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CONVEX HYPERSURFACES WITH PINCHED SECOND FUNDAMENTAL FORM

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1.

In this paper we prove the following result.

1.1. Main Theorem. *Let M^n be a smooth strictly convex complete hypersurface bounding a region in $\mathbb{R}^{n+1}$. Suppose that its second fundamental form is ε-pinched, in the sense that*

$$H_{ij} \geq \varepsilon H g_{ij}$$

where g_{ij} is the induced Riemannian metric, H_{ij} the second fundamental form, and its trace H is the mean curvature, for some $\varepsilon > 0$. Then M^n is compact.

This result can be used to simplify the proof of Huisken's theorem [Hu1] for the mean curvature flow of a convex hypersurface in Euclidean space. We would especially like to thank Mike Gage, who pointed out the relation of the pinching condition to quasi-conformal maps, and Burt Rodin, who showed us the basic estimates for quasi-conformal maps.

2.

Suppose now that $M^n \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}^1$ is written as the graph over a convex open set $U \subseteq \mathbb{R}^n$ of a strictly convex function

$$y = f(x_1, \dots, x_n)$$

such that $y \to +\infty$ as $X = (x_1, \dots, x_n)$ approaches the boundary of U. By translating upward if necessary, since y is bounded below we can assume $y \geq e$

everywhere, so that $\ell n \ell n\, y \geq 0$. Let

$$ds^2 = g_{ij} dx^i dx^j$$

be the Riemannian metric induced on M, so that

$$g_{ij} = I_{ij} + \frac{\partial y}{\partial x^i} \frac{\partial y}{\partial x^j} \, .$$

The following observation is fundamental to the proof.

Theorem 2.1. *The conformally equivalent metric*

$$d\tilde{s} = \frac{ds}{y \ell n\, y}$$

is complete with finite volume.

Proof. First we show $d\tilde{s}$ is complete. For any path going to ∞ we have $y \to +\infty$ and $ds \geq dy$. Therefore its length L satisfies

$$L \geq \int_b^\infty \frac{dy}{y \ell n\, y} = \ell n\, \ell n\, y \Big]_b^\infty = \infty$$

which is what we desire. □

To estimate the volume, we observe that, because y is a strictly convex function of x, outside of a compact set we must have

$$\left| \frac{\partial y}{\partial x^i} \right| \geq \delta$$

for some $\delta > 0$. Let dV denote the volume element on M in the induced metric ds, which in X coordinates is

$$dV = \sqrt{\det \left(I_{ij} + \frac{\partial y}{\partial x^i} \frac{\partial y}{\partial x^j} \right)} \, dx^1 dx^2 \cdots dx^n$$

and let dV_y denote the volume element of the part of M between height y and height $y + dy$. We can divide M into n pieces where $\partial y / \partial x^1$ is largest, or $\partial y / \partial x^2$, or so on, and estimate dV_y from above on each piece. Where $\partial y / \partial x^1$ is largest, we take y, $x^2, \ldots, x^n$ as coordinates. Then

$$\sqrt{\det \left(I_{ij} + \frac{\partial y}{\partial x^i} \frac{\partial y}{\partial x^j} \right)} \leq C \left| \frac{\partial y}{\partial x^1} \right|$$

since $\partial y/\partial x^1$ is larger than the other derivatives, and $|\partial y/\partial x^i| \geq \delta > 0$. This makes

$$dV \leq C dy dx^2 \cdots dx^n$$

on this patch. Moreover our gradient estimate shows that

$$|X| \leq Cy$$

for a suitably large constant. Therefore

$$dV_y \leq C y^{n-1} dy$$

since the region of integration over $x^2, \ldots, x^n$ to get dV_y from dV has its volume bounded by Cy^{n-1}. Then

$$d\widetilde{V}_y \leq \frac{C}{y \ln^n y}\, dy \qquad \text{and} \qquad \widetilde{V} \leq C \int \frac{dy}{y \ln^n y} < \infty$$

3.

By contrast we have the following result.

Theorem 3.1. *Let U be an open subset of the unit sphere S^n which is not empty and whose closure is not the whole sphere. Then there is no metric on U conformal to the round metric which is complete with finite volume.*

Proof. By hypothesis we can find some point N which is contained in U, and some point S which avoids the closure U. By a conformal transformation, we can take N to be the north pole and S the south pole. We can then find an $\varepsilon > 0$ so that the ε-ball around N lies in U, while the ε-ball around S avoids U. We can then find a conformal map of the sphere S^n minus these two balls to the cylinder $S^{n-1} \times [0, L]$, taking the boundary of the ε-ball around N to $S^{n-1} \times \{0\}$ and the boundary of the ε-ball around S to $S^{n-1} \times \{L\}$. The part of U outside the ε-ball around N will map to some relatively open subset W of the cylinder which contains $S^{n-1} \times \{0\}$ and avoids $S^{n-1} \times \{L\}$. The subset W will be a non-compact manifold with one compact boundary component S^{n-1}. Any complete metric on U wuth finite volume conformal to the round metric on S^n would give a complete metric with finite volume on W conformal to the product metric on $S^{n-1} \times [0, L]$. We show that such cannot exist.

We introduce coordinates $\Theta = (\theta_1, \dots, \theta_{n-1})$ on S^{n-1} and t on $[0, L]$. Let $d\sigma$ denote the metric on S^{n-1} and $d\mu$ the volume form. Then $ds^2 = d\sigma^2 + dt^2$ is the product metric on $S^{n-1} \times [0, L]$, and $dV = d\mu dt$ is the product volume form. For every $\Theta \in S^{n-1}$, there will be a first point $t = h(\Theta)$ where the pair is no longer in W, Of course h may not be a continuous function, and the pair may re-enter W for larger values of t. This does not matter. Any conformally equivalent metric on W is given by

$$d\tilde{s} = \rho(\Theta, t) ds$$

for some function ρ defined at least for $0 \leq t < h(\Theta)$. The corresponding volume form is

$$d\widetilde{V} = \rho^n d\mu dt\,.$$

If the total volume $\widetilde{V}$ of the conformally equivalent metric is finite, we have

$$\iint_W \rho^n d\mu dt = \widetilde{V} < \infty\,.$$

By Hölder's inequality

$$\iint_W \rho d\mu dt \leq \left\{ \iint_W \rho^n d\mu dt \right\}^{1/n} \left\{ \iint_W 1 d\mu dt \right\}^{\frac{n-1}{n}}$$

and surely

$$\iint_W 1 d\mu dt \leq L \operatorname{vol}(S^{n-1}) < \infty\,.$$

Therefore

$$\iint_{0 \leq t < h(\Theta)} \rho(\Theta, t) d\mu dt < \infty\,.$$

On the other hand, if we integrate first in t, we see that

$$\int_{S^{n-1}} \left\{ \int_0^{h(\Theta)} \rho(\Theta, t) dt \right\} d\mu \geq \operatorname{vol}(S^{n-1}) \inf_\Theta \int_0^{h(\Theta)} \rho(\Theta, t) dt$$

and therefore

$$\inf_\Theta \int_0^{h(\Theta)} \rho(\Theta, t) dt < \infty\,.$$

But along a path where Θ is constant we have $d\tilde{s} = \rho dt$. Thus there is some Θ where the path from $(\Theta, 0)$ to $(\Theta, h(\Theta))$ has finite length. This shows that the metric is not complete, and proves Theorem 3.1. □

4.

It only remains to observe that the Gauss map gives a diffeomorphism of the convex hypersurface M^n onto an open subset U of the sphere S^n which lies in a hemisphere. Thus U is not empty and its closure is not all of S^n. However, the pinching condition

$$H_{ij} \geq \varepsilon H g_{ij}$$

is equivalent to the assertion that the Gauss map is quasi-conformal. Since M has a conformally equivalent metric which is complete with finite volume, so must U. But this is impossible.

5.

Finally we indicate how to use this result to prove Huisken's result [Hu1] on compact convex hypersurfaces shrinking to a point under the Mean Curvature flow. Huisken shows that the pinching estimate

$$H_{ij} \geq \varepsilon H g_{ij}$$

is preserved by the flow. This is an easy consequence of the maximum principle. The usual derivative estimates guarantee that after a short time we can control all the covariant derivatives of the H_{ij} in terms of the size of H_{ij} itself, which in turn is controlled by H alone when $H_{ij} > 0$. As a result, the solution surely exists until H is unbounded as $t \to T$ for some $T < \infty$.

Moreover a standard argument allows us to "blow up" the singularity by taking a limit of a sequence of translations and dilations of solutions. We distinguish two cases. In Case I, where we have

$$\limsup\,(T-t)H^2 = A < \infty$$

the limit will be a solution to the Mean Curvature flow on $-\infty < t < A$ with $H = 1$ at the origin at time 0 and

$$0 < H \leq \frac{A}{A-t}$$

everywhere for all time. Because of the pinching condition our result shows the limit must be compact, and then Huisken's uniqueness result [Hu2] shows it must be the round sphere as desired. In Case II, where we have

$$\limsup\,(T-t)H^2=\infty$$

the limit will be a solution to the Mean Curvature flow on $-\infty<t<\infty$ with $H=1$ at the origin at time 0 and

$$0<H\le 1$$

everywhere for all time. The Harnack estimate applied with the strong maximum principle [Ha] shows this limit must be a translating soliton. All we need observe, however, is that it lasts forever and hence cannot be compact. Then the pinching estimate gives a contradiction, so this case is ruled out.

References

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