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ON PROJECTIVELY FLAT HERMITIAN MANIFOLDS

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Let (M^n, g) be a n-dimensional compact hermitian manifold, with $n \geq 2$. (M, g) will be called *projectively flat,* if its curvature matrix is of the form $\Theta = \alpha I_n$, where α is a $(1,1)$-form. Note that any metric conformal to g would also be projectively flat. In §1, we shall classify such manifolds, and in §2, we will give an application which may be considered as a generalization to higher dimensions of the Bogomolov's Theorem on VII_0 surfaces.

First of all, let us correct an error in our previous paper [L-Y-Z]. We found this error after the paper was in print. On page 220, the vanishing of c_1^2 and c_2 does not imply that the Hermitian-Einstein metric h is flat, but only projectively flat, i.e., $\Theta = \alpha I_2$. So the argument there is incomplete. However, this gap can be easily fixed by applying the results of P. Gauduchon ([G]) and D. Fried ([F]). Start from the projectively flat compact hermitian surface (S, h). By [G], h is locally conformally Kähler. That is, there exists a covering $\{U_\alpha\}$, and $f_\alpha \in \mathcal{C}^\infty(U_\alpha, R)$, such that each $e^{f_\alpha} g$ is a Kähler metric in U_α. Note $e^{f_\alpha} g$ is also projectively flat, hence flat, and in $U_\alpha \cap U_\beta$, $f_\alpha - f_\beta$ is a constant. Therefore, S is a complex similarity manifold. By Theorem 2 of [F], it is either covered by a complex 2-torus or a Hopf surface. This completes the proof of Bogomolov's theorem.[1]

1. Projectively flat manifolds

Now let us consider the projectively flat manifolds in general dimensions. First let us fix some notations. On a hermitian manifold (M^n, g), let $e =$

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[1]This error was also recently pointed out to us by A. Teleman, who found another interesting way to correct it.

$(e_1, \dots, e_n)$ be a local unitary frame and $\varphi = (\varphi_1, \dots, \varphi_n)$ its dual coframe. Let θ, Θ the $n \times n$ matrices of connection and curvature under e, and $\tau = (\tau_1, \dots, \tau_n)$ the torsion forms under e. Each τ_i is a $(2,0)$-form. The structure equations and the first Bianchi identity are:

$$d\varphi = \varphi \wedge \theta + \tau \ , \quad d\theta - \theta \wedge \theta = \Theta \ , \quad d\tau = \varphi \wedge \Theta - \tau \wedge \theta$$

Write $\tau_i = \frac{1}{2} \sum_{j,k=1}^{n} T^i_{jk} \varphi_j \wedge \varphi_k$, where $T^i_{jk} = -T^i_{kj}$, and denote by $\omega = \varphi \wedge \varphi^*$ the Kähler form of g (we omit the factor $\sqrt{-1}$).

Consider the Gauduchon torsion 1-form η ([G]) defined by

$$\eta = \frac{1}{n-1} \sum_{j,k=1}^{n} T^k_{jk} \varphi_j .$$

It is easy to check that $\partial(\omega^{n-1}) = (n-1)\eta \wedge \omega^{n-1}$, hence is uniquely determined and globally defined. First of all, one has:

Lemma 1. *If $\Theta = \alpha I_n$, then $\overline{\partial}\eta = \alpha$.*

Proof. Write the $(0,1)$ part of θ as $\theta''_{ij} = \sum_{l=1}^{n} A_{ij,\bar{l}} \overline{\varphi}_l$ and $\alpha = \sum_{i,j=1}^{n} \alpha_{i\bar{j}} \varphi_i \wedge \overline{\varphi}_j$. By the structure equation and the first Bianchi identity,

$$\overline{\partial}\varphi = \varphi \wedge \theta'' ; \qquad \overline{\partial}\tau = \varphi \wedge \alpha - \tau \wedge \theta''$$

Hence for each i, j, k, and l, one has:

$$\nabla_{\bar{l}} T^i_{jk} = \delta_{ij}\alpha_{k\bar{l}} - \delta_{ik}\alpha_{j\bar{l}} + \sum_{r=1}^{n} T^i_{rk} A_{jr,\bar{l}} - \sum_{r=1}^{n} T^i_{rj} A_{kr,\bar{l}} - \sum_{r=1}^{n} T^r_{jk} A_{ri,\bar{l}}$$

Therefore

$$\sum_{k=1}^{n} \nabla_{\bar{l}} T^k_{jk} = -(n-1)\alpha_{j\bar{l}} + \sum_{r,k=1}^{n} T^k_{rk} A_{jr,\bar{l}}$$

This leads to $\overline{\partial}\eta = \alpha$, and the lemma is proved. □

Next let $\sigma = (\tau - \eta \wedge \varphi) \otimes {}^t e$. Then one has $\overline{\partial}\sigma = (\alpha - \overline{\partial}\eta)\varphi \otimes {}^t e = 0$. So σ is a holomorphic section of $E = \Omega_M \otimes \Omega_M \otimes T_M$, where T_M, Ω_M denotes the holomorphic tangent and cotangent bundle of M. Let h be the hermitian metric on E induced from g on T_M. Fix a point $x \in M$, choose holomorphic frame $v_1, \dots, v_N$ of E near x, so that at x, $h_{i\bar{j}} = \delta_{ij}$, $dh_{i\bar{j}} = 0$. Write $\sigma = \sum_{i=1}^{N} \sigma_i v_i$, then at x:

$$\partial\bar{\partial}\|\sigma\|^2 = \sum_{i=1}^{N} \partial\sigma_i \wedge \overline{\partial\sigma_i} - \Theta_{\sigma\bar{\sigma}}(h) \geq \|\sigma\|^2 \alpha$$

Here we used the fact that $\Theta(h) = -\alpha I_N$. Note that if $\partial\bar{\partial}\omega^{n-1} = 0$, then $\bar{\partial}\eta \wedge \omega^{n-1} = \eta \wedge \bar{\eta} \wedge \omega^{n-1}$. When M is compact,

$$0 = \int \partial\bar{\partial}\|\sigma\|^2 \wedge \omega^{n-1} \geq \int \|\sigma\|^2 \eta \wedge \bar{\eta} \wedge \omega^{n-1}$$

therefore we have

Lemma 2. *If M is compact, $\Theta = \alpha I_n$, and $\partial\bar{\partial}\omega^{n-1} = 0$, then either $\eta = 0$ or $\tau = \eta \wedge \varphi$.*

When $\eta = 0$, ((M, g) is called balanced,) the first Ricci form r equals to the third Ricci form s:

$$r_{i\bar{j}} - s_{i\bar{j}} = \sum_{k=1}^{n} R_{k\bar{k}i\bar{j}} - R_{i\bar{k}k\bar{j}} = \nabla_{\bar{j}} \sum_{k=1}^{n} T_{ik}^{k} = 0$$

But for projectively flat metric, $r - s = (n-1)\alpha$, so $\eta = 0$ implies $\Theta = 0$ in this case.

While when $\tau = \eta \wedge \varphi$, $\partial\omega = \tau \wedge \varphi^* = \eta \wedge \omega$. Hence $\partial\eta \wedge \omega = 0$. When $n \geq 3$, this gives $\partial\eta = 0$, while when $n = 2$, since $\bar{\partial}\eta = \alpha$ is closed, $0 = \int \partial\eta \wedge \overline{\partial\eta}$ implies $\partial\eta = 0$. Therefore $d(\eta + \bar{\eta}) = \alpha + \bar{\alpha} = 0$, so locally g will be conformal to some Kähler metric, which is necessarily flat. So M is a complex similarity manifold. By [F], (M, g) is a finite undercover of either a flat complex torus, or a Hopf manifold of the form $(C^n \setminus 0)/Z\phi$, where $\phi(z) = azA$ is a complex expansion: $A \in U(n)$, $a > 1$ and $z = (z_1, \dots, z_n)$.

In conclusion, one has:

Theorem 1. *Let (M^n, g) be a compact projectively flat hermitian manifold, and suppose its Kähler form ω satisfies $\partial\bar{\partial}\omega^{n-1} = 0$. Then either (M, g) is flat and balanced ($d\omega^{n-1} = 0$), or M is a finite undercover of a quotient $C^n \setminus 0/Z\phi$ with ϕ a complex expansion.*

Note that by [G1], for any compact hermitian manifold (M, g), there exists an unique (up to homothety) metric h in the conformal class of g such that $\partial\bar{\partial}\omega_h^{n-1} = 0$; and if g is projectively flat, so is h. Therefore, *any projectively flat metric on M is conformal to one of the metrics in Theorem 1.*

For compact hermitian flat manifold (M, g), the torsion tensor is parallel. So the first Bianchi identity gives exactly the Poisson identities, and the universal covering space is a complex Lie group G equipped with an left invariant flat metric. That is, $M = \Gamma\backslash G$, where $\Gamma \subset G \cdot C$ is a discrete subgroup of the semidirect product of G with a compact subgroup $C \subset \mathrm{Aut}(G)$. See [Go] or [K-T] for example.

As a byproduct, we get the following

Corllary 2. *Any compact hermitian flat manifold (M^n, g) is balanced, i.e., $d\omega_g^{n-1} = 0$.*

This is because we can first conformally deform g to get a balanced and projectively flat metric $h = e^f g$. Since any connected Lie group G is either a $K(\pi, 1)$ or has $\pi_3(G) \neq 0$, so G can not be homotopic to S^{2n-1} if $n \geq 3$; while when $n = 2$, there are only two simply-connected complex Lie groups, both biholomorphic to C^2, so M can not be Hopf. Therefore, by Theorem 1, we know h is again flat, so f is pluriharmonic, hence a constant, and g is balanced.

In particular, for $n = 2$, we get the well-known fact that any compact hermitian flat surface has to be Kähler, namely, a complex 2-torus or a hyperelliptic surface.

2. An corollary

By the proof of [L-Y-Z], the theorem of Bogomolov on VII_0 surfaces ([B], [B1]) can now be stated in a slightly more general way, namely, if M^2 is a compact complex surface with stable tangent bundle T_M (with respect to a hermitian metric) and with $c_1^2 = c_2 = 0$, then M must be either flat or similarity Hopf.

In this section, we want to generalize this into higher dimensions by apply Theorem 1 in §1. First let us recall the definition of *refined Chern classes* by

Bott and Chern ([B-C]). Suppose E is a holomorphic vector bundle over a compact complex manifold M^n. Then for any two hermitian metrics h, h' on E, there always exists smooth functions f_k such that $\sqrt{-1}\partial\overline{\partial} f_k = C_k(h) - C_k(h')$, where C_k denotes the Chern forms. Hence the Chern forms define the refined Chern classes $\hat{c}_k(E)$ in $\hat{H}^{k,k}(M) = \mathrm{Ker}(d) \cap A^{k,k}/\,\mathrm{Im}(\sqrt{-1}\partial\overline{\partial})$. Here $A^{k,k}$ is the space of all smooth real (k,k) forms on M.

Next, let us recall the definition of *astheno-Kähler* from [J-Y]. A hermitian metric g on a compact complex n-manifold M is called astheno-Kähler, if its metric form ω_g satisfies $\partial\overline{\partial}\omega_g^{n-2} = 0$.

Any product manifold of curves and surfaces is astheno-Kähler. However, it would be more intersting to construct some "non-trivial" examples.

A necessary condition for the existence of such metrics is that, any semipositive $(2,2)$ current can not be $\partial\overline{\partial}$-exact (unless it is trivial). Note that for $n=3$, this is also a sufficient condition. (More generally, on a compact complex manifold, the non-existence of (non-trivial) $\partial\overline{\partial}$-exact positive $(n-1,n-1)$ current (acting on $(1,1)$ forms) always implies the existence of a hermitian metric g with $\partial\overline{\partial}\omega_g = 0$. Following the work of Harvey and Lawson ([H-L]), this is not hard to show.)

In particular, any global holomorphic 1-form φ on M must be closed, as $\partial\overline{\partial}(\varphi\wedge\overline{\varphi}) = -\partial\varphi\wedge\overline{\partial\varphi} \geq 0$. So, for example, a compact complex parallelizable manifold M (i.e., T_M holomorphicly trivial) can not be astheno-Kähler unless it is a complex torus. However, we believe that this definition has its potential in the future study of the non-Kähler geometry.

Now if we start off with a compact astheno-Kähler manifold (M^n, g). Let E be a g-polystable holomorphic vector bundle of rank r on M (i.e., E is the direct sum of g-stable bundles with the same g-slope (g-degree divided by rank)). Then by [L-Y] (when $n=2$, also by Buchdahl [Bu]), E admits a hermitian metric h which is g-Einstein: $tr_g(\Theta_h) = \mu I_r$ for some constant μ. By the Lübke-Kobayashi inequality, $(C_1^2(E,h) - \frac{2r}{r-1}C_2(E,h)) \wedge \omega_g^{n-2} \leq 0$ pointwisely. So if $\hat{c}_1^2(E) - \frac{2r}{r-1}\hat{c}_2(E) = 0$ in $\hat{H}^{2,2}(M)$ (or ≥ 0 in the obvious sense), then (E,h) is projectively flat: $\Theta_h = \alpha I_r$. In particular, when $E = T_M$, by Theorem1, we get the following:

Corllary 3. *Let (M^n, g) be a compact hermitian manifold which is astheno-kähler (i.e., $\partial\overline{\partial}\omega_g^{n-2} = 0$). Suppose that T_M is g-polystable and the refined Chern classes satisfy $\hat{c}_1^2 = \hat{c}_2 = 0$ in $\hat{H}^{2,2}(M)$. Then either M is similarity Hopf, or it admits a flat hermitian metric h.*

Obviously the condition on the refined Chern classes can be replaced by $\hat{c}_1^2 - \frac{2n}{n-1}\hat{c}_2 \geq 0$ in the sense that it can be represented by a pointwisely nonnegative $(2,2)$ form, or that its product with any $[\Omega]$ is nonnegative, for any $\partial\overline{\partial}$-closed nonnegative $(n-2, n-2)$ form Ω on M.

We also conjecture that the non-Kähler flat manifolds or similarity Hopf manifolds of dimension ≥ 3 do not admit astheno-Kähler metrics. This is true in some special cases, but at this moment we are unable to prove it in general. After this, the conclusion of Corollary 3 could be replaced by: "M is covered by a complex torus".

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References

[B] Bogomolov, F.A., *Classification of surfaces of class VII_0 with $b_2 = 0$,* Math. USSR-Izv. **10** (1976), 255–269.

[B1] Bogomolov, F.A., *Surfaces of class VII_0 and affine geometry,* Math. USSR-Izv. **21** (1983), 31–73.

[Bu] Buchdahl, N.P., *Hermitian-Einstein connections and stable vector bundles over compact complex surfaces,* Math. Ann. **280** (1988), 625–648.

[B-C] Bott, R. and Chern, S.S., *Hermitian vector bundles and the equidistribution of zeroes of their holomorphic sections,* Acta Math., **114** (1965), 71–112.

[F] Fried, D., *Closed similarity manifolds,* Comm. Math. Helv. **55** (1980), 576–582.

[G] Gauduchon, P., *La 1-forme de torsion d'une variété hermitienne,* Math. Ann. **267** (1984), 495–518.

[G1] Gauduchon, P., *Le théorème de l'excentricité nulle,* C.R. Acad. Sci. Paris **258** (1953), 387–390.

[Go] Goldberg, S., *Curvature and Topology,* Academic Press, New York and London 1962.

[H-L] Harvey, R. and Lawson, H.B. Jr., *An intrinsic characterization of Kähler manifolds,* Invent. Math. **74** (1983), 169–198

[J-Y] Jost, J. and Yau, S.-T., *A nonlinear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry,* to appear in Acta. Math..

[K-T] Kamber, F. and Tondeur, P., *Flat Manifolds,* Lecture Notes in Math. vol.**67**, Springer-Verlag, New York 1968.

[L-Y] Li, J. and Yau, S.-T., *Hermitian-Yang-Mills connections on non-Kähler manifolds,* in "Mathematical Aspects of String Theory", (ed. by S.T. Yau), World Sci. Publ. London, 1987, 560–573.

[L-Y-Z] Li, J., Yau, S.-T. and Zheng, F., *A simple proof of Bogomolov's theorem on class VII_0 surfaces with $b_2 = 0$,* Illinois J. Math. **34** (1990), 217–220.

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