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# REALIZATION IN $\mathbb{R}^3$ OF COMPLETE RIEMANNIAN MANIFOLDS WITH NEGATIVE CURVATURE

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ABSTRACT. This paper is concerned with the global smooth isometric immersion in $\mathbb{R}^3$ of complete simply connected negative curved surfaces. A sufficient condition involving the rate of decay of the curvature at infinity is imposed.

## INTRODUCTION

Since Weyl [W] and Lewy [L] proved that a two dimensional sphere with an arbitrary analytic positive curvature metric has an isometric analytic embedding in $\mathbb{R}^3$, there have been many papers to devoted to this field. Most of them involved positive curvature metrics. For example, Nirenberg [N] and Pogorelov [P] generalized the Weyl's result to the $C^\infty$ case; Heinz [He], Delanoe [D], and Hong [H] successively discussed the isometric embedding in $\mathbb{R}^3$ of a positive or nonnegative curvature metric given in some domains in $\mathbb{R}^2$ respectively. Also, a few papers are concerned with negative curvature metrics. Hilbert pointed out that any complete 2-dimensional surface with negative constant curvature has no $C^2$ isometric immersion in $\mathbb{R}^3$, and, later, Efimov [E] generalized the Hilbert Theorem and proved that any complete 2-dimensional surface with curvature bounded above by a negative constant has no $C^2$ isometric immersion in $\mathbb{R}^3$. As for the positive answer, except for the result due to Poznyak [Po], which states that any geodesic disk with smooth negative curvature has a smooth isometric immersion in $\mathbb{R}^3$, no more is known. So S.T.Yau [Y] raised the following question: Find a sufficient condition for a complete negative curved surface to be isometrically embedded in $\mathbb{R}^3$. He also pointed out that a reasonable sufficient condition might be the rate of decay of the curvature at infinity. In the present paper, a realization in $\mathbb{R}^3$ of such a surface is found, and in order to insure such a possibility of realization the rate of decay of curvature at infinity should be faster than the geodesic distance to the power $-2$.

Let $(M, g)$ be a complete simply connected 2-dimensional surface with negative curvature $-k$ for some positive smooth function $k$. The Hadamard theorem tells us that the exponential map $\exp_p$ is a global smooth diffeomorphism from $T_p(M)$ onto $M$. Denote by $(\rho, \theta)$ the corresponding geodesic polar coordinates. Then the main result in the present paper is as follows.

**Theorem A.** *Let $(M, g)$ be a smooth complete simply connected 2-dimensional surface with negative curvature $-k$. Assume that $k$ satisfies*

$$\partial_\rho \ln(k\rho^{2+\delta}) \leq 0 \text{ as } \rho \geq R,$$

*for some positive constants $\delta$ and $R$, and*

$$\partial_\theta^i \ln k \ (i = 1, 2), \quad \rho\partial_\theta\partial_\rho \ln k \text{ bounded}. \tag{0.1}$$

*Then $(M, g)$ has a smooth isometric immersion in $\mathbb{R}^3$.*

First of all, we establish a global geodesic coordinate system on $M$. Let $(e_1, e_2)$ be the orthonormal basis on $T_p(M)$ and define

$$X = \{q \in M \mid q = \exp_p xe_1, x \in \mathbb{R}^1\}. \tag{0.2}$$

Then any point on $X$ is identified by the coordinates $(x, 0)$. Now we introduce local geodesic coordinates $(x, t)$ with the base curve $X$, where $t$ is the oriented distance from the point $(x, 0)$ and $t > 0$ in the image $\{\exp_p(xe_1 + ye_2) | y > 0\}$ and $t < 0$ otherwise.

**Theorem B.** *The geodesic coordinates with the base curve $X$ mentioned above covers the whole manifold $M$.*

It seems that the conclusion in Theorem B should be well known. But I have not found the place including the verification. For the convenience of the reader in Section 2 the proof is given. Under these geodesic coordinates

$$g = dt^2 + B^2(x, t)dx^2 \tag{0.3}$$

with

$$B(x,0)=1 \text{ and } B_t(x,0)=0, \quad x\in\mathbb{R}^1 \tag{0.4}$$

Assume that the curvature of $g$ equals $-k$ for some smooth positive function $k$. Suppose that ($H_1$) $k>0$ and $t\partial_t \ln(k|t|^{2+\delta})\le 0$ as $|t|\ge T$ for some positive constants $\delta$ and $T$; ($H_2$) $k$, $\partial_x^i \ln k (i=1,2)$, $t\partial_x\partial_t \ln k$ are bounded and $\partial_t \ln k$ is locally bounded in $t$, namely, bounded for finite $t$; ($H_3$) $\inf_x \int_0^\infty k(x,t)dt$ and $\inf_x \int_{-\infty}^0 k(x,t)dt$ positive.

**Theorem C.** *Let $(M,g)$ be a complete simply connected smooth surface with negative curvature $-k$. Suppose that the hypotheses ($H_1$)- ($H_3$) are fulfilled. Then $(M,g)$ has a smooth isometric immersion in $\mathbb{R}^3$.*

Obviously, ($H_1$) implies that the rate of decay at infinity of $k$ is faster than $|t|^{-2}$ and $\ln k$ has some monotonity in $t$ at infinity. Using the method in this paper, without difficulty, one can conclude that if $(M,g)$ is $C^{4,1}$, then it can be realized by some $C^{3,1}$ surface in $\mathbb{R}^3$.

One wonders if the restriction on the rate of the decay of the curvature as fast as the geodesic distance from the base curve of $-(2+\delta)$ order at infinity, can be relaxed. It is still open.

Usually, there are two ways to realize in $\mathbb{R}^3$ a Riemannian manifolds. One way is to solve the Darboux equation involving its curvature as done in [N], and the other way is to solve, the Gauss- Codazzi system as done in [P]. It seems that the first way is more suitable for the positive curvature case and the second way for the negative curvature case. The scheme of the present paper is as follows. In Section 1, the proof of Theorem B is presented and some lemmas about the properties of the function B in (0.3), which are useful for the estimation of solutions to the Gauss-Codazzi system, are given. In Section 2 under the hypotheses ($H_1$)-($H_3$) some asymptotic behavior of solutions to the Gauss-Codazzi system is studied and the proof of the Theorem C is covered. Section 3 completes the proof of Theorem A.

## 1. Global geodesic coordinates

This section is concerned with the verification of Theorem B and some properties about the function B are given also.

*The proof of Theorem B.* We first prove that the following transformation:

$T$: $\mathbb{R}^2 \ni (x,t) \mapsto q \in M$ such that $q$ is a point on the geodesic $\gamma_1$ passing through $(x,0)$ and perpendicular to $X$ with the oriented geodesic distance $t$ from $q$ to $X = \{\exp_p(xe_1) \mid x \in \mathbb{R}^1\}$,

is surjective. In fact, for each $q \in M$ we can find a shortest geodesic $\gamma$ from the point $q$ to $X$ since the geodesic distance $t^*$ from $q$ to $X$ is less than or equal to the distance between $q$ and the polar $p$. Obviously, $\gamma$ is perpendicular to $X$. So $T(x^*, t^*)$ or $T(x^*, -t^*) = q$ where $T(x^*, 0)$ is the intersection of $\gamma$ and $X$. This proves the surjectivity of $T$.

To prove the injectivity, we first point out that $T$ is locally diffeomorphic. With $T_x = dT(\partial/\partial x)$ and $T_t = dT(\partial/\partial t)$, by the definition of the transformation $T$, we have $\langle T_t, T_t\rangle = 1$. By Gauss's theorem we know $\langle T_x, T_t\rangle = 0$, namely, $T_x$ and $T_t$ form an orthogonal frame on the tangent space. Twice differentiating $\langle T_x, T_x\rangle$ with respect to $t$ gives

$$\partial^2/\partial t^2 \langle T_x, T_x\rangle = |\nabla_{T_x} T_t|^2 - \langle R(T_t, T_x)T_x, T_t\rangle$$

The assumption on the negativity of the curvature guarantees that $\partial/\partial t\langle T_x, T_x\rangle$ is monotonely-increasing and

$$t\partial/\partial t\langle T_x, T_x\rangle > \partial/\partial t\langle T_x, T_x\rangle|_{t=0} = -2\langle T_t, \nabla_{T_x} T_x\rangle|_{t=0} = 0, \quad \text{if } t \neq 0$$

It turns out

$$\langle T_x, T_x\rangle \ \geq\ \langle T_x, T_x\rangle|_{t=0} = 1 \text{ for all } t \tag{1.1}$$

Summing up we have the map $dT$ being nonsingular everywhere and hence $T$ is locally diffeomorphic.

To prove the global diffeomorphism of $T$, in view of the Hadamard lemma, we only need to illustrate $T$ being proper since $T$ is surjective and $M$ is simply connected. Let $C$ be any compact set in $M$. Then we have

$$\begin{aligned} &\sup\{|t| \mid \text{there exists } x \in \mathbb{R}^1 \text{ such that} T(x,t) \in C\} \\ \leq\ &\sup\{\rho = |\exp_p^{-1}(T(x,t))| \text{ where } T(x,t) \in C\} < +\infty \end{aligned}$$

and

$$\begin{aligned} & \sup\{|x| \mid \text{there exists } t \in \mathbb{R}^1 \text{ such that} T(x,t) \in C\} \\ \leq \;& 2\sup\{\rho = |\exp_p^{-1}(T(x,t))| \text{ where } T(x,t) \in C\} < +\infty \end{aligned}$$

since $\exp_p$ is diffeomorphic from $T_p(M)$ onto $M$. This completes the proof of Theorem B. □

So far we have established global geodesic coordinates $(x, t)$ with the base curve $X$ for any complete simply connected Riemannian manifold with non-positive curvature. Under this coordinate system the metric $g$ is of the form as mentioned in (0.3) with (0.4) since $B^2 = < T_x, T_x > \geq < T_x, T_x > |_{t=0} = 1$ and $B_t(x, 0) = - < T_t, \nabla_{T_x} T_x > |_{t=0} = 0$.

In the sequel we often use the geodesic coordinates $(x, t)$ as mentioned above and the geodesic polar coordinates $(\theta, \rho)$. Suppose that under the geodesic polar coordinates, the metric is of the form.

$$g = d\rho^2 + G^2 d\theta^2.$$

Then from the Gauss equation we have

$$G_{\rho\rho} = kG \text{ with } G = 0 \text{ and } G_\rho = 1 \text{ at } \rho = 0$$

$$\text{and } B_{tt} = kB \text{ with (0.4)} \tag{1.2}$$

For the needs of the later discussion we investigate some properties about the functions $B$ and $G$.

**Lemma 1.1.** *Let* $(H_1)$*–*$(H_3)$ *be fulfilled. Then*

$$\frac{B_t}{B} = \frac{1}{t}(1 + O(1/|t|^\delta)) \tag{1.3}$$

*uniformly in* $x$ *for sufficiently large* $|t|$ *and*

$$\partial_t \ln B, B\partial_t\partial_x \ln B \text{ are bounded.} \tag{1.4}$$

*Proof.* Integrating the second equation in (1.2) gives

$$B_t = b_0(t) + \int_0^t B_s(s) N(t, s) ds \tag{1.5}$$

where

$$b_0(t) = \int_0^t kds \text{ and } N(t,s) = \int_s^t kd\xi.$$

This is an integral equation of Voterra type. By iteration we find a solution of explicit form in terms of Neumann series

$$B_t = \sum_{n=0}^{\infty} T^n b_0 \tag{1.6}$$

where

$$T^0 b_0 = b_0 \text{ and } T^n b_0 = \int_0^t (T^{n-1}b_0)(s)N(t,s)ds.$$

Let us prove

$$|T^n b_0| \le b_0(t)\left[\int_0^t skds\right]^n \frac{1}{n!} \tag{1.7$_n$}$$

by induction on $n$. Obviously the assertion $(1.7_0)$ is true. Suppose $(1.7_m)$ is valid for all $m \le n$. From (1.6) and integration by parts we can derive

$$\begin{aligned} 0 \le T^{n+1}b_0(t) &\le \int_0^t b_0(s)\left[\int_0^s \tau kd\tau\right]^n \frac{1}{n!}N(t,s)ds \\ &\le -b_0(t)\int_0^t s\frac{d}{ds}\left\{\left[\int_0^s \tau kd\tau\right]^n \frac{1}{n!}N(t,s)\right\}ds \\ &\le b_0(t)\int_0^t sk\left[\int_0^s \tau kd\tau\right]^n \frac{1}{n!}ds \\ &= b_0(t)\left[\int_0^t skds\right]^{n+1} \frac{1}{(n+1)!} \end{aligned}$$

The induction is completed. Thus we have proved

$$\int_0^t kds \le B_t \le \int_0^t kds \exp\left[\int_0^t skds\right] \tag{1.8}$$

From the hypothesis $(\mathrm{H}_1)$ $(\mathrm{H}_2)$ it follows that

$$0 < k \le C/(1+|t|)^{2+\delta} \tag{H$_1'$}$$

for some positive constant $C$ and hence

$$1 + \int_0^t ds \int_0^s kd\tau \le B \le 1 + C_1 t \tag{1.9}$$

for some positive constant $C_1$. By means of the Gauss equation we can derive

$$(B_t/B)_t = k - (B_t/B)^2$$

It turns out

$$|(B_t/B)| \le |\int_0^t k(x,s)ds|$$

and hence, $(B_t/B) = \partial_t \ln B$ is bounded since $(H_1')$ insures the right hand side of the last inequality bounded. A differentiation of the equation for $\partial_t \ln B$ with respect to $x$ provides the equation for $\partial_x\partial_t \ln B$. Integration of it using the hypotheses $(H_2)$ gives at once

$$|B\partial_x\partial_t \ln B| \le B^{-1}\int_0^t |k_x|B^2(x,s)ds \le C\int_0^t kB(x,s)ds$$

for some positive constant $C$. Since $B$ monotonely increases in $t > 0$ and the inequality: $|k_x| \le C|k|$, follows immediately from $(H_2)$. (1.9) ensures the last integral bounded above by a constant under control and hence (1.4) has been proved.

So far we have only used the hypotheses $(H_1)$ $(H_2)$. Combining (1.8) with $(H_3)$ one soon obtains the estimation of the lower bound for $B$

$$B_t(x,\infty) \ge \inf \int_0^\infty kds \tag{1.10}$$

and $B \ge C_2|t|$ if $|t| \ge T$ for some positive constants $C_2$ and $T$.

In what follows we shall study the asymptotic behavior of $B$ at infinity. From

$$B_t = B_t(x,\infty) - \int_t^\infty Bkds$$

and

$$B = 1 + B_t(x,\infty)t - t\int_t^\infty kBds - \int_0^t Bksds$$

we have

$$\frac{B_t}{B} = \frac{(1-(B_t(x,\infty))^{-1}\int_t^\infty Bkds}{t\left[1+(tB_t(x,\infty))^{-1}(1-t\int_t^\infty Bkds - \int_0^t Bksds)\right]}$$

From this expression, (1.3) follows immediately since

$$\int_t^\infty Bkds \quad \text{and} \quad t^{-1}\int_0^t Bksds$$

are controlled by $Ct^{-\delta}$ for some constant $C$ independent of $x$. The proof of Lemma 1.1 is completed. □

*Remark* 1.1. If ($H_1$) is fulfilled and $k$ is bounded, then the following inequality,

$$\left|t(\frac{B_t}{B} - \frac{1}{t})\right| \le C_3 \frac{1}{t^\delta \mu(x)} \quad \text{as } t^\delta \mu(x) \ge C_4$$

where $C_i (i = 3, 4)$ are some positive constants and $\mu(x)$ is an even function on $\mathbb{R}^1$ defined as follows:

$$\mu(x) = \text{ the minimum of } \int_0^\infty k(\xi, t)dt \text{ and } \int_{-\infty}^0 k(\xi, t)dt \text{ over all } |\xi| \le x,$$

is valid. This is the direct consequence of the argument in proving (1.3).

Define a function $h$ which satisfies

$$h_{tt} = k^* h \text{ with } h(0) = 1 \text{ and } h_t(0) = 0 \tag{1.11}$$

where

$$k^* = C(1 + |t|)^{-2-\delta/2} > k > 0 \tag{1.12}$$

where the constant $C$ is the same as mentioned in ($H_1'$). In a similar argument, since

$$\int_{-\infty}^\infty |s| k^* ds < \infty \text{ and } \int_0^\infty k^* ds \cdot \int_{-\infty}^0 k^* ds > 0,$$

we can obtain

$$\frac{h_t}{h} = \frac{1}{t} + O(\frac{1}{t^{1+\delta/2}}) \text{ and} \tag{1.13}$$

$$C_1|t| \le h \le (1 + C_2|t|) \text{ if } |t| \ge T$$

for some constants $C_1$, $C_2$ and $T$. Now we have

**Lemma 1.2.** *If ($H_1$)-($H_3$) are fulfilled, then*

$$B_t/B < h_t/h \text{ and } B < h, \quad \text{if } t > 0 \tag{1.14}$$

*Proof.* By (1.11) we have

$$(h_t/h)_t = k^* - (h_t/h)^2 \tag{1.15}$$

Subtracting the second equation of (1.2) from (1.15) provides, with $w = (h_t/h) - (B_t/B)$,

$$w_t = k^* - k - w[(h_t/h) + (B_t/B)]$$

with $w(0) = 0$. Integration of the last equation, noting $(k^* - k) > 0$ if $t \neq 0$, gives $tw > 0$ and hence the first part of (1.14) is proved. Integrating the first part of (1.14) again soon completes the proof of another part of (1.14). □

Now we proceed to deal with the geodesic polar coordinates.

**Lemma 1.3.** *Let* (0.1) *be fulfilled. Then we have*

$$1 \leq G_\rho \leq \exp \sup_\theta \int_0^\infty \rho k dt = \mu, \rho \leq G \leq \mu\rho, \tag{1.16}$$

$$\left|\frac{G_\rho}{G} - \frac{1}{\rho}\right| |\rho|^{\delta+1}$$

*uniformly bounded at infinity and*

$$\partial_\rho \ln G, \rho\partial_\rho\partial_\theta \ln G \text{ are bounded} \tag{1.17}$$

*Moreover if* $\widetilde{k}^* = C/(1+\rho)^{2+\delta/2} > k$, *and* $\widetilde{h}_{\rho\rho} = \widetilde{k}^*\widetilde{h}$ *with* $\widetilde{h} = 0$ *and* $\widetilde{h}_\rho = 1$ *at* $\rho = 0$, *then*

$$\rho\frac{G_\rho}{G} > 1, \quad \frac{G_\rho}{G} < \frac{\widetilde{h}_\rho}{\widetilde{h}} \quad \text{and } G < \widetilde{h} \text{ if } \rho > 0 \tag{1.18}$$

In proving (1.16) (1.17) there is no difficulty, so no detail need be repeated. [GW] also got the similar estimate (1.16) by another approach. The first part of (1.18) comes from an ordinary equation for $G/G_\rho$. Subtracting this equation from that for $h/h_\rho$ and using the comparison principle we can obtain the remaindering part of (1.18).

Next we shall study some behavior of the transformation from the geodesic coordinates $(x, t)$ to the geodesic polar coordinates $(\theta, \rho)$. Since $(M, g)$ has negative curvature, from the triangular inequality and the definition of both coordinate systems we have

$$t \text{ and } \frac{1}{2}|x| \leq \rho(x,t) \leq t + |x| \tag{1.19}$$

By (1.2) and (0.3) we have

$$\rho_t^2 + G^2\theta_t^2 = 1, \rho_x^2 + G^2\theta_x^2 = B^2 \tag{1.20}$$

$$\rho_t\rho_x + G^2\theta_t\theta_x = 0$$

On the other hand, by the geodesic equations of the $t$-curve we have $\rho_{tt} = GG_\rho\theta_t^2$. Solving $\theta_t$ from the first equation of (1.20) we can get

$$\rho_t = \text{tgh}\Phi \text{ where } \Phi = \int_0^t \frac{G_\rho}{G}d\tau \tag{1.21}$$

Therefore from (1.21) it is easy to get

$$\theta_t = \frac{m}{G\,\text{ch}\,\Phi},\ \rho_x = -\frac{mB}{\text{ch}\,\Phi} \text{ and } \theta_x = \frac{B}{G}\text{tgh}\,\Phi \tag{1.22}$$

where $m = 1$ or $-1$. From (1.19) and (1.18) we can see, if $t > 0$,

$$\Phi = \int_0^t \frac{G_\rho}{G}d\tau \geq \int_0^t \frac{1}{\rho}d\tau \geq \int_0^t \frac{1}{\tau + |x|}d\tau \geq \ln\frac{t + |x|}{|x|} \tag{1.23}$$

Hence

$$0 < 1 - \rho_t \leq \frac{1}{\text{ch}\Phi} \leq \frac{\frac{2|x|}{(t+|x|)}}{1 + \frac{|x|^2}{(t+|x|)^2}} \text{ if } t > 0 \tag{1.24}$$

## 2. The proof of Theorem C

From Theorem B it suffices to prove Theorem C for the surface $(\mathbb{R}^2, g)$ where the curvature $-k$ of $g$ satisfies ($H_1$)- ($H_3$). Later we often identify "0" and "1" with "$x$" and "$t$" respectively. Denote by $\overrightarrow{r}$ the position vector of the surface in $\mathbb{R}^3$ we require. As is well known, the coefficients of its second fundamental form, $L = \langle \overrightarrow{r}_{11}, \overrightarrow{n} \rangle$, $M = \langle \overrightarrow{r}_{12}, \overrightarrow{n} \rangle$ and $N = \langle \overrightarrow{r}_{22}, \overrightarrow{n} \rangle$ where $\overrightarrow{n}$ denotes the unit normal vector to $\overrightarrow{r}$, satisfy the Gauss-Codazzi system

$$L_2 - M_1 = L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2 \tag{2.1}$$

and

$$M_2 - N_1 = L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{21}^1) - N\Gamma_{21}^2 \tag{2.2}$$

with

(2.3) $LN - M^2 = -kg$ where $g$ is the determinant of the metric tensor.

Indeed, if $k > 0$ the Gauss-Codazzi system can be reduced to a hyperbolic system. Introducing Riemann invariants

$$r = (-M - \sqrt{kg})/L \quad s = (-M + \sqrt{kg})/L \tag{2.4}$$

through a computation we find

$$(2.5) r_2 + sr_1 = \frac{1}{2}(r-s)(Q_2 + rQ_1) + A + Br + Cs + Drs + Er^2 + Fr^2 s$$

and

$$(2.6) s_2 + rs_1 = \frac{1}{2}(s-r)(Q_2 + sQ_1) + A + Bs + Cr + Drs + Es^2 + Fs^2 r$$

where

$$(2.7) \qquad Q = \ln \sqrt{kg}$$

and $A, B, C, D, E$ and $F$ are only involved in christoffel symbols of the metric. It is not difficult to see that the system (2.5) (2.6) is equivalent to (2.1) (2.2) with (2.3) if $s > r$. This is a linearly degenerate hyperbolic system. There have been many papers to devoted to global smooth solutions with small initial data of Cauchy problems for linearly degenerate, strictly hyperbolic system. But these methods are not applicable to the present case since the system (2.5) (2.6) is not strictly hyperbolic in any neighbourhood of $r = 0$ and $s = 0$ although it is linearly degenerate. This is the main difficulty we ran across. To look clearly at the affects of $g$ and $k$, following [PO] we reintroduce new unknown variables

$$r \longmapsto Br \text{ and } s \longmapsto Bs$$

still denote them by $r$ and $s$ and rewrite (2.5) (2.6) in the form

(2.8)

$$\begin{aligned} L_1(r,s) &= r_t + \frac{s}{B}r_x + s(1+r^2)\partial_t(\ln B) - \frac{r-s}{4}(\partial_t + \frac{r}{B}\partial_x)(\ln k) \\ &= r_t + \frac{s}{B}r_x - f = 0 \end{aligned}$$

(2.9)

$$\begin{aligned} L_2(r,s) &= s_t + \frac{r}{B}s_x + r(1+s^2)\partial_t(\ln B) - \frac{s-r}{4}(\partial_t + \frac{s}{B}\partial_x)(\ln k) \\ &= s_t + \frac{r}{B}s_x - q = 0 \end{aligned}$$

We shall solve a Cauchy problem of (2.8) (2.9) with initial data $r(x,0)$ and $s(x,0)$ only in the upper plane $t > 0$ since a similar argument will give the solution in the lower plane.

Let $(u(x,t), v(x,t)) \in C^1(\mathbb{R}^1 \times [0,T])$ be the solution to the following system.

$$\frac{\partial u}{\partial t} + \lambda_1 \frac{\partial u}{\partial x} = a_{11}u + a_{12}v + R_1 \tag{2.10}$$

$$\frac{\partial v}{\partial t} + \lambda_2 \frac{\partial v}{\partial x} = a_{21}u + a_{22}v + R_2 \tag{2.11}$$

with

$$u(x,0) = \phi_1(x) \text{ and } v(x,0) = \phi_2(x) \tag{2.12}$$

$$\lambda_1 \le \lambda_2 \tag{2.13}$$

Here $\lambda_i$, $a_{ij}$ and $R_i (i,j = 1,2)$ are $C^1$ functions of $x, t$. With each $\lambda_i(x,t)$ we can associate a characteristic curve $X = \Gamma_i(\tau; t, x)$, i.e.,

$$\frac{dX}{d\tau} = \lambda_i(\tau, X), \tau < t, \text{ with } X(t) = x \tag{2.14}$$

where $X = \Gamma_i(\tau; t, x)$. Obviously, we have

$$\Gamma_2(\tau; t, x) \le \Gamma_1(\tau; t, x), \quad 0 \le \tau \le t \le T$$

For each $A(x^*, t^*)$, $t^* \in [0,T]$ we can assign $A$ to a characteristic triangle $\Delta_A$

$$\Delta_A = \{(x,t) | \Gamma_2(t; t^*, x^*) \le x \le \Gamma_1(t; t^*, x^*), 0 \le t \le t^*\} \tag{2.15}$$

Let us denote by $I(s) = [\Gamma_2(s; t^*, x^*), \Gamma_1(s; t^*, x^*)]$ and by

$$H = \text{the maximum}(|a_{ij}|, |\partial_x a_{ij}|, |\partial_x \lambda_i|) \text{ over } \Delta_A \text{ and } i = 1,2.$$

In the sequel, without otherwise stated, by "max" or "min" we always mean that the maximum or the minimum is taken over all $i = 1,2$.

**Lemma 2.1.** *Let $(u,v)$ be the $C^1$ solution in $\Delta_A(x^*, t^*)$ of (2.10) (2.11) where $R_i$ vanish, with (2.12). Then for $(x,t) \in \Delta_A$*

$$|u|, |v|, |u_x|, |v_x| \le \max_{I(0)}(|\phi_i| + |\partial_x \phi_i|) \exp 5Ht \tag{2.16}$$

*Proof.* Differentiation of (2.10) and (2.11) coupling with the original system gives the system satisfied by $u, v, u_x$ and $v_x$. Integrating along the characteristic curves and setting $m(s) = \max(|u(x,s)|, |v(x,s)|, |\partial_x u(x,s)|, |\partial_x v(x,s)|)$ where $x$ runs over the interval $I(s)$, we have,

$$m(t) \le \max_{I(0)}(|\phi_i| + |\partial_i \phi|) + 5H \int_0^t m(s) ds.$$

An application of the Gronwall inequality yields (2.16) soon. □

**Lemma 2.2.** *Let* $(u, v)$ *be a* $C^1$ *solution to the system* (2.10) (2.11). *If given* $A(x^*, t^*)$ *with* $t^* \in [0, T]$, *we have*

$$a_{12}(x,t) \text{ and } a_{21}(x,t) \geq 0 \text{ on } \Delta_A \tag{2.17}$$

*and*

$$\phi_1(\xi), \phi_2(\xi) \geq 0 (\leq 0) \text{ for all } \xi \in [\Gamma_2(0; t^*, x^*), \Gamma_1(0; t^*, x^*)]$$

$$\text{and } R_i(x,t) > 0 (< 0) \text{ on } \Delta_A \tag{2.18}$$

*then* $u(x,t)$, $v(x,t) > 0 (< 0)$ *for all* $(x,t) \in \Delta_A \cap \{t > 0\}$.

*Proof.* It suffices to study the positive case of $R_i$ since a transformation $u \mapsto -u$, $v \mapsto -v$, reduces the negative case to the positive case. First of all we claim that there is a positive $t_1$ such that $u(x,t)$, $v(x,t) > 0$ in $\Delta_A \cap \{0 < t \leq t_1\}$. Evidently it suffices to locally prove this assertion for each point $(x, 0)$. By continuity it needs only to investigate the case: $u(x_0, 0) = 0$ or $v(x_0, 0) = 0$. Without loss of generality, we assume $u(x_0, 0) = 0$. Let us look at the derivative $\partial u/\partial t + \lambda_1 \partial u/\partial x$ which is strictly positive at this point since $a_{12}v$ is nonnegative and $R_1$ is strictly positive there. Now by continuity we can derive the local assertion.

Next we shall complete the proof by contradiction. If the assertion in the present lemma was false, one can find $t_0 \in [0, t^*]$ such that $u(x_0, t_0)$ or $v(x_0, t_0) = 0$ for some $x_0 \in [\Gamma_2(t_0; t^*, x^*), \Gamma_1(t_0; t^*, x^*)]$ and $u(x,t), v(x,t) > 0$ in the region: $\Delta_{A(t^*,x^*)} \cap \{t < t_0\}$. Without loss of generality we may assume $u(x_0, t_0) = 0$. Integrating the first equation (2.10) and noting (2.18) we have

$$\begin{aligned} & u(x_0, t_0) \exp\left[-\int_0^{t_0} a_{11} d\tau\right] \\ \geq & \int_0^{t_0} (a_{12} v) \exp\left[-\int_0^{\tau} a_{11} d\tau\right] d\tau + \int_0^{t_0} R_i \exp\left[-\int_0^{\tau} a_{11} d\tau\right] d\tau > 0 \end{aligned}$$

where the variable $x$ in the above integrends is replaced by $\Gamma_1(\tau; t_0, x_0)$ since $R_i > 0$, $a_{12} \geq 0$ and $v > 0$ in the region considered. This contradicts the definition of $t_0$ and hence, completes the proof of the present lemma. □

Similarly we have the following remark.

*Remark* 2.1. Instead of $\Delta_A$, we restrict ourself to a curved triangle $\Delta_A$ which is enclosed by two characteristics emitting from $(x^*, t^*)$ and a space-like initial curve with respect to $t$. Let $(u, v)$ be a $C^1$ solution in $\Delta_A$ to the system (2.10) (2.11) and let (2.17) and (2.18) be fulfilled. Then the conclusion in the present lemma is continuous to be true.

As we know, the local smooth solution to (2.8) (2.9) exists if the initial data are smooth and bounded. Now we proceed to extend this local solution to the whole upper plane.

**The $C^0$ estimation.** As an initial step we first find a supersolution to (2.8) (2.9). It is evident that for any positive (or negative) constant $\psi_0(\psi_0, \psi_0)$ is a supersolution (or subsolution) to (2.8) (2.9). Indeed,

$$L_2(\psi_0, \psi_0) = L_1(\psi_0, \psi_0) = \tag{2.19}$$

$$\psi_0(1+\psi_0^2)\frac{B_t}{B} > 0(\text{or } < 0) \text{ if } t \geq 0 \text{ and } \psi_0 > 0(\text{or } < 0).$$

Another sub-supersolution to (2.8) (2.9) is also needed for the later estimates. Consider an initial value problem for an ordinary equation

$$\phi_t = \phi(1+\phi^2)\partial_t \ln h + \frac{\phi}{2}\partial_t \ln k^*, t > T \quad \text{with } \phi(T) = \psi_0 \tag{2.20}$$

where $h$ and $k^*$ are defined in (1.12) and (1.11). It is easy to find its solution

$$\phi = \frac{h\sqrt{k^*}b}{(1 - 2b^2\int_T^t hh_t k^* dt)^{1/2}} \quad \text{with } b^2 = \frac{\psi_0^2}{h^2(T)k^*(T)} \tag{2.21}$$

(1.13) tells us $\phi$ smooth in $[T, \infty)$ if $\psi_0$ is small enough, namely, one can find a small $\psi^*$ and a big $T$ such that

$$\phi(t) \in C^\infty([T, \infty)) \text{ if } 0 < \psi_0 \leq \psi^* \text{ and moreover,}$$

$$\phi(t) \text{ is monotonely decreasing} \tag{2.22}$$

since $(1+\phi^2)\partial_t \ln h + 2^{-1}\partial_t \ln k^* = -(\delta - 4\phi^2)/4t + O(1/t^{1+\delta/2})$.

Now we illustrate that $(r, s) = (-\phi, \phi)$ is the sub-supersolution to (2.8) (2.9), i.e.,

$$L_1(-\phi, \phi) < 0 \text{ and } L_2(-\phi, \phi) > 0 \text{ if } t \geq T \tag{2.23}$$

Indeed,

$$\begin{aligned} L_1(-\phi,\phi) &= -\phi_t + \phi(1+\phi^2)\partial_t \ln B + \frac{\phi}{2}(\partial_t \ln k - \frac{\phi}{B}\partial_x \ln k) \\ &= \phi(1+\phi^2)(\partial_t \ln B - \partial_t \ln h) + \frac{\phi}{2}(\partial_t \ln k - \partial_t \ln k^* - \frac{\phi}{B}\partial_x \ln k) \end{aligned}$$

From Lemma 1.2 it follows that the first term of the right-hand side of the last expression is negative. And the second term

$$= \frac{\phi}{2}\left[\partial_t \ln(kt^{2+\delta}) - \frac{\delta}{2t} + O(\frac{1}{t^2}) - \frac{\phi}{B}\partial_x \ln k\right]$$

and the hypotheses ($\mathrm{H}_1$) implies its first term nonpositive and the second part of (1.9) and the boundness of $\partial_x \ln k$ guarantee the remaindering part negative if $\psi^*$ is small enough. Hence we complete the first part of (2.23). Similarly, we can show the another part.

In the sequel, unless otherwise stated, we always assume that $T$ and $\psi^*$ are respectively so big and so small that (2.22) (2.23) hold for all $0 < \psi_0 \le \psi^*$, and moreover,

($\mathrm{H}_1''$)

$$-(\frac{1}{2} + \xi^2)\partial_t \ln B - \frac{1}{4}(\partial_t \ln k + \frac{\xi}{B}\partial_x \ln k) > 0, \quad \text{if } t \ge T \text{ for all } |\xi| \le \psi^*$$

**Lemma 2.3.** *Suppose that a $C^1$ solution $(r,s)$ to the system (2.8) (2.9) in $\Delta_A = \{(t,x) | \Gamma_2(t;t^*,x^*) \le x \le \Gamma_1(t;t^*,x^*), T \le t \le t^*\}$, satisfies*

$$(2.24) \qquad -\psi_0 \le r(x,T) < s(x,T) \le \psi_0 \text{ if } \psi_0 \le \psi^*$$

*and $x$ in $[\Gamma_2(T),\Gamma_1(T)]$ where $\Gamma_i(t) = \Gamma_i(t;t^*,x^*)$, $i = 1,2$, are the characteristic curves corresponding to $r/B$ and $s/B$ respectively, and passing through the point $A(t^*,x^*)$. Then in the region $\Delta_A$, we have,*

$$(2.25) \qquad -\psi_0 \le r(x,t) < s(x,t) \le \psi_0, \text{ if } t > T$$

*if $s > r$ in $\Delta_A$.*

*Proof.* Let $\tilde{r}$ and $\tilde{s}$ be two $C^1$ functions in the region considered. The difference $(r - \tilde{r}, s - \tilde{s})$ satisfies the following system, with $\partial_\alpha = \partial_t + (r/B)\partial_x$ and

$\partial_\beta = \partial_t + (s/B)\partial_x,$

(2.26)

$$\partial_\beta(r-\widetilde{r}) + \frac{s-\widetilde{s}}{B}\widetilde{r}_x = b(r-\widetilde{r}) + f_s(\widetilde{r},x,t)(s-\widetilde{s}) + +L_1(r,s) - L_1(\widetilde{r},\widetilde{s})$$

(2.27)

$$\partial_\alpha(s-\widetilde{s}) + \frac{r-\widetilde{r}}{B}\widetilde{s}_x = q_r(\widetilde{s},x,t)(r-\widetilde{r}) + b'(s-\widetilde{s}) + L_2(r,s) - L_2(\widetilde{r},\widetilde{s})$$

where

$$f_s(u,x,t) = q_r(u,x,t) = -(1+u^2)\frac{B_t}{B} - \frac{1}{4}(\partial_t + \frac{u}{B}\partial_x)\ln k$$

Here $b$ and $b'$ are continuous functions in $\Delta_A$. First of all we show that there is a constant $T_1 > T$ such that

$$-\psi_0 < r(x,t) < s(x,t) < \psi_0 \text{ on } \Delta_A \quad \{T < t \le T_1\} \tag{2.28}$$

If $x$ in $[\Gamma_2(T),\Gamma_1(T)]$ and $|r(x,t)|$, $|s(x,t)| < \psi_0$ then there is a neighbourhood of $(x,T)$ where (2.28) is valid. Hence one needs only to discuss the following case: $r(x,T) = -\psi_0$ or $s(x,T) = \psi_0$. If at this point $s(x,T) = \psi_0$ and $q_r(\psi_0,x,T) \ge 0$, then we replace $(\widetilde{r},\widetilde{s})$ in (2.26) (2.27) by $(\psi_0,\psi_0)$. Observe that $L_i(r,s) - L_i(\psi_0,\psi_0) = -L_i(\psi_0,\psi_0) < 0$ as $t > T$ and that $-\psi_0 \le r(\xi,T) < s(\xi,T) \le \psi_0$ for all $\xi$ under consideration. From (2.27) we derive $\partial_\alpha(s-\psi_0) < 0$ at the point in question and hence there is a neighbourhood of $(x,T)$ where $s(y,t) < \psi_0$ as $t > T$. If at the point in question, $q_r(\psi_0,x,T) < 0$, replacing $(\widetilde{r},\widetilde{s})$ by $(-\phi,\phi)$ and from (2.27) we can also derive $\partial_\alpha(s-\phi)$ strictly negative there and hence, the remainder proof is the same.

Next we shall complete the proof of (2.25) by contradiction. Suppose that there is a point $(x_1,t_1)$ such that $r(x_1,t_1) = -\psi_0$ or $s(x_1,t_1) = \psi_0$, (2.28) valid on $\Delta_A \cap \{T < t \le t_1\}$ and in any neighbourhood of $(x_1,t_1)$ there exists a point $(x,t)$ where $\max(|r(x,t)|,|s(x,t)|) > \psi_0$. Now repeated the same argument at $t = t_1$ as done at $t = T$ will soon lead the conclusion: $\psi_0 > s(x,t) > r(x,t) > -\psi_0$ near $t = t_1$ and $t > t_1$. This is the contradiction and hence, ends the proof of Lemma 2.3. □

The estimates of the lower bound for $s - r$ and the above bounds for $|\partial^\alpha r||\partial^\alpha s|, |\alpha| = 1$. This is the crucial point to the proof of the existence about global smooth solutions. If we can show $s - r$ strictly positive or the boundness of all first derivatives of $r$ and $s$ in any given set $\Omega_t = \mathbb{R}^1 \times [0,t]$, then its solution can be infinitely extended to $\Omega_\infty$. Subtracting (2.8) from (2.9) gives

$$\partial_\beta(s-r) = \frac{s-r}{B}s_x + Q(s-r) \tag{2.29}$$

and

$$\partial_\alpha(s-r) = \frac{s-r}{B}r_x + Q(s-r) \tag{2.30}$$

where

$$Q = (q-f)/(s-r) = (1-rs)\frac{B_t}{B} + \frac{1}{4}(\partial_\alpha \ln k + \partial_\beta \ln k) \tag{2.31}$$

Differentiating the system (2.8) (2.9) with respect to $x$ and solving $s_x$ and $r_x$ from (2.29) (2.30), after combining them we have, with $\widetilde{s} = (s-r)s_x/B$ and $\widetilde{r} = (s-r)r_x/B$,

$$\widetilde{L}_1(\widetilde{r},\widetilde{s}) = \partial_\beta\widetilde{r} - (Q + f_r - \frac{B_t}{B})\widetilde{r} - f_s\widetilde{s} - \frac{s-r}{B}\delta_x f = 0 \tag{2.32}$$

and

$$\widetilde{L}_2(\widetilde{r},\widetilde{s}) = \partial_\alpha\widetilde{s} - q_r\widetilde{r} - (Q + q_r - \frac{B_t}{B})\widetilde{s} - \frac{s-r}{B}\delta_x q = 0 \tag{2.33}$$

where $\delta_x$ denotes the differentiation only with respect to $x$.

**Lemma 2.4.** *Let the assumption in Lemma 2.3 be fulfilled. Then there exist three constants $\mu_*$, smaller $\psi^*$ and bigger $T$, and two positive monotonely-increasing functions $\theta_i(t)$ defined in $[T,t^*)i = 1,2$ such that in the region $\Delta_A$*

$$\theta_1(t)(s-r) \geq \inf_{x\in[\Gamma_2(T),\Gamma_1(T)]}(s(x,T) - r(x,T))$$

*and*

$$|s_x|, |r_x|, |s_t|, |r_t| \leq \theta_2(t) \quad \text{in } \Delta_A \tag{2.34}$$

*if for all $x$ in $[\Gamma_2(T),\Gamma_1(T)]$, $-\mu_* \leq r_x(x,T)$, $s_x(x,T)$, and $-\psi_0 \leq r(x,T) < s(x,T) \leq \psi_0 \leq \psi^*$. Here $\theta_i(t)$ depends on the minimum over $[\Gamma_2(T),\Gamma_1(T)]$ of $s-r$ as well as the $C^1$ bounds of $r$ and $s$ over $[\Gamma_2(T),\Gamma_1(T)]$.*

*Proof.* With $w_1 = (s-r)(r_x + \frac{1}{2}B_t)\frac{1}{B}$ and $w_2 = (s-r)(s_x + \frac{1}{2}B_t)\frac{1}{B}$ combining (2.32) (2.33) with (2.29) (2.30) we have

$$\partial_\beta w_1 = (Q + f_r - \frac{B_t}{B})w_1 + (f_s + \frac{B_t}{2B})w_2 + R_1 \tag{2.35}$$

$$\partial_\beta w_2 = (q_r + \frac{B_t}{2B})w_1 + (Q + q_s - \frac{B_t}{B})w_2 + R_2 \tag{2.36}$$

where

$$R_1 = \frac{s-r}{B}\left\{\frac{1}{2}[\frac{B_t^2}{B}(\frac{1}{2} + 2rs + r^2) + Bk + s\partial_t\partial_x \ln B + \frac{s-r}{4}\frac{B_t}{B}\partial_x \ln k] + \delta_x f\right\}$$

and

$$R_2 = \frac{s-r}{B}\left\{\frac{1}{2}[\frac{B_t^2}{B}(\frac{1}{2} + 2rs + s^2) + Bk + r\partial_t\partial_x \ln B - \frac{s-r}{4}\frac{B_t}{B}\partial_x \ln k] + \delta_x q\right\}$$

Now using Lemma 2.3, ($H_2$) and (1.3) we can find another smaller $\psi^*$ and a bigger $T$ such that if $|r(x,T)|$ and $|s(x,T)|$ are less than or equal to $\psi_0 \leq \psi^*$ we have $R_i > 0 (i = 1,2)$ and in view of ($H_1''$), $(f_s + B_t/2B) > 0$, $(q_r + B_t/2B) > 0$ also in the region considered. Set $\mu_* =$ the minimum of $B_t(x,T)/2$ over all $x$ in $\mathbb{R}^1$. The argument in Lemma 1.1 tells us $\mu_*$ positive for sufficiency large $T$. Then if $s_x(x,T)$, $r_x(x,T) \geq -\mu_*$ it follows that $w_1(x,T) \geq 0$, $w_2(x,T) \geq 0$. An application of Lemma 2.2 to (2.35) (2.36) provides $w_i \geq 0$ $i = 1,2$, namely, $(s-r)(r_x + B_t/2)/B \geq 0$ and $(s-r)(s_x + B_t/2)/B \geq 0$ in the region considered. Since $s - r > 0$, this means

$$r_x, s_x \geq -B_t/2, \quad \text{on } \Delta_A \tag{2.37}$$

Instead of $(r,s)$ directly dealing with $(-r,-s)$, in a similar way we can get another sided estimation provided that for all $|t| \geq T$, $\partial_t \ln(kt^{4+\delta}) \leq 0$. In order to weaken this restriction we turn to estimate the lower bound of $s - r$ in $\Delta_A$ and then, in view of the fact that the system (2.8) (2.9) is linearly degenerate, the $C^1$ bound for $(r,s)$ can be controlled by the lower bound on $\Delta_A$ of $s - r$, the $C^0$ bounds on $\Delta_A$ of $r, s$ and the $C^1$ bounds of $s$ and $r$ on $[\Gamma_2(T), \Gamma_1(T)]$. More precisely, solving $s - r$ from (2.29) or (2.30) and using (2.37) we have

$$\begin{aligned} s - r &\geq (\text{the minimum of } s - r \text{ over } [\Gamma_2(T), \Gamma_1(T)]) \exp[-(t-T)C_1(t)] \\ &= (\text{the minimum of } s - r \text{ over } [\Gamma_2(T), \Gamma_1(T)])/\theta_1(t) \end{aligned}$$

for some positive monotonely-increasing function $C_1(t)$ defined on $[T, t^*)$ and under control.

Now we are in a position to get the $C^1$ bounds. From (2.8) (2.9) it follows that

$$\begin{aligned} \partial_\beta \partial_\alpha r &= \partial_\alpha \partial_\beta r + [\partial_\beta, \partial_\alpha] r \\ &= \delta_\alpha f + f_r \partial_\alpha r + f_s q - (Q + \partial_t \ln B)(s-r) r_x / B \\ &= [f_r + (Q + \partial_t \ln B)] \partial_\alpha r + [(\delta_\alpha f + f_s q - (Q + \partial_t \ln B) f] \end{aligned} \tag{2.38}$$

In getting the last equality we have used (2.8) (2.9). The second term in the right hand side of the last equation is controlled. So integration of it yields $|\partial_\alpha r| \leq \theta_2'(t)$ on $\Delta_A$ for some positive monotonely-increasing function $\theta_2'(t)$ in $[T, t^*)$ under control. From $r_x = (\partial_\beta r - \partial_\alpha r)B/(s-r) = (f - \partial_\alpha r)B/(s-r)$, and $r_t = f - s r_x / B$, it is easy to find a positive monotonely increasing function $\theta_2(t)$ under control such that (2.34) holds. This completes the proof of lemma 2.4. □

**The end of the proof of Theorem C.** First of all, choose a sufficient large $T$ such that Lemma 2.3 and 2.4 hold. Let us consider the Cauchy problem for (2.8) (2.9) with the initial data $r(x,0) = -\varepsilon$ and $s(x,0) = \varepsilon$. Set

$$H = 1 + \sup_{|t|<T} \{ |\partial_x^i \partial_t \ln B|, |\partial_x^i \partial_t \ln k| \text{ and } |\partial_x^i \partial_x \ln k|, i = 0, 1 \}$$

Choose $\varepsilon$ so small that

$$\varepsilon \exp 7HT \leq \min(\psi^*, \mu_*)$$

Lemma 2.1 guarantees $|\partial^\alpha r(x,t)|$ and $|\partial^\alpha s(x,t)| \leq \min(\psi^*, \mu_*)$ for all $x$ in $\mathbb{R}^1$, $0 \leq t \leq T$ and $|\alpha| \leq 1$. And directly integrating (2.29) gives $s(x,T) - r(x,T) \geq 2\varepsilon / C(T)$ for a positive constant $C(T)$. Solve again the Cauchy problem for (2.8) (2.9) with the initial data $r(x,T)$ and $s(x,T)$ in the region $t > T$. Lemma 2.3 and 2.4 tells us that $|r(x,t)|$, $|s(x,t)| \leq \psi^*$, $|r_x(x,t)|$, $|s_x(x,t)|$, $|r_t(x,t)|$ and $|s_t(x,t)| \leq \theta_2(t)$ and $s - r \geq 2\varepsilon / \theta_1(t) C(t)$ in the region where $r$ and $s$ belong to $C^1$. So we can infinitely extend the solution $(r, s)$ to the whole upper plane. And the existence of global smooth solution to the Cauchy problem of (2.8) (2.9) with the initial data $r(x,0) = -\varepsilon$, $s(x,0) = \varepsilon$ has been proved. Moreover, $s - r > 0$ everywhere. Solving (2.3) (2.4) we can find the smooth coefficients

$L$, $M$ and $N$ of the second fundamental form of the required surface. The fundamental theorem on Differential Geometry enables us to find this surface with the prescribed metric (0.3). This completes the proof of Theorem C.

## 3. The proof of Theorem A

To find a smooth isometric immersion in $\mathbb{R}^3$ of the manifold mentioned in Theorem A, we have to prove the existence of the global smooth solutions to system (2.8) (2.9) under the hypothesis (0.1). In doing so we ran across two difficulties. First, if we directly solve the Gauss-Codazzi system under the geodesic polar coordinates, then its Riemann invariants are singular at the center of the exponential map $\exp_p$. Second, if we apply the result obtained in Theorem C to the present case, one can see that the hypotheses $(H_1)$-$(H_3)$ cease to be fulfilled. In order to bypass these two difficulties, a natural way is to split $R^2_+$ into two parts and discuss each separately. One contains a neighbourhood of the centre and denoted by $\Omega_1$, another is its complement in $R^2_+$ and denoted by $\Omega_2$. We shall solve the Gauss- Codazzi system in $\Omega_1$ and $\Omega_2$, respectively by using the geodesic coordinates and the geodesic polar coordinates.

First of all we consider the system for Riemann invariants of the Gauss-Codazzi equation under the geodesic polar coordinates and in the region $\rho > R$ for some sufficiently large $R$

$$\begin{aligned}L_1(u,v) &= u_\rho + \frac{v}{G}u_\theta + v(1+u^2)\partial_\rho(\ln G)\\ &\quad - \frac{u-v}{4}(\partial_\rho + \frac{u}{G}\partial_\theta)(\ln k) = u_\rho + \frac{v}{G}u_\theta - \widetilde{f} = 0\end{aligned} \tag{3.1}$$

$$\begin{aligned}L_2(u,v) &= v_\rho + \frac{u}{G}v_\theta + u(1+v^2)\partial_\rho(\ln G)\\ &\quad - \frac{v-u}{4}(\partial_\rho + \frac{v}{G}\partial_\theta)(\ln k) = v_\rho + \frac{u}{G}v_\theta - \widetilde{q} = 0\end{aligned} \tag{3.2}$$

Evidently, $(u,v) = (\psi_0, \psi_0)$ is the supersolution (or subsolution) of (3.1) (3.2) if $\psi_0 > 0$ (or $< 0$). In analogy with the argument in Section 2 we can find its sub-supersolution. Indeed, by Lemma 1.3 we can assume that there exist two constants $R$ and $\psi^*$ such that

(H$_1''$)

$$-(\frac{1}{2}+\xi^2)\partial_\rho \ln G - \frac{1}{4}(\partial_\rho \ln k + \frac{\xi}{G}\partial_\theta \ln k) > 0, \quad \text{if } \rho \geq R \text{ for all } |\xi| \leq \psi^*$$

Moreover the following Cauchy problem

$$\phi_\rho = \phi(1+\phi^2)\partial_\rho \ln \widetilde{h} + \frac{\phi}{2}\partial_\rho \ln \widetilde{k}^*, \rho > R \text{ with } \phi(R) = \phi_0 \tag{3.3}$$

where $\widetilde{h}$ and $\widetilde{k}^*$ are defined as in Lemma 1.3, has a global smooth solution $\phi(\rho)$

$$\phi = \frac{\widetilde{h}\sqrt{\widetilde{k}^*}b}{(1-2b^2\int_R^\rho \widetilde{h}\widetilde{h}_\rho \widetilde{k}^* dt)^{1/2}} \text{ with } b^2 = \frac{\psi_0^2}{\widetilde{h}^2(R)\widetilde{k}^*(R)}$$

which is monotonely decreasing in $[R,\infty)$ if $|\psi_0| \leq \psi^*$ and $(u,v) = (-\phi,\phi)$ is the sub-supersolution of (3.1) (3.2), that is,

$$L_1(-\phi,\phi) < 0, L_2(-\phi,\phi) > 0, \rho \geq R$$

In view of the Remark of Lemma 2.2, from line to line translating the argument as in proving Lemma 2.3, 2.4 and Theorem C, without difficulty, we have the existence of global smooth solutions to any generalized Cauchy problem in the region contained in $\{\rho \geq R\}$ for (3.1) (3.2). Precisely speaking, suppose that $\widetilde{\Omega}$ is a unbounded domain with a smooth boundary curve $\widetilde{S}$ denoted by

$$\widetilde{S} : \theta = \theta(\xi) \text{ and } \rho = \rho(\xi) \geq R \tag{3.4}$$

for some smooth function $\theta(\xi)$ and $\rho(\xi)$ with $\dot{\theta}^2 + \dot{\rho}^2 \neq 0$. And $\widetilde{\Omega}$ is always located the left hand-side of its boundary when $\xi$ runs along the increasing direction. Given the initial data

$$u = u_0(\xi) \text{ and } v = v_0(\xi), \text{ on } \widetilde{S} \tag{3.5}$$

such that

$$-\dot{\rho}(\xi)u_0(\xi) + \dot{\theta}(\xi)G > 0 \text{ and } \dot{\rho}(\xi)v_0(\xi) + \dot{\theta}(\xi)G > 0, \text{ on } \widetilde{S} \tag{3.6}$$

This means $\widetilde{S}$ space-like with respect to $\rho$. Now we have

**Lemma 3.1.** *There are two constants $R$ and $\psi^*$ such that the problem* (3.1) (3.2) *with* (3.5) *admits a global smooth solution $(u, v)$ in the closure of $\widetilde{\Omega}$ where $v - u > 0$ provided that*

$$-\psi^* \le u_0(\xi) < v_0(\xi) \le \psi^* \text{ on } \widetilde{S} \tag{3.7}$$

*and*

$$\frac{\dot{u}_0(\xi) - \dot{\rho}(\xi)\widetilde{f}}{\dot{\theta}(\xi)\dot{\rho}(\xi)v_0(\xi)G^{-1}} \text{ and } \frac{\dot{v}_0(\xi) - \dot{\rho}(\xi)\widetilde{g}}{\dot{\theta}(\xi) - \dot{\rho}(\xi)u_0(\xi)G^{-1}} \ge -\frac{1}{2}\inf_{\widetilde{S}} G_\rho = -\mu_* \tag{3.8}$$

(3.8) looks somewhat complicated. Indeed they are equal to $\partial_\theta u$ and $\partial_\theta v$ on $\widetilde{S}$. As our experience in Section 2, we can find smooth solution on $\Omega_1$ of (2.8) (2.9) as small as possible, as long as the initial data are sufficiently small. To connect this solution with Lemma 3.1 we have to evaluate the bounds on $\widetilde{S} = \partial F(\Omega_2)$ of $(u, v)$ and $(\partial_\theta u, \partial_\theta v)$ under the transformation $F$ from $(x, t)$ to $(\theta, \rho)$. From (1.20) we have

$$\begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \theta_x & \rho_x \\ \theta_t & \rho_t \end{bmatrix} \begin{bmatrix} \widetilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{bmatrix} \begin{bmatrix} \theta_x & \theta_t \\ \rho_x & \rho_t \end{bmatrix} \tag{3.9}$$

and

$$\begin{aligned} (\rho_t + \frac{r}{B}\rho_x)\frac{u}{G} &= (\theta_t + \frac{r}{B}\theta_x), \\ (\rho_t + \frac{s}{B}\rho_x)\frac{v}{G} &= (\theta_t + \frac{s}{B}\theta_x) \end{aligned} \tag{3.10}$$

Here

$$\det \begin{bmatrix} \theta_x & \theta_t \\ \rho_x & \rho_t \end{bmatrix} = \frac{B}{G}$$

Therefore if

$$(\rho_t + \frac{r}{B}\rho_x) > 0 \text{ and } (\rho_t + \frac{s}{B}\rho_x) > 0 \tag{3.11}$$

then the transformation in (3.10) make sense.

Next we illustrate the structure of the domains $\Omega_1$ and $\Omega_2$. Assume that

$$\Omega_1 = \{(x, t) \in R_+^2 | 0 < t < t_0(x) = R\sqrt{1 + x^2}/\mu_*^{1/\delta}(x)\}$$

where $R$ is a constant to be determined and $\mu_*(x)$ is smooth, in $\mathbb{R}^1$ and monotonely-decreasing in $R_+^1$. Moreover,

$$0 < \mu_*(x) \le \min(1, \mu(x)) \tag{3.12}$$

with $\mu(x)$ defined in Remark of Lemma 1.1. Such a function $\mu_*(x)$ can be constructed by mollifier without difficulty.

Let us now fix the constant $R$. In the sequel, we always assume $R$ so big that Lemma 3.1 is valid. Besides, by (1.22) (1.24) we may suppose

$$\rho_t + \frac{r}{B}\rho_x = (\text{tgh}\Phi - \frac{mr}{\text{ch}\Phi}) \geq 1 - \frac{4}{R} \geq \frac{1}{2}$$

and

$$(\rho_t + \frac{s}{B}\rho_x) \geq \frac{1}{2} \text{ if } (x,t) \text{ in } \bar{\Omega}_2 \text{ and } |r|, |s| \leq 1$$

Thus under the present circumstance, (3.11) is valid and the transformation (3.10) makes sense.

A vector field $V$ in $T_*(\Omega_2)$ is said to be normalized vector if $V$ is of the form $V = \xi(x,t)\partial_x + \partial_t$. The differential mapping $F_*$ transforms $V \to F_*(V) = (\theta_t + \xi\theta_x)\partial_\theta + (\rho_t + \xi\rho_x)\partial_\rho$. If $\rho_t + \xi\rho_x \neq 0$ in the region considered, it follows that $F_*(V) = (\rho_t + \xi\rho_x)(\widetilde{\xi}\partial_\theta + \partial_\rho) = (\rho_t + \xi\rho_x)\widetilde{V}$ with $\widetilde{\xi} = (\theta_t + \xi\theta_x)/(\rho_t + \xi\rho_x)$ and $\widetilde{V}$ is also normalized. Later we denote by $\widetilde{F}_*(V) = \widetilde{V}$ or $\widetilde{F}_*(\xi) = \widetilde{\xi}$ if it makes sense. Now we can rewrite (3.10) in the form $\widetilde{F}_*(r/B) = u/G$ and $\widetilde{F}_*(s/B) = v/G$. Furthermore, we have

$$v - u = \frac{s - r}{(\rho_t + \frac{r}{B}\rho_x)(\rho_t + \frac{s}{B}\rho_x)} \tag{3.13}$$

Generally speaking, for given vector fields $V_i = \xi_i\partial_x + \partial_t$, $i = 1, 2$

$$\widetilde{\xi}_1 - \widetilde{\xi}_2 = \frac{B(\xi_1 - \xi_2)}{G(\rho_t + \xi_1\rho_x)(\rho_t + \xi_2\rho_x)} \tag{3.14}$$

(3.13) (3.14) implies the transformation $\widetilde{F}_*$ keeping the order of the normalized vector fields.

Now we are in a position to fix the constant $R$. Denote by $\widetilde{\Omega}_2 = F(\Omega_2)$. In order to insure $(u,v)$ and $(\partial_\theta u, \partial_\theta v)$ small enough after the transformation $G\widetilde{F}_*(r/B)$ and $G\widetilde{F}_*(s/B)$, we evaluate $G\widetilde{F}_*(0)$ and $\partial_\theta G\widetilde{F}_*(0)$. By (1.22) we find,

$$G|\widetilde{F}_*(0)| = \left|G\frac{\theta_t}{\rho_t}\right| = \frac{1}{\text{sh}\Phi} \leq \frac{4}{R} \leq \frac{1}{2}\psi^* \text{ on } \partial\Omega_2 \tag{3.15}$$

provided that $R$ is chosen big enough. Differentiation of $G\widetilde{F}_*(0)$ yields

$$\partial_\theta G\widetilde{F}_*(0) = -\frac{m\text{ch}\Phi}{\text{sh}^2\Phi}\partial_\theta\Phi \tag{3.16}$$

From the formula $\rho_t = \text{tgh}\Phi$ and (1.20), it turns out that

$$\begin{aligned}\partial_\theta \Phi &= (\rho_{tt} t_\theta + \rho_{tx} x_\theta)\text{ch}^2\Phi \\ &= GG_\rho \theta_t^2 t_\theta \text{ch}^2\Phi + (GG_\rho \theta_x \theta_t - B_t G\theta_t) x_\theta \text{ch}^2\Phi\end{aligned}$$

From (1.22) it turns out that the first term $= mG_\rho/\text{ch}\Phi = O(1/R)$ on $\partial\Omega_2$. In view of the Remark of Lemma 1.1 and Lemma 1.3, the second term

$$\begin{aligned} &= m\left(\frac{G_\rho}{G}\text{tgh}\Phi - \frac{B_t}{B}\right) G\text{sh}\Phi \\ (3.17)\quad &= m\left[\left(\frac{1}{\rho} - \frac{1}{t}\right) - (1-\text{tgh}\Phi)\frac{G_\rho}{G} + O\left(\frac{1}{\rho^{1+\delta}}\right) + \frac{1}{t}O\left(\frac{1}{t^\delta \mu(x)}\right)\right] G\text{sh}\Phi \\ &= \left[O\left(\frac{1}{R}\right) + O\left(\frac{1}{R^\delta}\right)\right]\text{sh}\Phi, \text{ on } \partial\Omega_2 \end{aligned}$$

since $\rho \le t + |x|$, $\rho \le 2t$, $(\rho - t)/t \le 1/R$ and particularly, $t^\delta \mu(x) \ge R^\delta$ on $\partial\Omega_2$. Summing up we have $\partial_\theta \Phi = O(1/R^\delta)\text{sh}\Phi$ and $\partial_\theta G\widetilde{F}_*(0) = O(1/R^\delta)$. Thus we have proved that if $R$ is chosen sufficiently large

$$(3.18)\qquad |\partial_\theta G\widetilde{F}_*(0)| \le \mu_*/2, \quad \text{on } \partial\Omega_2$$

From now on we fix the constant $R$ such that Lemma 3.1, (3.15) (3.18) are all valid.

*The end of the proof of Theorem A.* The first step is to find the initial data

$$(3.19)\qquad r(x,0) = -\varepsilon s(x) \text{ and } s(x,0) = \varepsilon s(x)$$

such that the problem (2.8) (2.9) in $\Omega_1$ has a global smooth solution and moreover, after transformation $\widetilde{F}_*$, $(u,v)$ and $(\partial_\theta u, \partial_\theta v)$ on $\partial\widetilde{\Omega}_2$ satisfy the assumption in Lemma 3.1. More precisely, construct by mollifier two smooth even functions in $\mathbb{R}^1$

$$(3.20)\, H_1(\xi) \ge 1 + \sup_{\substack{(x,t)\in\Omega_1 \\ |x|\le\xi}} \{|\partial_x^i \partial_t \ln k|, |\partial_x^i \partial_x \ln k| \text{ and } |\partial_x^i \partial_t \ln B|, i = 0,1\}$$

and

$$(3.21)\qquad H_2(\xi) \ge 1 + \sup_{\substack{(x,t)\in\Omega_1 \\ |x|\le\xi}} \left\{\left|\frac{x}{1+x^2} - \frac{\dot{\mu}_*(x)}{\delta\mu_*(x)}\right|\right\}$$

Define

$$s(x) = \frac{1}{8\pi}\int_x^\infty \exp[-7H_1(3\xi)t_0(3\xi) - \xi^2]\frac{1}{t_0(3\xi)H_2(3\xi)}d\xi \tag{3.22}$$

if $|x| \geq 2$ and smooth if $|x| \leq 2$.

Now we claim that the system (2.8) (2.9) with (3.19) in $\Omega_1$ has a global smooth solution $(r, s)$ for each $\varepsilon$ in $(0, \varepsilon_0]$ for some $\varepsilon_0 \leq 1$. Indeed for each $|x_0| \geq 2$, for all $t$ in $[0, t_0(x_0)]$ the following assertions: ($\mathrm{A}_1$) for each $t$ in $(0, t_0(x_0)]$, $A = (x_0, t)$ is in the region where a $C^1$ solution $(r, s)$ to the problem (2.8) (2.9) exists. ($\mathrm{A}_2$) the dependent interval of $A$ is contained in $[x_0/2, 3x_0/2]$. ($\mathrm{A}_3$) in the characteristic $\Delta_A$

$$-\frac{1}{4t_0(x_0)H_2(x_0)} \leq |r|, |s|, |\partial_x r| \text{ and } |\partial_x s| \leq \frac{1}{4t_0(x_0)H_2(x_0)}$$

are valid. If it was false, then there exists a $t^* < t_0(x_0)$ such that for all $t \leq t^*$ the above assertions are true but some of them cease to be true if $t > t^*$. The local existence of solutions for the Cauchy problem of quasilinear hyperbolic system tells us $t^* > 0$ and ($\mathrm{A}_3$) guarantees $|r|$, $|s|$, $|\partial_x r|$ and $\partial_x s| \leq 1/4$. Therefore an application of Lemma 2.1 to the characteristic triangle $\Delta_A$, $A = (x_0, t)$, $t < t^*$, yields at once

$$\begin{aligned} &|r(x,t)|, |s(x,t)|, |\partial_x r(x,t)| \text{ and } |\partial_x s(x,t)| \\ &\leq \varepsilon[s(\frac{x_0}{2}) + |\dot{s}(\frac{x_0}{2})|] \exp[7H_1(\frac{3x_0}{2})t^*] \\ &\leq \varepsilon \exp[-7(t_0(x_0) - t^*)]\frac{1}{4t_0(x_0)H_2(x_0)} \\ &= \varepsilon q \frac{1}{4t_0(x_0)H_2(x_0)} \end{aligned} \tag{3.23}$$

for some constant $q$ strictly less than 1 and for all $A(x_0, t)$, $t < t^*$. Meanwhile, if $\varepsilon$ in $(0, \varepsilon_0]$ the dependent interval of $(x_0, t)$ as $t$ in $[0, t^*)$, is contained in $[x_0 - qt/4t_0(x_0), x_0 + qt/4t_0(x_0)] \subset (x_0/2, 3x_0/2)$ if $x_0 \geq 2$. This implies that this solution can be extended to a larger characteristic triangle than $\Delta_{(x_0,t^*)}$ and ($\mathrm{A}_1$)-($\mathrm{A}_3$) are continuous to be true. For the remaindering region: $(x, t)$ in $\Omega_1 \cap \{|x| \leq 2\}$, since this is a bounded one, it is easy to prove the existence of smooth solution if $\varepsilon_0$ is chosen small enough.

Finally the conclusion: $s-r>0$ everywhere on $\bar{\Omega}_1$ follows from the equation (2.29) or (2.30) and the fact that $r, s$ in $C^1(\bar{\Omega}_1)$.

Let us turn to study the behavior of the image $\partial F(\Omega_2) = \partial\widetilde{\Omega}_2$ of the boundary $\partial\Omega_2$ under the transformation $F$. It can be expressed in terms of parameter $x : \theta = \theta(x, t_0(x))$, $\rho = \rho(x, t_0(x))$ A direct computation convinces us that $\dot{\theta}^2 + \dot{\rho}^2 \neq 0$ for all $x$. In the meantime,

$$\begin{aligned} & -\dot{\rho}(x)\widetilde{F}_*(r/B) + \dot{\theta}(x) \\ = \; & -(\rho_x + \rho_t\dot{t}_0(x))\widetilde{F}_*(r/B) + (\theta_x + \theta_t\dot{t}_0(x)) \\ = \; & \frac{B(1 - \frac{r}{B}\dot{t}_0(x))}{G(\rho_t + \frac{r}{B}\rho_x)} \\ \geq \; & \frac{3B}{4G(\rho_t + \frac{r}{B}\rho_x)} \text{ if } x \text{ in } \mathbb{R}^1 \text{ and } |x| \geq 2 \end{aligned}$$

In getting the last inequality we have used (3.23), i.e.,

$$|\dot{t}_0(x)r| \leq \varepsilon|\dot{t}_0(x)|/4t_0(x)H_2(x) \leq 1/4,$$

for all $\varepsilon$ in $(0, \varepsilon_0]$. Similarly

$$-\dot{\rho}(x)\widetilde{F}_*(s/B) + \dot{\theta}(x) > 0, \qquad \text{if } x \text{ in } \mathbb{R}^1 \text{ and } |x| \geq 2$$

For the rest part: $|x| \leq 2$, choose a smaller $\varepsilon_0$ provides at once the desired result. So far we have verified (3.6) valid, and, hence, (3.13) and the fact that $s - r > 0$ on $\partial\Omega_2$ insure $v - u > 0$ on $\partial\widetilde{\Omega}_2$.

It remains only to evaluate the bounds on $\partial\widetilde{\Omega}_2$ of $(u, v)$ and $(\partial_\theta u, \partial_\theta v)$. By (1.20) and (3.10) we have

$$u = G\widetilde{F}_*(0) + \frac{r}{\text{tgh}\Phi(\text{tgh}\Phi - mr/\text{ch}\Phi)}$$

and

$$v = G\widetilde{F}_*(0) + \frac{s}{\text{tgh}\Phi(\text{tgh}\Phi - ms/\text{ch}\Phi)}$$

The following inequalities

$$\begin{aligned} |r_\theta| &= |r_t t_\theta + r_x x_\theta| \\ &\leq \varepsilon \frac{G}{4t_0(x)H_2(x)}(1 + 1/\text{ch}\Phi) \\ &\leq C\varepsilon \quad \text{on } \partial\widetilde{\Omega}_2 \end{aligned}$$

and

$$|s_\theta| \le C\varepsilon \quad \text{on } \partial\tilde{\Omega}_2$$

for some constant $C$ independent of $\varepsilon$, are valid since $G \le \mu\rho$ on $\partial\tilde{\Omega}_2$ from (1.16). With the aid of (3.15) (3.18) and the above inequalities,, it is not difficult to find another smaller $\varepsilon_0 > 0$ such that for $\varepsilon$ in $(0, \varepsilon_0]$ (3.7) (3.8) hold.

Thus we have proved that the problem (3.1) (3.2) admits a global smooth solution $(u, v)$ with the prescribed initial data $G\tilde{F}_*(r/B)$ and $G\tilde{F}_*(s/B)$ on $\partial\tilde{\Omega}_2$ and moreover, $v > u$ everywhere. Finally pulling $(\theta, \rho)$ and $(u, v)$ back to $(x, t)$ and $(r, s)$, matching the solution in $\Omega_1$ we can obtain the global smooth solution in $\bar{R}^2_+$ of (2.8) (2.9) with $r(x, 0) = -\varepsilon s(x)$, $s(x, 0) = \varepsilon s(x)$. (3.13) tells us $s - r > 0$ everywhere. This completes the proof of Theorem A. □

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