ , the row exact sequence follows from standard facts about relative Ext sheaves. To see the column exact sequence, we use Lemma 5.10 (ii) and apply the functor  $\pi_{2*}$  to the exact sequence

$$0 \rightarrow \pi_1^*\mathcal{O}_X(\zeta) \otimes I_{Z_{\ell_\zeta-k}} \rightarrow \pi_1^*\mathcal{O}_X(\zeta) \rightarrow \pi_1^*\mathcal{O}_X(\zeta) \otimes \mathcal{O}_{Z_{\ell_\zeta-k}} \rightarrow 0.$$

(ii) Note that  $Hom = I_{Z_{\ell_\zeta-k}}$  and that  $R^i\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Hom) = 0$  for  $i = 0, 2$ . By the Grothendieck-Riemann-Roch Theorem, we have

$$\begin{aligned} & - \text{ch}(R^1\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Hom)) \\ &= \pi_{2*} \left( \text{ch}(\pi_1^*\mathcal{O}_X(\zeta) \otimes I_{Z_{\ell_\zeta-k}}) \cdot \pi_1^* \text{Todd}(T_X) \right) \\ &= \pi_{2*} \left( \pi_1^* \text{ch}(\mathcal{O}_X(\zeta)) \cdot \text{ch}(I_{Z_{\ell_\zeta-k}}) \cdot \pi_1^* \text{Todd}(T_X) \right). \end{aligned}$$

Now, the conclusion follows by comparing the degree 1 terms and by the fact that

$$\begin{aligned} \text{ch}(I_{Z_{\ell_\zeta-k}}) = 1 - \text{ch}(\mathcal{O}_{Z_{\ell_\zeta-k}}) &= 1 - [Z_{\ell_\zeta-k}] - \frac{c_3(\mathcal{O}_{Z_{\ell_\zeta-k}})}{2} + \\ &+ (\text{terms with degree} \geq 4). \end{aligned}$$

(iii) We have  $R^i\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Ext^1) = 0$  for  $i = 1, 2$ . By Lemma 5.10 (iv),

$$\text{ch}(Ext^1) = [Z_k] + \frac{c_3(Ext^1)}{2} + (\text{terms with degree} \geq 4).$$

Again, using the Grothendieck-Riemann-Roch Theorem, we obtain

$$\begin{aligned} & \text{ch}(\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Ext^1)) \\ &= \pi_{2*} \left( \text{ch}(\pi_1^*\mathcal{O}_X(\zeta) \otimes Ext^1) \cdot \pi_1^* \text{Todd}(T_X) \right) \\ &= \pi_{2*} \left( \pi_1^* \text{ch}(\mathcal{O}_X(\zeta)) \cdot \text{ch}(Ext^1) \cdot \pi_1^* \text{Todd}(T_X) \right). \end{aligned}$$

Then, our conclusion follows by comparing the degree 1 terms.  $\square$

Now, we can compute the term (5.7) for  $j = 2\ell_\zeta - 1$ .

**Proposition 5.12.** *Let  $\alpha \in H_2(X; \mathbb{Z})$  and  $a = (\zeta \cdot \alpha)/2$ . Then,*

$$\begin{aligned} S_{2\ell_\zeta-1} &= \sum_{k=0}^{\ell_\zeta} ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^{2\ell_\zeta-1} \cdot s_1(\mathcal{E}_\zeta^{\ell_\zeta-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,\ell_\zeta-k})^\vee) \\ &= (-4) \cdot \frac{(2\ell_\zeta)!}{\ell_\zeta!} \cdot (\alpha^2)^{\ell_\zeta-1} \cdot a. \end{aligned}$$

*Proof.* By the symmetry between  $k$  and  $(\ell_\zeta - k)$ , we see that  $S_{2\ell_\zeta-1}$  is equal to

$$\begin{aligned} \sum_{k=0}^{\ell_\zeta} ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^{2\ell_\zeta-1} \cdot \\ \frac{s_1(\mathcal{E}_\zeta^{\ell_\zeta-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,\ell_\zeta-k})^\vee) + s_1(\mathcal{E}_\zeta^{k,\ell_\zeta-k} \oplus (\mathcal{E}_{-\zeta}^{\ell_\zeta-k,k})^\vee)}{2}. \end{aligned}$$

From Lemma 5.11, we conclude that  $c_1(\mathcal{E}_\zeta^{\ell_\zeta-k,k})$  is equal to

$$([\mathcal{Z}_{\ell_\zeta-k}] + [\mathcal{Z}_k]) / (\zeta - K_X/2) + \frac{\pi_{2*}[c_3(\mathcal{O}_{\mathcal{Z}_{\ell_\zeta-k}}) + c_3(\text{Ext}^1(I_{\mathcal{Z}_k}, I_{\mathcal{Z}_{\ell_\zeta-k}}))]}{2}.$$

Since  $s_1(\mathcal{E}_\zeta^{\ell_\zeta-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,\ell_\zeta-k})^\vee) = c_1(\mathcal{E}_{-\zeta}^{k,\ell_\zeta-k}) - c_1(\mathcal{E}_\zeta^{\ell_\zeta-k,k})$ , we see that

$$\frac{s_1(\mathcal{E}_\zeta^{\ell_\zeta-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,\ell_\zeta-k})^\vee) + s_1(\mathcal{E}_\zeta^{k,\ell_\zeta-k} \oplus (\mathcal{E}_{-\zeta}^{\ell_\zeta-k,k})^\vee)}{2} = (-2) \cdot ([\mathcal{Z}_{\ell_\zeta-k}] + [\mathcal{Z}_k]) / \zeta$$

where the  $c_3$ 's are cancelled out. Therefore, by Lemma 5.8,

$$\begin{aligned} S_{2\ell_\zeta-1} &= \sum_{k=0}^{\ell_\zeta} ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^{2\ell_\zeta-1} \cdot (-2) \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\zeta + [\mathcal{Z}_k]/\zeta) \\ &= (-2) \cdot \sum_{k=0}^{\ell_\zeta} \left[ \binom{2\ell_\zeta-1}{2k} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha)^{2\ell_\zeta-2k-1} \cdot ([\mathcal{Z}_k]/\alpha)^{2k} \cdot [\mathcal{Z}_{\ell_\zeta-k}]/\zeta \right. \\ &\quad \left. + \binom{2\ell_\zeta-1}{2k-1} \cdot ([\mathcal{Z}_{\ell_\zeta-k}]/\alpha)^{2\ell_\zeta-2k} \cdot ([\mathcal{Z}_k]/\alpha)^{2k-1} \cdot [\mathcal{Z}_k]/\zeta \right] \end{aligned}$$

$$\begin{aligned}
 &= (-4) \cdot \sum_{k=1}^{\ell_\zeta} \binom{2\ell_\zeta - 1}{2k - 1} \cdot ([\mathcal{Z}_{\ell_\zeta - k}]/\alpha)^{2\ell_\zeta - 2k} \cdot ([\mathcal{Z}_k]/\alpha)^{2k-1} \cdot [\mathcal{Z}_k]/\zeta \\
 &= (-4) \cdot \frac{(2\ell_\zeta)!}{\ell_\zeta!} \cdot (\alpha^2)^{\ell_\zeta - 1} \cdot a
 \end{aligned}$$

□

It is possible, but far more complicated, to compute (5.7) for  $j = 2\ell_\zeta - 2$ .

Next, we shall draw some consequences from our previous computations. Recall that  $q_X$  denotes the intersection form of  $X$ , and that

$$\delta(\Delta) = (-1)^{\frac{\Delta^2 + \Delta \cdot K_X}{2}}$$

is the difference between the complex orientation and the standard orientation on the instanton moduli space (see [6]). Adopting Kotschick's notation [17], we put

$$\varepsilon(\zeta, \Delta) = (-1)^{\frac{(\zeta - \Delta) \cdot K_X}{2}} = (-1)^{\left(\frac{\zeta - \Delta}{2}\right)^2}.$$

Theorem 5.13 below has already been obtained by Kotschick and Morgan [18] for any smooth 4-manifold with  $b_2^+ = 1$ .

**Theorem 5.13.** *Let  $\zeta$  define a wall of type  $(w, p)$ , and  $d = -p - 3$ . Then,*

$$\begin{aligned}
 &[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d \equiv \\
 &\equiv (-1)^{h(\zeta) + \ell_\zeta} \cdot \frac{d!}{\ell_\zeta! \cdot (d - 2\ell_\zeta)!} \cdot a^{d - 2\ell_\zeta} \cdot (\alpha^2)^{\ell_\zeta} \pmod{a^{d - 2\ell_\zeta + 2}}
 \end{aligned}$$

for  $\alpha \in H_2(X; \mathbb{Z})$ , where  $a = (\zeta \cdot \alpha)/2$ . Thus,

$$\begin{aligned}
 &\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+) \equiv \\
 &\equiv (-1)^{d + \ell_\zeta} \cdot \varepsilon(\zeta, \Delta) \cdot \frac{d!}{\ell_\zeta! \cdot (d - 2\ell_\zeta)!} \cdot \left(\frac{\zeta}{2}\right)^{d - 2\ell_\zeta} \cdot q_X^{\ell_\zeta} \pmod{\zeta^{d - 2\ell_\zeta + 2}}.
 \end{aligned}$$

*Proof.* By Theorem 5.4 and our notation (5.7), we have

$$[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d \equiv \sum_{j=2\ell_\zeta - 1}^{2\ell_\zeta} \binom{d}{j} \cdot (-1)^{h(\zeta) + \ell_\zeta + j} \cdot a^{d - j} \cdot S_j \pmod{a^{d - 2\ell_\zeta + 2}}.$$

By Proposition 5.12,  $S_{2\ell_\zeta-1}$  is divisible by  $a$ . Therefore,

$$[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d \equiv \binom{d}{2\ell_\zeta} \cdot (-1)^{h(\zeta)+\ell_\zeta} \cdot a^{d-2\ell_\zeta} \cdot S_{2\ell_\zeta} \pmod{a^{d-2\ell_\zeta+2}}.$$

Now, the first conclusion follows from Proposition 5.9, and the second follows from the facts that  $\gamma_\pm(\alpha^d) = \delta(\Delta) \cdot \mu_\pm(\alpha)^d$  and that

$$\delta(\Delta) \cdot (-1)^{h(\zeta)+\ell_\zeta} = (-1)^{\frac{\Delta^2+\Delta \cdot K_X}{2}} \cdot (-1)^{\frac{(\zeta \cdot K_X)}{2} - \frac{\zeta^2}{2} - 1 + \ell_\zeta} = (-1)^{d+\ell_\zeta} \cdot \varepsilon(\zeta, \Delta).$$

We remark that the sign in  $\delta_{w,p}^X(C_-, C_+)$  is indeed a homotopy invariant. (Note that the sign in Theorem 6.1.1 in [18] is not correct.)

The following is proved by using a similar method.

**Theorem 5.14.** *Let  $\zeta$  define a wall of type  $(w, p)$ . For  $\alpha \in H_2(X; \mathbb{Z})$ , let  $a = (\zeta \cdot \alpha)/2$ . Then, modulo  $a^{d-2\ell_\zeta}$ ,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to*

$$\frac{1}{4} \cdot (-1)^{h(\zeta)+\ell_\zeta-1} \cdot \frac{(d-2)!}{\ell_\zeta! \cdot (d-2-2\ell_\zeta)!} \cdot a^{d-2-2\ell_\zeta} \cdot (\alpha^2)^{\ell_\zeta}.$$

*Proof.* By Theorem 5.5,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to

$$\frac{1}{4} \cdot \sum_{j=2\ell_\zeta-1}^{2\ell_\zeta} \binom{d-2}{j} \cdot (-1)^{h(\zeta)+\ell_\zeta-1+j} \cdot a^{d-2-j} \cdot S_j$$

modulo  $a^{d-2\ell_\zeta}$ , where  $S_j$  is the notation introduced in (5.7). By Proposition 5.12,  $S_{2\ell_\zeta-1}$  is divisible by  $a$ ; by Proposition 5.9, we have

$$S_{2\ell_\zeta} = \frac{(2\ell_\zeta)!}{\ell_\zeta!} \cdot (\alpha^2)^{\ell_\zeta}.$$

Therefore, modulo  $a^{d-2\ell_\zeta}$ ,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to

$$\frac{1}{4} \cdot (-1)^{h(\zeta)+\ell_\zeta-1} \cdot \frac{(d-2)!}{\ell_\zeta! \cdot (d-2-2\ell_\zeta)!} \cdot a^{d-2-2\ell_\zeta} \cdot (\alpha^2)^{\ell_\zeta}.$$

□

## 6. The formulas when $\ell_\zeta = 0, 1, 2$ .

In this section, we shall compute  $[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d$  and

$$[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$$

by assuming that  $\ell_\zeta = 0, 1, 2$ . Our first result, Theorem 6.1 below, was first obtained by Mong and Kotschick [17].

**Theorem 6.1.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta = 0$ . Then,*

$$[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d = (-1)^{h(\zeta)} \cdot \left(\frac{\zeta \cdot \alpha}{2}\right)^d$$

for  $\alpha \in H_2(X; \mathbb{Z})$ . In other words,  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+) = (-1)^d \cdot \varepsilon(\zeta, \Delta) \cdot (\zeta/2)^d$ .

*Proof.* There are two cases:  $h(\zeta) > 0$  and  $h(\zeta) = 0$ . In the first case when  $h(\zeta) > 0$ , the formula follows immediately from Theorem 5.4. In the second case when  $h(\zeta) = 0$ , we must have  $\zeta^2 = p$  and  $\zeta \cdot K_X = \zeta^2 + 2 = p + 2$  by Corollary 2.7. Then  $\mathfrak{M}_+$  consists of  $\mathfrak{M}_-$  and an additional connected component  $E_{-\zeta}^{0,0} \cong \mathbb{P}^{-p-3}$ . We have constructed a universal sheaf  $\mathcal{U}$  over  $X \times E_{-\zeta}^{0,0}$ :

$$0 \rightarrow \pi_1^* \mathcal{O}_X(\Delta - F) \otimes \pi_2^* \lambda \rightarrow \mathcal{U} \rightarrow \pi_1^* \mathcal{O}_X(F) \rightarrow 0$$

where  $F$  is the unique divisor satisfying  $(2F - \Delta) = \zeta$ ,  $\lambda$  is the line bundle corresponding to a hyperplane in  $E_{-\zeta}^{0,0} \cong \mathbb{P}^{-p-3}$ , and  $\pi_1$  and  $\pi_2$  are the natural projections of  $X \times E_{-\zeta}^{0,0}$ . Thus for  $\alpha \in H_2(X; \mathbb{Z})$ , we have

$$\mu_+(\alpha) = \mu_-(\alpha) - \frac{1}{4} \cdot p_1(\mathcal{U})/\alpha = \mu_-(\alpha) + a\lambda$$

where  $a = (\zeta \cdot \alpha)/2$ . Since  $h(\zeta) = 0$ , we conclude that

$$\mu_+(\alpha)^d = \mu_-(\alpha)^d + \left(\frac{\zeta \cdot \alpha}{2}\right)^d = \mu_-(\alpha)^d + (-1)^{h(\zeta)} \cdot \left(\frac{\zeta \cdot \alpha}{2}\right)^d.$$

□

The proof of the next result is similar to the proof of Theorem 6.1.

**Theorem 6.2.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta = 0$ , let  $d = -p-3$ . Then, for  $\alpha \in H_2(X; \mathbb{Z})$ , we have*

$$[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_- = \frac{1}{4} \cdot (-1)^{h(\zeta)-1} \cdot \left(\frac{\zeta \cdot \alpha}{2}\right)^{d-2}.$$

□

Next, we shall study the difference  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)$  when  $\ell_\zeta = 1$ . In this case, we have to know (5.7) for  $j = 2, 1, 0$ . In view of Propositions 5.9 and 5.12, it suffices to calculate (5.7) for  $j = 0$ . The following lemma deals with this.

**Lemma 6.3.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta = 1$ . Then*

$$S_0 = \sum_{k=0}^1 s_2(\mathcal{E}_\zeta^{1-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,1-k})^\vee) = (6\zeta^2 + 2K_X^2).$$

*Proof.* First, we compute the Chern classes of  $\mathcal{E}_\zeta^{1,0}$ . Let notations be as in Lemma 5.11, and set  $\ell_\zeta = 1$  and  $k = 0$  in Lemma 5.11. Then  $Ext^1 = 0$ . Since  $(H_{\ell_\zeta-k} \times H_k) = X$ , the codimension 2 cycle  $\mathcal{Z}_1$  is exactly the diagonal in  $X \times (H_{\ell_\zeta-k} \times H_k) = X \times X$ . Thus,  $\pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes \mathcal{O}_{\mathcal{Z}_1}) = \mathcal{O}_X(\zeta)$ . By Lemma 5.11 (i), the bundle  $\mathcal{E}_\zeta^{1,0}$  sits in an exact sequence:

$$0 \rightarrow \mathcal{O}_X(\zeta) \rightarrow \mathcal{E}_\zeta^{1,0} \cong R^1 \pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes \mathcal{H}om) \rightarrow \mathcal{O}_X^{\oplus h(\zeta)} \rightarrow 0.$$

Thus,  $c_1(\mathcal{E}_\zeta^{1,0}) = \zeta$  and  $c_2(\mathcal{E}_\zeta^{1,0}) = 0$ .

Next, we compute the Chern classes of  $\mathcal{E}_\zeta^{0,1}$ . Let  $\ell_\zeta = 1$  and  $k = 1$  in Lemma 5.11. Then,  $Ext^1 = \det(N)$  where  $N$  is the normal bundle of  $\mathcal{Z}_1$  in  $X \times X$ . Thus,

$$\pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes Ext^1) = \mathcal{O}_X(\zeta - K_X).$$

By Lemma 5.11 (i), the bundle  $\mathcal{E}_\zeta^{0,1}$  sits in an exact sequence:

$$0 \rightarrow \mathcal{O}_X^{\oplus h(\zeta)} \rightarrow \mathcal{E}_\zeta^{0,1} \rightarrow \mathcal{O}_X(\zeta - K_X) \rightarrow 0.$$

Thus,  $c_1(\mathcal{E}_\zeta^{0,1}) = \zeta - K_X$  and  $c_2(\mathcal{E}_\zeta^{0,1}) = 0$ . Replacing  $\zeta$  by  $-\zeta$  gives  $c_1(\mathcal{E}_{-\zeta}^{0,1}) = -\zeta - K_X$  and  $c_2(\mathcal{E}_{-\zeta}^{0,1}) = 0$ . It follows that  $c_1(\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) = 2\zeta + K_X$  and that

$$c_2(\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) = \zeta \cdot (\zeta + K_X) = \zeta^2 + \zeta \cdot K_X.$$

So we conclude that the Segre class  $s_2(\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee)$  is equal to

$$c_1(\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee)^2 - c_2(\mathcal{E}_\zeta^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) = 3\zeta^2 + 3\zeta \cdot K_X + K_X^2.$$



Replacing  $\zeta$  by  $-\zeta$  gives  $s_2(\mathcal{E}_{-\zeta}^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) = 3\zeta^2 - 3\zeta \cdot K_X + K_X^2$ . Therefore,

$$\begin{aligned}
S_0 &= \sum_{k=0}^1 s_2(\mathcal{E}_{\zeta}^{1-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,1-k})^\vee) \\
&= s_2(\mathcal{E}_{\zeta}^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) + s_2(\mathcal{E}_{\zeta}^{0,1} \oplus (\mathcal{E}_{-\zeta}^{1,0})^\vee) \\
&= s_2(\mathcal{E}_{\zeta}^{1,0} \oplus (\mathcal{E}_{-\zeta}^{0,1})^\vee) + s_2((\mathcal{E}_{\zeta}^{0,1})^\vee \oplus \mathcal{E}_{-\zeta}^{1,0}) \\
&= (3\zeta^2 + 3\zeta \cdot K_X + K_X^2) + (3\zeta^2 - 3\zeta \cdot K_X + K_X^2) \\
&= 6\zeta^2 + 2K_X^2.
\end{aligned}$$

□

Now we can compute the difference  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)$  when  $\ell_\zeta = 1$ .

**Theorem 6.4.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta = 1$ . Then,*

$$\begin{aligned}
[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d &= \\
&= (-1)^{h(\zeta)+1} \cdot \left\{ d(d-1) \cdot a^{d-2} \cdot \alpha^2 + (2K_X^2 + 2d + 6) \cdot a^d \right\}
\end{aligned}$$

for  $\alpha \in H_2(X; \mathbb{Z})$ , where  $a = (\zeta \cdot \alpha)/2$ . In other words,  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)$  is equal to

$$(-1)^{d+1} \cdot \varepsilon(\zeta, \Delta) \cdot \left\{ d(d-1) \cdot \left(\frac{\zeta}{2}\right)^{d-2} \cdot q_X + (2K_X^2 + 2d + 6) \cdot \left(\frac{\zeta}{2}\right)^d \right\}.$$

*Proof.* From 5.4, 5.9, 5.12, and 6.3, we conclude that

$$\begin{aligned}
&[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d \\
&= (-1)^{h(\zeta)+1} \cdot d(d-1) \cdot a^{d-2} \cdot \alpha^2 + (-1)^{h(\zeta)+1} \cdot 8d \cdot a^d \\
&\quad + (-1)^{h(\zeta)+1} \cdot a^d \cdot (6\zeta^2 + 2K_X^2) \\
&= (-1)^{h(\zeta)+1} \cdot \left\{ d(d-1) \cdot a^{d-2} \cdot \alpha^2 + (2K_X^2 + 2d + 6) \cdot a^d \right\}.
\end{aligned}$$

□

For  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$ , we have the following.

**Theorem 6.5.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta = 1$ , let  $d = -p - 3$ . For  $\alpha \in H_2(X; \mathbb{Z})$ , let  $a = (\zeta \cdot \alpha)/2$ . Then,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to*

$$\frac{1}{4} \cdot (-1)^{h(\zeta)} \cdot \left[ (d-2)(d-3) \cdot a^{d-4} \cdot \alpha^2 + (2K_X^2 + 2d - 18) \cdot a^{d-2} \right].$$

*Proof.* By Theorem 5.5,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to

$$\begin{aligned} & \frac{1}{4} \cdot \sum_{j=0}^2 \binom{d-2}{j} \cdot (-1)^{h(\zeta)+j} \cdot a^{d-2-j} \cdot S_j - \\ & \quad - (-1)^{h(\zeta)} \cdot a^{d-2} \cdot \sum_{k=0}^1 ([\mathcal{Z}_{1-k}] + [\mathcal{Z}_k])/x \\ & = \frac{1}{4} \cdot \sum_{j=0}^2 \binom{d-2}{j} \cdot (-1)^{h(\zeta)+j} \cdot a^{d-2-j} \cdot S_j - (-1)^{h(\zeta)} \cdot 2a^{d-2}. \end{aligned}$$

By Proposition 5.9, Proposition 5.12, and Lemma 6.3, we have

$$S_2 = 2\alpha^2, S_1 = -8a, S_0 = 6\zeta^2 + 2K_X^2.$$

Therefore, we conclude that  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to

$$\frac{1}{4} \cdot (-1)^{h(\zeta)} \cdot \left[ (d-2)(d-3) \cdot a^{d-4} \cdot \alpha^2 + (2K_X^2 + 2d - 18) \cdot a^{d-2} \right].$$

□

In the rest of this section, we assume that  $\ell_\zeta = 2$ . The following standard facts about double coverings can be found in [2, 10].

**Lemma 6.6.** *Let  $\phi : Y_1 \rightarrow Y_2$  be a double covering between two smooth projective varieties with  $\phi_* \mathcal{O}_{Y_1} = \mathcal{O}_{Y_2} \oplus L^{-1}$  where  $L$  is a line bundle on  $Y_2$ .*

- (i)  $K_{Y_1} = \phi^*(K_{Y_2} \otimes L)$  and  $L^{\otimes 2} = \mathcal{O}_{Y_2}(B)$  where  $B$  is the branch locus in  $Y_2$  and is the image of the fixed set of the involution  $\iota$  on  $Y_1$ ;
- (ii) If  $D$  is a divisor on  $Y_1$ , then  $\phi_*(\mathcal{O}_{Y_1}(D))$  is a rank 2 bundle on  $Y_2$  with  $c_1(\phi_*(\mathcal{O}_{Y_1}(D))) = \phi_* D - L$  and

$$c_2(\phi_*(\mathcal{O}_{Y_1}(D))) = \frac{1}{2} \cdot [(\phi_* D)^2 - \phi_*(D^2) - \phi_* D \cdot L].$$

Next, we recall some standard facts about the Hilbert scheme  $H_2 = \text{Hilb}^2(X)$ . Let  $\Delta_0 \subset X \times X$  be the diagonal, and let  $\iota$  be the obvious involution on  $\tilde{H}_2 = \text{Bl}_{\Delta_0}(X \times X)$ , the blowup of  $X \times X$  along  $\Delta_0$ . Let  $E$  be the exceptional divisor of the blowup in  $\tilde{H}_2$ . Then,  $H_2 = \tilde{H}_2/\iota$  and the branch locus lies under  $E$ . Let  $\tilde{\mathcal{Z}}_2 \subset X \times \tilde{H}_2$  be the pullback of the codimension 2 cycle  $\mathcal{Z}_2 \subset X \times H_2$ . Then,  $\tilde{\mathcal{Z}}_2$  splits into a union of two cycles  $\tilde{H}_{12}$  and  $\tilde{H}_{13}$  in  $X \times \tilde{H}_2$ , which are the proper transforms in  $X \times \tilde{H}_2$  of the two morphisms of  $X \times X$  into  $X \times (X \times X)$ : the first maps the first factor in  $X \times X$  diagonally into  $X \times X$  which is the product of the first and second factors in  $X \times (X \times X)$ , while the second maps the first factor in  $X \times X$  diagonally into  $X \times X$  which is the product of the first and third factors in  $X \times (X \times X)$ . Thus each  $\tilde{H}_{1j}$  is isomorphic to  $\text{Bl}_{\Delta_0}(X \times X)$ , and the projection of each to  $\tilde{H}_2$  is an isomorphism. If  $\alpha \in H_2(X; \mathbb{Z})$ , then

$$(6.7) \quad [\tilde{\mathcal{Z}}_2]/\alpha = \alpha \otimes 1 + 1 \otimes \alpha = \alpha \otimes 1 + \iota^*(\alpha \otimes 1)$$

where  $\alpha \otimes 1$  and  $1 \otimes \alpha$  are the pull-backs of  $\alpha$  by the two projections of  $\tilde{H}_2$  to  $X$ . Fix  $x \in X$ . Let  $\tilde{X}_x$  be the pull-back of  $X \times x \subset X \times X$  to  $\tilde{H}_2$ . Then,  $\tilde{X}_x$  is isomorphic to the blow-up of  $X$  at  $p$  with the exceptional divisor  $(\tilde{X}_x \cap E)$ ; moreover,

$$(6.8) \quad [\tilde{\mathcal{Z}}_2]/x = \tilde{X}_x + \iota^* \tilde{X}_x.$$

It is known (see p. 685 in [9]) that  $\mathcal{Z}_2$  is smooth. Let  $B$  be the branch locus of the natural double covering from  $\mathcal{Z}_2$  to  $H_2$ . Then,  $B \sim 2L$  for some divisor  $L$  on  $H_2$ , and the pull-back of  $B \subset H_2$  to  $\tilde{H}_2$  is  $2E$ . Let  $i : \mathcal{Z}_2 \rightarrow X \times H_2$  be the embedding, and  $\pi_1$  and  $\pi_2$  be the natural projections of  $X \times H_2$  to  $X$  and  $H_2$  respectively.

In the following, we compute the Chern and Segre classes of  $\mathcal{E}_\zeta^{2-k,k}$  for  $k = 0, 1, 2$ . The method is to use Lemma 5.11 together with Lemma 6.6. We start with  $\mathcal{E}_\zeta^{2,0}$ .

**Lemma 6.9.**  $c_3(\mathcal{E}_\zeta^{2,0}) = c_4(\mathcal{E}_\zeta^{2,0}) = 0$ ,  $c_1(\mathcal{E}_\zeta^{2,0}) = [\mathcal{Z}_2]/\zeta - L$ , and

$$c_2(\mathcal{E}_\zeta^{2,0}) = \frac{1}{2} [([\mathcal{Z}_2]/\zeta)^2 - \zeta^2 \cdot X_x - [\mathcal{Z}_2]/\zeta \cdot L]$$

where  $x$  is any point on  $X$ , and  $X_x$  stands for  $[\mathcal{Z}_2]/x$ .

*Proof.* Let notations be as in Lemma 5.11, and let  $\ell_\zeta = 2$  and  $k = 0$ . Then,  $\text{Ext}^1 = 0$ . By Lemma 5.11 (i),  $\mathcal{E}_\zeta^{2,0}$  sits in an exact sequence

$$0 \rightarrow (\pi_2 \cdot i)_*(\pi_1 \cdot i)^* \mathcal{O}_X(\zeta) \rightarrow \mathcal{E}_\zeta^{2,0} \rightarrow [\mathcal{O}_{H_2}]^{\oplus h(\zeta)} \rightarrow 0.$$

Since  $(\pi_2 \cdot i)_*(\pi_1 \cdot i)^*\mathcal{O}_X(\zeta)$  has rank 2,  $c_3(\mathcal{E}_\zeta^{2,0}) = c_4(\mathcal{E}_\zeta^{2,0}) = 0$ . By Lemma 6.6 (ii),

$$c_1(\mathcal{E}_\zeta^{2,0}) = (\pi_2 \cdot i)_*(\pi_1 \cdot i)^*\zeta - L = [\mathcal{Z}_2]/\zeta - L$$

since  $(\pi_2 \cdot i)_*(\pi_1 \cdot i)^*\zeta = [\mathcal{Z}_2]/\zeta$ ; moreover, we have

$$\begin{aligned} c_2(\mathcal{E}_\zeta^{2,0}) &= \frac{1}{2} [((\pi_2 \cdot i)_*(\pi_1 \cdot i)^*\zeta)^2 - (\pi_2 \cdot i)_*((\pi_1 \cdot i)^*\zeta)^2 - \\ &\quad - (\pi_2 \cdot i)_*(\pi_1 \cdot i)^*\zeta \cdot L] \\ &= \frac{1}{2} [([\mathcal{Z}_2]/\zeta)^2 - \zeta^2 \cdot X_x - [\mathcal{Z}_2]/\zeta \cdot L] \end{aligned}$$

since  $(\pi_2 \cdot i)_*((\pi_1 \cdot i)^*\zeta)^2 = \zeta^2 \cdot (\pi_2 \cdot i)_*(\pi_1 \cdot i)^*x = \zeta^2 \cdot [\mathcal{Z}_2]/x = \zeta^2 \cdot X_x$ .  $\square$

The following follows from Lemma 6.9 and Remark 5.6.

**Corollary 6.10.** *The Segre classes of the bundle  $\mathcal{E}_\zeta^{2,0}$  are given by*

$$\begin{aligned} s_1(\mathcal{E}_\zeta^{2,0}) &= L - [\mathcal{Z}_2]/\zeta \\ s_2(\mathcal{E}_\zeta^{2,0}) &= \frac{1}{2} [([\mathcal{Z}_2]/\zeta)^2 - 3[\mathcal{Z}_2]/\zeta \cdot L + 2L^2 + \zeta^2 \cdot X_x] \\ s_3(\mathcal{E}_\zeta^{2,0}) &= [([\mathcal{Z}_2]/\zeta)^2 \cdot L - 2[\mathcal{Z}_2]/\zeta \cdot L^2 + L^3 - \zeta^2 \cdot X_x \cdot [\mathcal{Z}_2]/\zeta + \zeta^2 \cdot X_x \cdot L \\ s_4(\mathcal{E}_\zeta^{2,0}) &= \frac{(\zeta^2)^2}{2} - 5\zeta^2 - \frac{5}{2}\zeta \cdot K_X + (6\chi(\mathcal{O}_X) - K_X^2). \end{aligned}$$

Here we have identified degree 4 classes with the corresponding integers.

*Proof.* Since the computation is straightforward, we only calculate  $s_4(\mathcal{E}_\zeta^{2,0})$ . For simplicity, let  $c_i$  denote the  $i^{\text{th}}$  Chern class of  $\mathcal{E}_\zeta^{2,0}$ . Note that  $c_3 = c_4 = 0$  by Lemma 6.9. Thus,  $s_4(\mathcal{E}_\zeta^{2,0}) = c_1^4 - 3c_1^2c_2 + c_2^2$  by Remark 5.6. Therefore,

$$\begin{aligned} s_4(\mathcal{E}_\zeta^{2,0}) &= ([\mathcal{Z}_2]/\zeta - L)^4 - 3([\mathcal{Z}_2]/\zeta - L)^2 \cdot \frac{1}{2} [([\mathcal{Z}_2]/\zeta)^2 - \zeta^2 \cdot X_x \\ &\quad - [\mathcal{Z}_2]/\zeta \cdot L] + \frac{1}{4} [([\mathcal{Z}_2]/\zeta)^2 - \zeta^2 \cdot X_x - [\mathcal{Z}_2]/\zeta \cdot L]^2 \\ &= L^4 - \frac{5}{2} \cdot [\mathcal{Z}_2]/\zeta \cdot L^3 + \frac{7}{4} \cdot ([\mathcal{Z}_2]/\zeta)^2 \cdot L^2 + \frac{3}{2}\zeta^2 \cdot X_x \cdot L^2 \\ &\quad - \frac{1}{4}([\mathcal{Z}_2]/\zeta)^4 + \frac{1}{4}(\zeta^2)^2 \cdot X_x^2 + \zeta^2 \cdot ([\mathcal{Z}_2]/\zeta)^2 \cdot X_x \end{aligned}$$

since  $([\mathcal{Z}_2]/\zeta)^3 \cdot L = 0 = [\mathcal{Z}_2]/\zeta \cdot L \cdot X_x$ . Now, we need a claim.

**Claim.** Let  $\alpha, \beta \in H_2(X; \mathbb{Z})$ . Then, we have the following:

- (i)  $[\mathcal{Z}_2]/\alpha \cdot [\mathcal{Z}_2]/\beta \cdot X_x = \alpha \cdot \beta$ ;
- (ii)  $X_x^2 = 1$ ;
- (iii)  $X_x \cdot L^2 = -1$ ;
- (iv)  $L^4 = 6\chi(\mathcal{O}_X) - K_X^2$ ;
- (v)  $[\mathcal{Z}_2]/\alpha \cdot L^3 = \alpha \cdot K_X$ ;
- (vi)  $[\mathcal{Z}_2]/\alpha \cdot [\mathcal{Z}_2]/\beta \cdot L^2 = -2(\alpha \cdot \beta)$ .

*Proof.* Let  $\pi: \tilde{H}_2 \rightarrow H_2 = \tilde{H}_2/\iota$  be the quotient map. By (6.8), we have

$$\pi^* X_x = \pi^*([\mathcal{Z}_2]/x) = [\tilde{\mathcal{Z}}_2]/x = (\tilde{X}_x + \iota^* \tilde{X}_x).$$

(i) Recall from (6.7) that  $\pi^*([\mathcal{Z}_2]/\alpha) = [\tilde{\mathcal{Z}}_2]/\alpha = \alpha \otimes 1 + 1 \otimes \alpha$ . Thus,

$$\begin{aligned} [\mathcal{Z}_2]/\alpha \cdot [\mathcal{Z}_2]/\beta \cdot X_x &= \frac{1}{2} \cdot \pi^*([\mathcal{Z}_2]/\alpha) \cdot \pi^*([\mathcal{Z}_2]/\beta) \cdot \pi^* X_x \\ &= \frac{1}{2} \cdot (\alpha \otimes 1 + 1 \otimes \alpha) \cdot (\beta \otimes 1 + 1 \otimes \beta) \cdot (\tilde{X}_x + \iota^* \tilde{X}_x) \\ &= \alpha \cdot \beta. \end{aligned}$$

(ii) Let  $x_1 \in X$  be a point different from  $x$ . Then,

$$\begin{aligned} X_x^2 &= X_x \cdot X_{x_1} = \frac{1}{2} \cdot \pi^*(X_x) \cdot \pi^*(X_{x_1}) \\ &= \frac{1}{2} \cdot (\tilde{X}_x + \iota^* \tilde{X}_x) \cdot (\tilde{X}_{x_1} + \iota^* \tilde{X}_{x_1}) \\ &= 1. \end{aligned}$$

(iii) Since  $B \sim 2L$  and  $\pi^*(B) = 2E$ ,  $\pi^*(L) \sim E$ . Thus,

$$X_x \cdot L^2 = \frac{1}{2} \cdot (\tilde{X}_x + \iota^* \tilde{X}_x) \cdot E^2 = \tilde{X}_x \cdot E^2 = (\tilde{X}_x \cdot E)^2 = -1.$$

(iv) Since  $E = \mathbb{P}(N^\vee)$  where  $N$  is the normal bundle of  $\Delta_0$  in  $X \times X$ ,  $-E|E = \xi$  is the tautological line bundle on  $E$ . Since  $N = T_{\Delta_0}$ ,

$$\xi^2 = -(\pi|E)^* c_1(N) \cdot \xi - c_2(N) = (\pi|E)^* K_{\Delta_0} \cdot \xi + (K_X^2 - 12\chi(\mathcal{O}_X)).$$

It follows that  $\xi^3 = (2K_X^2 - 12\chi(\mathcal{O}_X)) \cdot \xi$ . Therefore,

$$L^4 = \frac{1}{2} \cdot E^4 = -\frac{1}{2} \cdot \xi^3 = 6\chi(\mathcal{O}_X) - K_X^2.$$

(v) Note that  $(\alpha \otimes 1)|E = (\pi|E)^* \alpha$  since  $\Delta_0 \cong X$ . Thus,

$$[\mathcal{Z}_2]/\alpha \cdot L^3 = \frac{1}{2} \cdot (\alpha \otimes 1 + 1 \otimes \alpha) \cdot E^3 = (\alpha \otimes 1) \cdot E^3 = (\pi|E)^* \alpha \cdot \xi^2 = \alpha \cdot K_X.$$

(vi) Again since  $(\alpha \otimes 1)|E = (\pi|E)^* \alpha = (1 \otimes \alpha)|E$ , we have

$$\begin{aligned} [\mathcal{Z}_2]/\alpha \cdot [\mathcal{Z}_2]/\beta \cdot L^2 &= \frac{1}{2} \cdot (\alpha \otimes 1 + 1 \otimes \alpha) \cdot (\beta \otimes 1 + 1 \otimes \beta) \cdot E^2 \\ &= -2 \cdot (\pi|E)^* \alpha \cdot (\pi|E)^* \beta \cdot \xi \\ &= -2(\alpha \cdot \beta). \end{aligned}$$

We continue the calculation of  $s_4(\mathcal{E}_\zeta^{2,0})$ . By Lemma 5.8 (i),  $([\mathcal{Z}_2]/\zeta)^4 = 3(\zeta^2)^2$ . It follows from the above Claim with a straightforward computation that

$$s_4(\mathcal{E}_\zeta^{2,0}) = \frac{(\zeta^2)^2}{2} - 5\zeta^2 - \frac{5}{2}\zeta \cdot K_X + (6\chi(\mathcal{O}_X) - K_X^2).$$

□

Next, we compute the Chern and Segre classes of  $\mathcal{E}_\zeta^{0,2}$  on  $H_2$ .

**Lemma 6.11.**  $c_3(\mathcal{E}_\zeta^{0,2}) = c_4(\mathcal{E}_\zeta^{0,2}) = 0$ ,  $c_1(\mathcal{E}_\zeta^{0,2}) = [\mathcal{Z}_2]/(\zeta - K_X) + L$ , and

$$c_2(\mathcal{E}_\zeta^{0,2}) = \frac{1}{2} [L \cdot [\mathcal{Z}_2]/(\zeta - K_X) + [[\mathcal{Z}_2]/(\zeta - K_X)]^2 - (\zeta - K_X)^2 \cdot X_x].$$

*Proof.* Let  $\ell_\zeta = 2$  and  $k = 2$  in Lemma 5.11. By Lemma 6.6 (i),

$$(\det T_{\mathcal{Z}_2})^{-1} = \mathcal{O}_{\mathcal{Z}_2}(K_{\mathcal{Z}_2}) = (\pi_2 \cdot i)^* \mathcal{O}_{H_2}(K_{H_2} + L).$$

Let  $N_{\mathcal{Z}_2}$  be the normal bundle of  $\mathcal{Z}_2$  in  $X \times H_2$ . Since  $\mathcal{Z}_2$  is smooth and has codimension 2 in  $X \times H_2$ ,  $\text{Ext}^1 = \text{Ext}^1(I_{\mathcal{Z}_2}, \mathcal{O}_{X \times H_2})$  is isomorphic to

$$\det N_{\mathcal{Z}_2} = i^* \det T_{X \times H_2} \otimes (\det T_{\mathcal{Z}_2})^{-1} = \mathcal{O}_{\mathcal{Z}_2}((\pi_2 \cdot i)^* L - (\pi_1 \cdot i)^* K_X).$$

By Lemma 5.11 (i),  $\mathcal{E}_\zeta^{0,2}$  sits in an exact sequence

$$0 \rightarrow [\mathcal{O}_{H_2}]^{\oplus h(\zeta)} \rightarrow \mathcal{E}_\zeta^{0,2} \rightarrow (\pi_2 \cdot i)_* \mathcal{O}_{\mathcal{Z}_2}((\pi_2 \cdot i)^* L + (\pi_1 \cdot i)^*(\zeta - K_X)) \rightarrow 0.$$

Note that  $(\pi_2 \cdot i)_*(\pi_2 \cdot i)^*L = 2L$ . Thus, by Lemma 6.6 (ii),

$$c_1(\mathcal{E}_\zeta^{0,2}) = (\pi_2 \cdot i)_*[(\pi_2 \cdot i)^*L + (\pi_1 \cdot i)^*(\zeta - K_X)] - L = [\mathcal{Z}_2]/(\zeta - K_X) + L.$$

Also, Lemma 6.6 (ii) together with a straightforward calculation gives

$$c_2(\mathcal{E}_\zeta^{0,2}) = \frac{1}{2} [L \cdot [\mathcal{Z}_2]/(\zeta - K_X) + [[\mathcal{Z}_2]/(\zeta - K_X)]^2 - (\zeta - K_X)^2 \cdot X_x]$$

where we have used the projection formula

$$(\pi_2 \cdot i)_*[(\pi_2 \cdot i)^*L \cdot (\pi_1 \cdot i)^*(\zeta - K_X)] = L \cdot (\pi_2 \cdot i)_*(\pi_1 \cdot i)^*(\zeta - K_X)$$

and the fact that  $(\pi_2 \cdot i)_*(\pi_2 \cdot i)^*L^2 = 2L^2$ .  $\square$

The following follows from Lemma 6.11 and Remark 5.6.

**Corollary 6.12.** *The Segre classes of  $\mathcal{E}_\zeta^{0,2}$  are given by*

$$\begin{aligned} s_1(\mathcal{E}_\zeta^{0,2}) &= [\mathcal{Z}_2]/(K_X - \zeta) - L \\ s_2(\mathcal{E}_\zeta^{0,2}) &= \frac{1}{2} [[[\mathcal{Z}_2]/(\zeta - K_X)]^2 + \\ &\quad + 3[\mathcal{Z}_2]/(\zeta - K_X) \cdot L + 2L^2 + (\zeta - K_X)^2 \cdot X_x] \\ s_3(\mathcal{E}_\zeta^{0,2}) &= -[[[\mathcal{Z}_2]/(\zeta - K_X)]^2 \cdot L - 2[\mathcal{Z}_2]/(\zeta - K_X) \cdot L^2 - L^3 \\ &\quad - (\zeta - K_X)^2 \cdot X_x \cdot [\mathcal{Z}_2]/(\zeta - K_X) - (\zeta - K_X)^2 \cdot X_x \cdot L \\ s_4(\mathcal{E}_\zeta^{0,2}) &= \frac{((K_X - \zeta)^2)^2}{2} - 5(K_X - \zeta)^2 - \\ &\quad - \frac{5}{2}(K_X - \zeta) \cdot K_X + (6\chi(\mathcal{O}_X) - K_X^2). \end{aligned}$$

*Proof.* The calculation of  $s_4(\mathcal{E}_\zeta^{0,2})$  is similar to that of  $s_4(\mathcal{E}_\zeta^{2,0})$  in Corollary 6.10.  $\square$

Note that  $s_4(\mathcal{E}_\zeta^{0,2})$  may be obtained from  $s_4(\mathcal{E}_\zeta^{2,0})$  by replacing  $\zeta$  by  $K_X - \zeta$ , and indeed this holds more generally for  $s_i$  when we add the sign  $(-1)^i$ .

Now we compute the Chern and Segre classes of  $\mathcal{E}_\zeta^{1,1}$  on  $X \times X$ .

**Lemma 6.13.** *Let  $\tau_1$  and  $\tau_2$  be the two natural projections of  $X \times X$  to  $X$ , let  $\Delta_0$  be the diagonal in  $X \times X$ , and let  $j : \Delta_0 \rightarrow X \times X$  be the inclusion.*

Then

$$\begin{aligned}
c_1(\mathcal{E}_\zeta^{1,1}) &= \tau_1^* \zeta + \tau_2^*(\zeta - K_X) \\
c_2(\mathcal{E}_\zeta^{1,1}) &= \tau_1^* \zeta \cdot \tau_2^*(\zeta - K_X) + \Delta_0 \\
c_3(\mathcal{E}_\zeta^{1,1}) &= \tau_1^* \zeta \cdot \Delta_0 - \tau_2^*(\zeta - K_X) \cdot \Delta_0 - j_* K_{\Delta_0} \\
c_4(\mathcal{E}_\zeta^{1,1}) &= -\frac{K_{\Delta_0}^2}{2}.
\end{aligned}$$

*Proof.* Let  $\ell_\zeta = 2$  and  $k = 1$  in Lemma 5.11. Recall that  $\pi_1$  and  $\pi_2$  are the natural projections of  $X \times (X \times X)$  to  $X$  and  $(X \times X)$  respectively.

**Claim 1.**  $\pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes Ext^1) \cong \tau_2^* \mathcal{O}_X(\zeta - K_X) \otimes I_{\Delta_0}$ .

*Proof.* Let  $\Delta_{12}$  be the diagonal in  $X \times X$  which is formed by the first and second factors in  $X \times (X \times X)$ , and let  $\Delta_{13}$  be the diagonal in  $X \times X$  which is formed by the first and third factors in  $X \times (X \times X)$ . Then,  $\Delta_{12} \times X$  and  $\Delta_{13} \times X$  are smooth codimension 2 subvarieties in  $X \times (X \times X)$ . Here it is understood that the factor  $X$  in  $\Delta_{13} \times X$  is embedded as the second factor in  $X \times (X \times X)$ . Moreover,  $\Delta_{12} \times X$  and  $\Delta_{13} \times X$  intersect properly along the diagonal  $\Delta_{123}$  in  $X \times X \times X$ . Thus, from Lemma 5.10 (iii), we conclude that

$$Ext^1 = Ext^1(I_{\Delta_{13} \times X}, I_{\Delta_{12} \times X}) \cong I \otimes \det N$$

where  $N$  is the normal bundle  $\Delta_{13} \times X$  in  $X \times (X \times X)$ , and  $I$  is the ideal sheaf of  $\Delta_{123}$  in  $\Delta_{13} \times X$ . Now, the restriction of  $\pi_2$  to  $\Delta_{13} \times X$  gives an isomorphism from  $\Delta_{13} \times X$  to  $X \times X$ . Via this isomorphism,  $\Delta_{123}$  in  $\Delta_{13} \times X$  is identified with the diagonal  $\Delta_0$  in  $X \times X$ ,  $\det N$  is identified with  $\tau_2^*(-K_X)$ , and the restriction  $\pi_1^* \mathcal{O}_X(\zeta)|_{(\Delta_{13} \times X)}$  is identified with  $\tau_2^*(\zeta)$ . Therefore,

$$\pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes Ext^1) \cong \pi_{2*}(\pi_1^* \mathcal{O}_X(\zeta) \otimes I \otimes \det N) = \tau_2^* \mathcal{O}_X(\zeta - K_X) \otimes I_{\Delta_0}.$$



□

Note that  $\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes \mathcal{O}_{\Delta_{12} \times X}) = \tau_1^*(\zeta)$ . Thus by Lemma 5.11 (i) and Claim 1, we have a row exact sequence and a column exact sequence (6.14)

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \tau_1^*(\zeta) & & & \\
& & & \downarrow & & & \\
0 \rightarrow & R^1\pi_{2*}(\pi_1^*\mathcal{O}_X(\zeta) \otimes \text{Hom}) & \rightarrow & \mathcal{E}_\zeta^{1,1} & \rightarrow & \tau_2^*\mathcal{O}_X(\zeta - K_X) \otimes I_{\Delta_0} & \rightarrow 0. \\
& & & \downarrow & & & \\
& & & [\mathcal{O}_{X \times X}]^{\oplus h(\zeta)} & & & \\
& & & \downarrow & & & \\
& & & 0 & & & 
\end{array}$$

In the next claim, we compute the Chern classes of  $I_{\Delta_0}$ . Clearly,  $c_0(I_{\Delta_0}) = 1$ .

**Claim 2.**  $c_1(I_{\Delta_0}) = 0$ ,  $c_2(I_{\Delta_0}) = \Delta_0$ ,  $c_3(I_{\Delta_0}) = -j_*K_{\Delta_0}$ ,  $c_4(I_{\Delta_0}) = K_{\Delta_0}^2/2$ .

*Proof.* Note that  $\text{Todd}(N_{\Delta_0})^{-1} = 1 + K_{\Delta_0}/2 + (K_{\Delta_0}^2/4 - \chi(\mathcal{O}_{\Delta_0}))$ . By a formula on p.288 of [12] (a special case of the Grothendieck-Riemann-Roch Theorem),

$$\begin{aligned}
\text{ch}(j!\mathcal{O}_{\Delta_0}) &= j_*(\text{Todd}(N_{\Delta_0})^{-1} \cdot \text{ch}(\mathcal{O}_{\Delta_0})) = j_*(\text{Todd}(N_{\Delta_0})^{-1}) \\
&= \Delta_0 + \frac{j_*K_{\Delta_0}}{2} + j_*\left(\frac{K_{\Delta_0}^2}{4} - \chi(\mathcal{O}_{\Delta_0})\right).
\end{aligned}$$

Since  $\text{ch}(j!\mathcal{O}_{\Delta_0})$  is just equal to  $\text{ch}(j_*\mathcal{O}_{\Delta_0})$ , we obtain

$$\text{ch}(I_{\Delta_0}) = \text{ch}(\mathcal{O}_{X \times X}) - \text{ch}(j_*\mathcal{O}_{\Delta_0}) = 1 - \Delta_0 - \frac{j_*K_{\Delta_0}}{2} - j_*\left(\frac{K_{\Delta_0}^2}{4} - \chi(\mathcal{O}_{\Delta_0})\right).$$

From this, the Chern classes of  $I_{\Delta_0}$  follows immediately. In particular,

$$c_4(I_{\Delta_0}) = \frac{\Delta_0^2}{2} + j_*\left(\frac{3K_{\Delta_0}^2}{2} - 6\chi(\mathcal{O}_{\Delta_0})\right) = \frac{K_{\Delta_0}^2}{2}$$

since  $\Delta_0^2 = c_2(T_X) = 12\chi(\mathcal{O}_X) - K_X^2$  (see the Example 8.1.12 in [12]).  $\square$

Now the calculation of the Chern classes of  $\mathcal{E}_\zeta^{1,1}$  follows from (6.14) and Claim 2. In particular,

$$\begin{aligned} c_4(\mathcal{E}_\zeta^{1,1}) &= -\tau_1^*\zeta \cdot \tau_2^*(\zeta - K_X) \cdot \Delta_0 - \tau_1^*\zeta \cdot j_*K_{\Delta_0} + \tau_2^*(\zeta - K_X)^2 \cdot \Delta_0 \\ &\quad + 2\tau_2^*(\zeta - K_X) \cdot j_*K_{\Delta_0} + \frac{j_*K_{\Delta_0}^2}{2} = -\frac{K_{\Delta_0}^2}{2} \end{aligned}$$

since  $\tau_1^*\zeta \cdot \tau_2^*(\zeta - K_X) \cdot \Delta_0 = \zeta \cdot (\zeta - K_X)$  and  $\tau_1^*\zeta \cdot j_*K_{\Delta_0} = \zeta \cdot K_X$ .  $\square$

The next result follows immediately from Lemma 6.13 and Remark 5.6.

**Corollary 6.15.** *Let notations be the same as in Lemma 6.13. Then*

$$\begin{aligned} s_1(\mathcal{E}_\zeta^{1,1}) &= -\tau_1^*\zeta - \tau_2^*(\zeta - K_X) \\ s_2(\mathcal{E}_\zeta^{1,1}) &= \tau_1^*\zeta^2 + \tau_1^*\zeta \cdot \tau_2^*(\zeta - K_X) + \tau_2^*(\zeta - K_X)^2 - \Delta_0 \\ s_3(\mathcal{E}_\zeta^{1,1}) &= -\tau_1^*\zeta^2 \cdot \tau_2^*(\zeta - K_X) - \tau_1^*\zeta \cdot \tau_2^*(\zeta - K_X)^2 \\ &\quad + \tau_1^*\zeta \cdot \Delta_0 + 3\tau_2^*(\zeta - K_X) \cdot \Delta_0 + j_*K_{\Delta_0} \\ s_4(\mathcal{E}_\zeta^{1,1}) &= (12\zeta \cdot K_X - 12\zeta^2 - 3K_X^2). \end{aligned}$$

$\square$

We can now work out (5.7) explicitly for  $\ell_\zeta = 2$  and  $j = 2, 1, 0$ . For simplicity, let

$$(6.16) \quad S_j = \sum_{k=0}^2 S_{j,k} = \sum_{k=0}^2 ([\mathcal{Z}_{2-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot s_{4-j}(\mathcal{E}_\zeta^{2-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,2-k})^\vee).$$

**Lemma 6.17.**  $S_2 = 64a^2 + (12\zeta^2 + 4K_X^2 - 20)\alpha^2$  where  $a = (\zeta \cdot \alpha)/2$ .

*Proof.* Note that  $s_i(\mathcal{E}_{-\zeta}^{k,2-k})$  (respectively,  $S_{j,k}$ ) can be obtained from  $s_i(\mathcal{E}_\zeta^{k,2-k})$  (respectively,  $(-1)^j \cdot S_{j,2-k}$ ) by replacing  $\zeta$  by  $-\zeta$ . Also,  $S_{2,2}$  is equal to

$$\begin{aligned} ([\mathcal{Z}_2]/\alpha)^2 \cdot s_2(\mathcal{E}_\zeta^{0,2} \oplus (\mathcal{E}_{-\zeta}^{2,0})^\vee) &= \\ &= ([\mathcal{Z}_2]/\alpha)^2 \cdot \left[ s_2(\mathcal{E}_\zeta^{0,2}) - s_1(\mathcal{E}_\zeta^{0,2}) \cdot s_1(\mathcal{E}_{-\zeta}^{2,0}) + s_2(\mathcal{E}_{-\zeta}^{2,0}) \right]. \end{aligned}$$

Therefore, by Corollary 6.10 and Corollary 6.12, we obtain

$$S_{2,2} + S_{2,0} = 32a^2 + (6\zeta^2 + 2K_X^2 - 12)\alpha^2 + 2(\alpha \cdot K_X)^2.$$

Let  $\tau_1$  and  $\tau_2$  be the projections of  $X \times X$  to  $X$ . Then, by Corollary 6.15,

$$\begin{aligned} S_{2,1} &= (\tau_1^* \alpha + \tau_2^* \alpha)^2 \cdot s_2(\mathcal{E}_\zeta^{1,1} \oplus (\mathcal{E}_{-\zeta}^{1,1})^\vee) \\ &= (\tau_1^* \alpha + \tau_2^* \alpha)^2 \cdot \left[ s_2(\mathcal{E}_\zeta^{1,1}) - s_1(\mathcal{E}_\zeta^{1,1}) \cdot s_1(\mathcal{E}_{-\zeta}^{1,1}) + s_2(\mathcal{E}_{-\zeta}^{1,1}) \right] \\ &= 32a^2 + (6\zeta^2 + 2K_X^2 - 8)\alpha^2 - 2(\alpha \cdot K_X)^2. \end{aligned}$$

It follows that  $S_2 = (S_{2,2} + S_{2,0}) + S_{2,1} = 64a^2 + (12\zeta^2 + 4K_X^2 - 20)\alpha^2$ .  $\square$

Next, adopting the same method as in the proof of Lemma 6.17, we compute the values of  $S_1$  and  $S_0$  in the next two lemmas respectively.

**Lemma 6.18.**  $S_1 = -(48\zeta^2 + 16K_X^2 - 120)a$  where  $a = (\zeta \cdot \alpha)/2$ .

*Proof.* In view of (6.16), we have to compute  $S_{1,2}$ ,  $S_{1,1}$ , and  $S_{1,0}$ . Note that  $S_{1,0}$  can be obtained from  $-S_{1,2}$  by replacing  $\zeta$  by  $-\zeta$ . Using Corollary 6.10 and Corollary 6.12, we see that  $(S_{1,2} + S_{1,0}) = -(24\zeta^2 + 8K_X^2 - 72)a - 6(\zeta \cdot K_X)(\alpha \cdot K_X)$ . Let  $\tau_1$  and  $\tau_2$  be the projections of  $X \times X$  to  $X$ . Then, by Corollary 6.15,

$$\begin{aligned} S_{1,1} &= (\tau_1^* \alpha + \tau_2^* \alpha) \cdot s_3(\mathcal{E}_\zeta^{1,1} \oplus (\mathcal{E}_{-\zeta}^{1,1})^\vee) \\ &= -(24\zeta^2 + 8K_X^2 - 48)a + 6(\zeta \cdot K_X)(\alpha \cdot K_X). \end{aligned}$$

It follows that  $S_1 = (S_{1,2} + S_{1,0}) + S_{1,1} = -(48\zeta^2 + 16K_X^2 - 120)a$ .  $\square$

**Lemma 6.19.**  $S_0 = 18(\zeta^2)^2 + (12K_X^2 - 105)\zeta^2 + [2(K_X^2)^2 - 50K_X^2 + 96]$ .

*Proof.* We need to compute  $S_{0,2}$ ,  $S_{0,1}$ , and  $S_{0,0}$ . Again,  $S_{0,0}$  can be obtained from  $S_{0,2}$  by replacing  $\zeta$  by  $-\zeta$ . Using Corollary 6.10 and Corollary 6.12, we see that

$$(S_{0,2} + S_{0,0}) = 9(\zeta^2)^2 + (6K_X^2 - 63)\zeta^2 + [(K_X^2)^2 - 43K_X^2 + 60].$$

By Corollary 6.15,  $S_{0,1} = 9(\zeta^2)^2 + (6K_X^2 - 42)\zeta^2 + [(K_X^2)^2 - 7K_X^2 + 36]$ . Therefore,

$$S_0 = (S_{0,2} + S_{0,0}) + S_{0,1} = 18(\zeta^2)^2 + (12K_X^2 - 105)\zeta^2 + [2(K_X^2)^2 - 50K_X^2 + 96].$$

□

Now we can calculate the difference  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)$  when  $\ell_\zeta = 2$ .

**Theorem 6.20.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta = 2$ . Then*

$$[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d = (-1)^{h(\zeta)} \cdot \left\{ g_0 \cdot a^d + g_1 \cdot a^{d-2} \cdot \alpha^2 + g_2 \cdot a^{d-4} \cdot (\alpha^2)^2 \right\}$$

for  $\alpha \in H_2(X; \mathbb{Z})$ , where  $a$  stands for  $(\zeta \cdot \alpha)/2$  and

$$\begin{aligned} g_2 &= \frac{d!}{2! \cdot (d-4)!} \\ g_1 &= \binom{d}{2} \cdot (4K_X^2 + 4d + 8) \\ g_0 &= 2d^2 + 4d \cdot K_X^2 + 2(K_X^2)^2 + 13d + 10K_X^2 + 21. \end{aligned}$$

In other words, the difference  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)$  is equal to

$$(-1)^d \cdot \varepsilon(\zeta, \Delta) \cdot \left\{ g_0 \cdot \left(\frac{\zeta}{2}\right)^d + g_1 \cdot \left(\frac{\zeta}{2}\right)^{d-2} \cdot q_X + g_2 \cdot \left(\frac{\zeta}{2}\right)^{d-4} \cdot q_X^2 \right\}.$$

*Proof.* In view of Theorem 5.4 and the notation (6.16), we have

$$[\mu_+(\alpha)]^d - [\mu_-(\alpha)]^d = \sum_{j=0}^4 \binom{d}{j} \cdot (-1)^{h(\zeta)+j} \cdot a^{d-j} \cdot S_j.$$

Now,  $S_4$  and  $S_3$  are given by Proposition 5.9 and Proposition 5.12 respectively;  $S_2, S_1$ , and  $S_0$  are computed in the previous three lemmas. So it follows that the coefficient of  $(-1)^{h(\zeta)} \cdot a^{d-4} \cdot (\alpha^2)^2$  is equal to

$$g_2 = \frac{d!}{2! \cdot (d-4)!}.$$

Similarly, also keeping in mind that  $\zeta^2 = (p+8) = (5-d)$ , we have

$$\begin{aligned} g_1 &= \binom{d}{2} \cdot (12\zeta^2 + 4K_X^2 + 16d - 52) = \binom{d}{2} \cdot (4K_X^2 + 4d + 8) \\ g_0 &= 64 \cdot \binom{d}{2} + (48\zeta^2 + 16K_X^2 - 120) \cdot d + \\ &\quad + [18(\zeta^2)^2 + 12\zeta^2 \cdot K_X^2 + 2(K_X^2)^2 - 105\zeta^2 - 50K_X^2 + 96] \\ &= 2d^2 + 4d \cdot K_X^2 + 2(K_X^2)^2 + 13d + 10K_X^2 + 21. \end{aligned}$$

**Corollary 6.21.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta \leq 2$ . Then, the difference  $\delta_{w,p}^X(\mathcal{C}_-, \mathcal{C}_+)$  of Donaldson polynomial invariants is a polynomial in  $\zeta$  and  $q_X$  with coefficients involving only  $\zeta^2$ , homotopy invariants of  $X$ , and universal constants.*

*Proof.* Follows from Theorems 6.1, 6.4, and 6.20.  $\square$

Finally, we compute the difference  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  for  $\ell_\zeta = 2$ .

**Theorem 6.22.** *Let  $\zeta$  define a wall of type  $(w, p)$  with  $\ell_\zeta = 2$ , and let  $d = -p - 3$ . Then,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to*

$$\frac{1}{4} \cdot (-1)^{h(\zeta)+1} \cdot \left\{ \tilde{g}_0 \cdot a^{d-2} + \tilde{g}_1 \cdot a^{d-4} \cdot \alpha^2 + \tilde{g}_2 \cdot a^{d-6} \cdot (\alpha^2)^2 \right\}$$

for  $\alpha \in H_2(X; \mathbb{Z})$ , where  $a$  stands for  $(\zeta \cdot \alpha)/2$  and

$$\begin{aligned} \tilde{g}_2 &= \frac{(d-2)!}{2! \cdot (d-6)!} \\ \tilde{g}_1 &= \binom{d-2}{2} \cdot (4K_X^2 + 4d - 40) \\ \tilde{g}_0 &= 2d^2 + 4d \cdot K_X^2 + 2(K_X^2)^2 - 35d - 38K_X^2 - 99. \end{aligned}$$

*Proof.* By Theorem 5.5,  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to

$$\frac{1}{4} \cdot \sum_{j=0}^4 \binom{d-2}{j} \cdot (-1)^{h(\zeta)+1+j} \cdot a^{d-2-j} \cdot S_j - \sum_{j=0}^2 \binom{d-2}{j} \cdot (-1)^{h(\zeta)+1+j} \cdot a^{d-2-j} \cdot T_j$$

where for simplicity we have defined  $T_j = \sum_{k=0}^2 T_{j,k}$  as

$$\sum_{k=0}^2 ([\mathcal{Z}_{2-k}]/\alpha + [\mathcal{Z}_k]/\alpha)^j \cdot ([\mathcal{Z}_{2-k}] + [\mathcal{Z}_k])/x \cdot s_{2-j}(\mathcal{E}_\zeta^{2-k,k} \oplus (\mathcal{E}_{-\zeta}^{k,2-k})^\vee).$$

Next, we compute  $T_0$ . Using Corollary 6.10 and Corollary 6.12, we obtain

$$\begin{aligned} T_{0,0} &= X_x \cdot s_2(\mathcal{E}_\zeta^{2,0} \oplus (\mathcal{E}_{-\zeta}^{0,2})^\vee) \\ &= X_x \cdot \left[ s_2(\mathcal{E}_\zeta^{2,0}) - s_1(\mathcal{E}_\zeta^{2,0}) \cdot s_1(\mathcal{E}_{-\zeta}^{0,2}) + s_2(\mathcal{E}_{-\zeta}^{0,2}) \right] \\ &= (3\zeta^2 + 3\zeta \cdot K_X + K_X^2 - 3). \end{aligned}$$

Note that  $T_{0,2}$  can be obtained from  $T_{0,0}$  by replacing  $\zeta$  by  $-\zeta$ . Thus,

$$T_{0,2} = (3\zeta^2 - 3\zeta \cdot K_X + K_X^2 - 3).$$

Similarly, using Corollary 6.15, we get  $T_{0,1} = (6\zeta^2 + 2K_X^2 - 4)$ . Therefore,

$$T_0 = \sum_{k=0}^2 T_{0,k} = (12\zeta^2 + 4K_X^2 - 10).$$

By similar but much simpler arguments, we conclude that  $T_1 = -16a$  and  $T_2 = 4\alpha^2$ .

From (5.9), (5.12), (6.17), (6.18), and (6.19), we have

$$\begin{aligned} S_4 &= 12(\alpha^2)^2, \\ S_3 &= -48a \cdot \alpha^2, \\ S_2 &= 64a^2 + (12\zeta^2 + 4K_X^2 - 20)\alpha^2, \\ S_1 &= -(48\zeta^2 + 16K_X^2 - 120)a, \\ S_0 &= 18(\zeta^2)^2 + (12K_X^2 - 105)\zeta^2 + [2(K_X^2)^2 - 50K_X^2 + 96]. \end{aligned}$$

Putting all these together, we see that  $[\mu_+(\alpha)]^{d-2} \cdot \nu_+ - [\mu_-(\alpha)]^{d-2} \cdot \nu_-$  is equal to

$$\frac{1}{4} \cdot (-1)^{h(\zeta)+1} \cdot \left\{ \tilde{g}_0 \cdot a^{d-2} + \tilde{g}_1 \cdot a^{d-4} \cdot \alpha^2 + \tilde{g}_2 \cdot a^{d-6} \cdot (\alpha^2)^2 \right\}$$

where  $\tilde{g}_0, \tilde{g}_1$ , and  $\tilde{g}_2$  are as defined in the statement of Theorem 6.22 above.

□

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*Added in the proof.* Recently, Ellingsrud and Göttsche obtained detailed information on the transition formula, and showed that there is an algorithm that computes the  $SU(2)$  and  $SO(3)$  polynomial invariants for the projective plane and other rational surfaces  $X$  as long as the corresponding chamber contains an ample divisor  $H$  with  $H \cdot K_X < 0$ .