COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 2, Number 3, 431–450, 1994

ON THE HARNACK INEQUALITIES OF PARTIAL DIFFERENTIAL EQUATIONS

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Dedicated to Elliott Lieb on his sixtieth birthday

In 1979, I was at the Institute for Advanced Study organizing the special year in geometry. I had many interactions with Elliot Lieb. I was very much interested in the log concavity theorem that Brascamp and Lieb [1] did on the eigenfunctions of the Laplacian. I was trying to see the meaning of this theorem for eigenfunctions defined on a curved manifold. Since the original argument of Brascamp-Lieb [1] cannot work on a curved space, I developed a continuity argument which was able to treat some cases. While I gave several seminar talks on this argument in 1980, it appeared partially in the joint paper with other coauthors [4]. Since I am invited to write an essay for Elliot, I have decided to give more complete discussion on this topic.

I also found some improvement of my previous work with Peter Li [3]. It is a curiosity that a similar calculation appears for the wave equation. In my previous work with Li, we derive a sharp estimate for the heat kernel based on the Harnack inequality. I expect that Harnack inequality for other equations should give some fundamental information about the equations.

1. Harnack inequality for semilinear elliptic equations

In this section, we recall our result with Peter Li. Theorem 1.1 is somewhat unnatural. But it shows how the argument can be used to treat semilinear equation. Theorem 1.2 does sharpen the work of [3].

Research supported by NSF Grant No. DMS-92-06938

Let ds^2 be a complete metric defined on a manifold M (with possibly nonempty convex boundary). Then we shall consider the equation

$$\frac{\partial u}{\partial t} = \Delta u - Vu$$

where $u > 0$ and $V = h(x) + k(u)$.

Let $\varphi = -\log u$. Then

$$\varphi_t = \Delta\varphi + V - |\nabla\varphi|^2. \tag{1.1}$$

As in my work with Peter Li [3],

(1.2)
$$\frac{d}{dt}(|\nabla\varphi|^2 + \varphi_t - V) = \Delta_t(|\nabla\varphi|^2 + \varphi_t - V) - 2\varphi_i(|\nabla\varphi|^2 + \varphi_t - V)_i$$
$$+ (\frac{d}{dt}\Delta_t)(\varphi) - 2\sum R_{ij}\varphi_i\varphi_j - 2\sum\varphi_{ij}^2 + \Delta h + \Delta(k(u)).$$

Then for any $c, \widetilde{c}$

$$\begin{aligned}(1.3)\quad & \frac{d}{dt}(|\nabla\varphi|^2 + \varphi_t - V + ck + \widetilde{c} - \frac{n}{2t}) \\ & \leq \Delta_t(|\nabla\varphi|^2 + \varphi_t - V + ck + \widetilde{c} - \frac{n}{2t}) \\ & - 2\varphi_i(|\nabla\varphi|^2 + \varphi_t - V + ck + \widetilde{c} - \frac{n}{(2-c_1)t})_i \\ & - 2\sum_{ij}\varphi_{ij}^2 + (-cu^2k_{uu} - 2cuk_u + uk_u + u^2k_{uu})|\nabla\varphi|^2 \\ & - 2\sum R_{ij}\varphi_i\varphi_j + \Delta h - uk_u(\Delta\varphi) - cVu\frac{\partial k}{\partial u} + f(\varphi) + \frac{n}{2t^2}.\end{aligned}$$

We shall assume that

$$-cu^2k_{uu} - 2cuk_u + uk_u + u^2k_{uu} \leq 2\inf_i R_{ii}. \tag{1.4}$$

For simplicity, we assume that M is compact with no boundary. (The general case can be treated as in [3]). Let (x_0, t_0) be the point so that $|\nabla\varphi|^2 + \varphi_t - V + ck + \widetilde{c} - \frac{n}{2t}) < 0$ in $M \times [0, t_0)$ and is equal to zero at (x_0, t_0). Then it follows from the maximum principle and (1.4) that

$$0 \leq -2\sum_{i,j}\varphi_{ij}^2 + \Delta h - cV\frac{\partial k}{\partial u} + f(\varphi) - uk_u(\Delta\varphi) + \frac{n}{2t^2}. \tag{1.5}$$

But

$$
\begin{aligned}
\sum_{i,j} \varphi_{ij}^2 &\geq \frac{1}{n}(\Delta\varphi)^2 \\
&= \frac{1}{h}(|\nabla\varphi|^2 + \varphi_t - V)^2 \\
&= \frac{1}{n}(-ck - \widetilde{c} + \frac{n}{2t})^2.
\end{aligned} \tag{1.6}
$$

Hence

$$
\begin{aligned}
0 \leq &\frac{2}{n}(ck + \widetilde{c})^2 + 2(ck + \widetilde{c})t^{-1} + uk_u(ck + \widetilde{c}) \\
&- \frac{nuk_u}{2t} + \Delta h - cVuk_u + f(e^{-u}).
\end{aligned} \tag{1.7}
$$

Hence if we assume that

$$
2(ck + \widetilde{c}) - \frac{nuk_u}{2} \leq 0 \tag{1.8}
$$

and

$$
\frac{2}{n}(ck + \widetilde{c})^2 \geq (ck + \widetilde{c})uk_u + \Delta h - cVuk_u + f(e^{-u}). \tag{1.9}
$$

then inequality (1.6) becomes equality and the maximum principle shows that

$$
|\nabla\varphi|^2 + \varphi_t - V + ck + \widetilde{c} - \frac{n}{2t} \leq 0 \tag{1.10}
$$

for all $t > 0$.

We have therefore proved the following theorem which generalizes the previous work with Peter Li [3].

Theorem 1.1. *Let M be a compact manifold. let u be a positive solution of the equation*

$$
\frac{\partial u}{\partial t} = \Delta_t u - Vu,
$$

where Δ_t is the Laplacian of metrics ds_t^2 and $V = h(x) + k(u)$. Assume that (1.4), (1.8) and (1.9) hold, then

$$
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - V + ck(u) + \widetilde{c} \leq \frac{n}{2t}.
$$

Corollary 1. *Let M be a compact manifold with nonnegative Ricci curvature. Let u be a positive solution of the equation $\frac{\partial u}{\partial t} = \Delta u + u^{\alpha+1}$ such that either $0 \le \alpha \le \frac{4}{n} - 1$ or $\frac{4}{n} - 1 \le \alpha \le 0$ or $\alpha \le -1$. Then $\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} + \frac{u^\alpha}{1+\alpha} \le \frac{n}{2t}$ for all $t > 0$.*

Corollary 2. *Let M be a compact manifold with nonnegative Ricci curvature. Let u be a positive solution of the equation $\frac{\partial u}{\partial t} = \Delta u - u^{\alpha+1}$ such that either $\alpha + 1 \ge \frac{4}{n}$ or $-1 \le \alpha \le \frac{4}{n} - 1 \le 0$. Then $\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - \frac{u^\alpha}{1+\alpha} \le \frac{n}{2t}$ for all $t > 0$.*

Let us now improve the basic estimate in [3]. For simplicity, assume that Δ_t is independent of t. We shall consider the function

$$\psi = |\nabla\varphi|^2 + \varphi_t - V - \alpha\sqrt{|\nabla\varphi|^2 + \beta} - \frac{nt^{-1}}{2}$$

where $\alpha, \beta \ge 0$ are constants.

By computation,

(1.11)
$$\begin{aligned}
&\frac{\partial\psi}{\partial t} - \Delta\psi + 2\sum_i \varphi_i\psi_i \\
&= -2\sum_{i,j} R_{ij}\varphi_i\varphi_j - 2\sum\varphi_{ij}^2 + \Delta V + \alpha(|\nabla\varphi|^2 + \beta)^{-1/2}\sum_{i,j} R_{ij}\varphi_i\varphi_j \\
&\quad - \alpha(|\nabla\varphi|^2 + \beta)^{-3/2}\sum_{i,j,k}\varphi_k\varphi_j\varphi_{ik}\varphi_{ij} - \alpha\frac{\sum\varphi_i V_i}{\sqrt{|\nabla\varphi|^2 + \beta}} + \frac{nt^{-2}}{2} \\
&\le [-2 + \alpha(|\nabla\varphi|^2 + \beta)^{-1/2}]\sum_{i,j} R_{ij}\varphi_i\varphi_j - 2\sum\varphi_{ij}^2 + \Delta V + \frac{\alpha n}{2}|\nabla V| + \frac{n}{2}t^{-2}.
\end{aligned}$$

As before, we conclude that when ψ achieves its maximum,

(1.12)
$$\begin{aligned}
2(|\nabla\varphi|^2 + \varphi_t - V)^2 &= 2(\Delta\varphi)^2 \\
&\le 2n\sum_{i,j}\varphi_{ij}^2 \\
&\le n\Delta V + \frac{\alpha n}{2}|\nabla V| \\
&\quad + n\,[-2 + \alpha(|\nabla\varphi|^2 + \beta)^{-1/2}]\sum_{i,j} R_{ij}\varphi_i\varphi_j + \frac{n^2}{2}t^{-2}.
\end{aligned}$$

Hence if $\alpha \leq 2\beta^{\frac{1}{2}}$,

(1.13)

$$|\nabla\varphi|^2 + \varphi_t - V \leq \frac{n}{2}t^{-1} + [\frac{n}{2}(\Delta V) + \frac{\alpha n}{2}|\nabla V| + (-n \inf R_{ii})|\nabla\varphi|^2]^{1/2}.$$

Hence if

$$\frac{n}{2}(\Delta V) + \frac{\alpha n}{2}|\nabla V| + n(-\inf R_{ii})|\nabla\varphi|^2 \leq \alpha^2(|\nabla\varphi|^2 + \beta) \tag{1.14}$$

we can prove $\psi \leq 0$ for all t.

Theorem 1.2. *Let M be a compact manifold and u be a positive solution of the equation $\frac{\partial u}{\partial t} = \Delta u - Vu$. Let α, β be positive constants so that*

$$\alpha \leq 4\beta$$
$$\alpha^2 \geq (-\inf R_{ii})$$
$$\alpha^2\beta \geq \frac{n}{2}(\Delta V) + \frac{\alpha n}{2}|\nabla V|.$$

Then

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - \alpha\sqrt{\frac{|\nabla u|^2}{u^2} + \beta} \leq \frac{n}{2t}$$

for all t.

Note. Theorem 1.2 should be used to improve the heat kernel estimate for manifolds with negative curvature in [3].

2. The heat equation for the Hessian of φ

From this section, we shall restrict ourselves to metrics independent of t for simplicity. We shall find a lower estimate of the Hessian of $-\log u$.

Let ψ be the minimum eigenvalue of the Hessian of φ and e_i be the corresponding eigenvector. Then $\psi = \varphi_{ii}$ and $\varphi_{ij} = 0$ for $e_j \perp e_i$. We can then conclude $\Delta\psi = \sum_j(\varphi_{ii})_{jj} = \sum_j\varphi_{iijj}$.

By the commutation formula for covariant derivatives, we have the following formula

(2.1)

$$
\begin{aligned}
(\Delta\varphi)_{ii} &= \sum \varphi_{jjii} \\
&= \sum_j \varphi_{jiji} - (\sum_k R_{ik}\varphi_k)_i \\
&= \sum \varphi_{ijji} - \sum_k R_{ik,i}\varphi_k - \sum R_{ik}\varphi_{ki} \\
&= \sum \varphi_{ijij} - \sum_{j,\ell} \varphi_{\ell j} R_{\ell iij} - \sum_{j,\ell} \varphi_{i\ell} R_{\ell jij} - \sum_k R_{ik,i}\varphi_k - \sum_k R_{ik}\varphi_{ki} \\
&= \sum \varphi_{iijj} + \sum_{k,j} (R_{ijik}\varphi_k)_j - \sum \varphi_{\ell j} R_{\ell iij} - \sum \varphi_{i\ell} R_{\ell jij} - \sum_k R_{ik}\varphi_{ki} \\
&= \sum \varphi_{iijj} + \sum_{k,j} (R_{ijik}\varphi_k)_j - \sum \varphi_{\ell j} R_{\ell iij} \\
&\qquad - \sum \varphi_{i\ell} R_{\ell jij} - \sum R_{ik,i}\varphi_k - \sum R_{ik}\varphi_{ki} \\
&= \Delta\psi + \sum_k \left(\sum_j R_{ijik,j} \right) \varphi_k + \sum_{k,j} R_{ijik}\varphi_{kj} - \sum_{\ell,j} \varphi_{\ell j} R_{\ell iij} \\
&\qquad - \sum \varphi_{i\ell} R_{\ell jij} - \sum_{\ell,j} \varphi_{\ell j} R_{\ell iij} - \sum R_{ik,i\varphi_k} - \sum R_{ik}\varphi_{ki}.
\end{aligned}
$$

By the Bianchi identity,

$$
\begin{aligned}
\sum_k R_{ijik,j} &= -\sum_j R_{ijkj,i} - \sum_j R_{ijji,k} \\
&= -R_{ik,i} + R_{ii,k}.
\end{aligned}
\tag{2.2}
$$

Hence

$$
\begin{aligned}
(\Delta\varphi)_{ii} = \Delta\psi - 2\sum_k R_{ik,i}\varphi + k + \sum_k R_{ii,k}\varphi_k \\
+ 2\sum_{k,j} R_{ijik}\varphi_{kj} - 2\sum R_{ik}\varphi_{ki}.
\end{aligned}
\tag{2.3}
$$

Similarly,

$$
\begin{aligned}
(|\nabla\varphi|^2)_{ii} &= 2\sum \varphi_{ji}^2 + 2\sum \varphi_j\varphi_{iij} + 2\sum_{j,.k} R_{ijik}\varphi_j\varphi_k \\
&= 2\psi^2 + 2\sum \varphi_j\psi_j + 2\sum R_{ijik}\varphi_j\varphi_k.
\end{aligned}
\tag{2.4}
$$

Hence from equation (1.1)

$$\frac{\partial\psi}{\partial t} = \Delta\psi - 2\sum_k R_{ik,i}\varphi_k + \sum_k R_{ii,k}\varphi_k + 2\sum R_{ijik}\varphi_{kj} - 2R_{ii}\psi + V_{ii} - 2\psi^2 - 2\sum \varphi_j\psi_j - 2\sum R_{ihik}\varphi_j\varphi_k. \tag{2.5}$$

By the second Bianchi identity,

$$2R_{ik,i} + R_{ii,k} = 0.$$

Hence

$$\frac{\partial\psi}{\partial t} = \Delta\psi + V_{ii} - 2\psi^2 - 2R_{ii}\psi + 2\sum R_{ijik}\varphi_{kj} - 2\sum R_{ijik}\varphi_j\varphi_k - 2\sum \varphi_j(\psi - \frac{3}{2}R_{ii})_j. \tag{2.6}$$

Let us now assume that M is flat so that $R_{ijik} = 0$. If we know that $\psi \geq \sqrt{\frac{1}{4t^2} + \inf_i V_{ii}}$ at $0 \leq t \leq t_0$ and that for some (x_0, t_0), $\psi = \sqrt{\frac{1}{4t^2} + \inf_i V_{ii}}$, then it follows easily from (2.6) and the minimal principle at (x_0, t_0), that

$$0 \geq \inf V_{ii} - 2\psi^2 + \frac{1}{2t^2} \tag{2.7}$$

which violates the sharp minimum principle.

Theorem 2.1. *Let V be a potential defined on a compact flat manifold and u be a positive solution of $\frac{\partial u}{\partial t} = \Delta u - Vu$. If $-(\log u)_{ii} \geq \sqrt{\frac{1}{4t^2} + \inf_i V_{ii}}$ for $t = 0$, then it is true everywhere as long as $\inf_i V_{ii} + \frac{1}{4t^2} \geq 0$.*

Note. Theorem 2.1 can be generalized easily to non compact manifold as in [3].

If we do not want to make any assumptions on the second derivatives of V, we can proceed as follows. Let $V = \Delta F$ and we find that

$$\frac{d}{dt}(\psi + F_{ii} - \frac{1}{2t}) = \Delta(\psi + F_{ii}) - 2\psi^2 + \frac{1}{2t^2}. \tag{2.8}$$

Suppose there is a function f so that

$$\frac{\partial f}{\partial t} \geq \Delta f + 2\sup_i(f + F_{ii})^2. \tag{2.9}$$

Then we can conclude that if $\psi + F_{ii} + f - \frac{1}{2t} \geq 0$ for $t = 0$ and $F_{ii} + f \geq 0$, then $\psi + F_{ii} + f - \frac{1}{2t} \geq 0$ for all t. (Note that when $\psi + F_{ii} + f - \frac{1}{2t} = 0, \psi \leq \frac{1}{2t}$.)

Theorem 2.2. *Let F and f be functions defined on R^n so that* $\inf_i(F_{ii} + f) \geq 0$ *and* $\frac{\partial f}{\partial t} \geq \Delta f + 2\sup_i(f + F_{ii})^2$. *Then for any positive solution u of* $\frac{\partial u}{\partial t} = \Delta u - (\Delta F)u$,

$$-(\log u)_{ij} + F_{ij} + (f - \frac{1}{2t})\delta_{ij} \geq 0$$

for all $t \geq 0$ as long as it is true for $t = 0$.

3. Concavity estimates on curved manifolds

In order to compensate the curvature terms in §2, we shall use the following equations

$$\frac{\partial}{\partial t}\varphi_i^2 = \Delta\varphi_i^2 - 2\sum \varphi_j(\varphi_i^2)_j - 2\sum_j \varphi_{ij}^2 - 2\sum_{j,k} R_{ijik}\varphi_j\varphi_k - 2V_i\varphi_i \tag{3.1}$$

$$\frac{\partial}{\partial t}\varphi_t = \Delta\varphi_t - 2\sum \varphi_j(\varphi_t)_j. \tag{3.2}$$

Let α, β, γ be contents. Then

$$\begin{aligned}\frac{\partial}{\partial t}(\psi - \alpha\varphi_i^2 - \beta\varphi_t + \gamma\varphi - \frac{3}{2}R_{ii} + c(t)) \\ &= \Delta(\psi - \alpha\varphi_i^2 - \beta\varphi_t + \gamma\varphi - \frac{3}{2}R_{ii} + c(t)) \\ &\quad - 2\sum_j \varphi_j(\psi - \alpha\varphi_i^2 - \beta\varphi_t + \gamma\varphi - \frac{3}{2}R_{ii} + c(t))_j \\ &\quad + V_{ii} + c'(t) - 2\alpha\varphi_i V_i + \frac{3}{2}\Delta R_{ii} - 2R_{ii}\psi \\ &\quad + 2\sum_{j,k} R_{ijik}\varphi_{kj} + (2\alpha - 2)\sum_j \varphi_{ij}^2 \\ &\quad + (2\alpha - 2)\sum R_{ijik}\varphi_j\varphi_k + \gamma|\nabla\varphi|^2 - \gamma V.\end{aligned} \tag{3.3}$$

We shall assume that

$$\varphi_{ij} - \alpha\varphi_i\varphi_j - \beta\varphi_t\delta_{ij} + \gamma\varphi\delta_{ij} - \frac{3}{2}R_{ij} + c(t)\delta_{ij} \geq 0 \tag{3.4}$$

for $0 \leq t \leq t_0$ and its minimal eigenvalue is zero at some positive (x_0, t_0). We shall denote its minimal eigenvector to be e_i. Let λ be the minimal eigenvalue of R_{ijik}. Then

(3.5)
$$\begin{aligned}\sum_{j,k} R_{ijik}\varphi_{kj} &\geq \sum_{j,k}(R_{ijik}-\lambda\delta_{jk})\varphi_{kj}+\lambda\Delta\varphi\\ &\geq \sum_{j,k}(R_{ijik}-\lambda\delta_{jk})(\alpha\varphi_j\varphi_k+\beta\varphi_t\delta_{jk}+\frac{3}{2}R_{jk}-\gamma\varphi\delta_{jk}-c(t)\delta_{jk})\\ &\quad +\lambda(|\nabla\varphi|^2+\varphi_t-V)\\ &=\alpha\sum_{j,k} R_{ijik}\varphi_j\varphi_k+(\lambda-\alpha\lambda)|\nabla\varphi|^2\\ &\quad +\beta R_{ii}\varphi_t+(\lambda-\lambda\beta n)\varphi_t+\frac{3}{2}\sum_{j,k}R_{ijik}R_{jk}-\frac{3}{2}\lambda R\\ &\quad -\gamma R_{ii}\varphi+\lambda\gamma n\varphi-(R_{ii}-n\lambda)c(t)-\lambda V.\end{aligned}$$

Choosing $\beta=\frac{1}{n}$ and $\alpha=1$, the above equation can be simplified. By the minimal principle, we conclude, by setting (3.4) to be zero, the following inequality

$$\begin{aligned}0 &\geq V_{ii}+c'(t)+n\lambda c(t)-2\varphi_i V_i+\frac{3}{2}\Delta R_{ii}-2R_{ii}\varphi_i^2\\ &\quad -3R_{ii}^2+3\sum_{j,k}R_{ijik}R_{jk}-3\lambda R+2\sum R_{ijik}\varphi_j\varphi_k\\ &\quad +2\lambda\gamma n\varphi-(\gamma+2\lambda)V+\gamma|\nabla\varphi|^2.\end{aligned} \tag{3.6}$$

If $\lambda\varphi\geq 0$, (3.7) implies the following inequality

$$\begin{aligned}0 &\geq V_{ii}+c'(t)-2\varphi_i V_i+\frac{3}{2}\Delta R_{ii}-2R_{ii}\varphi_i^2-3R_{ii}^2\\ &\quad +3\sum R_{ijik}R_{jk}+2\sum R_{ijik}\varphi_j\varphi_k-(\gamma+2\lambda)V+\gamma|\nabla\varphi|^2\end{aligned} \tag{3.7}$$

We can choose γ so that

$$\gamma|\nabla\varphi|^2+2\sum R_{ijik}\varphi_j\varphi_k-2R_{ii}\varphi_i^2-\varphi_i^2\geq 0. \tag{3.8}$$

Then (3.8) implies

$$0\geq V_{ii}+c'(t)-V_i^2+\frac{3}{2}\Delta R_{ii}-3R_{ii}^2+3\sum_{j\cdot k}R_{ijik}R_{jk}-(\gamma+2\lambda)V. \tag{3.9}$$

Theorem 3.1. *Let M be a complete manifold with curvature bounded from below by λ. Let u be a positive solution of the equation $\frac{\partial u}{\partial t}=\Delta u-Vu$ so that $-\lambda\log u\geq 0$. Suppose that $V_{ii}-V_i^2+\frac{3}{2}\Delta R_{ii}-3R_{ii}^2+3\sum_{j,k}R_{ijik}R_{jk}\geq(\gamma+2\lambda)V$*

for some constant γ which dominates the eigenvalues of the quadratic form $-2R_{ijik} + (2R_{ii} + 1)\delta_{ij}\delta_{ik}$. Then for $\varphi = -\log u$,

$$\varphi_{ij} \geq \varphi_i\varphi_j + \frac{1}{n}\varphi_t\delta_{ij} + \gamma\varphi\delta_{ij} + \frac{3}{2}R_{ij} - c(t)\delta_{ij} \tag{3.10}$$

if such an inequality holds for $t = 0$. Here c is any function such that $c'(t) \geq 0$.

By examining the proof, we can derive the following:

Corollary. *If M is an Einstein manifold with sectional curvature greater than λ and u is any solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$ with $-\lambda \log u \geq 0$, then (3.10) holds as long as it holds when $t = 0$. Here γ is any constant which dominates the eigenvalues of $-2R_{ijik} - 2R_{ii}\delta_{ij}\delta_{ik}$.*

Note that if we assume $\lambda = 0$, there is no assumption on u. For the general case, we need to put an assumption on a lower bound of u.

In fact, we can choose γ so that γ is greater than the eigenvectors of $-2R_{ijik} + (2R_{ii} + 1)\delta_{ji}\delta_{ki}$. Then it follows from (3.7) that

$$\begin{aligned} 0 \geq V_{ii} + c'(t) + n\lambda c(t) - |\nabla V|^2 + \frac{3}{2}\Delta R_{ii} - 3R_{ii}^2 \\ + 3\sum_{j,k} R_{ijik}R_{jk} - 3\lambda R + 2\lambda\gamma n\varphi - (\gamma + 2\lambda)V. \end{aligned} \tag{3.11}$$

Theorem 3.2. *Let M be a complete manifold whose sectional curvature is bounded from below by λ. Let $c(t)$ be any function so that $c'(t) + n\lambda c(t) \geq 0$. Let γ be any number greater than the eigenvalues of $-2R_{ijik} + (2R_{ii} + 1)\delta_{ji}\delta_{ki}$. Then if*

(3.12)

$$V_{ii} \geq |\nabla V|^2 + (\gamma + 2\lambda)V + 3R_{ii}^2 - 3\sum_{j,k} R_{ijik}R_{jk} - \frac{3}{2}\Delta R_{ii} - 3\lambda R - 2\lambda\gamma n\varphi,$$

the following inequality holds

(3.13)

$$-(\log u)_{ij} \geq (\log u)_i(\log u)_j - \frac{1}{n}(\log u)_t\delta_{ij} - \gamma(\log u)\delta_{ij} + \frac{3}{2}R_{ij} - c(t)\delta_{ij}$$

for all $t \geq 0$ as long as it holds also for $t = 0$. Here $\frac{\partial u}{\partial t} = \Delta u - Vu$.

When $\lambda = 0$, we can take β to be any number as long as $\alpha \geq 1$. In particular, the same argument shows the following

Theorem 3.3. *If M is a complete Einstein manifold with nonnegative curvature and if u is a positive solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$, then for $\alpha \geq 1$, $\gamma \geq 2\sup_i R_{ii}$,*

$$\begin{aligned} -(\log u)_{ij} \geq \alpha(\log u)_i(\log u)_j - \beta(\log u)_t\delta_{ij} \\ - \gamma(\log u)\delta_{ij} + \frac{3}{2}R_{ij} - c(t)\delta_{ij} \end{aligned} \tag{3.14}$$

for $t \geq 0$ as long as it holds for $t = 0$. Here $c(t)$ is any function so that $c'(t) \geq 0$.

We leave as an exercise to derive similar inequality when M is not Einstein and $V \neq 0$.

So far, we are assuming $\alpha \geq 1$. Let us now consider the case $\alpha = \beta = 0$ and $R_{ijik} \geq 0$. We also assume that the function $\psi + \gamma\varphi - \frac{3}{2}R_{ii}$ is greater than zero at $t = 0$. Then for the first time when it becomes zero,

$$\begin{aligned} 0 \geq V_{ii} + \frac{3}{2}\Delta R_{ii} - 3R_{ii}^2 + 3\sum R_{ijik}R_{jk} \\ - 2(-\gamma\varphi + \frac{3}{2}R_{ii})^2 - 2\sum R_{ijik}\varphi_j\varphi_k + \gamma|\nabla\varphi|^2 - \gamma V. \end{aligned} \tag{3.15}$$

Hence we choose

$$\gamma \geq \sup R_{ijij} \tag{3.16}$$

and

$$V_{ii} - \gamma V \geq -\frac{3}{2}\Delta R_{ii} + 3R_{ii}^2 - 3\sum R_{ijik}R_{jk} + 2(-\gamma\varphi + \frac{3}{2}R_{ii})^2, \tag{3.17}$$

We arrive at the following theorem.

Theorem 3.4. *Let M be a complete manifold with nonnegative curvature. Let γ be a constant greater than $\sup R_{ijij}$. Suppose that V is a function satisfying (3.17). Then for any positive solution u of $\frac{\partial u}{\partial t} = \Delta u - Vu$, the following inequality holds:*

$$-(\log u)_{ij} - \gamma(\log u)\delta_{ij} - \frac{3}{2}R_{ij} \geq 0 \tag{3.18}$$

for $t \geq 0$ as long as it holds at $t = 0$.

4. Gradient estimates for systems

In this section, we study gradient estimate for harmonic maps. Let $u : M \to N$ be a map which is a critical point of the functional $\frac{1}{2}\int_M |\nabla u|^2 + \int_M V(u)$ where V is a function defined on N. Using the normal coordinate for M and N, the critical map satisfies the equation

$$\Delta u^\alpha - \frac{\partial V}{\partial y^\alpha} = 0. \tag{4.1}$$

The gradient flow satisfies the equation

$$\frac{\partial u^\alpha}{\partial t} = \Delta u^\alpha - \frac{\partial V}{\partial y^\alpha}. \tag{4.2}$$

Let F be any smooth positive function defined on N. Then

(4.3)
$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)(\frac{|\nabla u|^2}{F^2} - c(t)) &= -2F^{-2}\sum (u^\alpha_{ij})^2 - 2F^{-2}\sum R^M_{ij} u^\alpha_i u^\alpha_j \\ &+ 2F^{-2}\sum R^N_{\alpha\beta\alpha\beta}(u^\alpha_i u^\beta_j u^\alpha_i u^\beta_j - u^\alpha_i u^\alpha_j u^\beta_i u^\beta_j) \\ &+ 2F^{-2}\sum \frac{\partial^2 V}{\partial y^\alpha \partial y^\beta} u^\alpha_j u^\beta_j + 2|\nabla u|^2 F^{-3}\sum F_\alpha \frac{\partial V}{\partial y^\alpha} \\ &+ 2\frac{\nabla F}{F}\cdot \nabla(\frac{|\nabla u|^2}{F^2} - c(t)) \\ &+ 2|\nabla u|^2 F^{-4}[\sum (FF_{\alpha\beta} - F_\alpha F_\beta)\nabla u^\alpha \cdot \nabla u^\beta] - c'(t).\end{aligned}$$

We shall assume that for some $c_1 > 0$,

$$FF_{\alpha\beta} - F_\alpha F_\beta \le -c_1. \tag{4.4}$$

Then at the maximum point of $\frac{|\nabla u|^2}{F^2} - c(t)$,

$$\begin{aligned}0 \le F^{-2}|\nabla u|^2[-2\inf R^M_{ii} + 2\sup_\alpha \frac{\partial^2 V}{(\partial y^\alpha)^2} + 2F^{-1}\sum F_\alpha \frac{\partial V}{\partial y^\alpha}] \\ + 2F^{-4}(\sup F^2 R^N_{\alpha\beta\alpha\beta})|\nabla u|^4 - 2c_1 F^4 |\nabla u|^4 - c'(t).\end{aligned} \tag{4.5}$$

If we assume

$$c_1 > \sup(F^2 R^N_{\alpha\beta\alpha\beta}) \tag{4.6}$$

then

(4.7)

$$(c_1 - \sup F^2 R^N_{\alpha\beta\alpha\beta})[F^2|\nabla u|^2$$
$$- \frac{1}{2}(c_1 - \sup(F^2 R_{\alpha\beta\alpha\beta}))^{-1}(-\inf R^M_{ii} + \sup_\alpha \frac{\partial^2 V}{(\partial y^\alpha)^2} + F^{-1}\sum F_\alpha \frac{\partial V}{\partial y^\alpha})^2$$
$$\leq \frac{1}{4}(c_1 - \sup(F^2 R_{\alpha\beta\alpha\beta}))^{-1}(-\inf R^M_{ii} + \sup_\alpha \frac{\partial^2 V}{(\partial y^\alpha)^2} + F^{-1}\sum F_\alpha \frac{\partial V}{\partial y^\alpha})^2 - c'(t).$$

If we choose

$$c(t) = \frac{1}{2}| - \inf R^M_{ii} + \sup_\alpha \frac{\partial^2 V}{(\partial y^\alpha)^2} + F^{-1}\sum F_\alpha \frac{\partial V}{\partial y^\alpha}| \quad (4.8)$$
$$[1 + (c_1 - \sup(F^2 R_{\alpha\beta\alpha\beta}))^{-1}] + (c_1 - \sup F^2 R_{\alpha\beta\alpha\beta})^{-1} t^{-1}$$

then we can prove that $F^{-2}|\nabla u|^2 - c(t) \leq 0$ for all t.

Theorem 4.1. *Let u be a map from a complete manifold M into another manifold N which satisfies equation (4.2). Suppose that for some function F defined on N, (4.4) and (4.6) hold. Then $F(u)^{-2}|\nabla u|^2 \leq c(t)$ where $c(t)$ is defined by (4.8).*

Note that if the curvature of N is nonpositive, (4.6) and (4.8) can be simplified by dropping the term $R^N_{\alpha\beta\alpha\beta}$.

5. Estimates for the wave equation

Let u be a positive solution for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u - Vu \quad (5.1)$$

where V may depend on u.

Then $\varphi = \log u$ satisfies the equation

$$\varphi_{tt} + \varphi_t^2 = \Delta\varphi + |\nabla\varphi|^2 - V. \quad (5.2)$$

Let

$$\psi = \varphi_{tt} + \varphi_t^2 + V. \quad (5.3)$$

Then using equation (5.2),

(5.4)
$$\begin{aligned}\psi_{tt} &= \Delta\varphi_{tt} + 2\sum(\varphi_t)_j^2 + 2\sum\varphi_j(\varphi_{tt})_j\\ &= \Delta(\varphi_{tt}+\varphi_t^2+V) - 2\varphi_t(\Delta\varphi)_t - \Delta V + 2\sum\varphi_j(\varphi_{tt})_j\\ &= \Delta(\varphi_{tt}+\varphi_t^2+V) - 2\varphi_t(\psi - |\nabla\varphi|^2)_t - \Delta V + 2\sum\varphi_j(\psi-\varphi_t^2-V)_j\\ &= \Delta\psi - 2\varphi_t\psi_t + 2\sum\varphi_j\psi_j - \Delta V - 2\sum\varphi_j V_j.\end{aligned}$$

Rewriting (5.4), we find the following fundamental equation

$$(e^{2\varphi}\psi_t)_t = \sum_j (e^{2\varphi}\psi_j)_j - \sum_j (e^{2\varphi}V_j)_j. \tag{5.5}$$

Let ρ be any function with compact support in M. Then we can form the energy

$$E_\rho(\psi) = \int_M \rho^2 e^{2\varphi}(\psi_t^2 + |\nabla\psi|^2). \tag{5.6}$$

By computation, we obtain

(5.7)
$$\begin{aligned}\frac{dE_\rho}{dt} &= 2\int_M (\rho\rho_t e^{2\varphi}(\psi_t^2+|\nabla\psi|^2) + 2\int_M \rho^2\frac{d}{dt}(e^{2\varphi}\psi_t)\psi_t\\ &\quad - 2\int \rho^2 e^{2\varphi}\varphi_t\psi_t^2 + 2\int \rho^2 e^{2\varphi}\varphi_t|\nabla\psi|^2 + 2\int\rho^2 e^{2\varphi}\nabla\psi\cdot\nabla\psi_t.\end{aligned}$$

Integrating by part, we obtain

(5.8)
$$\begin{aligned}&2\int_M \rho^2\frac{d}{dt}(e^{2\varphi}\psi_t)\psi_t + 2\int_M \rho^2 e^{2\varphi}\nabla\psi\cdot\nabla\psi_t\\ &= 2\int_M \rho^2[\frac{d}{dt}(e^{2\varphi}\psi_t) - \nabla\cdot(e^{2\varphi}\nabla\psi)]\psi_t - 4\int_M \rho e^{2\varphi}\psi_t\nabla\rho\cdot\nabla\psi\\ &= -2\int_M \rho^2\psi_t\nabla(e^{2\varphi}\nabla V) - 4\int_M \rho e^{2\varphi}\psi_t\nabla\rho\cdot\nabla\psi.\end{aligned}$$

Hence

(5.9)

$$\begin{aligned}\frac{d}{dt}E_\rho &= 2\int_M \rho\rho_t e^{2\varphi}(\psi_t^2+|\nabla\psi|^2) - 4\int_M \rho^2 e^{2\varphi}\psi_t\nabla\varphi\cdot\nabla V - 2\int_M \rho^2 e^{2\varphi}\psi_t\Delta V\\ &\quad - 4\int_M \rho e^{2\varphi}\psi_t\nabla\rho\cdot\nabla\psi + 2\int_M \rho^2 e^{2\varphi}\varphi_t(|\nabla\psi|^2-\psi_t^2)\\ &= 2\int_M \rho(\rho_t+\rho\varphi_t)e^{2\varphi}\left|\nabla\psi - \frac{\psi_t\nabla\rho}{\rho_t+\rho\varphi_t}\right|^2\\ &\quad + 2\int_M \rho e^{2\varphi}\left(\rho_t-\rho\varphi_t-\frac{|\nabla\rho|^2}{\rho_t+\rho\varphi_t}\right)\psi_t^2\\ &\quad - 4\int_M \rho^2 e^{2\varphi}\psi_t\nabla\varphi\cdot\nabla V - 2\int_M \rho^2 e^{2\varphi}\psi_t\Delta V.\end{aligned}$$

Assume that

$$\rho_t+\rho\varphi_t<0 \tag{5.10}$$

and

$$\rho_t^2 \ge \rho^2\varphi_t^2+|\nabla\rho|^2-\epsilon\rho(\rho_t+\rho\varphi_t), \tag{5.11}$$

where $\epsilon>0$. Then it follows from (5.9) that

$$\frac{d}{dt}E_\rho \le \frac{1}{4\epsilon}\int_M \rho^2|\nabla(e^{2\varphi}\nabla V)|^2. \tag{5.12}$$

In conclusion,

$$E_\rho(t)\le E_\rho(0)+\frac{1}{4\epsilon}\int_0^t\int_M \rho^2(\nabla(e^{2\varphi}\nabla V)|^2. \tag{5.13}$$

Note that if $V=0$, (5.13) says that

$$E_\rho(t)\le E_\rho(0). \tag{5.14}$$

Theorem 5.1. *Let u be a positive solution of the wave equation (5.1). Let $\varphi=\log u$. Then (5.5) holds. Furthermore, let ρ be any function which satisfies (5.10) and (5.11). Then (5.13) holds for E_ρ defined by (5.6).*

If V is independent of t, then $\psi - V$ satisfies an equation similar to (5.5) with $V=0$. Hence if we define the energy to be $\int_M \rho^2 e^{2\varphi}(\psi_t^2+|\nabla(\psi-V)|^2)$, it will decrease in time as long as (5.10) and (5.11) hold (with $\epsilon=0$).

Note that a function ρ can be constructed in the following way to satisfy (5.10) and (5.11): If we have constructed a function $\widetilde{\rho}$ which satisfies the inequalities $\widetilde{\rho}\widetilde{\rho}_t < 0$ and $\widetilde{\rho}_t^2 > |\nabla\widetilde{\rho}|^2$, then $\rho = \widetilde{\rho}\exp(-\varphi + C\widetilde{\rho})$, where C is a large constant, will satisfy (5.10) and (5.11). The function $\widetilde{\rho}$ can be constructed by taking a function with compact support composite with $(1-\epsilon)r + t$.

6. Harnack inequality for degenerate elliptic equations

Let us now consider a nonlinear equation of the type

$$\frac{\partial u}{\partial t} = \Delta(F(u)) \tag{6.1}$$

with $F' \geq 0$.

For simplicity, we shall assume that the equation is defined on a compact manifold. It is straightforward to generalize all the theorems to complete manifolds.

Let G be a function such that $G'(t) = t^{-1}F'(t)$. Then setting $\varphi = \frac{1}{2}\log u^2$, we have

$$\frac{d\varphi}{dt} = \frac{\Delta(F(u))}{u} = \Delta(G(u)) + \nabla G(u)\cdot\nabla\varphi. \tag{6.2}$$

We shall consider $G(u) = G(e^{\varphi})$ for $u > 0$ and $G(u) = G(-e^{\varphi})$ for $u < 0$. We shall write it as $G(\varphi)$. Then we are dealing with the equation

$$\frac{d\varphi}{dt} = \Delta G + \nabla G\cdot\nabla\varphi. \tag{6.3}$$

Let

$$\psi = |\nabla G|^2 - \alpha\frac{dG}{dt}. \tag{6.4}$$

Then

$$\begin{aligned}
\psi_t &= 2\nabla G_t\cdot\nabla G - \alpha(G_t)_t \\
\Delta\psi &= 2\sum G_{ij}^2 + 2\sum_j(\Delta G)_j G_j + 2\sum_{i,j} R_{ij}G_iG_j - \alpha(\Delta G)_t
\end{aligned}$$

and

(6.5)
$$\begin{aligned}\psi_t - G'\Delta\psi &= 2\nabla(G_t - G'\Delta G)\cdot\nabla G + 2G''(\Delta G)\nabla\varphi\cdot\nabla G - 2G'\sum_{i,j} G_{ij}^2\\ &\quad - 2G'\sum_{i,j} R_{ij}G_iG_j - \alpha(G_t - G'\Delta G)_t - \alpha G''\varphi_t\Delta G\\ &= 4\sum G_iG_jG_{ij} + 2G''\Delta G\nabla G\cdot\nabla\varphi - 2G'\sum G_{ij}^2\\ &\quad - 2G'\sum R_{ij}G_iG_j - 2\alpha\sum G_i(G_t)_i - \alpha G''\varphi_t\Delta G\\ &= 2\nabla G\cdot\nabla\psi - 2G'\sum G_{ij}^2 - 2G'\sum R_{ij}G_iG_j\\ &\quad + 2G''\Delta G\nabla G\cdot\nabla\varphi - \alpha G''\varphi_t\Delta G\\ &= 2\nabla G\cdot\nabla\psi - 2G'\sum G_{ij}^2 - 2G'\sum_{i,j} R_{ij}G_iG_j\\ &\quad - \alpha G''(\Delta G)^2 + (2-\alpha)G''\Delta G\nabla G\cdot\nabla\varphi.\end{aligned}$$

Let $c(t)$ be any smooth function of t. Then

(6.6)
$$\begin{aligned}&\frac{d}{dt}(\psi - c(t)) - G'\Delta(\psi - c(t))\\ &\quad = 2\nabla G\cdot\nabla(\psi - c(t)) - 2G'\sum G_{ij}^2 - 2G'\sum_{i,j} R_{ij}G_iG_j\\ &\qquad - \alpha G''(\Delta G)^2 + (2-\alpha)G''\Delta G\nabla G\cdot\nabla\varphi - c'(t).\end{aligned}$$

We assume that $\psi - c(t) \le 0$ for $t = 0$. If $\psi - c(t) \le 0$ for $t \le t_0$ and $\psi(x_0) - c(t_0) = 0$ for some x_0, we would have

(6.7)
$$\begin{aligned}0 &\le -2G'\sum G_{ij}^2 - 2G'\sum_{i,j} R_{ij}G_iG_j\\ &\quad - \alpha G''(\Delta G)^2 + (2-\alpha)G''\Delta G\nabla G\cdot\nabla\varphi - c'(t).\end{aligned}$$

Since $\Delta G = \psi + (\alpha - 1)\nabla G\cdot\nabla\varphi$, we obtain

(6.8)
$$\begin{aligned}0 &\le -2G'\sum_{i,j} G_{ij}^2 - 2G'\sum_{i,j} R_{ij}G_iG_j\\ &\quad - \alpha G''(\Delta G)^2 + (2-\alpha)G''c(t)\nabla G\cdot\nabla\varphi\\ &\quad + (2-\alpha)(\alpha-1)G''(\nabla G\cdot\nabla\varphi)^2 - c'(t).\end{aligned}$$

We shall assume $R_{ij} \geq 0$, $G' > 0$ and ψ is a smooth function. Then

$$\left(\frac{2G'}{n} - \alpha G''\right)(\Delta G)^2 \leq (2-\alpha)c(t)G''\nabla G \cdot \nabla\varphi + (2-\alpha)(\alpha-1)G''(\nabla G\cdot\nabla\varphi)^2 - c'(t). \tag{6.9}$$

At $(x_0, t_0), \Delta G = \psi + (\alpha-1)\nabla G\cdot\nabla\varphi = c(t_0) + (\alpha-1)\nabla G\cdot\nabla\varphi$. Hence

$$[(2-\alpha)(\alpha-1)G'' - (\frac{2G'}{n} - \alpha G'')(\alpha-1)^2](\nabla G\cdot\nabla\varphi)^2 + [(2-\alpha)G'' - (\alpha-1)(\frac{4}{n}G' - 2\alpha G'')]c\nabla G\cdot\nabla\varphi - c'(t) - (\frac{2G'}{n} - \alpha G'')c^2 \geq 0. \tag{6.10}$$

Note that $\nabla G\cdot\nabla\varphi \geq 0$. Hence we have the following theorem.

Theorem 6.1. *Let u be a solution of the equation $\frac{\partial u}{\partial t} = \Delta(F(u))$ with $F' > 0$. Let G be defined by $G' = t^{-1}F'(t)$. Then for $G = G(u)$,*

$$|\nabla G|^2 - \alpha G_t - c(t) \leq 0 \tag{6.11}$$

if the following quadratic inequality holds for all $x \geq 0$

$$[(2-\alpha)(\alpha-1)\widetilde{G}'' - (\frac{2}{n}\widetilde{G}' - \alpha\widetilde{G}'')(\alpha-1)^2]x^2 \\ [(2-\alpha)\widetilde{G}'' - (\alpha-1)(\frac{4}{n}\widetilde{G}' - 2\alpha\widetilde{G}'')]c(t)x - c'(t) - (\frac{2\widetilde{G}'}{n} - \alpha\widetilde{G}'')c^2 \leq 0 \tag{6.12}$$

and $|\nabla G|^2 - \alpha G_t - c(t)$ is smooth and ≤ 0 at $t = 0$. Here $\widetilde{G}(t)$ is either $G(e^t)$ or $\widetilde{G}(t) = G(-e^t)$.

Theorem 6.2. *Let u be a solution of the equation $\frac{\partial u}{\partial t} = \Delta(F(u))$ with $F' > 0$. Let $G(t)$ be defined by $G' = t^{-1}F'(t)$. Suppose $c(t)$ is a function such that $c'(t) \geq 0$, $uF''(u)c \leq 0$ and $(\frac{2}{n}F' + uF''(u))c^2 \geq 0$ for all u. Then*

$$|\nabla G|^2 - G_t - c(t) \leq 0 \tag{6.13}$$

if $|\nabla G|^2 - G_t - c(t)$ is a smooth function which is nonpositive at $t = 0$. In particular, if $c = 0$, the only condition on F is $F'(u) > 0$. On the other hand, if $c'(t) > 0$, we can replace the condition $F'(u) > 0$ by $F'(u) \geq 0$.

Theorem 6.3. *Let u be a solution of $\frac{\partial u}{\partial t} = \Delta(F(u))$. Suppose $F'(u) > 0$ and $\frac{2}{n}F'(u) + uF''(u) \geq 0$. Then for any nonnegative function $c(t)$ such that $c'(t) \geq 0$,*

$$|\nabla G|^2 - 2G_t - c(t) \leq 0 \tag{6.14}$$

where G is defined by $G'(t) = t^{-1}F'(t)$. We assume that $|\nabla G|^2 - 2G_t - c(t)$ is a smooth function which is nonpositive at $t = 0$. If either $c'(t) > 0$ or $\frac{2}{n}F(u) + uF''(u) > 0$, we can replace the condition $F'(u) > 0$ by $F'(u) \geq 0$.

The condition $F'(u) > 0$ is used to guarantee the strictly parabolicity of the equation (6.1). Suppose the solution u of (6.1) is *stable* in the following sense: Perturb F to $F_\epsilon(t) = F(t) + \epsilon t$. If there is a sequence of solutions u_ϵ of the equations $\frac{\partial u}{\partial t} = \Delta(F_\epsilon(u))$ so that u_ϵ approaches u in C^1-norm as $\epsilon \to 0$, we say that u is a stable solution of (6.1). In this case, Theorem 2 and Theorem 3 remain true for these stable solutions by assuming only $F' \geq 0$.

The estimates (6.13) and (6.14) should give certain descriptions of the behaviour of those points where $F'(u) = 0$. We shall come back to this later.

For the convextiy question, we can only understand the case when dim $M = 1$. As

$$\begin{aligned} \frac{d}{dt}G_{ii} &= G'''\varphi_i^2\Delta G + G''\varphi_{ii}\Delta G + 2G''\varphi_i(\Delta G_)i \\ &\quad + G'\Delta G_{ii} + 2\sum G_{ij}^2 + 2\sum G_j G_{jii} \end{aligned} \tag{6.15}$$

we can derive that at the maximum point of the function $G_{ii} - c(t)$,

$$\begin{aligned} 0 &\leq G'''\varphi_i^2\Delta G + G''\varphi_{ii}\Delta G + 2G_{ii}^2 - c'(t) \\ &= (G''' - \frac{(G'')^2}{G'})\varphi_i^2 G_{ii} + \frac{G''}{G'}(G_{ii})^2 + 2G_{ii}^2 - c'(t). \end{aligned} \tag{6.16}$$

If we assume $G_{ii} - c(t) = 0$ at the maximum point,

$$0 \leq (G''' - \frac{(G'')^2}{G'})c(t)\varphi_i^2 + (2 + \frac{G''}{G'})G_{ii}^2 - c'(t). \tag{6.17}$$

Hence if

$$(G''' - \frac{(G'')^2}{G'})c(t) \leq 0 \tag{6.18}$$

and

$$(2+\frac{G''}{G'})c^2 - c'(t) < 0 \tag{6.19}$$

we arrive at a contradiction.

Theorem 6.4. *Let u be a solution of the $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(F(u))$ such that $F' \geq 0$. Then for any $c(t)$ such that $[u^2F'''F' + uF''F' - u^2(F'')^2]c(t) \leq 0$ and $(2+\frac{u^2F''}{F'})c^2 < c'(t)$, we have the inequality $G(u)_{xx} \leq c(t)$ for all t if it is true at $t = 0$.*

Corollary. *If $F(u) = u^m$, we can choose $c(t) = -\frac{1}{(m+1)(t+t_0)}$.*

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Received March 12, 1994