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ENTROPY ESTIMATES FOR EVOLVING HYPRSURFACES

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ABSTRACT. We consider parabolic curvature flows for hypersurfaces in Euclidean space, defined in terms of the mixed discriminants of Aleksandrov. Dilation-invariant curvature integral estimates are proved for convex hypersurfaces evolving by these equations, generalising the so-called entropy estimates which have been proved for special cases by Richard Hamilton, Ben Chow, and the author. The earlier proof by contradiction is replaced here by a new proof which is more direct and applies more generally, and which demonstrates the intimate connection between entropy estimates and the Aleksandrov-Fenchel inequalities.

1. INTRODUCTION

In this paper we consider parabolic evolution equations for convex hypersurfaces in Euclidean space: Suppose $\varphi_0 : M^n \to \mathbf{R}^{n+1}$ is a strictly convex initial embedding. We allow this to evolve according to an equation of the following form:

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(x,t) &= F(\mathcal{W}(x,t),\nu(x,t))\nu(x,t); \\ \varphi(x,0) &= \varphi_0(x) \end{aligned} \tag{1–1}$$

for all x in M^n and t in $[0,T)$. Thus the function F gives the speed of motion of the hypersurface, in terms of the Weingarten curvature $\mathcal{W}$ and the outward normal direction ν. Equation (1–1) is parabolic provided that F is monotonic in the Weingarten curvature at each point—precisely, the derivative of F with respect to the curvature, $\dot{F}_\ell^k = \frac{\partial F}{\partial \mathcal{W}_k^\ell}$, is required to be negative definite.

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These evolution equations can also be written as evolution equations for the support function $s : S^n \to \mathbf{R}$ of the hypersurface (see section 2):

$$\frac{\partial}{\partial t} s(z,t) = F(\mathcal{W}, z) \tag{1–2}$$

for all z in S^n and t in $[0, T)$.

Several so-called *entropy estimates* have been proved for hypersurfaces evolving under parabolic equations of this kind. These estimates give dilation-invariant integrals of curvature which are monotonic in time; such estimates are useful in controlling the types of singularites which may occur under the flow. The first such estimate was proved by Richard Hamilton [Ha] for the curve-shortening flow, in which we take $n = 1$ and $F = -k$, the curvature of the curve. Hamilton showed that the following quantity decreases in time for a convex embedded solution to the flow:

$$A^{\frac{1}{2}} \exp\left\{ \frac{1}{2\pi} \int_{S^1} \ln k \, d\theta \right\}, \tag{1–3}$$

where A is the area enclosed by the curve. Here it is understood that a point on the curve is associated with a point on S^1 via the Gauss map ν.

This result was extended to higher dimensions by Chow [Ch] for the Gauss curvature flow, in which n is arbitrary and $F = -K$, where K is the Gauss curvature. The entropy estimate in this case states that the following quantity is decreasing:

$$V^{\frac{n}{n+1}} \exp\left\{ \frac{1}{|S^n|} \int_{S^n} \ln K \, d\mu \right\} \tag{1–4}$$

where V is the volume enclosed by the convex hypersurface, $|S^n|$ is the n-dimensional area of the unit hypersphere S^n in $\mathbf{R}^{n+1}$, and $d\mu$ is the area element on S^n.

These estimates have been generalised by the author [An1] to certain other flows, where the speed of the hypersurface is given by $F = -\frac{K}{S_{n-k}[\lambda]} = -S_k[\kappa]^{-1}$, for $k = 1, \ldots, n$. Here K is again the Gauss curvature, and $S_\ell[\lambda]$ is the ℓ^{th} elementary symmetric function of the principal curvatures of the hypersurface, for $\ell = 0, \ldots, n$ (interpreting $S_0[\lambda] \equiv 1$); $S_k[\kappa]$ is the k^{th} elementary symmetric function of the principal radii of curvature. In these cases we have the following decreasing quantity:

$$V_{k+1}^{\frac{k}{k+1}} \exp\left\{\frac{1}{|S^n|}\int_{S^n} \ln|F|\,d\mu\right\} \tag{1–5}$$

where $V_\ell = \int_{S^n} sS_{\ell-1}[\kappa]\,d\mu$ is the ℓ^{th} integral cross-sectional volume of the hypersurface, for $\ell = 1, \ldots, n+1$ (see section 2).

In all of these papers, the proofs of the entropy estimates are similar: Denoting by E the integral $\int_{S^n} \ln|F|\,d\mu$, one calculates the second time derivative of E, showing that it satisfies an estimate of the following form:

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}E\right) \geq \frac{k+1}{k|S^n|}\left(\frac{\partial}{\partial t}E\right)^2 \tag{1–6}$$

This estimate is closely related to the Harnack inequalities satisfied by the solutions. It is known that the interval of existence T of the solution to these equations is just proportional to $V_{k+1}[\varphi]$ (the area A in the case of the curve shortening flow, and the enclosed volume V in the case of the Gauss curvature flow). If $\frac{\partial}{\partial t}E$ were initially too large, then the estimate (1–6) would force it to become infinite before the final time was reached; the estimate obtained on $\frac{\partial}{\partial t}E$ from this argument is just enough to show that the entropy quantity decreases.

In this paper, we prove new entropy inequalities for a much wider class of evolution equations. Furthermore, the indirect proof used in the previous cases is replaced by a direct application of the Aleksandrov-Fenchel inequalities (see section 2). The results hold for a class of evolution equations defined in terms of the *mixed discriminants* of Aleksandrov, which are more general than those mentioned above, in several respects: Firstly, we allow flows which have speeds homogeneous of any degree in the curvature; secondly, the speed is allowed to be anisotropic (so that it depends nontrivially on the normal direction, as well as on the curvature). Precisely, we allow evolution equation with speed F defined as follows (see section 2 for the definitions):

$$F = \operatorname{sgn}\alpha\,\sigma(\nu)\left(\frac{\mathcal{Q}_k[s]}{\mathcal{Q}_k[\sigma]}\right)^\alpha \tag{1–7}$$

where σ is a positive, smooth support function of a fixed bounded convex region, and $\mathcal{Q}_k[f]$ is a mixed discriminant, defined for any function f in terms of smooth support functions $s_{k+1}, \ldots, s_n$ of fixed bounded convex regions:

$$\mathcal{Q}_k[f] = \mathcal{Q}[\underbrace{f, \ldots, f}_{k \text{ times}}, s_{k+1}, \ldots, s_n]. \tag{1–8}$$

s is the support function of the evolving region decribed by φ, and $\mathcal{Q}$ denotes the mixed discriminant. α is allowed to be any non-zero real number.

The main theorem is as follows:

Theorem 1–9. *For any strictly convex solution to equation* (1–1) *with speed given by* (1–7)*, the following quantity is decreasing in time:*

$$V_{k+1}^{\frac{k}{k+1}} Z[s] \tag{1–10}$$

where we define

$$V_\ell = V[\underbrace{s, \ldots, s}_{\ell \text{ times}}, \underbrace{\sigma, \ldots, \sigma}_{k+1-\ell \text{ times}}, s_{k+1}, \ldots, s_n] \tag{1–11}$$

for $\ell = 0, 1, \ldots, k+1$. $V[f_0, f_1, \ldots, f_n]$ *is the mixed volume of the functions* $f_0, \ldots, f_n$*, and* $Z[s]$ *is defined as follows:*

$$Z[s] = \begin{cases} \left(\frac{1}{V_0} \int_{S^n} \sigma \mathcal{Q}_k[\sigma] \left(\frac{\mathcal{Q}_k[s]}{\mathcal{Q}_k[\sigma]} \right)^{1+\alpha} d\mu \right)^{\frac{\operatorname{sgn} \alpha}{1+\alpha}} & \text{for } \alpha \neq -1; \\ \exp \left\{ \frac{1}{V_0} \int_{S^n} \sigma \mathcal{Q}_k[\sigma] \ln \left(\frac{\mathcal{Q}_k[s]}{\mathcal{Q}_k[\sigma]} \right) d\mu \right\}^{-1} & \text{for } \alpha = -1. \end{cases} \tag{1–12}$$

Note that the flows considered here include the flows where the speed is given by a power of the Gauss curvature (the case $k = n$, $\sigma \equiv 1$), and also the flows by powers of the harmonic mean curvature ($k = 1$, $\sigma, s_2, \ldots, s_n \equiv 1$).

Section 2 of the paper introduces the notation used in the paper, including that required for the description of hypersurfaces using the support function. The mixed volumes and mixed discriminants are introduced, along with some of their important properties, including the Aleksandrov-Fenchel inequalities.

The proof of the entropy estimate is given in section 3. With the machinery set in place in section 2, this requires only a short calculation. This section also proves some global lower bounds on the entropy quantities.

2. Notation for Convex Regions

This section introduces the basic notation and preliminary results required for the paper: We review the use of support functions to describe convex regions, and introduce the mixed volumes, the mixed discriminants, and the Aleksandrov-Fenchel inequalities. For a detailed exposition of the material in this section, see [BZ].

Support functions.

The support function $s : S^n \to \mathbf{R}$ of a convex region D in $\mathbf{R}^{n+1}$ is defined as follows:

$$s(z) = \sup_{y \in D} \langle y, z \rangle \tag{2–1}$$

for each z in S^n. This gives the distance of each supporting hyperplane of D from the origin. The region D can be recovered from s as follows:

$$D = \bigcap_{z \in S^n} \{ y \in \mathbf{R}^{n+1} : \langle y, z \rangle \leq s(z) \}. \tag{2–2}$$

If D is smooth and strictly convex one can construct an embedding φ of S^n to the boundary ∂D of D which associates to each direction z in S^n the point of ∂D with normal z. This is given by the following expression:

$$\varphi(z) = s(z) z + \bar{\nabla} s(z) \tag{2–3}$$

where $\bar{\nabla} s$ is the gradient vector of s with respect to the standard metric $\bar{g}$ on S^n. This leads to a simple expression for the Weingarten curvature $\mathcal{W}$ of ∂D: From any function f on S^n, we can construct a map $A[f]$ of TS^n, the tangent bundle of S^n, as follows:

$$A[f] = \bar{g}^* \operatorname{Hess}_{\bar{\nabla}} f + f \operatorname{Id}. \tag{2–4}$$

Here $\operatorname{Hess}_{\bar{\nabla}} f$ is the Hessian form of f with respect to the standard connection $\bar{\nabla}$ on S^n, and $\bar{g}^* \operatorname{Hess}_{\bar{\nabla}} f$ is the corresponding map of TS^n obtained using $\bar{g}$:

$$\bar{g}\,(u, \bar{g}^* \operatorname{Hess}_{\bar{\nabla}} f(v)) = d_u d_v f - d_{\bar{\nabla}_u v} f \tag{2–5}$$

for all vector fields u and v in TS^n. This definition gives the following expression for $\mathcal{W}$ (see [An1] for details):

$$\mathcal{W}^{-1} = A[s]. \tag{2–6}$$

For any two convex regions D_1 and D_2, we can take the Minkowski sum $D_1 + D_2$ given by $\{a+b : a \in D_1, b \in D_2\}$. This sum is again a convex region. If D_1 and D_2 have support functions s_1 and s_2 respectively, then $D_1 + D_2$ has support function s given by

$$s(z) = s_1(z) + s_2(z). \tag{2–7}$$

Mixed volumes.

The volume $V(D)$ of a convex region D can be calculated in terms of the support function by the following integral over S^n:

$$V(D) = \frac{1}{n+1}\int_{S^n} s \det\left(\mathcal{W}^{-1}\right) d\mu = \frac{1}{n+1}\int_{S^n} s \det A[s]\, d\mu \tag{2–8}$$

where $d\mu$ is the standard measure on S^n.

Now we consider a linear combination of convex regions (in the Minkowski sense): Let D_i, $i = 1, \ldots, N$ be convex regions with support functions s_i, and consider the Minkowski sum $\sum_{i=1}^N \epsilon_i D_i$ for arbitrary positive ϵ_i (here multiplication by a scalar corresponds to scaling about the origin). The expressions (2–4), (2–7) and (2–8) show that the volume $V(\sum \epsilon_i D_i)$ is a homogeneous polynomial of degree $n+1$ in the variables ϵ_i:

$$V\left(\sum \epsilon_i D_i\right) = \frac{1}{n+1}\sum_{1\le i_0,\ldots,i_n\le N} \epsilon_{i_0}\ldots\epsilon_{i_n} V\left(D_{i_0}, \ldots, D_{i_n}\right) \tag{2–9}$$

where the coefficient $V\left(D_{i_0}, \ldots, D_{i_n}\right)$ is called the *mixed volume* of the $n+1$ regions $D_{i_0}, \ldots, D_{i_n}$, and is given by the following expression in local coordinates:

(2–10)

$$V\left(D_0, \ldots, D_n\right) = \int_{S^n} s_0 \mathcal{Q}\left[s_1, \ldots, s_n\right] d\mu$$

(2–11)

$$\mathcal{Q}\left[f_1, \ldots, f_n\right] = \frac{1}{n!}\sum_{\sigma,\tau\in S_n} (-1)^{\operatorname{sgn}(\tau)+\operatorname{sgn}(\sigma)} A\left[f_1\right]_{\tau(1)}^{\sigma(1)} \ldots A\left[f_n\right]_{\tau(n)}^{\sigma(n)}.$$

The operator $\mathcal{Q}$ is a multilinear operator acting on n functions $s_1, \ldots, s_n$ on S^n; it is called the *mixed discriminant* of $s_1, \ldots, s_n$. It has several important properties:

Lemma 2–12.

(1). *$\mathcal{Q}$ is independent of the order of its arguments:*

$$\mathcal{Q}\left[f_1, \ldots, f_n\right] = \mathcal{Q}\left[f_{\sigma_1}, \ldots, f_{\sigma_n}\right]$$

for every permutation $\sigma \in S_n$.

(2). *$\mathcal{Q}[f_1, \ldots, f_n]$ is positive for any n functions $f_1, \ldots, f_n$ with each $A[f_i]$ positive definite.*

(3). *If $f_2, \ldots, f_n$ are fixed smooth functions, and $A[f_i]$ is positive definite for each i, then $\mathcal{Q}[f] := \mathcal{Q}\left[f, f_2, \ldots, f_n\right]$ is a nondegenerate second-order linear elliptic operator, given in local coordinates by an expression of the following form:*

$$\mathcal{Q}[f] = \sum_{i,j} \dot{\mathcal{Q}}^{ij}\left(\bar{\nabla}_i \bar{\nabla}_j f + \bar{g}_{ij} f\right) \tag{2–13}$$

where $\dot{\mathcal{Q}}$ is a positive definite matrix at each point of S^n, depending only on the functions $f_2, \ldots, f_n$.

(4). *The following identity holds for any $f_2, \ldots, f_n$ as above:*

$$\sum_i \bar{\nabla}_i \dot{\mathcal{Q}}^{ij} = 0. \tag{2–14}$$

The formula (2–14) follows from the definition (2–11) and the Codazzi equations, which imply the symmetry of $\bar{\nabla} A\left[f_i\right]$ for every i.

These properties of $\mathcal{Q}$ allow us to deduce some important properties of the mixed volumes:

Lemma 2–15. *Let $D_0, D_0', D_1, \ldots, D_n$ be bounded convex regions. Then we have the following facts:*

(1). *Symmetry: Let σ be a permutation of the set $\{0, \ldots, n\}$. Then*

$$V(D_0, \ldots, D_n) = V(D_{\sigma(0)}, \ldots, D_{\sigma(n)}).$$

(2). *Invariance under translations: If p is any point of $\mathbf{R}^{n+1}$, then*

$$V(D_0 + p, D_1, \ldots, D_n) = V(D_0, D_1, \ldots, D_n).$$

(3). *Positivity:*

$$V(D_0, \ldots, D_n) \geq 0.$$

(4). *Monotonicity: Suppose $D_0 \subseteq D_0'$. Then*

$$V(D_0, D_1, \ldots, D_n) \leq V(D_0', D_1, \ldots, D_n).$$

Property (1) follows from the identity (2–14), which allows us to integrate by parts. Property (2) then follows because we have $A[\langle z, e\rangle] = 0$ for all e in $\mathbf{R}^{n+1}$, so that $A[s_0]$ is not affected by translations. Positivity follows since we can choose the origin to make s_0 positive, and $\mathcal{Q}[s_1, \ldots, s_n]$ is positive. Monotonicity follows since $s_0 \leq s_0'$ and $\mathcal{Q}[s_1, \ldots, s_n]$ is positive.

It is interesting to note certain special cases of mixed volumes: For any strictly convex region D, define M_k to be the mixed volume of k copies of D with $n - k + 1$ copies of the unit ball B, for $k = 1, \ldots, n + 1$:

$$M_k(D) = V(\underbrace{D, \ldots, D}_{k \text{ times}}, \underbrace{B, \ldots, B}_{n+1-k \text{ times}}) \tag{2–16}$$

For $k = n + 1$ this gives the volume $V(D)$. For $k = n$ we have $M_n(D) = H^n(\partial D)$, the Hausdorff n-measure of the boundary of D. For $k < n$ these mixed volumes are called the integral cross-sectional volumes of D, and give the average k-measure of projections of D onto k- planes.

The Aleksandrov-Fenchel inequalities.

The Aleksandrov-Fenchel inequalities relate the different mixed volumes which can be formed from a collection of convex regions (see [Al1–2], [F]):

Theorem 2–17 (Aleksandrov, Fenchel). *Let $D_0, \ldots, D_n$ be bounded convex regions in $\mathbf{R}^{n+1}$. Then the following inequality holds:*

(2–18)

$$V(D_0, D_0, D_2, \ldots, D_n)\, V(D_1, D_1, D_2, \ldots, D_n) \leq V(D_0, D_1, D_2, \ldots, D_n)^2.$$

Certain of the flows given by (1–7) have been used by the author to give a simple proof of the Aleksandrov-Fenchel inequalities (see [An3]). The basic inequality has many useful consequences: For the special case of the integral cross-sectional volumes $M_k(D)$, we have the following:

$$M_k(D)^2 \geq M_{k-1}(D) M_{k+1}(D) \tag{2–19}$$

for $k = 1, \ldots, n$, where we interpret $M_0(D) = V(B)$, the volume of the unit ball. By applying this several times, we obtain the Minkowski inequalities:

$$M_k(D)^{a+b} \geq M_{k-a}(D)^b M_{k+b}(D)^a \tag{2–20}$$

for $0 \leq k - a < k < k + b \leq n + 1$. The case $k = a = n, b = 1$ is the isoperimetric inequality.

The Aleksandrov-Fenchel inequality can be interpreted as a kind of Poincaré inequality: For any function f in $W^{1,2}(S^n)$, and any σ, $s_2, \ldots, s_n$ the support functions of fixed smooth, strictly convex bounded regions, we have the following inequality:

(2–21)
$$\frac{1}{V_0}\int_{S^n} \dot{\mathcal{Q}}^{ij}\bar{\nabla}_i f\, \bar{\nabla}_j f\, d\mu \geq \frac{1}{V_0}\int_{S^n} \dot{\mathcal{Q}}(\mathrm{Id}) f^2\, d\mu - \left(\frac{1}{V_0}\int_{S^n} f\mathcal{Q}[\sigma]\, d\mu\right)^2.$$

Note that this inequality is invariant under the transformation $f \to f + C\sigma$ for any C. This allows us to prove the inequality without requiring that f be the support function of a convex region.

3. Decreasing Entropy

Now we can proceed to calculate the rate of change of the entropy under these evolution equations. To simplify the calculations, we use the following notation:

$$\mathcal{Q}_s[f] := \mathcal{Q}[f, \underbrace{s, \ldots, s}_{k-1 \text{ times}}, s_{k+1}, \ldots, s_n] \tag{3–1}$$

for any function f. In particular we have $\mathcal{Q}_k[s] = \mathcal{Q}_s[s]$. For the mixed volumes we use the corresponding notation

$$V_s[f_1, f_2] := V[f_1, f_2, \underbrace{s, \ldots, s}_{k-1 \text{ times}}, s_{k+1}, \ldots, s_n] \tag{3–2}$$

for any f_1 and f_2.

First consider the mixed volume $V_{k+1}[s] = V_s[s,s]$:

$$\begin{aligned}\frac{\partial}{\partial t}V_{k+1}[s] &= \frac{\partial}{\partial t}\int_{S^n} s\mathcal{Q}_s[s]\,d\mu \\ &= \int_{S^n} F\mathcal{Q}_s[s]\,d\mu + k\int_{S^n} s\mathcal{Q}_s[F]\,d\mu \\ &= (k+1)\int_{S^n} F\mathcal{Q}_s[s]\,d\mu \\ &= (k+1)V_s[F,s].\end{aligned} \tag{3–3}$$

Next we calculate the time derivative of the integral Z. First consider the case where $\alpha \neq -1$:

$$\begin{aligned}\frac{\partial}{\partial t}Z &= \frac{\partial}{\partial t}\left(\frac{1}{V_0}\int_{S^n}\sigma\mathcal{Q}_k[\sigma]\left(\frac{\mathcal{Q}_s[s]}{\mathcal{Q}_k[\sigma]}\right)^{1+\alpha}d\mu\right)^{\frac{\operatorname{sgn}\alpha}{1+\alpha}} \\ &= k\operatorname{sgn}\alpha Z^{1-\operatorname{sgn}\alpha(1+\alpha)}\frac{1}{V_0}\int_{S^n}\sigma\left(\frac{\mathcal{Q}_s[s]}{\mathcal{Q}_k[\sigma]}\right)^{\alpha}\mathcal{Q}_s[F]\,d\mu \\ &= kZ^{1-\operatorname{sgn}\alpha(1+\alpha)}\frac{1}{V_0}V_s[F,F].\end{aligned} \tag{3–4}$$

In the case $\alpha = -1$ we have:

$$\begin{aligned}\frac{\partial}{\partial t}Z &= \frac{\partial}{\partial t}\exp\left\{\frac{1}{V_0}\int_{S^n}\sigma\mathcal{Q}_k[\sigma]\ln\left(\frac{\mathcal{Q}_s[s]}{\mathcal{Q}_k[\sigma]}\right)d\mu\right\}^{-1} \\ &= \frac{k}{ZV_0}\int_{S^n}\sigma\frac{\mathcal{Q}_k[\sigma]}{\mathcal{Q}_s[s]}\mathcal{Q}_s[F,s]\,d\mu \\ &= \frac{k}{ZV_0}\int_{S^n}F\mathcal{Q}_s[F]\,d\mu\end{aligned} \tag{3–5}$$

The time derivative of the entropy quantity (1-12) can be found in the case $\alpha \neq -1$ by combining equations (3–3) and (3–4):

$$\frac{\partial}{\partial t}\left(V_k^{\frac{k}{k+1}}Z\right) = kZ^{-\frac{1+\alpha}{\operatorname{sgn}\alpha}+1}V_{k+1}^{-\frac{1}{k+1}}\left(V_s[s,s]\,V_s[F,F] - V_s[F,s]^2\right), \tag{3–6}$$

noting that $V_s[F,s] = V_0Z^{\frac{1+\alpha}{\operatorname{sgn}\alpha}}$. The term in brackets we recognize as the quantity in the Aleksandrov-Fenchel inequality (2–17), which is therefore negative.

In the case $\alpha = -1$ we have:

$$\frac{\partial}{\partial t}\left(V_{k+1}^{\frac{k}{k+1}}Z\right) = \frac{kV_{k+1}^{-\frac{1}{k+1}}}{ZV_0}\left(V_s[s,s]\,V_s[F,F] - V_0^2\right). \tag{3–7}$$

Since $V_0 = V_s[F,s]$ in this case, the Aleksandrov-Fenchel inequalities apply again to show that the bracket is negative. The result follows.

If F is isotropic and homogeneous of degree one ($\sigma, s_{k+1}, \ldots, s_n \equiv 1$, $\alpha = -\frac{1}{k}$), it is known that the solutions become spherical at the end of their interval of existence (see [An2]). Hence the entropy converges to that for the sphere; we have the following result:

Theorem 3–8. *For any smooth, strictly convex region with support function s, the following inequalities hold:*

$$V_{k+1}^{\frac{k}{k+1}} Z_{k,\alpha}[s] \geq |S^n|^{\frac{k}{k+1}} \tag{3–9}$$

for $k = 1, \ldots, n$ and $\alpha \leq \frac{1}{\alpha}$, where $Z_{k,\alpha}[s]$ is the isotropic entropy given by equation (1–12).

Corollary 3–10. *For any smooth $\sigma, s_{k+1}, \ldots, s_n$, $k = 1, \ldots, n$, and any $\alpha \leq -\frac{1}{k}$ there exists a constant $C(\sigma, k, s_{k+1}, \ldots, s_n, \alpha) > 0$ such that the following estimate holds for all support functions s of convex regions:*

$$V_{k+1}^{\frac{k}{k+1}} Z[s] \geq C. \tag{3–11}$$

Proof. The theorem follows in the case $\alpha = -\frac{1}{k}$ simply from the fact that the entropy decreases and converges to that for the sphere. The result for lower α follows from this by the Hölder inequality. The corollary follows because the entropy for any anisotropic flow is bounded above and below by multiples of the entropy for the isotropic flow. □

A simple application of the entropy estimates also shows that solutions do not in general converge to the homothetic solution σ for small negative exponents α: Observe that the limit as α tends to zero of the entropy is just the isoperimetric ratio $V_0 V_{k+1}^{\frac{k}{k+1}} V_k^{-1}$, which is less than or equal to $V_0^{\frac{k}{k+1}}$ by the Aleksandrov-Fenchel inequalities. Furthermore, if s is not a translated multiple of σ, the inequality is strict. Hence for any s, there is some $\varepsilon > 0$ such that the inequality $V_{k+1}^{\frac{k}{k+1}} Z < V_0^{\frac{k}{k+1}}$ holds for all α in the range $(-\varepsilon, 0)$. It follows that the flows with initial condition s and α in this range do not converge to σ after rescaling, because the entropy is already less than that for σ.

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