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A COUNTEREXAMPLE IN A UNIQUE CONTINUATION PROBLEM

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The purpose of this paper is to prove the following result.

Theorem 1. (a) *If $d \geq 4$ there is a smooth function $u : \mathbb{R}^d \rightarrow \mathbb{R}$, not identically zero, which vanishes to infinite order at the origin and satisfies $|\triangle u(x)| \leq C|x|^{-1}|\bigtriangledown u(x)|$ for a certain constant C.*

(b) *If $d \geq 5$ then the function u in (a) may be taken so that in addition $|\triangle u| \leq V|\bigtriangledown u|$ with $V \in L^d$.*

Part (b) complements work of Jerison-Kenig [3] and the author [6, 8]. Namely, the analogous question for Schrodinger type inequalities is whether there can be a function satisfying $|\triangle u| \leq V|u|$ with $V \in L^{d/2}$ and vanishing to infinite order at the origin, and in [3] this is shown to be impossible in all dimensions $d \geq 3$. In [8] it is shown that there is no function satisfying the inequality of (b) and vanishing on an open set, and in [6] it is shown that if $d \leq 4$ there is no function satisfying this inequality and vanishing to infinite order at a point. Analogous positive results are also known for the inequality in (a): see for example Pan [4] where it is shown that there is no function vanishing to infinite order at the origin and satisfying $|\triangle u| \leq C|x|^{-2}|u|$, and also that examples as in (a) are impossible when d=2. Note that we leave it open whether such examples exist when d=3 and more significantly do not answer the question of what is the best L^p exponent to replace d in (b). It's not clear to us whether $d + \epsilon$ should work or whether possibly the exponent $\frac{3d-4}{2}$ obtained in [6] is optimal. There is a procedure going back to Plis (e.g. [5]) and P. Cohen for constructing such counterexamples and we will follow this procedure here, at least in principle. It can be thought of as taking place in two stages: a finite construction followed by an iteration. Section 1 of this paper contains the finite construction and Section 2 contains the iteration. We will use the notation $x \lesssim y$ to mean that $x \leq Cy$ where C is a constant depending only on the dimension or other clearly specified quantities, and $x \approx y$ for "$x \lesssim y$ and $y \lesssim x$".

We are expending quite a lot of effort to gain comparatively little, since it is easy (in any $\mathbb{R}^d$) to find functions $u : \mathbb{R}^d \to \mathbb{R}$ vanishing to infinite order at the origin and such that $\frac{\Delta u}{|\nabla u|} \in L^p$ for all $p < d$. On the other hand, in order to prove Theorem 1 it is necessary to work with highly oscillatory functions, for reasons which are discussed at the end of [7], and it seems unlikely (to the author at least) that there is a way of doing this which does not involve a fair amount of calculation.

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1. Proof of Theorem 1, Part 1.

In this section we prove Lemma 1.1 stated below.

We always assume $d \geq 3$, and denote variables in $\mathbb{R}^d$ by $x = (\overline{x}, \overline{\overline{x}})$ with $\overline{x} \in \mathbb{R}^2$ and $\overline{\overline{x}} \in \mathbb{R}^{d-2}$. We identify $\overline{x}$ with the complex number $x_1 + ix_2$, and define r, θ and ρ by $\overline{x} = (r\cos\theta, r\sin\theta)$ and $|\overline{\overline{x}}| = \rho$. We let α and β be two small positive constants to be specified later and define $A_n = \{x \in \mathbb{R}^d : 1 - \alpha \leq r \leq 1 + \alpha, \rho \leq \beta n^{-1/2}\}$.

Lemma 1.1. *Assume $d \geq 4$. If n is sufficiently large then there is a smooth function $u_n : \mathbb{R}^d \to \mathbb{R}$ which is even if n is even and odd if n is odd, and such that*

(i) $u_n(x) = \mathcal{O}(|x|^n)$ *at* ∞, $u_n(x) = \mathcal{O}(|x|^{n+2})$ *at* 0.
(ii) *supp* $\Delta u_n \subset A_n$.
(iii) $|\Delta u_n| \lesssim r^n$ *on* A_n.
(iv) $|\nabla u_n| \gtrsim r^n$ *on* A_n.

Remark. 1) The constants in (iii) and (iv) are of course independent of n.

2) The "shape" of the A_n (two long sides, $d-2$ short sides) should be compared to known counterexamples involving Carleman inequalities and related oscillatory integrals, e.g. [1, 2].

3) An immediate consequence of (ii), (iii), (iv) is that $\|\frac{\Delta u_n}{|\nabla u_n|}\|_d \lesssim n^{-\frac{1}{2}(d-2)}$. This is best possible, in the sense that there is a constant C such that any function u_n with $\Delta u_n \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$ and satisfying (i) must also satisfy $\|\frac{\Delta u_n}{|\nabla un|}\|_d \geq Cn^{-\frac{1}{2}(d-2)}$. This is a consequence of a Carleman inequality proved in Lemma 3.1 of [6] - see [9] for further discussion.

4) Note that the functions in Lemma 1.1 as well as in Theorem 1 are real valued. It is slightly easier to obtain complex valued examples, but we regarded this as cheating, for several reasons, especially the fact that topological issues are in principle relevant when one wants a lower bound on the gradient such as (iv). On the other hand, all the positive results mentioned above including the one in remark 3) are proved using versions of the Carleman method and are therefore valid also for complex valued or for that matter vector valued functions (Added in proof: the two dimensional result in [4,Theorem 3] should be excepted here).

The proof of Lemma 1.1 is based on considerations involving certain explicit functions w_n which we now define. Assume $d \geq 3$ and let

$$Q_n(x) = \int_{-\pi}^{\pi} (|\overline{x} - e^{i\phi}|^2 + |\overline{\overline{x}}|^2)^{-(\frac{d-2}{2})} e^{in\phi} d\phi \tag{1}$$

where we are identifying $\overline{x}$ with $x_1 + ix_2$ as previously discussed. Up to a multiplicative constant Q_n is the potential of the measure $e^{in\theta}d\theta$, so is harmonic except on $\{x : r = 1, \rho = 0\}$ and vanishes at infinity. Q_n obeys the following symmetries:

$$Q_n(e^{i\theta}\overline{x}, \overline{\overline{x}}) = e^{in\theta} Q_n(\overline{x}, \overline{\overline{x}}) \tag{2}$$

$$Q_n(\overline{x}, \overline{\overline{x}}) = Q_n(\overline{x}, \overline{\overline{y}}) \quad \text{if} \quad |\overline{\overline{x}}| = |\overline{\overline{y}}|. \tag{3}$$

It follows that Q_n has the form

$$Q_n(x) = d_n(r, \rho) r^n e^{in\theta}. \tag{4}$$

The function d_n is real since the definition (1) shows Q_n is real on the x_1 axis. Let q_n be the degree n Taylor polynomial of Q_n at the origin. The symmetry (2) is valid for q_n and it follows that q_n has the form

$$q_n(x) = c_n^{(d)} r^n e^{in\theta} \tag{5}$$

where $c_n^{(d)}$ is a real constant. (To see this, write out q_n in the form

$$\sum_{j+k\leq n} p_{jk}(\overline{x})(re^{i\theta})^j(re^{-i\theta})^k,$$

and then compare coefficients of powers of $e^{i\theta}$ in 2). We will show below that $c_n^{(d)} \neq 0$.

Let ϕ be a nonnegative C_0^∞ function on $\mathbb{R}$ with support in (-1,1) and with $\int \phi = 1$ and (for a certain $\epsilon > 0$ to be determined later) let $\phi_\epsilon(s) = \epsilon^{-1}\phi(\frac{s-1}{\epsilon})$. Define $w_n : \mathbb{R}^d \backslash \{x : 1-\epsilon \leq r \leq 1+\epsilon, \rho = 0\} \to \mathbb{C}$ by

$$w_n(x) = (c_n^{(d)})^{-1} \int \phi_\epsilon(s) s^n (Q_n(\frac{x}{s}) - q_n(\frac{x}{s}))ds. \tag{6}$$

The properties we need for w_n are given in the next lemma.

Lemma 1.2. *Suppose that $\delta > 0$ is given. Then provided $\epsilon > 0$ is sufficiently small and $n > n(\delta, \epsilon)$ is sufficiently large, the function w_n will have the following properties: w_n is even if n is even and odd if n is odd and*

(i) $w_n = \mathcal{O}(|x|^n)$ *at* ∞, $w_n = \mathcal{O}(|x|^{n+2})$ *at 0.*

(ii) $\triangle w_n = 0$.

(iii) $w_n(x) = a_n(r,\rho) r^n e^{in\theta}$ *where a_n satisfies: let* $R_n = \{1-4\epsilon \leq r \leq 1+4\epsilon, \rho \leq n^{-1/2}\} \backslash \{1-2\epsilon \leq r \leq 1+2\epsilon, \rho \leq (\frac{\delta}{n})^{1/2}\}$. *Then*

$$|a_n| \leq C(\delta, \epsilon), \quad \textit{if } x \in R_n, \tag{7}$$

$$|\frac{da_n}{dr}| + n^{-1/2}|\frac{da_n}{d\rho}| \leq C(\delta,\epsilon), \quad x \in R_n, \tag{8}$$

$$\frac{1}{2} \leq |a_n| \leq 2, \quad \textit{if } x \in R_n \textit{ and } \rho > (\frac{\delta}{n})^{1/2}. \tag{9}$$

Remark. The set R_n is of course a rectangle containing the singularities of w_n with a smaller such rectangle deleted. The estimates (7)—(9) say that as long as one stays away from the singularities, a_n will be a slowly varying "amplitude" function. Furthermore, as indicated by (9) a_n will be bounded away from zero and actually will be a small perturbation of the constant function -1 provided ρ is kept bounded below by a constant multiple of $n^{-\frac{1}{2}}$. These properties give a lower bound on $|\nabla w_n|$, since $|\nabla(r^n e^{in\theta})|$ is of course large. This will eventually imply (iv) of Lemma 1.1 for the function u_n defined by (19) below.

In fact, it is clear from the definitions that

$$a_n(r,\rho) = (c_n^{(d)})^{-1} \int \phi_\epsilon(s) d_n(\frac{r}{s}, \frac{\rho}{s}) ds - 1$$

and $d_n(r,\rho)$ will turn out to be small when ρ is on the order of $n^{-\frac{1}{2}}$. This will follow from Lemma 1.6(i) and the bound for $(\frac{\omega}{r})^n$ given by Lemma 1.5(i).

Proof of Lemma 1.2. The symmetries (2) and (3) are clearly also valid for w_n and imply that w_n is even if n is even and odd if n is odd, since $\theta_{-x} = \theta_x + \pi$. Next $Q_n - q_n$ is $\mathcal{O}(|x|^n)$ at ∞ since Q_n is bounded and q_n is a polynomial of degree n. Consequently w_n is $\mathcal{O}(|x|^n)$ at ∞. $Q_n - q_n$ is $\mathcal{O}(|x|^{n+1})$ at 0 by Taylor's theorem and therefore w_n is $\mathcal{O}(|x|^{n+1})$ at 0. But w_n is even or odd depending on n and it follows that there are no degree $n+1$ terms in its Taylor expansion. So w_n is $\mathcal{O}(|x|^{n+2})$ at 0. This proves (i). To prove (ii), observe that q_n is harmonic since Q_n is harmonic near 0 (or by (4)). It follows that $Q_n - q_n$ is harmonic except on $r = 1, \rho = 0$ and therefore w_n is harmonic on its domain $\mathbb{R}^d \backslash \{1-\epsilon \leq r \leq 1+\epsilon, \rho = 0\}$. It is clear from (4) , (5) , (6) that w_n has the form $a_n(r,\rho) r^n e^{in\theta}$, so we only need to show that $c_n^{(d)} \neq 0$ and to prove the estimates (7)—(9).

We fix $x \in \mathbb{R}^d$, with $\theta = 0$ and $r \neq 0$, $(r,\rho) \neq (1,0)$.Then $\frac{1+r^2+\rho^2}{r} > 2$ so the equation

$$t^2 - \frac{1+r^2+\rho^2}{r} t + 1 = 0 \tag{10}$$

has two roots ω and ω^{-1} with $\omega \in (0,1)$. Our calculations will be based on the following contour integration formula.

Lemma 1.3. *If* $\theta = 0, r \neq 0, (r,\rho) \neq (1,0)$ *then*

$$Q_n(x) = \int_{\gamma_1} (\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})} (\omega^{-1} - \omega\zeta)^{-(\frac{d-2}{2})} \zeta^n \frac{d\zeta}{i\zeta} \, r^{-(\frac{d-2}{2})} \omega^n$$

for any contour $\gamma_1 \subset D(0,\omega^{-2}) \backslash [0,1]$ *with* $\operatorname{ind}(\gamma_1, 1) = 1$.

Remark. The function $(\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})}$ has a single valued branch on $\mathbb{C}\backslash[0,1]$ and the function $(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})}$ has a single valued branch on $D(0,\omega^{-2})$. We take the branches which are positive when $\zeta=-1$.

Proof of Lemma 1.3. We have

$$\begin{aligned} Q_n(x) &= \int_{-\pi}^{\pi} (|r-e^{i\phi}|^2+\rho^2)^{-(\frac{d-2}{2})} e^{in\phi}\,d\phi \\ &= \int_{|\zeta|=1} ((\zeta-r)(\zeta^{-1}-r)+\rho^2)^{-(\frac{d-2}{2})}\zeta^n \frac{d\zeta}{i\zeta}. \end{aligned}$$

Also $(\zeta-r)(\zeta^{-1}-r)+\rho^2 = -r\zeta^{-1}(\zeta^2-\frac{1+r^2+\rho^2}{r}+1) = -r\zeta^{-1}(\zeta-\omega)(\zeta-\omega^{-1})$, so in view of the remark before the proof we can write

$$Q_n(x) = \int_{|\zeta|=1} (\zeta^{-1}(\zeta-\omega))^{-(\frac{d-2}{2})}(\omega^{-1}-\zeta)^{-(\frac{d-2}{2})}\zeta^n\frac{d\zeta}{i\zeta}\,r^{-(\frac{d-2}{2})}.$$

By change of variables $\zeta\to\omega\zeta$

$$Q_n(x) = \int_{|\zeta|=\omega^{-1}} (\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})}(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})}\zeta^n\frac{d\zeta}{i\zeta}\,r^{-(\frac{d-2}{2})}\omega^n.$$

and the lemma now follows by Cauchy's theorem. □

Lemma 1.4. *If n is sufficiently large then $|c_n^{(d)}|\approx n^{(\frac{d}{2}-2)}$. In particular $c_n^{(d)}\neq 0$.*

Proof. Clearly $c_n^{(d)} = \lim_{r\to 0} r^{-n}Q_n(r,0,\dots,0)$. Since $\omega(r,0)=r$ we have

$$\begin{aligned} c_n^{(d)} &= \lim_{r\to 0}\int_{\gamma_1} (\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})}(1-r^2)^{-(\frac{d-2}{2})}\zeta^n\frac{d\zeta}{i\zeta} \\ &= \int_{\gamma_1} (\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})}\zeta^n\frac{d\zeta}{i\zeta}. \end{aligned} \tag{11}$$

If $d=3$, the integral can be moved onto the segment $[0,1]$, i.e.

$$c_n^{(3)} = 2\int_0^1 (x^{-1}(1-x))^{-1/2}x^n\frac{dx}{x}.$$

This easily implies $|c_n^{(3)}|\approx n^{-1/2}$ as claimed. Also $c_n^{(4)}$ can be evaluated using the residue theorem at the pole -1, leading to $c_n^{(4)}=2\pi$ for all n, again as claimed. When $d\geq 5$ the integral is less easy to work with, so we proceed as follows: denoting $Q_n(r,0,\dots,0)$ by $T_n^{(d)}(r)$, the definition (1) implies

$$\frac{dT_n^{(d)}}{dr} = (2-d)(rT_n^{d+2}-\frac{1}{2}T_{n-1}^{(d+2)}-\frac{1}{2}T_{n+1}^{(d+2)}).$$

Comparing terms of degree $n-1$ in the Taylor expansion at 0 gives

$$nc_n^{(d)} = (2-d)(-\frac{1}{2}c_{n-1}^{(d+2)})$$

and therefore also

$$c_n^{(d)} = (\frac{d}{2}-1)^{-1}(n+1)c_{n+1}^{(d-2)}.$$

The lemma now follows from the three and four dimensional cases by induction on d. $\square$

We want to use Lemma 1.3 to do asymptotics for Q_n. We need some bounds for ω, which we formulate as a lemma.

Lemma 1.5. *Assume* $\frac{1}{2} \le r \le 2$, $\rho \le 1$. *Then*

(i) $\omega \le \min(r, r^{-1})$ *and* $\min(r, r^{-1}) - \omega \approx \frac{\rho^2}{|1-r|+\rho}$.
(ii) $\omega^{-1} - \omega \approx \rho + |1-r|$.
(iii) $|\frac{\partial}{\partial r}(\frac{\omega}{r})| \lesssim 1$.
(iv) $|\frac{\partial}{\partial r}(\frac{\omega}{r})| \lesssim (\frac{\rho}{|1-r|+\rho})^2$ *when* $r < 1$.
(v) $|\frac{\partial}{\partial \rho}(\frac{\omega}{r})| \lesssim \frac{\rho}{|1-r|+\rho}$.

Proof. (i) By the quadratic formula

$$\omega = \frac{1+r^2+\rho^2-\sqrt{(1+r^2+\rho^2)^2-4r^2}}{2r}$$

which may be rewritten as

$$\omega = \frac{1+r^2+\rho^2-(|1-r^2|+\rho^2)}{2r} - \frac{|1-r^2|+\rho^2}{2r}\left(\sqrt{1+\frac{2\rho^2(1+r^2-|1-r^2|)}{(|1-r^2|+\rho^2)^2}}-1\right).$$

The first term on the right side is $\min(r, \frac{1}{r})$. Estimating the second term using $\sqrt{1+x}-1 \approx \min(x, \sqrt{x})$ gives

$$\min(r, \frac{1}{r}) - \omega \approx \frac{|1-r^2|+\rho^2}{2r}\min(x, \sqrt{x})$$

where $x = \frac{2\rho^2(1+r^2-|1-r^2|)}{(|1-r^2|+\rho^2)^2}$. The assumption on r implies $2r \approx 1$, $1+r^2-|1-r^2| \approx 1$, $|1-r^2| \approx |1-r|$, so

$$\min(r, \frac{1}{r}) - \omega \approx \min(\rho, \frac{\rho^2}{|1-r|+\rho^2}).$$

Considering cases ($\rho \leq |1-r|$, $\rho \geq |1-r|$) we see that this implies (i).

(ii) We record the fact that

$$|1-r| + \frac{\rho^2}{|1-r|+\rho} \approx |1-r| + \rho \tag{12}$$

which follows by considering cases $|1-r| \leq \rho$, $|1-r| \geq \rho$. Since $\omega \approx 1$, (i) implies $\omega^{-1} - \omega \approx 1 - \omega \approx |1-r| + \frac{\rho^2}{|1-r|+\rho}$.

(iii) and (iv) Differentiate equation (10) for r obtaining

$$(2\omega - \frac{1+r^2+\rho^2}{r})\frac{\partial \omega}{\partial r} = \omega \frac{\partial}{\partial r}(\frac{1+r^2+\rho^2}{r}).$$

Part (i) implies that

$$\begin{aligned} \frac{1+r^2+\rho^2}{r} - 2\omega &= \frac{1+r^2}{r} - 2\min(r, \frac{1}{r}) + E \\ &= |\frac{1}{r} - r| + E \end{aligned}$$

with $E \approx \frac{\rho^2}{|1-r|+\rho}$. Therefore

$$\begin{aligned} \omega^{-1}\frac{\partial \omega}{\partial r} - \frac{1}{r} &= \frac{\frac{1}{r} - r - |\frac{1}{r} - r| + \frac{\rho^2}{r} - E}{r(|\frac{1}{r} - r| + E)} \\ &\approx \frac{\frac{1}{r} - r - |\frac{1}{r} - r| + \frac{\rho^2}{r} - E}{|1-r|+\rho} \end{aligned}$$

by (12). If $r < 1$ it follows that

$$|\omega^{-1}\frac{\partial \omega}{\partial r} - \frac{1}{r}| \leq \frac{\rho^2 + E}{|1-r|+\rho} \lesssim \frac{\rho^2}{(|1-r|+\rho)^2}$$

by the bounds on E. This implies (iv) since $|\frac{\partial}{\partial r}(\frac{\omega}{r})| = \frac{\omega}{r}|\omega^{-1}\frac{\partial \omega}{\partial r} - \frac{1}{r}|$. If $r > 1$ we have instead

$$|\omega^{-1}\frac{\partial \omega}{\partial r} - \frac{1}{r}| \lesssim 1 + \frac{\rho^2}{(|1-r|+\rho)^2} \lesssim 1$$

which implies (iii).

(v) Differentiate (10) for ρ obtaining

$$\begin{aligned} (2\omega - \frac{1+r^2+\rho^2}{r})\frac{d\omega}{d\rho} &= \omega \frac{d}{d\rho}\frac{1+r^2+\rho^2}{r} \\ \omega^{-1}\frac{d\omega}{d\rho} &= \frac{2\rho/r}{|\frac{1}{r} - r| + E} \lesssim \frac{\rho}{|1-r|+\rho} \end{aligned}$$

by (12). This implies (v). □

Lemma 1.6. *With* d_n *defined by* (4) *we have*

(i) $|d_n(r,\rho) - c_n^{(d)} r^{-(\frac{d-2}{2})}(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}(\frac{\omega}{r})^n| \lesssim n^{\frac{d}{2}-3}(\omega^{-1}-\omega)^{-\frac{d}{2}}(\frac{\omega}{r})^n$

(ii) $|\frac{\partial d_n}{\partial r}| \lesssim n^{\frac{d}{2}-2}(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-\frac{d}{2}} + n^{\frac{d}{2}-1}(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}|\frac{\partial(\frac{\omega}{r})}{\partial r}|$

(iii) $|\frac{\partial d_n}{\partial \rho}| \lesssim n^{\frac{d}{2}-1}(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}|\frac{\partial(\frac{\omega}{r})}{\partial \rho}|$

provided $\frac{1}{2} \le r \le 2$, $\rho \le 1$ *and* $|1-r|+\rho \ge \frac{1}{n}$.

Proof. Lemma 1.3 implies that

$$(13)\quad d_n(r,\rho) = r^{-(\frac{d-2}{2})}(\frac{\omega}{r})^n \int_{\gamma_1} (\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})}(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})}\zeta^n \frac{d\zeta}{i\zeta}$$

where we take γ_1 to be the positively oriented rectangle with vertices at $\frac{-b}{n} - i\frac{b}{n}$, $1+\frac{b}{n}-i\frac{b}{n}$, $1+\frac{b}{n}+i\frac{b}{n}$, $\frac{-b}{n}+i\frac{b}{n}$, with b being a small positive constant. Using (11) to evaluate $c_n^{(d)}$ we obtain

$$\begin{aligned} &|d_n(r,\rho) - c_n^{(d)} r^{-(\frac{d-2}{2})}(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}(\frac{\omega}{r})^n| \\ &\quad\lesssim \int_{\gamma_1} |\zeta^{-1}(\zeta-1)|^{-(\frac{d-2}{2})}|(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})} - (\omega^{-1}-\omega)^{-(\frac{d-2}{2})}| \\ &\qquad\qquad \times |\zeta|^n \frac{|d\zeta|}{|\zeta|}\, r^{-(\frac{d-2}{2})}(\frac{\omega}{r})^n. \end{aligned}$$

Here

$$|(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})} - (\omega^{-1}-\omega)^{-(\frac{d-2}{2})}| \lesssim |\zeta-1| \max_{z\in\overline{\zeta 1}} |\omega^{-1}-\omega z|^{-\frac{d}{2}}$$

by the mean value theorem. The segment $\overline{\zeta 1}$ is inside the rectangle γ_1, which is contained in $D(0, 1+\frac{\sqrt{2}b}{n})$, hence in $D(0,\omega^{-1})$ if b is small, by assumption and (ii) of Lemma 1.5. We conclude that

$$\begin{aligned} |(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})} - (\omega^{-1}-\omega)^{-(\frac{d-2}{2})}| &\lesssim |\zeta-1|(\omega^{-1}-1)^{-\frac{d}{2}} \\ &\approx |\zeta-1|(\omega^{-1}-\omega)^{-\frac{d}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} &|d_n(r,\rho) - c_n^{(d)} r^{-(\frac{d-2}{2})}(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}(\frac{\omega}{r})^n| \\ &\lesssim \int_{\gamma_1} |\zeta^{-1}(\zeta-1)|^{-(\frac{d-2}{2})}|\zeta-1||\zeta|^n \frac{|d\zeta|}{|\zeta|}(\omega^{-1}-\omega)^{-\frac{d}{2}} r^{-(\frac{d-2}{2})}(\frac{\omega}{r})^n. \end{aligned}$$

The integral is easily seen to be $\lesssim n^{\frac{d}{2}-3}$ (the main contribution is when $|\zeta| \geq 1-\frac{1}{n}$) and (i) follows.

Note that (i) implies

$$|d_n| \lesssim n^{\frac{d}{2}-2}(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}(\frac{\omega}{r})^n \tag{14}$$

since $|c_n^{(d)}| \approx n^{\frac{d}{2}-2}$ and the right side of (i) is lower order since $\omega^{-1}-\omega \gtrsim \frac{1}{n}$ by Lemma 1.5(ii). The right side of (ii) dominates the right side of (14), so by the product rule it suffices to prove (ii) with d_n replaced by $r^{\frac{d}{2}-1}d_n$ on the left hand side. With the same γ_1 as before we have

(15)

$$\begin{aligned}\frac{\partial}{\partial r}&(r^{\frac{d-2}{2}}d_n)\\ &=(\frac{\omega}{r})^n\int_{\gamma_1}(\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})}\frac{\partial}{\partial r}(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})}\zeta^n\frac{d\zeta}{i\zeta}\\ &\quad+n(\frac{\omega}{r})^{n-1}\frac{\partial}{\partial r}(\frac{\omega}{r})\int_{\gamma_1}(\zeta^{-1}(\zeta-1))^{-(\frac{d-2}{2})}(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2})}\zeta^n\frac{d\zeta}{i\zeta}.\end{aligned}$$

Here $|\frac{\partial}{\partial r}(\omega^{-1}-\omega\zeta)^{-(\frac{d-2}{2}}| \lesssim |\frac{\partial\omega}{\partial r}|\ |\omega^{-1}-\omega\zeta|^{-\frac{d}{2}}$ which is $\lesssim |\frac{\partial\omega}{\partial r}|(\omega^{-1}-\omega)^{-\frac{d}{2}}$, as above. Using (iii) of Lemma 1.5 we may bound the first term on the right side of (15) by

$$(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-\frac{d}{2}}\int_{\gamma_1}|\zeta^{-1}(\zeta-1)|^{-(\frac{d-2}{2})}|\zeta|^n\frac{|d\zeta|}{|\zeta|} \lesssim n^{\frac{d}{2}-2}(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-\frac{d}{2}}.$$

The second term is similarly $\lesssim n^{\frac{d}{2}-1}(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}|\frac{\partial}{\partial r}(\frac{\omega}{r})|$ and (ii) follows. (iii) may be done essentially the same: taking the ρ derivative of (13) , then putting absolute values inside leads to

$$|\frac{\partial d_n}{\partial\rho}| \lesssim n^{\frac{d}{2}-2}(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-\frac{d}{2}}|\frac{\partial}{\partial\rho}(\frac{\omega}{r})| + n^{\frac{d}{2}-1}(\frac{\omega}{r})^n(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}|\frac{\partial}{\partial\rho}(\frac{\omega}{r})|.$$

Lemma 1.5 (ii) implies that the second term on the right hand side dominates the first and (iii) follows. □

Completion of proof of Lemma 1.2. We have (from the definitions)

$$a_n(r,\rho)=(c_n^d)^{-1}\int\phi_\epsilon(s)d_n(\frac{r}{s},\frac{\rho}{s})ds-1. \tag{16}$$

We assume (r,ρ) belongs to R_n and ϵ is small and n large. Then $(\frac{r}{s},\frac{\rho}{s})$ will satisfy the hypotheses of Lemmas 1.5 and 1.6, e.g. $\frac{r}{s}\in(\frac{1}{2},2)$. Formula

(14) and Lemma 1.4 imply that $|d_n(\frac{r}{s},\frac{\rho}{s})| \lesssim |c_n^{(d)}|(\omega^{-1}-\omega)^{-(\frac{d-2}{2})}(\frac{s\omega}{r})^n$, where $\omega = \omega(\frac{r}{s},\frac{\rho}{s})$. Therefore

$$|1+a_n(r,\rho)| \lesssim \int \phi_\epsilon(s)(\omega^{-1}-\omega)^{-(\frac{d-2}{2}}(\frac{s\omega}{r})^n ds.$$

We record the fact that

$$(17) \qquad (\frac{s\omega}{r})^n \le C_2 \min(1,\frac{s}{r})^{2n} e^{-C_1 \frac{|s-r|+n\rho^2}{|s-r|+\rho}}$$

when $(r,\rho) \in R_n$ and $1-\epsilon \le s \le 1+\epsilon$, for certain constants $C_1$and C_2. Inequality (17) is proved as follows: Lemma 1.5 (i) implies

$$\begin{aligned}\frac{s\omega}{r} &\le \min(1,(\frac{s}{r})^2) - C_0\frac{\rho^2}{|s-r|+\rho} \\ &\le \min(1,(\frac{s}{r})^2)(1-\frac{C_0}{4}\frac{\rho^2}{|s-r|+\rho})\end{aligned}$$

where the last line uses $\frac{s}{r} \le 2$. Consequently

$$(\frac{s\omega}{r})^n \le \min(1,\frac{s}{r})^{2n} e^{-\frac{C_0}{4}\frac{n\rho^2}{|s-r|+\rho}}$$

and (17) follows with $C_1 = \frac{C_0}{4}, C_2 = e^{C_1}$.

Define $E_k : \mathbb{R}^+ \to \mathbb{R}$ by $E_k(x) = x^{\frac{k}{2}} e^{-C_1 x}$. Lemma 1.5 (ii) implies that $\omega^{-1}-\omega \approx |s-r|+\rho$. So by (17)

$$(18) \qquad \begin{aligned}|1+a_n(r,\rho)| &\lesssim \int \phi_\epsilon(s)(|s-r|+\rho)^{-(\frac{d-2}{2})} e^{-C_1\frac{|s-r|+n\rho^2}{|s-r|+\rho}} ds \\ &= \int \phi_\epsilon(s)(|s-r|+n\rho^2)^{-(\frac{d-2}{2})} E_{d-2}(\frac{|s-r|+n\rho^2}{|s-r|+\rho})ds.\end{aligned}$$

We now prove (7). Namely, if $(r,\rho) \in R_n$, $1-\epsilon \le s \le 1+\epsilon$, then $|s-r|+n\rho^2 \gtrsim \min(\delta,\epsilon)$. Therefore

$$|1+a_n(r,\rho)| \le C(\delta,\epsilon)\int \phi_\epsilon(s) E_{d-2}(\frac{|s-r|+n\rho^2}{|s-r|+\rho})ds.$$

E_{d-2} is of course a bounded function, so (7) is proved. Next we prove (9). If $\rho \ge \sqrt{\frac{\delta}{n}}$ then by (18)

$$\begin{aligned}|1+a_n(r,\rho)| &\le C(\delta)\int \phi_\epsilon(s) E_{d-2}(\frac{|s-r|+n\rho^2}{|s-r|+\rho})ds \\ &\le C(\delta) \max_{1-\epsilon\le s\le 1+\epsilon} E_{d-2}(\frac{|s-r|+n\rho^2}{|s-r|+\rho}).\end{aligned}$$

Of course $E_{d-2}(x) \to 0$ as $x \to +\infty$. On the other hand, if A is any preassigned number then by taking ϵ small enough and n large enough we can insure that $\frac{|s-r|+n\rho^2}{|s-r|+\rho} \geq A$ for all $r \in (1-4\epsilon, 1+4\epsilon)$, $\rho \geq \sqrt{\frac{\delta}{n}}$ and $s \in (1-\epsilon, 1+\epsilon)$ If we do this with $A = \frac{1}{2C(\delta)}$ we have proved (9).

It remains to prove the derivative estimates (8). We first consider $\frac{\partial a_n}{\partial \rho}$. Differentiating (16),

$$\frac{\partial a_n}{\partial \rho} = (c_n^{(d)})^{-1} \int \phi_\epsilon(s) \frac{\partial d_n}{\partial \rho}(\frac{r}{s}, \frac{\rho}{s}) \frac{ds}{s}.$$

By Lemma 1.6(iii) and Lemma 1.4,

$$|\frac{\partial a_n}{\partial \rho}| \lesssim n \int \phi_\epsilon(s) (\frac{s\omega}{r})^n (\omega^{-1} - \omega)^{-(\frac{d-2}{2})} \lambda(s) ds$$

where

$$\lambda(s) = \left| \frac{\partial(\frac{\omega}{r})}{\partial \rho}(\frac{r}{s}, \frac{\rho}{s}) \right| \lesssim \frac{\rho}{|s-r| + \rho}$$

by Lemma 1.5 (v) and (ii). So

$$\begin{aligned} |\frac{\partial a_n}{\partial \rho}| &\lesssim n\rho \int \phi_\epsilon(s) (\frac{s\omega}{r})^n (|s-r| + \rho)^{-\frac{d}{2}} ds \\ &\lesssim n\rho \int \phi_\epsilon(s) (|s-r| + n\rho^2)^{-\frac{d}{2}} E_d(\frac{|s-r| + n\rho^2}{|s-r| + \rho}) ds \\ &\lesssim C(\delta, \epsilon) n\rho. \end{aligned}$$

In the next to last inequality we used (17) in the same way as in the proof of (7). Since $\rho \leq n^{-\frac{1}{2}}$ on R_n it follows that $n^{\frac{1}{2}} |\frac{\partial a_n}{\partial \rho}| \leq C(\delta, \epsilon)$ as claimed in (8).

We now prove the bound for $\frac{\partial a_n}{\partial r}$ in (8). We differentiate (16) for r:

$$\frac{\partial a_n}{\partial r} = (c_n^{(d)})^{-1} \int \phi_\epsilon(s) \frac{\partial d_n}{\partial r}(\frac{r}{s}, \frac{\rho}{s}) \frac{ds}{s}$$

We now proceed as before, estimating $\frac{\partial d_n}{\partial r}$ via Lemma 1.6(ii). This gives

$$|\frac{\partial a_n}{\partial r}| \lesssim \int \phi_\epsilon(s) (\frac{s\omega}{r})^n (\omega^{-1} - \omega)^{-\frac{d}{2}} ds + n \int \phi_\epsilon(s) (\frac{s\omega}{r})^n (\omega^{-1} - \omega)^{-(\frac{d-2}{2})} \kappa(s) ds.$$

where

$$\kappa(s) = \left| \frac{\partial(\frac{\omega}{r})}{\partial r}(\frac{r}{s}, \frac{\rho}{s}) \right| \lesssim \begin{cases} 1 & \text{if } s < r \\ (\frac{\rho}{|s-r|+\rho})^2 & \text{if } s \geq r \end{cases}$$

by Lemma 1.5 (iii), (iv). The first integral may be estimated by $C(\delta, \epsilon)$ just as in the proof of (7): the only difference is the exponent $-\frac{d}{2}$ instead of $-(\frac{d-2}{2})$.

We omit the details of this and will now consider the second integral. We split it into $\int_{s>r}$ and $\int_{s<r}$, and substitute in the bound for κ and the bound $\omega^{-1} - \omega \approx |s-r| + \rho$. This gives

$$|\frac{\partial a_n}{\partial r}| \lesssim C(\delta,\epsilon) + n\rho^2 \int_{s>r} \phi_\epsilon(s)(|s-r|+\rho)^{-(\frac{d+2}{2})}(\frac{s\omega}{r})^n ds$$
$$+ n\int_{s<r} \phi_\epsilon(s)(|s-r|+\rho)^{-(\frac{d-2}{2})}(\frac{s\omega}{r})^n ds$$

The $s > r$ integral here may be estimated by $C(\delta,\epsilon)$ as before (again the only difference is in the exponent of $|s-r|+\rho$). We therefore have

$$\begin{aligned}
|\frac{\partial a_n}{\partial r}| &\lesssim C(\delta,\epsilon)(1+n\rho^2) + n\int_{s<r} \phi_\epsilon(s)(|s-r|+\rho)^{-(\frac{d-2}{2})}(\frac{s\omega}{r})^n ds \\
&\leq C(\delta,\epsilon)(1+n\rho^2) \\
&+ Cn\int_{s<r} \phi_\epsilon(s)(|s-r|+n\rho^2)^{-(\frac{d-2}{2})} E_{d-2}(\frac{|s-r|+n\rho^2}{|s-r|+\rho})(\frac{s}{r})^{2n} ds
\end{aligned}$$

where we used (17). Each of the factors in the integrand is bounded by a constant $C(\delta,\epsilon)$ so we obtain

$$|\frac{\partial a_n}{\partial r}| \lesssim C(\delta,\epsilon)(1+n\rho^2 + n\int_{s<r}(\frac{s}{r})^{2n} ds).$$

The integral here is $\lesssim \frac{1}{n}$, so

$$|\frac{\partial a_n}{\partial r}| \leq C(\delta,\epsilon)(1+n\rho^2)$$

and now we are done, since by assumption $\rho \leq n^{-\frac{1}{2}}$ □

Proof of Lemma 1.1. We start by constructing the function u_n. Let γ be a sufficiently small positive constant to be specified later. Let e^1 and e^2 be the first two standard basis vectors and define e^1_* and e^2_* by

$$e^1_* = (1+\frac{\gamma}{n})^{-\frac{1}{2}}(1,0,(\frac{\gamma}{n})^{\frac{1}{2}},0,\dots,0)$$
$$e^2_* = (1+\frac{\gamma}{n})^{-\frac{1}{2}}(0,1,0,(\frac{\gamma}{n})^{\frac{1}{2}},0,\dots,0).$$

It is here that we use that $d \geq 4$. For given $x \in \mathbb{R}^d$ define coordinates r_*, θ_*, ρ_* via

$$x = r_* \cos\theta_* e^1_* + r_* \sin\theta_* e^2_* + \overline{\overline{x}}_* \quad , \; \rho_* = |\overline{\overline{x}}_*|$$

where $\overline{\overline{x}}_* \perp sp(e_*^1, e_*^2)$. In other words r_*, θ_*, ρ_* are defined like r, θ, ρ but using the vectors e_*^1 and e_*^2 instead of e^1 and e^2. Next let δ be small enough and choose ϵ and functions $w_n(x) = a_n(r,\rho)r^n e^{in\theta}$ by Lemma 1.2. Let p and q_n be C_0^∞ functions on $\mathbb{R}$ with $p(r) = 1$ when $|r-1| \le 2\epsilon$, $p(r) = 0$ when $|r-1| \ge 3\epsilon$ and $q_n(\rho) = 1$ when $\rho \le \sqrt{\frac{\delta}{n}}$, $q_n(\rho) = 0$ when $\rho \ge \sqrt{\frac{2\delta}{n}}$, and with $|\frac{d^j p}{dr^j}| \lesssim \epsilon^{-j}, |\frac{d^j q_n}{d\rho^j}| \le (\frac{\delta}{n})^{-\frac{j}{2}}$. Let $\chi_n(r,\rho) = 1 - p(r)q_n(\rho)$ and define

(19)
$$u_n(x) = \frac{1}{n}(\chi_n(r,\rho)a_n(r,\rho)r^n \cos n\theta \;+\; \chi_n(r_*,\rho_*)a_n(r_*,\rho_*)r_*^n \sin n\theta_*).$$

It is clear that u_n is even if n is even and odd if n is odd and has property (i) of Lemma 1.1. To prove (ii) - (iv) we need the following (elementary) lemma. In parts (iv) and (v), of course we are regarding $\frac{\partial}{\partial r}$ etc. as vector fields.

Lemma 1.7. *If $\frac{1}{2} \le r \le 2$ and ρ is sufficiently small (independently of γ and n) then*

(i) $|r - r_*| \le C(\rho\sqrt{\frac{\gamma}{n}} + \frac{\gamma}{n})$.
(ii) $|\theta - \theta_*| \le C(\rho\sqrt{\frac{\gamma}{n}} + \frac{\gamma}{n})$.
(iii) $|\rho - \rho_*| \le C\sqrt{\frac{\gamma}{n}}$.
(iv) $|\frac{\partial}{\partial r} - \frac{\partial}{\partial r_*}| \le C\sqrt{\frac{\gamma}{n}}$.
(v) $|\frac{1}{r}\frac{\partial}{\partial \theta} - \frac{1}{r_*}\frac{\partial}{\partial \theta_*}| \le C\sqrt{\frac{\gamma}{n}}$.

Furthermore if $\frac{1}{2} \le r \le 2$ then

(vi) *$max(\rho, \rho_*) \ge C^{-1}\sqrt{\frac{\gamma}{n}}$.*

Proof. For (i)-(v) the relevant properties of the e^i and e_*^i are that

$$|\langle e^i - e_*^i, e^j\rangle| \le \frac{C\gamma}{n}\ , \ i,j \in \{1,2\} \tag{20}$$

$$|e^i - e_*^i| \le C\sqrt{\frac{\gamma}{n}}\ , i \in \{1,2\} \tag{21}$$

and are easily checked.

To prove (i) and (ii) we use (20):

$$\begin{aligned} |r - r_*| \lesssim |r^2 - r_*^2| &= |\sum_{i=1}^{2} \langle x, e^i \rangle^2 - \langle x, e_*^i \rangle^2| \\ &\leq C \sum_{i=1}^{2} |\langle x, e^i - e_*^i \rangle|. \end{aligned}$$

Replacing x here by its projection on the orthogonal complement of $sp\{e_1, e_2\}$ introduces an error of $\lesssim \frac{\gamma}{n}$ by (20) and therefore $|\langle x, e^i - e_*^i \rangle| \leq C(\rho\sqrt{\frac{\gamma}{n}} + \frac{\gamma}{n})$, which gives (i).

To prove (ii) write

$$|\cos\theta - \cos\theta_*| = |\frac{\langle x, e^1 \rangle}{r} - \frac{\langle x, e_*^1 \rangle}{r_*}|.$$

For sufficiently small ρ the bound on $|r - r_*|$ implies $r_* \geq \frac{1}{4}$ so

$$\begin{aligned} |\cos\theta - \cos\theta_*| &\leq C|r_* \langle x, e^1 \rangle - r\langle x, e_*^1 \rangle| \\ &\leq C(|r - r_*|\, |\langle x, e^1 \rangle| + r|\langle x, e^1 - e_*^1 \rangle|) \\ &\leq C(|r - r_*| + |\langle x, e^1 - e_*^1 \rangle| \\ &\leq C(\rho\sqrt{\frac{\gamma}{n}} + \frac{\gamma}{n}). \end{aligned}$$

Likewise $|\sin\theta - \sin\theta_*| \leq C(\rho\sqrt{\frac{\gamma}{n}} + \frac{\gamma}{n})$ (replace e^1 and e_*^1 by e^2 and e_*^2 in the preceding argument) and therefore $|\theta - \theta_*| \leq C(\rho\sqrt{\frac{\gamma}{n}} + \frac{\gamma}{n})$, i.e. (ii) holds.

For (iii) we write

$$\begin{aligned} |\rho - \rho_*| &= |\, |x - r\cos\theta e^1 - r\sin\theta e^2| - |x - r_*\cos\theta_* e_*^1 - r_*\sin\theta_* e_*^2|\, | \\ &\leq |r\cos\theta e^1 - r_*\cos\theta_* e_*^1| + |r\sin\theta e^2 - r_*\sin\theta_* e_*^2|. \end{aligned}$$

(iii) now follows easily using (i), (ii) and (21) and the triangle inequality.

For (iv) and (v) write

$$|\frac{\partial}{\partial r} - \frac{\partial}{\partial r_*}| = |\cos\theta e^1 + \sin\theta e^2 - \cos\theta_* e_*^1 - \sin\theta_* e_*^2|,$$

$$|\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{1}{r_*}\frac{\partial}{\partial\theta_*}| = |-sin\theta e^1 + cos\theta e^2 + sin\theta_* e_*^1 - cos\theta_* e_*^2|.$$

Now argue as in the proof of (iii).

We now prove (vi). The definition of e_*^j shows that

$$\sum_{j=1}^{2} < x, e_*^j >^2 \leq (1+\frac{\gamma}{n})^{-1} \sum_{j=1}^{2} < x, e_j >^2 + (1+\frac{\gamma}{n})^{-1}\frac{\gamma}{n}\rho^2.$$

Therefore

$$\begin{aligned}\rho_*^2 &\geq |x|^2 - (1+\frac{\gamma}{n})^{-1}|x|^2 - (1+\frac{\gamma}{n})^{-1}\frac{\gamma}{n}\rho^2 \\ &= \frac{\gamma}{n}(1+\frac{\gamma}{n})^{-1}(|x|^2 - \rho^2)\end{aligned}$$

so that $\rho \leq \frac{1}{2}|x|$ implies $\rho_* \geq C^{-1}\sqrt{\frac{\gamma}{n}}$. This proves (vi). $\square$

Completion of proof of Lemma 1.1. We need to define α and β and prove (ii)—(iv). We can take $\alpha = \frac{7}{2}\epsilon$, $\beta = C\sqrt{\gamma} + 2\sqrt{\delta}$ where C is the constant in Lemma 1.7. Then (ii) is proved as follows: $\text{supp}(\Delta u_n) \subset B \cup B_*$ where $B = \{x : |r-1| \leq 3\epsilon\} \cap \{x : \rho \leq 2\sqrt{\frac{\delta}{n}}\}$ and B_* is defined similarly replacing r and ρ by r_* and ρ_*. If n is large, then (ii) follows using (i) and (iii) of Lemma 1.7.

Next (i) and (ii) of Lemma 1.7 imply

$$|r^n - r_*^n| \lesssim \gamma r^n \tag{22}$$

$$|e^{in\theta} - e^{in\theta_*}| \lesssim \gamma \tag{23}$$

on A_n, provided we have chosen $\delta < \gamma$.

We now prove (iii). Constants in this argument may depend on everything except n. We have

$$\begin{aligned} & \triangle(\chi_n(r,\rho)a_n(r,\rho)r^n \cos\, n\theta) \\ = \;& \triangle(\chi_n(r,\rho))a_n(r,\rho)r^n\cos(n\theta) + 2\bigtriangledown(\chi_n(r,\rho))\cdot\bigtriangledown(a_n(r,\rho)r^n\cos(n\theta)) \\ = \;& \triangle(\chi_n(r,\rho))a_n(r,\rho)r^n\cos\, n\theta + 2\bigtriangledown(\chi_n(r,\rho))\cdot\bigtriangledown(a_n(r,\rho))r^n\cos\, n\theta \\ & +2a_n(r,\rho)\frac{\partial\chi_n}{\partial r}(r,\rho)nr^{n-1}\cos\, n\theta. \end{aligned}$$

The various terms here may be bounded by nr^n using the derivative bounds on p and q_n and the estimates in Lemma 1.2 (iii). Namely, $|\triangle\chi_n| \lesssim n$ and $|a_n| \lesssim 1$ on R_n (note that R_n contains the set where $\chi_n \neq 0$), so the first term is $\lesssim nr^n$. Likewise $|\bigtriangledown\chi_n| \lesssim n^{\frac{1}{2}}$, and $|\bigtriangledown a_n| \lesssim n^{\frac{1}{2}}$ on R_n, so the second term is $\lesssim nr^n$. Finally the last term is $\lesssim nr^n$ since $|\frac{\partial\chi_n}{\partial r}| \lesssim 1$ and $|a_n| \lesssim 1$ on R_n.

Similar estimates can of course be made for the second term in (19), so we conclude that

$$|\triangle u_n| \lesssim r^n + r_*^n.$$

But this implies (iii) in view of (22).

To prove (iv) we first isolate the terms in ∇u_n where the derivative falls on $r^n \cos n\theta$ or $r_*^n \sin n\theta_*$, i.e.

$$\begin{aligned}\nabla u_n &= \chi_n(r,\rho)a_n(r,\rho)r^{n-1}(\cos n\theta\frac{\partial}{\partial r} - \sin n\theta\frac{1}{r}\frac{\partial}{\partial\theta}) \\ &\quad + \chi_n(r_*,\rho_*)a_n(r_*,\rho_*)r_*^{n-1}(\sin n\theta_*\frac{\partial}{\partial r_*} + \cos n\theta_*\frac{1}{r_*}\frac{\partial}{\partial\theta_*}) + E_1\end{aligned}$$

where

$$E_1 = \frac{1}{n}(\nabla(\chi_n(r,\rho)a_n(r,\rho))r^n \cos n\theta + \nabla(\chi_n(r_*,\rho_*)a_n(r_*,\rho_*))r_*^n \sin n\theta_*).$$

It is easily seen (using the bounds from Lemma 1.1 and the derivative bounds for χ_n, as in the proof of (iii)) that $|E_1| \leq C(\delta,\epsilon)n^{-\frac{1}{2}}(r^n + r_*^n)$, and therefore $|E_1| \leq C(\delta,\epsilon)n^{-\frac{1}{2}}r^n$ by (22). Next, (22) , (23) , and (iv), (v) of Lemma 1.7 imply that

$$|r_*^{n-1}(\sin n\theta_*\frac{\partial}{\partial r_*} + \cos n\theta_*\frac{1}{r_*}\frac{\partial}{\partial\theta_*}) - r^{n-1}(\sin n\theta\frac{\partial}{\partial r} + \cos n\theta\frac{1}{r}\frac{\partial}{\partial\theta})| \leq C\gamma r^n$$

with C an absolute constant. We therefore obtain

$$\begin{aligned}(24)\quad \nabla u_n &= \chi_n(r,\rho)a_n(r,\rho)r^{n-1}(cos\ n\theta\frac{\partial}{\partial r} - sin\ n\theta\frac{1}{r}\frac{\partial}{\partial\theta}) \\ &\quad + \chi_n(r_*,\rho_*)a_n(r_*,\rho*)r^{n-1}(sin\ n\theta\frac{\partial}{\partial r} + cos\ n\theta\frac{1}{r}\frac{\partial}{\partial\theta}) + E_1 + E_2\end{aligned}$$

with $|E_1| \leq C(\delta,\epsilon)n^{-\frac{1}{2}}r^n$ and $|E_2| \leq C\gamma r^n|\chi_n(r_*,\rho_*)a_n(r_*,\rho_*)|$. By the triangle inequality

$$|\nabla u_n| \geq J - C(\delta,\epsilon)n^{-\frac{1}{2}}r^n - C\gamma r^n|\chi_n(r_*,\rho_*)a_n(r_*,\rho_*)|,$$

where J, the absolute value of the first two groups of terms on the right side of (24), may be calculated explicitly using that $\sin^2 n\theta + \cos^2 n\theta = 1$. This gives

$$\begin{aligned}J &= \sqrt{2}r^{n-1}\sqrt{(\chi_n(r,\rho)a_n(r,\rho))^2 + (\chi_n(r_*,\rho_*)a_n(r_*,\rho_*))^2} \\ &\geq (|\chi_n(r,\rho)a_n(r,\rho)| + |\chi_n(r_*,\rho_*)a_n(r_*,\rho_*)|)r^{n-1}.\end{aligned}$$

Therefore, using the bounds for E_1 and E_2 and that $r \le 2$,

$$|\nabla u_n| \ge \Big(\frac{1}{4}(|\chi_n(r,\rho)a_n(r,\rho)| + |\chi_n(r_*,\rho_*)a_n(r_*,\rho_*)|) \\ - C\gamma|\chi_n(r_*,\rho_*)a_n(r_*,\rho_*)| - C(\delta,\epsilon)n^{-\frac{1}{2}}\Big)r^n.$$

The term involving γ may be dropped if γ has been chosen sufficiently small. Furthermore estimate (9) implies that $|\chi_n(r,\rho)a_n(r,\rho)|+|\chi_n(r_*,\rho_*)a_n(r_*,\rho_*)| \ge \frac{1}{2}$ provided δ is small enough, since then either ρ or ρ_* will be $\ge 2(\frac{\delta}{n})^{\frac{1}{2}}$ by Lemma 1.7(vi). Estimate (iv) now follows by taking n sufficiently large. □

2. Proof of Theorem 1, Part 2.

We first prove a certain genericity statement. Denote

$\mathcal{H}_k =$ homogeneous harmonic polynomials of degree k

$\mathcal{H}_k^* = \{Y \in \mathcal{H}_k :\ \nabla Y$ vanishes only at $0\}$

Let $\Gamma^e(x) = |x-e|^{2-d}$, let e_1 be the first standard basis vector and Z_k (a normalization of the kth zonal harmonic) the degree k term in the Taylor expansion of Γ^{e_1} at 0. Let $\mathcal{O}(d)$ be the orthogonal group. If $\rho \in \mathcal{O}(d)$ maps e to e_1 then $\Gamma^e = \Gamma^{e_1}\circ\rho$, so $Z_k\circ\rho$ is the degree k term in the expansion of Γ^e at 0.

Lemma 2.1. *If $Y \in \mathcal{H}_k$ then the set $\{(\rho_1,\dots,\rho_d,\alpha) \in \mathcal{O}(d)\times\ldots\times\mathcal{O}(d)\times\mathbb{R}^d : Y + \sum_j \alpha_j Z_k\circ\rho_j \in \mathcal{H}_k^*\}$ has full measure in $\mathcal{O}(d)\times\ldots\times\mathcal{O}(d)\times\mathbb{R}^d$.*

Proof. We first show that the set $E = \{(\rho_1,\dots,\rho_d) \in \mathcal{O}(d)\times\ldots\times\mathcal{O}(d) : Z_k\circ\rho_1,\dots,Z_k\circ\rho_d$ have no common zeroes except the origin $\}$ has full measure.

For this, consider the map

$$F : \mathcal{O}(d)\times\ldots\times\mathcal{O}(d)\times S^{d-1} \to \mathbb{R}^d,$$
$$F(\rho_1,\dots,\rho_d,x) = Z_k(\rho_1 x),\dots,Z_k(\rho_d x).$$

For $i,j \in \{1,\dots,d\}$ the ρ_i - derivative of $Z_k(\rho_j x)$ clearly vanishes identically if $i \ne j$, and if $i = j$ vanishes only when $\rho_j x$ is a critical point of $Z_k|_{S^{d-1}}$. We conclude: if $(\rho_1,\dots,\rho_d,x)$ is such that $\rho_j x$ is a regular point of Z_k for all $j \in \{1,\dots,d\}$, then $(\rho_1,\dots,\rho_d,x)$ is a regular point of F.

On the other hand 0 is a regular value of $Z_k|_{S^{d-1}}$ by uniqueness for ODE's since $Z_k|_{S^{d-1}}$ is a solution of a second order ODE in the variable $\angle \cdot 0e_1$. So if $F(\rho_1, \dots, \rho_d, x) = 0$ then $(\rho_1, \dots, \rho_d, x)$ is a regular point of F, i.e. F is transverse to zero. Define $F_{\rho_1 \dots \rho_d} : S^{d-1} \to \mathbb{R}^d$ by $F_{\rho_1 \dots \rho_d}(x) = F(\rho_1, \dots, \rho_d, x)$. By the transversality theorem 0 is a regular value of $F_{\rho_1 \dots \rho_d}$ for a.e. $(\rho_1, \dots, \rho_d)$. But by dimensional considerations 0 can only be a regular value of $F_{\rho_1 \dots \rho_d}$ if it is an omitted value. This proves the claim.

If $(\rho_1, \dots, \rho_d) \in E$ then the function

$$G : \mathbb{R}^d \times S^{d-1} \longrightarrow \mathbb{R}^d,$$
$$G(\alpha, x) = Y(x) + \sum_j \alpha_j Z_k(\rho_j x)$$

is transverse to zero, since its α_j derivative is $Z_k(\rho_j x)$. So 0 is a regular value of the function $G_\alpha : S^{d-1} \to \mathbb{R}$, $G_\alpha(x) = Y(x) + \sum_j \alpha_j Z_k(\rho_j x)$, for a full measure set of α. But if 0 is a regular value of G_α on S^{d-1} then (by Euler's identity for homogeneous functions) $Y + \sum_j \alpha_j Z_k \circ \rho_j$ has no critical points in $\mathbb{R}^d \backslash 0$. This finishes the proof. □

We will now make a certain technical modification in Lemma 1.1. Let A_n be as there.

Lemma 2.2. *Assume $d \geq 4$. Then there is $n_0 < \infty$ and a sequence of smooth functions $u_n : \mathbb{R}^d \to \mathbb{R}$ $(n_0 \leq n < \infty)$ such that*

(i) $u_n = \mathcal{O}(|x|^n)$ *at* ∞, $u_n = \mathcal{O}(|x|^{n+1})$ *at* 0. *Furthermore let p_n and q_n be respectively the degree n term in the expansion of u_n at ∞ and the degree n+1 term in the Taylor expansion at* 0. *Then $p_n \in \mathcal{H}_n^*$ and $q_n = p_{n+1}$.*

(ii) $\operatorname{supp}(\triangle u_n) \subset A_n$.

(iii) $|\triangle u_n| \lesssim \eta_n r^n$ *and* $|\nabla u_n| \gtrsim \eta_n r^n$ *on A_n, where the $\{\eta_n\}$ are constants.*

Remark. Perhaps we should be more precise about the definition of the function p_n. If u is a harmonic function defined on the complement of a compact set in $\mathbb{R}^d$ with $d \geq 4$ and $u(x) = \mathcal{O}(|x|^n)$ at ∞, for some n then it is not hard to show the

Claim 2.3. *There is a harmonic polynomial p such that $u-p$ is $\mathcal{O}(|x|^{2-d})$ at ∞.*

Given the claim, we may let p_n be the terms of degree n in the polynomial p. The claim is probably fairly well known but we will sketch the proof since we do not know a reference. We use induction on n, where n is the smallest integer such that $u(x) = \mathcal{O}(|x|^n)$ at ∞. If $n \leq 0$ the statement is standard and follows from the removable singularities theorem for harmonic functions by using the Kelvin transform. Now suppose $n \geq 1$ and $u(x) = \mathcal{O}(|x|^n)$ at ∞. Then $\nabla u(x) = \mathcal{O}(|x|^{n-1})$ so may be assumed to have the form of the claim. Now write $u(x) = \int_{x_0}^{x} \nabla u \cdot dx + u(x_0)$, x large, for a fixed $x_0 \neq 0$. The assumption $d \geq 4$ implies that $|x|^{2-d}$ is integrable on rays not containing the origin. It follows that $u(x)$ has the form polynomial + bounded and then the claim follows from the $n = 0$ case.

Proof of Lemma 2.2. We claim first that Lemma 1.1 is valid with the additional conclusion that the degree n term in u_n at ∞ and degree $n+2$ term at 0 belong to $\mathcal{H}_n^*$ and $\mathcal{H}_{n+2}^*$ respectively.

For this, let u_n be the function from Lemma 1.1 and p_n and q_n its degree n term at ∞ and degree $n+2$ term at 0. By Lemma 2.1 the set

$$\Big\{(\rho_1, \dots, \rho_d, \alpha \in \mathcal{O}(d) \times \dots \times \mathcal{O}(d) \times \mathbb{R}^d : \\ p_n - \sum_{j=1}^{d} \alpha_j \, Z_n \circ \rho_j \in \mathcal{H}_n^* \ , \ q_n + \sum_{j=1}^{d} \alpha_j \, Z_n \circ \rho_j \in \mathcal{H}_{n+2}^* \Big\}$$

has full measure. Hence, letting Z_k^e be the degree k term in Γ^e at 0, the set

$$J = \{(x_1, \dots, x_d, \alpha) \in S^{d-1} \times \dots \times S^{d-1} \times \mathbb{R}^d : \\ p_n - \sum_{j=1}^{d} \alpha_j Z_n^{x_j} \in \mathcal{H}_n^* \ , \ q_n + \sum_{j=1}^{d} \alpha_j Z_{n+2}^{x_j} \in \mathcal{H}_{n+2}^* \}$$

has full measure.

If n is even, then we choose $\{x_j\}_{j=1}^d \subset A_n$ so that $(x_1, \dots x_d, \alpha) \in J$ for a full measure set of $\alpha \in \mathbb{R}^d$. Let $\tilde{\Gamma}_{x_j}$ be a smooth function which agrees with Γ_{x_j} outside A_n, let $T_{x_j}^{(n)}$ be the degree n Taylor polynomial of Γ_{x_j} at 0 and consider the function

$$\tilde{u_n}(x) = u_n(x) + \frac{1}{2}\sum_{j=1}^{d} \alpha_j[\tilde{\Gamma}_{x_j}(x) - T_{x_j}^{(n)}(x) + \tilde{\Gamma}_{x_j}(-x) - T_{x_j}^{(n)}(-x)]$$

where the α_j are very small and $(x_1, \dots, x_d, \alpha) \in J$. If the α_j are small enough then clearly (iii) and (iv) of Lemma 1.1 will still hold. Moreover $\tilde{u}_n$ is $\mathcal{O}(|x|^n)$ at ∞, $\mathcal{O}(|x|^{n+1})$ at 0, and being even, must in fact be $\mathcal{O}(|x|^{n+2})$ at 0. The degree n term at ∞ and degree n+2 term at 0 are respectively $p_n - \sum_{j=1}^{d} \alpha_j Z_n^{x_j}$ and $q_n + \sum_{j=1}^{d} \alpha_j Z_{n+2}^{x_j}$ and therefore belong to $\mathcal{H}_n^*$ and $\mathcal{H}_{n+2}^*$ by construction. This proves the claim for n even. If n is odd we define instead

$$\tilde{u}_n(x) = u_n(x) + \frac{1}{2}\sum_{j=1}^{d} \alpha_j[\tilde{\Gamma}_{x_j}(x) - T_{x_j}^{(n)}(x) - (\tilde{\Gamma}_{x_j}(-x) - T_{x_j}^{(n)}(-x))]$$

and proceed the same.

To finish the proof of the lemma let u_n be as in the claim and denote its degree n term at ∞ and degree $n+2$ term at 0 by a_n and b_n respectively. Let $v_n = u_n - a_n$ which is $\mathcal{O}(|x|^{n-2})$ at ∞, $\mathcal{O}(|x|^n)$ at 0, with the order n term at 0 being $-a_n$. Consider the functions $w_n = u_n - \epsilon_n v_{n+1}$ where ϵ_n is a small positive constant. w_n is $\mathcal{O}(|x|^n)$ at ∞, $\mathcal{O}(|x|^{n+1})$ at 0, with the order n term at ∞ being a_n and the order $n+1$ term at 0 being $\epsilon_n a_{n+1}$. Also supp $\Delta w_n \subset A_n \cup A_{n+1} = A_n$, and if ϵ_n is small then clearly (iii) and (iv) of Lemma 1.1 will hold for w_n.

Now define the function u_n of Lemma 2.2 by $u_n = (\prod_{j<n} \epsilon_j) \cdot w_n$. Since u_n is a scalar multiple of w_n it has properties (ii) and (iii) of Lemma 2.2 and is $\mathcal{O}(|x|^n)$ at ∞, $\mathcal{O}(|x|^{n+1})$ at 0. The order $n+1$ term in u_n at 0 and order $n+1$ term in u_{n+1} at ∞ are both equal to $(\prod_{j<n+1} \epsilon_j) \cdot a_{n+1}$ so the proof is complete. □

We will now proceed more or less as in our earlier paper [7] , although the details are simpler in the present context.

Lemma 2.4 (gluing lemma). *Assume $d \geq 4$. Let u_n be as in Lemma 2.2 and suppose $\epsilon > 0$ is given. Then if $s > 0$ is small enough and $r < \infty$ large enough there is a function $g_n : \mathbb{R}^d \to \mathbb{R}$ such that*

(i) $g_n(x) = r^{n+1} u_n(r^{-1}x)$ *if* $|x| \geq 2$.

(ii) $g_n(x) = s^{n+1}u_{n+1}(s^{-1}x)$ *if* $|x| \leq \frac{1}{2}$.

(iii) $|\triangle g_n| \leq \epsilon |\nabla g_n|$ *if* $\frac{1}{2} \leq |x| \leq 2$.

Furthermore g_n *satisfies bounds* $|D^\alpha g_n| \leq C_{\alpha,n}|x|^{n+1-|\alpha|}$ *independently of* r *and* s *provided* r *and* s^{-1} *are sufficiently large.*

Proof. Simply choose a smooth function ψ with $\psi = 1$ when $|x| \leq \frac{1}{2}$, $\psi = 0$ when $x \geq 2$ and define

$$g_n(x) = \psi(x)s^n u_{n+1}(s^{-1}x) + (1-\psi(x))r^n u_n(r^{-1}x).$$

We need only prove the estimates. However, in the region $\frac{1}{4} \leq |x| \leq 4$ we have

$$|r^{n+1}u_n(r^{-1}x) - q_n(x)| \leq C_n r^{-1}$$

since $u_n - q_n = \mathcal{O}(|x|^{n+2})$ at 0. By the derivative estimates for harmonic functions we also have

$$|D^\alpha(r^{n+1}u_n(r^{-1}x) - q_n(x))| \leq C_{\alpha,n}r^{-1} \tag{25}$$

when $\frac{1}{2} \leq x \leq 2$, for any α. Using that $u_{n+1} - p_{n+1} = \mathcal{O}(|x|^n)$ at ∞ and arguing in the same way we obtain a similar bound

$$|D^\alpha(s^{n+1}u_{n+1}(s^{-1}x) - p_{n+1}(x))| \leq C_{\alpha,n}s, \quad \frac{1}{2} \leq |x| \leq 2.$$

Since $p_{n+1} = q_n$ this may of course be written as

$$|D^\alpha(s^{n+1}u_{n+1}(s^{-1}x) - q_n(x))| \leq C_{\alpha,n}s. \tag{26}$$

By (25) and (26)

$$|\nabla g_n| \geq |\nabla q_n| - C_{\alpha,n}(r^{-1}+s)$$
$$|\triangle g_n| \leq C_{\alpha,n}(r^{-1}+s)$$

when $\frac{1}{2} \leq x \leq 2$. Since $q_n \in \mathcal{H}^*_{n+1}$ the bound (iii) now follows immediately. Next by (i) of Lemma 2.2 and derivative estimates for harmonic functions we have $|D^\alpha u_n(x)| \leq C_{\alpha,n}|x|^{n+1-|\alpha|}$ for small $|x|$, $|D^\alpha u_{n+1}(x)| \leq C_{\alpha,n}|x|^{n+1-|\alpha|}$ for large $|x|$. The last statement in the lemma follows by scaling arguments, which we omit. □

Proof of Theorem 1. Let $\{r_n\}_{n=n_0}^{\infty}$ be a sequence decreasing rapidly to zero. Define g_n using Lemma 2.4 with $\epsilon = 2^{-n}, r = \sqrt{\frac{r_n}{r_{n+1}}}$, $s = \sqrt{\frac{r_{n+1}}{r_n}}$ and let $h_n(x) = (\sqrt{r_n r_{n+1}})^{n+1} g_n(\frac{x}{\sqrt{r_n r_{n+1}}})$. This is possible provided r_{n+1} is small enough compared with r_n. Then

$$h_n(x) = r_n^{n+1} u_n(r_n^{-1} x), \qquad |x| \geq 2\sqrt{r_n r_{n+1}}$$
$$h_n(x) = r_{n+1}^{n+1} u_{n+1}(r_{n+1}^{-1} x), \qquad |x| \leq \frac{1}{2}\sqrt{r_n r_{n+1}}$$

$$(27) \qquad |\triangle h_n| \leq 2^{-n}(r_n r_{n+1})^{-\frac{1}{2}} |\bigtriangledown h_n|\,, \quad \frac{1}{2}\sqrt{r_n r_{n+1}} \leq x \leq 2\sqrt{r_n r_{n+1}}.$$

The derivative bounds in Lemma 2.4 translate to bounds $|D^\alpha h_n| \leq C_{\alpha,n} |x|^{n+1-|\alpha|}$.

Define the function u of Theorem 1 by

$$u(x) = r_{n_0}^{-1} h_{n_0}(x) \quad \text{ if } |x| > r_{n_0}$$
$$u(x) = (\prod_{j\leq n} r_j)^{-1} \cdot h_n(x) \quad \text{ if } n \geq n_0, r_{n+1} < |x| \leq r_n.$$

Then u is smooth on the boundaries $|x| = r_n$, since when $|x| \in (2\sqrt{r_n r_{n+1}}, r_n)$ we have $u(x) = (\prod_{j\leq n} r_j)^{-1} \cdot r_n^{n+1} u_n(r_n^{-1}x)$ and when $|x| \in (r_n, 2\sqrt{r_{n-1} r_n})$ we have $u(x) = (\prod_{j\leq n-1} r_j)^{-1} \cdot r_n^n u_n(r_n^{-1}x)$ which is the same.

Furthermore all derivatives of u go to zero as $|x| \to 0$ provided the $\{r_j\}$ decrease fast enough. Namely, the derivative bounds on the h_n translate to bounds

$$|D^\alpha u(x)| \leq C_{\alpha,n} (\prod_{j\leq n} r_j)^{-1} \cdot |x|^{n+1-|\alpha|}, \qquad r_{n+1} \leq |x| \leq r_n.$$

If $n \geq |\alpha|$ this is $\leq C_{\alpha,n}(\prod_{j\leq n} r_j)^{-1} r_n^{n+1-|\alpha|} = C_{\alpha,n}(\prod_{j<|\alpha|} r_j)^{-1} \prod_{|\alpha|\leq j\leq n} \frac{r_n}{r_j}$ which goes to zero as $n \to \infty$ for all α provided (say) $\dfrac{r_n}{r_{n-1}} \leq \min_{\alpha:|\alpha|\leq\frac{n}{2}} C_{\alpha,n}^{-1} \prod_{j\leq|\alpha|} r_j$.

Next we show that $|\triangle u| \leq C|x|^{-1}|\bigtriangledown u|$. Define $B_n = \{x : \frac{1}{2}\sqrt{r_n r_{n+1}} < |x| < 2\sqrt{r_n r_{n+1}}\}$. Then Δu vanishes except on the sets $r_n A_n$ and the sets B_n. If $x \in B_n$ then $|\triangle u| \leq 2^{-n}|x|^{-1}|\bigtriangledown u|$ by (27). On the other hand, if $x \in r_n A_n$ then,using scaling and (iii) of Lemma 2.2, we obtain $\frac{|\triangle u(x)|}{|\bigtriangledown u(x)|} = r_n^{-1} \frac{|\triangle u_n(r_n^{-1}x)|}{|\bigtriangledown u_n(r_n^{-1}x)|} \leq C r_n^{-1}$, and therefore $|\triangle u(x)| \leq C|x|^{-1}|\bigtriangledown u(x)|$, as claimed.

This completes the proof of part (a) of Theorem 1. Now we prove (b). Namely,

$$\begin{aligned}\|\frac{\triangle u}{|\nabla u|}\|_d^d &\lesssim \sum_{n=n_0}^{\infty}\int_{r_nA_n}|x|^{-d}dx + \sum_{n=n_0}^{\infty}\int_{B_n}2^{-nd}|x|^{-d}dx\\ &\lesssim \sum_n |A_n| + \sum_n 2^{-nd}.\end{aligned}$$

The second sum is obviously finite, and $|A_n| \approx n^{-(\frac{d-2}{2})}$ so the first sum is finite as well, provided $d \geq 5$. □

We remark that in part (b) of the theorem, the L^d norm of V may be taken arbitrarily small: just start the construction at a high finite stage n_1 instead of at n_0.

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