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THE BRIDGE PRINCIPLE FOR UNSTABLE AND FOR SINGULAR MINIMAL SURFACES

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INTRODUCTION

Let S be a two dimensional minimal surface in R^N, and let $P \subset R^N$ be a thin curved rectangle whose two short sides lie along ∂S and that is otherwise disjoint from S. More generally S can be an m-dimensional minimal surface and P can be a set homeomorphic to $[0,1] \times B^{m-1}$. Typically S will have two connected components, and P will join one to the other. The **bridge principle** for minimal surfaces is the principle that it should usually be possible to deform $S \cup P$ slightly to make a minimal surface with boundary $\partial(S \cup P)$.

In a previous paper, we showed that it is possible provided that S is smooth and strictly stable, that P is sufficienty thin, and that, at each end of P, the angle between P and S is strictly between 0 and 2π. ("Strictly stable" means "stable and having no nonzero jacobi fields that vanish on the boundary" or, equivantly, "having index 0 and nullity 0 as a critical point for the area functional".) In §1 of this paper we extend that result to all unstable smooth surfaces S that have nullity 0. As a corollary we prove that a certain simple closed curve in ∂B^3 that is smooth except at one point has the following property. For every genus $g \leq \infty$, every area a in some interval $[L, +\infty]$, and every index $\iota \leq \infty$, there exist uncountably many (namely $2^{\aleph_0}$) embedded minimal surfaces with genus g, area a, and index ι.

In §2 we give two examples to show that the nullity 0 assumption is necessary. First, we show that if M is a catenoid of nullity 1 with boundary in ∂B^3, if N is a minimal surface with boundary in one of the simply connected

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components of $\partial B^3 \setminus (\partial M)$, and if P is a bridge in the region of ∂B^3 between ∂M and ∂N that joins them, then there is no connected minimal surface having the same boundary as $M \cup P \cup N$. We also give a similar counterexample when M and N are both disks.

In §3, we show that the bridge principle is true for any surface S, possibly singular, that is uniquely area minimizing (as a current or a flat chain mod p) in some open subset of the ambient space. The resulting minimal surface will be diffeomorphic to $S \cup P$ except near the singularities (if there are any) of S.

Now let C_1 and C_2 be smooth curves in R^3 and let C' be the connected curve formed by joining C_1 to C_2 with a thin bridge P. The theorems described so far assert (under suitable hypotheses) that given minimal surfaces S_i bounded by C_i, there exists a minimal surface S close to $S_1 \cup P \cup S_2$ and with boundary C'. One can ask if there is a converse: given a minimal surface S with boundary C', must there exist minimal surfaces S_i with boundary C_i such that S is close to $S_1 \cup P \cup S_2$? In other words, does the bridge principle describe all the minimal surfaces bounded by C'? In §4, we give two examples to show that the answer is, in general, no. However, we also prove that (roughly speaking) if C_1 and C_2 are not too close together and if the bridge is not too crooked, then the answer is yes.

This is useful for the following reason. Although there are various interesting theorems about the number of minimal surfaces bounded by a curve, there are rather few kinds of curves for which this number is known. Using the converse to the bridge principle described above, we construct for every k a connected curve that bounds exactly 3^k stationary integral varifolds; all are embedded disks of nullity 0, and exactly $\binom{k}{p}2^{k-p}$ have index p. This seems to be the first example of a connected curve for which the exact number of minimal surfaces is known and is greater than 3.

In §5 we offer some partial answers to the question: for which pairs of curves C_1 and C_2 are we guaranteed the existence of minimal surfaces S_i (with $\partial S_i = C_i$) that can be joined by a bridge?

We remark that §1 and §3 (which are about unstable and singular surfaces, respectively) are entirely independent of each other. We also mention that the main theorems (1.2 and 3) hold for arbitrary ambient riemannian manifolds and, except for the uniqueness assertion at the end of 1.2, for arbitrary smooth parametric elliptic functionals. (The proof of the uniqueness uses Allard's regularity theorems, which are not known to hold for elliptic functionals other than area.) No significant change is required in the proofs. (The device in [W4, §8] lets one deduce the results for general ambient manifolds from Euclidean case.)

The history of the bridge principle is discussed in [W1]. Until now, the only published proofs of bridge principles for unstable or for singular surfaces seem to be those of N. Smale. His first bridge theorem [SN1] applies to all smooth unstable surfaces of nullity 0, but the bridge has to be tailored to the surface in a way that precludes using it for nonexistence examples such as the one given in this paper. (Also if one wishes to bridge surfaces with boundaries in ∂B^n, [SN1] requires the bridge to extend out of B^n.) In a subsequent paper [SN2], he joined minimal surfaces with isolated cone-like singularities by bridges to obtain connected minimal hypersurfaces with many isolated singularities. His result is more general than the ones here in that his surfaces can be both singular and unstable. The results here are more general in that the bridges do not have to be tailored to the surface, and that the singularities need not be isolated.

This paper is a sequel to [W1], and the reader is referred there for definitions of terms such as *bridge, skillet*, and *shrinking nicely.* As in that paper, a nonzero stationary integral m-varifold V is said to have boundary C if $\delta V \leq \mathcal{H}^{m-1} \llcorner C$, where δV is the first variation measure associated with V.

1. Unstable Surfaces

For the next two theorems it is necessary to consider functionals of the form:

$$(*) \qquad S \mapsto \operatorname{area}(S) + \phi\Big(\int_S \vec{f}\,\Big)$$

where S is a submanifold of $U \subset R^N$, and $\vec{f} : U \to R^p$ and $\phi : R^p \to R^+$ are smooth functions such that

$$(\sup |D\phi|)(\sup |\vec{f}|) < 1$$

(This inequality implies that the functional is lower semicontinuous with respect to weak convergence and that the basic GMT regularity theory applies to minimizing surfaces: see [W2, §1 and §3].)

The first theorem simply says that the bridge principle for strictly stable surfaces [W1, §2] applies uniformly to families of functionals.

1.1. Theorem. *Let U be a bounded open subset of R^N and let C be a smooth compact $(m-1)$ manifold in U. Let $F^v, v \in \bar{B}^k(\epsilon)$ be a smooth k-parameter family of smooth functionals of the form (*). Let $S^v, v \in \bar{B}^k(\epsilon)$ be a k-parameter family of smooth surfaces in U such that each S^v has boundary C and is strictly stable and uniquely homologically minimizing in $\bar{U}$ for the functional F^v. Let $\Gamma \subset R^N$ be a smooth curve such that for each v,*

- $\Gamma \cap S^v = \Gamma \cap \partial S^v = \partial\Gamma$
- *At each of its endpoints, Γ makes a nonzero angle with the tangent halfplane to S^v at that endpoint.*

Then

(1) *there is a bounded open set U' containing $U \cup \Gamma$ such that each S^v is homologically F^v minimizing in $\overline{U'}$, and*

(2) *there is a sequence P^n of bridges on C shrinking to Γ nicely.*

If P_n is such a sequence of bridges, let T_n^v be an F^v-minimizing surface in $\overline{U'}$ with boundary $\partial(S^v \cup P_n)$. Then for sufficiently large n, T_n^v is unique and strictly stable for F^v and:

(3) *$F^v(T_n^v) \to F^v(S^v)$ uniformly in v,*

(4) *for all v, there is a diffeomorphism $f_n^v : S^v \cup P_n \to T_n^v$ with*

$$f_n^v(x) \equiv x \text{ for } x \in \partial(S^v \cup P_n)$$

and

$$\max_{x,v} |x - f_n^v(x)| = O(w_n)$$

where w_n is the width of P_n.

(5) *The maps f_n^v converge uniformly smoothly on compact subsets of $S^v \setminus \Gamma$ to the identity map $S^v \to S^v$.*

Proof. In [W1, §1.1 and §2] this was proved for $k = 0$. The general case is proved in exactly the same way. (See also [W1, §5.1].) □

Corollary. *The T_n^v depend smoothly on v.*

Proof. Continuous dependence (which is all that we will use) follows from the uniquely minimizing property of T_n^v. Smooth dependence then follows (by the implicit function theorem) from the strict stability of T_n^v. □

1.2. Theorem. *Let $S \subset R^N$ be a compact, smooth, embedded minimal submanifold with boundary C, and with index k and nullity 0. Let Γ be a smooth arc such that*

$$\Gamma \cap S = \Gamma \cap \partial S = \partial\Gamma$$

and such that at each of its two endpoints, Γ makes a nonzero angle with the tangent half-plane to S at that endpoint.

Let P_n be a sequence of bridges on ∂S that shrink nicely to Γ.

Then for sufficiently large n, there exists a minimal surface T_n with boundary $\partial(S \cup P_n)$ and a diffeomorphism $f_n : S \cup P_n \to T_n$ such that

- $\text{area}(T_n) \to \text{area}(S)$
- $f_n(x) \equiv x$ *for* $x \in \partial(S \cup P_n)$
- $|x - f_n(x)| = O(w_n)$ *where w_n is the width of P_n.*
- *The maps f_n converge smoothly on compact subsets of $S \setminus \Gamma$ to the identity map $S \to S$.*
- *T_n has index k and nullity 0.*

Furthermore, the T_n are unique in the following sense. If V_n is any stationary integral varifold with boundary ∂T_n and if the V_n converge as varifolds to S, then for large n, V_n is the varifold associated to T_n.

Proof. By theorem 6 of [W2] there exist smooth functions $\vec{f} : R^N \to R^k$ and $\phi : R^k \to R^+$, an open set $U \subset R^N$ containing S, and an $\epsilon > 0$ such that:

(1) $\phi(x) = K|x|^2$ for sufficiently small $|x|$.
(2) $(\sup|D\phi|)(\sup|\vec{f}|) < 1$.

(3) For each $v \in \bar{B}^k(\epsilon)$, there is a unique surface S^v that minimizes the functional

$$F^v(M) = \text{area}(M) + \phi\left(\int_M \vec{f} - v\right)$$

among surfaces in $\bar{U}$ with boundary ∂S.

(4) Each S^v is strictly stable for F^v,

(5) S^v depends smoothly on v,

(6) $S^0 = S$,

(7) The map

$$\Phi : v \mapsto \int_{S^v} \vec{f} - v$$

is a diffeomorphisms from $B^k(\epsilon)$ onto a neighborhood of 0.

By lemma 2.1 of [W1], we can assume that U contains Γ. We then let $U' = U$.

Now let T_n^v be the surfaces given by theorem 1.1. Let

$$\Phi_n : v \mapsto \int_{T_n^v} \vec{f} - v$$

By theorem 1.1 and its corollary, we know that the Φ_n are continuous (even smooth) and converge uniformly to Φ. Thus (7) implies that for sufficiently large n, there is a $v_n = v(n)$ such that

$$(*) \qquad \Phi_n(v_n) = 0$$

Let $T_n = T_n^{v(n)}$. If w is a smooth normal vectorfield on T_n that vanishes on ∂T_n, then by (*),

$$\begin{aligned} F^{v(n)}(T_n + tw) &= \text{area}(T_n + tw) + \phi(O(t)) \\ &= \text{area}(T_n + tw) + O(t^2) \end{aligned}$$

It follows that since T_n is stationary for $F^{v(n)}$, it must also be stationary for area.

To prove that T_n has index k and nullity 0, we will first show that it has index $\geq k$, and then that it has index plus nullity $\leq k$.

Since S has index k, there exist k linearly independent normal vectorfields $w_1, \ldots, w_k$ on S that vanish on ∂S and are such that

$$\left(\frac{d}{dt}\right)^2_{t=0} \text{area}(S + tw) < 0$$

if w is a nonzero linear combination of the w_i. Note if we perturb the w_i slightly in $L^{1,2}$, this remains true. Thus we may choose the w_i to be compactly supported in the interior of S. Extend each vectorfield so that it is a smooth vectorfield on R^N that vanishes on a neighborhood on $\Gamma \cup \partial S$. By the smooth convergence (on compact subsets of $R^N \setminus \Gamma$) of T_n to S,

$$\lim_{n\to\infty} \left(\frac{d}{dt}\right)^2_{t=0} \operatorname{area}(T_n + tw) = \left(\frac{d}{dt}\right)^2_{t=0} \operatorname{area}(S + tw) < 0$$

if w is a nonzero linear combination of the w_i. Also, for the same reason, for sufficiently large n, the vectorfields $w_i|_{T_n}$ $(i = 1, \dots, k)$ will be linearly independent. This proves that the index of T_n is $\geq k$.

We now show that the index plus the nullity of T_n is $\leq k$. Suppose not. By passing to a subsequence, we can assume that

$$\operatorname{index}(T_n) + \operatorname{nullity}(T_n) > k$$

for all n. It follows that for every n, there is an eigenfunction u_n of the jacobi operator on T_n that has eigenvalue ≤ 0 and that is orthogonal to each $w_i|_{T_n}$ $(i = 1, \dots, k)$, where the w_i are as above.

Let $|u_n|$ achieve its maximum at $z_n \in T_n$. We normalize $|u_n(z_n)|$ to be 1.

Claim: z_n is bounded away from ∂T_n. For suppose not. By passing to a subsequence we may assume that

$$(1/\operatorname{dist}(z_n, \partial T_n))_\#(T_n - z_n)$$

converges to a limit L, which must be either

(1) a halfspace, or
(2) an infinite strip, or
(3) a skillet-like minimal surface (cf. [W1, §1])

By the standard regularity theory, the convergence is smooth (on compact subsets of R^N.) Thus the u_n's converge to a bounded eigenfunction u on L with eigenvalue ≤ 0. But the surfaces (1)–(3) do not have such eigenfunctions. This proves the claim.

It follows from the claim that (after passing to a subsequence) the u_n converge smoothly on compact subsets of $R^N \setminus \Gamma$ to a bounded nonzero eigenfunction u on $S \setminus \partial\Gamma$. Since u is bounded, it in fact extends smoothly to all of S.

Now for each $i = 1, \dots, k$,

$$\int_S w_i \cdot u = \lim_{n\to\infty} \int_{T_n} w_i \cdot u_n = 0$$

Thus u is an eigenfunction on S that has an eigenvalue ≤ 0 and that is orthogonal to $w_1, \dots, w_k$. But that contradicts the fact that S has index k and nullity 0.

Finally, the uniqueness of the T_n is proved exactly as in [W1, 5.3]. $\square$

Remark. Note that the proof actually shows somwhat more: that there is a neighborhood G of Γ such that the index of $T_n \setminus G$ is k. In fact, if S consists of two components that Γ joins, then the index of each component of $T_n \setminus G$ is equal to the index of the corresponding component of S.

It follows that all the theorems in [W1], stated there for strictly stable surfaces, continue to hold for unstable surfaces of nullity 0. For theorem 3 of [W1], one should add the conclusion that there is a neighborhood G of Γ such that

(1) $\text{nullity}(S_n) = 0$,
(2) $\text{index}(S_n) = \text{index}(S_n \setminus G) = \text{index}(S)$,
(3) each connected component of $S_n \setminus G$ has the same index as the corresponding component of S.

This allows one to add

$$\text{index} \lim_{n\to\infty} \sigma(S_1, \dots, S_n) = \sum_{n=1}^{\infty} \text{index}(S_n)$$

to the conclusions of theorem 4.1.

We then also have the following improvement of [W1, 4.2]:

1.3. Theorem. *There is a connected embedded curve C in ∂B^3 that is smooth except at one point and there is a number L such that for every*

$$a \in [L, \infty],$$
$$g \in \{0, 1, 2, \dots\} \cup \{\infty\},$$
$$\iota \in \{0, 1, 2, \dots\} \cup \{\infty\}$$

there exist uncountably many ($2^{\aleph_0}$) embedded minimal surfaces with boundary C, area a, genus g, and index ι.

Proof. The proof is exactly as in the stable case [W1, 4.2], except that we let $p(i)$ be a sequence of 1's, 2's, 3's, and 4's such that 2 occurs exactly g times and 4 occurs exactly i times. □

Remark. If the reader examines the proof of [W1, 4.2], he or she will see that we appealed to minimax theory to produce an index 1 embedded minimal disk S_n^4 bounded by C_n (we already had two strictly stable disks S_n^1 and S_n^3.) If we allow C to lie in the boundary of a convex set other than the sphere, then we can give a more elementary proof that uses degree theory instead of minimax theory. We begin by choosing in the $z = -1$ plane a smooth convex closed curve Σ that includes a straight line segment. Now we let M be the cone $\{tp : t \in [0, \infty), p \in \Sigma\}$; this cone will take the place of the 2-sphere.

As before, for $n = 1, 2, \ldots$, we let C_n' and C_n'' be horizontal slices of M with C_n'' very close to C_n', and we join C_n' and C_n'' by a very thin bridge to get a connected curve C_n. Note we can choose bridges that lie in the planar portion of the cone M. We perturb the bridges slightly (keeping them in the planar portion of M) so that C_n bounds no regular minimal surfaces with jacobi fields. As before, C_n will bound two strictly stable embedded disks S_n^1 and S_n^3 and a strictly stable embedded surface S_n^2 of genus 1. By degree theory ([W5, 2.1] or [TA]), the number of embedded disks of even index minus the number of odd index is 1. Thus C_n must bound an odd index disk S_n^4. Note that the total curvature of C_n is 6π. (This is why we use the cone M; the corresponding curve in ∂B^3 has total curvature slightly greater than 6π.) By the Gauss-Bonnet formula, S_n^4 has total curvature less than 4π. Thus the nodal line of the second eigenfunction of the jacobi operator divides S_n^4 into two regions, at least one of which has total curvature less than 2π. By the theorem of Barbosa and do Carmo [BC], this region is strictly stable. Note that the second eigenvalue of the S_n^4 is equal to subregion's first eigenvalue, which we have just seen to be positive. Thus S_n^4 has index 1. □

2. Counterexamples: Why Nullity is Bad

In this section we give two examples to show that it is not always possible to connect minimal surfaces by bridges.

EXAMPLE 1. Consider an entire catenoid centered at the origin. As is well known, there is an $r > 0$ such that the intersection M of the catenoid with $B = B_r(0)$ has index 0 and nullity 1. Let P_1 and P_2 be the planes that contain the boundary circles of M. We may assume that the planes are horizontal and that P_1 lies below P_2. Now let N be any smooth embedded minimal surface below P_1 with boundary in ∂B.

Theorem. *Let P be a bridge in ∂B that joins ∂M to ∂N and that lies below P_1. Then there is no connected minimal surface having the same boundary as $M \cup P \cup N$.*

Proof. Let V be a minimal surface with the same boundary as $M \cup P \cup N$. Since $\partial V \in \partial B$, V cannot have boundary branch points [N, §366].

Let $C_0 = \partial M = \partial B_r \cap (P_1 \cup P_2)$, and for $t > 0$, let

$$C_t = \partial B_{r+t} \cap (P_1 \cup P_2)$$

Thus C_t is a pair of circles. Note that for $t > 0$, C_t bounds exactly two catenoids, one strictly stable and the other unstable. Let M_t be the stable one and M_{-t} be the unstable one. (One can obtain each M_t by dilating the original entire catenoid a suitable amount and then intersecting with the slab between P_1 and P_2. The asserted properties of the M_t follow readily from this description.) We let $M_0 = M$. Note that the M_t ($t \in R$) form a continuous one parameter family, and that the M_t with $t > 0$ foliate the region between P_1 and P_2 and outside of M. Hence by the maximum principle, V does not touch that region. In other words, V lies in the simply connected component of $B \setminus M$. By the Hopf boundary maximum principle, V is never tangent to M along the boundary. Thus for $t < 0$ near 0, M_t is disjoint from V. Hence M_t is disjoint from V for all $t < 0$ (since otherwise the maximum principle would be violated for the greatest $t < 0$ for which M_t intersects V.) Likewise, V must be disjoint from all translates $M_t + v$ where v is a horizontal vector of length less than $|t|$. But such translates fill up the entire slab between P_1 and P_2. Thus the slab separates V into two connected components. □

EXAMPLE 2. Similar situations arise even when one considers only disk-type surfaces. Recall ([W4, §8] and [W5, §1 and §2]) that if X is the space of all smooth closed embedded curves in the two sphere ∂B and if $\mathcal{M}$ is the space of all embedded minimal disks with boundaries in X, then $\mathcal{M}$ is a Banach manifold and the map

$$\Pi : \mathcal{M} \to X$$
$$\Pi(S) = \partial S$$

is a proper real-analytic Fredholm map of Fredholm index 0.

Let $C_0 \in X$ be a circle, and let $C_1 \in X$ be a curve that bounds more than one embedded minimal disk. Let

$$t \mapsto C_t \qquad (0 \leq t \leq 2)$$

be a real analytic curve in X (starting at C_0 and passing through C_1) that is transverse to Π. Then

$$\tilde{X} = \Pi^{-1}\{C_t : 0 \leq t \leq 1\}$$

is a compact real-analytic 1 manifold with boundary, and

$$\pi(= \Pi|_{\tilde{X}}) : \tilde{X} \to X$$

is real-analytic.

Let τ be the infimum of t for which $\pi^{-1}(C_t)$ contains more than one element. Note that $\tau > 0$. For $0 \leq t < \tau$, let D_t be the unique minimal disk bounded by C_t. Let $D_\tau = \lim_{t \to \tau-} D_t$.

If D_τ is a regular point of π, then π is strictly monotonic on a neighborhood of D_τ. If D_τ is a critical point, then by analyticity it is an isolated critical point, so there is a neighborhood I of D_τ in $\tilde{X}$ such that π is strictly monotonic in each component of $I \setminus (D_\tau)$. Thus if it were not monotonic on all of I, then π would "turn around" at D_τ, contradicting the fact that $\pi^{-1}(C_t)$ has only one element for $t < \tau$. Thus π must be strictly monotonic near D_τ.

It follows that $\pi^{-1}(C_\tau)$ contains more than one element. Now for any curve such as C_τ that bounds more than one minimal variety, there is a region W bounded by two stable disks in $\pi^{-1}(C_\tau)$ such that all stationary varieties bounded by C_τ lie in $\bar{W}$ (see [MY] or [W5, corollary 2.2] or [L]). Thus one of these two extreme disks, call it M, must be different from D_τ. Let U^+ be the connected component of $B \setminus M$ that does not include W, and let U^- be the other component. Note that M does not minimize area in any open subset of B, since if it did, it could not disappear under boundary perturbations such as $C_\tau \to C_{\tau-\epsilon}$.

Theorem. *Let M be the minimal disk described above. Let N be any regular minimal surface in $\overline{U^-}\backslash M$ with boundary in ∂B. Let P be a bridge in $\overline{U^-}\cap\partial B$ joining M to N, and let V be a minimal surface with boundary $\partial(M \cup P \cup N)$. Then V is bounded away from M (independently of P). That is, there is an open set O containing the interior of M such that no such V intersects O.*

Proof. Each stable embedded minimal surface $\neq M$ bounded by C_τ divides B into two connected components, one of which contains M (this is by the extreme property of M). Let O be the intersection of those components that contain M. Since C_τ can only bound finitely many stable embedded minimal surfaces [M], O is open.

First note that V cannot intersect U^+. For let M' be minimize area among all surfaces in $\overline{U^+}$ that have boundary C_τ and that are disjoint from V. Then M' is a minimal surface bounded by C_τ, so it must lie in $\overline{U^-}$ by the extreme property of M. Thus $M' \subset \overline{U^+} \cap \overline{U^-}$ so $M' = M$, which forces $V \subset U^-$ by the maximum principle. Since ∂V lies in the boundary of a convex set, V is a smooth embedded manifold at the boundary. Thus by the Hopf boundary maximum principle, V is never tangent to M along ∂M. Now let D' minimize area among all surfaces that are homologous to M in $B \setminus V$. Then D' is a minimal surface that does not (by the maximum principle) touch V except at the boundary. Thus D' is locally minimizing. Consequently, $D' \neq M$, so D' is disjoint from O by definition of O. It then follows that D' separates V from O. □

3. Singular Minimizing Surfaces

3.1. Theorem. *Let U be a bounded open subset of R^N and let S be a integral current (or flat chain modulo p) supported in U that uniquely minimizes area among currents supported in $\bar{U}$.*

Let Γ be a smooth embedded arc whose endpoints are regular boundary points of S and that is otherwise disjoint from S. Suppose also that at each of its two *endpoints,* Γ *makes a nonzero angle with the tangent half-plane to* S *at that endpoint.*

Then there is an open set W *containing* Γ *and a bounded open set* U' *containing* $U \cup W$ *such that if* P_n *is a sequence of bridges on* ∂S *shrinking nicely to* Γ *and if* T_n *is an surface that minimizes area among all surfaces in* $\overline{U'}$ *with boundary* $\partial(S \cup P_n)$*, then*

(1) $T_n \to S$ *weakly as currents,*
(2) $\operatorname{dist}(T_n, S \cup \Gamma) \to 0$,
(3) *for sufficiently large* n, T_n *does not touch* $\partial U'$ *and is therefore stationary,*
(4) T_n *converges smoothly to* S *on compact subsets of* $R^N \setminus (\Gamma \cup \operatorname{sing}(S))$*, where* $\operatorname{sing}(S)$ *is the set of singularities of* S*, and*
(5) *for sufficiently large* n, $T_n \cap W$ *is diffeomorphic to a strip* $\bar{B}^{m-1} \times (0,1)$.

Remark. For the purposes of this theorem, a singular point is any point that has no neighborhood in which T is a smooth manifold (or manifold with boundary) with multiplicity 1. (So a smooth surface with multiplicity > 1 is considered to be singular.)

Proof. By lemma 2.1 of [W1] there is a bounded open set U' that contains $U \cup \Gamma$ and in which S is uniquely area minimizing.

The first two conclusions are proved exactly as in the smooth strictly stable case [W1, 2.2]. The third follows immediately from the first two. The fourth conclusion is an immediate consequence of the basic GMT regularity theory.

Let $W \subset U'$ be a neighborhood of Γ such that $\bar{W} \cap S$ is diffeomorphic to the union of two closed m-balls, and such that ∂W is transverse to S. (Note that ∂W is not transverse to ∂S; otherwise $\bar{W} \cap S$ would have corners).

Let $T_n' = T_n \cap \bar{W}$ and $S = S \cap \bar{W}$. Note by (3) that T_n' converges smoothly to $T \cap \bar{W}$ away from Γ. Let

$$C_n = [(\partial W) \cap \operatorname{interior}(T_n)] \cup [\bar{W} \cap \partial S]$$

Then C_n converges smoothly to $\partial S'$. Let S_n' be the area minimizing (in $\overline{U'}$) surface with boundary C_n. Then by standard regularity theory, S_n' converges to S' smoothly. Note also that T_n' and $S_n' \cup P_n$ have the same boundary.

Thus we have:

(1) A smooth strictly stable and uniquely minimizing surface $S' = S \cap \bar{W}$ in an open set U',
(2) An arc $\Gamma \in U'$ joining two points of $\partial S'$ nicely,
(3) A sequence of smooth minimal surfaces S'_n converging smoothly to S' and such that $\partial S'_n$ coincides with $\partial S'$ near the endpoints of Γ
(4) a sequence of bridges P_n shrinking to Γ nicely, and
(5) for each n a surface T'_n with boundary $\partial(S'_n \cup P_n)$ that minimizes area (among surfaces in U' with the same boundary).

The last conclusion of the theorem now follows immediately from [W1, 2.2]. □

4. The Converse of the Bridge Principle

Let C' be the connected curve formed by joining two curves C_1 and C_2 by a thin bridge. We have proved various theorems showing that minimal surfaces bounded by C_1 and C_2 can be joined to form a minimal surface bounded by C'. One can ask whether all the minimal surfaces bounded by C' arise in this way. The answer is in general no, unless we impose fairly strong hypotheses on the bridge. Consider the following examples:

Example 1. Let C_1 and C_2 be horizontal circles in ∂B^3, one slighly above and one slightly below the equator. Let Γ be an arc in ∂B^3 joining C_1 to C_2, and let P_n be a sequence of bridges in ∂B^3 shrinking nicely to Γ. Let C'_n be the curve formed by joining C_1 to C_2 with P_n. Now $C_1 \cup C_2$ bounds exactly three classical minimal surfaces [SR], namely

(1) a pair of disks,
(2) a stable catenoid, and
(3) an unstable catenoid.

These have nullity 0, so the bridge theorem 1.2 gives us a corresponding set of three minimal surfaces bounded by C'_n, namely a disk and two genus 1 surfaces. Note that the area of the disk is approximately 2. However, C'_n bounds a disk with far less area (for instance, the component of $\partial B^3 \setminus C'_n$ that lies between C_1 and C_2). Thus the least area disk does not resemble a minimal surface with a bridge attached. (It looks like a thin ribbon near the equator.)

Example 2. Let C_1 and C_2 be the boundaries of a pair of disjoint disks. Let Γ be a "geometrically knotted" arc joining C_1 to C_2. In other words, there should be a convex open set U such that $U \cap \Gamma$ is a connected arc that is knotted in U; see [DW]. We may choose Γ so that U does not intersect either of the two disks. Let P_n be a sequence of bridges shrinking to Γ nicely, and let C'_n be the connected curve formed by joining C_1 to C_2 with P_n. Then by [DW], there will be a soap-film-like minimal variety bounded by a portion of one of the bridge arcs (and not touching C_1 or C_2!)

The problem in example 1 is that the curves C_1 and C_2 are two close together. The problem in example 2 is that the arc Γ is not straight enough. From now on we will avoid these problems by making the following hypotheses:

4.1. Hypotheses. *Let X be a catenoid centered at the origin, and let X^* be the union of all dilates of X. Note that $\overline{X^*}$ is a cone whose complement consists of two convex components. Let K_i $(i = 1, 2)$ be convex sets, one in each component of $R^3 \setminus \overline{X^*}$. Let C_i be a smooth embedded curve in K_i, and let p_i be a point in $C_i \cap \partial K_i$. Let Γ be a smooth arc such that*

(1) $\Gamma \cap K_i = \partial\Gamma \cap K_i = p_i$,

(2) $\Gamma \cap X^* = \emptyset$, *and*

(3) *there is a vector v in R^3 such that Γ is never perpendicular to v.*

Let P_n be a sequence of bridges on $C_1 \cup C_2$ that shrink to Γ nicely, and let C'_n be the curve obtained by joining C_1 to C_2 along P_n.

4.2. Lemma. *Assume the hypotheses 4.1. Let V_n be a nonzero stationary integral varifold bounded by C'_n. Then every sequence of the V_n has a subsequence that converges (in the sense of varifolds) to a limit $S_1 \cup S_2$, where $S_i \subset K_i$ is a stationary integral varifold with boundary C_i.*

Proof. Existence of subsequential limits follows from Allard's compactness theorem (see [AW1] or [SL, §42.8]). Let V' be any subsequential limit. We claim that $V' \cap X^* = \emptyset$. For if not, some of the dilates of X would intersect V'. Since V' is compactly supported, there would then be a largest such dilate X'.

But at the point of contact between V' and X', we would have a contradiction to the maximum principle [SW].

It follows that V' decomposes into two pieces S_1 and S_2, one in each of the connected components of $R^3 \setminus \overline{X^*}$. Furthermore, S_i is stationary with respect to $C_i \cup \Gamma$. By using catenoidal barriers as in [DW], one can show that the support of S_i does not touch Γ except at p_i. Thus S_i is stationary with respect to C_i. Finally, the convex hull property implies that S_i is in K_i. □

We remark that the same proof shows that any minimal variety bounded by $C_1 \cup C_2$ is the union of two components, one bounded by C_1 and the other by C_2.

4.3. Theorem. *For smooth closed curves C in R^3, let $K(C)$ denote the set of all genus 0 classical minimal surfaces bounded by C. Assume the hypotheses 4.1 and suppose there are only finitely many surfaces in $K(C_1 \cup C_2)$, and that each such surface has nullity 0 and has no branch points. Then for sufficiently large n, the surfaces in $K(C'_n)$ have nullity 0 and no branch points, and there is a one-to-one correspondence between $K(C'_n)$ and $K(C_1 \cup C_2) \equiv K(C_1) \times K(C_2)$. The correspondence preserves index, and pairs embedded surfaces with embedded surfaces.*

Proof. Suppose first that $K(C_i)$ contains only embedded disks. Theorem 1.2 guarantees the existence of a one-to-one map

$$\phi_n : K(C_1 \cup C_2) \to K(C'_n)$$

with all the desired properties except surjectivity.

Suppose the surjectivity failed for arbitrarily large n. Then (passing to a subsequence) we may assume that for each n, C'_n bounds a disk D_n that does not arise (as in theorem 1.2) from connecting disks in $K(C_1)$ and $K(C_2)$, and that the D_n converge as varifolds to $S_1 \cup S_2$ as in lemma 4.2. Because the genus and total curvature of D_n is bounded (the latter by the Gauss-Bonnet theorem), it follows (see theorem 3 of [W6]) that there is a finite set $Z \subset R^3$ such that the second fundamental form of D_n is uniformly bounded (for large n) on compact subsets of $R^3 \setminus (Z \cup \Gamma)$. Consequently S_i must be a regular immersed minimal surface except possibly at the points $Z \cup \partial\Gamma$. Now a finite genus minimal surface with smooth boundary that is regular except at finitely many points is in fact a classical minimal surface (possibly with branch points). Thus $S_i \in K(C_i)$ and therefore has no branch points. But now it follows from the uniqueness assertion in theorem 1.2 that $D_n = \phi_n(S_1 \cup S_2)$ for sufficiently large n.

The case of immersed disks can be reduced to the embedded case by the device given in [W1, §5.2]. □

4.4. Theorem. *Fix any $0 \leq g \leq \infty$ and any $0 \leq \iota \leq \infty$, and let $K(C)$ denote either*

(1) *all classical minimal surfaces of genus $\leq g$ and index $\leq \iota$ that are bounded bounded by C, or*
(2) *all classical minimal surfaces of genus $\leq g$ and index $\leq \iota$ that are bounded by C and that have no self-intersections, or*
(3) *all nonzero stationary integral varifolds bounded by C (If C is not connected, we require the varifold to have support that touches each connected component of C.)*

Assume the hypotheses 4.1. Suppose also that $K(C_1 \cup C_2)$ is finite, and that each of the surfaces in $K(C_1 \cup C_2)$ has nullity 0 and is a regular embedded surface with no boundary branch points. Then for all sufficiently large n, there is a bijection

$$\phi_n : K(C_1 \cup C_2) \mapsto K(C'_n)$$

For each $S \in K(C_1 \cup C_2)$, $\phi_n(S)$ is a regular embedded surface with the same index and genus as S.

Proof. Essentially the same as 4.3. □

EXAMPLE. Let $E \subset R^3$ be Enneper's surface as represented in [N, §88-93]. Let E_r be the portion of E inside the ellipsoid

$$x^2 + y^2 + \frac{4}{3}z^2 = \frac{1}{9}r^2(3 + r^2)^2$$

Then by a theorem of Ruchert [R], ∂E_1 bounds exactly one minimal disk, namely E_1, which has nullity 1. Furthermore, by work of Beeson and Tromba [BT], for every r slightly larger than 1, ∂E_r bounds exactly 3 minimal disks near E_r. Two are strictly stable and the other (namely E_r) has index 1 and nullity 0.

By [MY] (or [W5, corollary 2.2] or [L]) an extreme curve either bounds exactly one minimal variety or bounds more than one minimal disk. Consequently E_1 is the only stationary integral varifold bounded by ∂E_1. It follows (by the Allard regularity theorems [AW1, AW2]) that for r slightly larger than 1, every stationary integral varifold bounded by $C = E \cap \partial B_r$ is an embedded disk near E_r (and thus one of the three disks mentioned above).

Hence if we connect two translates of such a C by a suitable bridge as in theorem 1.2, we get a smooth unknotted curve that bounds exactly 9 stationary varifolds. Each of the 9 varifolds is a smooth embedded disk.

Similarly (proceeding inductively), we can connect k translates of C together to get a curve that bounds exactly 3^k stationary integral varifolds. All are embedded disks with nullity 0, and, for $0 \leq p \leq k$, exactly $\binom{k}{p}2^{k-p}$ have index p.

5. A Question Raised by Nitsche

Professor J. C. C. Nitsche raised the following question. Let C_1 and C_2 be simple closed curves in R^3. Can one join C_1 to C_2 by a thin bridge to get a curve C that bounds a minimal disk that can be perturbed slightly to the form $D_1 \cup D_2 \cup P$ where D_i is a minimal disk bounded by C_i and P is a thin ribbon? The problem is that one does not know if C_i bounds any minimal surface that satisfies the hypotheses of any of the theorems of this paper or its companion [W1]. For instance it is conceivable that there is a smooth curve C that bounds a continuous family of area minimizing disks (each with a boundary branch point) but no other minimal surfaces.

The first answer is yes, provided the curves C_1 and C_2 are real analytic. This implies that each C_i bounds only finitely many area minimizing disks (by [TF]), none of which have branch points (by [O] together with [G] or [AH1-2]) or boundary branch points [GL]. Let D_i be one such disk. If the proof of [W2, theorem 2] shows that there is an open set $U \subset R^3$ containing D_i in which D_i is minimizing as a current. The desired result follows immediately from theorem 3. If D_i is immersed, we use the trick in [W1, §5.2] to reduce to the embedded case.

The second answer is yes, provided C_1 and C_2 are not too close together and provided we can choose the arc Γ along which we put the bridge. That is, suppose we have the hypotheses 4.1 of the previous section. Let D_n be a least area disk bounded by C'_n. The argument in the proof of theorem 4.3 shows that a subsequence of the D_n converges (in the sense of varifolds) to $S_1 \cup S_2$, where S_i is a minimal disk bounded by C_i. That D_n is an embedded ribbon near Γ is proved by essentially the same argument used for theorem 3. Note also that D_n has no interior branch points, and has no boundary branch points except near boundary branch points of $S_1 \cup S_2$.

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