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REMARKS ON THE ENTROPY AND HARNACK ESTIMATES FOR THE GAUSS CURVATURE FLOW

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Consider a smooth compact strictly convex body $M^n \subseteq \mathbb{R}^{n+1}$ subject to wear under impact at a random angle, such as a stone being tossed on a beach. The probability of impact at any point P is proportional to the Gauss curvature K. Thus the surface evolves in time by the Gauss curvature flow

$$\frac{\partial P}{\partial t} = -KN$$

where N is the unit outward normal. This equation was first studied by Firey [Fi]. Later Tso [Ts] showed that the solution exists and stays smooth and strictly convex until $t \to T$ for some time T when the diameter L shrinks to zero. Thus we can assume M^n shrinks to the origin 0. Recently Chow [Ch] proved an entropy and a Harnack estimate for this flow. We use these results to derive two useful corollaries.

Main Theorem 1. *The diameter L satisfies a dilation-invariant bound*

$$L \leq C(T-t)^{1/(n+1)}$$

This follows by combining Chow's upper bound on the entropy with a lower bound in terms of the least shadow area.

Main Theorem 2. *The Gauss curvature K satisfies a dilation-invariant bound*

$$K \leq \left(\frac{T}{t}\right)^{\frac{n}{n+1}} \frac{S}{T-t}$$

where $S = \langle P, N\rangle$ is the support function around the limit point 0. This result follow immediately from considering the Harnack estimate along radial lines. Combining these gives us the following.

Corollary. *The Gauss curvature flow satisfies a dilation-invariant bound*

$$K \leq C/(T-t)^{n/(n+1)}\,.$$

This follows since $S \leq L$ everywhere.

Our appreciation to Ben Chow and Mike Gage for many useful conversations.

1.

In this section we study the entropy integral

$$E = \int K \ell n K da_{\text{M}}$$

where da_{M} is the area element on the surface M.

1a. In his important paper [Ch], Ben Chow proves the following result. Let E_t be the value of the entropy at time t, E_0 its initial value, and σ_n the area of the unit n-sphere.

1a1 Theorem. (Chow). *The entropy has an upper bound*

$$E_t \leq E_0 + \sigma_n \ell n (T/T-t)^{n/(n+1)}\,.$$

Proof. See [Ch] p. 481. □

1b. Now we give a useful lower bound on the entropy of any compact convex smooth hypersurface M^n in $\mathbb{R}^{n+1}$.

1b1 Definition. The least shadow area $A_{\#}$ of M is the least area of the image of M under any projection onto a hyperplane $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$.

Again we let σ_n be the area of the unit n-sphere S^n, and we define the constant $C_n < \infty$ by

$$C_n = \int_{S^n_+} \ell n \sec \psi da_{\text{s}}$$

where S^n_+ is the unit northern hemisphere, ψ is the angle of declination from the north pole, and da_{s} is the area element on the sphere. The improper integral is finite because sec ψ is inversely proportional to the distance from the equator, and the logarithm grows quite slowly.

1b2 Theorem. *The entropy E has lower bound in terms of the least shadow area $A_\#$ given by*

$$E \geq \sigma_n \ell n(\sigma_n/2A_\#) - 2C_n\,.$$

Proof. Let P be the projection whose shadow has the least area $A_\#$. The surface M^n divides into two regions B' and B'' such that the projection P is one-to-one on each region with image $A_\#$, while the orientation of the projection is opposite on their interiors. We then apply the following Lemma on both B' and B'' and add. Note that the images of B' and B'' in the Gauss sphere are two hemispheres of area $\sigma_n/2$. □

1b3 Lemma. *Let P be any projection of a convex surface M^n on a hyperplane R^n in R^{n+1}, let B be any subset of M^n with unit normal N lying in one hemisphere, let $A_P(B)$ be the area of the image of B under the projection P, and let $A_s(B)$ be the area of the image of B in the Gauss sphere under the Gauss map G. Then the entropy of B*

$$E(B) = \int_B K \ell n K da_M$$

has a lower bound

$$E(B) \geq A_s(B) \ell n[A_s(B)\,/A_P(B)] - C_n\,.$$

Proof. Let da_S, da_M, and da_P denote the area elements on the Gauss sphere S^n, the surface M^n and the plane R^n which is the image of the projection P. Rotate so the projection is vertical onto the horizontal hyperplane R^n, with the normal N on B always in the northern hemisphere S^n_+, and let ψ be the angle of declination from the north pole. Then

$$da_S = K da_M \quad \text{and} \quad da_M = \sec \psi da_P\,.$$

This makes

$$\int_{G(B)} 1 da_S = A_S(B) \quad \text{and} \quad \int_{G(B)} \frac{1}{K \sec \psi}\, da_S = A_P(B)$$

while

$$\int_{G(B)} \ell n \left(\frac{1}{K \sec \psi} \right) da_S = -E(B) - C_M\,.$$

Now Jensen's inequality tells us that the average value of the logarithm is less then or equal to the logarithm of the average, since the logarithm is concave. Hence

$$\int_{G(B)} \ell n\left(\frac{1}{K\sec\psi}\right)\frac{da_{\mathrm{S}}}{\int_{G(B)} 1 da_{\mathrm{S}}} \le \ell n\left(\int_{G(B)} \frac{1}{K\sec\psi}\frac{da_{\mathrm{S}}}{\int_{G(B)} 1 da_{\mathrm{S}}}\right)$$

and the result follows. □

1c. If we combine the upper bound and the lower bound on the entropy we get a lower bound on the least shadow area $A_\#$.

1c1 Theorem. *The least shadow area* $A_\#$ *satisfies*

$$A_\#(t) \ge C_\#(T-t)^{n/(n+1)}$$

where the constant $C_\#$ *is given by*

$$C_\# = \frac{\sigma_n}{2T^{n/(n+1)}}\, e^{-(E_0+2C_n)/\sigma_n}\,.$$

Proof. We have

$$\sigma_n \ell n(\sigma_n/2A_\#) - 2C_n \le E \le E_0 + \sigma_n \ell n(T/T-t)^{n/n+1}$$

and the result follows if we solve for $A_\#$. □

The bound on the least shadow area gives us a bound on the diameter.

1d1 Definition. The diameter L of a compact convex body M is the length of the longest line segment contained in M. The volume V of M is the volume of the region enclosed.

1d2 Lemma. *The volume* V *in the Gauss curvature flow is given by*

$$V = \sigma_n(T-t)\,.$$

Proof. We have

$$\frac{dV}{dt} = -\int_M K da_{\mathrm{M}} = -\sigma_n$$

and $V \to 0$ as $t \to T$. □

1d3 Theorem. *The diameter L in the Gauss curvature flow satisfies dilation-invariant bounds*

$$c(T-t)^{1/(n+1)} \leq L \leq C(T-t)^{1/(n+1)}$$

for some constants $c > 0$ and $C < \infty$ which depend only on n, T and E_0.

The proof depends on a geometric Lemma, which we give first.

1d4 Lemma. *For any convex body M^n, the volume V, diameter L and least shadow area $A_\#$ are related by*

$$V \geq \frac{1}{3} LA_\# .$$

Proof. Find a line segment in M^n with length L, let P be the projection orthogonal to it, and let A_D be the area of the shadow, the projection of M^n under P onto a hyperplane. Rotate so the projection P is vertical onto a horizontal hyperplane $\mathbb{R}^n$, and translate so the line of length L lies over the origin 0 in $\mathbb{R}^n$.

Introduce cylindrical coordinates r, θ and z, where r is the distance from the origin in $\mathbb{R}^n$, $\theta \in S^{n-1}$ the angle, and z the height in the perpendicular direction. The shadow of M^n under P will be a set given by an inequality

$$0 \leq r \leq f(\theta)$$

for some function $f(\theta)$, and the set M^n itself will be the graph over this set of two functions $h_1(r,\theta)$ and $h_2(r,\theta)$, while the region inside is given by

$$h_1(r,\theta) \leq z \leq h_2(r,\theta) .$$

Note that when $r = f(\theta)$ we have

$$h_1(f(\theta),\theta) = h_2(f(\theta),\theta) = h(\theta)$$

for some function $h(\theta)$. If we translate in the z-direction so that the line segment of length L starts at the origin in R^{n+1}, we get

$$h_1(0,\theta) = 0 \quad \text{and} \quad h_2(0,\theta) = L$$

for all θ.

Since h_1 is concave and h_2 is convex, we must have

$$h_1(r,\theta) \leq \frac{r}{f(\theta)}\, h(\theta)$$

and

$$h_2(r,\theta) \geq \frac{r}{f(\theta)}\, h(\theta) + \left[1 - \frac{r}{f(\theta)}\right] L\,.$$

The volume V inside M^n is given by

$$V = \int_{S^{n-1}} \int_{r=0}^{f(\theta)} \int_{z=h_1(r,\theta)}^{h_2(r,\theta)} r dz dr d\theta$$

in cylindrical coordinates. Using the above inequalities and doing the two inner integrals gives

$$V \geq \frac{L}{6} \int_{S^{n-1}} f(\theta)^2 d\theta\,.$$

On the other hand, the area of the shadow is given by

$$A_b = \int_{S^{n-1}} \int_{r=0}^{f(\theta)} r dr d\theta$$

in polar coordinates, so

$$A_b = \frac{1}{2} \int_{S^{n-1}} f(\theta)^2 d\theta$$

which makes

$$V \geq \frac{1}{3}\, L A_b\,.$$

Since $A_\#$ is the least shadow area, $A_b \geq A_\#$. This proves the Lemma. □

Now we can finish the proof of Theorem 1d3. We have

$$L \leq 3V/A_\#$$

by the previous Lemma,

$$V = \sigma_n(T-t)$$

by Lemma 1d2, and

$$A_\# \geq C_\#(T-t)^{n/(n+1)}$$

by Theorem 1c1. This gives

$$L \leq C(T-t)^{1/(n+1)}$$

for a constant C determined by $C_\#$ and n. Finally, M^n must be contained in a ball of radius L, so

$$V \leq \sigma_n L^{n+1}$$

which gives a lower bound on L of the form

$$L \geq c(T-t)^{1/(n+1)}$$

where c depends only on n.

2.

In this section we study the Harnack inequality for the Gauss curvature flow. It tells us that if the Gauss curvature K is large at some point after some time has elapsed, it must be comparably large at nearby points after some more time has elapsed. This is a common feature in heat equations where some function is always positive, as K is in this case.

2a. The Harnack estimate for the Gauss curvature flow was derived by Chow in [Ch]. He proves the following result (in his Theorem 3.7).

2a1 Theorem. (Chow). *For any points X_1 on M^n at time t_1 and X_2 on M^n at time t_2 with $0 < t_1 < t_2$ we have*

$$\frac{K(X_2,t_2)}{K(X_1,t_1)} \geq \left(\frac{t_1}{t_2}\right)^{\frac{n}{n+1}} e^{-\theta/4}$$

where θ is the integral over any path $X = X(t)$ on the surface at each time t with $X = X_1$ at $t = t_1$ and $X = X_2$ at $t = t_2$ given by

$$\theta = \int_{t_1}^{t_2} KH_{ij}^{-1}V^iV^j dt$$

where H_{ij}^{-1} is the inverse of the second fundamental form and V^i is the tangential component of the velocity of the path X, so that

$$\frac{dX^\alpha}{dt} = -KN^\alpha + V^iD_iX^\alpha$$

in local coordinates on M^n.

Chow obtains this result by integrating the following differential Harnack inequality over the path.

2a2 Theorem. (Chow). *In any solution to the Gauss curvature flow for* $t > 0$, *we have*

$$\frac{\partial K}{\partial t} \geq H_{ij}^{-1} D_i K D_j K - \frac{n}{(n+1)t} K \,.$$

Proof. See [Ch], p.478 and multiply by K. □

2b. A very useful version of the Harnack estimate can be obtained by integrating over a path in $\mathbb{R}^{n+1}$ which is just a straight line segment. Of course the velocity will not be constant, but must be adjusted to stay on the surface M^n at each time t. This requires that the line segment from beginning to end must always be transversal to the surface M^n.

2b1 Theorem. *Along any straight line segment transversal to* M^n *the function*

$$t^{n/(n+1)} K \sec \theta$$

always increases, where θ *is the angle between the line segment and the normal to the surface.*

Proof. Suppose the line segment has a unit velocity vector U. Then

$$\frac{dX^\alpha}{dt} = \sigma U^\alpha$$

where σ must be adjusted to stay on the surface. Dividing dX/dt into tangential and normal components

$$\frac{dX^\alpha}{dt} = -K N^\alpha + V^i D_i X^\alpha$$

as before. Since $N \cdot U = -\cos\theta$, we need to take

$$\sigma = K \sec \theta \quad \text{and} \quad V_i = U^\alpha \cdot D_i X^\alpha$$

as we see by dotting with U or $D_j X^\alpha$.

The space and time derivatives of the unit outward normal N are given by Chow in [Ch] as

$$D_i N^\alpha = H_{ij} D_j X^\alpha$$

and

$$\frac{\partial N^\alpha}{\partial t} = D_i K D_i X^\alpha \,.$$

Now along the line segment

$$\frac{dN^\alpha}{dt} = \frac{\partial N^\alpha}{\partial t} + V_i D_i N^\alpha$$

so

$$\frac{dN^\alpha}{dt} = D_i K D_i X^\alpha + H_{ij} V_i D_j X^\alpha\,.$$

Then

$$\frac{dN}{dt} \cdot \frac{dX}{dt} = V_i D_i K + H_{ij} V_i V_j\,.$$

Now since $\sigma = K \sec\theta$,

$$N \cdot U = -\cos\theta = -K/\sigma$$

and

$$\frac{d^2X}{dt^2} = \frac{d\sigma}{dt}\, U$$

so

$$N \cdot \frac{d^2X}{dt^2} = -\frac{K}{\sigma}\frac{d\sigma}{dt} = -K \frac{d}{dt}\, \ell n(K \sec\theta)\,.$$

Also

$$N \cdot \frac{dX}{dt} = -K$$

so

$$\frac{d}{dt}\left(N \cdot \frac{dX}{dt}\right) = -\frac{dK}{dt}$$

differentiating along the path, where

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} + V_i D_i K\,.$$

Finally, note that

$$\frac{d}{dt}\left(N \cdot \frac{dX}{dt}\right) = N \cdot \frac{d^2X}{dt^2} + \frac{dN}{dt} \cdot \frac{dX}{dt}\,.$$

Combining these results gives

$$\frac{\partial K}{\partial t} + 2V_i D_i K + H_{ij} V_i V_j = K \frac{d}{dt}\, \ell n(K \sec\theta)$$

along the straight line segment.

Now Chow's differential Harnack inequality (Theorem 2a2) is equivalent to the statement that

$$\frac{\partial K}{\partial t} + 2V_i D_i K + H_{ij} V_i V_j \geq -\frac{n}{(n+1)t} K$$

for any vector V_i, as the worst possible choice of V_i is

$$V_i = -H_{ij}^{-1} D_j K$$

and even this works. Therefore

$$K \frac{d}{dt} \ell n\,(K \sec \theta) \geq -\frac{n}{(n+1)t} K$$

or equivalently

$$\frac{d}{dt} \ell n(t^{n/(n+1)} K \sec \theta) \geq 0$$

which shows $t^{n/(n+1)} K \sec \theta$ increases along the line segment, as desired. □

2c. We can apply this result to radial paths out of the limit point 0, since they must all be transversal to the flow.

2c1 Theorem. *The Gauss curvature satisfies the bound*

$$K \leq \left(\frac{T}{t}\right)^{\frac{n}{n+1}} \frac{S}{T-t}$$

where $S = \langle P, W \rangle$ *is the support function around the limit point* 0.

Proof. The angle θ between the radial line and the unit normal N is given by

$$S = \langle P, W \rangle = r \cos \theta$$

where $r = |P|$ is the distance to 0. Now along the radial line

$$\frac{dr}{dt} = -K \sec \theta$$

and we know

$$t^{n(n+1)} K \sec \theta$$

is increasing. Thus for $t_0 \leq t < T$ we get

$$t^{n/(n+1)} K \sec \theta \geq t_0^{n/(n+1)} K_0 \sec \theta_0$$

and using $t < T$ and

$$\sec \theta_0 = r_0 / S_0$$

we get

$$r_0 (t_0/T)^{n/(n+1)} K_0 / S_0 \leq -\frac{dr}{dt}\,.$$

Now

$$\int_{t_0}^{T} -\left(\frac{dr}{dt}\right) dt = r_0$$

so we have

$$r_0(t_0/T)^{n/(n+1)}(T - t_0)K_0/S_0 \leq r_0$$

after integrating, or

$$K_0 \leq \left(\frac{T}{t_0}\right)^{\frac{n}{n+1}} \frac{S}{T - t_0}$$

which proves the Theorem if we replace t_0 by t. □

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