COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 1, Number 3, 347–414, 1993

NONUNIQUENESS AND HIGH ENERGY SOLUTIONS FOR A CONFORMALLY INVARIANT SCALAR EQUATION

DANIEL POLLACK

INTRODUCTION

The Yamabe problem asserts that given any compact Riemannian manifold $(M, \overline{g})$, without boundary and of dimension greater than or equal to three, there exists a conformally related metric g, with constant scalar curvature. A glimpse into the colorful history of this problem is given by [20, 19, 1, 14]. Following Schoen's resolution of the Yamabe problem [14], there has been great effort and success in better understanding the quantity and behavior of metrics which can occur as solutions. In this paper we prove the existence of arbitrarily many distinct solutions to the Yamabe problem with constant positive scalar curvature, all lying within a fixed conformal class which is arbitrarily close, in the C^0 topology, to the conformal class of any given metric of positive scalar curvature. Before stating the result precisely we recall some of what is know about the existence of metrics of constant scalar curvature. For a complete and accessible discussion of the Yamabe problem we refer the reader to the excellent survey article by J. Lee and T. Parker [9].

Constant scalar curvature metrics arise as the critical points of the total scalar curvature functional

$$\mathcal{R}(g) = \int_M R(g) dv_g \tag{0.1}$$

restricted to those $g \in [\overline{g}]_1$, the space of metrics which are conformally equivalent to $\overline{g}$ and have unit volume. If we write $g = u^{\frac{4}{n-2}}\overline{g}$, for some $u \in C^\infty(M)$, $u > 0$, then the scalar curvature $R(g)$ is given by

$$R(g) = -c(n)^{-1} u^{-\frac{n+2}{n-2}} L_{\overline{g}} u,$$

where $c(n) = \frac{n-2}{4(n-1)}$, and $L_{\overline{g}}$ is the linear operator given by

$$L_{\overline{g}} = \Delta_{\overline{g}} - c(n)R(\overline{g}). \tag{0.2}$$

$L_{\overline{g}}$ is called the conformal Laplacian of $(M, \overline{g})$, it is a conformally invariant operator in the sense that if $g = u^{\frac{4}{n-2}}\overline{g}$, then for any function $\phi \in C^\infty(M)$ we have $L_{\overline{g}}(u\phi) = u^{\frac{n+2}{n-2}} L_g \phi$. The Euler-Lagrange equation of (0.1) is $R(g) \equiv K$, for some constant K, or equivalently

$$L_{\overline{g}} u + c(n) K u^{\frac{n+2}{n-2}} \quad = \quad 0. \tag{0.3}$$

This is known as Yamabe's equation and will be the focus of our study.

The approach used in solving the Yamabe problem was to seek a metric g which minimizes (0.1). The Yamabe Invariant is defined by

$$I(\overline{g}) \quad = \quad \inf\{\mathcal{R}(g) : g \in [\overline{g}]_1\}. \tag{0.4}$$

Thus the solution of the Yamabe problem produces a metric $g \in [\overline{g}]_1$ with constant scalar curvature $R(g) = \mathcal{R}(g) = I(\overline{g})$. Clearly by the definition, $I(\overline{g})$ is a conformal invariant. We thus refer to the sign of a conformal class, $[\overline{g}]$ as being the sign of $I(\overline{g})$. The sign of $I(\overline{g})$ dictates, to a large degree, the behavior of (0.3) and the types of solutions which can arise. $I(\overline{g}) > 0$, (respectively $= 0$, < 0) is equivalent to the existence of a metric $g \in [\overline{g}]$ with scalar curvature $R(g) > 0$, (respectively $= 0$, < 0).

For nonpositive conformal classes, the Yamabe problem has a unique solution among unit volume metrics. We deal exclusively with positive conformal classes. Among positive conformal classes, uniqueness does hold under very special circumstances. Recall that an Einstein metric g is a metric whose Ricci curvature satisfies $Ric(g) - \frac{R(g)}{n} g = 0$. Einstein metrics arise as critical points of the total scalar curvature functional (0.1) taken over the space of all unit volume metrics on M. A theorem of M. Obata [11], asserts that in the conformal class of an Einstein metric, there exists a unique metric of constant scalar curvature and unit volume, namely the Einstein metric suitably normalized. This holds provided that the conformal class of the Einstein metric is not the conformal class of the round metric g_0, on S^n. The proviso that one avoid the conformal class of g_0, is necessary since there exists an $n+1$ dimensional family of metrics in $[g_0]_1$ with constant scalar curvature. These metrics arise as the pullback of g_0 by the nonisometric, conformal diffeomorphisms of S^n, the space of which may be identified with the interior of the unit ball $\mathbb{B}^{n+1} \subset \mathbb{R}^{n+1}$. This exceptional property of (S^n, g_0) is heavily exploited in this paper, to show that the uniqueness property exhibited by Einstein metrics which are conformally distinct from g_0, is not a general phenomenon.

Recently R. Schoen has completed a thorough analysis of solutions to (0.3) which shows that if $(M, \overline{g})$ is not conformally equivalent to (S^n, g_0), then the set of solutions form a compact family, the size of which is determined solely by the conformal class, $[\overline{g}]$. In [16, 17, 18] Schoen considers the family of equations, indexed by the exponent $p \in [1, \frac{n+2}{n-2}]$, given by

$$L_{\overline{g}} u + E(u) u^p \;=\; 0 \tag{0.5}$$

where $E(u)$ is a positive constant which depends on the solution. When $p < \frac{n+2}{n-2}$, these equations are referred to as the subcritical regularization of (0.3). This approach to the study of (0.3) was originally introduced by Yamabe. For exponents $p < \frac{n+2}{n-2}$ the Sobolev embedding $i_p : H^1(M) \hookrightarrow L^{p+1}(M)$, is actually compact by the Rellich-Kondrakov theorem. This accounts for the fact that the subcritical equations, (0.5) with $p < \frac{n+2}{n-2}$, are more easily analyzed then (0.3) which has the critical exponent $p = \frac{n+2}{n-2}$ for which the embedding is still continuous but fails to be compact. This loss of compactness is the source of much of the difficulty of the Yamabe Problem. By a careful analysis of the possible types of blow up which could occur as $p \to \frac{n+2}{n-2}$, Schoen has derived uniform estimates on (0.5) provided that $(M, \overline{g})$ is not conformally equivalent to (S^n, g_0). These estimates imply that there exists a constant $\Lambda = \Lambda(\overline{g})$, which depends only on the conformal class of $\overline{g}$, such that every solution of (0.5) for any $p \in [1, \frac{n+2}{n-1}]$, lies in the set Ω_Λ defined by $\Omega_\Lambda = \{u \in C^{2,\alpha}(M) : \|u\|_{2,\alpha} < \Lambda\}$. This shows that no blow up can occur, and all solutions of (0.5) converge as $p \to \frac{n+2}{n-2}$, to a solution of Yamabe's equation (0.3). In the case when all the solutions to (0.3) in $[\overline{g}]_1$ are nondegenerate, as will be true for a generic conformal class, the existence theory which follows from this estimate implies that there are at most a finite number of solutions within $[\overline{g}]$.

These results provide some motivation for why we need to to perturb the given conformal class in order to find arbitrarily many solutions with arbitrarily high energy. In particular, they explain why these solutions do not exist without a perturbation of the conformal class.

The energy of a solution is given by

$$E(u) = \int_M \left(|\nabla u|^2 + c(n)R(\overline{g})u^2\right) dv_{\overline{g}}.$$

In the geometric case of the critical exponent $p = \frac{n+2}{n-2}$, for $g = u^{\frac{4}{n-2}}\overline{g}$ we have $\mathcal{R}(g) = c(n)^{-1}E(u)$. If we normalize our solutions so that $R(g) = n(n-1)$, thus leaving the volume uncontrolled, the energy of our solutions is then dependent on the volume of (M, g)

$$E(u) = \frac{n(n-2)}{4}\mathrm{Vol}_g(M).$$

The solution to Yamabe's problem produced a global minimum for the variational problem (0.1). This minimum always has energy less than or equal to that of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ with the induced metric g_0, and equality occurs if and only if (M, g) is conformally equivalent to (S^n, g_0). Thus, if we let ω_n denote the volume of the unit sphere, then for a minimizing solution $g_{\min} = u_{\min}^{\frac{4}{n-2}}\overline{g}$ we have

$$E(u_{\min}) \leq \frac{n(n-2)}{4}\omega_n.$$

We note that the Morse index of the solution $g = u^{\frac{4}{n-2}}\overline{g}$ is equal to the number of eigenvalues of Δ_g in $(0, n)$. Thus if we assume that g is a minimal solution then we know that λ_1, the first nonzero eigenvalue for Δ_g on M, satisfies $\lambda_1 \geq n$. The futher assumption that the solution is nondegenerate insures that the strict inequality $\lambda_1 > n$ holds.

We are now in a position to give a precise statement of our result.

Theorem 0.1. *Let (M, g) be a compact Riemannian manifold without boundary and of dimension $n \geq 3$, with positive scalar curvature, $R(g) > 0$. Given any integer $N \geq 1$ and any number $\overline{\epsilon} > 0$, there exists a conformal class $[g_1]$ which satisfies*

$$||g' - g_1'||_{C^0} < \overline{\epsilon},$$

for some metric $g' \in [g]$ *and* ${g_1}' \in [g_1]$, *and which contains metrics* $g_1, \dots, g_N$ *such that for each* $k = 1, \dots, N$

$$g_k = u_k^{\frac{4}{n-2}} g_1 \quad \text{and} \quad R(g_k) \equiv n(n-1).$$

Moreover the energy $E(u_k)$, *of each of these solutions can be estimated by*

$$|E(u_k) - (\,Vol_g(M) + (k+1)\omega_n)| \leq c(\overline{\epsilon}^{\frac{n-2}{2}} + \overline{\epsilon}^{\frac{n-2}{q}}),$$

for any $q > \frac{n}{2}$ *and a constant* $c = c(n, g)$.

Thus there exists a sequence of metrics $g_{\overline{\epsilon}}$, each with at least N distinct solutions in their conformal classes, which converge to g in the C^0 topology, as $\overline{\epsilon} \to 0$. This is valid even though g may only have one solution in its conformal class, e.g. if g is Einstein, or if all solutions in g are minimizing, as is the case for $g = g_0$ on S^n. Note that the energy estimate in Theorem 0.1 implies that each of the solutions g_k are nonminimizing. If we let

$$\mathcal{M}_N = \{\, [g] : \exists \geq N \text{ solutions of Yamabe's equation in } [g] \,\}$$

then we have the following Corollary.

Corollary 0.2. *For any integer* $N \geq 1$ *the set* $\mathcal{M}_N$ *is dense in the* C^0 *norm on the space of positive conformal classes.*

One should note that Schoen's estimates, referred to above, will be stable in a sufficiently strong topology. In other words, the same set Ω_Λ should contain all the solutions to (0.5) with respect to any metric $\tilde{g}$, which is ϵ close to $\overline{g}$ in the C^k norm, for k sufficiently large. Thus one can view the two results as being in opposition to each other.

In section 2 we construct approximate solutions to (0.3), which take the form of M joined with a string of $k+1$ spheres, for $k = 1, \dots, N$, each attached to the next by thin necks. The resulting metrics g_{1_k} are approximate solutions in the sense that their scalar curvature is bounded and the set where the scalar curvature is not constant, has small volume with respect to the metric g_{1_k}. It is possible to construct N distinct approximate solutions $g_{1_1}, \dots, g_{1_N}$ within a fixed conformal class, each of the form of M joined to a string of $2, \dots, N+1$ spheres respectively.

If we treat g_{1_k} as our new background metric and seek $g = u^{\frac{4}{n-2}} g_{1_k}$ with $R(g) = n(n-1)$ then, (0.3) becomes

$$\Delta_{g_{1_k}} u - c(n) R(g_{1_k}) u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0. \tag{0.6}$$

The statement that g_{1_k} is an approximate solution, means that $u = 1$ is an approximate solution to (0.6). Since we wish to find a solution to (0.6) which is close to our approximate solution $u = 1$, we write $g = (1+\eta)^{\frac{4}{n-2}} g_{1_k}$ where η is small. By linearizing (0.6) about $u = 1$, our equation takes the final form

$$\Delta_{g_{1_k}} \eta + n\eta = c(n) \left(R(g_{1_k}) - n(n-1) \right) (1+\eta) + Q(\eta), \tag{0.7}$$

where for η small, $Q(\eta)$ is quadratically small. For simplicity, we write (0.7) as

$$\mathcal{L}\eta = F(x, \eta).$$

The initial analytic difficulty in solving (0.7) is the possible presence of a kernel for the linear operator $\mathcal{L}$. Such a kernel could arise from the linear functions on $S^n \subset \mathbb{R}^{n+1}$. In section 3 we show that these functions actually do give rise to a small eigenspace K which necessarily contains any functions η in the kernel of $\mathcal{L}$ on M. This is done by identifying an explicit approximate kernel K_0, and showing that there is a small eigenspace $K \subset L^2(M)$, very close to K_0. This explicit control on K allows us to identify and quantify the component of $F(x, \eta)$ lying in K. This is one of the central features which allow us to solve (0.7) exactly.

We then show that we can invert the operator $\mathcal{L}$ on $K^\perp$, and obtain precise estimates on the solution $\eta \in K^\perp$ of

$$\mathcal{L}\eta = f,$$

for $f \in K^\perp$. In section 4 these estimates and the structure of the approximate solution, g_{1_k}, are used to solve (0.7) by a contraction mapping argument, provided that $F(x, \eta) \in K^\perp$.

The high degree of flexibility in our construction insures that the approximate solution may be deformed in such a way as to guarantee that the corresponding $F(x, \eta)$ is orthogonal to the small eigenspace K of $\mathcal{L}$. It has been well known for some time that one can construct approximate solution metrics. In particular, O. Kobayashi [8] has constructed approximate solutions with arbitrarily high energy within any positive conformal class. However these are constructed by an ODE method and do not have enough parameters to be deformed into exact solutions. The fact that our approximate solution construction has a large space of parameters which determine the corresponding metric g_{1_i}, is the source of the flexibility which is necessary to solve the nonlinear equation exactly. A primary component of the deformations we use, is the recognition that an integral identity having its origins in works of S. Pohozaev [12], and J. Kazdan and F. Warner [7], provides us with a precise measurement of the component of $F(x, \eta)$ lying in K. (The referee and Jose F. Escobar have pointed out that his identity is related to the Rellich identity [13], which characterizes the eigenvalues for the Dirichlet problem in terms of a boundary integral involing the normal derivative.) Moreover we show that it is possible to adjust each approximate solution, according to the required deformation, so that the deformations preserve the conformal class of the approximate solution. By doing this for each of the $1, \dots, N$ approximate solutions we find N distinct exact solutions to (0.7). The energy estimate is then a consequence of our construction of the approximate solutions.

The analytical methods used in this paper originate in the work of R. Schoen. In [15], Schoen proves the existence of weak solutions of (0.3) on S^n which have prescribed singular behavior. In particular, he constructs metrics g, conformally equivalent to g_0, which have constant scalar curvature, $n(n-1)$ and are complete on $S^n \setminus \{q_1, \dots, q_k\}$, where $\{q_i\}_{i=1}^k$ is any prescribed set of k points, for $k \geq 2$. In the interest of avoiding undue repetition, we will we often refer to [15].

We avoid some of the analytic difficulties of [15] by the fact that our approximate solutions are compact, whereas in [15] they are noncompact and, in certain cases can have exponential volume growth. In the noncompact case, the small eigenspace K of $\mathcal{L}$, is infinite dimensional and the solutions of the linearized operator do not, in general, lie in $L^2(M)$. On the other hand, in our case we must keep track of the conformal class of the approximate solutions and the conformal class of their deformations to insure that we can find our N distinct solutions all within one fixed conformal class. We also need to guarantee that neither the original manifold (M, g) nor the final sphere which we attach to our string, give any contribution to the kernel of $\mathcal{L}$. This is accomplished by a generic perturbation of the metrics and, in the locally conformally flat case, accounts for our C^0 perturbation of the conformal class. In the case that (M, g) is nowhere conformally flat, we require an additional perturbation to make it conformally flat in the neighborhood of a point. N. Kapouleas has used similar techniques in constructing a multitude of constant mean curvature surfaces in $\mathbb{R}^3$ (see [6, 5]). His approximate solutions are constructed from Delaunay surfaces, a classical one parameter family of constant mean curvature surfaces. These surfaces serve the same role in his construction, as the $n+1$ parameter family of constant scalar curvature metrics on S^n serve in ours.

Acknowledgement. This work was part of the authors doctoral dissertation at Stanford University under the direction of Richard M. Schoen, our gratitude for his support during that time can not be overstated. We would also like to thank Rafe Mazzeo and Karen Uhlenbeck for many helpful and enlightening discussion concerning this work.

1. Approximate Solutions

1.1. The Initial Background Metric. If we assume that $R(\overline{g}) = n(n-1)$, the scalar curvature of the unit sphere $S^n \subset \mathbb{R}^{n+1}$ and seek metrics $g \in [\overline{g}]$ satisfying $R(g) = n(n-1)$, then writing $g = u^{\frac{4}{n-2}}\overline{g}$ for some $u > 0$, $u \in C^\infty(M)$, u must satisfy

$$L_{\overline{g}}u + \frac{n(n-2)}{4}u^{\frac{n-2}{n+2}} = 0, \tag{1.1}$$

where $L_{\overline{g}}$ is the conformal Laplacian with respect to the backround metric $\overline{g}$,

$$L_{\overline{g}}u = \Delta_{\overline{g}}u - c(n)R(\overline{g})u = \Delta_{\overline{g}}u - \frac{n(n-2)}{4}u,$$

here $c(n) = \frac{n-2}{4(n-1)}$. We will require that $\overline{g}$ lie in a nondegenerate conformal class. This is equivalent to the invertibility of the operator $\mathcal{L}$, obtained by linearizing (1.1) about any solution u. Among positive conformal classes of metrics, this condition is generically satisfied, i.e. it is satisfied by an open and dense set of conformal classes. We will employ this condition in a very explicit manner. Since $u \equiv 1$ is a solution to (1.1), and the linearization about about this solution is

$$\mathcal{L} = \Delta_{\overline{g}} + n, \tag{1.2}$$

the nondegeneracy of the conformal class $[\overline{g}]$ implies that n is not an eigenvalue of $-\Delta_{\overline{g}}$. This fact will be of central importance in our construction of solutions to (1.1). Recall that on (S^n, g_0), n is an eigenvalue of $-\Delta_{g_0}$, with an $n+1$ dimensional eigenspace consisting of the restrictions of the linear functions in $\mathbb{R}^{n+1}$ to S^n. Thus $[g_0]$ is a degenerate conformal class. Since the kernel of $\mathcal{L}$ on S^n can be explicitly identified, it will be possible for us to use S^n in our construction of approximate solutions. We will also need to consider nondegenerate conformal classes of metrics on S^n. Since the nondegeneracy of $[g]$, for metrics g on S^n, is a generic condition, we may consider nondegenerate metrics which are arbitrarily close to g_0. We will let $\overline{g}_0$ denote a metric whose conformal class is positive and nondegenerate, and which satisfies for $\epsilon > 0$ and for some integer $k > 0$, $|\overline{g}_0 - g_0|_k \leq \epsilon$, where $|\cdot|_k$ denotes the C^k norm on tensors, taken with respect to the fixed metric g_0.

The first task will be to construct N approximate solutions $g_{1_1}, \dots, g_{1_N}$ to (1.1) all lying within a fixed conformal class. Toward this end it is necessary to deform g to make it conformally flat in the neighborhood of some point $x_0 \in M$. We deform g as follows. Let $\|W\|$ denote the norm of the Weyl tensor $W = W_{ijkl}$, taken with respect to g; this is a continuous function on M, so let x_0 be a point at which $\|W\|$ attains its minimum value. Let (r, θ) denote polar normal coordinates centered around $x_0 = 0$. In these coordinates, g takes the form $g = dr^2 + r^2 h_r$ where $h_r = h_{ij}(r,\theta) d\theta^i d\theta^j$ is a metric on S^{n-1}, and h_0 denotes the standard metric on S^{n-1}. Let $\Psi(r)$ be a smooth nondecreasing cut-off function which satisfies $\Psi(r) = 0$ for $r \leq 1$, $\Phi(r) = 1$ for $r \geq 2$ and $|\Psi'(r)| + |\Psi''(r)| \leq c$ for some constant c. We then define $\Psi_\rho(r) = \Psi(\rho^{-1} r)$, so that $\rho|\Psi_\rho'(r)| + \rho^2 |\Psi_\rho''(r)| \leq c$. For any $\rho > 0$ such that $2\rho < i(M,g) =$ the injectivity radius of (M, g), we deform our metric g to a new metric g_ρ defined by

$$g_\rho = \begin{cases} dr^2 + r^2((1 - \Psi_\rho(r))h_0 + \Psi_\rho(r) h_r) & \text{on } B_{2\rho}(x_0) \\ g & \text{on } M \setminus B_{2\rho}(x_0) \end{cases}$$

Note that g_ρ is Euclidean in $B_\rho(x_0)$, and hence conformally flat in a neighborhood of x_0. R. Schoen has observed [14] that for ρ sufficiently small, the conformal class of g_ρ, $[g_\rho]$ is positive. For such ρ, we find, by appealing to the resolution of the Yamabe problem, a new metric, again denoted by g, in $[g_\rho]$ with constant scalar curvature $n(n-1)$, and which is a minimal solution for the variational problem. This metric is conformally flat in a neighborhood of x_0, and will serve as our new background metric. Since we can also apply such a deformation to the metric $\overline{g}_0$ on S^n, we shall assume that $\overline{g}_0$ is non-degenerate, conformally flat in a neighborhood of a point $x_0 \in S^n$, satisfies $R(\overline{g}_0) \equiv n(n-1)$, and is a minimal solution for the variational problem. Before proceeding with the construction of our approximate solutions, we need to develop some facts concerning conformally flat metrics and the standard sphere (S^n, g_0).

1.2. Stereographic Coordinates and Conformally Flat Metrics. We will always regard S^n as the unit sphere in $\mathbb{R}^{n+1}$. If q is any point in S^n, let $\mathbf{P}(q) \subset \mathbb{R}^{n+1}$ denote the hyperplane passing through the origin and orthogonal to q, and let $x = (x^1, \dots, x^n)$ denote Euclidean coordinates in $\mathbf{P}(q)$. $\mathbb{R}^n = \mathbf{P}(q)$ can be identified with $S^n \setminus \{q\}$ by stereographic projection from q. For $p \in \mathbb{R}^{n+1}$, let $\xi(p) = p \cdot q$, we may then view $(x, \xi) = (x^1, \dots, x^n, \xi)$ as Euclidean coordinates for $\mathbb{R}^{n+1}$. Stereographic projection $\pi : \mathbb{R}^n \to S^n$ is then given by

$$\pi(x) = \left(\frac{2x}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1} \right).$$

The standard metric g_0 on $S^n \subset \mathbb{R}^{n+1}$, is then

$$g_0 = \frac{4}{(1+|x|^2)^2} \sum_{i=1}^{n} (dx^i)^2$$

when expressed in these coordinates. Let $B_r(q)$ denote the geodesic ball on S^n of radius $r < \pi$ centered at q. A little elementary trigonometry then shows that

$$\pi \left(\{x : |x| = (\tan \tfrac{1}{2} r)^{-1}\} \right) = \partial B_r(q)$$

and

$$|r^{-1} - (\tan \tfrac{1}{2} r)^{-1}| \le cr.$$

The fact that (M, g) is conformally flat in a neighborhood, $B_{r_0}(x_0)$ of $x_0 \in M$, is equivalent to the existence of a conformal diffeomorphism, Φ between $B_{r_0}(x_0)$ and a domain in (S^n, g_0), provided r_0 is sufficiently small. If $\Phi(x_0) = q$, for some $q \in S^n$, we may write $g = \phi^{-2} g_0$ in $\Omega = \Phi(B_{r_0}(x_0)) \subset S^n$, for some $\phi \in C^\infty(\Omega)$, $\phi > 0$. Let $\Lambda > 0$ satisfy $\phi > \Lambda^{-1}$ on Ω, and $\|\phi\|_2 < \Lambda$, where $\|\cdot\|_2$ denotes the C^2 norm taken with respect to the standard metric g_0 on S^n. In stereographic coordinates on $S^n \setminus \{q\}$, g takes the form,

$$g = 4(\phi(\pi(x)) + \phi(\pi(x))|x|^2)^{-2} \sum_{i=1}^{n} (dx^i)^2.$$

This expression holds in $\mathbb{R}^n \setminus B_{R_0}(0)$, for some $R_0 >> 1$. If we let $v^{\frac{4}{n-2}}(x) = (\phi(\pi(x)) + \phi(\pi(x))|x|^2)^{-2}$, then we have,

$$v(x) = |x|^{2-n}(\phi(\pi(x)) + \phi(\pi(x))|x|^{-2})^{\frac{2-n}{2}}.$$

We now show that by making a fixed linear change of coordinates in $\mathbb{R}^n$, we can rewrite this as,

$$\overline{v}(x) = |x|^{2-n}(a_0 + h(x))^{\frac{2-n}{2}}, \tag{1.3}$$

where $a_0 = \phi(q) > 0$ and $h(x) = O(|x|^{-2})$. This type of adjustment was used by Gidas, Ni, and Nirenberg [3] in showing that all global solutions on $\mathbb{R}^n$ to (1.1) are spherically symmetric.

Lemma 1.1. *Given any conformally flat metric g on $B_{r_0}(x_0) \subset M$, there exists an $R_0 >> 1$, and coordinates on $B_{r_0}(x_0) \setminus \{x_0\}$ such that*

$$g = 4\overline{v}^{\frac{4}{n-2}}(x) \sum_{i=1}^{n} (dx^i)^2,$$

for $|x| > R_0$, and $\overline{v}(x)$ takes the form

$$\overline{v}(x) = |x|^{2-n}(a_0 + h(x))^{\frac{2-n}{2}},$$

where $a_0 > 0$ and $h(x) = O(|x|^{-2})$, as $|x| \to \infty$.

Proof. We first extend ϕ from $\Omega \subset S^n$ to a small neighborhood of Ω in $\mathbb{R}^n$ so that $\frac{\partial^k \phi}{\partial \xi^k}(q) = 0, \forall k > 1$. Writing the Taylor expansion for $\phi(x, \xi)$ about

$q=(0,\dots,0,1)\in S^n$, as a function on $\mathbb{R}^{n+1}$, we have

$$\phi(x,\xi)=a_0+\sum_{i=1}^{n}\phi_i(q)x^i+\frac{1}{2}\sum_{i,j=1}^{n}\phi_{ij}(q)x^ix^j+O(|x|^3),$$

where $\phi_i=\frac{\partial\phi}{\partial x^i}$, $\phi_{ij}=\frac{\partial^2\phi}{\partial x^i\partial x^j}$. In stereographic coordinates this expression is transformed into an Laurent expansion about infinity, if we only keep track of those terms up to order $|x|^{-1}$, we then have,

$$\phi(\pi(x))=a_0+2\sum_{i=1}^{n}\frac{\phi_i(q)x^i}{|x|^2+1}+O(|x|^{-2}).$$

Thus,

$$v(x)^{\frac{4}{n-2}} = \frac{1}{a_0^2|x|^4}\left(1-\frac{2}{a_0}\sum_{i=1}^{n}\frac{\phi_i(q)x^i}{|x|^2+1}+O(|x|^{-2})\right).$$

If we were to replace x with $x-x_0$, for some $x_0=(x_0^1,\dots,x_0^n)\in\mathbb{R}^n$, then since

$$\frac{1}{|x-x_0|^4}=\frac{1}{|x|^4}\left(1+\frac{4}{|x|^2}\sum_{i=1}^{n}x_0^ix^i+O(|x|^{-2})\right),$$

this would have the net effect of introducing a new term of order $|x|^{-1}$ into our expansion. Explicitly for $\ell_{x_0}(x)=x'=x-x_0$ we have,

$$v(x')^{\frac{4}{n-2}} = \frac{1}{a_0^2|x|^4}\left(1+\frac{2}{a_0}\sum_{i=1}^{n}\frac{(2a_0x_0^i-\phi_i(q))x^i}{|x-x_0|^2+1}+O(|x|^{-2})\right).$$

Therefore, choosing $x_0^i=\frac{\phi_i(q)}{2a_0}$, we have

$$\overline{v}(x)^{\frac{4}{n-2}}=\frac{1}{a_0^2|x|^4}\left(1+O(|x|^{-2})\right),$$

or simply

$$\overline{v}(x)^{\frac{4}{n-2}}=|x|^{-4}(a_o+h(x))^{-2},$$

where $h(x)=O(|x|^{-2})$ as claimed. The same arguments easily show that $|\nabla h(x)|=O(|x|^{-3})$ and $\Delta h(x)=O(|x|^{-4})$ as well. Note that this expression is valid on $\mathbb{R}^n\setminus B_R(0)$, for any $R\geq R_0+C\geq R_0+|x_o|$, where $C=C(n,\Lambda)$. □

1.3. Conformal Transformations of (S^n, g_0). The group of conformal transformations of S^n is isomorphic to the group $O(n+1,1)$ of linear isometries of Minkowski $(n+2)$ space. If $G : S^n \to S^n$ is a nonisometric conformal transformation with precisely two fixed points on S^n, then we refer to G as a dilation. If the fixed points are antipodal points, say $\pm q$, and the differential $dG_p : T_pS^n \to T_{G(p)}S^n$ is a multiple of the identity at $\pm q$, then we refer to G as a centered dilation. Suppose that G is a centered dilation. We then let $|G'|$ denote the function whose value at $p \in S^n$ is the linear stretch factor of G at p, that is $|dG(v)| = |G'(p)|$ for any unit vector $v \in T_pS^n$, and assume that $|G'(q)| \geq |G'(-q)|$ We shall refer to q as the source of G and $-q$ as the sink of G. In stereographic coordinates we have $\pi^{-1}(-q) = 0 \in \mathbb{R}^n$, and G has the form

$$G(x) = \mu x$$

where $\mu = |G'(0)|$. It is easy to see that $|G'(q)| = \mu^{-1}$. This shows that $|G'(q)||G'(-q)| = 1$, moreover $\mu = 1$ if and only if $|G'| \equiv 1$, in which case G is an isometry of S^n.

Let $\lambda > 0$, be defined by $\lambda = |G'(q)|$, and thus $\lambda^{-1} = \mu = |G'(-q)| < 1$. We will think of λ as a large number (i.e. we will only work with strong dilations), and define a related small quantity ϵ by $\epsilon = 2\lambda^{-1/2}$.

The hypersurface in S^n upon which $|G'(p)| = 1$, corresponds under stereographic projection to the set Σ_1 of $x \in \mathbb{R}^n$, satisfying

$$\mu(1 + |x|^2) = 1 + \mu^2|x|^2.$$

It then follows easily that

$$\Sigma_1 = \{x : |x| = \lambda^{1/2}\}.$$

A general dilation G has a unique decomposition $G = RD$, where R is a rotation and D is a centered dilation. Note that $G^*g_0 = D^*R^*g_0 = D^*g_0$, so that $|G'| \equiv |D'|$. We let q and $-q$ denote the source and sink of D, respectively, and λ and ϵ are given by $\lambda = |D'(q)|$, and $\epsilon = 2\lambda^{-1/2}$.

1.4. The Bubble Construction. Given a metric g on M such that $R(g) = n(n-1)$ and g is locally conformally flat in a neighborhood of a point x_0, we now construct a metric g_1 conformal to g which agrees with $G^*(g_0)$ near x_0, and with g away from x_0. This new metric will have scalar curvature which is bounded and differs from $n(n-1)$ on a set whose volume with respect to g_1 is small. This is a generalization of [15, Prop. 1.1], which established this result in the case that $(M,g) = (S^n, g_0)$.

Proposition 1.2. *Let (M,g) be given such that $R(g) = n(n-1)$ and g is conformally flat in a neighborhood of $x_0 \in M$, with Φ and Λ as defined above. Suppose G is a dilation with source $q = \Phi(x_0)$ and $G = RD$. Let λ and ϵ be as above, and let $\sigma = 2\epsilon^{-1}$. There exists a metric $g_1 = u^{\frac{4}{n-2}} g$ on M with the following properties:*

(i) *Let $\epsilon_1 = \epsilon^{1+\frac{2}{n}}$ and $\epsilon_2 = \epsilon^{1-\frac{2}{n}}$. The metric g_1 is equal to $G^*(g_0)$ in $B_{c\epsilon_1}(x_0)$, and is equal to g on $M \setminus B_{c\epsilon_2}(x_0)$, for some constant $c = c(\Lambda)$.*

(ii) *Let R_1 denote the scalar curvature function of g_1. There exists a constant C depending only on n and Λ, such that the following inequalities hold:*

$$\max\{|R_1(p)| : p \in M\} \le C,$$
$$Vol_{g_1}\{p : R_1(p) \ne n(n-1)\} \le C\epsilon^{n-2},$$

where $Vol_{g_1}(\cdot)$ denotes the volume taken with respect to g_1.

(iii) *The metric g_1 can be described as follows near $\partial B_\epsilon(q) \subset S^n$. Let $x = (x^i, \cdots, x^n)$ be stereographic coordinates on $S^n \setminus \{q\}$. For $|x|$ near σ, g_1 is given by $g_1 = 4v_1^{\frac{4}{n-2}} \sum_{i=1}^n (dx^i)^2$, where*

$$v_1(x) = \sigma^{2-n} + |x|^{2-n}.$$

In particular, $R_1 \equiv 0$ near $\partial B_\epsilon(q)$.

(iv) *The metric g_1 varies continuously under deformations of G.*

Proof. Since (M,g) is locally conformally flat in a neighborhood of x_0, there exists an $r > 0$ such that $B_r(x_0)$, the geodesic ball of radius r with respect to g, is locally conformally flat, i.e. there exists a conformal embedding

$$\Phi : (B_r(x_0), g) \longrightarrow (S^n, g_0)$$

with $\Phi(x_0) = q \in S^n$.

As was shown above, there exists translated stereographic coordinates

$$\pi \cdot \ell_{-x_0} : \mathbb{R}^n \to S^n \setminus \{q\}$$

so that g has an expression of the form

$$g = 4\overline{v}(x)^{\frac{4}{n-2}} \sum_{i=1}^{n} (dx^i)^2,$$

for some $\overline{v}(x)$ as in (1.3). There exists an $R >> 1$ such that this expression for g is valid in $\mathbb{R}^n \setminus B_R(0) \subset \pi^{-1}(\Phi(B_r(x_0)))$.

Our new metric will be of the form $g_1 = 4v_1^{\frac{4}{n-2}} \sum_{i=1}^n (dx^i)^2$ on $\mathbb{R}^n \setminus B_R(0)$, for some $v_1 \in C^\infty(\mathbb{R}^n \setminus B_R(0)), v_1 > 0$ which we shall now construct explicitly. The function v_1 shall satisfy

$$v_1(x) = \begin{cases} \overline{v}(x) & \text{for } R \leq |x| \leq \sigma_2 \\ \sigma^{2-n} + (a_0^{1/2}|x|)^{2-n} & \text{for } 4\sigma_2 \leq |x| \leq \sigma_1 \\ (\lambda(1+\lambda^{-2}|x|^2))^{\frac{(2-n)}{2}} & \text{for } 4\sigma_1 \leq |x| \end{cases}$$

where σ_1 and σ_2 are to be chosen so that $\sigma_2 << \sigma << \sigma_1$.

We shall require that

$$(1.4) \qquad (\tan \frac{1}{2}\epsilon_2)^{-1} \leq \sigma_2, \qquad \sigma_1 \leq \tfrac{1}{4}(\tan \tfrac{1}{2}\epsilon_1)^{-1}, \qquad R \leq (\tan \frac{1}{2}\epsilon_2)^{-1}.$$

The third inequality will be satisfied provided λ is sufficiently large, i.e. provided that G is a sufficiently strong dilation. These three inequalities guarantee that $g_1 = G^*(g_0)$ on $B_{\epsilon_1}(q)$ and $g_1 = g$ on $\Omega \setminus B_{\epsilon_2}(q)$. Moreover we may choose a constant $c = c(n, \Lambda) > 0$ such that $B_{c\epsilon_1}(x_0) \subset \Phi^{-1}(B_{\epsilon_1}(q))$ and $\Phi^{-1}(\Omega \setminus B_{\epsilon_2}(q)) \subset M \setminus B_{c\epsilon_2}(x_0)$. Thus for this constant c, these inequalities will guarantee that property (i) holds.

For $a > 0$ we define a patching function $\Psi_a(x)$ on $\mathbb{R}^n$ by setting $\Psi_a(x) = \Psi(a^{-1}|x|)$, where $\Psi_a(x)$ satisfies

$$\Psi_a(x) = \begin{cases} 0 & \text{for } |x| \leq a \\ 1 & \text{for } |x| \geq 2a \end{cases}$$

and $a|\nabla\Psi_a| + a^2|\nabla\nabla\Psi_a| \leq C$.

We now define $v_1(x)$ explicitly for $\sigma_2 \leq |x| \leq 4\sigma_2$ and calculate the scalar curvature of the corresponding metric, $g_1 = 4v_1(x)^{\frac{4}{n-2}} \sum_{i=1}^n (dx^i)^2$. Define

$$v_1(x) = \begin{cases} |x|^{2-n}(a_0 + (1-\Psi_{\sigma_2}(x))h(x))^{\frac{2n}{2}} & \text{for } \sigma_2 \leq |x| \leq 2\sigma_2 \\ \Psi_{2\sigma_2}(x)\sigma^{2-n} + (a_0^{1/2}|x|)^{2-n} & \text{for } 2\sigma_2 \leq |x| \leq 4\sigma_2. \end{cases}$$

The scalar curvature of $g_1 = 4v_1^{\frac{4}{n-2}} \sum_{i=1}^n (dx^i)^2$ is given by

$$R_1 = -\frac{n-1}{n-2} v_1^{-\frac{n+2}{n-2}} \Delta v_1.$$

To show that R_1 is bounded for $\sigma_2 \leq |x| \leq 2\sigma_2$, we write

$$v_1(x) = |x|^{2-n} f(x)^{\frac{2-n}{2}},$$

where

$$f(x) = a_0 + (1 - \Psi_{\sigma_2}(x))\, h(x).$$

A straightfoward calculation yields the formula

$$R_1(x) = -\frac{n-1}{4} |x|^3 [|x| \,(n|\nabla f(x)|^2 - 2f(x)\Delta f(x)) \\ + 4(n-2) f(x)\,(\nabla f(x) \cdot \nabla |x|)].$$

Using the bounds noted above on $\Psi(x)$, $h(x)$ and their derivatives, it follows immediately that $R_1(x)$ is bounded for $\sigma_2 \leq |x| \leq 2\sigma_2$, for any $\sigma_2 \geq R \geq 1$. On $2\sigma_2 \leq |x| \leq 4\sigma_2$, $\Delta v_1(x) = \sigma^{2-n} \Delta \Psi_{2\sigma_2}$, so we have $|\Delta \Psi_{2\sigma}| \leq c\sigma^{n-2}(2\sigma_2)^{-2}$. A lower bound for $v_1(x)$ here, implies $v_1(x)^{-\frac{n+2}{n-2}} \leq (a_0^{1/2}|x|)^{n+2} \leq c\sigma_2^{n+2}$, where $c = c(n, a_0)$. These bounds yield $|R_1(x)| \leq c\sigma^{2-n}\sigma_2^n$, which is bounded provided $\sigma_2 \leq c\sigma^{1-\frac{2}{n}}$. Since $\epsilon_2 = \epsilon^{1-\frac{2}{n}}$ and $|\sigma(\tan \frac{1}{2}\epsilon)^{-1}| \leq c\epsilon$, we can choose a constant $c > 0$ so that this holds without violating (1.4).

For $\sigma_1 \leq |x| \leq 2\sigma_1$ we define $v_1(x)$ by

$$v_1(x) = \sigma^{2-n} + (1 - \Psi_{\sigma_1})(a_0^{1/2}|x|)^{2-n}.$$

We then get the estimate $|R_1(x)| \leq c\sigma^{n+2}\sigma_1^{-n}$, which is bounded provided that $\sigma_1 \geq c\sigma^{1+\frac{2}{n}}$. As above, since $\epsilon_1 = \epsilon^{1+\frac{2}{n}}$ we may choose a constant $c > 0$ so that this holds without violating (1.4).

Since $\sigma = \lambda^{1/2}$, we may rewrite the definition of $v_1(x)$ for $|x| \leq 4\sigma_1$, as

$$v_1(x) = \sigma^{2-n}(1 + \lambda^{-2}|x|^2)^{\frac{2-n}{2}}.$$

If we expand this out we may then write

$$v_1(x) = \sigma^{2-n} + \alpha(x),$$

and for $2\sigma_1 \leq |x| \leq 4\sigma_1$ one can easily check that the following bounds on $\alpha(x)$ hold

$$|\alpha(x)| + \sigma_1 |\nabla \alpha(x)| + \sigma_1^2 |\nabla\nabla \alpha(x)| \leq c\sigma^{-2-n}\sigma_1^2,$$

where c=c(n). With this in mind we define $v_1(x)$ for $2\sigma_1 \leq |x| \leq 4\sigma_1$ by

$$v_1(x) = \sigma^{n-2} + \Psi_{2\sigma_1}(x)\alpha(x).$$

An easy calculation using the bounds above then shows that $|R_1(x)|$ is bounded for $2\sigma_1 \leq |x| \leq 4\sigma_1$.

This completes the definition of g_1 on $\mathbb{R}^n \setminus B_R(0)$. Clearly by the construction of v_1, g_1 extends to a metric on all of M which agrees with g in $M \setminus B_{c\epsilon_2}(x_0)$ and with $G^*(g_0)$ in $B_{c\epsilon_1}(x_0)$ and satisfies $|R_1(x)| \leq C$ on all of M. Moreover properties (iii) and (iv) can be verified immediately from the construction of g_1. It remains to verify the second inequality of property (ii), which is equivalent to the inequality

$$\int_{\{x:\,\sigma_2\leq|x|\leq 4\sigma_1\}} v_1(x)^{\frac{2n}{n-2}}dx \leq C\epsilon^{n-2}.$$

For $\sigma_2 \leq |x| \leq \sigma$, $v_1(x) \leq C|x|^{2-n}$, so that $v_1(x)^{\frac{2n}{n-2}} \leq C|x|^{-2n}$. Hence

$$\int_{\{x:\,\sigma_2\leq|x|\leq\sigma\}} v_1(x)^{\frac{2n}{n-2}}dx \leq C\int_{\{x:\,\sigma_2\leq|x|\leq\sigma\}} |x|^{-2n}dx \leq C\sigma_2^{-n} \leq C\epsilon^{n-2}.$$

For $\sigma \leq |x| \leq 4\sigma_1$, $v_1(x) \leq c\sigma^{2-n}$, so that

$$\int_{\{x:\,\sigma\leq|x|\leq 4\sigma_1\}} v_1(x)^{\frac{2n}{n-2}}dx \leq C\int_{\{x:\,\sigma\leq|x|\leq 4\sigma_1\}} \sigma^{-2n}dx \leq C\sigma^{-2n}\sigma_1^n.$$

Since $\sigma_1 \leq \frac{1}{4}(\tan\frac{1}{2}\epsilon_1)^{-1}$ and $|\epsilon_1^1 - (\tan\frac{1}{2}\epsilon_1)^{-1}| \leq c\epsilon_1$, we can choose a constant $c > 0$ such that $\sigma^{-2n}\sigma_1^n \leq c\epsilon^{n-2}$. Thus $Vol_{g_1}\{x : R_1(x) \neq n(n-1)\} \leq c\epsilon^{n-2}$. This completes the proof of Proposition 1.2. □

1.5. Conformal Structures and Approximate Solutions. Our N approximate solutions $g_{1_1}, \dots, g_{1_N}$ can be distinguished from each other by the number of spherical regions (defined below) that each one possesses, for example, g_{1_k} will have k spherical regions. We will first show how to construct an approximate solution with k spherical regions, for any $k \geq 1$. Once this is done it will be easy to construct a conformal class with N approximate solutions.

The conformal data which we will use to construct our approximate solution consists of a chain $\mathcal{T}_k$, of k vertices $(1, \dots, k)$, each of which is labeled with a conformal transformation of S^n, F_i for $1 \leq i \leq k$. We will denote our maps either with subscripts i, corresponding to the vertex i, or with subscripts $\pm i$,

$+i$ denoting the edge between i and $i+1$ and $-i$ denoting the edge between i and $i-1$. We require that the transformations satisfy, for $2 \leq i \leq k-1$,

$$G_{\pm i} = F_{i\pm 1} \cdot F_i^{-1} = R_{\pm i} D_{\pm i}$$

and for $i = 1$ and $i = k$

$$G_{+1} = F_2 \cdot F_1^{-1} = R_{+1} D_{+1} \text{ and } G_{-k} = F_{k-1} \cdot F_k^{-1} = R_{-k} D_{-k}$$

are strong dilations whose sources are sufficiently separated for each i, $2 \leq i \leq k-1$ (this will be made precise momentarily). We also associate to each terminal vertex another such dilation which we denote

$$G_{-1} = R_{-1} D_{-1} \text{ and } G_{+k} = R_{+k} D_{+k}.$$

Thus for $1 \leq i \leq k$, we have associated to each edge $\pm i$, corresponding dilations $G_{\pm i}$. We let $q_{\pm i}$ denote the source of $G_{\pm i}$, and $\lambda_{\pm i} = |G'_{\pm i}(q_{\pm i})| = |D'_{\pm i}(q_{\pm i})|$ denote its strength. As before we also define $\epsilon_{\pm i} = 2\lambda_{\pm i}^{-\frac{1}{2}}$. This conformal data will be *admissible* provided that there are constants $\beta > 0$ and $\epsilon > 0$ such that, for every i,

$$|q_{+i} - q_{-i}| \geq \beta^{-1} \tag{1.5}$$

$$\beta^{-1}\epsilon \leq \epsilon_{\pm i} \leq \beta\epsilon. \tag{1.6}$$

Following [15], such a labeling of the chain $\mathcal{T}_{k+1}$ will be called an admissible conformal structure, and is denoted by σ.

We will now define using the conformal data above, a domain $\Omega \subset S^n$ which will be composed of k almost spherical regions, $\Omega_1, \ldots, \Omega_k$. For each $i \in [1, k]$, let $B_{\pm i}$ be the small ball such that

$$\partial B_{\pm i} = S_{\pm i} = \{p : |G_{\pm i}(p)| = 1\}.$$

Since we are assuming that q_{+i} and q_{-i} are sufficiently separated and that G_{+i} and G_{-i} are strong dilations, it follows that B_{+i} and B_{-i} are disjoint. Thus we may let $\mathcal{O}_i = S^n \setminus \{B_{+i} \cup B_{-i}\}$ and define

$$\Omega_i = F_i^{-1}(\mathcal{O}_i).$$

By their definitions $G_{-i+1} = G_{+i}^{-1}$, thus, since $|G^{-1\prime}(G(p))| = |G'(p)|^{-1}$ we have

$$|G'_{-i+1}(G_{+i}(p))| = |G'_{+i}(p)|^{-1}.$$

Now if we observe that $S^n \setminus B_{-i+1} = \{p : |G'_{-i+1}(p)| < 1\}$ and $B_{+i} = \{p : |G'_{+i}(p)| \geq 1\}$, then since

$$|G'_{+i}(G_{-i+1}(S^n \setminus B_{-i+1}))| = |G'_{-i+1}(S^n \setminus B_{-i+1})|^{-1},$$

we have

$$G_{-i+1}(S^n \setminus B_{-i+1}) = \overline{B}_{+i}.$$

Therefore $F_i(\Omega_{i+1}) \subset B_{+i}$ which implies that $F_i(\Omega_{i+1}) \cap \mathcal{O}_i = \emptyset$. Applying F_i^{-1} then shows that

$$\Omega_{i+1} \cap \Omega_i = \emptyset.$$

Combining this with the above equality gives us

$$\overline{\Omega}_i \cap \overline{\Omega}_{i+1} = F_i^{-1}(S_{+i}) = F_{i+1}^{1}(S_{-i+1}),$$

and moreover, we clearly have $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for any i, j such that $|i - j| \geq 2$. Thus we may define the open set $\Omega \subset S^n$ by

$$\Omega = \bigcup_{i=1}^{k} \Omega_i.$$

We now construct a metric g_1 on Ω in such a way that each (Ω_i, g_1) will be an *almost spherical region*, or asr for short. This will mean that outside of a small neighborhood of its two boundary components Ω_i will be isometric to S^n with two small disjoint disks removed. We first define a metric g_i on $\mathcal{O}_i$ which will be used to define g_1 on Ω_i. Let g_{+i} (respectively g_{-i}) denote the metric associated to the dilation G_{+i} (respectively G_{-i}) by Proposition 1.2, applied to the manifold $(M, g) = (S^n, g_0)$, with $\Phi \equiv$ the identity map on S^n. Since g_{+i} and g_{-i} both agree outside of two small disjoint balls, we can define a new metric g_i on S^n by

$$g_i = \begin{cases} g_{+i} & \text{in } B_\delta(q_{+i}) \\ g_{-i} & \text{in } B_\delta(g_{-i}) \\ g_0 & \text{in } S^n \setminus \{B_\delta(q_{+i}) \cup B_\delta(q_{-i})\} \end{cases}$$

where δ is a small radius, larger than $\epsilon_{+i}^{1-2/n}$ and $\epsilon_{-i}^{12/n}$. We then define g_1 in Ω_i by

$$g_1 = F_i^*(g_i) \text{ on } \Omega_i = F_i^{-1}(\mathcal{O}_i).$$

We must show that this definition fits together to give us a smooth metric on all of Ω. This is expressed in the compatibility conditions

$$\begin{array}{ll} F_i^*(g_i) = F_{i+1}^*(g_{i+1}) & \text{near } F_i^{-1}(S_{+i}) = F_{i+1}^{-1}(S_{-i+1}) \\ F_i^*(g_i) = F_{i-1}^*(g_{i-1}) & \text{near } F_i^{-1}(S_{-i}) = F_{i1}^{-1}(S_{+i-1}) \end{array}$$

or equivalently,

$$\begin{array}{l} g_i = G_{+i}^*(g_{i+1}) \text{ near } S_{+i} \\ g_i = G_{-i}^*(g_{i-1}) \text{ near } S_{-i}. \end{array}$$

To see that g_i extends smoothly across $\partial\overline{\Omega}_i \cap \partial\overline{\Omega}_{i+1} = F_i^{-1}(S_{+i}) = F_{i+1}^{-1}(S_{-i+1})$, recall that near S_{+i} in $\mathcal{O}_i$, we have

$$g_i = g_{+i} = 4v_{+i}^{\frac{4}{n-2}}(x) \sum_{j=1}^{n} (dx^j)^2,$$

in stereographic coordinates on $S^n \setminus q_{+i}$, where $v_{+i}(x) = \sigma_{+i}^{2-n} + |x|^{2-n}$. Since $S_{+i} = G_{+i}^{-1}(S_{-i+1})$, the metric $G_{+i}^*(g_{i+1})$ near S_{+i} in $\mathcal{O}_i$, is the same as g_{i+1} near S_{-i+1} in $\mathcal{O}_{i+1}$. Near S_{-i+1} in $\mathcal{O}_{i+1}$, we have

$$g_{i+1} = g_{-i+1} = 4v_{-i+1}^{\frac{4}{n-2}}(y) \sum_{j=1}^{n} (dy^j)^2,$$

in stereographic coordinates on $S^n \setminus q_{-i+1}$, where $v_{-i+1}(y) = \sigma_{-i+1}^{2-n} + |y|^{2-n}$. Finally, since $G_{+i} = G_{-i+1}^1$, $-q_{+i} = q_{-i+1}$ and $\sigma_{+i}^{-1} = \sigma_{-i+1}$, we see that we have two different coordinate descriptions for the same metric. The argument above clearly applies near S_{-i} as well, so we have shown that g_1 extends to a smooth metric on all of Ω, and, by construction, each Ω_i is an asr, the isometry being given by F_i.

To complete our construction of an approximate solution, we must attach (M, g) to one end of Ω, and $(S^n, \overline{g}_0)$ to the other. Recall both g and $\overline{g}_0$ are metrics whose conformal classes are nondegenerate and which are conformally flat in the neighborhood of some point. Since (M, g) is conformally flat in a neighborhood of x_0, we have a conformal map

$$\Phi : (B_r(x_0), g) \to (S^n, g_0)$$

for some $r > 0$. By applying Proposition 1.2 to (M, g) with the dilation $G = G_{-1}^{-1}$ whose source is $-q_{-1} = \Phi(x_0)$ we have a metric $\overline{g}$ on M which satisfies each of the properties $(i) - (iv)$. Let $B_0 = G_{-1}(S^n \setminus B_{-1})$ be the small ball such that $\partial B_0 = G_{-1}(S_{-1}) = S_{+0} = \{p : |G_{-1}^{-1\prime}(p)| = 1\}$. We define M_0 by

$$M_0 = M \setminus \Phi^{-1}(B_0).$$

We attach $(M_0, \overline{g})$ to (Ω, g_1) as follows. Define a new conformal transformation of S^n, F_0 by $F_0 = G_{-1} \cdot F_1$. We extend the definition of g_1 as follows. Let $g_1 = F_0^*(\overline{g})$ on $F_0^{-1}\left((S^n \setminus B_0) \cap \Phi(B_r(x_0))\right)$ and let

$$\Omega^+ = \Omega \cup F_0^{-1}\left((S^n \setminus B_0) \cap \Phi(B_r(x_0))\right).$$

From the construction of $\overline{g}$ on $(S^n \setminus B_0) \cap \Phi(B_r(x_0))$, of g_1 on Ω, we see as above, that g_1 thus defined extends smoothly across $\partial\Omega \cap F_0^1(S_{-1})$ into Ω. Moreover, the map $F_0^{-1} \cdot \Phi$ is an isometry from a neighborhood of $\partial M_0 \subset (M_0, \overline{g})$ onto a neighborhood of $\partial\Omega \cap F_0^{-1}(S_{-1}) \subset (\Omega, g_1)$. Thus we have a manifold diffeomorphic to $M \setminus B_{r_0}(x_0)$, having k asr's and which we write as

$$M_0 \cup_{F_0} \Omega.$$

To form the closed manifold (M, g_1) we attach $(S_0^n, \overline{g}_0)$ to the end of Ω represented by the terminal vertex k. This is done as above by applying Proposition 1.2 to $(S^n, \overline{g}_0)$ with the dilation $G = G_{+k}^{-1}$ whose source is $-q_{+k} = \Phi(x_0)$, for $x_0 \in S^n$. This gives us a closed manifold M, and a smooth metric g_1 on M which has k asr's and a nondegenerate spherical cap. We can represent this decomposition of (M, g_1) by writing

$$M = M_0 \cup_{F_0} \Omega \cup_{F_{k+1}} S_0^n. \tag{1.7}$$

We end this section with an observation concerning the volumes of our approximate solutions which follows immediately from the construction. This will allow us to give precise estimates for the energies of the solutions we construct. Let $V_1 = \mathrm{Vol}_g(M)$, $V_2 = \mathrm{Vol}_{\overline{g}_0}(S^n)$ and $\omega_n = \mathrm{Vol}_{g_0}(S^n)$, then for some $c = c(n, g, \overline{g}_0)$

$$\left|\mathrm{Vol}_{g_{1_k}}(M) - (V_1 + V_2 + k\omega_n)\right| \leq c\epsilon^{n-2}. \tag{1.8}$$

1.6. A Conformal Class With N Approximate Solutions. The construction of N distinct approximate solutions $(g_{1_1}, \dots, g_{1_N})$, having $1, \dots, N$ asr's respectively, is based upon the fact that an annular region

$$A = \{x : 1 < |x| < R\}$$

can be decomposed into k subannuli, for each $k = 1, \dots, N$, so that each subannulus has conformal modulus $R^{1/k}$. The conformal modulus, $cm(A)$, of an annulus is defined to be the ratio of the outer radius to the inner radius. Two annuli are conformally equivalent if and only if they have the same conformal modulus. The construction of the N approximate solutions will be done using appropriate powers of a fixed centered dilation G. The powers will be chosen so that each of the subdomains, Ω_i^k corresponding to the decomposition of a fixed domain $\Omega \subset S^n$ into k subdomains, as done above, will have conformal modulus (in stereographic coordinates)

$$cm(\pi^{-1}(\Omega_i^k)) = R^{1/k}$$

where $R = cm(\pi^{-1}(\Omega))$. Note that the metrics g_{1_k} constructed from these conformal structures will be spherically symmetric on $\pi^{-1}(\Omega)$.

We fix $R >> 1$ and let G be a centered dilation with source $q \in S^n$ and strength $\lambda = |G'(q)| = R$. For any k, $1 \leq k \leq N$, we let for each i, $1 \leq i \leq k$,

$$\begin{aligned} G_{-i} &= G^{1/k} \\ G_{+1} &= G^{-1/k}. \end{aligned}$$

If k is even, we set $F_i = G^{\frac{k-2i+1}{2k}}$ for $1 \leq i \leq k/2$, and $F_i = F_{k-i+1}^{-1}$ for $1 + k/2 \leq i \leq k$. If k is odd we set $F_i = G^{\frac{k-2i+1}{2k}}$ for $1 \leq i \leq \frac{k-1}{2}$, $F_i = F_{k-i+1}^{-1}$ for $\frac{k+3}{2} \leq i \leq k$ and $F_{\frac{k+1}{2}} =$ the identity. It is easily checked that this then defines a conformal structure which satisfies

$$cm(\pi^{-1}(\Omega_i^k)) = R^{1/k}$$

for each i, $1 \leq i \leq k$.

Let g_{1_k} denote the metric constructed canonically from the conformal structure above. Each (Ω_i^k, g_{1_k}) is conformally equivalent to a Euclidean annulus. We define $cm(\Omega_i^k, g_{1_k})$ to be the conformal modulus of this Euclidean annulus.

It is immediate from the construction of g_{1_k} that for $1 \le i, j \le k$

$$(\Omega_i^k, g_{1_k}) \cong (\Omega_j^k, g_{1_k}),$$

where $\cong$ denotes conformal equivalence. Finally the fact that for each k,

$$cm(\pi^1(\Omega)) = \prod_{i=1}^{k} cm(\pi^{-1}\Omega_i^k) = \prod_{i=1}^{k} R^{1/k} = R,$$

allows us to conclude that

$$cm(\Omega, g_{1_k}) = \prod_{i=1}^{k} cm(\Omega_i^k, g_{1_k}).$$

and that the right hand side is independent of k. Thus each (Ω, g_{1_k}) has the same conformal modulus, independent of the metric g_{1_k}. Extending these metrics to approximate solutions on all of M as in (1.7) we easily see that $g_{1_1}, \dots, g_{1_k}$ all lie in the same conformal class. Any collection of admissible conformal structures $\sigma_1, \dots, \sigma_N$ on chains $\mathcal{T}_1, \dots, \mathcal{T}_N$ for which the metrics $g_{1_1}, \dots, g_{1_k}$ all lie within the same conformal class, will be called an *admissible N structure* and will be denoted by $\sigma(N)$. Note that each g_{1_k} is uniquely determined from $(\mathcal{T}_k, \sigma_k)$ by the construction given above.

We close this section with a summary of our construction.

Theorem 1.3. *Given an integer $N \ge 1$ and metrics g on M, and $\overline{g}_0$ on S^n, such that $R(g) = R(\overline{g}_0) = n(n-1)$, there exists a constant c depending only on $n, \beta, g, \overline{g}_0$ and a admissible N structure $\sigma(N)$ with metrics $g_{1_1}, \dots, g_{1_N}$ such that, for each k*

$$\begin{aligned} \max\{x \in M : |R(g_{1_k})|(x)\} &\le c \\ Vol\{x \in M : R(g_{1_k})(x) \ne n(n-1)\} &\le c\epsilon^{n-2} \\ |\, Vol_{g_{1_k}}(M) - (V_1 + V_2 + k\omega_n)| &\le c\epsilon^{n-2}. \end{aligned}$$

Moreover, (M, g_{1_k}) has a decomposition

$$M = M_0 \cup_{F_0} \Omega^k \cup_{F_{k+1}} S_0^n$$

where g_{1_k} has the explicit description in Ω^k given above.

2. The Linear Analysis

2.1. The Linearized Equation. Associated to each admissible conformal structure σ on $\mathcal{T}_k$, there is a uniquely determined approximate solution, g_{1_k}, constructed in section 2. We refer to $(\mathcal{T}_k, \sigma)$, or just σ as an approximate solution. We define $[\sigma]$ to be the conformal class of any metric g_{1_k} constructed from an admissible N-structure, $\sigma(N)$. We work with an approximate solution with k asr's, for any $k \in [1, N]$. Since k will be fixed for much of our study, we write g_1 for g_{1_k}, omitting the subscript wherever convenient.

We want to find $g \in [\sigma]$ which satisfy $R(g) = n(n-1)$ and are close to g_1. With this in mind we write

$$g = (1+\eta)^{\frac{4}{n-2}} g_1,$$

regarding η as a small perturbation of the approximate solution 1. Equation (1.1) can be written as

$$\mathcal{L}\eta = \frac{n-2}{4(n-1)} \left(R_1 - n(n-1)\right)(1+\eta) + Q(\eta) \quad = F(x, \eta), \tag{2.1}$$

where $R_1 = R(g_1)$, $\mathcal{L} = \Delta_{g_1} + n$ is the linear operator appearing in (1.2), and

$$Q(\eta) = \frac{n(n-2)}{4} \left(1 + \frac{n+2}{n-2}\eta - (1+\eta)^{\frac{n+2}{n2}}\right). \tag{2.2}$$

Since we are interested in small η, $Q(\eta)$ is quadratically small.

In this section we show that we can find a k(n+1) dimensional small eigenspace K, consisting of all the $L^2(M)$ eigenfunctions of $\mathcal{L}$ with appropriately small eigenvalues. K will be very close, in $L^2(M)$, to an approximate kernel K_0, which we explicitly construct. We then find a bounded inverse of $\mathcal{L}$ off of K and show that we can uniquely solve the linear equation $\mathcal{L}\eta = f$, provided that f and η are orthogonal to K, and obtain precise ϵ-independent estimates on the solutions. The closeness of K to the explicit space K_0 is part of what will allow us to identify the component of $F(x, \eta)$ lying in K. Inverting $\mathcal{L}$ off of some abstract subspace without having such explicit control would not help us to solve the nonlinear problem.

The idea behind the identification of the small eigenspace is that for ϵ sufficiently small, the annular region Ω^k, composed of k asr's, behaves spectrally like the disjoint union of k standard spheres. As has been pointed out, the kernel of $\mathcal{L}$ can be identified explicitly on S^n as the span of the linear coordinate functions. It is from these functions that we construct our approximate kernel by carefully cutting off the coordinate functions on each $\mathcal{O}_i$ and using F_i to pull these functions back to Ω_i. The assumption that the metrics g and $\overline{g}_0$ on the ends, M_0 and S^n_0 of Ω, are nondegenerate insures that these ends do not support an approximate kernel for $\mathcal{L}$, in $L^2(M)$.

2.2. The Approximate Kernel. We shall construct our approximate kernel by cutting off the coordinate functions in the $k+1$ neck regions, which join the asr's Ω^k_i to each other, and to the ends M_0 and S^n_0. To do this we use a smoothed out harmonic cutoff function, which we find by first solving a Dirichlet problem on each neck region.

For each vertex $i \in [2, k-1]$ there are two neck regions between Ω_i and its two adjacent asr's. Fix δ to be a small radius which is substantially larger than ϵ, and define

$$\begin{aligned} N^{(\delta)}_{+i} &= F_i^{-1}(B_\delta(q_{+i})) \cap F_{i+1}^{-1}(B_\delta(q_{i+1})) \\ N^{(\delta)}_{-i} &= F_i^{-1}(B_\delta(q_{-i})) \cap F_{i-1}^{-1}(B_\delta(q_{+i1})). \end{aligned}$$

Note that $N^{(\delta)}_{+i} = N^{(\delta)}_{-i+1}$, and $N^{(\delta)}_{-i} = N^{(\delta)}_{+i-1}$. For the terminal vertices 1 and k, $N^{(\delta)}_{+1}$ and $N^{(\delta)}_{-k}$ are defined as above, and the terminal neck regions $N^{(\delta)}_{-1}$ and $N^{(\delta)}_{+k}$ are defined by

$$\begin{aligned} N^{(\delta)}_{-1} &= F_1^{-1}(B_\delta(q_{-1})) \cap F_0^{-1}(B_\delta(-q_1)) \\ N^{(\delta)}_{+k} &= F_k^{-1}(B_\delta(q_{+k})) \cap F_{k+1}^{-1}(B_\delta(-q_{+k})), \end{aligned}$$

The fact which distinguishes these terminal neck regions from the other $k-1$ neck regions, is that near the boundary component not contained in Ω, the metric g_1 is not the standard spherical metric g_0 as it is near both boundary components of the other neck regions. Here g_1 is isometric to one of the conformally flat metrics g and $\overline{g}_0$, respectively. We also note that for each $i \in [1, k]$

$$\Omega_i^{(-\delta)} = \Omega_i \setminus \{N^{(\delta)}_{-i} \cup N^{(\delta)}_{+i}\}$$

with the metric g_1, is isometric to the standard sphere with two balls removed

$$\mathcal{O}_i^{(-\delta)} = (S^n \setminus \{B_\delta(q-i) \cup B_\delta(q+i)\}, g_0),$$

provided that $\delta > \epsilon_{\pm i}^{1-\frac{2}{n}}$. Similarly,

$$M_0^{(-\delta)} = M_0 \setminus N_{-1}^{(\delta)} \quad \text{and} \quad S_0^{n(\delta)} = S_0^n \setminus N_{+k}^{(\delta)}$$

with the metric g_1, are isometric to

$$\left(M \setminus \Phi^{-1}\left(B_\delta(-q_{-1})\right), g\right) \quad \text{and} \quad \left(S^n \setminus \Phi^{-1}\left(B_\delta(-q_{+k})\right), \overline{g}_0\right),$$

respectively.

Fix a vertex i, $1 \le i \le k$ and a neck region, say $N_{+i}^{(\delta)}$. Let h_{+i} be the solution of the following Dirichlet problem on $N_{+i}^{(\delta)}$.

$$\begin{aligned} \Delta_{g_1} h_{+i} &= 0 \quad \text{on} \quad N_{+i}^{(\delta)} \\ h_{+i} &= 1 \quad \text{on} \quad F_i^{-1}(\partial B_\delta(q_{+i})) \\ h_{+i} &= 0 \quad \text{on} \quad F_{i+1}^{-1}(\partial B_\delta(q_{-i+1})). \end{aligned} \tag{2.3}$$

We choose an orthonormal basis for $\mathbb{R}^{n+1}$, $\omega_1, \dots, \omega_{n+1}$, and let q^α denote the α coordinate function on S^n with respect to this basis. Thus for any point $q \in S^n$, we write $q = \sum_{\alpha=1}^{n+1} q^\alpha \omega_\alpha$. For each $i \in [1, k]$, and $\alpha \in [1, n+1]$ we define functions q_i^α on S^n by,

$$q_i^\alpha(q) = \begin{cases} q^\alpha + \Psi_\delta(\rho(q, q_{+i}))(q_{+i}^\alpha - q^\alpha) + \Psi_\delta(\rho(q, q_{-i}))(q_{-i}^\alpha - q^\alpha) \\ \qquad\qquad \text{for} \quad q \in S^n \setminus \{B_\delta(q_{+i}) \cup B_\delta(q_{-i})\} \\ q_{+i}^\alpha \cdot \overline{h}_{+i} \cdot F_i^{-1} \quad \text{for} \quad q \in B_\delta(q_{+i}) \\ q_{-i}^\alpha \cdot \overline{h}_{-i} \cdot F_i^{-1} \quad \text{for} \quad q \in B_\delta(q_{-i}) \end{cases} \tag{2.4}$$

Here $\rho(\cdot, \cdot)$ is the distance function on (S^n, g_0), and $\overline{h}_{+i}$ is the function h_{+i}, smoothed out to be globally defined as follows

$$\overline{h}_{+i} = \begin{cases} 1 & \text{on} \quad A_{+i} = F_i^{-1}(B_\delta(q_{+i}) \setminus B_{\frac{3\delta}{4}}(q_{+i})) \\ (h_{+i} - 1)(\Psi_{+i} \cdot F_i) + 1 & \text{on} \quad B_{+i} = F_i^{-1}(B_{\frac{3\delta}{4}}(q_{+i}) \setminus B_{\frac{\delta}{2}}(q_{+i})) \\ h_{+i} & \text{on} \quad C_{+i} = F_i^{-1}(B_{\frac{\delta}{2}}(q_{+i})) \cap F_{i+1}^{-1}(B_{\frac{\delta}{2}}(q_{-i+1})) \\ h_{+i}(\Psi_{-i+1} \cdot F_{i+1}) & \text{on} \quad B_{-i+1} = F_{i+1}^{-1}(B_{\frac{3\delta}{4}}(q_{-i+1}) \setminus B_{\frac{\delta}{2}}(q_{-i+1})) \\ 0 & \text{on} \quad A_{-i+1} = F_{i+1}^{-1}(B_\delta(q_{-i+1}) \setminus B_{\frac{3\delta}{4}}(q_{-i+1})) \end{cases}$$

Here $\Psi_{+i} = \Psi(4\rho(\cdot, q_{+i}) - \delta)$ and $\Psi_{-i+1} = \Psi(4\rho(\cdot, q_{-i+1}) - \delta)$. The function $\overline{h}_{-1}$ is defined similarly about the neck region $N_{-i}^{(\delta)}$. The approximate kernel K_0 is formed from the functions q_i^α by defining for $1 \le i \le k$ and $1 \le \alpha \le n+1$

$$\eta_i^\alpha = q_i^\alpha \cdot F_i$$

and then letting

$$K_0 = \text{ the linear span of the } \eta_i^\alpha \text{ in } L^2(M).$$

Note that for each i, $\eta_i^\alpha \in C^\infty(M)$ and has support in $\Omega_i^{(+\delta)}$.

In establishing pointwise and L^2 bounds for $\mathcal{L}\eta_i^\alpha$ we need to have strong estimates on h_{+i} and h_{-i} and their derivatives near the boundaries of $N_{+i}^{(\delta)}$ and $N_{-i}^{(\delta)}$ respectively. This is expressed by the following Lemma.

Lemma 2.1. *The functions h_{+i} (and similarly h_{-i}) satisfy the bounds*

$$\begin{aligned}\sup\{(1-h_{+i}(x))+\delta|\nabla h_{+i}(x)| : x\in F_i^{-1}(B_\delta(q_{+i})\setminus B_{\frac{\delta}{2}}(q_{+i}))\} &\leq c(\epsilon/\delta)^{n-2}\\ \sup\{h_{+i}(x)+\delta|\nabla h_{+i}(x)| : x\in F_{i+1}^{-1}(B_\delta(q_{-i+1})\setminus B_{\frac{\delta}{2}}(q_{-i+1}))\} &\leq c(\epsilon/\delta)^{n-2}.\end{aligned}$$

We refer to [15, Lemma 3.1] for the proof, these bounds are established there when $N_{+i}^{(\delta)}$ is a neck region separating two asr's. In the case of the extreme neck regions, e.g. $N_{+k}^{(\delta)} = F_k^{-1}\left(B_\delta(q_{+k})\cap G_{+k}^{-1}\left(B_\delta(-q_{+k})\right)\right)$, we have an explicit description of g_1 in stereographic coordinates $x=(x^1,\dots,x^n)$ on $S^n\setminus -q_{+k}$. In these coordinates $g_1 = 4v_1^{\frac{4}{n-2}}\sum_{i=1}^n (dx^i)^2$, and the region above is contained in $\{x : R\leq |x|\leq \sigma_2\}$, where $\sigma=\lambda_{+1}^{1/2}$ and σ_2 is defined as in Proposition 1.2. Thus, the conformal factor takes the form of (1.3). A partial description of $v_1(x)$ on a larger set is then given by,

$$v_1(x)=\begin{cases}|x|^{2-n}(a_0+h(x))^{\frac{2-n}{2}} & \text{for}\quad R\leq |x|\leq\sigma_2\\ \sigma^{2-n}+(a_0^{1/2}|x|)^{2-n} & \text{for}\quad 4\sigma_2\leq|x|\leq\sigma_1\end{cases}$$

where $h(x)=O(|x|^{-2})$. Consider the Kelvin transform $y=K(x)=\frac{x}{|x|^2}$, if we now write $g_1=4w_1(y)\sum_{i=1}^n(dy^i)^2$, then $w_1(y)$ is partially given by

$$w_1(y)=\begin{cases}\sigma^{2-n}|y|^{2-n}+a_0^{\frac{2-n}{2}} & \text{for}\quad \sigma_1^{-1}\leq|y|\leq\frac{1}{4}\sigma_2^{-1}\\ (a_0+k(y))^{\frac{2-n}{2}} & \text{for}\quad \sigma_2^{-1}\leq|y|\leq R^{-1}.\end{cases}$$

where $k(y)=O(|y|^2)$. It then follows that $w_1(y)$ satisfies

$$\begin{aligned}&|w_1(y)-a_0^{\frac{2-n}{2}}|\leq c(|y|^2+\epsilon^{n-2}|y|^{2-n}),\\ &\sum_{i=1}^n\left|\frac{\partial w_1}{\partial y^i}(y)\right|\leq c(|y|+\epsilon^{n-2}|y|^{1-n}),\end{aligned}\tag{2.5}$$

for y such that $\epsilon\leq|y|\leq R^{-1}$, where $\epsilon=2\lambda_{+k}^{1/2}$. The proof of Lemma 2.1 then follows precisely as in [15, Lemma 3.1]. Lemma 2.1 allows us to prove the relevant pointwise and L^2 estimates on K_0, in exactly the same manner as [15, Lemma 3.2]. We refer there for the proof of

Proposition 2.2. *The functions* η_i^α *satisfy the bound*

$$\sup_M(\delta|\mathcal{L}\eta_i^\alpha| + \epsilon|\nabla\eta_i^\alpha|) \leq c,$$

and any $\eta \in K_0$ *satisfies*

$$\|\mathcal{L}\eta\|_{L^2(M)} \leq c\delta^{\frac{n-2}{2}}\|\eta\|_{L^2(M)}.$$

2.3. Conformally Invariant Sobolev Inequality. It is necessary for us to consider approximate solutions constructed from arbitrarily strong dilations, or equivalently, we must allow ϵ to be arbitrarily small. This fact makes the analysis of (2.1) difficult because the geometry of the manifold (M, g_{1_k}) degenerates as $\epsilon \to 0$. In particular, the injectivity radius of the neck region $N_{+i}^{(\delta)}$ tends to zero as the strength of the dilation, G_{+i}, tends to infinity. This difficulty is overcome by exploiting the fact that

$$\Omega^{(+\delta)} = N_{-1}^{(\delta)} \cup \Omega \cup N_{+k}^{(\delta)}$$

is conformally equivalent to a subdomain of S^n, and has scalar curvature which is bounded independent of ϵ. It is this property of our approximate solutions which allows us to use the following Sobolev inequality.

Lemma 2.3. *If* (M, h) *has bounded scalar curvature,* $|R(h)| \leq c_1$, *and is conformally equivalent to a subdomain of a compact manifold* (N, g), *with positive scalar curvature, then for any* $\phi \in C_c^\infty(M)$, *the inequality*

$$\left(\int_M \phi^{\frac{2n}{n-2}} dv_h\right)^{\frac{n-2}{n}} \leq c\int_M \left(|\nabla\phi|^2 + \phi^2\right) dv_h$$

holds for $c = c(c_1, I(g))$, *where* $I(g) > 0$ *is the Yamabe invariant of* (N, g).

Proof. Recall that the definition of the Yamabe invariant given in (0.4) is equivalent to

$$I(g) = \inf_{\phi \in C^\infty(N)} \frac{-\int \phi L_g \phi dv_g}{\left(\int \phi^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}} \tag{2.6}$$

where L_g is the conformal Laplacian taken with respect to g and the integrals are taken over all of N. Let

$$\Phi : U \to M$$

be the conformal diffeomorphism of a subdomain U of N onto M, so that

$$\Phi^*(h) = u^{\frac{4}{n-2}} g = \overline{g}$$

for some $u \in C^\infty(U)$. By the conformal invariance of L_g, we have

$$L_{\overline{g}}(\psi) = u^{-\frac{n+2}{n-2}} L_g(\psi u),$$

for any $\psi \in C_c^\infty(U)$. Thus given any $\phi \in C_c^\infty(M)$, $\psi = \phi \cdot \Phi \in C_c^\infty(U)$ so we have

$$I(g) \leq \frac{-\int (\psi u) L_g(\psi u) dv_g}{\left(\int (\psi u)^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}} = \frac{-\int \psi L_{\overline{g}}(\psi) dv_{\overline{g}}}{\left(\int \psi^{\frac{2n}{n-2}} dv_{\overline{g}}\right)^{\frac{n-2}{n}}},$$

where the integrals are taken over U. After a change of variables this implies that

$$\left(\int_M \phi^{\frac{2n}{n-2}} dv_h\right)^{\frac{n-2}{n}} \leq \frac{1}{I(g)} \int_M \left(|\nabla\phi|^2 + c(n)R(h)\phi^2\right) dv_h.$$

Choosing $c = \max(1, c(n)c_1)$ then completes the proof of Lemma 2.3. □

2.4. $L^2(M)$ Estimates for $K_0^\perp$. To prove the existence of a small eigenspace K close to K_0, we need prove estimates for functions $\eta \in K_0^\perp$, which indicate that $\mathcal{L}$ is bounded below in $K_0^\perp \subset L^2(M)$, in the operator norm on $L^2(M)$. Here $K_0^\perp$ denotes the orthogonal complement of K_0 in $L^2(M)$. We define the orthogonal projection operators p_0 and q_0 by

$$\begin{aligned} p_0 &: L^2(M) \to K_0 \\ q_0 &: L^2(M) \to K_0^\perp. \end{aligned}$$

In addition, for $1 \leq i \leq k$, we define $K_0^{(i)}$ to be the linear span of η_i^α for $\alpha = 1, \ldots, n+1$, and let $p_0^{(i)}$ and $q_0^{(i)}$ denote the orthogonal projections of $L^2(M)$ onto $K_0^{(i)}$ and $K_0^{(i)^\perp}$, respectively. For $2 \leq i \leq k-1$, we define

$$\widehat{\Omega}_i = \Omega_{i-1} \cup \Omega_i \cup \Omega_{i+1}.$$

For the terminal vertices, and the ends M_0 and S_0^n, defined in the decomposition (1.7) we define

$$\begin{aligned} \widehat{\Omega}_1 &= M_0 \cup \Omega_1 \cup \Omega_2, & \widehat{\Omega}_k &= \Omega_{k-1} \cup \Omega_k \cup S_0^n, \\ \widehat{M}_0 &= M_0 \cup \Omega_1, & \widehat{S}_0^n &= S_0^n \cup \Omega_k. \end{aligned}$$

The following lemma then holds.

Lemma 2.4. *Suppose $\eta \in K_0^{(i)^\perp} \cap C^\infty(M)$, then the estimate*

$$\|\eta\|_{L^2(\Omega_i)} \leq \|\mathcal{L}\eta\|_{L^2(\widehat{\Omega}_i)} + c\,(\log(1/\epsilon))^{\frac{(1-n)}{n}} \|\eta\|_{L^2(\widehat{\Omega}_i)}$$

holds. Moreover, for $\eta \in C^\infty(M)$ we have the following estimates on the ends, M_0 and S_0^n of M,

$$\|\eta\|_{L^2(M_0)} \leq \|\mathcal{L}\eta\|_{L^2(\widehat{M}_0)} + c\,(\log(1/\epsilon))^{\frac{(1-n)}{n}} \|\eta\|_{L^2(\widehat{M}_0)}$$
$$\|\eta\|_{L^2(S_0^n)} \leq \|\mathcal{L}\eta\|_{L^2(\widehat{S}_0^n)} + c\,(\log(1/\epsilon))^{\frac{(1-n)}{n}} \|\eta\|_{L^2(\widehat{S}_0^n)}$$

Proof. This is virtually identical to [15, Lemma 3.4], the distinction being that to derive the estimates on the ends we must obtain the inequality

$$(2.7) \quad \int_M (\Psi_{M_0}\eta)^2 dv_1 \leq c(\overline{\eta}_1)^2 + \int_M \left(|\nabla(\Psi_{M_0}\eta)|^2 - n(\Psi_{M_0}\eta)^2\right) dv_1,$$

where $\eta \in C^\infty(M)$, and Ψ_{M_0} is a cut off function whose support is contained in $M_0^{(-\delta_1)}$, and $\overline{\eta}_1$ denotes the average of $\Psi_{M_0}\eta$ over (M, g). On the asr's this is identical to [15, (3.6)].

From inequality (2.7) the derivation of the estimates in Lemma 2.4 is identical to the argument given in [15, 356- 359]. The inequality for the function $\Psi_{S_0^n}\eta$ is derived in exactly the same manner. Since $\Psi_{M_0}\eta$ is supported in $M_0^{(-\delta_1)}$, where g_1 is isometric to the nondegenerate metric g, we may use spectral information on (M, g) instead of (S^n, g_0).

We obtain the estimate (2.7) as follows. Since

$$\overline{\eta}_1 = Vol_g(M)^{-1} \int_M \Psi_{M_0}\eta dv_1,$$

the variational characterization of λ_1 allows us to conclude that

$$\lambda_1 \int_M (\Psi_{M_0}\eta - \overline{\eta}_1)^2 dv_g \leq \int_M |\nabla(\Psi_{M_0}\eta)|^2 dv_g.$$

Since g is a minimal, nondegenerate solution we have $\lambda_1 > n$, this then gives us

$$\int_M (\Psi_{M_0}\eta - \overline{\eta}_1)^2 dv_g \leq (\lambda_1 - n)^{-1} \int_M \left(|\nabla(\Psi_{M_0}\eta)|^2 - n(\Psi_{M_0}\eta - \overline{\eta}_1)^2\right) dv_g$$

From this inequality we immediately derive (2.7) for the function $\Psi_{M_0}\eta$. The proof then proceeds as in [15, 356-359]. We refer to there for the remainder of the proof. □

Lemma 2.4 easily gives the following Corollary which is fundamental to the linear analysis of (2.1).

Corollary 2.5. *Suppose $\eta \in K_0^\perp \cap C^\infty(M)$. We then have the bound*

$$\|\eta\|_{L^2(M)} \leq c\|\mathcal{L}\eta\|_{L^2(M)}$$

provided ϵ is small enough.

Proof. By applying Lemma 2.4 over each component of the decomposition

$$M = M_0 \cup \Omega \cup S_0^n, \qquad \Omega = \cup_{i=1}^k \Omega_i,$$

we have

$$\begin{aligned} \int_M \eta^2 dv_1 &= \int_{M_0} \eta^2 dv_1 + \sum_{i=1}^k \int_{\Omega_i} \eta^2 dv_1 + \int_{S_0^n} \eta^2 dv_1 \\ &\leq c \int_M (\mathcal{L}\eta)^2 dv_1 + c \log(1/\epsilon)^{\frac{2(1-n)}{n}} \int_M \eta^2 dv_1. \end{aligned}$$

Choosing ϵ sufficiently small then completes the proof. □

2.5. The Small Eigenspace K. If ϕ is an eigenfunction of $\mathcal{L}$ with eigenvalue λ, then ϕ is also an eigenfunction for Δ_{g_1} with eigenvalue $\lambda + n$. In particular, there is a one-to-one correspondence between the spectum of Δ_{g_1} on M, that is, the set of eigenvalues for Δ_{g_1}, and the spectrum of $\mathcal{L}$ on M, which we denote by $\sigma(\mathcal{L})$. A consequence of the basic spectral theory for the self adjoint operator Δ_{g_1} on $L^2(M)$ (see [2]), is that $\sigma(\mathcal{L}) = \{\lambda_i\}_{i=0}^\infty$ is a discrete set tending to $+\infty$,

$$\lambda_0 = -n < \lambda_1 \leq \lambda_2 \leq \cdots,$$

and the corresponding eigenfunctions $\phi_i \subset L^2(M) \cap C^\infty(M)$ (normalized so that $\|\phi_i\|_{L^2} = 1$), form an orthonormal basis for $L^2(M)$. It follows that for any $\eta \in L^2(M)$ we can write $\eta = \sum_{i=0}^\infty a_i \phi_i$ and $\mathcal{L}\eta = -\sum_{i=0}^\infty a_i \lambda_i \phi_i$. When $\eta \in K_0$, the inequality $\|\mathcal{L}\eta\|_{L^2(M)} \leq c_{K_0} \delta^{\frac{n-2}{2}} \|\eta\|_{L^2(M)}$ of Proposition 2.2 is then equivalent to

$$\sum_{i=0}^\infty a_i^2 (\lambda_i^2 - c_{K_0}^2 \delta^{n-2}) \;\; \leq \;\; 0$$

The corresponding inequality $\|\eta\|_{L^2(M)} \le c_{K_0^\perp}\|\mathcal{L}\eta\|_{L^2(M)}$ of Corollary 2.5 for $\eta \in K_0^\perp$ is equivalent to

$$\sum_{i=0}^{\infty} a_i^2(1 - c_{K_0^\perp}^2 \lambda_i^2) \quad \le \quad 0.$$

We define the subspace $K \subset L^2(M)$ of small eigenfunctions by

$$K \quad = \quad \mathrm{span}\{\phi_i : |\lambda_i| \le c_{K_0}\delta^{\frac{n-2}{2}}\}. \tag{2.8}$$

If there were an eigenvalue λ_j such that $\lambda_j > c_{K_0}\delta^{\frac{n-2}{2}}$ which also satisfied $\lambda_j < 1/c_{K_0^\perp}$ then the corresponding eigenfunction, ϕ_j would belong to neither K_0 nor $K_0^\perp$, since each of the necessary inequalities would be violated. Since $L^2(M) = K_0 \oplus K_0^\perp$ no such eigenvalue can exist.

The above consequences of Proposition 2.2 and Corollary 2.5, indicate that the closed subspace $K \subset L^2(M) \cap C^\infty(M)$ is invariant with respect to the operator $\mathcal{L}$, i.e. $\mathcal{L} : K \to K$, and has the properties

$$\begin{array}{llll} \|\mathcal{L}\eta\|_{L^2(M)} & \le & c\delta^{\frac{n-2}{2}}\|\eta\|_{L^2(M)} & \text{for} \quad \eta \in K, \\ \|\eta\|_{L^2(M)} & \le & c\|\mathcal{L}\eta\|_{L^2(M)} & \text{for} \quad \eta \in K^\perp \cap C^\infty(M). \end{array} \tag{2.9}$$

Let

$$\begin{array}{rcl} \mathbf{p} : L^2(M) & \to & K \\ \mathbf{q} : L^2(M) & \to & K^\perp \end{array}$$

be the orthogonal projection operators. The following Lemma is analogous to [15, Lemma 3.6], and reflects the extent to which we can use the explicit nature of K_0 to control the behavior of K.

Lemma 2.6. *For any $\eta \in L^2(M)$,*

$$\|\mathbf{p}(\eta) - \mathbf{p}_0(\eta)\|_{L^2(M)} = \|\mathbf{q}(\eta) - \mathbf{q}_0(\eta)\|_{L^2(M)} \le c\delta^{\frac{n-2}{2}}\|\eta\|_{L^2(M)}.$$

There is a basis ϕ_i^α, $i = 1, \dots, k$, $\alpha = 1, \dots, n+1$, for K satisfying for $i \neq j$

$$\begin{aligned} \|\phi_i^\alpha - \eta_i^\alpha\|_{L^2(M)} &\leq c\delta^{\frac{n-2}{2}}, \\ \|\phi_i^\alpha\|_{L^2(M_0^{(+\delta)})} &\leq c\delta^{\frac{n-2}{2}}, \\ \|\phi_i^\alpha\|_{L^2(\Omega_j^{(+\delta)})} &\leq c\delta^{\frac{n-2}{2}}, \\ \|\phi_i^\alpha\|_{L^2(S_0^n{}^{(+\delta)})} &\leq c\delta^{\frac{n-2}{2}}, \\ \left|\langle \phi_i^\alpha, \phi_j^\beta \rangle_{L^2(M)}\right| &\leq c\delta^{\frac{n-2}{2}}. \end{aligned}$$

Remark 2.1. In [15, Lemma 3.6], Schoen establishes this result by proving exponential decay in $D(i,j)$, the graph distance between the vertices i and j, for the latter two quantities. This more difficult result is necessary, in part, due to the exponential volume growth that Schoen's approximate solutions may exhibit. Since our approximate solutions are compact and have volumes which may be estimated as in Theorem 1.3, the estimates which we prove here are sufficient. We refer to [15] for the proof of the $L^2(M)$ estimates on the difference of the projection operators, and here establish the estimates on the basis $\{\phi_i^\alpha\}$ for K.

Proof. We first define the basis element ϕ_i^α to be the component of the orthogonal projection of η_i^α into K which arises solely from η_i^α. Explicitly, for each $i = 1, \dots, k$ let $\hat{K}_i$ denote the linear span of $\mathbf{p}(\eta_j^\alpha)$, $j \neq i$, $\alpha = 1, \dots, n+1$. We define ϕ_i^α by

$$\phi_i^\alpha = \mathbf{p}(\eta_i^\alpha) - \hat{\mathbf{p}}^{(i)}(\eta_i^\alpha),$$

where $\hat{\mathbf{p}}^{(i)} : L^2(M) \to \hat{K}_i$ is the orthogonal projection operator. If $\phi = \sum_{j\neq i,\beta} a_j^\beta \mathbf{p}(\eta_j^\beta) \in \hat{K}_i$ with $\|\phi\|_{L^2(M)} = 1$, then $\phi = \mathbf{p}(\eta)$, where $\eta = \sum_{j\neq i,\beta} a_j^\beta \eta_j^\beta \in K_0$. We then have

$$\|\phi - \eta\|_{L^2(M)} \leq c\delta^{\frac{n-2}{2}} \|\eta\|_{L^2(M)},$$

and hence

$$\|\eta\|_{L^2(M)} \leq c.$$

Since $\eta = \mathbf{p}_0(\eta)$ we may write

$$\langle \eta_i^\alpha, \phi \rangle_{L^2(M)} = \langle \eta_i^\alpha, \eta \rangle_{L^2(M)} + \langle \eta_i^\alpha, \mathbf{p}(\eta) - \mathbf{p}_0(\eta) \rangle_{L^2(M)}.$$

Therefore, we have

$$|\langle \eta_i^\alpha, \phi\rangle_{L^2(M)}| \leq \left|\sum_{\beta=1}^{n+1}\left(a_{i-1}^\beta\langle \eta_i^\alpha, \eta_{i1}^\beta\rangle_{L^2(M)} + a_{i+1}^\beta\langle \eta_i^\alpha, \eta_{i+1}^\beta\rangle_{L^2(M)}\right)\right| + c\delta^{\frac{n-2}{2}}.$$

By the definition of the functions η_i^α we immediately have

$$\begin{aligned}\left|\sum_{\beta=1}^{n+1}\left(a_{i-1}^\beta\langle \eta_i^\alpha, \eta_{i1}^\beta\rangle_{L^2(M)} + a_{i+1}^\beta\langle \eta_i^\alpha, \eta_{i+1}^\beta\rangle_{L^2(M)}\right)\right| &\leq c\delta^n\sum_{\beta=1}^{n+1}(|a_{i+1}^\beta| + |a_{i-1}^\beta|)\\ &\leq c\delta^n\sum_{\beta=1}^{n+1}(|a_{i+1}^\beta|^2 + |a_{i-1}^\beta|^2)^{\frac{1}{2}}\end{aligned}$$

On the other hand, since

$$\begin{aligned}\sum_{\beta,\gamma,j\neq i}|a_j^\beta|(|a_{j-1}^\gamma| + |a_{j+1}^\gamma|) &\leq c\sum_{\beta,j}(a_j^\beta)^2 + \sum_{\beta,\gamma,j}((a_{j-1}^\gamma)^2 + (a_{j+1}^\gamma)^2)\\ &\leq c\sum_{\beta,j}(a_j^\beta)^2,\end{aligned}$$

we can estimate

$$\begin{aligned}\langle \eta, \eta\rangle_{L^2(M)} &= \sum a_j^\beta\langle \eta_j^\beta, \sum a_l^\gamma\eta_l^\gamma\rangle\\ &\geq c\sum(a_j^\beta)^2 - c\delta^n\sum_{\beta,\gamma,j\neq i}|a_j^\beta|(|a_{j-1}^\gamma| + |a_{j+1}^\gamma|)\\ &\geq c\sum(a_j^\beta)^2.\end{aligned}$$

Thus using this estimate above and the $L^2(M)$ bound on η, we have

$$|\langle \eta_\alpha^i, \phi\rangle| \leq c\delta^{\frac{n-2}{2}}.$$

If we let

$$\phi = \frac{\hat{\mathbf{p}}^{(i)}(\eta_i^\alpha)}{\|\hat{\mathbf{p}}^{(i)}(\eta_i^\alpha)\|_{L^2(M)}},$$

then it follows that

$$\|\hat{\mathbf{p}}^{(i)}(\eta_i^\alpha)\|_{L^2(M)} \leq c\delta^{\frac{n-2}{2}}.$$

Thus,

$$\begin{aligned}\|\phi_i^\alpha - \eta_i^\alpha\|_{L^2(M)} &\leq \|\mathbf{p}(\eta_i^\alpha) - \mathbf{p}_0(\eta_i^\alpha)\|_{L^2(M)} + \|\hat{\mathbf{p}}^{(i)}(\eta_i^\alpha)\|_{L^2(M)}\\ &\leq c\delta^{\frac{n-2}{2}}.\end{aligned}$$

This establishes the first property of the functions ϕ_i^α. The other properties follow easily from this. The $L^2(M)$ estimates for ϕ_i^α on each asr and on the ends M_0 and S_0^n are a direct consequence, as follows

$$\begin{aligned}\|\phi_i^\alpha\|_{L^2(\Omega_j^{(+\delta)})} &\leq \|\phi_i^\alpha - \eta_i^\alpha\|_{L^2(M)} + \|\eta_i^\alpha\|_{L^2(\Omega_j^{(+\delta)})} \\ &\leq c\delta^{\frac{n-2}{2}} + c\delta^n \\ &\leq c\delta^{\frac{n-2}{2}}.\end{aligned}$$

The corresponding estimates on the ends M_0 and S_0^n follow in exactly the same way. The final estimate is then derived from this. First note that since $\hat{\mathbf{p}}^{(j)}(\eta_j^\beta)$ is orthogonal to ϕ_i^α, we have

$$\begin{aligned}\langle \phi_i^\alpha, \phi_j^\beta \rangle_{L^2(M)} &= \langle \phi_i^\alpha, \mathbf{p}(\eta_j^\beta) \rangle_{L^2(M)} \\ &= \langle \phi_i^\alpha, \eta_j^\beta \rangle_{L^2(M)}.\end{aligned}$$

Thus since η_j^β has its support contained within $\Omega_j^{(+\delta)}$, we have

$$\begin{aligned}\left|\langle \phi_i^\alpha, \phi_j^\beta \rangle_{L^2(M)}\right| &= \left|\langle \phi_i^\alpha, \eta_j^\beta \rangle_{L^2(M)}\right| \\ &\leq \|\eta_j^\beta\|_{L^2(\Omega_j^{(+\delta)})} \|\phi_i^\alpha\|_{L^2(\Omega_j^{(+\delta)})} \\ &\leq c\delta^{\frac{n-2}{2}}.\end{aligned}$$

This completes the proof of Lemma (2.6). □

2.6. Pointwise Estimates for the Solution of the Linear Problem. To study the solution of the linear equation, $\mathcal{L}\eta = f$, on the orthogonal complement of the small eigenspace K, we first derive pointwise estimates for $\mathcal{L}$. The main task is to derive an ϵ- independent C^0 estimate. Recall that the condition for our conformal structure to be admissible ensures that the parameters $\epsilon_{\pm i} = 2\lambda_{\pm i}^{\frac{1}{2}}$ are comparable to ϵ as in (1.6). As ϵ tends to zero the geometry of our solutions degenerate. In light of this, we make a choice of the $C^{r,\alpha}$ norm on M which will encode the ϵ dependence of our estimates. As in [15] we first make a choice of the $C^{r,\alpha}$ norm on (S^n, g_0). For a domain $\mathcal{O} \subset S^n$, and $\eta \in C^{r,\alpha}(\mathcal{O})$, we let

$$\|\eta\|_{r,\alpha,\mathcal{O}} = \sum_{s=0}^{r} \epsilon^s \sup_{\mathcal{O}} |\partial^{(s)}\eta| + \epsilon^{r+\alpha} \|\partial^{(r)}\eta\|_{(\alpha),\mathcal{O}},$$

where $|\partial^{(s)}\eta|$ is the length with respect to g_0 of covariant derivatives of η of order s. The quantity $\|\partial^{(r)}\eta\|_{(\alpha),\mathcal{O}}$ is the Hölder exponent of r-th derivatives of η, which we define by using a fixed coordinate covering of S^n. Similarly we make a choice of $C^{r,\alpha}$ norm on domains in (M,g) and $(S^n,\overline{g})$, defined with respect to the fixed background metrics g and $\overline{g}$. For any $\eta \in C^{r,\alpha}(M)$ we define

$$\|\eta\|_{r,\alpha} = \max_{1\le i\le k}\{\|\eta\cdot F_i^{-1}\|_{r,\alpha,\mathcal{O}_i}, \|\eta\|_{r,\alpha,M_0}, \|\eta\|_{r,\alpha,S_0^n}\},$$

where $\mathcal{O}_i = F_i(\Omega_i)$ and M_0 and S_0^n are the ends of our approximate solution given by the decomposition (1.7). The following Theorem is analogous to [15, Theorem 3.9] and establishes the basic pointwise estimate for $\eta \in K$.

Theorem 2.7. *For any non-negative integer r and any $\alpha \in (0,1)$*

$$\|\eta\|_{r,\alpha} \le c\|\eta\|_{L^2(M)}$$

for every $\eta \in K$, where $c = c(r,\alpha)$ is independent of ϵ. Thus the inclusion $K \subset C^{r,\alpha}(M)$ has bounded norm.

Proof. Due to its central role we sketch the proof of Theorem 2.7 here, referring to [15, Theorem 3.9] for more details.

Note that $K \subset C^\infty \cap L^2(M)$, follows immediately from elliptic regularity since K is constructed from eigenfunctions of Δ_{g_1} on M. We first derive the supremum estimate for $\eta \in K$. For any point $p \in M$, $B_{\delta/2}(p)$ is contained in either $M_0^{(\delta)}$, $S_0^{n^{(\delta)}}$ or $\Omega_i^{(\delta)}$ for some i, where $\delta > c\epsilon^{1-\frac{2}{n}}$. Since $\delta \to 0$ as $\epsilon \to 0$, this does not allow us to apply the Sobolev inequality, Lemma 0.4, in balls whose radius is uniformly bounded below independent of ϵ. We avoid this difficulty by noting that both (M,g) and $(S^n,\overline{g}_0)$ are locally conformally flat in a ball of radius $r = \min(r(g), r(\overline{g}_0))$ about the point at which $\Omega = \cup_{i=1}^k \Omega_i$ is attached. Thus we may choose some $\rho < r/2$ independent of ϵ, so that $B_\rho(p)$ is contained in either $\Omega^{(r)}$, $M_0^{(r)}$ or $S_0^{n^{(r)}}$. These domains are conformally equivalent to a domain in (S^n, g_0), (M,g) or $(S^n,\overline{g}_0)$, respectively. Thus we may apply the Sobolev inequality in $B_\rho(p)$, for any $p \in M$. This allows us to use the De Giorgi-Nash-Moser theory (see [4, Theorem 8.17]) to locally estimate the supremum norm of η

$$\sup_{B_{\rho/2}(p)} |\eta| \leq c\|\eta\|_{L^2(B_\rho(p))} + c\|\mathcal{L}\eta\|_{L^q(B_\rho(p))}, \tag{2.10}$$

where $q > \frac{n}{2}$, and c depends on q, ρ and the Sobolev constant. Since we may choose ρ to be bounded from below, say $\rho > r/4$ and we may bound the Sobolev constant in terms of the maximum of the Yamabe constants of (S^n, g_0), (M, g) and $(S^n, \overline{g}_0)$, we regard c as depending on q alone. To estimate the $L^q(M)$ norms of $\mathcal{L}\eta$, we again make repeated use of the Sobolev inequality. Let $\zeta \in C^\infty$ have support contained in either $M_0^{(r)}$, $S_0^{n^{(r)}}$ or $\Omega^{(r)}$ and let $\beta \geq 2$ and any $\eta_0 \in C^\infty(M)$ be given. After integrating by parts, we have

$$\begin{aligned} -\int_M \zeta^2 \eta_0^{\beta-1} \mathcal{L}\eta_0 dv_1 &= \int_M \zeta^2 \left((\beta-1)\eta_0^{\beta-2}|\nabla\eta_0|^2 - n\eta_0^\beta\right) dv_1 \\ &\quad +2\int_M \left(\zeta\eta_0^{\beta-1}\nabla\zeta\cdot\nabla\eta_0\right) dv_1. \end{aligned}$$

The supremum estimate is then established by applying the Sobolev inequality to $\phi = \zeta|\eta_0|^{\beta/2}$ with $\eta_0 = \mathcal{L}\eta$ and iterating a finite number of times to obtain

$$\|\mathcal{L}\eta\|_{L^{\beta_s}(B_\rho(p))} \leq c\|\eta\|_{L^2(M)}. \tag{2.11}$$

where c depends on s, $\beta_s = 2\kappa^s$, and $\kappa = \frac{n}{n-2}$. Combining this with the De Giorgi-Nash-Moser estimate (2.10) (with $q = \beta^s$) and summing over a fixed covering of M by balls of radius ρ we get the supremum estimate

$$\sup_M |\eta| \leq c\|\eta\|_{L^2(M)}. \tag{2.12}$$

We now use the interior Schauder estimates (see [4]) to derive the higher derivative estimates for η. About each point $p \in M$ the metric g_1 is uniformly equivalent to one of the background metrics g_0 and $\overline{g}_0$ on S^n, or g on M, in a ball of radius proportional to ϵ, the proportionality constant being $c_1 = \beta$ from (1.6). This follows immediately from the bubble construction §2.4 and the fact that g_1 is constructed from an admissible conformal structure. If $p \in \overline{\Omega}_i$ then we may compare the metric $\hat{g}_1 = (F_i^{-1})^* g_1$ to the spherical metric g_0 in a ball centered at $q = F_i(p)$. If $x^1, \ldots, x^n$ denote normal coordinates for g_0 centered at q, then for $|x| \leq \hat{\epsilon} \equiv c_1^{-1}$, $\hat{g}_1 = \sum h_{\alpha\beta}\, dx^\alpha\, dx^\beta$, where

$$c^{-1} \sum \xi_\alpha^2 \leq \sum h_{\alpha\beta}\xi_\alpha\xi_\beta \leq c \sum \xi_\alpha^2,$$
$$\epsilon^r |\partial^{(r)} h_{\alpha\beta}| \leq c,$$

$|\partial^{(r)} h_{\alpha\beta}|$ denoting the absolute value of partial derivatives of order r. A similar estimate holds for p in either $\overline{M}_0$ or $\overline{S_0^n}$. For example, if $p \in \overline{M}_0 \subset M_0^{(-\delta)} \cup N_{-1}^{(\delta)}$ then we distinguish between which region p lies in as follows. If $p \in M_0^{(-\delta)}$ then we use the identity map and note that g_1 is isometric to g in a ball of radius $\hat{\epsilon}$ about p. If $p \in N_{-1}^{(\delta)}$ then the metric $\hat{g}_1 = (F_0^{-1} \cdot \Phi)^* g_1$ is uniformly equivalent to the background metric g on M in a ball of radius $\hat{\epsilon}$ about $q = \Phi^{-1} \cdot F_0\,(p)$, and if we let $x^1, \ldots, x^n$ denote normal coordinates for g then the above inequalities are also valid in this ball.

If $p \in \overline{\Omega}$, say $p \in \overline{\Omega}_i$ then the Schauder estimate implies that

$$\|\eta \cdot F_i^{-1}\|_{2,\alpha,D_{\hat{\epsilon}/2}} \leq c \left(\epsilon^2 \|\mathcal{L}\eta \cdot F_i^{-1}\|_{0,\alpha,D_{\hat{\epsilon}}} + \sup_{D_{\hat{\epsilon}}} |\eta \cdot F_i^{-1}| \right),$$

where $D_{\hat{\epsilon}} = \{x : |x| \leq \hat{\epsilon}\}$ with respect to normal coordinates and the norms are the ϵ weighted norms defined above. A similar estimate holds for $p \in \overline{M}_0$ and $p \in \overline{S_0^n}$. Because of the ϵ weighting of the norms and the uniform equivalence of the metric g_1 to one of the three fixed metrics g_0, g or $\overline{g}_0$ in balls of radius proportional to ϵ, we have

$$\|\eta \cdot F_i^{-1}\|_{r,\alpha,\mathcal{O}_i} \leq c \sup_{q_0 \in \mathcal{O}_i} \|\eta \cdot F_i^{-1}\|_{r,\alpha,B_{\hat{\epsilon}}(q_0)},$$

where c depends on c_1. Note that this estimate also holds on M_0 and S_0^n. The $C^{2,\alpha}$ estimate now follows by applying the fundamental theorem of calculus and standard estimates for the Poisson equation. We refer to [15, Theorem 3.9] for the remainder of the proof of Theorem 2.7. □

We now prove the main theorem concerning solvability of the linear equation $\mathcal{L}\eta = f$.

Theorem 2.8. (i) *Suppose $\eta \in C^{r+2,\alpha}(M) \cap K^\perp$, where r is a non- negative integer and $\alpha \in (0,1)$. We then have $\mathcal{L}\eta \in C^{r,\alpha}(M) \cap K^\perp$, and*

$$\|\eta\|_{r+2,\alpha} \leq c\|\mathcal{L}\eta\|_{r,\alpha},$$

where $c = c(r, \alpha)$.

(ii) *Given* $f \in C^{r,\alpha}(M) \cap K^{\perp}$, *there is a unique* $\eta \in C^{r+2,\alpha}(M) \cap K^{\perp}$ *satisfying* $\mathcal{L}\eta = f$.

Proof. To prove part (i), let $\eta \in C^{r+2,\alpha}(M) \cap K^{\perp}$. By applying the Schauder estimate as in the previous proof we get

$$\|\eta\|_{r+2,\alpha} \leq c\left(\epsilon^2\|\mathcal{L}\eta\|_{r,\alpha} + \|\eta\|_{C^0(M)}\right),$$

where c depends on c_1. The De Giorgi-Nash-Moser estimate gives us

$$\|\eta\|_{C^0(M)} \leq c\|\eta\|_{L^2(M)} + c\|\mathcal{L}\eta\|_{C^0(M)}.$$

Since $\eta \in K^{\perp}$ and the volume of (M, g_1) can be estimated by a fixed constant, independent of ϵ, we get

$$\|\eta\|_{L^2(M)} \leq c\|\mathcal{L}\eta\|_{L^2(M)} \leq c\|\mathcal{L}\eta\|_{C^0(M)}.$$

Therefore

$$\begin{aligned}\|\eta\|_{r+2,\alpha} &\leq c\left(\epsilon\|\mathcal{L}\eta\|_{r,\alpha} + c\|\mathcal{L}\eta\|_{C^0(M)}\right)\\ &\leq c\|\mathcal{L}\eta\|_{r,\alpha}\end{aligned}$$

as claimed.

To establish the second claim we note that since $f \in K^{\perp}$, there exists a unique $\eta \in K^{\perp}$ with $\mathcal{L}\eta = f$, namely the function $\eta = \mathcal{L}^{-1}f$. By part (i) of our theorem, since $f \in C^{r,\alpha}(M)$ we have

$$\|\eta\|_{r+2,\alpha} \leq c\|f\|_{r,\alpha}.$$

Thus $\eta \in C^{r+2,\alpha}(M)$ This completes the proof of Theorem 2.8. □

3. The Projected Problem

We define the projected problem as

$$\mathcal{L}\eta = \mathbf{q}\left(F(x, \eta)\right), \tag{3.1}$$

where $\mathbf{q} : L^2(M) \to K^{\perp}$ is the orthogonal projection operator discussed in Lemma 2.6. The linear estimates of section 3 combined with a contraction mapping argument allows us to prove that (3.1) has a unique small solution

$\eta \in K^{\perp}$. The Theorem is nearly identical to [15, Proposition 4.1], and we refer there for its proof.

Theorem 3.1. *There exists a $\delta > 0$ depending only on n, β such that (3.1) has a unique solution $\eta \in C^{2,\alpha}(M) \cap K^{\perp}$ satisfying $\|\eta\|_{C^0(M)} \leq \delta$.*

We need more precise $C^0(M)$ estimates on both the solution η to the projected problem (3.1) and on the scalar curvature R, of the resulting metric $g = (1+\eta)^{\frac{4}{n-2}} g_1$. These can be easily derived from the proof of Theorem 3.1. In [15] it is shown that

$$\|\eta\|_{C^0(M)} \leq c(\epsilon^{\frac{n-2}{2}} + \epsilon^{\frac{n-2}{q}}) \tag{3.2}$$

for any $q > \frac{n}{2}$, and $c = c(q)$.

The scalar curvature R of the metric $g = (1+\eta)^{\frac{4}{n-2}} g_1$ is given by

$$R = -\frac{4(n-1)}{n-2}(1+\eta)^{-\frac{n+2}{n-2}} \left(\Delta_{g_1}(1+\eta) \frac{n-2}{4(n-1)} R_1(1+\eta) \right).$$

Therefore by using the definitions of $F(x,\eta)$ and $Q(\eta)$ in (2.2), we see that

$$R - n(n-1) = -\frac{4(n-1)}{n-2}(1+\eta)^{-\frac{n+2}{n-2}} [\mathcal{L}\eta - F(x,\eta)],$$

and hence since $\mathcal{L}\eta = \mathbf{q}(F(x,\eta))$, this implies that

$$R - n(n-1) = \frac{4(n-1)}{n-2}(1+\eta)^{-\frac{n+2}{n-2}} \mathbf{p}(F(x,\eta)).$$

Thus by either (3.2) or simply $\|\eta\|_{C^0(M)} \leq 1$, and Theorem 2.7 we have

$$\begin{aligned} \|R - n(n-1)\|_{C^0(M)} &\leq c\|\mathbf{p}(F(x,\eta))\|_{C^0(M)} \\ &\leq c\|F(x,\eta)\|_{L^2(M)} \\ &\leq c\|R_1 - n(n-1)\|_{L^2(M)}. \end{aligned}$$

Hence we conclude that

$$\|R - n(n-1)\|_{C^0(M)} \leq c\epsilon^{\frac{n-2}{2}}. \tag{3.3}$$

The solution $g = (1+\eta)^{\frac{4}{n-2}} g_1$, to the projected problem (3.1) satisfies the energy estimate of Theorem (0.1). This follows immediately from the volume estimate (1.8) on the approximate solution and the $C^0(M)$ estimate (3.2) on η. Since

$$\mathrm{Vol}_g(M) = \int_M (1+\eta)^{\frac{2n}{n-2}} dv_{g_1},$$

we have, for any $q > \frac{n}{2}$,

$$\begin{aligned} |\mathrm{Vol}_g(M) - \mathrm{Vol}_{g_1}(M)| &\leq c\|\eta\|_{C^0(M)} \mathrm{Vol}_{g_1}(M) \\ &\leq c(\epsilon^{\frac{n-2}{2}} + \epsilon^{\frac{n-2}{q}}) \mathrm{Vol}_{g_1}(M). \end{aligned}$$

Therefore by the triangle inequality and (1.8) we obtain

$$|\mathrm{Vol}_g(M) - (\mathrm{Vol}_{g_1}(M) + (k+1)\omega_n)| \leq c(\epsilon^{\frac{n-2}{2}} + \epsilon^{\frac{n-2}{q}}). \tag{3.4}$$

4. The Generalized Pohozaev Identity

4.1. The Identity. The solution $g = (1+\eta)^{\frac{4}{n-2}} g_1$ to the projected problem (3.1), will be an exact solution provided that the function $F(x,\eta)$ lies orthogonal to the small eigenspace K, i.e.

$$\mathbf{q}(F(x,\eta)) = F(x,\eta). \tag{4.1}$$

This will not be true in general. It will be necessary to perturb each of the admissible conformal structures $\sigma_1, \dots, \sigma_N$ individually, to produce a new admissible N structure $\sigma'(N)$ so that (4.1) holds for each of the new admissible conformal structures $\sigma'_1, \dots, \sigma'_N$. Fortunately we can exhibit an identity which will both provide necessary and sufficient conditions for (4.1) to be satisfied by a given admissible conformal structure and will provide the means to insure that we can find deformations of certain $\sigma(N)$, to a new conformal N structure $\sigma'(N)$ which will satisfy these conditions.

The generalized Pohozaev identity is given by the following proposition.

Proposition 4.1. *Let (N,g) be a compact Riemannian manifold of dimension n with smooth boundary ∂N. Let R denote the scalar curvature of N, and suppose that X is a conformal Killing vector field on N. We than have the identity*

$$\int_N (\mathcal{L}_X R) dv = \frac{2n}{n-2} \int_{\partial N} (Ric_g - n^{-1} R g)(X,\nu) d\sigma. \tag{4.2}$$

where $Ric_g(\cdot,\cdot)$ is the Ricci tensor for (N,g), $\mathcal{L}_X$ denotes the Lie derivative, ν denotes the outward unit normal vector to ∂N, and dv,$d\sigma$ are volume and surface measure (with respect to g), respectively.

We refer to [15] for the proof of (4.1). This identity provides an obvious necessary condition for the metric g to have constant scalar curvature R, namely the vanishing of the boundary integrals

$$\int_{\partial\Omega_i} (Ric_g - n^{-1}Rg)(X,\nu)d\sigma, \tag{4.3}$$

for all conformal Killing fields X on Ω_i. There is a natural class of Conformal killing fields on Ω_i for which we can test this necessary condition. Any vector $\mathbf{w} \in \mathbb{R}^{n+1}$ gives a conformal Killing vector field W on S^n by

$$W(q) = \mathbf{w} - (\mathbf{w}\cdot q)q.$$

W is the tangential projection of $\mathbf{w}$ onto S^n. These vector fields are gradient vector fields on S^n, i.e. they arise as the gradient of a globally defined function on S^n,

$$W = \mathrm{grad}_{g_0}(l_{\mathbf{w}}), \quad l_{\mathbf{w}}(q) = \mathbf{w}\cdot q.$$

We get a conformal Killing field X on Ω_i by defining

$$X = dF_i^{-1}(W).$$

Since F_i is a conformal diffeomorphism from (Ω_i, g_1) to $(\mathcal{O}_i, g_0)$, X is conformal Killing for $g = (1+\eta)^{\frac{4}{n-2}} g_0$. For $\alpha = 1, \dots, n+1$ we let X_α denote the vector field arising from $\mathbf{w}_\alpha$, where $\mathbf{w}_1, \dots, \mathbf{w}_{n+1}$ is an orthonormal basis for $\mathbb{R}^{n+1}$.

4.2. Balanced Admissible Conformal Structures. In [15], Schoen computes the boundary integrals (4.3) for the approximate solutions g_1 constructed in section 2. Recall that g_1 is scalar flat in a small neighborhood of $\partial\Omega_i$. The results of this computation are given by the following Proposition (see [15, Prop. 1.5]).

Proposition 4.2. *Let X be the conformal Killing field on Ω_i, arising from a vector $\mathbf{w} \in \mathbb{R}^{n+1}$. We have the formula*

$$\int_{\partial\Omega_i} Ric_{g_1}(X,\nu)d\sigma_1 = -2(n-1)(n-2)\omega_{n-1}\mathbf{w}\cdot C_i,$$

where ω_{n-1} is the volume of S^{n-1}, and $C_i \in \mathbb{R}^{n+1}$ is given by

$$C_i = \epsilon_{-i}^{n-2} q_{-i} + \epsilon_{+i}^{n-2} q_{+i}.$$

If $X = dF_i^{-1}(X_0)$ for a rotation vector field X_0 on (S^n, g_0), then the boundary integral on each component of $\partial\Omega_i$ vanishes.

The vector C_i is regarded as a "center of mass" attached to each vertex i. C_i vanishes if and only if the two dilations G_{-i} and G_{+i} have the same strength and have sources q_{-i} and q_{+i} which are antipodal points on S^n. An admissible conformal structure whose approximate solution satisfies $C_i = 0$ for each i, will be called a *balanced* admissible conformal structure. We note that the admissible N structure, $\sigma(N)$ constructed in §2.6 consists of balanced admissible conformal structures.

We use the Generalized Pohozaev identity (4.1) in two ways. We first show that it provides a sufficient criteria for determining when the solution to the projected problem is actually an exact solution.

4.3. A Sufficient Criterion for Solutions. If we assume that our approximate solution g_1 arises from a balanced admissible conformal structure then the vanishing of the boundary integrals (4.3) for those vector fields X arising from a vector $\mathbf{w} \in \mathbb{R}^{n+1}$ is actually sufficient to guarantee that $\mathbf{p}(F(x,\eta)) = 0$. This result is identical to [15, Prop. 4.4], which we state here for reference.

Proposition 4.3. *Suppose $g = (1+\eta)^{\frac{4}{n-2}} g_1$ is the metric constructed in Theorem 3.1. Suppose that, for every vertex i,*

$$\int_{\partial\Omega_i} (Ric_g - n^{-1} Rg)(X,\nu)\, d\sigma_g \quad = \quad 0$$

for any conformal vector field X associated with a vector $\mathbf{w} \in \mathbb{R}^{n+1}$. Then η is a solution of (2.1); that is, g has scalar curvature identically $n(n-1)$.

Remark 4.1. The idea of the proof is to use the Generalized Pohozaev identity (4.1) to rewrite the vanishing of the boundary integrals above as the vanishing of the integral of a Lie derivative of the scalar curvature, $R(g)$, over each Ω_i. One then uses the equation for R and the estimates on η found in section 4 to show that this implies that $\mathbf{p}(F(x,\eta)) = 0$, i.e. $F(x,\eta) \in K^{\perp}$. The Generalized Pohozaev identity thus measures the extent to which $F(x,\eta)$ lies orthogonal to K. In the last step of the proof Schoen employs the exponential decay estimates for the basis $\{\phi_i^\alpha\}$ of K, since our solutions are compact and have controlled volume, the estimates we've established in Lemma 2.6 are sufficient here. Thus, if the metric $g = (1+\eta)^{\frac{4}{n-2}} g_1$ satisfies the hypothesis of Proposition 4.3, then η is actually a solution to (2.1). We refer to [15] for the details of the proof of Proposition 4.3.

In the next section we use the Generalized Pohozaev identity to assign values to deformation parameters which are used to perturb an initial balanced admissible conformal structure to a nearby conformal structure for which the hypothesis of Proposition 4.3 are satisfied.

5. Deformations of Approximate Solutions

5.1. Admissible Conformal Structures Revisited. Recall that our admissible conformal N structures, $\sigma(N)$, consist of an assignment of conformal maps to the vertices of each of the N chains, $\mathcal{T}_1, \dots, \mathcal{T}_N$, so that the corresponding approximate solutions $g_{1_1}, \dots, g_{1_N}$ all lie within a fixed conformal class. In §2.6 we constructed a conformal N structure, $\sigma(N)$, which had the additional property that each of the admissible conformal structures, σ_k, was balanced, in the sense of §4.2. We need to consider deformations of $\sigma(N)$, which will consist of deforming each approximate conformal structure, σ_k, separately and then showing that the necessary deformations may be done without changing the conformal class of the corresponding approximate solution metric. The deformations which we consider in this section are exactly the same as those used in [15], however all the approximate solutions considered there were in the conformal class $[g_0]$, of the round metric g_0 on S^n, hence the adjustments which we make to the approximate solutions of the new conformal structures to insure that they remain in a fixed conformal class, were unnecessary there. Much of the material in this section can be found in [15], we include it nonetheless, for the sake of completeness. We deform the approximate conformal structures by considering deformations of the conformal maps F_i, which make up σ_k. Our aim in considering deformations of σ_k is to show that we can find a nearby admissible conformal structure for which the hypothesis of Proposition 4.3 are satisfied. Thus by 4.3 such an admissible conformal structure will produce an exact solution to the main equation (2.1).

Before deforming the σ_k we need to introduce a topology on the space of admissible conformal structures. For $G_1 = R_1 D_1$ and $G_2 = R_2 D_2$, two dilations uniquely decomposed into rotational and centered dilation factors, we define the distance between G_1 and G_2 to be

$$d(G_1, G_2) = \max\{\|R_1 - R_2\|, |q_1 - q_2|, |\log \frac{\lambda_1}{\lambda_2}|\},$$

where q_i is the source and λ_i is the strength of G_i, for $i = 1, 2$ respectively. The quantity $\|R_1 - R_2\|$ denotes the operator norm of $R_1 - R_2$ thought of as a linear transformation of $\mathbb{R}^{n+1}$. If σ_k and $\overline{\sigma}_k$ are two admissible conformal structures on $\mathcal{T}_k$ then we define the distance between σ_k and $\overline{\sigma}_k$ to be

$$d(\sigma_k, \overline{\sigma}_k) = \max_i \left(\max\{d(G_{-i}, \overline{G}_{-i}), d(G_{+i}, \overline{G}_{+i})\}\right), \tag{5.1}$$

where the dilations $\{G_{\pm i}\}$, (respectively $\{\overline{G}_{\pm i}\}$) are those which correspond to σ_k, (respectively $\overline{\sigma}_k$). Note that σ_k is defined either by specifying the dilations $G_{\pm i}$ for $i = 1, \ldots, k$ directly, or by specifying the conformal maps F_i for $i = 1, \ldots, k$ whose compositions generate most of the dilations and in addition specifying the dilations G_{-1} and G_{+k} independently. Thus one can easily see that $d(\sigma_k, \overline{\sigma}_k) = 0$ if and only if $\sigma_k = \overline{\sigma}_k$, in the sense that $G_{\pm i} = \overline{G}_{\pm i}$ for all $i = 1, \ldots, k$. It is then clear that $d(\cdot, \cdot)$ defines a distance function on the space of all conformal structures on $\mathcal{T}_k$.

Let $\mathcal{S}_k$ denote the space of admissible conformal structures on $\mathcal{T}_k$, with the topology that $\mathcal{S}_k$ inherits from the distance function $d(\cdot, \cdot)$ defined above. If we fix k and suppress the index, then for any $\sigma_1 \in \mathcal{S}$ and any $\delta > 0$, we define the set $\mathcal{S}_\delta(\sigma_1) \subset \mathcal{S}$ by

$$\mathcal{S}_\delta(\sigma_1) = \{\sigma \in \mathcal{S} : d(\sigma, \sigma_1) < \delta\}.$$

In §3.5 we constructed a specific metric $g_1(\sigma_k)$ corresponding to a given $\sigma_k \in \mathcal{S}_k$, which has served as our approximate solution. Given an admissible conformal structure $\sigma_k \in \mathcal{T}_k$ we will call a metric g on M compatible with σ_k if it is conformally equivalent to the metric $g_1(\sigma_k)$ constructed in §3.5. For each

$\sigma \in \mathcal{S}_k$ let $\mathcal{M}_\sigma$ denote the space of compatible metrics. We then define $\mathcal{M}$ by

$$\mathcal{M} = \bigcup_{\sigma \in \mathcal{T}_k} \mathcal{M}_\sigma.$$

Thus for any metric $g \in \mathcal{M}$ there exists an admissible conformal structure $\sigma = \sigma(g)$ and a metric $g_1(\sigma)$ which is conformally equivalent to g, such that $g_1(\sigma)$ is an approximate solution. Given any $g \in \mathcal{M}$ we use the generalized Pohozaev identity to associate to g a set of parameters which measure the extent to which g fails to satisfy the hypothesis of Theorem 4.3. These parameters will then be used to specify deformations of given admissible conformal structures. Before defining our parameter space we need to review some facts about the conformal group of S^n. The following section is essentially from [15], for completeness and in order to establish our notation we include it here.

5.2. The Conformal Group of S^n. Recall that Minkowski (n+2)-space, $\mathbf{M}^{n+2}$, is $(\mathbb{R}^{n+2}, \langle\langle \cdot, \cdot \rangle\rangle)$, where $\langle\langle \cdot, \cdot \rangle\rangle$ is the Lorentz inner product

$$\langle\langle y, z \rangle\rangle = \sum_{i=1}^{n+1} y_i z_i - y_{n+2} z_{n+2}.$$

We view S^n as the set of points y in $\mathbf{M}^{n+2}$ satisfying $\sum_{i=1}^{n+1} y_i^2 = y_{n+2}^2 = 1$. The Lie group $O(n+1,1) \subset GL(n+2,\mathbb{R})$ is the linear isometry group of $boldM^{n+2}$, i.e. it is the set of invertible $(n+2) \times (n+2)$ matrices which preserve the Lorentz inner product

$$O(n+1,1) = \{A \in GL(n+2,\mathbb{R}) : \langle\langle Ay, Az \rangle\rangle = \langle\langle y, z \rangle\rangle\}.$$

The group $O(n+1,1)$ acts on the sphere S^n by

$$A(y) = (Ay)_{n+2}^{-1} Ay.$$

Under this action $O(n+1,1)$ represents the conformal group of S^n. The Lie algebra $o(n+1,1)$ of $O(n+1,1)$, consists of $(n+2) \times (n+2)$ matrices X of the form

$$X = \begin{pmatrix} X_0 & \mathbf{w}^t \\ \mathbf{w} & 0 \end{pmatrix},$$

where $X_0 \in o(n+1)$, the Lie algebra of the orthogonal group $O(n+1)$, $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_{n+1}) \in \mathbb{R}^{n+1}$ is an vector, and $\mathbf{w}^t$ is its transpose. Thus X_0 is a skew symmetric matrix which we refer to as the rotational part of X and the vector $\mathbf{w} \in \mathbb{R}^{n+1}$ is referred to as the pure dilational part of X. By an abuse of notation we write $X = X_0 + \mathbf{w}$.

The Killing form $B : o(n+1,1) \times o(n+1,1) \to \mathbb{R}$ is the nondegenerate symmetric quadratic form given by

$$\begin{aligned} B(X, \hat{X}) &= -\sum_{1 \le i < j \le n+1} x_{ij}\hat{x}_{ij} + \sum_{i=1}^{n+1} x_{in+2}\hat{x}_{in+2} \\ &= \frac{1}{2}\mathrm{Tr}(X_0\hat{X}_0) + \mathbf{w} \cdot \hat{\mathbf{w}}, \end{aligned}$$

where we have written X and $\hat{X}$ in terms of their rotational and pure dilational components, and $\mathbf{w} \cdot \hat{\mathbf{w}}$ denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. If we decompose $o(n+1,1) = o(n+1) \oplus \mathbb{R}^{n+1}$, as above, then we see that $B(\cdot,\cdot)$ is positive definite on $\mathbb{R}^{n+1}$ and negative definite on $o(n+1)$. Since B is nondegenerate, it provides an identification of $o(n+1,1)$ with its dual space. If X is an element of $o(n+1,1)$, we let X^* denote the linear functional given by

$$X^*(Y) = B(X, Y).$$

It is through this identification that we shall use the boundary integrals (4.3) to attach to a each oriented edge of $\mathcal{T}_k$ an element of the dual of $o(n+1,1)$. These elements will in turn give rise to a collection of parameters corresponding to a particular compatible metric for some admissible conformal structure.

5.3. The Parameter Space and Evaluation Map. We define the parameter space $\mathcal{X}$ as follows. First fix a background balanced admissible conformal structure, σ_0 on $\mathcal{T}_k$. We may, for example let σ_0 be the conformal structure constructed in §2.6. Let E denote the set of oriented edges of $\mathcal{T}_k$. E has cardinality $2k$ and is given explicitly by

$$E = \{\pm 1, \dots, \pm k\}.$$

For each vertex $i = 1, \dots, k$ we define a hyperplane $W_i \subset \mathbb{R}^{n+1}$ by

$$W_i = \{p \in \mathbb{R}^{n+1} : p \cdot q_{+i} = 0\},$$

where q_{+i} is the source of the dilation G_{+i} in the balanced admissible conformal structure σ_0. Note that the balancing condition implies that $q_{-i} + q_{+i} = 0$, thus W_i could be defined with respect to q_{-i} as well. We define two spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ by

$$\mathcal{X}_1 = \prod_{i=1}^{k} W_i \qquad \mathcal{X}_2 = \prod_{e \in E} \mathbb{R} = \mathbb{R}^{|E|}.$$

The parameter space $\mathcal{X}$ is then defined by

$$\mathcal{X} \quad = \quad \mathcal{X}_1 \times \mathcal{X}_2.$$

A point $(b, \mu) \in \mathcal{X}$ has components $b_i \in W_i \subset \mathbb{R}^{n+1}$ for $i = 1, \dots k$, and $\mu_e \in \mathbb{R}$ for $e \in E$. Given a pair $\delta = (\delta_1, \delta_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ we define $\mathcal{X}_\delta(b, \mu)$ by

$$\mathcal{X}_\delta(b, \mu) \quad = \quad \{(c, \nu) \in \mathcal{X} : |b_i - c_i| \leq \delta_1, \forall i = 1, \dots, k, \ |\nu_e - \mu_e| \leq \delta_2, \forall e \in E\}.$$

Given any admissible conformal structure $\sigma \in \mathcal{S}_k$, and any compatible metric $g \in \mathcal{M}_\sigma$ we define a linear functional $X^*(\sigma, g)_e$, on $o(n+1, 1)$, for each $e \in E$ by

$$\text{(5.2)} \quad X^*(\sigma, g)_{+i}(X) = -k(n)^{-1} \int_{F_i^{-1}(S_{+i})} (\mathrm{Ric}_g - n^{-1} R g)(dF_i^{-1}(X), \nu) d\sigma_g,$$

for any $X \in o(n+1, 1)$, where $k(n) = 2(n-1)(n-2)$, ν denotes the outward unit normal vector, with respect to g, from Ω_i, and we've chosen $e = +i$. $X^*(\sigma, g)_{-i}(X)$ is defined similarly. This defines for each edge $e \in E$ an element of the dual of $o(n+1, 1)$. We let $X(\sigma, g)_e$ denote the element of $o(n+1, 1)$ which is associated to $X^*(\sigma, g)_e$ under the identification given by $B(\cdot, \cdot)$. Thus $X(\sigma, g)_e$ is defined by the property that for all $X \in o(n+1, 1)$

$$X^*(\sigma, g)_e(X) = B(X(\sigma, g)_e, X).$$

Proposition 4.2 shows that $X(\sigma, g_1)_{+i} = \epsilon_{+i}^{n-2} q_{+i}$, where g_1 is the compatible metric constructed in §2.5. Thus $X(\sigma, g_1)_{+i}$ has no rotational component, and its pure dilational component is $\epsilon_{+i}^{n-2} q_{+i}$. For any compatible metric g the elements $X(\sigma, g)_{+i}$ and $X(\sigma, g)_{-i}$ are determined by boundary integrals over the same hypersurface. The following Lemma shows that they are related to one another by the adjoint representation. We refer to [15, Lemma 2.2] for the proof.

Lemma 5.1. *For any $\sigma \in S$, and any compatible metric g we have*

$$X(\sigma, g)_{+i} = -Ad(G_{-i})\,(X(\sigma, g)_{-i})\,.$$

In particular, $B(X(\sigma,g)_{+i}, X(\sigma,g)_{+i}) = B(X(\sigma,g)_{-i}, X(\sigma,g)_{-i})$.

For each vertex $i = 1, \ldots, k$ we define $X(\sigma,g)_i \in o(n+1,1)$ by

$$X(\sigma,g)_i = X(\sigma,g)_{-i} + X(\sigma,g)_{+i}.$$

Note that if σ_0 is the balanced admissible conformal structure constructed in §2.6 and g_1 the compatible metric constructed in §2.5 with respect to σ_0, then $X(\sigma_0, g_1)_i = 0$ for each $i = 1, \ldots, k$. We express $X(\sigma,g)_i$ in terms of its rotational and pure dilational components by writing

$$X(\sigma,g)_i = X^0(\sigma,g)_i + C(\sigma,g)_i,$$

where $X^0(\sigma,g)_i \in o(n+1)$ and $C(\sigma,g)_i \in \mathbb{R}^{n+1}$.

We define the evaluation map

$$\Pi : \mathcal{M} \to \mathcal{X}$$

by setting $\Pi(\sigma,g) = (b, \mu)$, where b and μ are described as follows. The components $b_i \in W_i$ of $b \in \mathcal{X}_1$ are defined by

$$b_i = \epsilon^{n-2}\mathbf{P}_i(C(\sigma,g)_i),$$

where $\mathbf{P}_i : \mathbb{R}^{n+1} \to W_i$ is the orthogonal projection operator. To define the components μ_e of $\boldsymbol{\mu} \in \mathcal{X}_2$, we first attach a weight $\lambda_e(\sigma,g)$ to each edge $e \in E$ by setting

$$\lambda_e(\sigma,g) = \left[B\left(X(\sigma,g)_e, X(\sigma,g)_e\right)\right]^{\frac{1}{2-n}},$$

provided the term in brackets is positive, and zero otherwise. In all the situations we encounter, $B(X(\sigma,g)_e, X(\sigma,g)_e)$ will be positive, this is because will consider only conformal maps with strong dilational components and weak rotational components. Observe that Lemma 5.1 implies that $\lambda_{-i}(\sigma,g) = \lambda_{+i}(\sigma,g)$. Proposition 4.2 implies that for $e = +i$ $\lambda_e(\sigma_0, g_1) = \lambda_{+i}$, the strength of G_{+i}. The element $\boldsymbol{\mu} \in \mathcal{X}_2$ is then defined by setting

$$\mu_e = \log\left(\frac{\lambda_e(\sigma,g)}{\lambda_e(\sigma_0,g_1)}\right),$$

for each $e \in E$, where (σ_0, g_1) is the fixed balanced admissible conformal structure and compatible metric constructed in §2.5-2.6. Notice that $\Pi(\sigma_0, g_1) = (0,0)$.

5.4. Deformations of Conformal Structures. The deformation map will assign to any element in the parameter space, (b, μ), which is close to $(0,0)$, the parameters which come from $(\sigma_0, g_1) \in \mathcal{M}$, a new admissible conformal structure $\mathcal{D}(b, \mu)$ which is close to σ_0. Thus if δ_1, δ_2 and δ_3 are small and $\delta = (\delta_1, \delta_2)$, then $\mathcal{D}$ will satisfy

$$\mathcal{D} : \mathcal{X}_\delta(0,0) \quad \rightarrow \quad \mathcal{S}_{\delta_3}(\sigma_0).$$

The map $\mathcal{D}$ will have the additional property that if $\mathcal{D}(b, \mu) = \sigma$ and g_1 denotes the metric canonically constructed from σ as in §2.6 then $\Pi(\mathcal{D}(b, \mu), g_1) = \Pi(\sigma, g_1)$ is very close to (b, μ). This says that when we evaluate the parameters arising from the deformed conformal structure, with the canonical metric associated to it, we get parameters which are close to the parameters we used to specify the deformation. Before defining the deformation map itself we first discuss how to deform the centers of mass and the edge weights independently.

5.4.1. *Deformations of Centers of Mass.* Given a set of points $q_1, \dots, q_l$ in S^n and weights $a_1, \dots, a_l$ in $\mathbb{R}^+$, the weighted center of mass is the sum

$$C \quad = \quad \sum_{j=i}^{l} a_j q_j.$$

The balancing condition is that the weighted center of mass vanishes, here q_{-i} and q_{+i} are the sources of the dilations G_{-i} and G_{+i}, and the weights are ϵ_{-i}^{n-2} and ϵ_{+i}^{n-2}. So the balancing condition becomes

$$\epsilon_{-i}^{n-2} q_{-i} + \epsilon_{+i}^{n-2} q_{+i} \quad = \quad 0,$$

so that $\epsilon_{-i} = \epsilon_{+i}$ and $q_{-i} = -q_{+i}$. Therefore to specify a balanced admissible conformal structure we need only specify at each vertex i a point $q_i \in S^n$ and a weight $a_i \in R^+$ satisfying $\beta_1^{-1} \leq a_i \leq \beta_1$ for a constant β_1.

Fix $q_1 \in S^n$, and a_1 such that $\beta^{-1} \leq a_1 \leq \beta$, and let $W = \{p \in \mathbb{R}^{n+1} : p \cdot q_1 = 0\}$. For any $p \in B^{n+1}$ let $F_p : B^{n+1} \rightarrow B^{n+1}$ be the unique centered dilation with $F_p(0) = p$. We define a map $C : B^{n+1} \cap W \to W$ by

$$C(p) = a_1 F_p(q_1) + a_1 F_p(-q_1).$$

The following lemma will allow us to use this map to produce deformations of the centers of mass which are approximately prescribed by the parameters $b_i \in \mathcal{X}_1$. We refer to [15, Lemma 2.3] for the proof.

Lemma 5.2. *The map $C : B^{n+1} \cap W \to W$ is smooth and defines a diffeomorphism from a neighborhood $B^{n+1}_{\sigma_1} \cap W = \{p \in W : |p| < \sigma_1\}$ of 0 onto a neighborhood of 0 containing $B^{n+1}_{\sigma_2} \cap W$, where σ_1, σ_2 depend only on n, β.*

5.4.2. *Deformations of Edge Weights.* Given any assignment of weights to the edges of $\mathcal{T}$, we achieve this exact assignment by a deformation of σ_0. Let $\overline{\lambda} = \{\overline{\lambda}_e\} \in \mathcal{X}_2$ be an assignment of edge weights with $\beta_2 \overline{\lambda}_e \leq \lambda_e \leq \beta_2 \overline{\lambda}_e$ for a constant β_2 and for all edges $e \in E$. Here $\lambda_e = \lambda_e(\sigma_0, g_1)$ is weight assigned to the edge e by the balanced conformal structure σ_0. We define $\sigma(\overline{\lambda})$ by defining the dilations $G_e(\overline{\lambda})$ by defining for $G_{-i}(\overline{\lambda})$ by

$$G_{-i}(\overline{\lambda}) \quad = \quad R_{-i} D_{-i}(\overline{\lambda}),$$

where $G_{-i} = R_{-i} D_{-i}$ is the corresponding dilation for σ_0, and $D_{-i}(\overline{\lambda})$ is the centered dilation with the same fixed points as D_{-i} but with strength $\overline{\lambda}_{-i}$. The dilations $G_{+i}(\overline{\lambda})$ are defined in the same way. Thus if $\overline{\lambda} = \lambda = \{\lambda_{\pm i}\}$ are the weights of σ_0, we have $D_{\pm i}(\lambda_{\pm i}) = D_{\pm i}$, and $\sigma(\lambda) = \sigma_0$.

5.4.3. *The Deformation Map.* We are now in a position to define the deformation map $\mathcal{D}$. We shall initially deform the background balanced admissible conformal structure σ_0 on $\mathcal{T}_k$, as dictated by a given set of parameters. Given $(b, \mu) \in \mathcal{X}_\delta(0,0)$, we define $\sigma = \mathcal{D}(b, \mu)$ as follows. From each of the parameters $b_i \in b$ we specify a centered dilation in the following way. Let $a_i = \epsilon^{n-2}\epsilon_{-i}^{n-2}$ (thus by the balancing condition $a_i = \epsilon^{n-2}\epsilon_{+i}^{n-2}$ as well). By applying Lemma 5.2 to the vectors q_{-i} and q_{+i} with the weight a_i we find that there is a unique point $p_i \in B^{n+1}_{\sigma_1} \cap W_i$ such that $C(p_i) = b_i$. Thus we get a unique (small) centered dilation $D_i = F_{p_i}$, such that

$$\begin{aligned} b_i &= a_i[D_i(q_{-i}) + D_i(q_{+i})] \\ &= \epsilon^{n-2}\epsilon_{-i}[D_i(q_{-i}) + D_i(-q_{-i})]. \end{aligned}$$

The parameters $\{\mu_e\} = \mu$ are used to produce a change in the dilation strength along each edge, which when coupled with the change in the center of mass at each vertex i prescribed by $\{b_i\} = b$ as above will define the deformation $\mathcal{D}$. The admissible conformal structure σ will be specified by defining the dilations $G_{\pm i}^{(\sigma)}$ by

$$(5.3) \quad G_{-i}^{(\sigma)} = D_{i-1} \cdot G_{-i}^{(\sigma(\overline{\lambda}))} \cdot D_i^{-1} \quad \text{and} \quad G_{+i}^{(\sigma)} = D_{i+1} \cdot G_{+i}^{(\sigma(\overline{\lambda}))} \cdot D_i^{-1},$$

where $G_{\pm i}^{(\sigma(\overline{\lambda}))} = G_{\pm i}(\overline{\lambda})$ as above, and $\overline{\lambda}$ is an assignment of edge weights chosen so that

$$(5.4) \qquad \mu_{-i} = \log \frac{\lambda_{-i}(\sigma)}{\lambda_{-i}(\sigma_0)} \quad \text{and} \quad \mu_{+i} = \log \frac{\lambda_{+i}(\sigma)}{\lambda_{+i}(\sigma_0)}.$$

Here $\lambda_{\pm i}(\sigma)$ are the strengths of the dilations $G_{\pm i}^{(\sigma)}$ from our new conformal structure σ, and $\lambda_{\pm i}(\sigma_0)$ are the strengths of the dilations $G_{\pm i}^{(\sigma_0)}$ from our background balanced conformal structure σ_0. In [15, 345-346] it is shown that there is a unique choice of $\overline{\lambda}_{-i}$ and $\overline{\lambda}_{+i}$ which are close in ratio to $\lambda_i(\sigma_0)$ and $\lambda_{+i}(\sigma_0)$ respectively, such that (5.4) holds. Note that the deformation $\sigma_0 \rightarrow \sigma(\overline{\lambda})$ only produces a change in the dilation strength along each edge. Our choice for the dilations $G_{\pm i}^{(\sigma)}$ insures that the conformal transformations $F_i^{(\sigma)}$ attached by σ to each vertex i of $\mathcal{T}_k$ satisfy

$$F_i^{(\sigma)} \;=\; D_i \cdot F_i^{(\sigma(\overline{\lambda}))}.$$

This completes the definition of the deformation map $\mathcal{D}$. It is immediate from the construction that $\mathcal{D}$ is continuous with respect to the product topology on $\mathcal{X}$ and the topology induced on $\mathcal{S}$ by the distance function d defined above. The degree to which $\Pi(\mathcal{D}(b,\mu), g_1)$ is close to (b,μ) for any $(b,\mu) \in \mathcal{X}_\delta(0,0)$ is expressed by the following Proposition.

Proposition 5.3. *Let $\delta = (\delta_1, \delta_2)$, where δ_1, δ_2 are small positive numbers. We then have $\mathcal{D}(\mathcal{X}_\delta(0,0)) \subset \mathcal{S}_{\delta_3}(\sigma_0)$, where $\delta_3 = c\max\{\delta_1^{1/2}, \delta_2\}$, c depending only on n and β. If $(b,\mu) \in \mathcal{X}_\delta(0,0)$, and we denote $\mathcal{D}(b,\mu) = \sigma$, then we have $\Pi(\sigma, g_1) = (b(\sigma), \mu(\sigma))$, where $\mu(\sigma) = \mu$, and $b(\sigma)$ satisfies*

$$|b_i(\sigma) - b_i| \;\leq\; c(\epsilon^2 \delta_1^{1/2} + \delta_2)$$

for a constant c depending only on n and β.

This is identical (verbatim) to [15, Proposition 2.4], and hence we refer there for the proof.

5.5. Deforming Within a Fixed Conformal Class. For each approximate solution g_{1_k}, we have constructed a solution of the projected problem (3.1). The N distinct approximate solution metrics $g_{1_1}, \dots, g_{1_N}$, derived from the background balanced conformal N structure, $\sigma_0(N)$, all lie within a fixed conformal class. Thus the N distinct solutions to each of the projected problems arising from the metrics $g_{1_1}, \dots, g_{1_N}$ also lie within this fixed conformal class. In the next section we show that for each of these projected problems we may find an appropriate deformation of the background balanced admissible conformal structure σ_{0_k} so that the hypothesis of Proposition 4.3 are satisfied, and hence the solution of the projected problem with respect to this new admissible conformal structure is actually an exact solution to the main equation (2.1). In order for this procedure to produce N solutions to (2.1) in a fixed conformal class, as we require, it will be necessary to show that the required deformations actually preserve the conformal class. This is done by associating to each deformation $\sigma_{0_k} \to \sigma_k$ a conformal transformation of the nondegenerate end sphere $(S^n, \overline{g}_0)$ and then by adjusting the decomposition (1.7) accordingly, we show that the two approximate solution metrics $g_{1_k} = g_1(\sigma_{0_k})$ and $g_1(\sigma_k)$ are conformally equivalent.

To emphasize the dependence of the decomposition (1.7) on the conformal structure we fix k, $1 \leq k \leq N$ and write

$$(5.5) \qquad M = M_0 \cup_{F_0^{(\sigma_0)}} \Omega_{\sigma_0} \cup_{F_{k+1}^{(\sigma_0)}} S_0^n$$

for the decomposition of M with respect to the conformal structure σ_0 on $\mathcal{T}_k$, and

$$(5.6) \qquad M = M_0 \cup_{F_0^{(\sigma)}} \Omega_{\sigma} \cup_{F_{k+1}^{(\sigma)}} S_0^n$$

for the decomposition of M with respect to the conformal structure σ on $\mathcal{T}_k$. The subdomains M_0 and S_0^n appearing in the decompositions (5.5) and (5.6) also depend on the conformal structure. We obtain M_0 by removing from M the inverse image of a small ball B_0 in S^n under the conformal map Φ_M from a domain in (M, g) to a domain in (S^n, g_0),

$$M_0 \;=\; M \setminus \Phi_M^{-1}(B_0).$$

The ball B_0 is determined directly by the dilation G_{-1}, namely

$$\partial B_0 \;=\; \{p : |G_{-1}^{-1'}(p)| = 1\}.$$

Thus the subdomain M_0 in (5.5) is determined by $G_{-1}^{(\sigma_0)}$, while the corresponding subdomain in (5.6) is determined by $G_{-1}^{(\sigma)}$. In view of the definition of our deformation map $\mathcal{D}$, we need only consider conformal structures σ which are close to σ_0, in particular we assume that $d(\sigma, \sigma_0) < \delta_3 = c \max\{\delta_1^{1/2}, \delta_2\}$. This implies that the balls $B_0(\sigma_0)$ and $B_0(\sigma)$ are close to each other, since their centers $-q_{-1}(\sigma_0)$ and $-q_{-1}(\sigma)$ satisfy

$$|(-q_{-1}(\sigma_0)) - (-q_{-1}(\sigma))| \;\leq\; \delta_3,$$

and the strength of the dilations $G_{-1}^{(\sigma_0)}$ and $G_{-1}^{(\sigma)}$ which determine them satisfy

$$\left|\log\left(\lambda_{-1}^{(\sigma_0)}/\lambda_{-1}^{(\sigma)}\right)\right| \;\leq\; \delta_3.$$

The subdomain S_0^n is obtained in the same manner as above, with the corresponding small balls $B_{k+1}(\sigma_0)$ and $B_{k+1}(\sigma)$ being determined by the dilations $G_{+k}^{(\sigma_0)}$ and $G_{+k}^{(\sigma)}$ respectively. We now use the closeness of these balls to rewrite the decompositions (5.5) and (5.6) in such a way that the ends are independent of the conformal structure, and thus isometric to one another when endowed with the approximate solution metrics $g_1(\sigma_0)$ and $g_1(\sigma)$.

Recall that for some $r > 0$, (M, g) and $(S^n, \overline{g}_0)$ are locally conformally flat in a ball of radius r about some point $x_0 \in M$, and $y_0 \in S^n$. It is about these distinguished points that we've attached the string of k asr's, Ω. We shall assume that δ_3 is arbitrarily small relative to r, this will hold provided the same is true of δ_1 and δ_2, i.e. if we restrict our deformations to a neighborhood of $(0,0) \in \mathcal{X}$, which is small relative to r. Thus there exists a small constant c_1 such that

$$\Phi_M^{-1}(B_0) \subset B_{c_1}(x_0) \subset B_r(x_0) \subset M \text{ and } \Phi_{S^n}^{-1}(B_{k+1}) \subset B_{c_1}(y_0) \subset B_r(y_0) \subset S^n,$$

where in M these are geodesic balls with respect to g, and in S^n with respect to $\overline{g}_0$, and B_0 and B_{k+1} are the balls described above for any admissible conformal structure $\sigma \in \mathcal{S}_{\delta_3}(\sigma_0)$. We define the subdomains M_0^- and $(S_0^n)^-$ by

$$M_0^- = M \setminus B_{c_1}(x_0) \quad \text{and} \quad (S_0^n)^- = S^n \setminus B_{c_1}(y_0).$$

If we use these subdomains in place of M_0 and S_0^n, the decompositions (5.5) and (5.6) become

$$M = M_0^- \cup \Omega_{\sigma_0}^+ \cup (S_0^n)^-, \tag{5.7}$$

for the decomposition with respect to σ_0 and

$$M = M_0^- \cup \Omega_{\sigma}^+ \cup (S_0^n)^-, \tag{5.8}$$

for the decomposition with respect to σ. The domains $\Omega_{\sigma_0}^+$ and Ω_σ^+ which contain the k asr's are given by

$$\begin{aligned} \Omega_{\sigma_0}^+ &= \{M_0(\sigma_0) \setminus M_0^-\} \cup_{F_0^{(\sigma_0)}} \Omega_{\sigma_0} \cup_{F_{k+1}^{(\sigma_0)}} \{S_0^n(\sigma_0) \setminus (S_0^n)^-\} \\ \Omega_{\sigma}^+ &= \{M_0(\sigma) \setminus M_0^-\} \cup_{F_0^{(\sigma)}} \Omega_{\sigma} \cup_{F_{k+1}^{(\sigma)}} \{S_0^n(\sigma) \setminus (S_0^n)^-\} \end{aligned}$$

In the decompositions (5.7) and (5.8) the subdomains M_0^- and $(S_0^n)^-$ do not depend on the admissible conformal structure. Moreover, since c_1 is arbitrarily large relative to δ_3 we see that when endowed with the approximate solution metrics, $(M_0^-, g_1(\sigma_0))$ and $(M_0^-, g_1(\sigma))$ are isometric as are $((S_0^n)^-, g_1(\sigma_0))$ and $((S_0^n)^-, g_1(\sigma))$. We have omitted the conformal maps F_0 and F_{k+1} from the decompositions (5.7) and (5.8) because their role is in defining $\Omega_{\sigma_0}^+$ and Ω_σ^+, while the ends M_0^- and $(S_0^n)^-$ are attached by the identity map.

The issue then becomes whether $\Omega_{\sigma_0}^+$ and Ω_σ^+ are conformally equivalent. In general, they are not conformally equivalent, however we now show that by making a conformal deformation of $(S^n, \overline{g}_0)$ we can exhibit a final decomposition of M with respect to σ as

$$M = M_0^- \cup \tilde{\Omega}_{\sigma}^+ \cup (\tilde{S}_0^n)^-, \tag{5.9}$$

such that

$$\begin{aligned} (M_0^-, g_1(\sigma_0)) &= (M_0^-, g_1(\sigma)) \\ (\Omega_{\sigma_0}^+, g_1(\sigma_0)) &\cong (\tilde{\Omega}_{\sigma}^+, g_1(\sigma)) \\ ((S_0^n)^-, g_1(\sigma_0)) &\cong ((\tilde{S}_0^n)^-, g_1(\sigma)), \end{aligned}$$

where $=$ denotes an isometry and $\cong$ denotes a conformal equivalence. Note that the isometry is immediate since we do not change the role of M_0^- in the decomposition. The conformal equivalences are possible because $(\Omega^+_{\sigma_0}, g_1(\sigma_0))$ and $(\Omega^+_\sigma, g_1(\sigma))$ are each conformally equivalent to Euclidean annuli, $A_R = \{x : 1 \leq |x| \leq R\}$, and any two annuli are conformally equivalent if and only if they have the same conformal modulus. Let R_0 and $R(\sigma)$ be the two radii such that

$$\begin{aligned}(\Omega^+_{\sigma_0}, g_1(\sigma_0)) &\cong A_{R_0} \\ (\Omega^+_\sigma, g_1(\sigma)) &\cong A_{R(\sigma)}.\end{aligned}$$

Since $d(\sigma, \sigma_0) < \delta_3$, we have $|R(\sigma) - R_0| < c\delta_3$, where $c = c(k)$. Any deformation $\sigma_0 \to \sigma$, where $\sigma \in \mathcal{S}_{\delta_3}(\sigma_0)$ consists of deforming each dilation, say $G^{(\sigma_0)}_{-i}$ to one whose source is close to $q_{-i}(\sigma_0)$ and whose strength is close to $\lambda_{-i}(\sigma_0)$. Thus the net affect of these deformations is at most a small increase or decrease (according to whether the strengths of the dilations increase or decrease on average) in the conformal modulus, $cm\,(\Omega^+_{\sigma_0})$.

On $(S^n, \overline{g}_0)$ there exists a one parameter family of metrics $\overline{g}_\tau \in [\overline{g}_0]$ which are each isometric to $\overline{g}_0$ outside of a neighborhood of $y_0 \in S^n$. These are constructed by using weak centered dilations on (S^n, g_0). Consider the conformal map

$$\Phi_{S^n} : (B_r(y_0), \overline{g}_0) \to (S^n, g_0).$$

Let $q = \Phi_{S^n}(y_0)$, and let D_τ denote the centered dilation with source q and strength $|D'_\tau(q)| = \tau$. We will use the maps D_τ when $cm(\Omega^+_{\sigma_0}) > cm(\Omega^+_\sigma)$ and when $cm(\Omega^+_{\sigma_0}) < cm(\Omega^+_\sigma)$ we use the maps $D_{-\tau} = D_\tau^{-1}$. Assume that our deformation is such that $cm(\Omega^+_{\sigma_0}) > cm(\Omega^+_\sigma)$, i.e. $R_0 > R(\sigma)$, the other case will be dealt with in a similar manner. The dilations D_τ give rise to a conformal map Ψ_τ of $(B_{2c_1}(y_0), \overline{g}_0)$ into $(B_r(y_0), \overline{g}_0)$ defined by

$$\Psi_\tau = \Phi_{S^n}^{-1} \cdot D_\tau \cdot \Phi_{S^n} \quad : \quad B_{2c_1}(y_0) \to B_r(y_0).$$

Define the metric g_τ by

$$g_\tau = \Psi^*_\tau(\overline{g}_0) \quad \text{in} \quad B_{2c_1}(y_0).$$

We require that τ be sufficiently small so that $\Psi_\tau(B_{2c_1}(y_0)) \subset B_r(y_0)$. Since Ψ_τ is conformal $g_\tau = \psi_\tau \overline{g}_0$, for some $\psi_\tau \in C^\infty(B_{2c_1}(y_0))$, $\psi_\tau > 0$. Let ξ be a smooth nonincreasing radial function about y_0 on S^n, i.e. $\xi(y) = \xi(d_{\overline{g}_0}(y, y_0))$ which satisfies

$$\xi(y) = \begin{cases} 1 & \text{for} \quad y \in B_{c_1}(y_0) \\ 0 & \text{for} \quad y \in B_{2c_1}(y_0), \end{cases}$$

Here $d_{\overline{g}_0}(\cdot, y_0)$ denotes the distance function from y_0 taken with respect to $\overline{g}_0$, we assume that $2c_1 < i(S^n, \overline{g}_0)$, the injectivity radius, so that ξ is well defined in $B_{2c_1}(y_0)$ and hence on all of S^n. Define $\overline{\psi}_\tau$ by

$$\overline{\psi}_\tau = \xi\psi_\tau + (1 - \xi),$$

then $\overline{\psi}_\tau \in C^\infty(S^n)$ and $\overline{\psi}_\tau > 0$. Therefore the metric

$$\overline{g}_\tau = \overline{\psi}_\tau \overline{g}_0 \quad \text{on} \quad S^n$$

is conformally equivalent to $\overline{g}_0$ on S^n and satisfies

$$\overline{g}_\tau = \begin{cases} g_\tau & \text{on} \quad B_{c_1}(y_0) \\ \overline{g}_0 & \text{on} \quad S^n \setminus B_{2c_1}(y_0). \end{cases}$$

Therefore we have

$$(B_{c_1}(y_0), g_\tau) = (\Psi_\tau(B_{c_1}(y_0)), \overline{g}_0).$$

Thus since $\overline{g}_\tau$ and $\overline{g}_0$ are pointwise conformal on S^n, we let

$$(\tilde{S}_0^n)^- = S^n \setminus \Psi_\tau(B_{c_1}(y_0))$$

and obtain

$$((S_0^n)^-, \overline{g}_\tau) = ((\tilde{S}_0^n)^-, \overline{g}_0).$$

Thus

$$\begin{aligned} ((\tilde{S}_0^n)^-, g_1(\sigma)) &\cong ((S_0^n)^-, \overline{g}_0) \\ &= ((S_0^n)^-, g_1(\sigma_0)) \end{aligned}$$

as required. Let $A(\tau) = \Psi_\tau(B_{c_1}(y_0)) \setminus B_{c_1}(y_0)$ and define $\tilde{\Omega}_\sigma^+$ by

$$\begin{aligned} \tilde{\Omega}_\sigma^+ &= \{M_0(\sigma) \setminus M_0^-\} \cup_{F_0^{(\sigma)}} \Omega_\sigma \cup_{F_{k+1}^{(\sigma)}} \{S_0^n(\sigma) \setminus (S^n \setminus \Psi_\tau(B_{c_1}(y_0)))\} \\ &= \Omega_\sigma^+ \cup A(\tau). \end{aligned}$$

For an appropriate choice of τ this will give the decomposition (5.9). When endowed with the approximate solution metric $g_1(\sigma)$, $\tilde{\Omega}_\sigma^+$ is the union

$$(\tilde{\Omega}_\sigma^+, g_1(\sigma)) = (\Omega_\sigma^+, g_1(\sigma)) \cup (A(\tau), \overline{g}_0).$$

Since $(A(\tau), \overline{g}_0)$ is conformally equivalent to a Euclidean annulus we have

$$cm(\tilde{\Omega}_\sigma^+, g_1(\sigma)) = R(\sigma) + R(\tau),$$

where $R(\tau) = cm(A(\tau), \overline{g}_0)$. Finally since $R_0 - R(\sigma) < c\delta_3$ and c_1 while large relative to δ_3 may be chosen arbitrarily small relative to r, we see that τ may be chosen in a sufficiently large continuum about zero to guarantee the existence of a τ such that

$$R(\tau) = R_0 - R(\sigma).$$

With this choice of τ we thus have

$$cm(\tilde{\Omega}_\sigma^+, g_1(\sigma)) = R_0,$$

and therefore

$$(\tilde{\Omega}_\sigma^+, g_1(\sigma)) \cong (\Omega_{\sigma_0}^+, g_1(\sigma_0))$$

as required.

The following Theorem summarizes what we have shown.

Theorem 5.4. *Given any admissible conformal structure $\sigma \in \mathcal{S}_{\delta_3}(\sigma_0)$, there exist decompositions of M,*

$$\begin{aligned} M &= M_0^- \cup \Omega_{\sigma_0}^+ \cup (S_0^n)^- \\ M &= M_0^- \cup \tilde{\Omega}_\sigma^+ \cup (\tilde{S}_0^n)^- \end{aligned}$$

such that

$$\begin{aligned} (M_0^-, g_1(\sigma_0)) &= (M_0^-, g_1(\sigma)) \\ (\Omega_{\sigma_0}^+, g_1(\sigma_0)) &= (\tilde{\Omega}_\sigma^+, g_1(\sigma)) \\ ((S_0^n)^-, g_1(\sigma_0)) &= ((\tilde{S}_0^n)^-, g_1(\sigma)) \end{aligned}$$

provided that $\delta_3 < c$ where $c = c(N, r)$ is a small constant. In particular $g_1(\sigma) \in [g_1(\sigma_0)]$, and thus the required deformations preserve the conformal class.

6. Deformation to Exact Solutions

6.1. Controlled Deformations. In this section we solve the main equation (2.1) by finding a unique small deformation of the background balanced admissible conformal structure σ_0 on $\mathcal{T}_k$ to a new admissible conformal structure $\sigma \in \mathcal{S}_{\delta_3}(\sigma_0)$, such that the solution to the projected problem (3.1) corresponding to σ is actually an exact solution of (2.1). This is done by choosing the deformation so that the solution g_k, of (3.1) corresponding to σ satisfies the hypothesis of Proposition 4.3 and is therefore an exact solution of (2.1). Since it was shown in §5.5 that the deformations preserve the conformal class, by finding such a deformation for each $k = 1, \ldots, N$ we produce N distinct solutions in a fixed conformal class. It remains to show that these solutions lie in a conformal class which is an arbitrarily small perturbation in C^0 of our initial background conformal class g on M. This is shown in the final section of this section, and thus completes the proof of our Main Theorem (0.1).

Before showing that such deformations to exact solutions exist we need to recall some results from [15] which indicate the extent to which the deformations we employ are "controlled". We fix a k, $1 \leq k \leq N$, and let $g_k = (1+\eta)^{\frac{4}{n-2}} g_1$ denote the solution of the projected problem (3.1). We suppress the dependence on k in g_1 and η but maintain it in the metric g_k to avoid confusion with the background metric g on M. To indicate the dependence of g_k on the conformal structure $\sigma \in \mathcal{S}_{\delta_3}(\sigma_0)$ (which determines the approximate solution $g_1 = g_{1_k}(\sigma)$) we write $(\sigma, g_k) \in \mathcal{M}$. We need to show that there exists an admissible conformal structure σ whose parameters $\Pi(\sigma, g_k) = (b(\sigma, g_k), \mu(\sigma, g_k))$ obtained from σ with the solution to the corresponding projected problem g_k, are the null parameters $(0,0) \in \mathcal{X}$. This is equivalent to the requirement that g_k satisfy the hypothesis of Proposition 4.3. Proposition 5.3 shows that the parameters $\Pi(\sigma, g_1) = (b(\sigma, g_1), \mu(\sigma, g_1))$ obtained from the deformed conformal structure with its approximate solution metric are close to the parameters which prescribe the deformation. We need to relate these parameters $(b(\sigma, g_1), \mu(\sigma, g_1))$ with the parameters obtained from σ endowed with the projected solution metric g_k, $(b(\sigma, g_k), \mu(\sigma, g_k))$.

This is done in [15, Proposition 4.2]. Since the parameters are obtained by evaluating certain integrals over the boundaries of each asr, comparing the parameters of a fixed conformal structure (which determines the asr's) with two different compatible metrics is purely a local problem in each of the neck regions $N_{\pm i}^{(\delta)}$. Schoen does this by carefully examining the structure of the metric g_k on the neck regions and using the estimates he obtains to compare the Lie algebra elements $X_e(\sigma, g_1)$ and $X_e(\sigma, g_k)$, where $e = \pm i$ is any edge. We write $X_e = X_e^0 + C_e$ for the decomposition into rotational and pure dilational parts. Schoen's result is the following

Proposition 6.1. *There is a constant $\tau_4 > 0$ such that at each vertex i, $1 \leq i \leq k$, and each edge $e = +i, -i$, the following inequalities hold:*

$$\begin{aligned} \|X_e^0(\sigma, g_k)\| &\leq \epsilon^{n-2+\tau_4}, \\ |C_e(\sigma, g_k) - C_e(\sigma, g_1)| &\leq \epsilon^{n-2+\tau_4}. \end{aligned}$$

Remark 6.1. Although it might appear unnecessary, we have retained Schoen's notation for consistancy with [15] and in order that the constant τ_4 is not confused with the parameter τ used in §5.5. We refer to [15, pages 381-385] for the proof of Proposition 6.1.

The other result we need in order to prove our main Theorem is [15, Proposition 4.5], which asserts that the rotational parts of the Lie algebra elements $X_i(\sigma, g_k)$ derived from (σ, g_k), are controlled in terms of their pure dilational parts. Let $\mathbf{X}$, $\mathbf{X}^0$, $\mathbf{C}$ denote the k-tuples of the Lie algebra elements X_i, the rotational parts X_i^0, and the pure dilation parts C_i respectively. We define norms for the latter two quantities by

$$\|\mathbf{X}^0\| = \max_i \|X_i^0\| \quad \text{and} \quad \|\mathbf{C}\| = \max_i |C_i|,$$

where $\|X_i^0\|^2 \equiv -\frac{1}{2}\mathrm{Tr}((X_i^0)^2)$. The following Proposition is then equivalent to [15, Proposition 4.5].

Proposition 6.2. *Suppose that g_k is the metric constructed in Theorem* 3.1 *with respect to the admissible conformal structure σ. There is a small constant ϵ_{10} such that*

$$\|\mathbf{X}^0(\sigma, g_k)\| \leq \epsilon_{10}\|\mathbf{C}(\sigma, g_k)\|.$$

We refer to [15, pages 388-390] for the proof of Proposition 6.2.

These two Propositions now enable us to prove the existence of the required deformations.

6.2. Achieving an Exact Solution.

Theorem 6.3. *Given a balanced admissible conformal structure* σ_0 *on* $\mathcal{T}_k$*, there exists an* $\sigma \in \mathcal{D}(\mathcal{X}_\delta(0,0)) \subset \mathcal{S}_{\delta_3}(\sigma_0)$*, where* $\delta = (\delta_1, \delta_2)$ *and* δ_1*,* δ_2*,* δ_3 *are small positive numbers, such that for the metric* g_k *constructed in Theorem* 3.1 *with respect to* σ*,* $R(g_k) \equiv n(n-1)$*.*

Remark 6.2. Theorem 6.3 is almost identical to [15, Theorem 4.6]. Since it is central to our construction we include its proof here.

Proof. We define a map of the parameter space into itself by evaluating the deformed conformal structure with the metric g_k found by solving the associated projected problem. Explicitly, define

$$\mathbf{T} : \mathcal{X}_\delta(0,0) \to \mathcal{X}$$

by

$$\mathbf{T}(b,\mu) = \Pi(\mathcal{D}(b,\mu), g_k).$$

We write $\sigma = \mathcal{D}(b,\mu)$, so that

$$\mathbf{T}(b,\mu) = \Pi(\sigma, g) = (b(\sigma, g_k), \mu(\sigma, g_k)) \in \mathcal{X}.$$

We want to show that these parameters are close to the parameters, (b,μ) which specify the deformation. We first estimate the difference $|b_i(\sigma, g_k) - b_i|$ for each vertex i. Since Proposition 5.3 shows that $(b(\sigma, g_1), \mu(\sigma, g_1))$ are close to (b,μ), we have

$$\begin{aligned} |b_i(\sigma, g_k) - b_i| &\leq |b_i(\sigma, g_k) - b_i(\sigma, g_1)| + |b_i(\sigma, g_1) - b_i| \\ &\leq |b_i(\sigma, g_k) - b_i(\sigma, g_1)| + c(\epsilon^2 \delta_1^{1/2} + \delta_2). \end{aligned}$$

We use Proposition 6.1 to estimate the first term, as follows

$$\begin{aligned} |b_i(\sigma, g_k) - b_i(\sigma, g_1)| &= \epsilon^{n-2}|\mathbf{P}_i(C(\sigma, g_k)_i - C(\sigma, g_1)_i)| \\ &\leq \epsilon^{n-2}\left(|C(\sigma, g_k)_{-i} - C(\sigma, g_1)_{-i}| + |C(\sigma, g_k)_{+i}C(\sigma, g_1)_{+i}|\right) \\ &\leq 2\epsilon^{\tau_4}. \end{aligned}$$

Therefore we have

$$|b_i(\sigma, g_k) - b_i| \leq c(\epsilon^2\delta_1^{1/2} + \delta_2 + \epsilon^{\tau_4}). \tag{6.1}$$

We use Proposition 6.1 again to estimate the difference $|\mu_e(\sigma, g_k) - \mu_e|$, for each edge $e = -i, +i$. Notice that from the definition of the parameters $\mu_e(\sigma, g_k)$ and μ_e, we have

$$|\mu_e(\sigma, g_k) - \mu_e| = |\log(\frac{\lambda_e(\sigma, g_k)}{\lambda_e(\sigma, g_1)})|.$$

Thus since

$$\lambda_e(\sigma, g_k) = [-\|X^0(\sigma, g_k)_e\|^2 + |C(\sigma, g_k)_e|^2]^{\frac{1}{2-n}},$$

and

$$\lambda_e(\sigma, g_1) = |C(\sigma, g_1)_e|^{\frac{2}{2-n}},$$

we find that

$$\begin{aligned} |\mu_e(\sigma, g_k) - \mu_e| &= \left|\log\left(\frac{-\|X^0(\sigma, g_k)_e\|^2 + |C(\sigma, g_k)_e|^2}{|C(\sigma, g_1)_e|^2}\right)^{\frac{1}{2-n}}\right| \\ &\leq \frac{2}{n-2}\left|\log\left(\frac{|C(\sigma, g_k)_e|}{C(\sigma, g_1)_e} + \frac{\|X^0(\sigma, g_k)_e\|}{|C(\sigma, g_1)_e|}\right)\right|. \end{aligned}$$

By applying Proposition 6.1 we can estimate

$$\left|\frac{C(\sigma, g_k)_e}{C(\sigma, g_1)_e}\right| \leq 1 + \frac{\epsilon^{n-2+\tau_4}}{|C(\sigma, g_1)_e|},$$

and

$$\frac{\|X^0(\sigma, g_k)_e\|}{|C(\sigma, g_1)_e|} \leq \frac{\epsilon^{n-2+\tau}}{|C(\sigma, g_1)_e|},$$

which gives us

$$|\mu_e(\sigma, g_k) - \mu_e| \leq \frac{2}{n-2}\left|\log(1 + \frac{2\epsilon^{n2+\tau_4}}{|C(\sigma, g_1)_e|})\right|.$$

Proposition 4.2 explicitly calculates $|C(\sigma, g_k)_e| = \epsilon_{\sigma_e}^{n-2}$, thus since σ is an admissible conformal structure we have

$$
(6.2) \qquad \begin{aligned} |\mu_e(\sigma, g_k) - \mu_e| &\leq \tfrac{2}{n-2}|\log(1 + 2\beta^{n2}\epsilon^{\tau_4})| \\ &\leq c\epsilon^{\tau_4}. \end{aligned}
$$

It is clear from the definitions of $\mathcal{D}$ and Π that $\mathbf{T}$ is continuous with respect to the product topology $\mathcal{X}_\delta(0,0)$. Moreover if we choose $\delta_2 \geq c\epsilon^{\tau_4}$ and $\delta_1 \geq c(\epsilon^2\delta_1^{1/2} + \delta_2 + \epsilon^{\tau_4})$, then the estimates (6.1) and (6.2) imply that the map $\mathbf{I} - \mathbf{T}$ takes the compact, contractible set $\mathcal{X}_\delta$ into itself. Therefore we may apply the Schauder fixed point theorem (see [10]) to find parameters $(b, \mu) \in \mathcal{X}_\delta(0,0)$ such that

$$(b, \mu) - \mathbf{T}(b, \mu) = (b, \mu),$$

and hence

$$\mathbf{T}(b, \mu) = (0, 0).$$

If we again let σ denote $\mathcal{D}(b, \mu)$ and let g_k denote the solution of the projected problem with respect to σ, then we have

$$\Pi(\sigma, g_k) = (b(\sigma, g_k), \mu(\sigma, g_k)) = (0, 0).$$

We now show that this implies that $R(g_k) \equiv n(n-1)$. Since $\mu(\sigma, g_k) = 0$, for each edge e, we have

$$\lambda_e(\sigma, g_k) \quad = \quad \lambda_e(\sigma_0, g_1),$$

thus since σ_0 is balanced we conclude that for each i, $1 \leq i \leq k$,

$$\lambda_{-i}(\sigma, g_k) \quad = \quad \lambda_{+i}(\sigma, g_k).$$

Therefore by the definitions of the parameters $\lambda_e(\sigma, g_k)$ in terms of the Killing form $B(\cdot, \cdot)$, we have

$$|C_{-i}(\sigma, g_k)| - \|X^0_{-i}(\sigma, g_k)\| \quad = \quad |C_{+i}(\sigma, g_k)| - \|X^0_{+i}(\sigma, g_k)\|.$$

This can be rewritten as,

$$|(C_{-i} + C_{+i}) \cdot (C_{-i} - C_{+i})| \quad = \quad B(X^0_{-i} + X^0_{+i}, X^0_{-i}X^0_{+i}),$$

where each of the elements of $o(n+1,1)$ corresponds to the pair $(\sigma, g_k) \in \mathcal{M}$. We write

$$\begin{aligned} C(\sigma,g_k)_{-i} - C(\sigma,g_k)_{+i} &= C(\sigma,g_1)_{-i} - C(\sigma,g_1)_{+i} \\ &\quad +(C(\sigma,g_k)_{-i} - C(\sigma,g_1)_{-i}) + (C(\sigma,g_k)_{+i}C(\sigma,g_1)_{+i}), \end{aligned}$$

and use Propositions 4.2 and 6.1 to derive

$$C(\sigma,g_k)_{-i} - C(\sigma,g_k)_{+i} = \epsilon_{\sigma-i}^{n-2} q_{\sigma-i} \epsilon_{\sigma+i}^{n-2} q_{\sigma+i} + O(\epsilon^{n-2+\tau_4}),$$

which implies that

$$|(C_{-i} + C_{+i}) \cdot (q_{\sigma-i} - q_{\sigma+i})| = \|X_{-i}^0 + X_{+i}^0\| \, \|X_{-i}^0 - X_{+i}^0\| + c\epsilon^{n2+\tau_4}|C_{-i} + C_{+i}|.$$

Therefore, since σ is an admissible conformal structure, for each vertex i we have

$$|C(\sigma,g_k)_i \cdot (q_{\sigma-i} - q_{\sigma+i})| \leq c\epsilon^{n-2}\|X_i^0\| \, \|X_{-i}^0 - X_{+i}^0\| + c\epsilon^{\tau_4}|C_i|.$$

Thus, by again applying Proposition 6.1 we obtain

$$|C(\sigma,g_k)_i \cdot (q_{\sigma-i} - q_{\sigma+i})| \leq c\epsilon^{\tau_4}(\|X^0(\sigma,g_k)_i\| + |C(\sigma,g_k)_i|).$$

Since $\sigma \in \mathcal{S}_{\delta_3}(\sigma_0)$ we know that for each edge e, $|q_{\sigma e} - q_e| < \delta_3$, therefore

$$|C(\sigma,g_k)_i \cdot q_{+i}| \leq c\epsilon^{\tau_4}(\|X^0(\sigma,g_k)_i\| + |C(\sigma,g_k)_i|) + \delta_3|C(\sigma,g_k)_i|.$$

We now use the fact that $b_i(\sigma,g_k) = \epsilon^{2-n}\mathbf{P}_i(C(\sigma,g_k)_i) = 0$, or equivalently

$$|C(\sigma,g_k)_i| = |C(\sigma,g_k)_i \cdot q_{+i}|.$$

Thus we have the estimate

$$|C(\sigma,g_k)_i| \leq c\epsilon^{\tau_4}(\|X^0(\sigma,g_k)_i\| + |C(\sigma,g_k)_i|) + \delta_3|C(\sigma,g_k)_i|,$$

and by choosing ϵ sufficiently small we conclude that

$$|C(\sigma,g_k)_i| \leq c\epsilon^{\tau_4}\|X^0(\sigma,g_k)_i\|.$$

Therefore by taking the maximum over all vertices i, $1 \leq i \leq k$ we have

$$\|\mathbf{C}\| \leq c\epsilon^{\tau_4}\|\mathbf{X}^0\|.$$

By Proposition 6.2 this implies that

$$\mathbf{C} \equiv 0 \quad \text{and} \quad \mathbf{X}^0 \equiv 0,$$

which implies that for each vertex i,

$$X(\sigma, g_k)_i = 0.$$

Thus (σ, g_k) satisfy the hypothesis of Proposition 4.3, which we apply to conclude that

$$R(g_k) \equiv n(n-1).$$

This completes the proof of Theorem 6.3. □

6.3. The Degree of the Metric Perturbation. To complete our proof of Theorem 0.1 it remains to show that the initial metric $\overline{g}$ on M with positive scalar curvature $R(\overline{g}) > 0$, belongs to a conformal class which is arbitrarily close in C^0 to the conformal class of any of our N solutions, g_k. The initial perturbation of $\overline{g}$ which we performed in §1.1, was to find a new metric g, which agrees with $\overline{g}$ outside of a small neighborhood of some point x_0 and which is locally conformally flat in a slightly smaller neighborhood. This perturbation produces such metrics which are arbitrarily close in C^1 to $\overline{g}$, and converge to $\overline{g}$ in C^1 as the size of the neighborhood decreases. The more delicate perturbation is that of the metric on the spherical end S^n_0. While we may find nondegenerate (in the sense of §1.1) metrics $\overline{g}_0$, on S^n, which are arbitrarily close to g_0 in as strong a topology as we would like, e.g. $C^{k,\alpha}$, the effect of such a perturbation will not generally result in the conformal class of the corresponding approximate solution $[g_1]$ being arbitrarily close to $[g]$ in $C^{k,\alpha}$. This is because the nondegenerate metric $\overline{g}_0$, on S^n, occurs in (M, g_1) as the pullback $G^*_\lambda(\overline{g}_0)$, of $\overline{g}_0$ under an arbitrarily strong dilation G_λ. Since $\overline{g}_0$ is not in the conformal class of g_0, $G^*_\lambda(\overline{g}_0)$ will not, in general, be in the conformal class of $\overline{g}_0$.

We now show that the metric g_1 may be constructed so that it is arbitrarily close in C^0 to a metric $g' \in [g]$. Here g is our nondegenerate minimal solution metric on M, $R(g) = n(n-1)$ and we let g_1 denote the approximate solution metric which is constructed as in §1.5 from the balanced conformal structure σ_0 on $\mathcal{T}_N$ by using a particular perturbation $\overline{g}_0$, of the round metric g_0 on S^n. Note that by §1.6 and §5.5 each of our N solutions g_k where $k = 1, \ldots, N$, are in the conformal class of g_1.

Proposition 6.4. *Given any $\overline{\epsilon} > 0$, let $\overline{g}_0$ be a metric on S^n which is locally conformally flat in the neighborhood of a point, which belongs to a nondegenerate conformal class, and which satisfies*

$$\begin{aligned} R(\overline{g}_0) &= n(n-1) \\ \|g_0 - \overline{g}_0\|_{C^0} &< \overline{\epsilon}, \end{aligned}$$

and let g_1 be the metric constructed from σ_0, the balanced admissible conformal structure on $\mathcal{T}_N$, using the metric $\overline{g}_0$ as in §1.5. Then there exists a metric $g' \in [g]$ which satisfies

$$\|g_1 - g'\|_{C^0} < \overline{\epsilon}.$$

Proof. We take g' to be the metric constructed in the identical manner as g_1 via σ_0, except that we use the standard round metric g_0 on S^n rather than the metric $\overline{g}_0$ on the spherical end S^n_0. The fact that $g' \in [g]$ is immediate from the construction of the approximate solutions, g' differs from g only in a small neighborhood $B_r(x_0) \subset M$, where g is conformally flat. The metrics g_1 and g' are isometric in $M \setminus S^n_0$. The result does not follow immediately from the assumption on the metrics $\overline{g}_0$ and g_0 since S^n_0 and M_0 are separated by an annulus of large conformal modulus. Thus the effect of the construction of g_1 and g' on these metrics is conformally equivalent to taking the pullback of the metrics g_0 and $\overline{g}_0$ under a strong centered dilation G_λ. The strength λ of this dilation depends on N, the number of solutions we construct and on the number ϵ used in the definition of an admissible conformal structure (1.6). In particular λ tends to infinity as either N tends to infinity or as ϵ tends to zero. Consider the metrics g_λ and $\overline{g}_\lambda$ on S^n defined by

$$g_\lambda = G^*_\lambda(g_0) \quad \text{and} \quad \overline{g}_\lambda = G^*_\lambda(\overline{g}_0),$$

and let h and h_λ be the symmetric $(0,2)$ tensors on S^n defined by

$$h = g_0 - \overline{g}_0 \quad \text{and} \quad h_\lambda = g_\lambda - \overline{g}_\lambda.$$

Notice that G_λ is a conformal diffeomorphism of (S^n, g_0),

$$g_\lambda = |G'_\lambda|^2 g_0,$$

where $|G'_\lambda|$ is the linear stretch factor of the dilation G_λ. This will not be the case for $\overline{g}_\lambda = G^*_\lambda(\overline{g}_0)$. We may compute the C^0 norms of h and h_λ with respect to a fixed metric, for example g_0, by writing these tensors in stereographic coordinates on $S^n \setminus \{q\}$, where q is the source of G_λ. This computation easily shows that

$$\|h_\lambda(x)\| = \|h(\lambda x)\|,$$

therefore

$$\|h_\lambda\|_{C^0} = \|h\|_{C^0}. \tag{6.3}$$

Thus since $h = g_0 - \overline{g}_0$, we have $\|h\|_{C^0} < \overline{\epsilon}$, the desired estimate follows from (6.3). This completes the proof of Proposition 6.4. □

References

1. Aubin, T., *The scalar curvature*, Differential Geometry and Relativity (Cahen and Flato, eds), Reider, 1976.
2. Chavel, I., *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.
3. Gidas, B., Ni, W.M. and Nirenberg, L., *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979) 209–243.
4. Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, second edition, 1983.
5. Kapouleas, N., *Compact constant mean curvature surfaces in Euclidean three-space*, J. Differential Geom. **33** (1991) 683–715.
6. Kapouleas, N., *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math. (2) **181** (1990) 239–330.
7. Kazdan, J., and Warner, F., *Curvature functions for compact 2-manifolds*, Ann. of Math. (2) **99** (1974) 14–74.
8. Kobayashi, O., *Scalar curvature of a metric with unit volume*, Math. Annal.**279** (1987) 253- 265.
9. Lee, J.M. and Parker, T.H., *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.) **17** (1987) 37–91.
10. Nirenberg, L., *Topics in Nonlinear Functional Analysis*, NYU Lecture Notes, 1973.
11. Obata, M., *The conjectures on conformal transformations of Riemannian manifolds*, J. Differential Geom. **6** (1972) 247–258.
12. Pohozaev, S., *Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$*, Soviet Math. Doklady, **6** (1965) 1408–1411.
13. Rellich, R., *Über das asymptotische Verhalten der Lösugen von$\Delta u + \lambda u = 0$in* unendlichen Gebieten, Jahresber. Deutsch. Math. verein **53** (1943), 57–65.

14. Schoen, R., *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geometry **20** (1984) 479–495.
15. Schoen, R., *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. **41** (1988) 317–392.
16. Schoen, R., *On the number of constant scalar curvature metrics in a conformal class*, preprint.
17. Schoen, R., *A report on some recent progress on nonlinear problems in geometry*, Surveys in Differential Geometry, Supplement to J. Differential Geom. **1** (1991) 201–241.
18. Schoen, R., *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, Topics in Calculus of Variations (M. Giaquinta, ed.), Lecture Notes in Math., Vol. 1365, Springer-Verlag, Berlin, 1987.
19. Trudinger, N., *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1968) 265–274.
20. Yamabe, H., *On a deformation of Riemannian structures on compact manifolds*, Osaka J. Math. **12** (1960) 21–37.

The University of Texas at Austin, U. S. A.

Received October 26, 1992 Revised January 25, 1993