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LEGENDRIAN COBORDISM AND CHERN-SIMONS THEORY ON 3-MANIFOLDS WITH BOUNDARY

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1. Introduction and Main Theorem

In Floer's instanton homology theory for homology 3-spheres, the Chern-Simons function (or a suitable perturbation of it) is used as a sort of Morse function on the space of $SU(2)$ connections modulo gauge equivalence. The chain complex for the homology theory is generated by the critical points, namely the equivalence classes of flat connections, of this function (excepting the trivial connection).

In this paper we consider $\mathcal{M}$, the critical set of the Chern-Simons function on the space of $SU(2)$ connections modulo gauge equivalence for arbitrary oriented 3-manifolds with boundary. Generically, $\mathcal{M}$ is a smooth manifold of dimension $3g-3$ except for several types of singularities. These singularities arise from connections which either are reducible on Y or are reducible when restricted to ∂Y. We describe the structure of $\mathcal{M}$ near these singularities.

There is a natural restriction map from the flat moduli space on the 3-manifold to the $(6g-6)$-dimensional flat moduli space for its boundary $\mathcal{M}_\Sigma$. Goldman described a symplectic structure on the latter in [G], and with respect to this structure the restriction map is Lagrangian. In fact, there is a Legendrian lift of the restriction to a $U(1)$ bundle with a contact structure over $\mathcal{M}_\Sigma$. The main theorem of the paper, roughly stated, is the following.

Theorem 1. *Under perturbation of the Chern-Simons function, the flat moduli space for a 3-manifold changes by oriented Legendrian cobordism.*

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This paper involves both gauge theory and symplectic geometry. We include in the first few sections some basic material from both areas with the hope that the reader who is well-versed in one area but not the other will have an easier time.

This paper is organized as follows. In Section 2, we make some basic definitions from symplectic geometry which will be needed in the remainder of the paper. Section 3 contains basic gauge theory results for manifolds with boundary. Section 4 reviews the topology of the flat moduli space for the boundary 2-manifold and the symplectic structure on it.

Our situation differs from the case of closed 3-manifolds in that here the Chern-Simons function is not quite gauge invariant (even modulo the integers). The problem is a boundary integral which arises from an integration by parts. To remedy this situation, we construct a $U(1)$ bundle of which the Chern-Simons function determines a gauge invariant section, following [RSW]. This is described in Section 5.

In Section 6, we describe an admissible class of perturbations. We prove that for generic perturbations the flat moduli space is a compact set. In Section 7, we describe the various strata of the flat moduli space.

The main theorem of the paper is proved in Section 8. We show the existence of the Legendrian lift of $r : \mathcal{M} \to \mathcal{M}_\Sigma$ and show that any two such Legendrian lifts, corresponding to different perturbations, differ by a Legendrian cobordism. Orientations for the $\mathcal{M}$ and for the cobordisms are discussed in Section 9. Section 10 contains a number of technical results about the perturbations used elsewhere in the paper.

Finally, Section 11 contains some remarks about the topological invariant of a 3-manifold Y given by the equivalence class of $\mathcal{M}$ up to perturbation. In particular, we give examples to show that this equivalence class contains elements outside of the equivalence class determined by Hamiltonian flows and is strictly smaller than the one determined by oriented Legendrian cobordism.

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2. The Basics

In this section we outline the basic framework for gauge theory on 3-manifolds with boundary and the symplectic geometry definitions needed in the paper. We also give a general definition of stratified spaces.

2.1. Gauge Theory. We begin by establishing some notation. Let Y be a compact, smooth, oriented 3-dimensional manifold with boundary and let $\Sigma = \partial Y$ denote its boundary. We assume that Σ is connected, although many of the results hold in the general case. We let $P(Y) = Y \times SU(2)$ and $P(\Sigma) = \Sigma \times SU(2)$ be the respective trivial bundles and, for concreteness, we fix once and for all trivializations and an identification of $P(\Sigma)$ with $P(Y)|_\Sigma$.

Let the respective gauge groups be denoted by $\mathcal{G}$ and $\mathcal{G}_\Sigma$. Using the fixed trivializations these may be identified with $C^\infty(Y, SU(2))$ and $C^\infty(\Sigma, SU(2))$.

The space of connections on the respective bundles will be denoted by $\mathcal{A}$ and $\mathcal{A}_\Sigma$. By fixing the trivial connection coming from the trivialization of $P(Y)$ we obtain identifications $\mathcal{A} = \Omega^1(Y, su(2))$ and $\mathcal{A}_\Sigma = \Omega^1(\Sigma, su(2))$. We will take completions using Sobolev norms. This will be made explicit in Section 3. The gauge group acts on connections by

$$(g, A) \mapsto g^*A = g^{-1}dg + g^{-1}Ag,$$

where $g \in \mathcal{G}$ and $A \in \mathcal{A}$. The stabilizer $\mathrm{Stab}(A)$ of a connection A on either Y or Σ is isomorphic to $\mathbf{Z}_2, U(1)$, or $SU(2)$. If the stabilizer is $\mathbf{Z}_2$ the connection is called irreducible, if $U(1)$ it is called abelian, if $SU(2)$ it is called central. This is more restrictive than the standard terminology. A central connection is usually considered to be abelian. To avoid ambiguity in this paper we reserve the term *abelian* for noncentral abelian connections.

The curvature of a connection A, which we denote by $F(A)$, is defined to be the $su(2)$ valued 2-form $dA + A \wedge A$. A is called flat if $F(A) = 0$.

We let $\mathcal{B}$ and $\mathcal{B}_\Sigma$ denote the quotients $\mathcal{A}/\mathcal{G}$ and $\mathcal{A}_\Sigma/\mathcal{G}_\Sigma$, respectively. We will denote by $\mathcal{A}_Y^*$ the space of irreducible connections on Y and we will similarly adorn other spaces of connections or equivalence classes of connections whenever we wish to refer to only the irreducibles.

2.2. Sobolev Norms. Let M^n be a smooth manifold. We will be concerned mainly with the quotient space of connections modulo gauge equivalence. In order to define a Hilbert manifold structure on this infinite dimensional space we must take Sobolev completions of the gauge group and the space of connections.

Let E be a vector bundle over M. We define $L^p_s(M,E)$ to be the $\| \; \|_{L^p_s}$ completion of the space of C^∞ sections of E. When $p = 2$, $L^p_s(M,E)$ is a Hilbert space.

We will need several standard results about Sobolev spaces (see [P],[H]).

Rellich Lemma. *If $t < s$ then the inclusion of $L^p_s(M,E)$ into $L^p_t(M,E)$ is compact.*

Sobolev Theorems. (a) *$L^p_s(M,E) \subset L^{p'}_{s'}(M,E)$ if $s - \frac{n}{p} \geq s' - \frac{n}{p'}$ and $s > s'$. If $p' = \infty$ or $p = 1$ then strict inquality is required.*

(b) *If $s - \frac{n}{p} > r$ then $L^p_s(M,E) \subset C^r$.*

Multiplication Theorems. (a) *If $(s_1 - \frac{n}{p_1}) + (s_2 - \frac{n}{p_2}) \geq (s - \frac{n}{p})$ with $s_1 - \frac{n}{p_1} < 0$, $s_2 - \frac{n}{p_2} < 0$ and $s - \frac{n}{p} < 0$, then $L^{p_1}_{s_1}(M,E) \otimes L^{p_2}_{s_2}(M,E) \to L^p_s(M,E)$ is defined and continuous.*

(b) *If $ps > n$ and $p's' > n$ and if $L^p_s(M,E) \subset L^{p'}_{s'}(M,E)$ then $L^{p'}_{s'}(M,E)$ is an $L^p_s(M,E)$ algebra.*

Composition Lemma. *If $ps > n$ then composition on the left by a smooth function maps $L^p_s(M,E)$ to itself. Composition on the right by a smooth function is always a linear map from $L^p_s(M,E)$ to itself.*

We will also need the following theorem.

Theorem (([H). *, Theorem B.1.9] If $N = \partial M$ and $i : N \to M$ is the inclusion, then the restriction map $i^* : L^2_s(M,E) \to L^2_{s-\frac{1}{2}}(N,E)$ is continuous and surjective as long as $s > \frac{1}{2}$. Furthermore, i^* has a continuous right inverse.*

2.3. Definitions from Symplectic Geometry. In this subsection we recall the necessary definitions and properties from symplectic geometry. We begin by defining symplectic manifolds, contact structures and Lagrangian and Legendrian submanifolds.

Let M be a smooth Hilbert manifold. A symplectic structure on M is a closed, nondegenerate 2-form Ω on M. The nondegeneracy requirement on Ω may be described as the requirement that

$$(a, b) \mapsto \Omega(a, b)$$

is a bounded linear functional on $T_pM \otimes T_pM$ and for every tangent vector $a \in T_pM$, the map $b \mapsto \Omega(a, b)$ is a nonzero linear functional on T_pM. The existence of such a structure implies that M is either even dimensional or infinite dimensional.

We will be interested in only a special type of contact manifold, namely a $U(1)$ bundle with connection over a symplectic manifold. The connection is required to have curvature 2-form equal to i times the symplectic form on the base manifold. Let $p : L \to M$ be a smooth $U(1)$ bundle over the symplectic manifold (M, Ω). Let ω be a connection on L with $F(\omega) = i\Omega$. Note that the holonomy of ω around the boundary ∂S of any surface S is then given by

$$\mathrm{hol}_{\partial S}\, \omega = \exp(-i \int_S \Omega).$$

We next define Lagrangian and Legendrian immersions. Let L and (M, Ω) be as above, but assume M has finite dimension $2n$. A Lagrangian immersion into M is an immersion $g : N^n \to M$ with the property that $g^*\Omega = 0$. By the nondegeneracy condition on Ω, n is the largest dimension for which such an immersion is possible. A Legendrian immersion into L is a horizontal lift of a Lagrangian immersion into M, or, equivalently, an immersion $\tilde{g} : N^n \to L$ such that for each $p \in N$ $g_* T_pN$ is contained in the horizontal subspace of $T_{g(p)}M$ determined by ω. In this $U(1)$ bundle context, a Lagrangian submanifold of M which has a Legendrian lift is sometimes called a Bohr-Sommerfeld orbit (see [JW]).

We finish the subsection by defining Legendrian cobordism (see [A], [Au]). Let (M, Ω) be a symplectic manifold and let (L, ω) be a $U(1)$ bundle with contact structure over M. We define a contact structure on $L \times T^*[0,1]$ as follows. We will use π_1 and π_2 to denote the projections onto the first and second factors in both $M \times T^*[0,1]$ and $L \times T^*[0,1]$. Let $(t,\nu) \in [0,1] \times \mathbf{R}$ be coordinates on $T^*[0,1]$. A connection on $L \times T^*[0,1]$ is given by the connection 1-form $\pi_1^*\omega - \pi_2^*(\nu dt)$. This determines a contact structure compatible with the product symplectic structure on $M \times T^*[0,1]$ given by the symplectic form $\pi_1^*\Omega + \pi_2^*(dt \wedge d\nu)$.

For $i = 0,1$, let $\tilde{g}_i : N_i \to L$ be immersed Legendrian submanifolds. A Legendrian cobordism between $\tilde{g}_0$ and $\tilde{g}_1$ is an immersed Legendrian submanifold $\tilde{g} : N \to L \times T^*[0,1]$ which is transverse to $\partial(L \times T^*[0,1])$ and

$$p \circ \tilde{g}|_{\partial N} = \tilde{g}_1(N_1) \times \{1\} \cup \tilde{g}_0(N_0) \times \{0\}$$

where $p : T^*[0,1] \to [0,1]$ is projection. Two oriented Legendrian submanifolds N_0 and N_1 are oriented Legendrian cobordant if there is an oriented Legendrian cobordism N with $\partial N = N_1 - N_0$ as oriented manifolds.

2.4. Stratified Spaces. In this subsection we define stratified spaces. The existing definitions of stratified spaces in the literature are many and varied. We adopt a definition that is only as general as necessary for the purposes of this paper. For any smooth manifold M, denote the cone on M by $c(M)$.

A *stratified space* is a topological space S which may be partitioned into a collection of locally closed C^∞ manifolds

$$S = S_1 \cup \cdots \cup S_n$$

satisfying three requirements. The manifolds S_i are called the *strata*. For each stratum, let ∂S_i denote $\bar{S}_i \setminus S_i$. The requirements are:

(a) For each $1 \le i \le n$, $\partial S_i \subset \bigcup_{j<i} S_j$.
(b) For each $j < i$, $\partial S_i \cap S_j$ is a smooth submanifold of S_j.
(c) For each $j < i$, $\partial S_i \cap S_j$ has a neighborhood in $S_i \cup (\partial S_i \cap S_j)$ homeomorphic to a bundle over $\partial S_i \cap S_j$ with fiber $c(M)$ for some smooth manifold M and the homeomorphism is a diffeomorphism on each stratum.

We call the neighborhood described in (c) the normal bundle of S_i in S_j. We shall say a homeomorphism between two stratified spaces is a *diffeomorphism* *in the stratified sense* if it preserves the stratification and is a diffeomorphism on each stratum.

3. Connections Modulo Gauge Equivalence

We begin by making precise our definitions of the spaces of connections and gauge transformations. In particular, we define the Sobolev completions necessary for our analysis as follows. For simplicity of notation, we will denote $\Gamma(M; \Lambda^p(T^*M) \otimes su(2))$ by $\Omega^p(M, su(2))$.

$$\begin{aligned}
\mathcal{A} &= L^2_2\Omega^1(Y, su(2))\\
\mathcal{A}_\Sigma &= L^2_{\frac{3}{2}}\Omega^1(\Sigma, su(2))\\
\mathcal{G} &= \{g \in L^2_3(\mathrm{End}(P \times_{SU(2)} \mathbf{C}))|\ g^*g = 1 \text{ a. e. } \}\\
\mathcal{G}_\Sigma &= \{g \in L^2_{\frac{5}{2}}(\mathrm{End}(P|_\Sigma) \times_{SU(2)} \mathbf{C})|\ g^*g = 1 \text{ a. e. } \}
\end{aligned}$$

The following are standard facts. Proofs may be found in [FU] or [L].

Proposition 2. *The curvature map $F : \mathcal{A} \to L^2_1\Omega^2(Y, su(2))$ is a smooth function. Likewise, $F : \mathcal{A}_\Sigma \to L^2_{\frac{1}{2}}\Omega^2(\Sigma, su(2))$ is smooth.*

Proposition 3. *$\mathcal{G}$ and $\mathcal{G}_\Sigma$ are Hilbert Lie groups with Lie algebras*

$$L^2_3(\Omega^0(Y, su(2))) \text{ and } L^2_{\frac{5}{2}}(\Omega^0(\Sigma, su(2))), \text{ respectively.}$$

Proposition 4. *$\mathcal{G}$ acts smoothly on $\mathcal{A}$ and $\mathcal{G}_\Sigma$ acts smoothly on $\mathcal{A}_\Sigma$.*

Lemma 5. *The following are equivalent conditions for any $SU(2)$ connection A on P_Y (the corresponding statement for Σ with appropriate changes in Sobolev norms is also true).*

(1) $d_A : L^2_3\Omega^0(Y, su(2)) \to L^2_2\Omega^1(Y, su(2))$ *is not injective.*
(2) $\mathrm{Stab}(A) \neq \{\pm \mathrm{id}\}$.

If these conditions hold, then we call A reducible. Furthermore, if A is reducible, then either $\mathrm{Stab}(A) \cong U(1)$ *or* $\mathrm{Stab} \cong SU(2)$. *In either case, the kernel of the map in* (1) *is equal to the Lie algebra of* $\mathrm{Stab}(A)$.

We fix a Riemannian metric on Y and hence on Σ. This allows us to define an L^2 inner product on $\Omega^p(Y, su(2))$ by the formula

$$\langle a, b\rangle = -\int_Y \operatorname{tr}(a \wedge *b),$$

where $* : \Omega^p \to \Omega^{n-p}$ is the Hodge star operator and $\operatorname{tr} : su(2) \otimes su(2) \to \mathbf{R}$ is the ordinary trace of the product. For this particular group, the bilinear form given by trace of the product coincides with the Killing form. When it is necessary to refer to the Hodge star operator on Σ, we will denote it by $*_\Sigma$.

For each $A \in \mathcal{A}$ we define the slice

$$X_A = \{A + a \mid a \in \ker d_A^* \cap L_2^2\Omega_\nu^1(Y, su(2))\}$$

where $d_A^* = - * d_A *$ is the adjoint of $d_A : L_3^2\Omega^0 \to L_2^2\Omega^1$, and

$$\Omega_\nu^1(Y, su(2)) = \{a \in \Omega^1(Y, su(2)) \mid \ * a|_\Sigma = 0\}.$$

This is motivated by the following proposition.

Proposition 6. *The slice X_A is the L^2 orthogonal complement to the tangent space $T_A(\mathcal{G}(A))$ of the gauge orbit $\mathcal{G}(A)$ through A.*

Proof. We begin by noting that

$$T_A(\mathcal{G}(A)) = d_A(L_3^2\Omega^0(Y, su(2))).$$

The condition that

$$\alpha \perp T_A(\mathcal{G}(A)) \subset T_A\mathcal{A}$$

is equivalent to the condition that for all $\gamma \in L_3^2\Omega^0(Y, su(2))$ we have

$$\begin{aligned} 0 = \langle \alpha, d_A\gamma\rangle &= -\int_Y \operatorname{tr}(*\alpha \wedge d_A\gamma) \\ &= -\int_\Sigma \operatorname{tr}(*\alpha \wedge \gamma) + \int_Y \operatorname{tr}(d_A * \alpha \wedge \gamma). \end{aligned}$$

By choosing γ to be zero near the boundary, we see that $d_A^*\alpha = 0$, which then implies that $*\alpha|_\Sigma = 0$. □

Let M equal either Y or Σ.

Theorem 7. *The quotient $\mathcal{B}_M^*$ is a smooth manifold.*

Proof. The only difference between this and the cases described in [FU] and [L] is the fact that Y has boundary. The only thing to check is that the second partial of the map

$$Q_A : \mathcal{A} \times \mathcal{G} \to L^2_1\Omega^0(Y, su(2)) \oplus L^2_{\frac{3}{2}}\Omega^1(\Sigma, su(2))$$

given by

$$Q_A(A + a, g) = (d^*_A(g^{-1}d_A g + g^{-1}ag), *g^{-1}d_A g|_\Sigma)$$

is surjective. The derivative of Q is the map

$$(\delta A, \delta g) \mapsto$$
$$(d^*_A(-g^{-1}\delta g g^{-1}d_A g + g^{-1}d_A\delta g - g^{-1}\delta g g^{-1}Ag + g^{-1}\delta A g + g^{-1}A\delta g), *g^{-1}d_A\delta g|_\Sigma).$$

At $(0, \mathrm{id})$ this simplifies to

$$(\delta A, \delta g) \mapsto (d^*_A(d_A\delta g + \delta A), *d_A\delta g|_\Sigma).$$

Lemma 8. $d^*_A d_A \oplus *d_A|_\Sigma : L^2_3\Omega^0(Y, su(2)) \to L^2_1\Omega^0(Y, su(2)) \oplus L^2_{\frac{3}{2}}\Omega^1(\Sigma, su(2))$ *is an elliptic boundary value problem.*

It follows from the lemma that (with the prescribed boundary conditions) $d^*_A d_A$ is Fredholm, and hence has closed range. Thus to show that $d^*_A d_A$ is surjective, it is sufficient to show that the image of $d^*_A d_A$ contains all smooth sections. To do this, we begin by showing that $d^*_A d_A$ is injective. Suppose some smooth $\psi \in L^2_3\Omega^0(Y, su(2))$ satisfying $*d_A\psi|_\Sigma = 0$ is in $\ker(d^*_A d_A)$. Then

$$\begin{aligned} 0 = \langle d^*_A d_A\psi, \psi\rangle &= -\int_\Sigma \mathrm{tr}(*d_A\psi \wedge \psi) + \langle d_A\psi, d_A\psi\rangle \\ &= \|d_A\psi\|_{L^2}, \end{aligned}$$

which implies that $d_A\psi = 0$. Since A is irreducible, this implies that $\psi = 0$.

Suppose now that some smooth $\phi \in L^2_3\Omega^0(Y, su(2))$ has the property that $\langle \phi, d^*_A d_A\psi\rangle = 0$ for every $\psi \in L^2_3\Omega^0(Y, su(2))$. Then, integrating by parts,

$$\begin{aligned} 0 = \langle \phi, d^*_A d_A\psi\rangle &= -\int_\Sigma \mathrm{tr}(\phi \wedge *d_A\psi) + \langle d_A\phi, d_A\psi\rangle \\ &= 0 - \int_\Sigma \mathrm{tr}(*d_A\phi \wedge \psi) + \langle d^*_A d_A\phi, \psi\rangle. \end{aligned}$$

By choosing ψ with support in the interior of Y, this shows $d_A^* d_A \phi = 0$. Thus ϕ also satisfies the boundary condition $*d_A\phi|_\Sigma = 0$. Therefore, $\phi = 0$, and $d_A^* d_A$ is surjective. □

Let M be either Y or Σ. For A reducible, we can describe a neighborhood of $[A] \in \mathcal{B}_M$ in a similar fashion, but we must take into account the gauge symmetry of A.

Theorem 9. *For each reducible A in $\mathcal{A}_M$, there is a neighborhood of $[A]$ in $\mathcal{B}_M$ homeomorphic to $O_A/\operatorname{Stab}(A)$ where $O_A \subset X_A$ is a neighborhood of A in the slice.*

4. The Topology of the Flat Moduli Space for a Riemann Surface

In this section we review the structure of the moduli space of flat connections modulo gauge equivalence for a surface Σ of genus g.

Let

$$\mathcal{F}_\Sigma = \{A \in \mathcal{A}_\Sigma \mid F(A) = 0\},$$

and define

$$\mathcal{M}_\Sigma = \mathcal{F}_\Sigma / \mathcal{G}_\Sigma$$

to be the flat moduli space. We will denote the irreducible part of $\mathcal{M}_\Sigma$ by $\mathcal{M}_\Sigma^{\mathbf{Z}_2}$, the abelian part by $\mathcal{M}_\Sigma^{U(1)}$, and the central part by $\mathcal{M}_\Sigma^{SU(2)}$.

Fix a basepoint $x_0 \in \Sigma$. If $\gamma : S^1 \to M$ is a loop based at x_0 and P is an $SU(2)$ bundle over M, then let $\operatorname{hol}_\gamma(A)$ denote the holonomy of A around γ. The condition that A is flat is equivalent to the condition that $\operatorname{hol}_\gamma(A)$ depends only on the homotopy class $[\gamma] \in \pi_1(M)$.

The association of each flat connection to its holonomy representation gives an identification of $\mathcal{M}_\Sigma$ with

$$\operatorname{Hom}(\pi_1(\Sigma), SU(2))/SU(2),$$

where $SU(2)$ acts on representations by conjugation.

The space $\operatorname{Hom}(\pi_1(\Sigma), SU(2))/SU(2)$ is also called the character variety of Σ. It was studied extensively by Atiyah and Bott and by Goldman (see [AB], [G2]; see also [Wa] for more details). We review the relevant results now.

For any tensor product $V \otimes W$, we call an element *decomposable* if it is of the form $v \otimes w$ for some $v \in V$ and $w \in W$. We let $(V \otimes W)^\flat$ denote the set of decomposable elements and $(V \otimes W)^\sharp = (V \otimes W) \setminus (V \otimes W)^\flat$ denote the its complement.

Theorem 10. *$\mathcal{M}_\Sigma$ is a stratified space. The strata are $\mathcal{M}_\Sigma^{\mathbf{Z}_2}, \mathcal{M}_\Sigma^{U(1)}$, and $\mathcal{M}_\Sigma^{SU(2)}$, which have dimensions $6g-6$, $2g$, and 0, respectively. $\mathcal{M}_\Sigma^{U(1)}$ has normal bundle fiber $c(S^{2g-3} \times S^{2g-3})/U(1)$ in $\mathcal{M}_\Sigma^{\mathbf{Z}_2}$. ($U(1)$ acts by the diagonal action on $S^{2g-3} \times S^{2g-3} \subset \mathbf{C}^{2g-2}$.) $\mathcal{M}_\Sigma^{SU(2)}$ has normal bundle fiber in $\mathcal{M}_\Sigma^{\mathbf{Z}_2}$ equal to $((\mathbf{R}^{2g} \otimes su(2))^\sharp)/SU(2)$. The normal bundle of $\mathcal{M}_\Sigma^{SU(2)}$ in $\mathcal{M}_\Sigma^{U(1)}$ has fiber $((\mathbf{R}^{2g} \otimes su(2))^\flat)/SU(2) = \mathbf{R}^{2g}/\mathbf{Z}_2$.*

For any flat connection A, the Zariski tangent space to

$$\mathrm{Hom}(\pi_1(\Sigma), SU(2))/SU(2)$$

is $H^1(\Sigma; ad\rho)$, where ρ is the holonomy representation of A. Since we work with flat connections instead of representations, we identify this with the de Rham cohomology for the flat connection A (we will always identify this with the space of harmonic $su(2)$ valued forms). We denote this cohomology by $\mathcal{H}_A^1(\Sigma)$. In the gauge theory context, instead of being the Zariski tangent space, this is the space of tangent vectors to paths along which the curvature vanishes to first order.

The main step in proving this theorem, first done by Goldman, is to identify which elements of $\mathcal{H}_A^1(\Sigma)$ are in fact tangent vectors to paths of flat connections. Let $[\cdot \wedge \cdot] : \mathcal{H}_A^1(\Sigma) \times \mathcal{H}_A^1(\Sigma) \to \mathcal{H}_A^2(\Sigma)$ denote the combined Lie bracket on $su(2)$ elements and the wedge product on forms. Note that $[\alpha \wedge \alpha] = 2\alpha \wedge \alpha$, since both the wedge product and the Lie bracket anticommute.

Theorem 11. *(Theorem 3 of [G2]; see also [MMR], Prop, 13.2.3) An element $\alpha \in \mathcal{H}_A^1(\Sigma)$ is tangent to a curve in $\mathcal{M}_\Sigma$ if and only if $[\alpha \wedge \alpha] = 0$.*

The proof of Theorem 11 falls into three parts, corresponding to the different strata. The irreducible case is the easiest, and follows immediately from the Kuranishi picture outlined in Section 6. The proof in the abelian case will be sketched in Section 7.2. With that as a template, the central case can safely be left as an exercise for the reader.

All that remains to prove Theorem 10 is to identify the zero set of the map $\alpha \mapsto [\alpha \wedge \alpha]$ in each case. For the irreducible case, $\mathcal{H}^2_A(\Sigma) \cong \mathcal{H}^0_A(\Sigma)$, which is zero since A is irreducible. The abelian case is worked out in [Wa]. We leave the central case as an exercise for the reader. □

There is a symplectic structure (see [G1], [AB]) on $\mathcal{M}_\Sigma$, i.e. a closed non-degenerate exterior 2-form on each Zariski tangent space $\mathcal{H}^1_A(\Sigma)$, given by

$$\Omega(\alpha, \beta) = \frac{1}{2\pi} \int_\Sigma \operatorname{tr}(\alpha \wedge \beta).$$

Atiyah and Bott [AB] showed that this symplectic structure can be constructed by symplectic reduction from the symplectic structure on $\mathcal{A}_\Sigma$ given by exactly the same pairing on $su(2)$ valued 1-forms on Σ. (The reduced symplectic structure is then the same form restricted to the space of harmonic 1-forms.)

The symplectic structure is compatible with the stratification in the following sense. If $[A_i] \in \mathcal{M}_\Sigma$ is a sequence converging to $[A]$, then there is a natural inclusion $\mathcal{H}^1_{A_i}(\Sigma) \to \mathcal{H}^1_A(\Sigma)$ for i sufficiently large. The symplectic form on $\mathcal{H}^1_{A_i}(\Sigma)$ is the pullback of the symplectic form on $\mathcal{H}^1_A(\Sigma)$ under this inclusion.

Notice that the symplectic form is related to the L^2 inner product on forms by

$$\Omega(\alpha, \beta) = \frac{1}{2\pi} \langle \alpha, *_\Sigma \beta \rangle.$$

5. Chern-Simons Theory

In this section we review Chern-Simons gauge theory for 3-manifolds with boundary. In the first subsection, we define a $U(1)$ bundle with connection over $\mathcal{B}$ of which the Chern-Simons function induces a section. In the second subsection, we show the gradient of this section is $\frac{1}{2\pi}$ times the Hodge dual of the curvature.

5.1. Construction of the $U(1)$ bundle.

We begin by recalling the definition of the Chern-Simons function. Define the map $CS : \mathcal{A} \to \mathbf{R}$ by

$$CS(A) = \frac{1}{4\pi} \int_Y \operatorname{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

On a closed manifold Y, changing the connection by a gauge transformation changes the value of the Chern-Simons function by 2π times an integer. The proof of this fact involves an integration by parts. If Y has boundary there is a correction term which is an integral over the boundary. Following [RSW], we use this correction term to define a $U(1)$ bundle over $\mathcal{B}$ of which $\exp(iCS)$ determines a section. The $U(1)$ bundle is a special case of a general construction outlined in [PS].

We define a map $\Theta : \mathcal{A}_\Sigma \times \mathcal{G}_\Sigma \to U(1)$ by the formula

$$\Theta(A, g) = \exp(i(CS(\tilde{g}^*\tilde{A}) - CS(\tilde{A}))), \tag{1}$$

where $\tilde{A}$ and $\tilde{g}$ are any extensions of A and g to Y. Θ is independent of these extensions and is given explicitly by

$$\Theta(A, g) = \exp(i\frac{1}{4\pi}\int_\Sigma \mathrm{tr}(g^{-1}Adg)). \tag{2}$$

It is clear from equation 1 that Θ is a cocycle, i.e.

$$\Theta(A, g)\Theta(g^*A, h) = \Theta(A, gh).$$

Note that if g is in the stabilizer of any connection, then $\Theta(A, g) = 1$ for all $A \in \mathcal{A}_\Sigma$. Thus we get a topological $U(1)$ bundle $\mathcal{L}_\Sigma$ over $\mathcal{B}_\Sigma$ by dividing out by Θ. In other words, we define

$$\mathcal{L}_\Sigma = \mathcal{A}_\Sigma \times_\Theta U(1) = \mathcal{A}_\Sigma \times U(1)/\sim,$$

where the equivalence relation is

$$(A, u) \sim (g^*A, \Theta(A, g)u)$$

for all $g \in \mathcal{G}_\Sigma$.

Lemma 12. Θ *is smooth.*

Proof. This follows immediately from the Multiplication Theorems and the formula (2). □

In [RSW] a connection is defined on $\mathcal{A}_\Sigma \times U(1)$ as follows. We define a global connection 1-form using the trivialization $\mathcal{A}_\Sigma \times \{1\}$ by the formula

$$\omega(\alpha) = \frac{i}{4\pi}\int_\Sigma \mathrm{tr}(A \wedge \alpha)$$

for $\alpha \in T_A\mathcal{A}_\Sigma$ and extend $U(1)$ equivariantly.

The curvature of the connection ω, evaluated on two tangent vectors $\alpha, \beta \in T_A\mathcal{F}_\Sigma$, is

$$\begin{aligned} d\omega(\alpha,\beta) &= \beta(\omega(\alpha)) - \alpha(\omega(\beta)) \\ &= \frac{i}{2\pi}\int_\Sigma \mathrm{tr}(\alpha\wedge\beta). \end{aligned}$$

This 2-form, without the factor of i, defines a symplectic structure on $\mathcal{A}_\Sigma$. Thus the connection ω determines a contact structure on $\mathcal{A}_\Sigma \times U(1)$. The symplectic structure on $\mathcal{H}^1_A(\Sigma; su(2))$ induced from this one coincides with that described in Section 4.

Lemma 13. *$\mathcal{G}_\Sigma$ acts, using Θ, by contactomorphism, i.e. $\mathcal{G}_\Sigma$ preserves the connection ω.*

Proof. First, we compute the derivative

$$\Theta^{-1}\delta\Theta = \frac{i}{4\pi}\int_\Sigma \mathrm{tr}(d\delta g g^{-1}\wedge A - dgg^{-1}\delta gg^{-1}\wedge A + dgg^{-1}\wedge\delta A).$$

Consider a tangent vector $(\delta A, u\delta u) \in T_{(A,u)}\mathcal{A}_\Sigma \times U(1)$. Its image under the action of $\Theta(A; g)$ is

$$(g^{-1}\delta A g, \Theta(A,g)u(\frac{i}{4\pi}\int_\Sigma \mathrm{tr}(g^{-1}\delta A\wedge dg) + \delta u)),$$

a tangent vector at $(g^*A, \Theta(A,g)u)$. It is easy to check that ω evaluated on the second tangent vector is $\frac{i}{4\pi}\int_\Sigma \mathrm{tr}(A\wedge\delta A) + \delta u$, as required. □

Corollary 14. *The restrictions $\mathcal{L}_\Sigma|_{\mathcal{M}_\Sigma^{\mathbf{Z}_2}}$ and $\mathcal{L}_\Sigma|_{\mathcal{M}_\Sigma^{U(1)}}$ are smooth bundles which inherit connections induced by ω.*

Proof. That the bundles are smooth follows from Lemma 12. After the last lemma, all that remains to check is that the connection ω evaluates to zero in directions tangent to the Θ action orbits in $\mathcal{A}_\Sigma \times U(1)$. Given $\xi \in T_{\mathrm{id}}\mathcal{G}_\Sigma$, the corresponding orbit tangent vector at (A,u) is $(d_A\xi, u(\frac{i}{4\pi}\int_\Sigma \mathrm{tr}(A\wedge d\xi)))$. Plugging this into the connection ω gives

$$\begin{aligned} \frac{i}{4\pi}\int_\Sigma \mathrm{tr}(A\wedge d_A\xi) + \frac{i}{4\pi}\int_\Sigma \mathrm{tr}(A\wedge d\xi) &= \frac{i}{4\pi}\int_\Sigma \mathrm{tr}(2dA\wedge\xi + A\wedge[A,\xi]) \\ &= \frac{i}{2\pi}\langle F(A), *_\Sigma\xi\rangle. \quad \square \end{aligned}$$

We next define the pullback bundle

$$\mathcal{L}_Y = r^* \ \mathcal{L}_\Sigma = \mathcal{A} \times_{r^*\Theta} U(1).$$

The connection ω pulls back to give a connection ω_Y on $\mathcal{A} \times U(1)$.

5.2. The Chern-Simons Function as a Section. We next study the section of $\mathcal{L}_Y$ induced by the Chern-Simons function. We will see that its critical set (where the derivative is taken with respect to ω_Y) is the moduli space of flat connections on $Y \times SU(2)$.

From the definition of $\mathcal{L}_Y$, it is clear that the map

$$\tilde{s} : \mathcal{A} \to \mathcal{A} \times U(1)$$

given by

$$\tilde{s}(A) = (A, \exp(iCS(A)))$$

is Θ equivariant. Thus it descends to a continuous section s of the quotient bundle $\mathcal{L}_Y$.

We compute the derivative of $\tilde{s}$ using the connection ω_Y. Let $\alpha \in T_A\mathcal{A}$. Then

(3)

$$\begin{aligned}
D_{\omega_Y}\tilde{s}(A)(\alpha) &= \tilde{s}^{-1}d\tilde{s}(\alpha) + \omega(\alpha) \\
&= \frac{i}{4\pi}\int_Y \operatorname{tr}(\alpha \wedge dA + d\alpha \wedge A \\
&\quad + \frac{2}{3}(\alpha \wedge A \wedge A - A \wedge \alpha \wedge A + A \wedge A \wedge \alpha)) + \omega(\alpha) \\
&= \frac{i}{2\pi}\int_Y \operatorname{tr}(\alpha \wedge F(A)) + \frac{i}{4\pi}\int_Y \operatorname{tr}(d(\alpha \wedge A)) + \omega(\alpha) \\
&= \frac{i}{2\pi}\int_Y \operatorname{tr}(\alpha \wedge F(A)) + \frac{i}{4\pi}\int_\Sigma \operatorname{tr}(\alpha \wedge A) + \frac{i}{4\pi}\int_\Sigma \operatorname{tr}(A \wedge \alpha) \\
&= \frac{i}{2\pi}\int_Y \operatorname{tr}(\alpha \wedge F(A)) \\
&= i\langle \alpha, \frac{1}{2\pi} * F(A)\rangle_{L^2}.
\end{aligned}$$

Thus the gradient vector field of the section is

$$\nabla_{\omega_Y}\tilde{s}(A) = \frac{1}{2\pi} * F(A)$$

Here the subscript ω_Y is a reminder that the derivative of s was taken using the connection ω_Y. In the next section, we will describe a method of perturbing CS. We will replace $\tilde{s}$ by $\tilde{s}_h(A) = (A, \exp(i(CS(A) + h(A)))$ where h is a suitably chosen function on the quotient space $\mathcal{B}_Y$, or, equivalently, a $\mathcal{G}$ equivariant function on $\mathcal{A}$. Repeating the calculation (3) gives

$$\begin{aligned} D_{\omega_Y}\tilde{s_h}(A)(\alpha) &= i(d(CS(A))(\alpha) + dh(A)) + \omega(\alpha) \\ &= i\langle \alpha, \frac{1}{2\pi} * F(A) + \nabla h(A)\rangle_{L^2} \end{aligned}$$

where ∇h is the ordinary gradient of h with respect to the L^2 metric.

We will denote the gradient vector field of $\tilde{s_h}$ by ζ_h. We define

$$\mathcal{M}_h = \zeta_h^{-1}(0)/\mathcal{G}.$$

to be the zero set of this vector field, modulo gauge equivalence. When we have a fixed perturbation h in mind, we will call connections in $\zeta_h^{-1}(0)$ perturbed flat, and we will call $\mathcal{M}_h$ the perturbed flat moduli space.

6. Perturbations of the Chern-Simons Function

The aim of the next two sections is to describe $\mathcal{M}_h$ for generic perturbations h. We begin this section with a statement of the main result. In the next two subsections, we define a class $\mathcal{H}$ of admissible perturbation functions h and show that $\mathcal{M}_h$ is compact for any admissible function h. Section 6.3 contains a brief review of Hodge theory for manifolds with boundary. The notation and results explained there are the main tools used in the proof of the later theorems. Section 6.4 outlines the Kuranishi technique for reducing the infinite dimensional problem of describing $\mathcal{M}_h$ to a finite dimensional model.

There is one technical point that should be made regarding what is meant by generic $h \in \mathcal{H}$. The space $\mathcal{H}$ is not connected; its components are in bijective correspondence with the isotopy classes of links in Y. When we use the term *generic* in the following statements, we will always mean generic in the component of $\mathcal{H}$ corresponding to a sufficiently large link. Proving that an arbitrary link may be made sufficiently large by adding a finite number of components is the main purpose of Section 10.

Let g denote the genus of the boundary surface Σ. We divide our structure theorem into the cases $g \geq 3, g = 2, g = 1$, and $g = 0$. Note that if $g = 0$ the perturbed flat moduli space for Y is naturally identified with that for the closed 3-manifold $Y \cup_\Sigma D^3$, since any flat connection on S^2 extends in a unique (up to gauge equivalence) way over the 3-ball.

For each ordered pair (G, H) of elements of $\{\mathbf{Z}_2, U(1), SU(2)\}$ with $G \subset H$ we define

$$\mathcal{A}^{(G,H)} = \{A \in \mathcal{A} |\ \mathrm{Stab}(A) \cong G \text{ and } \mathrm{Stab}(A|_\Sigma) \cong H\}.$$

Similarly, we define

$$\mathcal{B}^{(G,H)} = \mathcal{A}^{(G,H)}/\mathcal{G} \text{ and } \mathcal{M}_h^{(G,H)} = \mathcal{M}_h \cap \mathcal{B}^{(G,H)}.$$

Theorem 15. *For generic h, $\mathcal{M}_h$ is a compact stratified space with the following smooth strata:*

(a) *If $g \geq 3$, then*

$$\mathcal{M}_h = \mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)} \coprod \mathcal{M}_h^{(U(1),U(1))} \coprod \mathcal{M}_h^{(SU(2),SU(2))}.$$

(Note that this means the $(\mathbf{Z}_2, U(1))$-, $(\mathbf{Z}_2, SU(2))$-, and $(U(1), SU(2))$-strata are empty.) The $(\mathbf{Z}_2, \mathbf{Z}_2)$-stratum is $(3g-3)$-dimensional. The $(U(1), U(1))$-stratum is g-dimensional and has a normal bundle in the $(\mathbf{Z}_2, \mathbf{Z}_2)$-stratum with fiber $c(\mathbf{C}P^{(g-1)})$, the cone on $(g-1)$-dimensional complex projective space. The $(SU(2), SU(2))$-stratum is 0-dimensional; it has a normal bundle in the $(\mathbf{Z}_2, \mathbf{Z}_2)$-stratum with fiber

$$(\mathbf{R}^g \otimes su(2))^\sharp / SU(2)$$

and normal bundle in the $(U(1), U(1))$-stratum with fiber

$$(\mathbf{R}^g \otimes su(2))^\flat / SU(2) = \mathbf{R}^g / \mathbf{Z}_2.$$

(b) *If $g = 2$, then*

$$\mathcal{M}_h = \mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)} \coprod \mathcal{M}_h^{(\mathbf{Z}_2,U(1))} \coprod \mathcal{M}_h^{(U(1),U(1))} \coprod \mathcal{M}_h^{(SU(2),SU(2))}.$$

The $(\mathbf{Z}_2,\mathbf{Z}_2)$*-,* $(U(1),U(1))$*-, and* $(SU(2),SU(2))$*-strata have the structure described in (a). The* $(\mathbf{Z}_2,U(1))$*-stratum is 0-dimensional. It has normal bundle in the* $(\mathbf{Z}_2,\mathbf{Z}_2)$*-stratum with fiber* $c(S^1\times S^1)$*.*

(c) *If* $g=1$*, then*

$$\mathcal{M}_h=\mathcal{M}_h^{(\mathbf{Z}_2,U(1))}\coprod\mathcal{M}_h^{(U(1),U(1))}\coprod\mathcal{M}_h^{(SU(2),SU(2))}.$$

The $(\mathbf{Z}_2,U(1))$*- and* $(U(1),U(1))$*-strata are both 1-dimensional, the* $(SU(2),SU(2))$*-stratum 0-dimensional. The* $(SU(2),SU(2))$*-stratum has normal bundle in the* $(U(1),U(1))$*-stratum with fiber* $c(pt)=\mathbf{R}/\mathbf{Z}_2$*. There is a 0-dimensional submanifold of the* $(U(1),U(1))$*-stratum which makes up the boundary of the* $(\mathbf{Z}_2,U(1))$*-stratum, and this submanifold has normal bundle in the* $(\mathbf{Z}_2,U(1))$*-stratum with fiber* $c(pt)=\mathbf{R}/\mathbf{Z}_2$*.*

(d) *If* $g=0$*, then*

$$\mathcal{M}_h=\mathcal{M}_h^{(\mathbf{Z}_2,SU(2))}\coprod\mathcal{M}_h^{(U(1),SU(2))}\coprod\mathcal{M}_h^{(SU(2),SU(2))}.$$

All these strata are 0-dimensional.

6.1. Admissible Perturbation Functions. We now define a class $\mathcal{H}$ of admissible perturbation functions h. We shall use the same class of functions as was used by Floer and Taubes in their work on closed 3-manifolds having the homology of S^3. We will prove various technical results concerning these functions in Section 10. See also [T], [F] and [DFK].

Let $\phi=\{\gamma_i\}_{i=1}^k$ be a finite collection of disjoint embeddings of solid tori, $\gamma_i : S^1\times D^2\to Y$. Choose a corresponding collection of functions $\bar{h}_i\in C^2([-2,2],\mathbf{R})$, and let $h_i=\bar{h}_i\circ\mathrm{tr} : SU(2)\to\mathbf{R}$. Let $\bar{\mathcal{H}}$ denote the space of such functions $\bar{h}$. We give $\bar{\mathcal{H}}$ the compact-open C^2 topology.

Let $\eta(x) : D^2\to\mathbf{R}$ be a radially symmetric bump function on D^2 with support away from the boundary. Given the collections $\{\gamma_i\}_{i=1}^k$ and $\{h_i\}_{i=1}^k$ we define a function $h:\mathcal{A}\to\mathbf{R}$ (we suppress from notation the dependence on ϕ when ϕ is thought of as fixed) by

$$h(A)=\sum_{i=1}^k\int_{D^2}h_i(\mathrm{hol}_{\gamma_i}(x,A))\eta(x)dx,$$

where x is a pair of coordinates on D^2 and d^2x is the standard measure on D^2. Of course, a basepoint must be chosen before the holonomy around a loop $\gamma_i(S^1 \times \{x\})$ is defined. The Ad-invariance of the $\{h_i\}$ insures that the resulting definition of h does not depend on such choices. We define functions constructed in this manner to be *admissible*. We shall denote the space of admissible functions by $\mathcal{H}$. The gauge invariance of admissible functions is immediate.

The components of $\mathcal{H}$ are in bijective correspondence with isotopy classes of links in Y. Let

$$\Gamma_n = \coprod_{i=1}^{n} S^1 \times D^2.$$

Given an embedding $\phi : \Gamma_n \to Y$, we use $\mathcal{H}_\phi$ to denote the corresponding component. Let

$$\mathrm{Emb}_{[\phi]}(\Gamma_n, Y)$$

denote the space of C^2 embeddings isotopic to ϕ, also with the compact-open C^2 topology. The component $\mathcal{H}_\phi$ is parametrized by $\bar{\mathcal{H}}^n \times \mathrm{Emb}_{[\phi]}(\Gamma_n, Y)$.

For the purpose of this section and the next, namely finding sufficient perturbations to make $\mathcal{M}_h$ nondegenerate, we need only vary the $\bar{\mathcal{H}}^n$ component of our perturbations. By abuse of notation, we will leave off the $\mathrm{Emb}_{[\phi]}(\Gamma_n, Y)$ factor since we mean for ϕ to be fixed.

If $\phi : \Gamma_n \to Y$ and $\psi : \Gamma_m \to Y$ are disjoint embeddings, we denote by $\mathcal{H}_{(\phi\cup\psi)}$ the component of $\mathcal{H}$ corresponding to the union.

The following lemma and corollary summarize the other important properties of these functions. The lemma follows from the results in Section 10. Recall that the L^2 gradient of a function on $\mathcal{A}$ is an $su(2)$ valued 1-form ∇h with the property that for any tangent vector $a \in T_A\mathcal{A}$,

$$Dh(A)(a) = \langle \nabla h(A), a \rangle_{L^2}.$$

Lemma 16. *Let h be an admissible function.*

1. *The 1-form ∇h is zero outside of $\bigcup \gamma_i(S^1 \times D^2)$ and is given at a point $\gamma_i(s, x)$ by*

$$\nabla h(A) = \bar{h}_i'(\mathrm{tr}(\mathrm{hol}_{\gamma_i}(x, A)))\Pi_{\mathrm{Im}}(\mathrm{hol}_{\gamma_i}(s; x, A))\eta(x)ds.$$

2. *The Hessian of h is given by the formula*

$$\operatorname{Hess} h(A)(a,b) = \int_{[0,1]\times[0,1]\times D^2} \operatorname{tr}(P_{\gamma_i}[t,s,x;A]a(s,x)P_{\gamma_i}[s,t,x;A]b(t,x))dt\wedge ds\wedge\eta(x)dx$$

where $P_\gamma[t,s,x;A]$ *is the* $SU(2)$ *element determined by parallel transport along* $\gamma_i(S^1\times\{x\})$ *from* t *to* s *with respect to the connection* A.

3. h *is a smooth function on* $\mathcal{A}$ *with respect to the* L^2_k *topology for any* k.

In the first assertion of the lemma, $\Pi_{\text{Im}} : SU(2) \to su(2)$ is defined as follows. Any $SU(2)$ element g can be decomposed uniquely into a constant real multiple of the identity matrix plus an $su(2)$ element. $\Pi_{\text{Im}}(g)$ is that $su(2)$ element. Also, $\operatorname{hol}_{\gamma_i}(s;x,A)$ is the $SU(2)$ matrix which defines parallel translation around $\gamma_i(S^1\times\{x\})$ starting at $\gamma_i(s,x)$.

The following properties of $\nabla h(A)$ and $\operatorname{Hess} h(A)$ are direct consequences of Corollary 57 and Lemma 58 along with the Rellich Lemma.

Corollary 17. *Let h be an admissable function.*

1. *The map* $A\mapsto\nabla h(A)$ *is a smooth map from* $\mathcal{A}$ *to* $L^2_2\Omega^1(Y,su(2))$.
2. $\operatorname{Hess} h(A) : L^2_2 \to L^2_1$ *is a compact operator.*

The following proposition, which will be needed for the compactness result in the next subsection, follows immediately from the formula for ∇h and Corollary 62.

Proposition 18. *Let h be an admissible function. Then there is a constant C depending only on h such that* $\|\nabla h(A)\|_{L^2_2} \le C$ *for each A in* $\zeta_h^{-1}(0)$.

6.2. Compactness of the Perturbed Critical Set. We next adapt Uhlenbeck's compactness results [U] to our situation to show that, for admissible perturbation functions h, $\mathcal{M}_h$ is compact.

We first establish some notation. For a connection A we define a Sobolev norm $\|\ \|_{L^2_k,A}$ by using the associated covariant derivative $\nabla_A : \xi \to \nabla\xi + A\otimes\xi$ in place of the standard covariant derivative ∇ in the definition. A standard argument using the Sobolev Embedding Theorem, the Multiplication Theorem, and Hölder's inequality shows that this norm is equivalent to the standard one by a constant C depending on $\|A\|_{L^2_k}$ which goes to zero as $\|A\|_{L^2_k} \to 0$. Note that $\| \ \|_{L^2_k,A}$ is gauge invariant in the sense that for any gauge transformation g and any $\xi \in \Omega^p(M, su(2))$,

$$\|g^{-1}\xi g\|_{L^2_k, g^*A} = \|\xi\|_{L^2_k,A}.$$

We begin with a local theorem. Let D^3 denote the unit 3-ball.

Theorem 19. *([U], Theorem 2.1) There exist numbers $\kappa > 0$ and c such that every $A \in L^2_2\mathcal{A}_{D^3}$ with $\|F(A)\|_{L^2} < \kappa$ is gauge equivalent to a connection A' where A' satisfies*

(a) $d^*A' = 0$
(b) $*A'|_{\partial D^3} = 0$
(c) $\|A'\|_{L^2_1} \leq c\|F(A)\|_{L^2}$.

Corollary 20. *For κ small enough, there exists a constant C such that if $\|F(A)\|_{L^2_2,D} < \kappa$ then the connection A' in Theorem 19 also has the property that*

$$\|A'\|_{L^2_3} \leq C\|F(A')\|_{L^2_2}.$$

Proof. The proof is a standard bootstrapping argument. By condition (a), $\|A'\|_{L^2_2,A'}$ is bounded by a constant times $\|dA'\|_{L^2_1,A'}$. We write $dA' = F(A') - A' \wedge A'$. Then, using Hölder's inequality and the Sobolev Embedding Theorem, we get

$$\|dA'\|_{L^2_1,A'} \leq \|F(A')\|_{L^2_1,A'} + c_1\|A'\|_{L^2_1,A'}\|A'\|_{L^2_2,A'}.$$

When $\kappa < \frac{1}{2c_1}$, this implies

$$\|A'\|_{L^2_2,A'} \leq c''\|F(A')\|_{L^2_1,A'}.$$

By the above comment about equivalence of norms, this implies that

$$\|A'\|_{L^2_2} \leq C\|F(A)\|_{L^2_1,A}.$$

One more application of the same argument gives the result. □

Next we show that Uhlenbeck's global (weak) compactness result, along with our additional assumption of L^2_2 bounded curvature, implies strong compactness.

Lemma 21. *([U], Lemma 3.5) Let A_i be a sequence of connections in $\mathcal{A}$ with $\|F(A_i)\|_{L^2} \leq B$. Then there exists a fixed open cover $\{U_\alpha\}$ of Y and trivializations $\sigma_{\alpha,i}$ of $P|_{U_\alpha}$ which induce connection forms $\sigma_{\alpha,i} A_i \sigma_{\alpha,i}^{-1} = A_{\alpha,i}$ satisfying*

(a) *The $A_{\alpha,i}$ satisfy (a)-(c) of Theorem* 19.

(b) *The overlap functions $g_{\alpha,\beta,i} = \sigma_{\alpha,i}\sigma_{\beta,i}^{-1}$ are uniformly bounded in $L_2^2(U_\alpha \cap U_\beta, SU(2))$.*

(c) *For a subsequence, we have weak convergence*

$$A_{\alpha,i'} \rightharpoonup A_\alpha \text{ in } L_1^2$$
$$g_{\alpha,\beta,i} \rightharpoonup g_{\alpha,\beta} \text{ in } L_2^2.$$

(d) *The A_α represent a connection A on a bundle (trivial, by obstruction theory) presented in terms of a trivialization with overlap maps $g_{\alpha,\beta}$.*

Corollary 22. *In the situation of Lemma* 21, *if $\|F(A_i)\|_{L^2_{2,A_i}} \leq B$, then for a subsequence we have strong convergence*

$$A_{\alpha,i'} \to A_\alpha \text{ in } L_2^2$$
$$g_{\alpha,\beta,i} \to g_{\alpha,\beta} \text{ in } L_3^2.$$

Proof. By Corollary 20, the connections actually converge weakly in L_3^2, hence strongly in L_2^2. The proof of the second assertion is a standard argument to get one more L^2 bounded derivative on a gauge transformation between two connections than one has on the connections themselves (see [U] Lemma 1.2, for example). □

Finally, Uhlenbeck's weak compactness theorem, combined with the above corollaries, gives the following theorem.

Theorem 23. *Let $A_i \in \mathcal{A}$ be a sequence of connections with $\|F(A_i)\|_{L^2_{2,A_i}}$ bounded. Then there is a subsequence which, after L_3^2 gauge transformation, converges strongly in $\mathcal{A}$.*

Corollary 24. *For $h \in \mathcal{H}$, $\zeta_h^{-1}(0)/\mathcal{G}$ is compact.*

Proof. Fix any perturbation $h \in \mathcal{H}$. For any connection A, $\zeta_h(A) = 0$ implies $F(A) = -2\pi \nabla h(A)$, and so

$$\|F(A)\|_{L^2_{2,A}} = \|2\pi \nabla h(A)\|_{L^2_{2,A}} \leq C.$$

The last bound comes from Proposition 18. □

Let $\pi : \mathcal{B} \times \mathcal{H} \to \mathcal{H}$ be projection and define $P : \mathcal{A} \times \mathcal{H} \to \Omega^1(Y, su(2))$ to be the $\mathcal{G}$ equivariant map $P(A,h) = \zeta_h(A)$.

Corollary 25. $\pi|_{P^{-1}(0)/\mathcal{G}}$ *is a proper map.*

6.3. Hodge Theory. In this subsection, we recall some basic facts from Hodge theory for manifolds with boundary and give two orthogonal decompositions of $L^2_p\Omega^1(Y, su(2))$.

Let $h \in \mathcal{H}$ and $A \in \zeta_h^{-1}(0)$ be fixed throughout this subsection. We consider the deformation complex for the moduli space of perturbed flat connections near A.

The linearization of ζ_h at A is $*\frac{1}{2\pi}d_A + \operatorname{Hess} h(A)$, which we denote by $*d_{A,h}$. Since ζ_h satisfies $\zeta_h(g^*A) = g^{-1}\zeta_h(A)g$, if $\xi \in \Omega^0(Y, su(2))$,

$$*d_{A,h}(d_A\xi) = [\zeta_h(A), \xi] = 0.$$

Consider the complex

$$(4) 0 \to \Omega^0(Y, su(2)) \xrightarrow{d_A} \Omega^1(Y, su(2)) \xrightarrow{*d_{A,h}} \Omega^1(Y, su(2)) \xrightarrow{d_A^*} \Omega^0(Y, su(2)) \to 0.$$

Note that this differs from the standard twisted de Rham complex only in that we have identified Ω^2 with Ω^1 and Ω^3 with Ω^0 by the Hodge star isomorphism. To make this an elliptic complex, suitable boundary conditions must be imposed.

Let $\Delta_{A,h} = d_A d_A^* + (*d_{A,h})^* d_{A,h}$ be the corresponding Laplacian on 1-forms. It differs from $dd^* + d^*d$ by a lower order, and hence compact, operator. Integration by parts gives

$$\begin{aligned} \langle \Delta_{A,h}\alpha, \beta\rangle &= \langle \alpha, \Delta_{A,h}\beta\rangle - \int_\Sigma \operatorname{tr}(*d_{A,h}\alpha \wedge \beta) - \int_\Sigma \operatorname{tr}(d_A^*\alpha \wedge *\beta) \\ &\quad - \int_\Sigma \operatorname{tr}(\alpha \wedge *d_{A,h}\beta) - \int_\Sigma \operatorname{tr}(*\alpha \wedge d_A^*\beta) \end{aligned}$$

Define

$$\mathcal{V}_p = \{\alpha \in L^2_p\Omega^1(Y, su(2)) | \ *\alpha|_\Sigma = 0, *d_{A,h}\alpha|_\Sigma = 0\}$$

and

$$\mathcal{W}_p = \{\alpha \in L^2_p\Omega^1(Y, su(2)) | \ \alpha|_\Sigma = 0, d_A^*\alpha|_\Sigma = 0\}.$$

The operators $\Delta_{A,h} : \mathcal{V}_p \to L^2_{p-2}\Omega^0(Y, su(2))$ and $\Delta_{A,h} : \mathcal{W}_p \to L^2_{p-2}$ are elliptic boundary value problems with kernel (= cokernel) equal to

$$\mathcal{H}^1_{A,h}(Y) = \{\alpha \in \Omega^1_\nu(Y, su(2)) | d^*_A\alpha = 0, *d_{A,h}\alpha = 0\}$$

and

$$\mathcal{H}^1_{A,h}(Y, \partial Y) = \{\alpha \in \Omega^1_\tau(Y, su(2)) | d^*_A\alpha = 0, *d_{A,h}\alpha = 0\},$$

respectively, where $\Omega^1_\nu(Y, su(2))$ $(\Omega^1_\tau(Y, su(2)))$ is the space of 1-forms whose normal (tangential) components vanish along the boundary.

Lemma 26. *$L^2_p\Omega^1(Y, su(2))$ can be decomposed into the following two orthogonal decompositions:*

(a)
$$L^2_p\Omega^1(Y, su(2)) = d_A(L^2_{p+1}\Omega^0(Y, su(2))) \oplus \mathcal{H}^1_{A,h}(Y) \oplus *d_{A,h}(L^2_{p+1}\Omega^1_\tau(Y, su(2)))$$

(b)
$$L^2_p\Omega^1(Y, su(2)) = d_A(L^2_{p+1}\Omega^0_\tau(Y, su(2))) \oplus \mathcal{H}^1_{A,h}(Y, \partial Y) \oplus *d_{A,h}(L^2_{p+1}\Omega^1(Y, su(2))).$$

Proof. We demonstrate the first decomposition; the second follows as easily with the other choice of boundary conditions.

Given $a \in L^2_p\Omega^1(Y, su(2))$, we simply write $a = \alpha + d_Ad^*_Ab + *d_{A,h} * d_{A,h}b$ for $\alpha \in \mathcal{H}^1_{A,h}(Y)$ and $b \in \mathcal{V}_p$. By the choice of boundary conditions for $\mathcal{V}_p$, $*d_{A,h}b|_\Sigma = 0$. □

6.4. The Kuranishi Picture Near a Point in $\mathcal{M}_h$. The purpose of this subsection is to describe a finite-dimensional local model for $\mathcal{M}_h$. The basic tool is the Kuranishi deformation complex. This is described nicely, for example, in [MMR]. The main theorem is the following:

Theorem 27. *Fix $h \in \mathcal{H}$ and fix a smooth connection $A \in \zeta_h^{-1}(0)$. Then there are:*

(a) *a* Stab(A) *equivariant neighborhood V_A of* 0 *in $\mathcal{H}^1_{A,h}(Y)$,*
(b) *a $\mathcal{G}$ equivariant neighborhood U_A of A in $\mathcal{A}$,*

(c) *a* Stab(A) *equivariant real analytic embedding*

$$\phi_A : V_A \to X_A \cap U_A$$

whose differential at 0 *is just the inclusion of* $\mathcal{H}^1_{A,h}(Y)$ *into* $\ker d_A^* \cap L^2_1\Omega^1_\nu(Y, su(2))$,

(d) *and a* Stab(A) *equivariant map*

$$\Phi_A : V_A \to \mathcal{H}^1_{A,h}(Y, \partial Y)$$

such that ϕ_A *maps* ${\Phi_A}^{-1}(0)$ *homeomorphically onto the zero set of* $\zeta_h|_{X_A \cap U_A}$.

The only thing nonstandard about the proof of the theorem in this case is that the linearization

$$*d_{A,h} : L^2_2\Omega^1(Y, su(2)) \to L^2_1\Omega^1(Y, su(2))$$

of

$$\zeta_h : \mathcal{A} \to L^2_1\Omega^1(Y, su(2))$$

at A has infinite-dimensional cokernel, and hence is not elliptic, even after taking into account gauge symmetry. We therefore replace the condition

$$\zeta_h(A') = 0 \tag{5}$$

with the condition

$$\Pi_A\zeta_h(A') = 0 \tag{6}$$

where

$$\Pi_A : L^2_1\Omega^1(Y, su(2)) \to \ker d_A^* \cap L^2_1\Omega^1(Y, su(2))$$

is the orthogonal projection onto $\ker d_A^* \cap L^2_1\Omega^1(Y, su(2))$. The following two facts insure that maps (5) and (6) have the same zero set near A.

Lemma 28. *(see [MMR]) Let* $h \in \mathcal{H}$ *and let* A *be a smooth connection with* $\zeta_h(A) = 0$. *There exists an* $\epsilon > 0$ *such that if* $A' \in \mathcal{A}$ *and* $\|A' - A\|_{L^2_2} < \epsilon$ *then*

$$\Pi_A|_{\ker d^*_{A'} \cap L^2_1\Omega^1(Y, su(2))} : \ker d^*_{A'} \cap L^2_1\Omega^1(Y, su(2)) \to \ker d^*_A \cap L^2_1\Omega^1(Y, su(2))$$

is injective.

The second fact is a corollary to the Bianchi identity.

Lemma 29. *Let h be any gauge invariant function on $\mathcal{A}$. Then if $A \in \mathcal{A}$,*

$$\zeta_h(A) \in \ker d_A^* \cap \Omega^1_\nu(Y, su(2)).$$

Proof. The equation

$$0 = \langle \nabla h(A), d_A u\rangle = \langle d_A^* \nabla h(A), u\rangle - \int_\Sigma \operatorname{tr}(*\nabla h(A) \wedge u)$$

holds for all $u \in \Omega^0(Y, su(2))$. Thus $d_A^* \nabla h(A) = 0$ and $*\nabla h(A)|_\Sigma$. The proof now reduces to the Bianchi identity. □

We now complete the proof of Theorem 27. We begin by decomposing $\ker d_A^* \cap L_2^2\Omega^1_\nu(Y, su(2))$ into

$$\ker d_A^* \cap L_2^2\Omega^1_\nu(Y, su(2)) = \mathcal{H}^1_{A,h}(Y) \oplus *d_{A,h}L_3^2\Omega^1_\tau(Y, su(2))$$

as in Proposition 26. Define the map

$$B_{A,h} : \mathcal{H}^1_{A,h}(Y) \oplus *d_{A,h}L_3^2\Omega^1_\tau(Y, su(2)) \to *d_{A,h}L_2^2\Omega^1(Y, su(2))$$

by

$$B_{A,h}(\alpha, a) = \Pi'_A \zeta_h(A + \alpha + a)$$

where $\Pi'_A : \ker d_A^* \cap L_1^2\Omega^1(Y, su(2)) \to *d_{A,h}L_2^2\Omega^1(Y, su(2))$ is orthogonal projection.

The second partial $\frac{\partial B_{A,h}}{\partial a}(0,0) = \Pi'_A \circ *d_{A,h}$ of this map is surjective. By the implicit function theorem, there is an open neighborhood V_A of $0 \in \mathcal{H}^1_{A,h}(Y)$ and a map $\psi_A : V_A \to *d_{A,h}L_3^2\Omega^1_\tau(Y, su(2))$ with the properties that

$$\Pi'_A \zeta_h(A + \alpha + \psi_A(\alpha)) = 0$$

and that this equation parametrizes the zero set of $\Pi'_A\zeta_h(A + \alpha + a)$ in some neighborhood N_A of $(0,0)$. We define the map ϕ_A to be

$$\phi_A(\alpha) = A + \alpha + \psi_A(\alpha).$$

Finally, we set

$$\Phi_A(\alpha) = \Pi_A \zeta_h(A + \alpha + \psi_A(\alpha)). \quad \square$$

7. The Generic Perturbed Flat Moduli Space

In this section we use the tools outlined in the last section to prove a series of local results about the structure of $\mathcal{M}_h$ for generic perturbations. Section 7.1 discusses the $(\mathbf{Z}_2, \mathbf{Z}_2)$-stratum of $\mathcal{M}_h$. This argument is the basic model for the ones that follow in Sections 7.2-7.5, where we modify it as necessary to handle the other strata. In Section 7.6 we restate Theorem 15 in a slightly more precise fashion and prove it by combining the local results into a global one.

We should make one comment about the notation. The setup in each of the first five subsections is slightly different. The approach is always roughly the same, however. We apply the implicit function theorem to a map from a space of connections times a space of perturbations to show that the universal zero set Z is smooth in a neighborhood $V \times U$. To stress the similarities and yet avoid confusion between the slightly different frameworks in the different subsections, we label the objects (for example, Z, U, and V) with subscripts which coincide with the subsections in which the objects are used. For any product of spaces, let π_1 and π_2 denote the projections onto the first and second factors.

We also make this suggestion to the reader. An overview of the proof of Theorem 15 can be obtained by perusing the theorems in Sections 7.1-7.5. The proof of each of these, however, depends on one or two technical results which have been relegated to Section 10. Hence, once the overview is understood, the careful reader may wish to read Sections 7 and 10 in parallel.

7.1. The $(\mathbf{Z}_2, \mathbf{Z}_2)$-Stratum. In this subsection we will show that $\mathcal{M}_{h'}^{(\mathbf{Z}_2,\mathbf{Z}_2)}$ is a smooth $(3g-3)$-dimensional manifold near any fixed $[A] \in \mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)}$ for generic perturbation h'.

Fix an arbitrary admissible perturbation function (h, ϕ) and a point $[A]$ in the $(\mathbf{Z}_2, \mathbf{Z}_2)$ part of $\mathcal{M}_h$. Enlarge the collection of solid tori ϕ to a collection $\phi \cup \psi$ which satisfies the conclusion of Lemma 63 and let $\mathcal{H}_1 = \mathcal{H}_{(\phi\cup\psi)}$. We now identify h with the corresponding perturbation function in $\mathcal{H}_1$ which has $\bar{h}_i = 0$ for all the new components.

Recall that $P : \mathcal{A} \times \mathcal{H} \to \Omega^1(Y, su(2))$ was defined to be the map $P(A, h) =$

$\zeta_h(A)$. We consider its restriction to $X_A \times \mathcal{H}_1$.

Theorem 30. *In the situation described above, there exist a neighborhood $V_1 \subset X_A \cap \mathcal{A}^{(\mathbf{Z}_2,\mathbf{Z}_2)}$ of A and a neighborhood $U_1 \subset \mathcal{H}_1$ of h such that $Z_1 = P^{-1}(0) \cap (V_1 \times U_1)$ is a smooth submanifold.*

Proof. By Lemma 28 and the implicit function theorem, it is sufficient to check that $\Pi_A \circ P$ restricted to $X_A \times \mathcal{H}_1$ is a submersion at (A, h).

The linearization $\Pi_A \circ P$ at (A, h) is given by $\delta(\Pi_A \circ P) = *d_{A,h}(\delta A) + \nabla\delta h(A)$. By Lemma 26, the first partial derivative has cokernel $\mathcal{H}^1_{A,h}(Y, \partial Y)$. We must check that $\delta h \mapsto \nabla\delta h(A)$ is transverse to $\mathcal{H}^1_{A,h}(Y, \partial Y)^\perp$. For this it is enough to show that for any $\alpha \in \mathcal{H}^1_{A,h}(Y, \partial Y)$, there is a tangent vector δh at h such that $\langle \nabla\delta h(A), \alpha\rangle \neq 0$.

Suppose $\alpha \in \mathcal{H}^1_{A,h}(Y, \partial Y)$ and $\alpha \perp \nabla\delta h(A)$ for all $\delta h \in T_h\mathcal{H}_1$. Using the decomposition of Lemma 26 we get $\alpha = \beta + d_A b$ for some $\beta \in \mathcal{H}^1_{A,h}(Y)$ and $b \in \Omega^0(Y, su(2))$. If $\beta \neq 0$, then by Lemma 63 there is a tangent vector $\delta h \in T_h\mathcal{H}_K$ with $\langle \delta h(A), \beta\rangle \neq 0$. The gauge invariance of the admissable functions implies that $\langle \delta h(A), d_A b\rangle = 0$, so $\langle \delta h(A), \alpha\rangle \neq 0$, and this contradicts our hypothesis. On the other hand, suppose $\beta = 0$. In the long exact sequence of the pair (Y, Σ), the map $\mathcal{H}^1_{A,h}(Y, \partial Y) \to \mathcal{H}^1_{A,h}(Y)$ is simply the orthogonal projection $\Omega^1(Y, su(2)) \to \mathcal{H}^1_{A,h}(Y)$ restricted to $\mathcal{H}^1_{A,h}(Y, \partial Y)$. Thus α is in the image of $\mathcal{H}^0(\Sigma)$. Since $A|_\Sigma$ is irreducible, this implies $\alpha = 0$, and the theorem is proved. □

Let $\bar{V}_1$ be the image in $\mathcal{B}$ of V_1.

Theorem 31. *For an open dense set of $h' \in U_1 \subset \mathcal{H}_1$, $\mathcal{M}_{h'} \cap \bar{V}_1$ is a smooth submanifold of dimension $3g - 3$.*

Remark. In the cases $g = 0$ and $g = 1$, $\zeta_h(A) = 0$ implies $A|_\Sigma$ is flat and hence reducible, so $\mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)}$ is empty.

Proof. This is essentially the same argument as the proof of Theorem 3.17 in [FU]. Consider the restriction of the projection $\pi_2 : V_1 \times U_1 \to U_1$ to Z_1. By the Sard-Smale Theorem, the regular values of this map form an open dense set. Let h' be a regular value. Then $(\pi_2|_{Z_1})^{-1}(h')$, which corresponds to $\mathcal{M}_{h'} \cap \bar{V}_1$ under the quotient map, is a smooth submanifold. Its dimension, by a standard argument, is the index of the Kuranishi complex, namely

$$\dim \mathcal{M}_{h'} \cap (V_1/\mathcal{G}) = \dim \mathcal{H}^1_{A,h}(Y) - \dim \mathcal{H}^1_{A,h}(Y, \partial Y) = 3g - 3.$$

The latter equality follows from Poincaré Duality and the long exact sequence at the pair (Y, Σ) and the fact that $\mathcal{H}^0_A(\Sigma) = 0$. □

The long exact sequence and Poincaré Duality also imply that the image of $\mathcal{H}^1_{A,h}(Y)$ in $\mathcal{H}^1_A(\Sigma)$ has dimension $3g-3$. This gives the following important corollary.

Corollary 32. *For generic perturbations $h' \in U_1$, the restriction map $r : \mathcal{M}_{h'} \cap \bar{V}_1 \to \mathcal{M}^{\mathbf{Z}_2}_\Sigma$ is an immersion.*

We conclude this subsection by proving that $Z_1/\mathcal{G}$ maps submersively to $\mathcal{M}^{\mathbf{Z}_2}_\Sigma$. Let $\pi_1 : \mathcal{B}_Y \times \mathcal{H} \to \mathcal{B}_Y$ denote the projection.

Lemma 33. *The composition $r \circ \pi_1 : (Z_1/\mathcal{G}) \cap (\bar{V}_1 \times U_1) \to \mathcal{M}^{\mathbf{Z}_2}_\Sigma$ is a submersion.*

Proof. We have chosen V_1 and U_1 small enough that P is a submersion at every point in $Z_1 \cap (V_1 \times U_1)$. In other words, for any $(A', h') \in Z_1 \cap (V_1 \times U_1)$, the image of the map $\nabla : T_{h'}U_1 \to \Omega^1(Y, su(2))$ given by $\delta h' \mapsto \nabla \delta h'(A')$ orthogonally projects onto $\mathcal{H}^1_{A',h'}(Y)$. Let $\alpha_0 \in \mathcal{H}^1(\Sigma)$. Extend α_0 to a 1-form $a \in \Omega^1(Y, su(2))$. Note that $*d_{A',h'}a \in \ker d^*_{A'} \cap \Omega^1_\nu(Y, su(2))$, since α_0 is harmonic.

We decompose $*d_{A',h'}a$ orthogonally into $*d_{A',h'}a = \beta + *d_{A',h'}b$ where $b \in \Omega^1_\tau(Y, su(2))$ and $\alpha \in \mathcal{H}^1_A(Y)$. Choose a tangent vector $\nabla \delta h' = \beta + *d_{A',h'}c$ for some $c \in \Omega^1_\tau(Y, su(2))$. Let $a' = a + c - b$. Then $*d_{A',h'}a' = \nabla \delta h'$ (which means $(a', \delta h') \in T_{(A',h')}Z_1 \cap (V_1 \times U_1))$ and $a'|_\Sigma = \alpha_0$. □

7.2. The $(\mathbf{Z}_2, U(1))$-Stratum. We shall call a connection A *boundary-abelian* if $A|_\Sigma$ is abelian, *boundary-central* if $A|_\Sigma$ is central (recall that according to the conventions of this paper, the adjectives *abelian* and *central* are mutually exclusive). In this subsection we will explore the structure of $\mathcal{M}_h$ near equivalence classes of irreducible connections that are boundary-abelian. We will use a similar approach for the irreducible boundary-central case in the next subsection.

Our task is complicated by the fact that, at an irreducible boundary-abelian connection A, the linearization of the map $\Pi_A \circ P_1$ used in the last section is no longer surjective. The reason for this is purely topological. In the long exact sequence of the pair, the inclusion of the relative cohomology of Y into the absolute cohomology is not injective. In terms of the harmonic representatives for the cohomology, this means that part of $\mathcal{H}^1_{A,h}(Y, \partial Y)$ is orthogonal to $\mathcal{H}^1_{A,h}(Y)$, namely the image of the coboundary map $\mathcal{H}^0_A(\Sigma) \to \mathcal{H}^1_{A,h}(Y, \partial Y)$ in the long exact sequence of the pair. This fact prevents us from applying the implicit function theorem to $\Pi_A \circ P_1$ directly.

We get around this difficulty by considering, instead, the composition of $\Pi_A \circ P_1$ with the projection onto the orthogonal complement of this "extra cokernel." This composition is surjective, and the implicit function theorem therefore applies to it, giving a smooth universal zero set. We must then cut this zero set down by setting the "extra cokernel" component of ζ_h equal to zero.

Before stating and proving the local result about the $(\mathbf{Z}_2, U(1))$-stratum of $\mathcal{M}_h$, we make a brief digression to prove, using the Kuranishi technique, that $\mathcal{M}_\Sigma$ has the structure described in Theorem 10 near the abelian stratum. The framework and notation which we establish during the digression will be useful later in this subsection.

Fix an abelian flat connection A on Σ. Consider the Kuranishi picture for $\mathcal{M}_\Sigma$ near $[A]$. Let X^Σ_A denote the slice to the gauge group action at A. Choose a nonzero $\gamma_0 \in \mathcal{H}^0_A(\Sigma)$. Notice that $*_\Sigma\gamma_0$ generates the cokernel $\mathcal{H}^2_A(\Sigma) \cong \mathbf{R}$ of the linearization of the curvature map $F : X^\Sigma_A \to \Omega^2(\Sigma, su(2))$.

Define the map $\Pi_0 : \Omega^2(\Sigma, su(2)) \to \Omega^2(\Sigma, su(2)) \cap (*_\Sigma\gamma_0)^\perp$ be the orthogonal projection. We identify X^Σ_A with $\mathcal{H}^1_A(\Sigma) \oplus d^*_A\Omega^2(\Sigma, su(2))$. The second partial derivative of $\Pi_0 \circ F$ is surjective. Thus there are neighborhoods $U^\Sigma_A \subset X^\Sigma_A$ of A and $V^\Sigma_A \subset \mathcal{H}^1_A(\Sigma)$ of zero and a map $\phi^\Sigma_A : V^\Sigma_A \to d^*_A\Omega^2(\Sigma, su(2))$ such that the zero set

$$Z^\Sigma_0 = (\Pi_0 \circ F)^{-1}(0) \cap U^\Sigma_A$$

is the graph of ϕ^Σ_A. If we identify $\mathcal{H}^2_A(\Sigma)$ with $\mathbf{R}$ using $*_\Sigma\gamma_0$ as a basis vector, then the Kuranishi map $\Phi_A^\Sigma : \mathcal{H}_A^1(\Sigma) \to \mathbf{R}$ is

$$\Phi_A^\Sigma(\alpha) = \langle F(\phi_A^\Sigma(\alpha)), *_\Sigma\gamma_0\rangle_{L^2(\Sigma)}.$$

Using integration by parts, this can be simplified to

$$\Phi_A^\Sigma(\alpha) = \langle [\alpha \wedge \alpha], *_\Sigma\gamma_0\rangle_{L^2(\Sigma)}.$$

Having completed our digression, we return to the problem at hand. Let $A \in \mathcal{A}$ and (h, ϕ) be fixed with A irreducible and boundary-abelian and with $\zeta_h(A) = 0$. Let $\gamma_0 \in \mathcal{H}_A^0(\Sigma)$ be a generator and let $\tilde{\gamma}_0$ be an extension of γ_0 to Y. Let $\Pi_2 : \ker d_A^* \cap \Omega^1(Y, su(2)) \to \ker d_A^* \cap \Omega^1(Y, su(2)) \cap (d_A\tilde{\gamma}_0)^\perp$ be the orthogonal projection.

We enlarge the collection ϕ so that the conclusion of Lemma 63 is satisfied at A. Call the component of $\mathcal{H}$ corresponding to the new collection $\mathcal{H}_2$.

One slight technicality is that the zero set of $\Pi_2 \circ \Pi_A \circ P_1 : \mathcal{A} \times \mathcal{H}_2 \to \ker d_A^* \cap \Omega^1(Y, su(2)) \cap (d_A\tilde{\gamma}_0)^\perp$ is not gauge invariant. For this reason, we consider its restriction to $X_A \times \mathcal{H}_2$, where X_A is the slice to the gauge group action. (Alternatively, we could have made a connection-dependent γ_0 which changed by $ad(g^{-1})$ when we moved away from the slice by a gauge transformation g. Then the zero set would have been gauge invariant.)

Let $Z_2 = (\Pi_2 \circ \Pi_A \circ P_1)^{-1}(0) \cap (X_A \times \mathcal{H}_2)$. Let Z_0^Σ be as defined in our digression above.

Proposition 34. *There is a neighborhood* $(V_2 \times U_2) \subset (X_A \cap \mathcal{A}_Y^*) \times \mathcal{H}_2$ *of* (A, h) *satisfying the following four conditions:*

1. $Z_2 \cap (V_2 \times U_2)$ *is a submanifold.*
2. *For* $A' \in V_2$, $\langle \Pi_A(d'_A\tilde{\gamma}_0), \Pi_A(d_A\tilde{\gamma}_0)\rangle \neq 0$.
3. $Z_2 \cap (V_2 \times U_2)$ *maps to* Z_0^Σ *under the map* $r \circ \pi_1$.
4. *The map in* (3) *is a submersion.*

Proof. The implicit function theorem argument in the last subsection implies that Z_2 is a smooth submanifold in some neighborhood of (A, h). That the second condition can be met follows from the fact that $\Pi_A(d_A\tilde{\gamma}_0) \neq 0$ and $d_{A'}\tilde{\gamma}_0$ varies continuously in A'.

Suppose $(A', h') \in Z_2$ and $\langle F(A')|_\Sigma, *_\Sigma \gamma_0\rangle_{L^2(\Sigma)} = 0$. By Stokes' Theorem this last equality is equivalent to $\langle \zeta_{h'}(A'), d_{A'}\tilde{\gamma}_0\rangle_{L^2(Y)} = 0$, which implies that $\langle \Pi_A(\zeta_{h'}(A')), d_{A'}\Pi_A(\tilde{\gamma}_0)\rangle_{L^2(Y)} = 0$. Since $\Pi_2 \circ P_A(\zeta_{h'}(A')) = 0$, using condition (2), this means that $\Pi_A(\zeta_{h'}(A')) = 0$. By Lemma 28, for A' close enough to A this implies that $\zeta_{h'}(A') = 0$; in particular, it implies $F(A')|_\Sigma = 0$. This shows that the third condition will be satisfied if V_2 is small enough.

For the fourth condition, we simply note that $r \circ \pi_1 : Z_2 \to Z_0^\Sigma$ is a submersion at (A, h), by the proof of Lemma 33. □

Theorem 35. *There exist neighborhoods $\bar{V}_2 \subset \mathcal{B}_Y^*$ of $[A]$ and $U_2 \subset \mathcal{H}_2$ of h such that for generic $h' \in U_2$, $\mathcal{M}_{h'} \cap \bar{V}_2$ is a stratified space with the following structure:*

(a) *If $g > 2$, then $\mathcal{M}_{h'} \cap \bar{V}_2$ is empty.*

(b) *If $g = 2$, then*

$$\mathcal{M}_{h'} \cap \bar{V}_2 = (\mathcal{M}_{h'}^{(\mathbf{Z}_2,\mathbf{Z}_2)} \cap \bar{V}_2) \coprod (\mathcal{M}_{h'}^{(\mathbf{Z}_2,U(1))} \cap \bar{V}_2).$$

The $(\mathbf{Z}_2, \mathbf{Z}_2)$-stratum is $(3g-3)$-dimensional and the $(\mathbf{Z}_2, U(1))$-stratum is 0-dimensional. The latter has normal bundle in the former with fiber $c(S^1 \times S^1)$.

(c) *If $g = 1$, then $\mathcal{M}_{h'} \cap \bar{V}_2$ is a smooth 1-dimensional submanifold of $\bar{V}_2$.*

Proof. The standard argument used in the preceding subsection now shows that for regular values h' of $\pi_2 : Z_2 \to U_2$, $Z_2 \cap (V_2 \times \{h'\})$ is a smooth manifold of dimension

$$\dim \mathcal{H}^1_{A,h}(Y) - \dim(\mathcal{H}^1_{A,h}(Y, \partial Y) \cap (d_A\gamma_0)^\perp).$$

This dimension is easily calculated from the long exact sequence of the pair to be $3g - 2$.

Let U_2 and V_2 be as in Proposition 34 and let $\bar{V}_2$ to be the image in $\mathcal{B}_Y^*$ of V_2. By the Stokes' Theorem argument in the proof of that proposition,

$$P_1^{-1}(0) \cap (U_2 \times V_2) = (r \circ \pi_1)^{-1}(Z_0^\Sigma \cap \mathcal{F}_\Sigma).$$

In other words, the final equation $\langle \zeta_{h'}(A'), d_A\tilde{\gamma}_0\rangle_{L^2(Y)} = 0$ by which we must cut down is, for $(A', h') \in Z_2 \cap (U_2 \times V_2)$, equivalent to the condition that $\Phi_A^\Sigma(A'|_\Sigma) = 0$.

Define $Z_0^{\Sigma,U(1)} \subset Z_0^\Sigma$ to be the graph of $\phi_A^\Sigma|_{\mathcal{H}_A^1(\Sigma;\mathbf{R})}$. This is exactly the preimage in Z_0^Σ of the abelian stratum of $\mathcal{M}_\Sigma$ in the slice tangent space. Therefore,

$$\mathcal{M}_{h'}^{(\mathbf{Z}_2,U(1))} \cap \bar{V}_2 \cong Z_2^{U(1)} \cap (V_2 \times \{h'\})$$

where $Z_2^{U(1)} = Z_2 \cap (\pi_{Z_0^\Sigma})^{-1}(Z_0^{\Sigma,U(1)})$. $Z_2^{U(1)}$ has codimension $4g-4$ in Z_2. The index of the restriction $\pi_2 : Z_2^{U(1)} \to U_2$ is therefore $2-g$. This proves the assertions about the $(\mathbf{Z}_2, U(1))$-stratum in all cases. It remains to prove the assertion about the normal bundle in the genus 2 case.

Assume $g=2$. Let h' be a regular value of $\pi_2 : Z_2^{U(1)} \to U_2$ and let $(A', h') \in Z_2 \cap (V_2 \times \{h'\})$. Since (A', h') is a regular point of both $\Pi_2 \circ \Pi_A \circ P_1$ and $\pi_2|_{Z_2}$, it follows that $\mathcal{H}^1_{A',h'}(Y, \partial Y) \cap (d_A \tilde{\gamma}_0)^\perp = \{0\}$. Thus $\mathcal{H}^1_{A',h'}(Y) \to \mathcal{H}^1_{A'}(\Sigma) \cong \mathcal{H}^1_A(\Sigma) \cong \mathbf{R}^{2g} \oplus \mathbf{C}^{2g-2} = \mathbf{R}^4 \oplus \mathbf{C}^2$ is an injection (and hence by Poincaré duality, $\mathcal{H}^1_{A',h'}(Y)$ is 4-dimensional). Since Z_2 meets $V_2 \times \{h'\}$ transversely, $r_*(\mathcal{H}^1_{A',h'}(Y))$ is transverse to $\mathbf{R}^4$. The zero set of $\alpha \mapsto \langle \int_\Sigma [\alpha \wedge \alpha], \gamma_0 \rangle$ on any Lagrangian subspace of $\mathcal{H}^1_A(\Sigma)$ transverse to $\mathbf{R}^4$ is homeomorphic to $c(S^1 \times S^1)$. □

Remark. The calculation of the zero set above differs from the calculation of the structure of $\mathcal{M}_\Sigma$ near the abelian stratum in that there is no $U(1)$ stabilizer by which to divide out once we have found the zero set in the slice.

7.3. The $(\mathbf{Z}_2, SU(2))$-Stratum and the $(U(1), SU(2))$-Stratum. In this section we state an analogue of Theorem 35 for the irreducible boundary-central stratum. The formal dimension of the this stratum is $-3g$. Thus regardless of the genus of the boundary, this stratum is generically empty (unless $g=0$). The proof is so similar to that of Theorem 35 that we omit it. One simply must repeat the digression, this time for a central flat connection on Σ, replacing the single $*_\Sigma \gamma_0$ with a basis for the 3-dimensional $\mathcal{H}^2_A(\Sigma)$. One then repeats the rest of the argument, recalculating all the dimensions.

We again assume we have fixed a component $\mathcal{H}_3$ of $\mathcal{H}$ for which Lemma 63 is satisfied for the connection A in question.

Theorem 36. *If $g > 0$, then for any irreducible boundary-central $A \in \zeta_h^{-1}(0)$ there exist neighborhoods $\bar{V}_3 \subset \mathcal{B}_Y^*$ of $[A]$ and $U_3 \subset \mathcal{H}_3$ of h such that for generic $h' \in U_3$ the $(\mathbf{Z}_2, U(1))$-stratum of $\mathcal{M}_{h'}$ does not intersect $\bar{V}_3$.*

A similar argument gives the following proposition. For the basic setup, see the proof of Theorem 50, where we prove a similar result for 1-parameter families of perturbations.

Proposition 37. *The formal dimension of the $(U(1), SU(2))$-stratum is $-g$. Therefore, for generic perturbations, this stratum is empty unless $g = 0$, in which case it is 0-dimensional.*

7.4. The $(U(1), U(1))$-Stratum and its Normal Bundle. In this subsection we describe the abelian stratum and its normal bundle.

Let $U(1) \subset SU(2)$ be fixed and decompose $su(2) = \mathbf{R} \oplus \mathbf{C}$ where $\mathbf{R} = T_{\mathrm{id}}U(1)$. Consider an abelian connection A. After gauge transformation we can assume $A \in \Omega^1(Y, \mathbf{R})$.

Each admissable function h is $\mathcal{G}$ equivariant and hence, in particular, $\mathrm{Stab}(A)$ equivariant. Since $\mathrm{Stab}(A)$ acts with weight two on the coefficients of $\Omega^1(Y, \mathbf{C})$ and fixes $\Omega^1(Y, \mathbf{R})$, this implies that $\nabla\delta h(A)$ is orthogonal to $\Omega^1(Y, \mathbf{C})$ for any tangent vector $\delta h \in T_h\mathcal{H}$. In particular $\nabla\delta h(A) \perp \mathcal{H}_A^1(Y, \partial Y; \mathbf{C})$. It follows that

$$\mathrm{Hess}\, h : T_A\mathcal{A} = \Omega^1(Y, su(2)) \to \Omega^1(Y, su(2))$$

preserves the splitting

$$\Omega^1(Y, su(2)) = \Omega^1(Y, \mathbf{R}) \oplus \Omega^1(Y, \mathbf{C}).$$

The maps d_A and $*d_A$ also preserve the splitting.

Let Π_4 denote the orthogonal projection from $\Omega^1(Y, su(2))$ to $\ker d_A^* \cap \Omega^1(Y, \mathbf{R})$. We denote by $\mathrm{Sym}^{U(1)}(\mathcal{H}_{A,h}^1(Y; \mathbf{C}))$ the set of all symmetric $U(1)$ invariant bilinear forms on $\mathcal{H}_{A,h}^1(Y; \mathbf{C})$.

Lemma 38. *Let (h, ϕ) be an admissible perturbation. For any abelian $[A] \in \mathcal{M}_h$, the collection ϕ may be enlarged to a finite collection $\phi \cup \psi = \{\gamma_i\}_{i=1}^M$ such that, if $\mathcal{H}_4 = \mathcal{H}_{(\phi\cup\psi)}$ the map*

$$P_4 : X_A \cap \Omega^1(Y, \mathbf{R}) \times \mathcal{H}_4 \to \ker d_A^* \cap \Omega^1(Y, \mathbf{R}) \times \mathrm{Sym}^{U(1)}(\mathcal{H}_{A,h}^1(Y; \mathbf{C}))$$

given by

$$P_4(A', h') = (\Pi_4 \circ \Pi_A(\zeta_{h'}(A')), \operatorname{Hess} h'(A'))$$

is a submersion near (A, h).

Proof. Corollaries 64 and 66 guarantee that we can find a collection of solid tori such that the two components of the linearization of P_4

$$(\delta A, \delta h) \mapsto *d_{A,h}\delta A + \Pi_4 \circ \Pi_A \nabla \delta h(A)$$

and

$$(\delta A, \delta h) \mapsto \operatorname{Hess} \delta h(A)$$

are surjective. To get a submersion onto the product, consider the loops required for the two proofs. The only potential problem is that the loop with noncentral holonomy used to prove Corollary 66 may be among the list of curves needed for the proof of Corollary 64. In case this is true, we simply note that for any $x \in [-2, 2]$ we can vary the first and second derivatives of the component functions $\delta \bar{h}_i(x)$ independently. □

To study the abelian stratum itself, it would suffice to study the first component of P_4, and the argument in the proof of Theorem 31, with slight modification, proves that for generic h' the $(U(1), U(1))$-stratum of $\mathcal{M}_h$ is a smooth manifold near $[A]$ of dimension

$$\dim \mathcal{H}^1_A(Y; \mathbf{R}) - \dim \mathcal{H}^1_A(Y, \partial Y; \mathbf{R}) = g.$$

By considering both components, we will achieve this result and simultaneously obtain a description of the normal bundle. The key to describing the normal bundle is to show that $\mathcal{H}^1_{A',h'}(Y, \partial Y; \mathbf{C})$ vanishes everywhere along the abelian stratum for $g > 1$ and everywhere except at a finite number of points for $g = 1$.

Theorem 39. *Let* $[A]$ *be an abelian boundary-abelian point of* $\mathcal{M}_h$. *Let* $\mathcal{H}_4$ *be as in Lemma* 38. *Then there exist neighborhoods*

$$\bar{V}_4 \subset \mathcal{B}^{(\mathbf{Z}_2, \mathbf{Z}_2)} \bigcup \mathcal{B}^{(\mathbf{Z}_2, U(1))} \bigcup \mathcal{B}^{(U(1), U(1))}$$

of $[A]$ *and* $U_4 \subset \mathcal{H}_4$ *of* h *such that for generic* $h' \in U_4$, $\mathcal{M}_{h'} \cap \bar{V}_4$ *is a stratified space with the following structure:*

(a) *If $g > 1$, then*

$$\mathcal{M}_{h'} \cap \bar{V}_4 = \mathcal{M}_{h'}^{(\mathbf{Z}_2,\mathbf{Z}_2)} \cap \bar{V}_4 \coprod \mathcal{M}_{h'}^{(U(1),U(1))} \cap \bar{V}_4.$$

The $(\mathbf{Z}_2, \mathbf{Z}_2)$-stratum is a $(3g-3)$-dimensional manifold and the $(U(1), U(1))$-stratum is g-dimensional. The latter has normal bundle in the former with fiber $c(\mathbf{C}P^{(g-1)})$.

(b) *If $g = 1$, then*

$$\mathcal{M}_{h'} \cap \bar{V}_4 = \mathcal{M}_{h'}^{(\mathbf{Z}_2,U(1))} \cap \bar{V}_4 \coprod \mathcal{M}_{h'}^{(U(1),U(1))} \cap \bar{V}_4.$$

Both strata are 1-dimensional. The boundary of the $(\mathbf{Z}_2, U(1))$-stratum in the $(U(1), U(1))$-stratum is a 0-dimensional submanifold, with normal bundle in the $(\mathbf{Z}_2, U(1))$-stratum with fiber $c(pt) = \mathbf{R}/\mathbf{Z}_2$.

These identifications are diffeomorphisms in the stratified sense.

Proof. Let $n = \dim \mathcal{H}^1_{A,h}(Y, \partial Y; \mathbf{C})$. Since $*d_{A,h}$ is already a Fredholm isomorphism from the orthogonal complement of its kernel to the orthogonal complement of its cokernel, the effect on the Kuranishi map of adding a small perturbation δh to h is determined by $\operatorname{Hess} \delta h|_{\mathcal{H}^1_{A,h}(Y;\mathbf{C})}$. (We will consider the Hessian sometimes as a bilinear form and other times as a linear map, opting for whichever simplifies the notation and terminology at the moment.) Thus we will be interested not in the corresponding element B of $\operatorname{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y; \mathbf{C}))$ but in the composition $\Pi'_4 \circ B$ where $\Pi'_4 : \Omega^1(Y, su(2)) \to \mathcal{H}^1_{A,h}(Y, \partial Y; \mathbf{C})$ is orthogonal projection. Such a composition can be any complex linear map from $\mathcal{H}^1_{A,h}(Y; \mathbf{C})$ to $\mathcal{H}^1_{A,h}(Y, \partial Y; \mathbf{C})$ which, when precomposed with the inclusion of $\mathcal{H}^1_{A,h}(Y, \partial Y; \mathbf{C})$ into $\mathcal{H}^1_{A,h}(Y; \mathbf{C})$, is Hermitian.

For each $k = 1, \ldots, n$, let

$$N_k = \{B \in \operatorname{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y; \mathbf{C})) |\ \Pi'_4 \circ B \text{ has complex rank } n-k\}.$$

For each k, N_k is a submanifold of $\operatorname{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y; \mathbf{C}))$ of codimension $k^2 + 2k(g-1)$. Note that for the Hessian of a small perturbation to be less than full rank is equivalent to the existence of nontrivial $\mathcal{H}^1(Y, \partial Y; \mathbf{C})$.

By Lemma 38, P_4 is a submersion in some neighborhood of (A, h) which we may as well take to be a product neighborhood $V_4 \times U_4$.

$$Z_4 = P_4^{-1}(\{0\} \times \operatorname{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y; \mathbf{C}))) \cap (V_4 \times U_4)$$

is a smooth submanifold of $V_4 \times U_4$. By the argument sketched before the statement of this theorem, the projection from this submanifold to U_4 has index g.

Consider now the smaller set $P_4^{-1}(\{0\} \times N_k)$. This must also be a submanifold, and its projection to U_4 has index $g - (k^2 + 2k(g-1)) = (1-2k)(g-1) + 1 - k^2$, which is negative if $k > 1$. If $k = 1$, this index is negative unless $g = 1$, in which case it is 0.

In the $g > 1$ case, an application of the Sard-Smale Theorem now implies that for generic $h' \in U_4$ the cohomology $\mathcal{H}^1_{A',h'}(Y;\mathbf{C})$ vanishes for all $[A'] \in Z_4 \cap (V_4 \times \{h'\})$. In the $g = 1$ case, there will only be finitely many points in $Z_4 \cap (V_4 \times \{h'\})$ where $\mathcal{H}^1_{A',h'}(Y;\mathbf{C}) \cong \mathcal{H}^1_{A',h'}(Y,\partial Y;\mathbf{C})$ is nonzero, and at those points $\mathcal{H}^1_{A',h'}(Y;\mathbf{C}) = \mathbf{C}$. Transversality of the intersection of $P_4(Z_4 \cap (V_4 \times \{h'\})) \cap (\{0\} \times N_1)$ means that the Kuranishi picture for $\mathcal{M}_{h'} \cap V_4$ near such a point where the complex cohomology jumps is that of the zero set of the map $\mathbf{R} \times \mathbf{C} \to \mathbf{C}$ given by $(t,z) \mapsto tz$. Dividing out by the $U(1)$ stabilizer completes proof of the theorem. □

7.5. The $(SU(2), SU(2))$-Stratum and its Normal Bundle. In this subsection we describe a neighborhood of the central stratum for generic perturbations. The ideas in the proof are very similar to parts of the arguments in the previous subsection. We will highlight the differences. One difference simplifies our work. That is that the central part of $\mathcal{M}_h$ does not change under perturbation. This is due to the fact that ∇h vanishes at any central connection.

Let h be any admissible perturbation. Enlarge the collection of solid tori used to define h until the cores of the solid tori span $H_1(Y;\mathbf{R})$. Call the component of $\mathcal{H}$ corresponding to this new collection $\mathcal{H}_5$.

Lemma 40. *Let A be a flat central connection. Then there is a neighborhood $U_5 \subset \mathcal{H}_5$ of h such that for an open dense set of $h' \in U_5$, $\mathcal{H}^1_{A,h'}(Y,\partial Y) = 0$.*

Proof. Let $\Pi_5' : \Omega^1(Y, su(2)) \to \mathcal{H}^1_{A,h}(Y,\partial Y)$ denote the orthogonal projection. Then $\mathcal{H}^1_{A,h'}(Y,\partial Y) = 0$ if and only if $\Pi_5' \circ \mathrm{Hess}\, h'$ is surjective when restricted to $\mathcal{H}^1_{A,h}(Y)$. In the discussion preceding Proposition 67, we describe the space of all maps $B : \mathcal{H}^1_A(Y) \to \mathcal{H}^1_A(Y)$ which can arise as these Hessians. By analogy to the previous subsection, the composition of one of these maps with Π_5' can be any $SU(2)$ equivariant map from $\mathcal{H}^1_A(Y)$ to $\mathcal{H}^1_A(Y, \partial Y)$ whose precomposition with the inclusion $\mathcal{H}^1_A(Y, \partial Y) \to \mathcal{H}^1_A(Y)$ is symmetric. An easy dimension count shows that the codimension of the stratum of maps which don't have full rank is $g+1$. Now an easier version of the proof of Theorem 39 gives the result. (Here everything is much simpler because we don't actually perturb the central stratum, just the Hessian at the central point.) □

Corollary 41. *For generic $h' \in \mathcal{H}_5$, $[A]$ has a neighborhood in $\mathcal{M}_{h'}$ homeomorphic to $\mathbf{R}^g \otimes su(2)/SU(2)$. The abelian stratum corresponds to the image of the set of decomposable elements of $\mathbf{R}^g \otimes su(2)$ in the quotient. Thus, the abelian stratum near $[A]$ is homeomorphic to $\mathbf{R}^g/\mathbf{Z}_2$.*

7.6. Proof of Theorem 15. In this subsection we combine all the earlier local results in this section to prove a global theorem, Theorem 15. We first restate the genericity hypothesis in this theorem more precisely.

Clarification of Theorem 15. *Let (h, ϕ) be an arbitrary admissible function. Then ϕ can be enlarged to a finite collection $\phi \cup \psi$ such that there is a neighborhood $U_0 \subset \mathcal{H}_{(\phi\cup\psi)}$ of h with the following property. For any element h' of an open dense subset of U_0, $\mathcal{M}_{h'}$ has the structure described in Theorem 15.*

Proof of Theorem 15. The idea of the proof is straightforward. The compactness of $\mathcal{M}_h$ insures that we can find a finite collection of points $\{[A_i]\}_{i=1}^n \in \mathcal{M}_h$ such that the corresponding neighborhoods $\bar{V}_{[A_i]} \subset \mathcal{B}$ constructed in the local results cover $\mathcal{M}_h$. The only thing to check is that, in the local results, adding more solid tori to ϕ does not force us to take smaller neighborhoods $\bar{V}_{[A]}$.

Lemma 42. *For any of the local results in this section, the neighborhood $\bar{V} \subset \mathcal{B}$ may be chosen so that, if ϕ is enlarged to any $\phi \cup \psi$, there is still some neighborhood $U' \subset \mathcal{H}_{(\phi\cup\psi)}$ of h so that $\bar{V} \times U'$ satisfies the result.*

Proof of Lemma 42. The general setup in all the local results is the same. We have manifolds W_1 and W_2 and a submanifold $W_3 \subset W_2$ and we consider a map $P : W_1 \times \mathcal{H}_{(\phi\cup\psi)} \to W_2$ (or P_i for some i) which is a submersion at one point $(A, h) \in W_1 \times \mathcal{H}_{(\phi\cup\psi)}$. We let $Z = P^{-1}(W_3)$. The implicit function theorem shows that Z is a submanifold in some neighborhood $V \times U$ of (A, h). By taking a slightly smaller neighborhood V' with closure$(V') \subset V$, we can get a uniform lower bound ϵ on the norm of a one-sided inverse for the Hessian of P restricted to the normal directions to Z on the compact set $Z \cap (\text{closure}(V') \times \{h\})$.

If a new solid torus γ_0 is added to ϕ and the function

$$\int_{D^2} \bar{h}_0(\text{tr hol}_{\gamma'}(x, A))\eta(x)$$

is added to the old perturbation function h, the difference between the new Hessian of the function P and the old Hessian is bounded by the C^2 norm of $\bar{h}_0$. With this in mind, we define for each $\delta > 0$ the neighborhood

$$U(\delta, \bar{h}) = \{\bar{h}' \in \bar{\mathcal{H}} |\ \|\bar{h} - \bar{h}'\|_{C^2([-2,2])} < \delta\}.$$

For an admissable function $h = (\bar{h}_1, \ldots, \bar{h}_m)$ defined using solid tori $\phi = \{\gamma_i\}_{i=1}^m$, we define a neighborhood

$$U(\epsilon, h) = \prod_{i=1}^{m} U(\frac{\epsilon}{2^i}, \bar{h}_i) \subset \bar{\mathcal{H}}^m = \mathcal{H}_\phi.$$

It follows that P is a submersion along $Z \cap (V' \times U_\epsilon(h))$, regardless of the number n of components in Γ. Note that Lemma 33 and its generalizations in the other subsections depend only on this property of P. This proves the lemma. □

Now we finish the proof of Theorem 15. The local results together with Lemma 42 imply that we can enlarge ϕ once and for all (combining the enlargements of the n applications of the local results to $[A_1], \ldots, [A_n]$) to a collection $\phi \cup \psi = \{\gamma_i\}_{i=1}^m$. There are neighborhoods $\bar{V}_0 = \cup_{i=1}^n \bar{V}_{[A_i]} \subset \mathcal{B}$ of $\mathcal{M}_h$ and $U_0 = U(\epsilon_0, h) \subset \mathcal{H}_{(\phi\cup\psi)}$ of h, where ϵ_0 is the minimum of the lower bounds ϵ on the sets $\bar{V}_{[A_i]} \cap \mathcal{M}_h$, such that for generic $h' \in U_0$, $\mathcal{M}_{h'} \cap \bar{V}_0$ has the structure described in the theorem.

It is now sufficient to show that for h' close enough to h, $\mathcal{M}_{h'} \subset \bar{V}_0$. Suppose that this were not true. Then we could find a sequence $h_i' \to h$ and a sequence $[A_i] \in \mathcal{M}_{h_i'} \backslash \bar{V}_0$. By Corollary 25, a subsequence of $[A_{i(j)}]$ of the latter sequence must converge to some $[A_\infty]$, which, by the continuity of P, must lie in $\mathcal{M}_h$. This implies that for $i(j)$ large enough $[A_{i(j)}] \in \bar{V}_0$, giving a contradiction. □

8. Legendrian Submanifolds and Cobordisms

In this section we examine the symplectic properties of the restriction map $r : \mathcal{M}_h \to \mathcal{M}_\Sigma$. In the first subsection, we show that $r \circ s_h : \mathcal{M}_h \to \mathcal{L}_\Sigma$ is Legendrian. In the second subsection, we show that when the perturbation h is varied, $\mathcal{M}_h$ changes by a Legendrian cobordism. Of course, technically, these results must be stated with greater care due to the singularities.

8.1. Symplectic Properties of the map $r : \mathcal{M}_h \to \mathcal{M}_\Sigma$. We begin by making an observation which follows immediately from the definition of ζ_h as the gradient of $\tilde{s}$ with respect to the connection ω_Y.

Proposition 43. *If $\gamma : [0,1] \to \mathcal{A}$ is a smooth curve such that $\zeta_h(\gamma(t)) = 0$ for all $t \in [0,1]$, then $\tilde{s}_h \circ \gamma$ is a horizontal lift of γ to $\mathcal{A} \times U(1)$. Consequently,*

$$r \circ \tilde{s}_h \circ \gamma : [0,1] \to \mathcal{A}_\Sigma \times U(1)$$

is a horizontal lift of $r \circ \gamma$.

Combining Proposition 43 with the immersion results of Section 7 gives:

Corollary 44. *For generic h, the compositions $r \circ s_h : \mathcal{M}_h \to \mathcal{L}_\Sigma$ are Legendrian when restricted to the $(\mathbf{Z}_2, \mathbf{Z}_2)$- and $(U(1), U(1))$-strata and, in the $g = 1$ case, also the $(\mathbf{Z}_2, U(1))$-stratum.*

Corollary 45. *Let $\gamma : [0,1] \to \mathcal{M}_h$ be a closed continuous curve which is the image of a piecewise smooth path $\tilde{\gamma} : [0,1] \to \zeta_h^{-1}(0)$ with $g^*\tilde{\gamma}(0) = \tilde{\gamma}(1)$. Choose a smooth path $g_t : [1,2] \to \mathcal{G}$ with $g_1 = \mathrm{id}$ and $g_2 = g$, and define $\bar{\gamma} : [0,2] \to \zeta_h^{-1}(0)$ to be*

$$\bar{\gamma}(t) = \begin{cases} \tilde{\gamma}(t) & \text{if } 0 \le t \le 1 \\ g_t^*\tilde{\gamma}(1) & \text{if } 1 < t \le 2. \end{cases}$$

Then $r \circ \bar{\gamma} : S^1 \to \mathcal{F}_\Sigma$ extends to a map $S : D^2 \to \mathcal{F}_\Sigma$, and for any such map

$$\int_S \Omega = 0 \text{ modulo } 2\pi.$$

Proof. The fact that $\mathcal{F}_\Sigma$ is simply connected is known [D]. The basic argument is that if $W \subset \mathcal{A}_\Sigma$ is the set of higher index critical points of the Yang-Mills functional, then W has codimension greater than 2. Thus $\mathcal{A}_\Sigma \setminus W$ is simply connected. The Yang-Mills flow gives a deformation retraction onto $\mathcal{F}_\Sigma$. Since $D_{\omega_Y}\tilde{s} = 0$ along $\tilde{\gamma}$,

$$\tilde{s}(\tilde{\gamma}(0)) = \tilde{s}(\tilde{\gamma}(2)) = \mathrm{hol}_{\tilde{\gamma}}(\omega_Y)\tilde{s}(\tilde{\gamma}(0)).$$

But

$$\mathrm{hol}_{\tilde{\gamma}}(\omega_Y) = \mathrm{hol}_{r\circ\tilde{\gamma}}(\omega) = \exp(-i\int_S \Omega). \quad \square$$

Remark. If S maps the complement of a measure zero set in D^2 to one stratum of $\mathcal{F}_\Sigma$, then by symplectic reduction the integral can be performed in $\mathcal{M}_\Sigma$.

If $g = 1$, then $\mathcal{M}_\Sigma$ is equal to $T^2/\mathbf{Z}_2$. This is topologically 2-sphere but with 4 special points, the fixed points of the involution. It is often referred to as the *pillowcase.*

Corollary 46. *If $g = 1$, and $\gamma : [0,1] \to \mathcal{M}_h$ is a closed curve, then $r \circ \gamma$ bounds zero symplectic area modulo the symplectic area of $\mathcal{M}_\Sigma$.*

Proof. It is sufficient to verify that the symplectic area of $\mathcal{M}_\Sigma$ equals -2π. For simplicity, we perform the calculation on the double cover of $\mathcal{M}_\Sigma$, a torus, which we denote by $\tilde{\mathcal{M}}_\Sigma$. We may think of the $\tilde{\mathcal{M}}_\Sigma$ as the set of equivalence classes of based flat connections with holonomy in a prescribed circle subgroup $U(1) \subset SU(2)$, modulo $U(1)$ gauge equivalence. The lift of the symplectic form Ω on $\tilde{\mathcal{M}}_\Sigma$, which will also be denoted by Ω, is still given by

$$\Omega(\lambda, \mu) = \frac{1}{2\pi}\int_\Sigma \mathrm{tr}(\lambda \wedge \mu)$$

for λ and μ in $\Omega^1(\Sigma, su(2))$.

We will parametrize $\tilde{\mathcal{M}}_\Sigma$ by the set of constant 1-forms with Lie algebra parts lying in the Lie algebra of the prescribed circle subgroup, i.e.

$$\tilde{\mathcal{M}}_\Sigma = \left\{\left(\begin{bmatrix} e^{i2\pi a} & 0 \\ 0 & e^{-i2\pi a}\end{bmatrix} dx, \begin{bmatrix} e^{i2\pi b} & 0 \\ 0 & e^{-i2\pi b}\end{bmatrix} dy\right)\right\}.$$

Here (x, y) are coordinates on the circle such that

$$\int_{S^1} dx = \int_{S^1} dy = 1.$$

Consider the universal cover $f : \mathbf{R}^2 \to \tilde{\mathcal{M}}_\Sigma$ given by

$$f(a,b) = \left(\begin{bmatrix} e^{i2\pi a} & 0 \\ 0 & e^{-i2\pi a}\end{bmatrix} dx, \begin{bmatrix} e^{i2\pi b} & 0 \\ 0 & e^{-i2\pi b}\end{bmatrix} dy\right).$$

Then

$$\begin{aligned}
\int_{\mathcal{M}_\Sigma} \Omega &= \frac{1}{2}\int_{\tilde{\mathcal{M}}_\Sigma} \Omega \\
&= \frac{1}{2}\int_{[0,1]\times[0,1]} f^*(\Omega)(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}) da \wedge db \\
&= \frac{1}{2}\int_{[0,1]\times[0,1]} \Omega\left(\begin{bmatrix} i2\pi & 0 \\ 0 & -i2\pi \end{bmatrix} dx, \begin{bmatrix} i2\pi & 0 \\ 0 & -i2\pi \end{bmatrix} dy\right) da \wedge db \\
&= \frac{1}{2}\left(\int_{[0,1]\times[0,1]} da \wedge db\right) \Omega\left(\begin{bmatrix} i2\pi & 0 \\ 0 & -i2\pi \end{bmatrix} dx, \begin{bmatrix} i2\pi & 0 \\ 0 & -i2\pi \end{bmatrix} dy\right) \\
&= \frac{1}{2}\cdot 1 \cdot \frac{1}{2\pi}\int_\Sigma \operatorname{tr}\left(\begin{bmatrix} i2\pi & 0 \\ 0 & -i2\pi \end{bmatrix}\begin{bmatrix} i2\pi & 0 \\ 0 & -i2\pi \end{bmatrix}\right) dx \wedge dy \\
&= \frac{1}{4\pi}(-8\pi^2) = -2\pi. \quad \square
\end{aligned}$$

8.2. The Legendrian Cobordism Equivalence Relation. In [A], oriented Legendrian curves in $\mathbf{R}^3$ (with the contact form $dz - ydx$) are completely classified up to oriented Legendrian cobordism (which we will from now on refer to simply as cobordism). This involves describing a series of cobordisms which generate the entire group of cobordisms.

By the Darboux Theorem any 3-dimensional contact manifold is locally contactomorphic to $\mathbf{R}^3$, and so the group of cobordisms in an arbitrary 3-dimensional contact manifold is generated by the same list of local operations.

Theorem 47. *(see Section F of [A]) Two oriented Legendrian submanifolds of a 3-dimensional contact manifold are oriented Legendrian cobordant if and only if they differ by a sequence of the following moves:*

(a) *Isotopy through immersed Legendrian curves.*
(b) *Birth or death of small immersed Legendrian circle components.*
(c) *The switch of a nontransverse crossing described in Figure 1, performed when the two parts of (possibly the same component of) the Legendrian curve intersect and their oriented tangents agree at the point of intersection.*

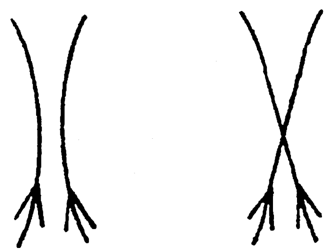

Figure 1

In our $g = 1$ case, where the contact manifold is a $U(1)$ bundle $p : L \to M$ over a symplectic 2-manifold M, then these moves may be described in terms of the projection of the Legendrian curve into M. Note that if $\tilde{g} : N \to L$ is Legendrian then $g = p \circ \tilde{g}$ is Lagrangian and each null homologous closed component of $g(N)$ bounds zero symplectic area modulo 2π. All other Legendrian lifts of such a g differ by $U(1)$ rotations.

The corresponding moves on Lagrangian submanifolds are

(a') *Isotopy through immersed Lagrangian submanifolds such the annulus traced out by the isotopy has zero symplectic area.*

(b') *Birth or death of small figure-eights which bound zero symplectic area.*

(c') *Oriented band connect sum by a band with a half twist when every null homologous components created by the switch bounds zero symplectic area modulo* 2π.

The integrality condition on the symplectic area bounded by any closed component generalizes to higher dimensions in the following sense. If $\tilde{g} : N \to L$ is any Legendrian submanifold of a contact $U(1)$ bundle $p : L \to M$, then for any curve $\gamma \subset N$ whose image $g(\gamma)$ is null homologous $g(\gamma)$ bounds zero symplectic area modulo 2π. Thus there is an immediate generalization of move (a') to higher dimensions. The remaining moves have yet to be classified for contact manifolds of dimensions $6g - 5$ when $g \geq 2$. These are the other dimensions with which we will be concerned in this paper.

8.3. The Cobordism Theorem. To define a Legendrian cobordism between Legendrian submanifolds of $\mathcal{L}_\Sigma$, we consider the larger bundle $\mathcal{L}_\Sigma \times T^*[0,1]$ over $\mathcal{M}_\Sigma \times T^*[0,1]$. We obtain a natural contact structure on $\mathcal{L}_\Sigma \times T^*[0,1]$ corresponding to the connection 1-form $\pi_1^*\omega_\Sigma - i\pi_2^*(\nu dt)$.

To form cobordisms, we must consider paths of perturbations. The basic paths we work with will be C^2 paths $h_t : [0,1] \to \mathcal{H}$. Given any two admissible perturbations defined using the same collection of solid tori, it is easy to find such a path connecting them. In fact, if h_0 and h_1 are defined using ϕ_0 and ϕ_1, respectively, where $\phi_0 \cup \phi_1$ is a disjoint collection solid tori, we can identify them with corresponding perturbations in a single larger component $\phi_0 \cup \phi_1$. A problem arises, however, if a solid torus in ϕ_0 intersects one in ϕ_1 but the two solid tori are not equal.

To handle this situation, we must consider a slightly more general type of path. Given any two collections ϕ_0 and ϕ_1, there is a smooth isotopy of one which makes it disjoint from the other. Combining the functions of trace for the corresponding perturbation with this isotopy gives a path.

Lemma 48. *If (h, ϕ) is nondegenerate in the sense that it is a regular value of all the projections to $\mathcal{H}$ used in the proof of Theorem* 15, *then for any ϕ' isotopic to ϕ, $\mathcal{M}_{(h,\phi)}$ and $\mathcal{M}_{(h,\phi')}$ are canonically diffeomorphic in the stratified sense by a map Φ such that $r \circ \Phi = r$.*

The proof of this lemma requires a straightforward generalization of the argument to follow, which covers the basic type of paths.

Remark. Alternatively, we could choose a slightly different perturbation space, namely $\coprod_{i=1}^\infty \mathrm{Emb}(S^1 \times D^2, Y) \times \bigoplus_{i=1}^\infty \bar{\mathcal{H}}$. Then we would have to arrange our paths to stay within the subset where the solid tori for which the functions $\bar{h}_i$ don't vanish are disjoint.

A path of perturbations $h_t : [0,1] \to \mathcal{H}$ determines a section $s_{\{h_t\}}$ of $\mathcal{L}_Y \times [0,1]$, namely the one induced by the $\mathcal{G}$ equivariant map

$$\tilde{s}_{\{h_t\}}(A,t) = e^{i(CS(A)+h_t(A))} : \mathcal{A} \times [0,1] \to U(1).$$

Let

$$\mathcal{M}_{\{h_t\}} = \{([A],t) \in \mathcal{B}_Y \times [0,1] |\ \zeta_{h_t}(A) = 0\}.$$

As an immediate consequence of Corollary 25 we have the following fact.

Proposition 49. *The set $\mathcal{M}_{\{h_t\}}$ is compact.*

To prove nondegeneracy results about paths near a fixed path h_t, h_t must sit inside a complete Banach manifold of paths so that arguments analogous to the proof of Theorem 15 can be used. For basic paths, where the collection ϕ remains fixed, we can take for this space of paths $C^2([0,1],\mathcal{H}_\phi)$. (For the paths of the isotopy type, where the functions $\bar{h}_i$ are don't vary, we may take the space of paths of perturbations in which the functions $\bar{h}_i$ are constant in t.)

Let

$$\mathcal{M}^{(G,H)}_{\{h_t\}} = \mathcal{M}_{\{h_t\}} \cap (\mathcal{B}^{(G,H)} \times [0,1]).$$

Theorem 50. *Any two admissible perturbations h_0 and h_1 which are generic in the sense of Theorem 15 can be connected by a path h_t such that $\mathcal{M}_{\{h_t\}}$ is a stratified space with the following structure:*

(a) *If $g > 3$, then*

$$\mathcal{M}_{\{h_t\}} = \mathcal{M}^{(\mathbf{Z}_2,\mathbf{Z}_2)}_{\{h_t\}} \coprod \mathcal{M}^{(U(1),U(1))}_{\{h_t\}} \coprod \mathcal{M}^{(SU(2),SU(2))}_{\{h_t\}}.$$

The $(\mathbf{Z}_2,\mathbf{Z}_2)$-stratum is $(3g-2)$-dimensional. The $(U(1),U(1))$-stratum is $(g+1)$-dimensional and has normal bundle in the $(\mathbf{Z}_2,\mathbf{Z}_2)$-stratum with fiber $c(\mathbf{C}P^{(g-1)})$. The $(SU(2),SU(2))$-stratum is 1-dimensional. It has normal bundle in the $(\mathbf{Z}_2,\mathbf{Z}_2)$-stratum with fiber $(\mathbf{R}^g \otimes su(2))^\sharp/SU(2)$ and normal bundle in the $(U(1),U(1))$-stratum with fiber $(\mathbf{R}^g \otimes su(2))^\flat/SU(2)$.

(b) *If $g = 3$ or $g = 2$, then*

$$\mathcal{M}_{\{h_t\}} = \mathcal{M}^{(\mathbf{Z}_2,\mathbf{Z}_2)}_{\{h_t\}} \coprod \mathcal{M}^{(\mathbf{Z}_2,U(1))}_{\{h_t\}} \coprod \mathcal{M}^{(U(1),U(1))}_{\{h_t\}} \coprod \mathcal{M}^{(SU(2),SU(2))}_{\{h_t\}}.$$

The $(\mathbf{Z}_2,\mathbf{Z}_2)$-, $(U(1),U(1))$-, and $(SU(2),SU(2))$-strata are as in (a). The $(\mathbf{Z}_2,U(1))$-stratum is $(3-g)$-dimensional with normal bundle in the $(\mathbf{Z}_2,\mathbf{Z}_2)$-stratum with fiber $c(S^{(2g-3)} \times S^{(2g-3)})$.

(c) *If $g = 1$, then*

$$\mathcal{M}_{\{h_t\}} = \mathcal{M}^{(\mathbf{Z}_2,U(1))}_{\{h_t\}} \coprod \mathcal{M}^{(U(1),U(1))}_{\{h_t\}} \coprod \mathcal{M}^{(SU(2),SU(2))}_{\{h_t\}} \coprod \mathcal{M}^{(U(1),SU(2))}_{\{h_t\}}.$$

The $(\mathbf{Z}_2, U(1))$*- and the* $(U(1), U(1))$*-strata are* 2*-dimensional. The boundary of the* $(\mathbf{Z}_2, U(1))$*-stratum is a smooth* 1*-dimensional submanifold of the* $(U(1), U(1))$*-stratum, with normal bundle fiber* $c(pt)$*. The* $(SU(2), SU(2))$*-stratum is a union of* 1*-dimensional arcs which are constant in* t*. It has normal bundle in the* $(U(1), U(1))$*-stratum with fiber* $c(pt)$*. The* $(U(1), SU(2))$*-stratum is* 0*-dimensional. Each point in this stratum has a neighborhood in* $\mathcal{M}_{\{h_t\}}$ *consisting of a* 2*-dimensional half-ball whose straight edge lies in another* 2*-dimensional ball. The interior of the half-ball lies in the* $(\mathbf{Z}_2, U(1))$*-stratum. One point in the arc of intersection is the* $(U(1), SU(2))$ *point, and the remainder of the* 2*-dimensional ball lies in the* $(U(1), U(1))$*-stratum.*

(d) *If* $g = 0$*, then*

$$\mathcal{M}_{\{h_t\}} = \mathcal{M}_{\{h_t\}}^{(\mathbf{Z}_2, SU(2))} \coprod \mathcal{M}_{\{h_t\}}^{(U(1), SU(2))} \coprod \mathcal{M}_{\{h_t\}}^{(SU(2), SU(2))}.$$

The $(\mathbf{Z}_2, SU(2))$*-,* $(U(1), SU(2))$*- and* $(SU(2), SU(2))$*-stratum are all* 1*-dimensional. The* $(\mathbf{Z}_2, SU(2))$*-stratum meets the* $(U(1), SU(2))$*-stratum at isolated interior points. The* $(U(1), SU(2))$*-stratum meets the* $(SU(2), SU(2))$*-stratum at isolated interior points.*

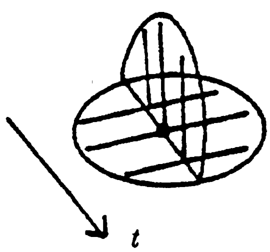

Figure 2a

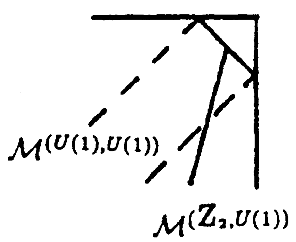

Figure 2b

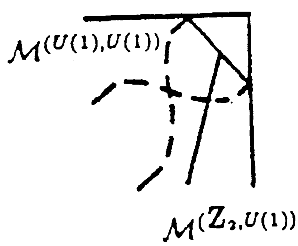

Figure 2c

Remark on the genus 1 case. Figure 2a illustrates a neighborhood of a point in the $(U(1), SU(2))$-stratum. Figures 2b and 2c show the pictures of the image in the pillowcase for t-values just before and just after this point in the cobordism.

Remark on the genus 0 case. If $g = 0$ the theorem basically says that just keeping track of the cobordism class of $\mathcal{M}_h$, i.e. the number of points in the different strata, yields no information. We will describe an example in Section 11 of a closed 3-manifold on which we can explicitly describe a path of perturbations during which a $(\mathbf{Z}_2, SU(2))$ point is cobordant to two $(U(1), SU(2))$ points, and then one of the $(U(1), SU(2))$ points and an $(SU(2), SU(2))$ point are cobordant to a single $(SU(2), SU(2))$ point.

Proof. The proof works just like in the single perturbation case. The only difference now is that there is one more dimension in the domain, and thus the index computations all change by 1.

For example, let $[A] \in \mathcal{M}_{h_t}$ be irreducible and boundary-irreducible. The linearization of

$$\Pi_A \zeta_{h_t} : X_A \times [0,1] \to \ker d_A^* \cap L_1^2\Omega^1(Y, su(2))$$

at (A, t) is

$$(\delta A, \delta t) \mapsto *d_{A,h_t}(\delta A) + \nabla \frac{\partial}{\partial t} h_t(A)\delta t.$$

The index of

$$\Pi_A \circ (*d_{A,h_t} \oplus 0) : T_A X_A \oplus T_t\mathbf{R} \to \ker d_A^* \cap L_1^2\Omega^1(Y, su(2)),$$

is $3g - 2$. The map

$$\delta t \mapsto \nabla \frac{\partial}{\partial t} h_t(A)\delta t$$

has finite dimensional range and hence is a compact operator and does not affect the index.

The abelian boundary-central singularities in the genus 1 case require more explanation, since they did not arise in the single perturbation theorem. We provide this explanation next.

For the moment, we consider $U(1)$ connections modulo $U(1)$ gauge transformations, as in Section 7.4. Let A be a perturbed flat abelian boundary-central connection for a fixed (h_{t_0}, ϕ) in a fixed path (h_t, ϕ). Assume that ϕ has been enlarged as in Corollary 64.

We roughly follow the proof of Theorem 39, replacing the function P_4 by the corresponding function

$$P_6 : (X_A \cap \Omega^1(Y,\mathbf{R})) \times [0,1] \times C^2([0,1],\mathcal{H}) \\ \to \ker d_A^* \cap \Omega^1(Y,\mathbf{R}) \times \mathrm{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y;\mathbf{C}))$$

given by

$$(A',t',h_t') \mapsto (\Pi_4 \circ \Pi_A \circ \zeta_{h'(t')}(A'), \mathrm{Hess}\, h'_{t'}(A')).$$

Exactly the same argument shows that P_6 is a submersion near (A,t_0,h_t).

We will, as before, obtain results about nearby irreducibles by first determining what the complex cohomology along the abelian stratum is. In the present case, however, we must deal with extra cokernel coming from the image of $\mathcal{H}^0(\Sigma) \cong \mathbf{R}\oplus\mathbf{C} \xrightarrow{d_A} \mathcal{H}^1_{A,h}(Y,\partial Y)$. Let $\gamma_1 \in \Omega^0(Y,\mathbf{R})$ be a generator for $\mathcal{H}^0_{A,h}(Y)$ and let $\gamma_2,\gamma_3 \in \Omega^0(Y,\mathbf{C})$ be forms such that $\{\gamma_2|_\Sigma, \gamma_3|_\Sigma\}$ span $\mathcal{H}^0(\Sigma;\mathbf{C})$. Let Π_6 denote the orthogonal projection $\Omega^1(Y,su(2)) \to \mathcal{H}^1(Y,\partial Y;\mathbf{C}) \cap (d_A\gamma_2)^\perp \cap (d_A\gamma_3)^\perp$. Instead of the singular strata $N_k \subset \mathrm{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y;\mathbf{C}))$ discussed in Theorem 39, we consider

$$N_k' = \{B \in \mathrm{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y;\mathbf{C})) |\ \Pi_6 \circ B \text{ has complex rank } n-k\}.$$

(Here n again denotes the complex dimension of $\mathcal{H}^1(Y;\mathbf{C})$.) N_k' is a submanifold of codimension k^2+2k (recall that we are in the $g=1$ case; $\Pi_6 \circ B$ is a map from $\mathbf{C}^n \oplus \mathbf{C}$ to $\mathbf{C}^n$ which is Hermitian on $\mathbf{C}^n$). Let

$$Z_6 = P_6^{-1}(\{0\} \times \mathrm{Sym}^{U(1)}(\mathcal{H}^1_{A,h}(Y;\mathbf{C}))).$$

(For ease of notation, we will not bother to keep track of the neighborhoods in which Z_6, for example, is a manifold. These details are completely analogous to the single perturbation case.) The standard argument now shows that for generic paths h_t' near h_t,

$$Z_6(h_t') = Z_6 \cap ((X_A \cap \Omega^1(Y,\mathbf{R})) \times [0,1] \times \{h_t'\})$$

is a smooth 2-dimensional submanifold near (A,t_0,h_t) whose image misses $\{0\} \times N_k'$ for all $k \geq 1$. In other words, for (A',t',h_t') in this submanifold, $\mathcal{H}^1_{A',h'_{t'}}(Y,\partial Y;\mathbf{C}) \cap (d_A\gamma_2)^\perp \cap (d_A\gamma_3)^\perp = \{0\}$.

Next we show that for generic paths the boundary-central points in this 2-dimensional submanifold are isolated. This amounts to observing that Z_6 maps surjectively onto $\mathcal{H}^1(\Sigma;\mathbf{R})$ (by the argument in Lemma 33 applied to the $\mathbf{R}$-valued forms). This implies that the boundary-central points of Z_6 form a submanifold of Z_6 of codimension 2.

Fix a path h'_t which is generic in both the above senses. Choose any abelian boundary-central connection, which we now call A, with $[A] \in \mathcal{M}_Y^{\{h'_t\}}$. Redefine Z_6, etc., for this choice of A.

We first verify that the entire 2-manifold $Z_6(h'_t)$ lies in the cobordism $\mathcal{M}_Y^{\{h'_t\}}$. We must check that the remaining equations $\langle \zeta_{h'_t}, d_A\gamma_2\rangle = \langle \zeta_{h'_t}, d_A\gamma_3\rangle = 0$ are already satisfied by everything in this set. This follows from the fact that $\zeta_{h'_t}(A) \in \Omega^1(Y,\mathbf{R})$ and $d_A\gamma_2, d_A\gamma_3 \in \Omega^1(Y,\mathbf{C})$.

We still need to determine the nearby irreducibles. To this end, we broaden our attention now to the entire slice at A. It follows from the above arguments that the zero set in $X_A \times [0,1]$ of the projection of $\zeta_{h'_t}$ to $\ker d_A^* \cap \Omega^1(Y, su(2)) \cap (d_A\gamma_2)^\perp \cap (d_A\gamma_3)^\perp$ is a smooth 4-dimensional manifold, modelled on the 2-manifold $Z_6(h'_t)$ crossed with $\mathbf{C}$. The stabilizer of A acts on the second factor with weight two.

Finally, we examine the zero set in this 4-manifold of the two remaining components of $\zeta_{h'_t}$. By an argument similar to that in Section 7.2, in a neighborhood of (A,t) this zero set corresponds to the set of tangent vectors (α,τ) at (A,t) such that

$$[\alpha \wedge \alpha]|_\Sigma \in \mathbf{R} \subset \mathcal{H}^2(\Sigma) \cong \mathbf{R} \oplus \mathbf{C}.$$

This implies that near abelian boundary-central points the cobordism has the structure described in the theorem. □

We next define a lift

$$\lambda : \mathcal{M}_{\{h_t\}} \to \mathcal{L}_\Sigma \times T^*[0,1]$$

of r as follows. First, we let

$$\tilde{\lambda} : \mathcal{A} \times [0,1] \to \mathcal{A}_\Sigma \times U(1) \times T^*[0,1]$$

be given by

$$\tilde{\lambda}(A,t,u) = (A|_\Sigma, \tilde{s}_{h_t}(A), t, \frac{\partial}{\partial t} h_t(A)).$$

Since $\tilde{\lambda}$ is equivariant with respect to the gauge group actions, it induces a map on the quotient, and we let λ be the restriction of this induced map to $\mathcal{M}_{\{h_t\}}$.

Theorem 51. *If $\mathcal{M}_{\{h_t\}}$ is a nondegenerate cobordism as in Theorem 50, then $\lambda : \mathcal{M}_{\{h_t\}} \to \mathcal{L}_\Sigma$ is a Legendrian cobordism on the $(\mathbf{Z}_2, \mathbf{Z}_2)$- and $(U(1), U(1))$-strata, and in the $g = 1$ case the $(\mathbf{Z}_2, U(1))$-stratum. The other strata are also horizontal.*

Proof. We must show that λ is an immersion on each of these strata and that λ is horizontal. We will prove that the restriction of λ to the $(\mathbf{Z}_2, \mathbf{Z}_2)$-stratum is an immersion. The proofs for the other strata are easy generalizations.

Let $([A], t_0) \in \mathcal{M}_{\{h_t\}}$. Either

$$T_{([A],t_0)}\mathcal{M}_{\{h_t\}} = (\mathcal{H}^1_{A,h_{t_0}}(Y) \oplus \{0\}) + \operatorname{span}(a, \delta t)$$

for some $(a, \delta t) \in T_A X_A \oplus T_{t_0}[0,1]$ with $\delta t \neq 0$ which satisfies

$$*d_{A,h_{t_0}} a + \nabla \frac{\partial}{\partial t}|_{t=t_0} h_t(A) \delta t = 0,$$

or else

$$T_{([A],t_0)}\mathcal{M}_{\{h_t\}} = \mathcal{H}^1_{A,h_{t_0}}(Y) \oplus \{0\}.$$

In the first case, $r_* : \mathcal{H}^1_{A,h_{t_0}}(Y) \to \mathcal{H}^1(\Sigma)$ is injective and $\lambda_*(a, \delta t)$ has nontrivial δt component. In the second case, the kernel of $r_* : \mathcal{H}^1_{A,h_{t_0}}(Y) \to \mathcal{H}^1(\Sigma)$ is 1-dimensional, say spanned by α. Then $\alpha = \beta + d_A b$ for some nonzero $\beta \in \mathcal{H}^1_{A,h_{t_0}}(Y, \partial Y)$ and some $b \in \Omega^0(Y, su(2))$. But $0 \neq \langle \nabla \frac{\partial}{\partial t}|_{t=t_0} h_t(A), \beta \rangle = \langle \nabla \frac{\partial}{\partial t}|_{t=t_0} h_t(A), \alpha \rangle$. Thus,

$$\tilde{\lambda}_*(\alpha, 0) = (0, 0, 0, \langle \nabla \frac{\partial}{\partial t}|_{t=t_0} h_t(A), \alpha \rangle) \neq 0.$$

Next we show that the tangent vectors to any stratum are horizontal. This has already been done for those in $\mathcal{H}^1_{A,h_{t_0}}(Y) \oplus \{0\}$. For any tangent vector

$(a, \delta t) \in T_{(A,t_0)}(\mathcal{A} \times [0,1])$,

$$\tilde{\lambda}_*(a, \delta t) = (a|_\Sigma, \tilde{s}_{h_{t_0}}(A) i[\langle \zeta_{h_{t_0}}(A), a\rangle + \frac{1}{4\pi}\int_\Sigma \operatorname{tr}(a, A) + \nabla \frac{\partial}{\partial t}|_{t=t_0} h_t(A)\delta t],$$
$$\delta t, \frac{\partial^2}{\partial t^2}|_{t=t_0} h_t(A)\delta t).$$

If $([A], t_0) \in \mathcal{M}_{\{h_t\}}$ then the $\langle \zeta_{h_{t_0}}(A), a\rangle$ term vanishes. Now suppose $(a, \delta t) \in T_{([A],t_0)}\mathcal{M}_{\{h_t\}}$.

Evaluating the connection 1-form on its image gives

$$(u^{-1}du + \omega - i\nu dt)(\lambda_*(a, \delta t)) = 0. \quad \square$$

9. Orientations

In this section we show that all the nondegenerate flat perturbed moduli spaces (for different $h \in \mathcal{H}$) and the cobordisms between them inherit natural orientations from a choice of orientation for $H^*(Y; \mathbf{R})$ and an orientation for $su(2)$. (These two together induce an orientation for $H^*(Y; su(2))$).

Theorem 52. *An orientation on $H^*_d(Y; \mathbf{R})$ along with an orientation for $su(2)$ induces a natural orientation on any nondegenerate $\mathcal{M}_h^{(\mathbf{Z}_2, \mathbf{Z}_2)}$.*

Proof. We outline a standard argument (see Chapter 5 of [FU], for example, for details). We associate to a family of Fredholm maps $K : X \to \operatorname{Fred}(V, W)$ between real Hilbert spaces its index bundle, a virtual vector bundle over X defined as the formal difference

$$\operatorname{Ind} K_x = \ker K(x) - \operatorname{coker} K(x)$$

(where $\dim\ker K(x)$ is not necessarily continuous but

$$\dim\ker K(x) - \dim\operatorname{coker} K(x)$$

is). An orientation of a virtual bundle is defined to be a section of the determinant line bundle

$$\det(\xi_1 - \xi_2)(x) = \Lambda^{\dim \xi_1(x)}(\xi_1(x)) \otimes (\Lambda^{\dim \xi_2(x)}(\xi_2(x)))^*.$$

When $\xi_2 = 0$ this is equivalent to the usual definition.

The family of Fredholm maps we have in mind is

$$K : \mathcal{A}_Y \times \mathcal{H} \to \operatorname{Fred}(L_2^2\Omega_\tau^0(Y, su(2)) \oplus L_2^2\Omega_\nu^1(Y, su(2)), L_1^2\Omega^0(Y, su(2)) \oplus L_1^2\Omega^1(Y, su(2)))$$

given by

$$K_{(A,h)}(\sigma, \tau) = (d_A^*\tau, d_A\sigma + *d_{A,h}\tau).$$

Notice that for a nondegenerate perturbation h and $A \in \zeta_h^{-1}(0)$,

$$\ker K_{(A,h)} = T_{[A]}\mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)} \text{ and } \operatorname{coker} K_{(A,h)} = 0$$

for each $[A]$ in $\mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)}$.

Since $\mathcal{A} \times \mathcal{H}$ is simply connected, $\operatorname{Ind} K$ is an orientable virtual bundle. A choice of an orientation at one point, for example at the trivial connection A_0 and the zero perturbation, determines one on all of $\operatorname{Ind} K$. Since $\operatorname{Ind} K$ is $\mathcal{G}$ equivariant it descends to a (virtual) bundle over $\mathcal{B}_Y^*$, which we will also call $\operatorname{Ind} K$. Thus we have an inclusion of virtual bundles

$$\begin{array}{ccc} T\mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)} & \longrightarrow & \operatorname{Ind} K \\ \downarrow & & \downarrow \\ \mathcal{M}_h^{(\mathbf{Z}_2,\mathbf{Z}_2)} & \longrightarrow & \mathcal{B}_Y^* \times \mathcal{H}. \end{array}$$

To finish the proof, it is sufficient to show that $\mathcal{G}$ is connected, since this implies that $\mathcal{B}_Y^* \times \mathcal{H}$ is simply connected. A standard argument shows that $\pi_0(\mathcal{G}) = [Y, SU(2)]$ (see [FU], Theorem 5.13) and $[Y, SU(2)] = [Y, S^3]$ consists of a point since Y is has nonempty boundary. □

Next we wish to obtain similar results for the $(\mathbf{Z}_2, U(1))$-stratum and the $(U(1), U(1))$-stratum. In the first case, $\ker K$ is the tangent space, but $\operatorname{coker} K$ is nontrivial (this was the "extra cokernel" we had to project away). The long exact sequence of the pair gives an isomorphism of this cokernel with $\mathcal{H}_A^0(\Sigma)$. In the second case, $\mathcal{H}_A^0(Y) \cong \mathcal{H}_A^0(\Sigma)$. Either $\mathcal{H}_{A,h}^1(Y) \cong \mathbf{R}$ and $\mathcal{H}_{A,h}^1(Y, \partial Y) = 0$ or $\mathcal{H}_{A,h}^1(Y) \cong \mathbf{R} \oplus \mathbf{C}$ and $\mathcal{H}_{A,h}^1(Y, \partial Y) \cong \mathbf{C}$. In either situation, the real part of $\mathcal{H}_{A,h}^1(Y)$ is the tangent space to the abelian stratum. In the latter situation, as we will see below, the complex structures on these copies of $\mathbf{C}$ are canonical given an orientation on $\mathcal{H}_A^0(Y)$. Thus to orient either stratum it is sufficient to orient $\operatorname{Ind} K$ and $\mathcal{H}^0_A(\Sigma)$. This leads us to the following construction.

We begin by fixing an oriented maximal torus $U(1) \subset SU(2)$. This induces a decomposition $su(2) = \mathbf{R} \oplus \mathbf{C}$ where $\mathbf{R} = T_{\mathrm{id}}U(1)$. Next we define

$$\hat{\mathcal{A}} = \{A \in \mathcal{A} |\ A|_\Sigma \in \Omega^1(\Sigma, \mathbf{R})\}$$

$$\hat{\mathcal{G}} = \{g \in \mathcal{G} |\ g|_\Sigma \in \operatorname{Map}(\Sigma, U(1))\}.$$

Several comments are in order here. First, any boundary-abelian $A \in \mathcal{A}$ is gauge equivalent to one in $\hat{\mathcal{A}}$. Second, the only gauge transformations leaving $\hat{\mathcal{A}}$ invariant which are not in $\hat{\mathcal{G}}$ restrict to be the involution on the boundary values of connections in $\hat{\mathcal{A}}$ coming from multiplication of the $\mathbf{R}$ valued forms by -1. By choosing the smaller gauge group, we will obtain the branched double cover $\tilde{\mathcal{M}}_h$ of the union of the boundary-abelian and central strata of $\mathcal{M}_h$ with a restriction map to the branched double cover $\tilde{\mathcal{M}}_\Sigma$ of the abelian and central strata of $\mathcal{M}_\Sigma$. Both are branched along the central stratum.

We first set up the deformation complex for the perturbed flat moduli space in this boundary-abelian framework. We drop the Sobolev norms; they are the same as before. We define for $p = 0, 1$,

$$\hat{\Omega}^p_\nu(Y) = \{(a_1, a_2) \in \Omega^p(Y, \mathbf{R}) \oplus \Omega^p(Y, \mathbf{C}) |\ a_2|_\Sigma = 0\}$$

$$\hat{\Omega}^p_\tau(Y) = \{(a_1, a_2) \in \Omega^p(Y, \mathbf{R}) \oplus \Omega^p(Y, \mathbf{C}) |\ *a_2|_\Sigma = 0\}.$$

For $A \in \zeta_h^{-1}(0) \cap \hat{\mathcal{A}}$, consider the complex

$$0 \to \hat{\Omega}^0_\nu(Y) \xrightarrow{d_A} \hat{\Omega}^1_\nu(Y) \xrightarrow{*d_{A,h}} \hat{\Omega}^1_\tau(Y) \xrightarrow{d_A} \hat{\Omega}^0_\tau(Y) \to 0.$$

This complex could have been used to analyze the boundary-abelian strata of $\mathcal{M}_h$. In fact it has the same cohomology groups as the standard one so long as $A|_\Sigma$ is not central. For the problem of orientations on these strata, it has the benefit that for any perturbed flat connection here $\mathcal{H}^0_A(\Sigma)$ is oriented.

We argue in analogy with the last theorem. Denote by $\hat{K}$ the correponding family of Fredholm operators. With the same initial data as before, we get an orientation on $\operatorname{Ind} \hat{K}$. The group $\hat{\mathcal{G}}$ is no longer simply connected. Thus, in order to argue that the orientation descends to $\tilde{\mathcal{M}}_h$, we must show the following:

Lemma 53. *The boundary-abelian gauge group $\hat{\mathcal{G}}$ preserves the orientation of* $\operatorname{Ind}\hat{K}$.

Proof. The standard argument shows $\pi_0(\hat{\mathcal{G}}) = \pi_0[(Y,\Sigma),(SU(2),U(1))] = \pi_0([\Sigma, U(1)])$. The latter equality holds since $\pi_0(SU(2)) = \pi_1(SU(2)) = \pi_2(SU(2)) = 0$. In fact, we can find a representative g of each homotopy class $[g] \in \pi_0(\hat{\mathcal{G}})$ which is the identity off of a collar neighborhood $\Sigma \times [0,1] \subset Y$ of Σ. Let g be such a representative.

It is sufficient to show that when g acts on the standard trivial connection $A_0 = 0$ (taking it to another trivial connection) the induced map $ad(g) : \mathcal{H}^*_{A_0}(Y,) \to \mathcal{H}^*_{g^*A_0}(Y)$ is orientation preserving with respect to the orientation on $\operatorname{Ind}\hat{K}$. By the excision property of orientations of index bundles (see [Do]), the problem reduces to considering a solid handlebody, where the necessary property is easily verified. For example, let $Y = S^1 \times D^2$, and consider the gauge transformation $g(e^{i\theta}, x) = \exp(i2\pi k\theta)$.

Consider the path of connections $A_t = it2\pi k d\theta$ connecting A_0 to g^*A_0. One easily checks that during the jumps in cohomology at $t = 0$ and $t = 1$ the induced orientations coincide with the following identifications:

$$\mathcal{H}^*_{A_0}(Y, su(2)) \cong H^*(Y;\mathbf{R}) \otimes (\mathbf{R} \oplus \mathbf{C}),$$

$$\mathcal{H}^*_{A_t}(Y, su(2)) \cong H^*(Y;\mathbf{R}) \text{ for } 0 < t < 1,$$

and

$$\mathcal{H}^*_{A_1}(Y, su(2)) \cong H^*(Y;\mathbf{R}) \otimes (\mathbf{R} \oplus \mathbf{C}). \quad \square$$

This proves the following generalization of Theorem 52 to the boundary-abelian case. Let $\tilde{\mathcal{M}}_h$ denote the double cover of

$$\mathcal{M}_h^{(\mathbf{Z}_2,U(1))} \bigcup \mathcal{M}_h^{(U(1),U(1))} \bigcup \mathcal{M}_h^{(SU(2),SU(2))}$$

branched along the centrals.

Theorem 54. *$\tilde{\mathcal{M}}_h$ inherits a preferred orientation from the one given on* $\mathcal{H}^*_d(Y;\mathbf{R})$.

Note. The $\mathbf{Z}_2$ action on $\tilde{\mathcal{M}}_h$ does not preserve this orientation. This argument easily generalizes to give orientations on the cobordisms. Let $\tilde{\mathcal{M}}_{\{h_t\}}$ denote the double cover of the $(\mathbf{Z}_2, U(1))$-, $(U(1), U(1))$- and $(SU(2), SU(2))$- strata of $\tilde{\mathcal{M}}_{\{h_t\}}$ branched along the central points.

Theorem 55. *An orientation on $\mathcal{H}^*_d(Y; \mathbf{R})$ and one on $[0,1]$ induce one on $\mathcal{M}^{(\mathbf{Z}_2,\mathbf{Z}_2)}_{\{h_t\}}$ and one on $\tilde{\mathcal{M}}_{\{h_t\}}$.*

Remark. In a future paper we will use this result to give a gauge theoretic proof of the generalization of Lin's theorem [Li] pointed out by D. Ruberman. The theorem states roughly that the number of trace-free irreducible representations of a knot group into SU(2) is the signature of the knot, counted with signs, equals one half of the signature of the knot. The generalization relates the number of representations where all the meridians go to group elements with arbitrary fixed trace to the corresponding equivariant knot signature.

Note. There is no continuous lift of a neighborhood of an abelian boundary-central point in $\mathrm{M}_{\{h_t\}}$ to $\tilde{\mathrm{M}}_\Sigma$. This may be seen as follows.

Let $A_1(t)$, $0 \le t \le 1$, be a smooth path of perturbed flat boundary-abelian connections (where the perturbation depends on t) whose image in $\mathrm{M}_{\{h_t\}}$ lies in the boundary of the $(\mathbf{Z}_2, U(1))$-stratum and passes through the $(U(1), SU(2))$-stratum. Similarly, let $A_2(t)$, $1 \le t \le 2$, be a smooth path of perturbed flat connections such that $A_2(1) = A_1(1)$ and $A_2(2)$ is gauge equivalent to $A_1(0)$, and such that for $1 < t < 2$, $[A_2(t)]$ is in the $(\mathbf{Z}_2, U(1))$-stratum of $\mathrm{M}_{\{h_t\}}$. Then the composition $A(t) = (A_2 \circ A_1)(t)$, $0 \le t \le 2$, is a path with $[A(t)]$ tracing out the loop in the cobordism shown in Figure 3.

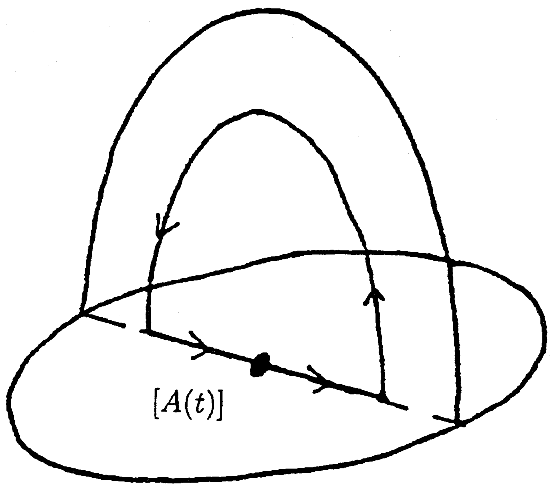

Figure 3

If the loop $[A(t)]$ is sufficiently small then there is some loop $\gamma \subset \Sigma$ such that $\operatorname{hol}_\gamma A(0)$ and $\operatorname{hol}_\gamma A(1)$ are in different components of $U(1)\setminus\{\pm \mathrm{Id}\}$. The path $\operatorname{hol}_\gamma A(t) \subset SU(2)$, $1 \le t \le 2$ connects $\operatorname{hol}_\gamma A(0)$ to $\operatorname{hol}_\gamma A(1)$ without hitting $\pm\mathrm{Id}$. Thus it is impossible to conjugate the path $\operatorname{hol}_\gamma A(t) \subset SU(2)$, $1 \le t \le 2$ into $U(1)$ by a path of $SU(2)$ elements fixing the endpoints. In other words, the path $A(t)$, $0 \le t \le 2$, is not gauge equivalent (by a continuous family of gauge transformations) to a closed loop of boundary-abelian connections.

This implies that the end of the $(\mathbf{Z}_2, U(1))$-stratum of $\tilde{\mathrm{M}}_{h_0}$ limiting to a point in part of the $(U(1), U(1))$-stratum of $\tilde{\mathrm{M}}_{h_0}$ becomes, after the boundary central point in the cobordism, an end of the $(\mathbf{Z}_2, U(1))$-stratum of $\tilde{\mathrm{M}}_{h_1}$ limiting to the opposite lift of the $(U(1), U(1))$-stratum. In other words, the way that the $(\mathbf{Z}_2, U(1))$- and $(U(1), U(1))$-strata of the $\tilde{\mathrm{M}}_h$ connect changes. Figure 4a and Figure 4b show the image of $\tilde{\mathcal{M}}_h$ in the pillowcase before and after the boundary-central abelian point in the cobordism.

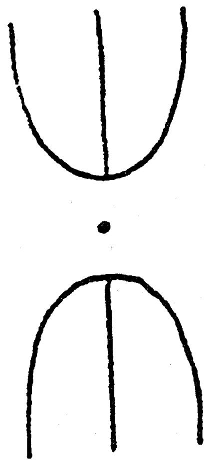

Figure 4a

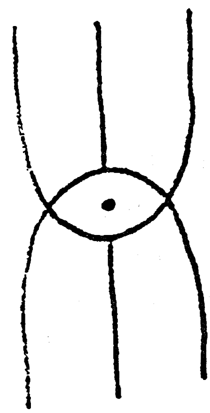

Figure 4b

This phenomenon can be realized by enlarging the bump functions in Example 2 (in Section 11) sufficiently that the top abelian arc goes over the central point in $\tilde{\mathrm{M}}_\Sigma$.

10. Technical Results about the Perturbations

In this section, we prove some technical lemmas concerning properties of the admissable perturbation functions. The results are necessary for the proofs of the genericity results in Sections 6 and 7. We begin by examining functions of the type $\mathrm{tr\ hol}_\ell(A)$ where $\ell : S^1 \to Y$ is an embedding. Let ℓ be such a loop, and let a be any 1-form. We will think of ℓ as a map $\ell(s) : [0,1] \to Y$. The definitions of $\mathrm{hol}_\ell(s, A)$ and $P_\ell[s,t;A]$ are given in section 6.1.

Lemma 56. *(This is equation 8.3 in [T].) The first and second derivatives of* $\mathrm{tr\ hol}_\ell A$ *are given by*

$$D\,\mathrm{tr\ hol}_\ell(A)(a) = \int_{[0,1]} \mathrm{tr}(\mathrm{hol}_\ell(s,A)a(\ell(s))(\ell_*(\frac{d}{ds})))ds$$

and

$$D^2 \operatorname{tr} \operatorname{hol}_\ell A(a,b)$$
$$= \int_{[0,1]\times[0,1]} \operatorname{tr}(P_\ell[t,s;A]a(\ell(s))(\ell_*(\frac{d}{ds}))P_\ell[s,t;A]b(\ell(t))(\ell_*(\frac{d}{dt})))dt \wedge ds.$$

Proof. We follow the proof in [DF]. By pulling back the connection and the 1-form a along ℓ, we reduce the problem to the case of an $SU(2)$ bundle over the circle. Fix a trivialization of this bundle, and let $P(s,A)$ denote the parallel translation by A from 0 to s. $P(s,D)$ satisfies the equation

$$\frac{\partial}{\partial s}P(s,A) = -A(s)P(s,A).$$

In terms of this notation, the problem reduces to showing that

$$\frac{d}{dt}|_{t=0} \operatorname{tr} P(1,A+ta) = \int_0^1 \operatorname{tr}(P(s,A)P(1,A)P(s,A)^{-1}a(s))$$

for any 1-form a on the circle.

First consider the case where A and a take their values in a 1-dimensional subspace $u(1) \subset su(2)$. Then

$$P(1,A+ta) = \exp(\int_0^1 A(s) + ta(s)ds)$$

Therefore,

$$\begin{aligned}\frac{d}{dt}|_{t=0} \operatorname{tr} P(1,A+ta) &= \operatorname{tr} \int_0^1 a(s)ds \exp(\int_0^1 A(s)ds) \\ &= \int_0^1 \operatorname{tr}(P(1,A)a(s)ds) \\ &= \int_0^1 \operatorname{tr}(P(s,A)P(1,A)P(s,A)^{-1}a(s)ds).\end{aligned}$$

This proves the lemma in the abelian case.

Next we note that there is a (t-dependent) gauge transformation $g_t = g(s,t)$ on the bundle $S^1 \times SU(2)$ such that $g_t^*(A+ta)$ is abelian. Let $\xi(s) = g(s,0)^{-1}\frac{d}{dt}|_{t=0}g(s,t)$, and let $A' = g_0^*A$. A gauge transformation doesn't change the trace of the holonomy of a connection. Thus

$$\frac{d}{dt}|_{t=0} \operatorname{tr}\ P(1,g_t^*(A+ta)) = \frac{d}{dt}|_{t=0} \operatorname{tr}\ P(1,A+ta).$$

On the other hand, since $g_t^*(A+ta)$ is abelian, we can compute

$$\begin{aligned}
&\frac{d}{dt}|_{t=0} \operatorname{tr}\ P(1, g_t^*(A+ta)) \\
&= \frac{d}{dt}|_{t=0} \operatorname{tr}\ \exp\Big(\int_0^1 g_t^{-1}\frac{d}{ds}g_t ds + g_t^{-1}Ag_t + tg_t^{-1}ag_t\Big) \\
&= \operatorname{tr}\Big(\exp\Big(\int_0^1 (g_0^{-1}\frac{d}{ds}g_0 ds + g_0^{-1}Ag_0 + tg_0^{-1}ag_0\Big) \\
&\quad \frac{d}{dt}|_{t=0}(g_t^{-1}\frac{d}{ds}g_t ds + g_t^{-1}Ag_t + tg_t^{-1}ag_t)\Big) \\
&= \int_0^1 \operatorname{tr}(\operatorname{hol}_\ell(s, g_0^*A)g_0^{-1}ag_0) + \int_0^1 \operatorname{tr}(\operatorname{hol}_\ell(s, g_0^*A)g_0^*(A)\xi) \\
&= \int_0^1 \operatorname{tr}(g_0^{-1}P(s,A)P(1,A)P(s,A)^{-1}g_0g_0^{-1}ag_0) + \int_0^1 \operatorname{tr}(\Pi_{\mathrm{Im}} \operatorname{hol}_\ell(s, A')A'\xi) \\
&= \int_0^1 \operatorname{tr}(P(s,A)P(1,A)P(s,A)^{-1}a(s)) \\
&= \int_0^1 \operatorname{tr}(\operatorname{hol}_\ell(s,A)a(s)) \\
&= \int_0^1 \operatorname{tr}(\Pi_{\mathrm{Im}}\ \operatorname{hol}_\ell(s,A)a(s)).
\end{aligned}$$

The term $\int_0^1 \operatorname{tr}(\Pi_{\mathrm{Im}} \operatorname{hol}_\ell(s,A')A'\xi)$ vanishes because, since A' is abelian, $\Pi_{\mathrm{Im}} \operatorname{hol}_\ell(s,A')A'$ is a multiple of the identity, which means $\Pi_{\mathrm{Im}} \operatorname{hol}_\ell(s,A')A'\xi$ is in $su(2)$ and hence is traceless.

The formula for the second derivative, like that for the first, is easily verified in the abelian case, and the general case is handled by noting that the formula is gauge invariant. □

The above computations immediately give the following bounds. (See [T] for some more details.)

Corollary 57. *For any smooth 1-forms a, b and embedded loop ℓ,*

$$|D \operatorname{tr}\ \operatorname{hol}_\ell(A)(a)| \leq C\|a\|_{L^2(S^1)}$$

and

$$|D^2 \operatorname{tr}\ \operatorname{hol}_\ell(A)(a,b)| \leq C\|a\|_{L^2(S^1)}\|b\|_{L^2(S^1)}$$

for a constant C independent of A.

If $\gamma : S^1 \times D^2 \to Y$ is an embedded solid torus, we can mollify the function $\mathrm{tr}\ \mathrm{hol}_{\gamma(S^1\times\{0,0\})}(A)$ by averaging over the disk D^2. Let $\eta(x)$ be a radially symmetric bump function on D^2 with support away from the boundary satisfying $\int_{D^2}\eta(x)d^2x = 1$ where d^2x is the standard measure on D^2. Define

$$p_\gamma(A) = \int_{D^2} \mathrm{tr}(\mathrm{hol}_{\gamma(S^1\times\{x\})}(A))\eta(x)d^2x.$$

Lemma 58. *Given any embedded loop $\ell : S^1 \to Y$ and any smooth 1-forms a, b on Y, the function $D\,\mathrm{tr}\ \mathrm{hol}_\ell A(a) : \mathcal{A} \to \mathbf{R}$ can be approximated by the function $Dp_\gamma(A)(a)$ for a suitably chosen thickening $\gamma : S^1 \times D^2 \to Y$ of ℓ.*

Proof. For $a \in \Omega^1(S^1 \times D^2, su(2))$, let $\|a\|_{L^2_{0,2}(S^1\times D^2)}$ denote the anisotropic Sobolev norm (see [H]) which measures only second derivatives of the form $\frac{\partial^2}{\partial x_i \partial x_j}a$ for coordinates $x = (x_1, x_2)$ on D^2. Then

$$\|a(x)\|_{L^2_2(D^2)} = \|a\|_{L^2_{0,2}(S^1\times D^2)} \leq \|a\|_{L^2_2(S^1\times D^2)},$$

where $a(x) : D^2 \to L^2(S^1, su(2))$ is the map corresponding to a. Combining this with Lemma 57, we get that

$$D\,\mathrm{tr}\ \mathrm{hol}_{\gamma(S^1\times\{x\})} A(a) \in L^2_2(D^2).$$

It is now a standard fact that $D\,\mathrm{tr}(\mathrm{hol}_{\gamma(S^1\times\{(u,v)\})}(A))(a)$ can be approximated in $L^2_2(D^2)$ by

$$\int_{D^2} D\,\mathrm{tr}(\mathrm{hol}_{\gamma(S^1\times\{x\})}(A))(a)\frac{\eta(t(x_1-u), t(x_2-v))}{t^2}d^2x$$

(see, for example, [GT]), letting $t \to 0$. Since by the Sobolev Embedding Theorem $L^2_2 \to C^0$, the convergence is pointwise in the disk, which means that we can approximate $D\,\mathrm{tr}\ \mathrm{hol}_\ell A(a)$ for any loop ℓ. The effect of shrinking t may also be accomplished, however, by choosing γ to be a narrower thickening of ℓ. Essentially the same proof works for the second derivative. □

Before showing that there are sufficient perturbations available to prove the genericity results in this paper, we recall some relevant facts about twisted cohomology and homology with coefficients in a flat bundle.

Let A be a flat connection and consider the homology of Y with coefficients in the flat $su(2)$ bundle, which we will denote by adA, determined by A. The chain groups are generated by covariantly constant lifts of simplices in Y.

(One could just as well take covariantly constant lifts of singular chains in Y.) By the de Rham Theorem for twisted coefficients, $\mathcal{H}^1_A(Y) \cong H^1(Y; adA) \cong H_1(Y; adA)^*$, where the last isomorphism is Hom duality. Note that for any loop ℓ, the lift of ℓ to adP given at the point $\ell(s)$ by $\Pi_{\mathrm{Im}} \operatorname{hol}_\ell(A, s)$ is covariantly constant. We denote this lift simply by $\Pi_{\mathrm{Im}} \operatorname{hol}_\ell A$.

Proposition 59. *If A is an irreducible flat connection, then there exists a basis for the twisted homology consisting of covariantly constant lifts of loops in Y each of which has noncentral holonomy.*

Proof. Since A is irreducible, we can find 3 loops a_1, a_2, and a_3 with a common basepoint such that $\Pi_{\mathrm{Im}} \operatorname{hol}_{a_i} A$ are 3 covariantly constant lifts of these loops whose values at the basepoint are linearly independent. Reorder, if necessary, so that the first k of these horizontal lifts are linearly independent in homology and the remainder are linearly dependent with the first k in homology.

Now consider a loop ℓ around which A has central holonomy. The space V_ℓ of covariantly constant lifts of ℓ is 3-dimensional (note that some lifts might be homologically nontrivial). The span in $\mathcal{H}^1_A(Y)$ of

$$\{\Pi_{\mathrm{Im}} \operatorname{hol}_{a_1} A, \dots, \Pi_{\mathrm{Im}} \operatorname{hol}_{a_k} A, \Pi_{\mathrm{Im}} \operatorname{hol}_{a_1 * \ell} A, \Pi_{\mathrm{Im}} \operatorname{hol}_{a_2 * \ell} A, \Pi_{\mathrm{Im}} \operatorname{hol}_{a_3 * \ell} A\}$$

is the same as that of

$$\operatorname{Span}\{\Pi_{\mathrm{Im}} \operatorname{hol}_{a_1} A, \dots, \Pi_{\mathrm{Im}} \operatorname{hol}_{a_k} A\} + V_\ell.$$

Using this fact the construction of a basis satisfying the proposition is straightforward. □

Choose a basis $\{\tilde{a}_i\}_{i=1}^k$ for $H_1(Y; adA)$ as in the Proposition. Then we can find a Hom dual basis $\{\alpha_i\}_{i=1}^k$ for $\mathcal{H}^1_A(Y)$ satisfying

$$\int_{a_i} \operatorname{tr}(\tilde{a}_i \alpha_j) = -\delta_{ij}.$$

Here $\tilde{a}_i$ is a horizontal lift of a_i. This choice of bases will simplify our computations enormously.

Consider a loop $a_i : [0,1] \to Y$ in this collection. Use parallel translation to trivialize the pulled back bundle $a_i^* ad(P) \to [0,1]$. With respect to this trivialization, $\tilde{a}_i$ is a constant element of $su(2)$, and $\int_{a_i} \operatorname{tr}(\tilde{a}_i \alpha_i) = -\langle \tilde{a}_i, \int_{a_i} \alpha_i \rangle$.

Since $A|_{a_i}$ has noncentral holonomy, then the only possible covariantly constant sections $\tilde{a}_i$ are multiples of $\Pi_{\text{Im}} \text{hol}_{a_i}(A)$.

Lemma 60. *Given any flat irreducible connection A, there is a finite collection $\Lambda = \{\ell_i\}_{i=1}^m$ of disjoint loops $\ell : S^1 \to Y$ such that the map*

$$(D \text{tr } \text{hol}_{\ell_1}, \ldots, D \text{tr } \text{hol}_{\ell_m}) : \mathcal{H}^1_A(Y) \subset T_A\mathcal{A} \to \mathbf{R}^m$$

is injective.

Proof. Choose bases $\{\tilde{a}_i\}_{i=1}^k$ and $\{\alpha_i\}_{i=1}^k$ as above. It is then clear that

$$\int_{a_i} \text{tr}((\text{hol}_{a_i} A)\alpha_j) = -\langle \tilde{a}_i, \int_{a_i} \alpha_j \rangle C_{i,j}\delta_{ij}.$$

with $C_{i,j} \neq 0$. (This is because $\tilde{a}_i$ and $\Pi_{\text{Im}} \text{hol}_{a_i} A$ are both nonzero sections of the 1-dimensional space of covariantly constant sections over a_i.) □

In order to generalize this result to perturbed flat connections, we need the following explicit description of the perturbed flat connections on one of the solid tori $\gamma_i(S^1 \times D^2)$. Let $\lambda_i = \gamma_i(S^1 \times \{1\})$ and $\mu_i = \gamma_i(\{1\} \times \partial D^2)$.

Lemma 61. *If $\zeta_h(A) = 0$ then A is flat on $Y \setminus \bigcup_{i=1}^n \gamma_i(S^1 \times D^2)$. If $\text{hol}_{\lambda_i}(A) = \exp(i\theta_{\lambda_i})$, then*

$$\text{hol}_{\mu_i}(A) = \exp(ig_i(\theta_{\lambda_i}))$$

where

$$g_i(\theta_{\lambda_i}) = -2\pi \bar{h}_i'(2\cos\theta_{\lambda_i})\sin\theta_{\lambda_i}.$$

Remark. Any arbitrary smooth odd 2π periodic function $g_i(\theta)$ with $g_i(0) = 0$ can be achieved by a suitable choice of $\bar{h}_i$.

Proof. From the formula for ∇h, $\text{hol}_{\gamma_i(S^1 \times \{x\})} A$ is independent of x, since the curvature has no ds component. Since the curvature is abelian on each solid torus, we can exponentiate the integral over $\{s\} \times D^2$ to determine the meridinal holonomy. □

This relation between the longitudinal and meridinal holonomies makes it possible to explicitly calculate the perturbed flat solutions in certain cases. We describe some examples in Section 11. It also implies that we could define the perturbations solely in terms of the representations of π_1. It is possible to reinterpret the problem as follows. We can view $\mathcal{M}_h(Y\setminus\coprod_{i=1}^n \gamma_i(S^1\times D^2))$ as a symplectic relation in $\mathcal{M}_\Sigma\times\prod_{i=1}^n \mathcal{M}_{\partial(\gamma_i(S^1\times D^2))}$. $\mathcal{M}_h$ is then the composition of this relation with a well understood symplectic submanifold of $\mathcal{M}_{\partial(\gamma_i(S^1\times D^2))}$ (the perturbed solutions on the solid tori). There is some trickiness when $\mathrm{hol}_{\lambda_i} = 1$. See Example 2 in the next section.

The above description of perturbed solutions A makes it also possible to write down, up to gauge equivalence, what the connection 1-form of A is.

Corollary 62. *For $A \in \zeta_h^{-1}(0)$, $A|_{\gamma_j(S^1\times D^2)}$ is gauge equivalent to a connection of the form*

$$A_0(s,x) = i\theta ds - [\int_{-\sqrt{1-x_1^2}}^{x_2} ig_j(\exp i\theta)\eta(x_1,\xi)d\xi]dx_1 \tag{7}$$

for some θ, where $x = (x_1, x_2)$ are coordinates on the disk.

Lemma 63. *Let (h,ϕ) be a fixed admissible perturbation and let $A \in \zeta_h^{-1}(0)$ be irreducible. The collection ϕ may be enlarged to a collection $\phi\cup\psi$ of disjoint embedded solid tori $\gamma_i : S^1\times D^2 \to Y$ such that the subspace of $L_2^2\Omega^1(Y, su(2))$ spanned by $\{\nabla \int_{D^2} \mathrm{tr}(\mathrm{hol}_{\gamma_i(x)} A)\eta(x)d^2x\}_{i=1}^k$ orthogonally projects onto $\mathcal{H}^1_{A,h}(Y)$.*

Proof. By Lemma 58, we may as well consider loops instead of solid tori. Lemma 60 proves the desired result in the case when $h = 0$. Suppose that $\alpha \in \mathcal{H}^1_{A,h}(Y)$ and $D\,\mathrm{tr}\ \mathrm{hol}_\gamma A(\alpha) = 0$ for all ℓ disjoint from ϕ. Then the proof of Lemma 60 shows that $\alpha = 0$ on the complement of ϕ. By the Mayer-Vietoris sequence, it suffices to show that if $\alpha|_{\gamma_i(S^1\times D^2)}$ is a nonzero element of $\mathcal{H}^1_{A,h_i}(\gamma_i(S^1\times D^2))$ then

$$D\,\mathrm{tr}(\mathrm{hol}_{\gamma_i}(A)(\alpha)) \neq 0,$$

which can be seen directly from the explicit description of the perturbed flat connections on the perturbation solid tori. □

Consider now the case of an abelian perturbed flat connection A. The action of the stabilizer of A gives a decomposition of $\mathcal{H}^1_A(Y)$ into $\mathcal{H}^1_A(Y;\mathbf{R})\oplus\mathcal{H}^1_A(Y;\mathbf{C})$. The proof of Lemma 60 has the following corollary.

Corollary 64. *For any abelian connection* $A \in \zeta_h^{-1}(0)$ *there is a finite collection* $\Lambda = \{\ell_i\}_{i=1}^m$ *of disjoint loops* $\ell : S^1 \to Y$ *such that for each* $A \in K$ *the map*

$$(D\,\mathrm{tr}\ \mathrm{hol}_{\ell_1}, \ldots, D\,\mathrm{tr}\ \mathrm{hol}_{\ell_m}) : \mathcal{H}^1_{A,h}(Y; \mathbf{R}) \to \mathbf{R}^m$$

is injective.

Suppose for the moment that A is an abelian flat connection. We decompose $H_1(Y; adA)$ into $H_1(Y; \mathbf{R}) \oplus H_1(Y; adA_{\mathbf{C}})$ where $adA_{\mathbf{C}}$ is the corresponding flat $\mathbf{C}$ bundle. Notice that it is impossible to find a loop ℓ with $\int_\ell \mathrm{tr}(\mathrm{hol}_\ell(A)\beta) \neq 0$ for any $\beta \in \mathcal{H}^1_A(Y, \mathbf{C})$. Because of this, a different approach is needed to understand the normal bundle of the abelian stratum. We lay the groundwork for this approach with the following lemma.

We choose a complex basis $\{\tilde{a}_n\}_{n=1}^N$ for $H_1(Y; adA_{\mathbf{C}})$ and a curve a_0 around which A has noncentral holonomy. We may assume that the holonomy is $e^{i\theta_0}$, for some $0 < \theta_0 < \pi$, and that the curves a_n are disjoint and all have the same basepoint y_0.

Let $S = [0,1] \cup \wedge_{n=1}^N S^1 / \sim$ where the wedge point is identified with $1 \in [0,1]$. Let $\bar{a} = (a_0, a_1, \ldots, a_N) : S \to Y$ and let $\bar{a}^*(adP)$ be identified with $S \times SU(2)$ by picking an identification at y_0 and using parallel transport to extend it. Note that under the map $a_0 : S \to Y$ the fibers at $0 \in [0,1]$ and $1 \in [0,1]$ are identified by the action of $\mathrm{hol}_{a_0} A$ on $su(2)$.

In particular, we will be interested in the complex line sub-bundle of adP (and its pullback to S) arising from the decomposition of the $\mathrm{Stab}\, A$ action into irreducible representations. The holonomy subgroup of $U(1)$ acts with weight two on this complex line bundle. We identify sections of the complex line bundle with maps to the $j-k$ plane, using the trivialization above. By modifying the basis for $H_1(Y; adA_{\mathbf{C}})$ chosen at the outset, we can assume that the $\{\tilde{a}_n\}_{n=1}^N$ have no component in the k direction, which will simplify some of our computation.

Let $\{\alpha_1, i\alpha_1, \ldots, \alpha_N, i\alpha_N\}$ be the real basis for $\mathcal{H}^1_A(Y; \mathbf{C})$ which is Hom dual to the real basis $\{\tilde{a}_1, i\tilde{a}_1, \ldots, \tilde{a}_N, i\tilde{a}_N\}$ for $H_1(Y; adA_{\mathbf{C}})$.

Let $\{\ell_i\}_{i=1}^M$ be the following set of loops in Y:

$$\{a_n\}_{0 \le n \le N} \cup \{a_n * a_m\}_{0 \le m < n \le N} \cup \{a_n * a_m * a_0\}_{1 \le m < n \le N}.$$

Here $a_n * a_m$ means the product of a_n and a_m as fundamental group elements. Let $\mathrm{Sym}^{U(1)}(\mathcal{H}^1_A(Y;\mathbf{C}))$ be the set of all symmetric, $U(1)$ invariant (here $U(1)$ acts by the diagonal action on pairs of complex-valued forms), real-valued bilinear forms on $\mathcal{H}^1_A(Y;\mathbf{C})$, which we identify with the set of all Hermitian complex matrices by using the basis $\{\alpha_n\}$.

Proposition 65. *Given a flat abelian connection A, if $\{\ell_i\}_{i=1}^M$ are chosen as above, then the map $\mathbf{R}^M \to \mathrm{Sym}^{U(1)}(H^1_A(Y;\mathbf{C}))$ given by*

$$(b_1,\dots,b_M) \to (\sum_{m=1}^M b_m D^2 \operatorname{tr} \operatorname{hol}_{\ell_m} A)|_{\mathcal{H}^1_A(Y;\mathbf{C})}$$

is a submersion.

Proof. We begin with the loops of the form a_n. In this case, since there is no holonomy and we have trivialized using parallel transport, it is easy to check that

$$D^2 \operatorname{tr} \operatorname{hol}_{a_n} A(\alpha_p, \alpha_q) = \operatorname{tr}((\int_{a_n} \alpha_p)(\int_{a_n} \alpha_q)),$$

and by our choice of basis, $(\int_{a_n} \alpha_p)$ is nonzero exactly when $n = p$. Furthermore,

$$D^2 \operatorname{tr} \operatorname{hol}_{a_n} A(\alpha_p, i\alpha_q) = 0$$

for all i, p, and q. Thus loops of this form allow construction of arbitrary real diagonal matrices.

Next we consider loops of the form $a_n * a_m$. Again there is no holonomy and one easily checks that the matrix corresponding to

$$D^2 \operatorname{tr} \operatorname{hol}_{a_n * a_m} A$$

has (the same) nonzero real entries in the nm and mn entries. The condition that they are real is again verified by noting that the second derivative evaluates to zero on $(\alpha_p, i\alpha_q)$ for all p and q.

It remains to show that by using curves of the third type, we can achieve arbitrary imaginary parts to the off-diagonal matrix entries (subject to the Hermitian condition). We do that now.

Consider the loop $a_n * a_m * a_0$, where $n < m$. We break up the interval domain of $a_n * a_m * a_0$ into thirds (corresponding to the three factors) and integrate over the 9 resulting regions separately. Note that the parallel transport factors in the formula for $D^2 \operatorname{tr}\ \operatorname{hol}_\ell A$ simplify to

$$P_\ell[s,t;A] = \begin{cases} 1 \text{ for } s < t \\ \exp(i\theta_0) \text{ for } s > t. \end{cases}$$

when we compute in the trivialized pulled-back bundle over S.

One can check, in this manner, that

$$\begin{aligned} & D^2 \operatorname{tr}\ \operatorname{hol}_{a_n * a_m * a_0} A(\alpha_p, i\alpha_q) - D^2 \operatorname{tr}\ \operatorname{hol}_{a_n * a_0} A(\alpha_p, i\alpha_q) \\ & -D^2 \operatorname{tr}\ \operatorname{hol}_{a_m * a_0} A(\alpha_p, i\alpha_q) + D^2 \operatorname{tr}\ \operatorname{hol}_{a_0} A(\alpha_p, i\alpha_q) \\ = & \int_{[0,1]\times[0,1]} \operatorname{tr}(\exp(i\theta_0)\alpha_p(a_n(s)) i\alpha_q(a_m(t)))ds \wedge dt \\ & + \int_{[0,1]\times[0,1]} \operatorname{tr}(\alpha_p(a_m(s)) \exp(i\theta_0) i\alpha_q(a_n(t)))ds \wedge dt \\ = & \operatorname{tr}(i \exp(i\theta_0)(\int_{a_n} \alpha_p) i (\int_{a_m} \alpha_q) + (\int_{a_m} \alpha_p) \exp(i\theta_0) i (\int_{a_n} \alpha_q)) \\ = & -\sin\theta_0 \operatorname{tr}\left((\int_{a_m} \alpha_p)(\int_{a_n} \alpha_q) - (\int_{a_n} \alpha_p)(\int_{a_m} \alpha_q) \right), \end{aligned}$$

which is nonzero exactly when the unordered pairs $\{p,q\}$ and $\{m,n\}$ are equal. □

Corollary 66. *For any abelian $A \in \zeta_h^{-1}(0)$, there is a finite collection $\Lambda = \{\ell_i\}_{i=1}^L$ of disjoint loops such that the map $\mathbf{R}^L \to \operatorname{Sym}(H^1_{A,h}(Y;\mathbf{C}))$ given by*

$$(b_1, \ldots, b_L) \to \sum_{i=1}^{L} b_i D^2 \operatorname{tr}\ \operatorname{hol}_{\ell_i} A$$

is a submersion.

Proof. The generalization to the perturbed case follows from a Mayer-Vietoris argument and the fact that $\mathcal{H}^0_{A,h}(\gamma_i(S^1 \times D^2); su(2))$ is either 1- or 3-dimensional, the latter only when $A|_{S^1\times D^2}$ is central, in which case A is actually flat on $S^1 \times D^2$. □

Finally we need a version of the last result for central connections. Note that the stratum of flat central connections does not change under perturbation, because the gradients of the perturbation functions vanish if the holonomy along the perturbation curves is central. Our problem is to understand the normal bundle of these points.

Let A now be a central flat connection. If we pick a basepoint in Y and an identification of the fiber of adP with $su(2)$ and also pick a 1-dimensional subspace $\mathbf{R} \subset su(2)$, then there is a canonical isomorphism $\mathcal{H}^1_{A,h}(Y) \cong \mathcal{H}^1_{A,h}(Y;\mathbf{R}) \otimes su(2)$.

There is one difference from the abelian case considered above. Since $\Pi_{\mathrm{Im}} \operatorname{hol}_\ell A = 0$ for every loop ℓ, one can check from the formula for the second derivative that for any orthogonal $su(2)$ elements v and w and any $\alpha, \beta \in \mathcal{H}^1_{A,h}(Y;\mathbf{R})$,

$$D^2 \operatorname{tr} \operatorname{hol}_\ell A(\alpha \otimes v, \beta \otimes w) = 0.$$

The bilinear form $D^2 \operatorname{tr} \operatorname{hol}_\ell A$ is invariant under the action of Stab $A \cong SU(2)$.

For simplicity choose a basis $\{\alpha_i\}_{i=1}^k$ for $H^1(Y;\mathbf{R})$ and let $\{v_1, v_2, v_3\}$ be an orthonomal basis for $su(2)$. In terms of the basis

$$\{\alpha_1 \otimes v_1, \alpha_1 \otimes v_2, \alpha_1 \otimes v_3, \ldots, \alpha_k \otimes v_1, \alpha_1 \otimes v_2, \alpha_1 \otimes v_3\}$$

the bilinear form $D^2 \operatorname{tr} \operatorname{hol}_\ell A$ has the form of a symmetric matrix built up of 3×3 blocks each of which is a multiple of the identity matrix. We denote the space of such matrices by $\mathrm{Sym}^{SU(2)}(\mathcal{H}^1_{A,h}(Y))$. The following proposition is an easier analogy of Proposition 65.

Proposition 67. *For any collection of loops $\{\ell_i\}_{i=1}^k$ which generate $H_1(Y;\mathbf{R})$, the map $\mathbf{R}^k \to \mathrm{Sym}^{SU(2)}(\mathcal{H}^1_{A,h}(Y))$ given by*

$$(b_1, \ldots, b_k) \mapsto \sum_{i=1}^k b_i D^2 \operatorname{tr} \operatorname{hol}_{\ell_i} A$$

is surjective.

11. Some Remarks and Examples

One should like to understand the equivalence class of $r \circ s_h : \mathcal{M}_h \to \mathcal{L}_\Sigma$ up to perturbation in its own right. This equivalence class is a topological invariant of Y. Theorem 50 provides an upper bound for this equivalence class.

Theorem 68. *Not every oriented stratified Legendrian cobordism of $r \circ s_h(\mathcal{M}^h_Y)$ can be achieved by a perturbation of the Chern-Simons function. In other* *words, the equivalence relation defined by the cobordisms in Theorem* 50 *is in fact larger than the equivalence class defined by perturbations. On the other hand, the equivalence defined by area preserving ambient isotopy or Hamiltonian flow is smaller.*

Proof. We prove the first statement by showing that there are Legendrian cobordisms which, if possible, would violate the invariance of Floer homology under perturbation. For the second statement, we demonstrate a perturbation which changes the topology of a stratum of the moduli space.

The unperturbed flat moduli space for the right-handed trefoil knot complement consists of an abelian arc together with an arc of irreducibles which hits the abelian arc at two interior points. The image in the pillowcase $\mathcal{M}_{T^2}$ is drawn in Figure 5a.

When a +1 surgery is performed on the right-handed trefoil knot, a homology 3-sphere is obtained which admits exactly two irreducible flat connections (up to gauge equivalence), namely the two intersections of the $r(\mathcal{M}_Y)$ with the straight arc of abelian connections which extend over the Dehn filling. See Figure 5b.

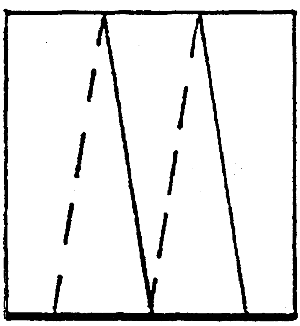

Figure 5a

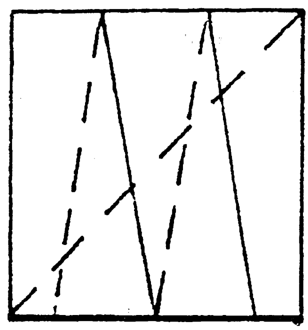

Figure 5b

The Floer graded chain complex (from which the homology is defined) is generated by these two flat connections. By [Y], the difference in gradings between them can be calculated as follows. Let γ_1 and γ_2, respectively, be the irreducible arc of $\mathcal{M}(S^3 \setminus \text{trefoil})$ and the abelian arc of $\mathcal{M}(S^1 \times D^2)$ between the intersection points. The difference in grading is equal to a relative Maslov index of the pair (γ_1, γ_2) plus twice the number of corners (fixed points the $\mathbf{Z}_2$ action) bounded by $\gamma_1 \circ \gamma_2$. For our purposes, the sign conventions are not important. The gradings of the generators are easily seen to differ by 4. Hence the boundary operators are trivial and the Floer homology consists of two copies of $\mathbf{Z}$ whose gradings differ by 4.

Now suppose we perform the series of oriented Legendrian cobordisms shown in Figure 5c and 5d. The result is that we now have two generators whose gradings differ by 2, and so this cobordism has altered the Floer homology. Floer showed the invariance of the homology under perturbation, so this Legendrian cobordism cannot be possible by perturbation. Notice that it is also possible to create this pair of cancelling kinks by a homotopy through immersions preserving the integrality condition (without a birth of a figure-eight). □

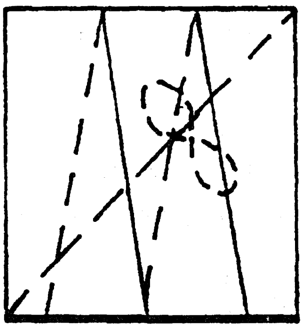

Figure 5c

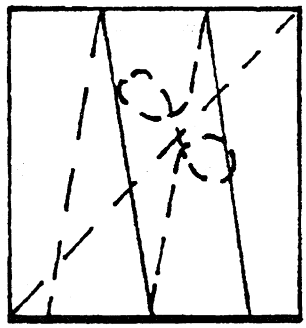

Figure 5d

Remark. It follows from a similar argument that the Maslov index plus twice the number of pillowcase corners enclosed must be zero modulo 8 for any smooth closed irreducible component in $\mathcal{M}_h(S^3 \setminus \text{knot})$ for any knot. A similar statement can be made about smooth closed abelian components there is a correction term involving the number of bifurcation points counted with sign.

Next we give some examples of paths of perturbations during which changes in the topology of the strata of $\mathcal{M}_h$ take place. The first will demonstrate that births of figure-eights sometimes occur and that at least sometimes moves of type (c) in Theorem 47 sometimes occur. In this example, the changes take place in the $(U(1), U(1))$-stratum of $\mathcal{M}_h$ for $Y = S^1 \times D^2$, but we explain at the end how to generalize to get the same changes in the $(\mathbf{Z}_2, U(1))$-stratum for other knot complements. The second example will demonstrate an instance when $g = 1$ in which a cancelling pair of bifurcation points are born along the $(U(1), U(1))$-stratum, with a $(\mathbf{Z}_2, U(1))$ arc connecting them.

We first make one observation regarding the framings chosen for the perturbation solid tori. The map $\gamma_i : S^1 \times D^2 \to Y$ determines a natural choice of longitude $\lambda_i = \gamma_i(S^1 \times \{1\})$ and meridian $\mu_i = \gamma_i(\{1\} \times \partial D^2)$ for the image solid torus. As described in Lemma 61, the perturbed flat solutions on $\gamma_i(S^1 \times D^2)$, up to gauge equivalence, have the form

$$\mathrm{hol}_{\lambda_i} = \exp(i\theta_{\lambda_i}) \text{ and } \mathrm{hol}_{\mu_i} = \exp(i\theta_{\mu_i})$$

where $\theta_{\mu_i} = g_i(\theta_{\lambda_i})$, so they form the graph of the fuction g_i (in the appropriate angle coordinates). If we describe these solutions in terms of a different framing $\lambda = \lambda_i - k\mu_i$ (and the same meridian μ_i), the solutions solve the equation

$$\theta_{\mu_i} = g_i(\theta_\lambda + k\theta_{\mu_i}).$$

This is no longer the graph of a function of θ_λ. It is, however, the image of the earlier graph $\theta_{\mu_i} = g_i(\theta_{\lambda_i})$ under the linear shearing homomorphism

$$(\theta_{\mu_i}, \theta_\lambda) = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{pmatrix} \theta_{\mu_i} \\ \theta_{\lambda_i} \end{pmatrix}.$$

EXAMPLE 1. Let $Y = S^1 \times D^2$. Let $\lambda = S^1 \times \{1\}$ and $\mu = \{1\} \times \partial D^2$ be a longitude and meridian for Y. Let γ_1 and γ_2 be disjoint thickened curves in Y which are parallel to λ but whose framings differ from those coming from the trivialization of Y by -1 and $+1$, respectively. The effect of choosing these framings is that the solutions will satisfy

$$\theta_{\mu_i} = g_i(\theta_{\lambda_i}) = g_i(\theta_\lambda \pm \theta_{\mu_i}).$$

Next we choose functions $g_{i,t}$ as follows. Let $g_{1,t}(\theta)$ be a fixed small positive bump function with support, say, in $[\frac{\pi}{8}, \frac{\pi}{4}]$, independent of t (t is a 1-dimensional parameter which will parametrize our path of perturbations). Let $g_{2,t}(\theta)$ be a slightly taller positive bump function centered at t for $\frac{\pi}{16} \leq t \leq \frac{15\pi}{16}$.

Figure 6 shows solutions $(\theta_{\mu_i}, \theta_\lambda)$ (we only show the solutions with $\theta_{\mu_i} \neq 0$) and the resulting solutions $(\theta_\mu, \theta_\lambda)$ for descending values of t. Note that $\mathrm{hol}_\mu(A) = \mathrm{hol}_{\mu_1}(A)\,\mathrm{hol}_{\mu_2}(A)$.

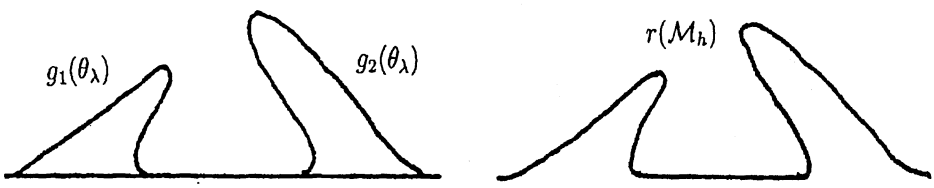

Figure 6*a*

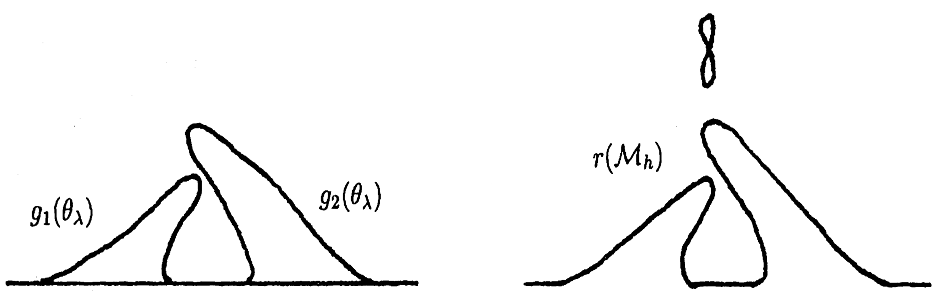

Figure 6*b*

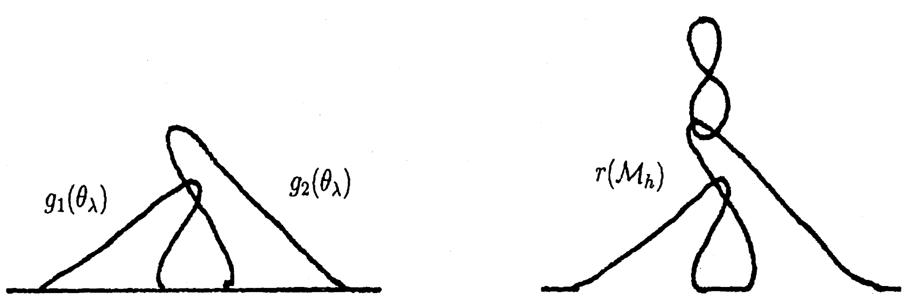

Figure 6*c*

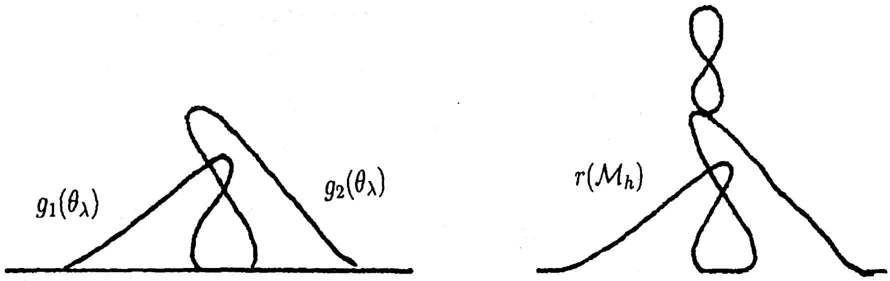

Figure 6*d*

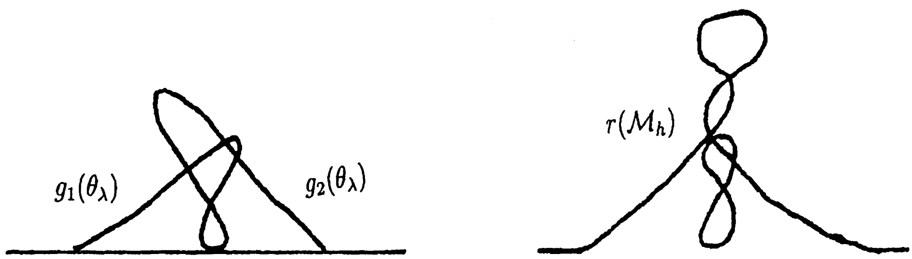

Figure 6*e*

Remark. To accomplish the same topological changes in of the $(\mathbf{Z}_2, U(1))$-stratum of, for example, the trefoil knot complement, we can simply do the same perturbation on the same pair of solid tori γ_i contained in a neighborhood N of a linking circle to the knot. See Figure 7. Let λ_{in} be a longitude for the knot which passes just inside of N (further from $\partial(S^3 \setminus \text{knot})$) and let λ_{out} be the usual longitude on ∂Y. Then λ_{out} and λ_{in} differ by

$$\lambda_{\text{out}} = \lambda_{\text{in}} \mu_N$$

and $\mu_{\text{knot}} = \lambda_N$. The net effect of doing the perturbation on N is to superimpose the pictures in Figure 6 onto each section of $r(\mathcal{M}(S^3 \setminus \text{knot}))$ which passes through the band $-2 < \text{tr}(\mu_{\text{knot}}) < 2$.

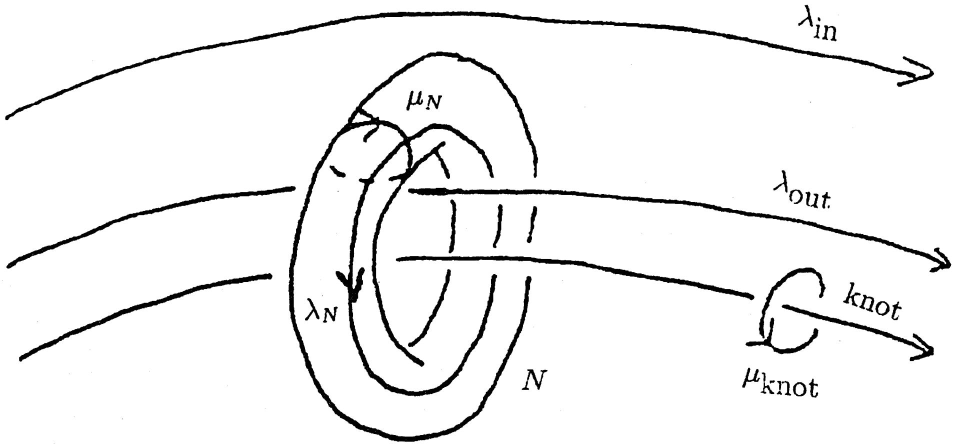

Figure 7

Remark. It appears from the computer generated pictures made by Mathias Rogel that for the 10_8 knot complement $\mathcal{M}$ has a figure-eight component before perturbing. This was pointed out to me by Eric Klassen.

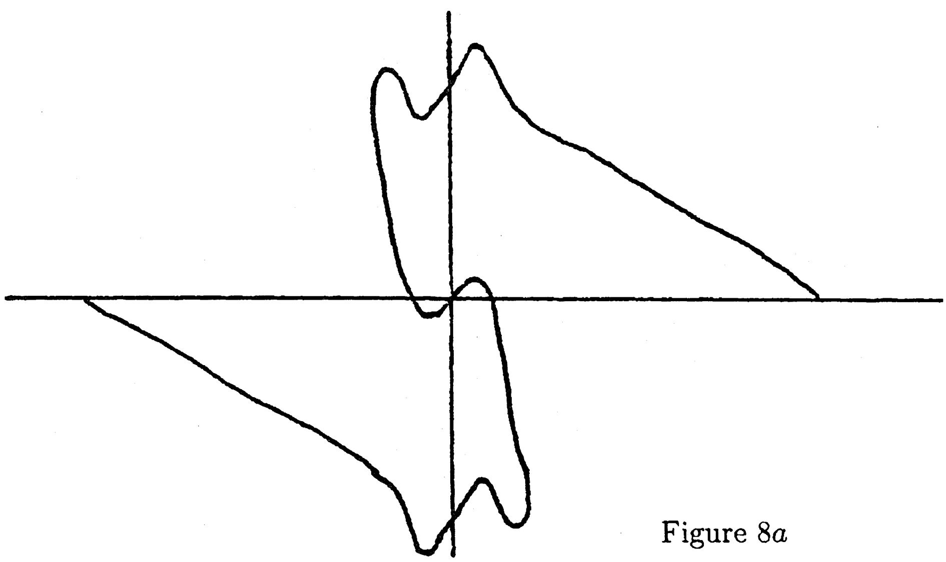

Figure 8*a*

EXAMPLE 2. For simplicity we situate our second example on $Y = S^1 \times D^2$. Choose γ_1, γ_2 parallel to λ, both with framing -1. In this example, the odd parity of the functions g_i will be crucial. Thus we draw the solutions for $\theta_\lambda \in [-\pi, \pi]$. Let $g_{1,t}(\theta) = 2\sin(\theta)$ be fixed (independent of t). Note that the sheared graph of $g_{1,t}$ hits the vertical line $\theta_\lambda = 0$. Let $g_{2,t}$ be a small bump function with $g_{2,t}(\theta) = 2t\theta$ when $0 \le \theta \le \epsilon$ and $g_{2,t}(\theta) = 0$ when $\theta \ge 2\epsilon$, extended to be odd and periodic. When t is slightly less than 1, the picture in the double cover of the pillowcase looks like Figure 8a. When $t > 1$ there are two new intersections of the graph with the vertical line $\theta_\lambda = 0$ near each previous intersection, having the form $\mathrm{hol}_\mu(A) = \exp(\theta_{\mu_1} \pm \theta_{\mu_2})$ where the $\theta_{\mu_i} \neq 0$ satisfy $\theta_{\mu_i} = g_{i,t}(\theta_{\mu_i})$. These two abelian perturbed flat connections are connected by an arc of irreducible perturbed flat connections, namely

$$\mathrm{hol}_\lambda(A) = 1 \text{ and } \mathrm{hol}_{\mu_1}(A) = \exp(i\theta_{\mu_1}),$$

and

$$\mathrm{hol}_{\mu_2}(A) = \exp((\cos\phi i + \sin\phi j)\theta_{\mu_2}).$$

The image of this $(\mathbf{Z}_2, U(1))$ arc in the pillowcase is vertical. See Figure 8b.

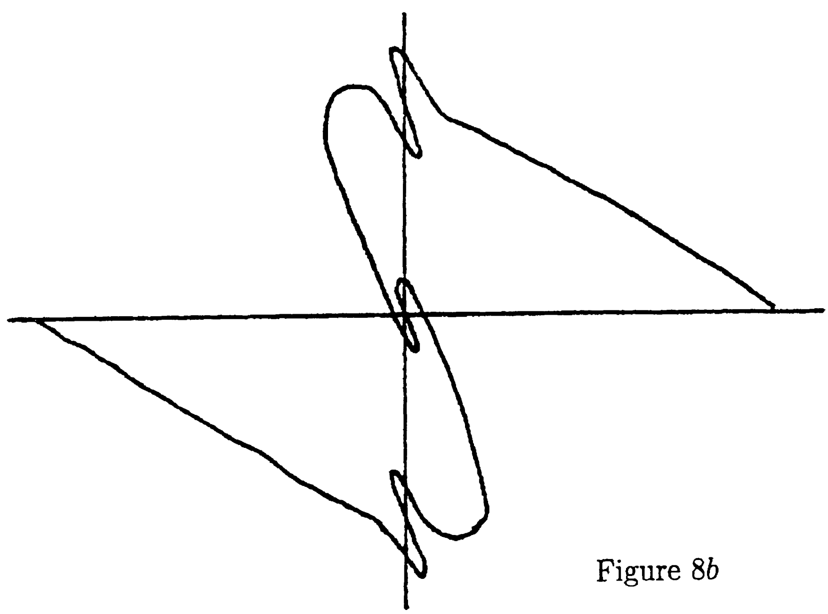

Figure 8b

Theorem 69. *There is a perturbation along curves contained in the unknot complement so that there are irreducible perturbed flat connections on* S^3.

Proof. If we consider S^3 to be the $\frac{1}{0}$-surgery on the unknot, then the perturbed flat solutions on the unknot complement which extend over the Dehn filling are exactly those whose images in the pillowcase lie on the vertical edge $\{\text{hol}_\lambda(A) = \text{id}\}$. Therefore, the perturbation in the preceding example provides whole arcs of irreducible perturbed flat solutions on S^3. This situation is degenerate, of course, since the $(\mathbf{Z}_2, U(1))$-stratum for a 3-manifold with empty boundary should be 0-dimensional. It can easily be made nondegenerate by adding another small perturbation along the curve μ. This is left to the reader. □

Finally we give the example promised in the remark after Theorem 50.

EXAMPLE 3. We use consider again the +1 surgery on the trefoil knot. We consider a family of perturbations using a curve γ contained in the Dehn filling. We draw the images of the flat moduli space for the knot complement and the perturbed flat moduli space for the Dehn filling, and the cobordism for the closed manifold is traced out by their intersections. See Figure 9.

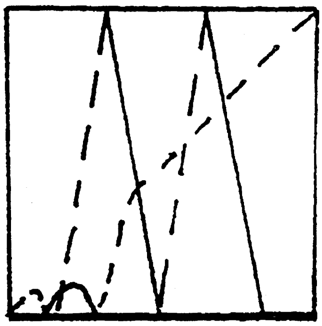

Figure 9a

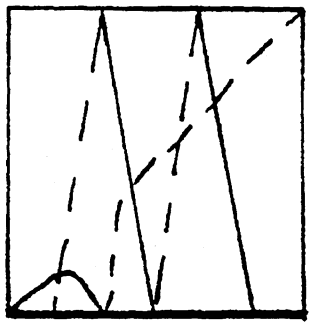

Figure 9b

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