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# On Harder-Narasimhan filtration of the tangent bundle

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## 1. Introduction.

Given a vector bundle $V$ on a polarized smooth projective variety, or more generally on a compact Kähler manifold, there is a natural filtration of $V$, called the Harder-Narasimhan filtration, such that the subsequent quotients are semi-stable sheaves satisfying a numerical condition. This filtration was introduced in [HN]. Here we are concerned with the Harder-Narasimhan filtration of the tangent bundle.

If the canonical bundle is ample is then from a Theorem of Yau it follows that the tangent bundle is semi-stable with respect to the polarization $K$. In general, of course, the tangent bundle is not semi-stable. In Theorem 2.2 we give a criterion for the length of the Harder-Narasimhan filtration of the tangent bundle to be at most two.

There is a well-known question of whether any holomorphic bundle admitting a holomorphic connection actually admits a flat connection. In Theorem 3.1 we produce a class of compact Kähler manifolds with the property that any holomorphic bundle on them with a holomorphic connection admits a flat connection. The proofs of Theorem 2.2 and Theorem 3.1 are quite similar in spirit and involve, among other things, a systematic use of the Leibniz identity.

## 2. A criterion for bounding the length.

Let $X$ be an irreducible smooth projective variety over $\mathbb{C}$ of dimension $d$. Fix an ample line bundle $L$ on $X$.

For an $\mathcal{O}_X$ coherent sheaf $F$ on $X$, the degree of $X$ is defined by

$$\deg(F) \ := \ \int_X c_1(F) \cup c_1(L)^{d-1} .$$

For a torsion-free coherent sheaf $F$, the quotient $\deg(F)/\text{rank}(F)$ is called the *slope* of $F$ and is denoted by $\mu(F)$.

**Definition 2.1.** A torsion-free $\mathcal{O}_X$ coherent sheaf $V$ on $X$ is called *semi-stable* (resp. *stable*) if for any proper sub-sheaf $0 \neq F \subset V$ with $V/F$ being torsion-free,

$$\mu(F) \leq \mu(V) \quad (\textit{resp.}\ \mu(F) < \mu(V)).$$

We will use the following convention. By a sub-sheaf of a torsion-free $\mathcal{O}_X$ coherent sheaf we will always mean a $\mathcal{O}_X$ coherent proper nonzero sub-sheaf such that the quotient is torsion-free. And by a quotient sheaf will mean a quotient by a sub-sheaf of the above type.

For any torsion-free coherent sheaf $V$ there is a unique filtration by sub-sheaves, called the *Harder-Narasimhan filtration* [Ko, Ch. V, Theorem 7.15]

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{k-1} \subset V_k = V$$

such that $V_i/V_{i-1}$ is the maximal semi-stable sub-sheaf of $V/V_{i-1}$. The integer $k$ will be called the *length* of the filtration. We will denote the number $\mu(V_i/V_{i-1})$, $1 \leq i \leq k$, by $\mu_i(V)$.

Let $T_0 \subset T_1 \subset T_2 \subset \ldots \subset T_l = T$ be the Harder-Narasimhan filtration of the tangent bundle $T$ of $X$.

**Theorem 2.2.** *If the following three conditions are satisfied*

**(i)** *the first two sub-sheaves $T_1$ and $T_2$ are locally free;*

**(ii)** $\mu_1(T) := \mu(T_1) \geq 0$;

**(iii)** *The rank of the Neron-Severi group of $X$ is one, i.e.* $H^{1,1}(X) \cap H^2(X, \mathbb{Q}) = \mathbb{Q}$;

*then either $T_1 = T$ (i.e. $T$ is semi-stable) or $T_2 = T$, i.e. $T$ is an extension of a semi-stable bundle by a semi-stable bundle with higher $\mu$.*

*Proof.* For two local sections $s$ and $t$ of $T$, let $[s,t]$ denote their Lie bracket. We want to deduce from the condition (ii) above that $T_1$ is closed under Lie bracket. We will use the notation $\mathcal{T}_1$ to denote the vector bundle given by the sheaf $T_1$; similarly for $T_2$. Consider the following homomorphism of sheaves

$$T_1 \otimes_{\mathbb{C}} T_1 \longrightarrow T/T_1 \tag{2.3}$$

which assigns to $s \otimes t$ the image of $[s,t]$ in $T/T_1$. The Leibniz formula for Lie bracket, namely

$$[fs,t] \;=\; f.[s,t] - \, < df, t > .s$$

and the identity $[X,Y] = -[Y,X]$ together imply that the homomorphism in (2.3) is $\mathcal{O}_X$-linear. In other words, it induces a homomorphism of bundles

$$\psi \,:\, \mathcal{T}_1 \otimes \mathcal{T}_1 \;\longrightarrow\; T/\mathcal{T}_1$$

The tensor product of two semi-stable sheaves is known to be semi-stable. Thus, since $\mathcal{T}_1$ is semi-stable, the tensor product $\mathcal{T}_1 \otimes \mathcal{T}_1$ is also semi-stable. So, the $\mu$ of any quotient sheaf of $\mathcal{T}_1 \otimes \mathcal{T}_1$ is greater than or equal to $\mu(\mathcal{T}_1 \otimes \mathcal{T}_1)$. But

$$\mu(\mathcal{T}_1 \otimes \mathcal{T}_1) \;=\; 2.\mu(\mathcal{T}_1) \;\geq\; \mu(\mathcal{T}_1)$$

The last inequality follows from the fact that $\mu(\mathcal{T}_1) \geq 0$. Since the $\mu$ of any sub-sheaf of $T/\mathcal{T}_1$ is less that or equal to $\mu_2(T)$ and $\mu_2(T) < \mu_1(T)$, considering the image of the homomorphism $\psi$ we conclude that the homomorphism $\psi$ must be zero. So $T_1$ is closed under Lie bracket. In other words, $\mathcal{T}_1$ is a holomorphic foliation on $X$. Now from the Theorem 3.2 and the subsequent identity (3.6) of [L] it follows that, for any integer $a > \dim_{\mathbb{C}}(T/\mathcal{T}_1)$, the characteristic class $(c_1(T/\mathcal{T}_1))^a = 0$. (Extend the partial connection along the foliation $\mathcal{T}_1$ given by the Lie-bracket on the normal bundle, $T/\mathcal{T}_1$, to a $GL(q,\mathbb{C})$ connection ($q = \dim_{\mathbb{C}}(T/\mathcal{T}_1)$) on $T/\mathcal{T}_1$. Use this connection to calculate $c_1(T/\mathcal{T}_1)^a$ by Chern-Weil theory (as done in Theorem 3.2 of [L]). The identity (3.6) in [L] would imply that the Chern form for $(c_1(T/\mathcal{T}_1))^a$, $a > \dim_{\mathbb{C}}(T/\mathcal{T}_1)$, vanishes identically.)

Since $H^{1,1}(X) \cap H^2(X,\mathbb{Q}) = \mathbb{Q}$, the class $c_1(T/\mathcal{T}_1)$ is a scalar multiple of the polarization class. So $c_1(T/\mathcal{T}_1)^a = 0$ implies that $c_1(T/\mathcal{T}_1) = 0$. Since $T_2/T_1$ is the maximal semi-stable sub-sheaf of $T/\mathcal{T}_1$, if $\mathcal{T}_2 \neq T$ then

$$(2.4) \quad \mu_2(T) \;>\; 0, \quad \textit{and} \;\; deg(T/\mathcal{T}_2) \;=\; deg(T/T_1) \;-\; deg(T_2/T_1) \;<\; 0$$

We want to show that $T_2$ in also closed under the Lie-bracket operation. This is obvious if $T_2 = T$; assume that $T_2 \neq T$. Using the homomorphism of sheaves, $\psi : T_2 \otimes_{\mathbb{C}} T_2 \longrightarrow T/T_2$, given by the Lie bracket, and the Leibniz rule together, we have a homomorphism of bundles $\psi : \mathcal{T}_2 \otimes \mathcal{T}_2 \longrightarrow T/\mathcal{T}_2$ as before.

Since $0 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ is the Harder-Narasimhan filtration of the bundle $\mathcal{T}_2$, the $\mu$ of any quotient of $\mathcal{T}_2 \otimes \mathcal{T}_2$ is greater than or equal to $2.\mu_2(T)$. The

$\mu$ of any sub-sheaf of $T/T_2$ is less that or equal to $\mu_3(T)$, and from the properties of the Harder-Narasimhan filtration we have

$$\mu_3(T) < \mu_2(T)$$

Since $\mu_2(T) > 0$ (see (2.4)), we have $2.\mu_2(T) > \mu_2(T)$. Thus from the above observations we get that the homomorphism $\psi$ must be zero. In other words, $T_2$ is closed under Lie-bracket. But, as earlier, that would imply that $\deg(T/T_2) = 0$, which in turn would contradict (2.4). This completes the proof. □

**Remark 2.5.** (i) The condition (ii) is satisfied, for example when $\deg(T) \geq 0$.

(ii) If $X$ is a smooth projective variety with ample canonical bundle then $X$ admits a Kähler-Einstein metric [Y]. This implies that $T$ is semi-stable with respect to the polarization $K_X$.

If the canonical bundle of $X$ is negative then there is an obstruction, known as the Futaki invariant, for the existence of Kähler-Einstein metric on $X$ [F].

## 3. Bundles with holomorphic connections.

In this section we assume $X$ to be a compact Kähler manifold equipped with a Kähler form $\omega$. As before, let

$$T_0 \subset T_1 \subset T_2 \subset \ldots \subset T_l = T$$

be the Harder-Narasimhan filtration of the holomorphic tangent bundle of $X$.

We will use the following convention. For an holomorphic bundle $V$ on $X$, $\Omega^i(V)$ will denote the holomorphic bundle $V \otimes \Omega^i$; and $\Gamma^{i,j}(X,V)$ will denote the space of all $C^\infty$ $(p,q)$-forms with values in $V$. We will not distinguish between a holomorphic bundle and the locally free $\mathcal{O}_X$-coherent sheaf corresponding to it.

A holomorphic structure on a $C^\infty$ bundle $V$ on $X$ is given by a first order operator $\overline{\partial}_V : \Gamma(X,V) \longrightarrow \Gamma^{0,1}(X,V)$ satisfying the Leibniz condition and the integrability condition $\overline{\partial}_V^2 = 0$. A holomorphic connection on a holomorphic bundle $V$ is a first order differential operator

$$\partial : V \longrightarrow \Omega^1(V)$$

satisfying the Leibniz condition $\partial(f.s) = d(f).s + f\partial(s)$, where $f$ is a holomorphic function and $s$ is a (local) holomorphic section of $V$. It is easy to check that $\partial$ being a holomorphic connection is equivalent to the condition that the operator $\partial + \overline{\partial}_V$ is a connection with a holomorphic End$(V)$ valued 2-form as curvature.

**Example 1.** Let $\nabla$ be a flat connection on a $C^\infty$ bundle on $X$. Then the operator $\nabla^{0,1}$, the $(0,1)$ part of $\nabla$, gives a holomorphic structure on $V$, and $\nabla^{1,0}$ gives a holomorphic connection on the holomorphic bundle.

From the Chern-Weil construction of characteristic classes it is obvious that all the Chern classes of a bundle with holomorphic connection vanish.

A holomorphic vector bundle $V$ is said to admit a flat connection if there is a flat connection $\nabla$ on $V$ such that $\nabla^{0,1} = \overline{\partial}_V$.

**Theorem 3.1.** *Let the Kähler manifold $X$ satisfy the condition that $\mu_l(T) > 0$. Let $V$ be a holomorphic bundle on $X$ admitting a holomorphic connection. Then $V$ admits a flat connection; also, the bundle $V$ is a direct sum of stable bundles of slope zero.*

*Proof.* Let $V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = T$ be the Harder-Narasimhan filtration of the bundle $V$. Let $\partial$ be a holomorphic connection on $V$. Note that using the duality between $T$ and $\Omega^1$, $\partial$ induces a $\mathbb{C}$-linear homomorphism from the tensor product (over $\mathbb{C}$) of the sheaves, $T \otimes V$, to the sheaf $V$. Consider the following map of sheaves $T \otimes_{\mathbb{C}} V_1 \longrightarrow V/V_1$ given by mapping $\theta \otimes s$ to the projection of $\partial_\theta s$ on $V/V_1$. Let $\mathcal{O}_X$ denote the sheaf of germs of holomorphic functions on $X$. The Leibniz condition implies that the above map is $\mathcal{O}_X$ linear, and hence it induces a homomorphism

$$\psi \; : \; T \otimes_{\mathcal{O}_X} V_1 \longrightarrow V/V_1 \tag{3.2}$$

Since $V_1$ is semi-stable and $\mu_l(T) \geq 0$, the $\mu$ of any quotient sheaf of $T \otimes_{\mathcal{O}_X} V_1$ is at least $\mu(V_1)$. This implies that the homomorphism $\psi$ is zero.

Since $\psi = 0$, the holomorphic connection $\partial$ induces a holomorphic connection on $V_1$. Any $\mathcal{O}_X$ coherent sheaf with a holomorphic connection is locally free [B, p. 211, Proposition 1.7]. (Though this proposition in [B] is stated for integrable connections ($D$-modules), the proof uses only the Leibniz rule (in particular, does not use vanishing of curvature). The Leibniz rule is valid for a holomorphic connection.)

Since the bundle $V_1$ admits a holomorphic connection, $c_1(V_1) = 0$, and hence $\mu(V_1) = 0$. Since $\mu(V) = 0 = \mu(V_1)$, we have $V = V_1$. This implies that $V$ is semi-stable.

If $V$ is actually a stable bundle then from [UY] it follows that $V$ admits a Hermitian-Yang-Mills connection; this metric is determined up-to a global scalar. We noted prior to Theorem 3.1 that the given condition that $V$ admits a holomorphic connection implies that all the Chern classes of $V$ vanish. This implies that the Hermitian-Yang-Mills connection is actually a flat unitary connection.

If $V$ is not stable then there is a filtration [K, Ch. V, §7, Theorem 7.18]

$$0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_{m-1} \subset W_m = V \tag{3.3}$$

such that $W_i/W_{i-1}$ is a stable sheaf with $\mu(W_i/W_{i-1}) = \mu(V)$.

In view of the above mentioned result, [UY], that any stable vector bundle with vanishing Chern classes admits an unitary flat connection, Theorem 3.1 is implied by the following proposition.

**Proposition 3.4.** *Each sub-sheaf $W_i$, $1 \leq i \leq m$, in (3.3) is of the form $W_{i-1} \oplus U_i$, where $U_i$ is a stable bundle with vanishing Chern classes.*

*Proof.* First we want to show that $W_1$ is a sub-bundle of $V$. In order to prove that we use the same method that has repeated been used in this paper : show that the holomorphic connection $\partial$ induces a holomorphic connection on $W_1$, and then using [B, p. 211, Proposition 1.7] (as used earlier in proving that $V$ is semi-stable) we may conclude that $W_1$ is locally free; we then show that $V/W_1$ has a quotient connection, and hence, using [B, p. 211, Proposition 1.7], $V/W_1$ is locally free. So $W_1$ must be a sub-bundle of $V$. We give the details of the argument below.

As in (3.2), the holomorphic connection $\partial$ induces a homomorphism of bundles $\phi : T \otimes W_1 \longrightarrow V/W_1$. Since $W_1$ is semi-stable with $\mu(W_1) = 0$, and $\mu_l(T) > 0$, the $\mu$ of any quotient sheaf of $T \otimes W_1$ is strictly positive. This implies that the homomorphism $\phi$ is zero. So the connection $\partial$ induces a holomorphic connection on $W_1$. Let $\nu : V \longrightarrow V/W_1$ denote the quotient projection. For a local section $s$ of $V/W_1$, consider $s \longmapsto \nu(\partial(\bar{s}))$, where $\bar{s}$ is a lift of $s$ to a local section of $V$. Since the sheaf $W_1$ is closed under the connection $\partial$, the section $\nu(\partial(\bar{s}))$ does not depend upon the choice of the lift $\bar{s}$ of $s$. Using this it is easy to see that the map $s \longmapsto \nu(\partial(\bar{s}))$ defines a holomorphic connection on $V/W_1$. So the sheaf $V/W_1$ is locally free. This implies that $W_1$ is a sub-bundle of $V$. As we already noted that the Chern classes of a bundle admitting a holomorphic connection vanish, since $W_1$ admits a holomorphic connection, all the Chern classes of $W_1$ vanish. Since $W_1$ is a stable bundle with vanishing Chern classes, from the Theorem of Uhlenbeck-Yau it admits a flat unitary structure.

Actually the same argument as above implies that any $W_i$, $0 \leq i \leq m$, is a sub-bundle of $V$. To see this, first check that for any $W_i$, the analogue of $\phi$ is zero. So the connection $\partial$ induces a connection on $W_i$. Then the rest of the argument is identical.

Now we want to show that $W_2 = W_1 \oplus U_2$, where $U_2$ is a stable bundle with the property that all its Chern classes vanish. Since all the Chern classes of both $W_1$ and $W_2$ vanish, from the general properties of Chern classes it follows that the Chern classes of $W_2/W_1$ vanish. So, for a bundle $U_2$ satisfying the condition that $W_2 = W_1 \oplus U_2$, all the Chern classes of $U_2$ must vanish.

For notational simplicity we will denote $W_2/W_1$ by $W$. Note that since both $W_1$ and $W_2$ are sub-bundles of $V$, the sheaf $W$ is locally free. Let $c \in H^1(X, W^* \otimes W_1)$ be the element corresponding to the extension, namely $W_2$, of $W$ by $W_1$. Let $C^{0,1}(X, W^* \otimes W_1) \subset \Gamma^{0,1}(X, W^* \otimes W_1)$ be the space of all $\overline{\partial}$-closed $(0,1)$-forms with values in $W^* \otimes W_1$. Let $\rho : C^{0,1}(X, W^* \otimes W_1) \longrightarrow H^1(X, W^* \otimes W_1)$ denote the surjection given by the Dolbeault resolution of $W^* \otimes W_1$. Define $I := \rho^{-1}(c)$.

Since $W$ and $W_1$ are stable bundles with vanishing Chern classes, they admit flat unitary connections, namely the Hermitian-Yang-Mills connections. The Hermitian metrics on $W$ and and $W_1$ induce a Hermitian structure on $W^* \otimes W_1$. Let

$$H \subset C^{0,1}(X, W^* \otimes W_1)$$

be the subspace consisting of all harmonic forms. The map $\rho$ identifies $H$ with $H^1(X, W^* \otimes W_1)$. Consider $\hat{c} := I \cap H$, the harmonic representative of the class $c$. Define the adjoint $\hat{c}^*$ of $\hat{c}$ as follows : Using the unitary structures on $W$ and $W_1$, for a (local) $C^\infty$ section $s$ of $W^* \otimes W_1 = \mathrm{Hom}(W, W_1)$ we may consider its adjoint, $s^*$, which is a $C^\infty$ section of $\mathrm{Hom}(W_1, W)$, defined by $< s^*(a), b > = < a, s(b) >$, where $a$ is a local section of $W_1$ and $b$ is a local section of $W$. Now, if in some local holomorphic co-ordinate chart $(z_1, \ldots, z_d)$ on $X$ the section $\hat{c}$ is equal to $\sum_k \theta_k \otimes d\bar{z}_k$ then

$$\hat{c}^* := \sum_{k=1}^{d} \theta_k^* \otimes dz_k \tag{3.5}$$

Since Hodge identities hold for Hermitian flat connections, the harmonicity of the section $\hat{c}$ implies that $\hat{c}^*$ is a holomorphic section of $\Omega^1 \otimes \mathrm{Hom}(W_1, W)$. In other words,

$$\hat{c}^* \in H^0(X, \mathrm{Hom}(T \otimes W_1, W))$$

But, since $\mu_l(T) > 0$, there is no non-zero homomorphism from $T \otimes W_1$ to $W$ (both $W$ and $W_1$ are stable bundles with slope zero). So $\hat{c}^* = 0$, which in turn implies that $\hat{c} = 0$. Since the extension class vanishes, the exact sequence

$$0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W \longrightarrow 0$$

splits. In other words, $W_2$ is isomorphic to $W_1 \oplus W$.

Now we will use induction to complete the proof of the Proposition 3.4.

Assume that $W_i = W_{i-1} \oplus U_i$ for all $1 \leq i \leq j < m$ (as in the statement of the proposition). We want to show that $W_{j+1} = W_j \oplus U_{j+1}$, where $U_{j+1}$ is a stable bundle with vanishing Chern classes. From the assumption,

$$W_j = \sum_{k=1}^{j} U_k \tag{3.6}$$

where each $U_k$ is a stable bundle with vanishing Chern classes. Let $W$ denote the quotient $W_{j+1}/W_j$. Since both $W_{j+1}$ and $W_j$ are sub-bundles of $V$, the sheaf $W$ is locally free. Since the Chern classes of both $W_{j+1}$ and $W_j$ vanish (they admit holomorphic connections induced by $\partial$), the Chern classes of $W$ must also vanish. Corresponding to the short exact sequence of bundles

$$0 \longrightarrow W_j \longrightarrow W_{j+1} \longrightarrow W := W_{j+1}/W_j \longrightarrow 0$$

we have the extension class $\xi \in H^1(X, \mathrm{Hom}(W, W_j))$. Using the decomposition (3.6) we have

$$\xi = \sum_{k=1}^{j} \xi_k$$

where $\xi_k \in H^1(X, \mathrm{Hom}(W, U_k))$. For any $1 \leq k \leq j$, both $W$ and $U_k$ are stable bundles with vanishing Chern classes. Recall the argument for proving the extension class $c$ corresponding to $W_2$ to be zero. Repeating this argument we get that the class $\xi_k$ must be zero for any $1 \leq k \leq j$. (In the argument for vanishing of the class $c$, $W_1$ should be replaced by $U_k$ to get the vanishing of $\xi_k$). So the extension class $\xi$ must vanish. This implies that the bundle $W_{j+1}$ is isomorphic to $W_j \oplus W$. This completes the proof of the Proposition 3.4. □

**Remark 3.7.** (i) The flat connection on $V$ that we obtained in Theorem 3.1 is actually an unitary flat connection.

(ii) Form the proof of Theorem 3.1 it follows that if $\mu_l(T) \geq 0$ and $V$ is a holomorphic bundle on $X$ with a holomorphic connection, then $V$ is semi-stable. The strict inequality, $\mu_l(T) > 0$ in Theorem 3.1 was used to prove that $V$ is a direct sum of stable bundles of slope zero (which is a stronger statement than semi-stability).

(iii) Let $X$ be a compact Riemann surface of genus at least 2. Consider the flat $PSL(2, \mathbb{R})$ connection on $X$ given by the uniformization theorem. This connection can be lifted to a flat $SL(2, \mathbb{R})$ connection, and the corresponding holomorphic bundle $V$ is given by the extension

$$0 \longrightarrow K^{1/2} \longrightarrow V \longrightarrow K^{-1/2} \longrightarrow 0$$

where $K^{1/2}$ is a square-root of the canonical bundle, and the extension class is $1 \in H^1(X, K^{1/2} \otimes K^{1/2}) = H^1(X, K) = \mathbb{C}$. This bundle $V$ is clearly not semi-stable. This example shows that in the statement of 3.7 (ii), the condition that $\mu_l(T) \geq 0$ is essential.

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