COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 1, Number 3, 473–486, 1993

A SPLITTING THEOREM AND AN ALGEBRAIC GEOMETRIC CHARACTERIZATION OF LOCALLY HERMITIAN SYMMETRIC SPACES

SHING TUNG YAU

In [6], Tian and I generalized the work of [1] by Cheng and myself. We proved the following:

Theorem 0.1. *Let M be an n-dimensional projective manifold and $D \subset M$ be a divisor with normal crossings. Let K be the canonical divisor of M.*

Suppose that $K + D$ is numerically effective, big and ample modulo D. Then:

$$C_1^n(\Omega_M(\log D)) \leq \frac{2(n+1)}{n} C_1^{n-2}(\Omega_M(\log D)) C_2(\Omega_M(\log D))$$

and equality holds iff $M \setminus D$ is an unramified quotient of the ball.

In this note, we shall derive more corollaries from the construction of canonical Einstein metric ds^2 on $M \setminus D$. They have been known to the author since 1985. All these theorems are easier to prove when D and C are empty, and in this case, they were known to the author much earlier. A new result on the canonical Einstein metric is that it is complete.

1. Description of the Almost Complete Kähler Einstein Metrics

Recall that Tian-Yau [6] proved the existence of a unique almost complete Kähler Einstein metric on $M \setminus D$ if $K_M + D$ is numerical effective, big and ample modulo D. In this section, we demonstrate that the metric is in fact complete. In fact, we shall have a description of the asymptotic behaviour of the metric.

Research supported by NSF Grant No. DMS-92-7609-2

We assume that D is a divisor with normal crossing. Let $D = D_1 + \cdots + D_m$ where each D_i is the zero section of some holomorphic section s_i of a holomorphic line bundle L_i. Then we can form a (1.1) form

$$\omega = \sum_i \frac{h_i D_i s_i \wedge \overline{D_i s_i}}{|s_i|^2(\log|s_i|^2)^2}$$

where h_i is a Hermitian metric of L_i and D_i is the associated covariant differentiation.

The form ω can be degenerate in some direction. However, it makes sense to discuss its holomorphic sectional curvature. It is easy to prove that in a neighborhood N of D, ω has strongly negative holomorphic sectional curvature. Hence by applying the Schwarz lemma (see [9]) in N we can prove that ds^2 dominates ω up to a constant. For any point in $M \setminus D$, it takes infinite distance (measured with respect to ω) to D. Hence ds^2 is complete.

For $0 < \mu_i < 1$ so that $K_M + \sum \mu_i D_i$ is represented by a positive (1,1) form ω_μ, we can form a metric $\omega_\mu + \omega = \omega_g$. Then it was proved in [6] that ds^2 is representable by $\omega_g + \partial\bar\partial\varphi$ where

$$(\omega_g + \partial\bar\partial u)^n = e^{f+u}\omega_g^n.$$

It was proved in [6] that $f + \varphi$ has an upper bound and φ has a lower bound on $M \setminus D$. Furthermore,

$$\omega_g + \partial\bar\partial\varphi \le c_1 e^{c_2\varphi}\omega_g$$

where c_2 can be chosen to be independent of μ_i.

The function f has the form $\sum_i(1-\mu_i)\ln|s_i|^2$. Hence the Kähler Einstein metric is dominated by $(\prod|s_i|^{2(\mu_i-1)})\omega_g$ from above and by $\prod(|s_i|^{2(1-\mu_i)})\omega_g$ from below.

Theorem 1.1. *Let M be a Kähler manifold with a divisor $D = D_1 + \cdots + D_m$ with normal crossing. Assume that $K_M + D$ is numerically effective, big and ample modulo D. Then $M \setminus D$ admits a complete Kähler Einstein metric ds^2. Furthermore in a neighborhood of D, ds^2 dominates Poincaré metric in the normal direction. For any $\epsilon > 0$, it is bounded asymptotically from above by $\prod|s_i|^{-2\epsilon}\omega_g$ and below by $\prod|s_i|^{2\epsilon}\omega_g$ of D depending on the constants $\epsilon_i > 0$.*

2. Theorem for Kähler Manifolds

Theorem 2.1. *Let M and D be as in Theorem* 0.1. *Suppose the bundle $\Omega_M(\log D)$ admits an endomorphism which has nontrivial kernel somewhere in $M \setminus D$. Then $M \setminus D$ is covered by the product of two manifolds M_1 and M_2 so that $M \setminus D$ is biholomorphic to the quotient of $M_1 \times M_2$ by a discrete group of automorphisms which preserve the product structure of $M_1 \times M_2$.* ***Note that,*** *in general, M_1 and M_2 are noncompact even when D is empty. This happens for example when M is a Hilbert modular surface.*

In order to state the next theorem, let us introduce some notation. Let $GL(n, \mathbb{C})$ acts on $V = \mathbb{C}^n$ in a standard manner. Then $GL(n, \mathbb{C})$ acts on the vector space which is the tensor product of several copies of V and V^* together. The action splits into many irreducible components. Associated to each of these splittings and a holomorphic vector bundle, we can define a new holomorphic vector bundle. Let us call bundles obtained in this way irreducible bundles associated to the given bundle.

Theorem 2.2. *Let M and D be as in Theorem* 0.1. *Suppose that $\Omega(\log D)$ can be splitted holomorphically as direct sum of irreducible holomorphic vector bundles $V_1 \oplus V_2 \oplus \ldots \oplus V_k$ such that for each i, either*

(1) *there is a non-trivial irreducible bundle associated to V_i which admits non-zero holomorphic section*

or

(2) $2(m_i+1)C_1^{m_i-2}(V_i)C_1^{n-m_i}(\Omega(\log D))C_2(V_i) = m_i C_1^{m_i}(V_i)C_1^{n-m_i}(\Omega(\log D))$.

Then $M \setminus D$ is a quotient of a bounded symmetric domain by a discrete group with finite volume.

Note that this theorem gives a complete algebraic geometric characterization of non-singular quotients of Hermitian symmetric domain with finite volume. It is possible to give similar characterization when the quotient has orbit fold singularities.

For any discrete group action on a Hermitian symmetric domain, the quotient space can be written as finite quotient of a non-singular manifold M' by a group G. The orbit M'/G acquires several kind of singularities. Let F_i

be the set of points x in M' so that for some $g \in G$, the fixed point set of g contains x and the codimension of this set is i. The images of F_i in M'/G for $i > 1$ form singular sets of M'/G. (Note that the F_i are not necessarily disjoint from each other.) The image of $F_1 \setminus (\bigcup_{i>1} F_i)$ is nonsingular.

Conversely, we assume that M is a projective variety whose singular set consists of quotient singularities only. Let $F_1 + D$ be a divisor with normal crossing. Let C_i be the irreducible components of F_1, then we define a sheaf $\overset{r}{\otimes} T \otimes \overset{s}{\otimes}\Omega[\log D, \sum_i \frac{m_i - 1}{m_i} C_i]$ on M in the following way. For any open set with empty intersection with $F_1 + D$, the sheaf is defined in the ordinary way as holomorphic sections of $\overset{r}{\otimes} T \otimes \overset{s}{\otimes}\Omega$. If the coordinate chart is chosen in such a way that $F_1 = \left\{\prod_{i=1}^{p} z_i = 0\right\}$ and $D = \left\{\prod_{i=p+1}^{q} z_i = 0\right\}$, then we use

$$(z_1)^{-\frac{m_1-1}{m_1}} dz_1, (z_2)^{-\frac{m_2-1}{m_2}} dz_2, \dots, (z_p)^{-\frac{m_p-1}{m_p}} dz_p, \frac{dz_{p+1}}{z_{p+1}}, \cdots, \frac{dz_q}{z_q}, dz_{q+1}, \cdots, dz_k$$

to form a basis for Ω in the complement of $F_1 \cup D$. It also provides a dual basis for Ω^*. Their tensor product defines a basis for $\overset{r}{\otimes} T \otimes \overset{s}{\otimes}\Omega$. The basis is defined only up to products of m_i-th roots of unity. An element in $\overset{r}{\otimes} T \otimes \overset{s}{\otimes}\Omega(\log D, \sum_r \frac{m_i-1}{m_i} C_i)$ is a holomorphic section of $\overset{r}{\otimes} T \otimes \overset{s}{\otimes}\Omega$ over $M \setminus D \setminus F_1$ such that when it is written in terms of the above basis, the coefficient is bounded.

Theorem 2.3. *Let M be a projective variety whose singularities are locally defined by the quotient of the ball by a finite group. Let $D + F$ be normal crossing divisors such that $K + D + \sum_i \frac{m_i-1}{m_i} C_i$ is numerically effective, big and ample modulo D. (Here C_i are irreducible components of F). Suppose $T \otimes \Omega(\log D, \sum_r \frac{m_i-1}{m_i} C_i)$ has holomorphic sections P_i so that ΣP_i is the identity and $P_i P_j = 0$ for $i \neq j$ at a point in $M \setminus D$. Then $M \setminus D$ is the orbit space of some product manifolds $M_1 \times \cdots \times M_k$ where $F = \Sigma C_i$ is the part of the branched locus of the group action where the fixed point set has codimension one with branching order m_i at each C_i. Furthermore if none of the projections P_i can be written as non-trivial sums of projections again and for each bundle defined by P_i, there is an irreducible holomorphic bundle associated to this* *bundle which admits non-trivial holomorphic endomorphism. Then $M \setminus D$ is the orbit space of a Hermitian symmetric domain with rank* > 1.

The algebraic geometric characterization of quotients of the ball by a discrete group where the fixed point set consists of divisors only was carried out by Tian and the author in [6]. The quotients with arbitrary singularities whose fixed point set consists of subvarieties with codimension ≥ 2 could also be done as in Tian-Yau [6]. One has to define the second Chern class in the right way so that it corresponds to the second Chern form that comes from the connections of the orbitfold metric. For algebraic surfaces, this was computed in detail by Cheng-Yau [2].

For completeness we state here the theorem of Tian-Yau.

Theorem 2.4. *Let M be an n-dimensional projective manifold. Let D and C be divisors so that $D + C$ have normal crossings. Suppose that $K_M + D + \sum_i \frac{m_i - 1}{m_i} C_i$ is numerically effective, big and ample modulo D. Here C_i are irreducible components of C. Then*

$$
\begin{aligned}
&[C_1(\Omega_M(\log(D + C))) - \sum_i \frac{1}{m_i} C_1(C_i)]^n \\
\leq\ & \frac{2(n+1)}{n} [C_1(\Omega_M(\log(D + C))) - \sum_i \frac{1}{m_i} C_1(C_i)]^{n-2} [C_2(\Omega_M(\log(D + C)))] \\
& - C_1(\Omega_M(\log(D + C))) \cdot \sum_i \frac{1}{m_i} C_1(C_i) + \sum_i \frac{1}{m_i^2} C_1^2(C_i)
\end{aligned}
$$

and the equality holds iff $X \setminus D$ is a branched quotient of the unit ball with branches along C.

Let us now give the demonstration of these theorems. The proof of Theorem 0.1 is contained in [6] and we shall omit it here.

The proof of Theorem 1.1 can be seen as follows. A holomorphic splitting of $\Omega(\log D)$ gives rise to a non-trivial endomorphism s of $\Omega(\log D)$. When D is empty, we compute the Laplacian of the norm of s with respect to the Kähler-Einstein metric of M. The Bochner formula shows that the curvature part of the formula vanishes when the metric is Kähler-Einstein. Hence by integration, we obtain the fact that s is covariant constant with respect to the

Kähler-Einstein metric. The standard de Rham theorem then implies that the universal cover of M must split metrically and holomorphically according to the rank of s. This proves Theorem 1.1 when D is empty.

When D is non-empty, we have to use the existence theorem of Tian-Yau [6]. The unique almost Kähler-Einstein metric constructed in [6] can be approximated by a sequence of Kähler-Einstein metric ds_m^2 compatible with certain cover which is branched along D in a neighborhood of D. The metric ds_m^2 has the property that if locally $\prod_{i=1}^{\ell} z_i = 0$ defines D, we can choose local branched cover along D so that if $z_i = w_i^m$, the metric ds_m^2 written in terms of w_i is a smooth metric. Note that $\frac{dz_i}{z_i}$ becomes $m\frac{dw_i}{w_i}$ and $w_i \to z_i$ is a branched cover of order m.

Hence if s is the holomorphic endomorphism of $\Omega(\log D)$, s is smooth with respect to the local holomorphic frame defining $\Omega(\log D)$. Let Δ_m be the Laplacian of ds_m^2 and $|s|_m^2$ be the norm of s with respect to the metric ds_m^2. Then if the zero set of s has codimension ≥ 2, the integral of $\Delta_m(\log |s|_m^2)$ over $M \setminus D$ is non-positive. If the zero set of s has codimension 1, then using the description of the metric in section 1 and the fact that $w_i \mapsto z_i$ has degree m, we conclude that the integral of $\Delta_m(\log |s|_m^2)$ is not greater than $\frac{c}{m}$ where c is a constant independent of m.

Since

$$|s|_m^2 \Delta_m(\log |s|_m^2) = |\partial_m s|_m^2 - |\langle s, \partial_m s\rangle|^2 |s|_m^{-2} \geq 0$$

we conclude that

$$\lim_{m\to\infty} (|\partial_m s|^2 - |\langle s, \partial_m s\rangle|^2 |s|_m^{-2}) = 0$$

must hold and there is a one form ω so that

$$\partial s = \omega s.$$

Considering s an endomorphism of $\Omega(\log D)$, the kernel of s can then be shown to be invariant under parallel transport. (Let e be any vector by $\ker(s)$). Then the above equation shows that ∂e is still a vector in $\ker(s)$.

Locally each point in M has a (holomorphic and metric) product neighborhood $D_1 \times D_2$ so that D_1 is part of the leaves of $\ker(s)$ and D_2 is part of the leaves defined by the orthogonal complement of $\ker(s)$. Since the metric ds^2 is complete, we know that the foliations defined on the complement of D give rise to a product structure on the universal cover of $M \setminus D$. The fundamental group of $M \setminus D$ acts by isometries on this product $M_1 \times M_2$. This gives the proof of Theorem 1.1.

Let us now prove Theorem 2.1. Under the assumption of Theorem 2.1, each holomorphic vector subbundle V_i which arises from some holomorphic foliations of $M \setminus D$ and $M \setminus D$ is covered by product manifolds $M_1 \times \cdots \times M_k$ with respect to the canonical Kähler-Einstein metric of Tian-Yau.

Under the hypothesis (1), the non-zero holomorphic automorphism of V_i would be parallel with respect to the Kähler-Einstein metrics as in the proof of Theorem 1.1. It cannot split V_i further and it has to reduce the group of holonomy of M_i which implies that M_i is locally Hermitian metric. The same argument as in [6] shows that the induced metric on M_i is complete and M_i is globally Hermitian symmetric.

Case (2) was treated in [6].

Note. Theorem 2.1 gives an algebraic characterization of locally Hermitian symmetric space. Hence it provides an alternate proof of Kazhdan's theorem about Galois conjugation of these manifolds.

Note that in Theorem 2.1, we can replace the assumption (1) by the existence of a non-trivial holomorphic section of $S^{2k}V_i \otimes (\det V_i)^{-k}$ over M. The proof is the same. One observes that the Bochner method also shows that the holomorphic section is parallel with respect to the Kähler-Einstein metric. (The study of these bundles goes back to Bogomolov. The Bochner argument was used independently by Kobayashi and Ochiai.)

Let us illustrate the theorem for the simplest case when M is a compact algebraic surface of general type. (This was a work that I did with J. Li.)

Theorem 2.5. *Let M be a compact algebraic surface of general type. If T_M is the sum of two line bundles or, more generally, $S^{2k}T \otimes K^k$ admits a non-trivial holomorphic section, then M is either a finite quotient of the product of two compact curves.*

Proof. Let us assume that M is minimal. If M has no rational curves with self-intersection number -2, then M has positive canonical line bundle and a Kähler-Einstein metric. The previous argument then applies. If M has -2 rational curves, then M still admits a Kähler-Einstein metric on the complement of these rational curves whose metric behavior near the curves can be described as "orbitfold" singularity. (See Cheng-Yau [2].) It is easy to see that the Bochner argument still works for this kind of metric and the holomorphic seciton s cannot vanish anywhere.

However, we claim that if -2 curve exists, s must vanish somewhere. To see this, we choose a standard model -2 curve in the following way.

Let V be the surface $xy - z^2 = 0$ inside the unit ball B^3. Let U be the proper transform of V in $\widetilde{B}^3$ which is the blowing-up of B^3 at the origin. Let C be the exceptional curve and $f : B^2 \to V$ be the map $(s_1, s_2) \to (s_1^2, s_2^2, s_1 s_2)$. Then the Kähler-Einstein metric on V can be lifted to be a non-singular metric under f. Hence we shall lift s to B^2 by f.

First of all, $U \subset \mathbb{C}^3 \times \mathbb{P}^2$ in the coordinate $(x, y, z) \times [u, v, w]$ defined by $xv - yu = 0$, $xw - zu = 0$, $yw - zv = 0$, $x^2 + y^2 - w^2 = 0$.

If we use a coordinate (x, v) on U by taking $u = 1$, then

$$y = xv, \;\; z = xw, \;\; v = w^2.$$

Any holomorphic section s of $S^{2k}T \otimes K^k$ can be written as

$$\begin{aligned}\widetilde{s} &= \sum h_i\Big(s_1^2, \frac{s_2^2}{s_1^2}\Big)\Big(\frac{1}{2s_1}\frac{\partial}{\partial s_1} + \frac{s_2}{2s_1^2}\frac{\partial}{\partial s_2}\Big)^i\Big(\frac{s_1^2}{2s_2}\frac{\partial}{\partial s_2}\Big)^{2k-i}\Big(4\frac{s_2}{s_1}ds_1 \wedge ds_2\Big)^k \\ &= \sum h_i\Big(s_1^2, \frac{s_2^2}{s_1^2}\Big)s_1^{3k-3i}s_2^{i-k}\Big(\frac{s_1}{2}\frac{\partial}{\partial s_1} + \frac{s_2}{2}\frac{\partial}{\partial s_2}\Big)^i\Big(\frac{1}{2}\frac{\partial}{\partial s_2}\Big)^{2k-i}(4ds_1 \wedge ds_2)^k.\end{aligned}$$

If we write

$$\widetilde{s}(s_1, s_2) = \sum g_i(s_1, s_2)\Big(\frac{\partial}{\partial s_1}\Big)^i\Big(\frac{\partial}{\partial s_2}\Big)^{2k-i}(ds_1 \wedge ds_2)$$

where g_i are holomorphic in B^2, then it is clear that $g_i(0,0) = 0$ for $i > k$.

For $i < k$, we could argue using the other coordinate chart with $v = 1$ for the blowing up. (So far we use $u = 1$.)

$$\begin{aligned} g_k(0,0) &= \lim_{s_1\to 0}\lim_{s_2\to 0} g_k(s_1,s_2) \\ &= \lim_{s_1\to 0} h_k(s_1^2,0)s_1^k \\ &= 0. \end{aligned}$$

Hence $\tilde{s}$ is zero at the origin of B^2 and s is identically zero.

Hence under the assumption that M is a minimal surface of general type, the canonical line bundle is ample and M is covered by the product of two unit disks.

If M is not minimal, we can blow down -1 rational curves over M to obtain a minimal surface M'. The section s over M can be pushed down to M' by Hartog's theorem. Same argument as above shows that the pushed down section has to vanish somewhere. This finishes the proof of the theorem. □

Note. It is not clear that algebraic manifolds M of general type with dimension ≥ 3 is covered by the polydisk if the tangent bundle of M is the direct sum of line bundles. This is of course true if the canonical line bundle is ample by the previous arguments.

3. Some splitting theorems for non-Kähler manifolds

In this section, we consider complex manifolds which are not Kähler. As was discussed in J. Li and myself in [10] (see also Buchdahl [11] when dim $M = 2$), we can replace Kähler metrics by Hermitian Yang-Mills connections.

Let ω be the Kähler form of a Hermitian metric defined on M^n which satisfies the equation

$$\partial\overline{\partial}(\omega^{n-1}) = 0.$$

The degree of a coherent sheaf is well deifned and is given by

$$\int_M C_1(\mathcal{F}) \cup \omega^{n-1}.$$

This number depends only on the "cohomology group". $\tilde{H}^{n-1,n-1}(M) = \{\Omega | \Omega \text{ is an } (n-1,n-1) \text{ form such that } \partial\overline{\partial}\Omega = 0\}/Im\partial + Im\overline{\partial}$.

A holomorphic bundle V is called stable iff for all coherent subsheaves $\mathcal{F}$ of V

$$\deg(\mathcal{F})(\dim(\mathcal{F}))^{-1} < (\deg V)(\dim V)^{-1}.$$

In [10], we proved that stable bundles V admit a Hermitian Yang-Mills connection.

Let r be the rank of V. Then the standard argument of using Chern form computation shows that

$$[2r \cdot c_2(V) - (r-1)c_1^2(V)] \cup \omega^{n-2} \geq 0.$$

(For the case of Kähler metrics, see [3]. For the case of vector bundles, see Lübke [4] and S. Kobayashi [5]). The equality holds only if the curvature form of V is diagonal.

There are several remarks about these theorems. One may wonder how to check the condition of stability for a bundle V. When M is rather "non-algebraic", there are less coherent sheaves and the stability condition can be checked easily. This was shown in my paper with J. Li and F.Y. Zheng [10] when M is a two-dimensional surface of class VII_0.

There is another condition which makes the proof of stability easier. Suppose that there is a group G which acts holomorphically on $\tilde{M}$, a regular cover of M and on $\tilde{V}$, the lift of V to $\tilde{M}$. Suppose $\Gamma \subset G$ and G preserve $\tilde{\omega}$ and a Hermitian metric on $\tilde{V}$, then the proof of the existence theorem in [10] shows that the coherent subsheaf $\mathcal{F}$ produced there can be lifted to $\tilde{M}$ to be a sheaf which is invariant under the action of G. This fact can be explained as follows:

In the proof of the existence theorem, we have to solve several equations on M. For example, we have to solve

$$\Delta u = f$$

where f is given and is G-invariant if we lift it to $\tilde{M}$.

In order to prove that the solution u is also G-invariant when lifted to $\tilde{M}$, we look at the equation

$$\Delta u = \epsilon u + f$$

with $\epsilon > 0$.

The lifted equation would satisfy the uniqueness property because if $\tilde{u}$ and $\tilde{u}'$ are two different liftings, $\tilde{u} - \tilde{u}' = v$ would satisfy

$$\Delta v = \epsilon v.$$

The maximal principle developed by Cheng and the author will then show $v = 0$. Hence $\tilde{u}$ is G-invariant. In this way, we can demonstrate the G-invariance of the solutions of all equations in the proof of the main theorem in [10]. As $\mathcal{F}$ is constructed from the solutions of these equations, it has to be G-invariant also.

In conclusion, we have proved the following theorem.

Theorem 3.1. *Let G be a group which acts on a complex manifold $\tilde{M}$ and a holomorphic vector bundle $\tilde{V}$ over $\tilde{M}$ so that both actions are compatible. Let $\Gamma \subset G$ be a subgroup such that $\tilde{M}/\Gamma$ is compact and $\tilde{V}$ descends to be a vector bundle V over M. Let $\tilde{\omega}$ be a G-invariant Hermitian form such that $\partial\bar{\partial}\tilde{\omega}^{n-1} = 0$. Suppose that for all G-invariant coherent subsheaf $\tilde{\mathcal{F}}$ of $\tilde{V}$, $\deg(\mathcal{F})\operatorname{rank}(\mathcal{F})^{-1} < \deg(V)\operatorname{rank}(V)^{-1}$, where $\mathcal{F}$ is the sheaf on M obtained from $\tilde{\mathcal{F}}$. Then if $\tilde{V}$ admits a G-equivariant Hermitian connection, it also admits a G-equivariant Yang-Mills connection.*

Note. If G has a biinvariant measure, we do not have to make the assumption that G preserves a Hermitian metric on $\tilde{V}$. In fact, let h be any Hermitian metric on V. Then it can be lifted to a Hermitian metric $\tilde{h}$ on $\tilde{V}$ which is Γ invariant. Since $\Gamma \setminus G$ has finite measure and $\tilde{h}$ is G-invariant, we can average the metric $\tilde{h}$ over the space $\Gamma \setminus G$. In this way, we obtain a metric which is invariant under G (and Γ). (Note that we can find a Hermitian form $\tilde{\omega}$ equivariant under G in a similar manner.)

Corollary 3.2. *Suppose for some point $x \in \tilde{M}$, the isotropic group of G at x acts irreducible on the fiber of $\tilde{V}$ at x. Then $\tilde{V}$ admits a G-invariant Hermitian Yang-Mills connection.*

Once we know how to produce a Hermitian Yang-Mills connection on V, most of the arguments in section one can be applied. Either a holomorphic endomorphism of V or a holomorphic section of $S^{2k}(V) \otimes (\det V)^{-k}$ will be a non-where zero section which is parallel with respect to the Hermitian Yang-Mills connection. It will reduce the holonomy group of V to a proper subgroup. Unfortunately, we do not have a theorem similar to Berger's theorem in the Riemannian category to conclude that M is special if $V = TM$.

There is an argument of H.C. Wang which enables one to conclude that if TM admits a flat connection, then M is the quotient of a complex Lie group G by some discrete group which acts on G by affine transformations. We can find characterizations of M such that TM admits a Hermitian flat connection.

For example, we have the following

Theorem 3.3. *Let M be a compact complex manifold such that $c_1(M)$ is $\partial\overline{\partial}$-exact. Suppose that $T(M)$ can be splitted as a direct sum of line bundles L_i. If the automorphism group of $\tilde{M}$ acts irreducibly on $T_x(M)$ for some $x \in M$, then M is covered by a complex Lie group.*

Proof. The assumption makes sure for every Hermitian form ω over M, $T(M)$ admits a Hermitian Yang-Mills connection. Hence

$$n \int_M c_1(L_i) \wedge \omega^{n-1} \leq \int_M c_1(M) \wedge \omega^{n-1}.$$

As $c_1(M) = \sum_i c_1(L_i) = 0$, we conclude easily that $c_1(L_i) \wedge \omega^{n-1} = 0$ for all Hermitian form ω with $\partial\overline{\partial}\omega^{n-1} = 0$. One easily derives that $c_1(L_i) = 0$ and TM admits a Hermitian flat metric. This finishes the proof. $\square$

It is tempted to use the Chern class inequality between c_2 and c_1^2. However, unless $\partial\overline{\partial}(\omega^{n-2}) = 0$, it is not easy to give a more topological meaning of the inequality. We shall come back to this at a later occasion.

4. Bundles with Singularities

Previously, we have only considered the holomorphic vector bundle over closed manifold. But in general, holomorphic vector bundles over an open manifold or a singular holomorphic vector bundle over a closed manifold are interesting. (We will specify it later.) In the following, we will encounter this situation and try to demonstrate that in some restricted case, one still can find a nice canonical connection over it which is the generalization of Hermitian-Yang-Mills connection.

Let us consider the following situation, a smooth complex manifold X and a normal crossing divisor D. We denote by X^0 the open manifold $X \setminus D$. In general, a holomorphic vector bundle V over X_0 can never be extended holomorphically (even smoothly) over X. What we are going to consider are those which can be extended in some broader sense.

We call a bundle over $X \setminus D$ extendable over D by orbifold if there is an open covering of a neighborhood of D in X, say $\{U_i\}$, and ramified covering $\pi_i : \tilde{U}_i \to U_i$ such that the pull-back bundle $\pi_i^* V$ over $\tilde{U}_i \setminus \tilde{D}_i$ ($\tilde{D}_i = \pi_i^{-1}(U_i \cap D)$) is extendable as a holomorphic vector bundle and two nearby extensions can be pieced together nicely. In other words, if we denote by G_i the finite holomorphic transformation group acting on $\tilde{U}_i \setminus \tilde{D}_i$ and $\tilde{U}_i \setminus \tilde{D}_i / G_i \cong U_i \setminus D$, then over $U_i \cap U_j \neq \phi$, $\pi_i^{-1}(U_i \cap U_j) \cong \pi_j^{-1}(U_i \cap U_j)$ biholomorphically and the corresponding group action G_i coincide.

In the following, we only consider those bundles which are extendable by orbifold.

In fact, there are quite a few of them. For example, any representation of $\pi_1(X \setminus D)$ into $GL(n, \mathbb{C})$ gives a flat bundle over $X \setminus D$. In some special case, the induced representation $\pi_1(U \setminus D) \to \pi_1(X \setminus D) \to GL(n, \mathbb{C})$ for some neighborhood U of D is torsion. After taking a branched covering, the pull-back bundle is easily shown to be extendable over the branched cover. So the flat bundle over $X \setminus D$ constructed by this representation is exactly the bundle we call extendable by orbifold.

Now we are in the position to demonstrate that we still can solve the Hermitian-Yang-Mills connection on those specific bundles if they are stable in the sense we will make clear now.

For any bundle V over $X \setminus D$ which is extendable over D by orbifold, we choose once and for all the corresponding resolution, $\pi_i, \tilde{U}_i, U_i, G_i$, etc. We call an object defined over $X \setminus D$ admissible if it can be extended smoothly with the corresponding properties after being lifted to the covering $\tilde{U}_i \setminus \tilde{D}_i$.

DEFINITION 4.1. A holomorphic vector bundle over $X \setminus D$ which is extendable over D by orbitfold is said to be stable with respect to an admissible Hermitian form ω (with $\partial\bar{\partial}\omega^{n-1} = 0$) if for any admissible coherent sheaf $\mathcal{F} \subset \mathcal{V}$,

$$\frac{1}{\operatorname{rank}(\mathcal{F})} \int c_1(\mathcal{F}) \wedge \omega^{n-1} < \frac{1}{\operatorname{rank}(V)} \int c_1(V) \wedge \omega^{n-1}.$$

(The meaning of $c_1(\mathcal{F})$ and $c_1(V)$ are represented by Chern forms which are calculated based on admissible connection.)

As before, we can show

Theorem 4.1. *Let V be a stable vector bundle over $X \setminus D$ which is extendable by orbifold with respect to hermitian form ω with $\partial\bar{\partial}\omega^{n-1} = 0$. Then there exists an admissible Hermitian-Yang-Mills connection on V.*

By establishing a suitable norm on the function space and establish some key inequalities, the proof of the theorem is quite straightforward.

References

1. Cheng, S.Y. and Yau, S.T., *On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation*, Comm. Pure App. Math. **33** (1980), 507–544.
2. Cheng, S.Y. and Yau, S.T., *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of $SU(2,1)$*, Contemporary Math. **49** (1986), 31–43.
3. Yau, S.T., *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci., USA **74** (1977), 1798–1799.
4. Yau, S.T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation* I, Comm. Pure Appl. Math. **31** (1978), 339–411.
5. Yau, S.T., *Métriques de Kähler-Einstein sur les variétés ouvertes*, in "Première Classe de Chern et courbure de Ricci: Preuve de la conjecture de Calabi, Séminaire Palaisean, France (1978), 163–167.
6. Tian, G. and Yau, S.T., *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, in "Mathematical Aspects of String Theory", edited by S.T. Yau, World Scientific, 1986.
7. Lübke, M., *Stability of Einstein-Hermitian vector bundles*, Manuscripta Math. **42** (1983), 245–257.
8. Kobayashi, S., *Curvature and stability of vector bundles*, Proc. Japan Acad. **58** (1982), 158–162.
9. Yau, S.T., *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), 197–203.
10. Li, J. and Yau, S.T., *Hermitian-Yang-Mills connections on non-Kähler manifolds*, in "Mathematical Aspects of String Theory", edited by S.T. Yau, World Scientific Publ. 1987, 560–573.
11. Buchdahl, N.P., *Hermitian-Einstein connections and stable vector bundles over compact complex surfaces*, Math. Ann. **280** (1988), 625–648.
12. Li, J., Yau, S.T. and Zheng, F., *A simple proof of Bogomolov's theorem on Class VII_0 surfaces with $b_2 = 0$*, Illinois Journal of Math. **34** (1990), 217–220.

Harvard University, U. S. A.

Received March 4, 1993.