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EVOLUTION OF HARMONIC MAPS WITH DIRICHLET BOUNDARY CONDITIONS

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INTRODUCTION

In this paper we shall study a left over problem concerning the heat flow of harmonic maps on manifolds with boundary. Let (M, g) be a compact smooth m-dimensional Reimannian manifold with nonempty smooth boundary ∂M, and let (N, h) be a compact smooth n-dimensional Reimannian manifold without boundary. We denote $M \cup \partial M$ by $\overline{M}$. Since (N, h) can be isometrically embedded into an Euclidean space $\mathbb{R}^k$, for some $k > n$, we may view N as a submanifold of $\mathbb{R}^k$.

In local coordinates on M, the energy of a map $u : M \to N \hookrightarrow \mathbb{R}^k$ is given by

$$E(u) = \frac{1}{2} \int_M g^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^i}{\partial x^\beta} \sqrt{g}\, dx \ , \tag{0.1}$$

here and here after $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}, g = \det(g_{\alpha\beta}), 1 \leq \alpha, \beta \leq m$ and a summation convention is employed.

The Euler-Lagrange equation associated with the functional (0.1) is

$$\Delta u = A(u)(du,\, du) \ , \tag{0.2}$$

where Δ denotes the Laplace-Beltram operator on M and $A(u)$ is the second fundamental form of N in $\mathbb{R}^k$ at u.

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We shall concern with the following evolution problem for a map $u : M \times \mathbb{R}_+ \to N$:

$$(0.3) \qquad \frac{\partial u}{\partial t} = \Delta u - A(u)(du, du), \quad \text{for } (x,t) \in M \times \mathbb{R}_+ ,$$

$$(0.4) \qquad u(x,0) = u_0(x), \quad \text{for } x \in M, \quad \text{and} \quad u(\cdot,t)\Big|_{\partial M} = u_0(\cdot)\Big|_{\partial M} .$$

For simplicity, we assume also that u_0 is smooth on $\overline{M}$. It will be clear later on in the paper that the $C^{2,\alpha}$ smoothness of $M, \partial M, u_0$ and N are sufficient for all purposes.

It is well known that (0.3)-(0.4) admits a unique smooth solution locally. The global existence of a smooth solution to (0.3)-(0.4) can be shown in the case that the Reimannian curvature of N is nonpositive. (see, e.g., [H] and references therein). Without such curvature hypothesis on N, one can, in general, construct examples of finite-time blow-up solutions of (0.3)-(0.4) even in the case that $m = 2$; see [CDY].

On the other hand, Chen and Struwe [CS] established the global existence and partial regularity of weak solutions of (0.3) and (0.4), under an additional hypothesis that $\partial M = \phi$ (cf. also [S]). Here we have the following generalization of their result to the case that ∂M is nonempty.

Theorem. *There is a global weak solution $u : M \times \mathbb{R}_+ \to N$ to (0.3)-(0.4) with $\partial_t u \in L^2(M \times \mathbb{R}_+)$ and $\nabla u \in L^\infty(\mathbb{R}_+, L^2(M))$ which is smooth off a singular set Σ. Set Σ is closed in $\overline{M} \times \mathbb{R}_+$ and has a locally finite m-dimensional Hausdorff measure with respect to the parabolic metric ($\delta((x,t),(y,s)) = |x - y| + \sqrt{|t-s|}$).*

Moreover, as $t \to +\infty$ suitably, $u(\cdot,t)$ converges weakly in $H^1(M,N)$ to a harmonic map $u_\infty : M \to N$ with $u_\infty|_{\partial M} = u_0|_{\partial M}$, which is smooth off a set $\Sigma_\infty \subset \overline{M}$ whose $(m-2)$-dimensional Hausdorff measure can be bounded in terms of c^2-norm of u_0 and $E(u_0)$.

As in [Ch], one can show that $\Sigma_t = \{(x,t) \in \Sigma\}$ has finite $(m-2)$-dimensional Hausdorff measure for each $t \in \mathbb{R}_+$.

The proof of the above theorem follows from the same line of argument as that in [S] and [CS]. There are two principal difficulties. The first one is to establish the monotonicity inequality near the boundary $\partial M \times \mathbb{R}_+$. Here we use, besides the integration by parts trick from [C], some careful estimates on approximate solutions. The second difficulty is to prove the small energy regularity theorem; see [S]. In order to use the Bochner-type inequality for the energy density of the map and mean-value inequality for subsolutions of the heat equations to derive L^∞-estimates on the gradient of maps at those points near the boundary $\partial M \times \mathbb{R}_+$, we go back to the original equations for approximate solutions and obtain first the gradient estimates at boundary $\partial M \times \mathbb{R}_+$.

To simplify the presentation, we consider first the case N is a standard sphere in an Euclidean space. The monotonicity inequality and the small energy regularity theorem are proven in Section 2 and Section 3, respectively. The general N can be handled after some necessary modifications, and this is done in the final section.

1. Monotonicity Inequality

When N is the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we consider, as in [CS], the following approximate solutions: $u = u^k, k = 1, 2, \dots$,

$$(1.1) \qquad \begin{cases} u_t - \Delta u + k(|u|^2 - 1)u = 0 & (x,t) \in M \times \mathbb{R}_+ \\ u(\,\cdot\,,t)|_{\partial M} = u_0|_{\partial M} & t \in \mathbb{R}_+ \\ u(x,0) = u_0(x) & x \in \overline{M} \end{cases}$$

For any fixed $k = 1, 2, \dots$, problem (1.1) has a unique smooth solutions $u = u^k$ with $\partial_t u^k, \nabla^2 u^k \in L^p((0,\infty) \times M)$ for all $1 < p < \infty$. Note that since $|u_0|(x) = 1$, then $|u^k|(x,t) \le 1$ by the maximum principle for parabolic equations; see[LSU]. But we do not need such precise estimates. In general, any uniform L^∞-bound on u^k is sufficient for our purpose.

For fixed k, $u = u^k$ satisfies also the following energy estimate:

Lemma 1.1. *Let $u_0 \in H^1(M,N)$. Then*

$$(1.2) \qquad \int_M |\nabla u|^2 \, dM + \int_M \frac{k}{2}(|u|^2 - 1)^2 \, dM + \int_0^t \int_M |u_t|^2 \, dM \, dt = \int_M |\nabla u_0|^2 \, dM \ , \quad \textit{for all } t > 0 \ .$$

Let ρ_0 be a suitably small positive constant such that for any $p_0 \in \partial M$, one can choose a coordinate system $\{x_\alpha\}$ in such a way that the set $B^M_{\rho_0}(p_0) = \{p \in \overline{M} : \mathrm{dist}_M(p, p_0) < \rho_0\}$ corresponds to the half ball $B^+_{\rho_0} = \{x \in \mathbb{R}^m, |x| < \rho_0, x_m \geq 0\}$. For a regular solution $u = u^k$ of (1.1), we define

$$e_k(u) = \frac{1}{2} g^{\alpha\beta} u_{x_\alpha} u_{x_\beta} + \frac{k}{4}(|u|^2 - 1)^2 \; ;$$
$$G_{x_0}(x,t) = \{4\pi(t_0 - t)\}^{-m/2} \exp\left\{-\frac{|x - x_0|^2}{4(t_0 - t)}\right\} ,$$

where $t < t_0, z_0 = (x_0, t_0) \in \overline{M} \times (0, \infty)$;

$$G(x,t) = G_{\underline{0}}(x,t) \; ;$$
$$T^+_R = \{(x,t) : x \in \mathbb{R}^m_+, -4R^2 < t < -R^2\} \; ;$$
$$\Psi^+(R) = \int_{T^+_R} e_k(u) G \phi^2(x) \sqrt{g(x)}\, dx\, dt \; ,$$

here $\phi \in C^\infty_0(B_{\rho_0}), 0 \leq \phi \leq 1$, $\phi(x) \equiv 1$ for $|x| \leq \rho_0/2$. Thus ϕ may be chosen so that $\|\phi\|_{C^2} \leq C(M)$.

Theorem 1.2 (Monotonicity Inequality). *Suppose that*

$$u = u^k : B^+_{\rho_0}(0) \times [-T, 0] \to \mathbb{R}^{n+1}$$

is a regular solution of (1.1) *(we may assume also that $T \leq \rho_0^2$). Then, for any $0 < R < R_0 \leq \sqrt{T}/2$, we have*

$$\text{(1.3)} \quad \Psi^+(R) \leq \exp[c_*(R_0^{1-\varepsilon} - R^{1-\varepsilon})]\Psi^+(R_0) + c_*(R_0^{1-\varepsilon} - R^{1-\varepsilon})(E_0 + 1) \; , \quad \textit{for any } \varepsilon \in \left(0, \frac{1}{2}\right) .$$

where $E_0 = E(u_0)$, and c_* is a constant depending only on $\overline{M}, N$ and c^2-norm of u_0 on ∂M. Here c_* may depend also on $\mathbb{C}^1$-norm of ϕ which, after suitable choices of ϕ, is a constant depending only on M.

Proof. For simplicity we present the proof for the case that $M = R^m_+ = \{x \in \mathbb{R}^m : x_m > 0\}$. In this case, we may choose ϕ to be identically equal to 1. As in [CS], the general case follows easily.

Let $u_R(x,t) = u(Rx, R^2t)$ and $h_R(x) = h_R(x') = u_0(Rx')$, where $x' = (x_1, \dots, x_{m-1})$. Denote $V_R = \frac{d}{dR}u_R = \frac{(x \cdot \nabla u_R + 2t\,\partial_t u_R)}{R}$. Then,

$$\Psi^+(R) = \frac{1}{2}\int\limits_{T_R^+} \left\{|\nabla u|^2 + \frac{k}{2}(|u|^2-1)^2\right\} G\,dx\,dt$$
$$= \frac{1}{2}\int\limits_{T_1^+} \left\{|\nabla u_R|^2 + \frac{kR^2}{2}(|u_R|^2-1)^2\right\} G\,dx\,dt$$

($\phi \equiv 1$ in this case). Thus

$$\begin{aligned}\frac{d}{dR}\Psi^+(R) &= \int\limits_{T_1^+} \nabla V_R \nabla u_R G\,dx\,dt + \int\limits_{T_1^+} kR^2(|u_R|^2-1)u_R V_R G\,dx\,dt \\ &\qquad + \int\limits_{T_1^+} \frac{kR}{2}(|u_R|^2-1)^2 G\,dx\,dt \\ &\triangleq I + II + III\,.\end{aligned} \tag{1.4}$$

It is obvious that $III \geq 0$. For the first term, we have

(1.5)
$$\begin{aligned}I &= \int\limits_{T_1^+} \nabla u_R \nabla\left(V_R - \frac{x'}{R}(\nabla_{x'}h_R)\right) G\,dx\,dt + \int\limits_{T_1^+} \nabla u_R \cdot \nabla\left(\frac{x'}{R}\nabla_{x'}h_R\right) G\,dx\,dt \\ &= \int\limits_{T_1^+} \nabla u_R \cdot \nabla\left(\frac{x'}{R}\nabla_{x'}h_R\right) G\,dx\,dt - \int\limits_{T_1^+} \Delta u_R\left(V_R - \frac{x'}{R}\nabla_{x'}h_R\right) G\,dx\,dt \\ &\quad - \int\limits_{T_1^+} \frac{x \cdot \nabla u_R}{2t}\left(V_R - \frac{x'}{R}\nabla_{x'}h_R\right) G\,dx\,dt\,.\end{aligned}$$

Here we have used the fact that $\nabla G = \frac{x}{2t}G$. Hence by equation (1.1), one has

$$\begin{aligned}I + II = &-\int\limits_{T_1^+} \left(\partial_t u_R + \frac{x \cdot \nabla u_R}{2t}\right)\left(V_R - \frac{x'}{R}\nabla_{x'}h_R\right) G\,dx\,dt \\ &+ \int\limits_{T_1^+} \nabla u_R \cdot \nabla\left(\frac{x'}{R}\nabla_{x'}h_R\right) G\,dx\,dt \\ &+ \int\limits_{T_1^+} kR^2\left(|u_R|^2-1\right)u_R\left(\frac{x'}{R} \cdot \nabla_{x'}h_R\right) G\,dx\,dt\end{aligned}$$

$$\begin{aligned}
(1.6)\qquad &= -\int_{T_1^+} \frac{R}{2t} V_R^2 G\,dx\,dt + \int_{T_1^+} \frac{R}{2t} V_R \left(\frac{x'}{R} \cdot \nabla_{x'} h_R\right) G\,dx\,dt \\
&\quad + \int_{T_1^+} kR^2(|u_R|^2 - 1)u_R \left(\frac{x'}{R} \cdot \nabla_{x'} h_R\right) G\,dx\,dt \\
&\quad + \int_{T_1^+} \nabla u_R \cdot \nabla \left(\frac{x'}{R} \cdot \nabla_{x'} h_R\right) G\,dx\,dt \\
&\triangleq A + B + C + D\,.
\end{aligned}$$

We have $A = -\int_{T_1^+} \frac{R}{2t} V_R^2 G\,dx\,dt \geq 0$,

$$\begin{aligned}
B &= \int_{T_1^+} \frac{R}{2t} V_R \left(\frac{x'}{R} \cdot \nabla_{x'} h_R\right) G\,dx\,dt \\
&\geq -\frac{A}{4} + R\|\nabla u_0\|^2_{L^\infty(M)} \int_{T_1^+} \frac{|x|^2}{2t} G\,dx\,dt \\
&\triangleq -\frac{A}{4} - c_1\,,
\end{aligned}$$

$$\begin{aligned}
\text{and}\quad D &\geq -\int_{T_1^+} |\nabla u_R|^2 G\,dx\,dt - \int_{T_1^+} \left|\nabla \left(\frac{x'}{R} \cdot \nabla_{x'} h_R\right)\right|^2 G\,dx\,dt \\
&\geq -\Psi^+(R) - c_2\,.
\end{aligned}$$

where $c_1 \leq c(m) R \|\nabla u_0\|^2_{L^\infty(M)} \leq c(m)\|\nabla u_0\|^2_{L^\infty(M)}$ (we shall assume also that $R \leq 1$), and

$$c_2 \leq c(m)\left(\|\nabla u_0\|^2_{L^\infty(M)} + R^2\|\nabla^2 u_0\|^2_{L^\infty(M)}\right)\,.$$

To prove Theorem 1.2, it suffices to show

$$(1.7)\qquad C \geq -\frac{A}{4} - \frac{c_3}{R^\varepsilon}(\Psi^+(R) + 1 + E_0), \quad \text{for } \varepsilon \in \left(0, \frac{1}{2}\right)\,,$$

and for some constant c_3 depending on $\overline{M}, N$, and u_0. In fact, (1.7) and above calculations imply that

$$(1.8)\qquad \frac{d}{dR}\Psi^+(R) \geq -\frac{c_4}{R^\varepsilon}(\Psi^+(R) + 1 + E_0)\,, \quad \text{for } \varepsilon \in \left(0, \frac{1}{2}\right)$$

and with $c_4 = \max\{c_1 + c_3,\ c_3 + 1\}$. The conclusion of Theorem 1.2 follows form (1.8) by a simple integration. □

The remainder of this section is devoted to showing (1.7) or equivalently the following estimate:

(1.9)
$$\left| \int_{T_1^+} kR^2 \left(|u_R|^2 - 1\right) u_R \left(\frac{x'}{R} \cdot \nabla_{x'} h_R\right) G\, dx\, dt \right| \le \frac{A}{4} + \frac{c_3}{R^\varepsilon}\left(\Psi^+(R) + 1 + E_0\right) .$$

Lemma 1.3. *There is a constant c_5 depending only on $\overline{M}, N$ and u_0 such that, for any $\lambda \in (0,1)$,*

$$\int_{T_1^+} kR^2 \left|\, |u_R|^2 - 1\right| |u_R|^2 G\, dx\, dt \le \frac{\lambda}{4} A + \frac{c_5}{\lambda}\left(\Psi^+(R) + 1\right) . \tag{1.10}$$

Proof. Multiplying the equation(1.1) by $u_R\phi(|u_R|^2 - 1)G$, where $\phi \in C^\infty(\mathbb{R})$, $\phi(0) = 0$ and

$$\phi(s) = \begin{cases} 1 & \text{if } s \ge \frac{1}{k} \\ -1 & \text{if } s \le -\frac{1}{k} \end{cases} , \quad \phi'(s) \ge 0 ,$$

we obtain that

$$\int_{T_1^+} \partial_t u_R \cdot u_R\phi(|u_R|^2 - 1)G\, dx\, dt + \int_{T_1^+} \nabla u_R \nabla (u_R\phi(|u_R|^2 - 1)G)\, dx\, dt$$
$$+ \int_{T_1^+} kR^2(|u_R|^2 - 1)|u_R|^2\phi(|u_R|^2 - 1)G\, dx\, dt = 0 .$$

(Note that $\phi(|u_R|^2 - 1) = 0$ on the boundary $x_m = 0$).

Also we have

$$\int_{T_1^+} \nabla u_R \nabla (u_R\phi(|u_R|^2 - 1)G)\, dx\, dt$$
$$= \int_{T_1^+} |\nabla u_R|^2 \phi(|u_R^2 - 1)G\, dx\, dt + 2\int_{T_1^+} |u_R \nabla u_R|^2 \phi'(|u_R|^2 - 1)G\, dx\, dt$$
$$+ \int_{T_1^+} \frac{x \cdot \nabla u_R}{2t} \cdot u_R\phi(|u_R|^2 - 1)G\, dx\, dt .$$

Thus

$$\int_{T_1^+} kR^2 \left| |u_R|^2 - 1 \right| |u_R|^2 G \, dx \, dt = \int_{T_1^+ \cap \{ ||u_R|^2-1| \le \frac{1}{k} \}} + \int_{T_1^+ \cap \{ ||u_R|^2-1| > \frac{1}{k} \}}$$

$$\le \int_{T_1^+} |u_R|^2 R^2 G \, dx \, dt + \int_{T_1^+} kR^2 (|u_R|^2 - 1) \phi(|u_R|^2 - 1) |u_R|^2 G \, .$$

Since $|u_R| \le 1$ (bounded by a constant will be sufficient), the first term on the right-hand side is bounded by $c(m) R^2 \le c(m)$.

The second term is, by above calculations, given by

$$- \int_{T_1^+} \frac{2t \, \partial_t u_R + x \cdot \nabla u_R}{2t} u_R \phi(|u_R|^2 - 1) G \, dx \, dt$$

$$- \int_{T_1^+} |\nabla u_R|^2 \phi(|u_R|^2 - 1) G \, dx \, dt - 2 \int_{T_1^+} |u_R \cdot \nabla u_R|^2 \phi'(|u_R|^2 - 1) G \, dx \, dt$$

$$\le - \int_{T_1^+} \frac{R}{2t} V_R \cdot u_R \phi(|u_R|^2 - 1) G \, dx \, dt - \int_{T_1^+} |\nabla u_R|^2 \phi(|u_R|^2 - 1) G \, dx \, dt \, ,$$

and hence, it is bounded by, for any $\lambda \in (0,1)$,

$$\Psi^+(R) + \frac{\lambda}{4} \int_{T_1^+} \frac{R V_R^{\,2}}{-2t} G + \frac{1}{\lambda} \int_{T_1^+} \frac{R}{-2t} |u_R|^2 G \, dx \, dt = \frac{\lambda}{4} A + \Psi^+(R) + \frac{1}{\lambda} c(m) R \, .$$

This proves (1.10). □

Lemma 1.4. *There is a constant c_6 depending only on $\overline{M}, N$ and u_0 such that*

(1.11)

$$\int_{T_1^+} kR^2 \left| |u_R|^2 - 1 \right| G \, dx \, dt \le \frac{\lambda A}{4} + c_6 \lambda^{-1} (1 + \Psi^+(R)), \quad \textit{for any } \lambda \in (0,1) \, .$$

Proof. Since$(|u_R|^2 - 1)^2 = (|u_R|^2 - 1)|u_R|^2 - (|u_R|^2 - 1)$, thus

$$\left| |u_R|^2 - 1 \right| \le (|u_R|^2 - 1)^2 + \left| |u_R|^2 - 1 \right| |u_R|^2 \, .$$

Therefore (1.11) follows from (1.10) and the definition of $\Psi^+(R)$. □

Lemma 1.5. *For any $\varepsilon \in (0, \frac{1}{2})$, there is a constant c_7 depending on M, N and u_0 such that*

$$(1.12) \quad \int_{T_1^+} kR^2|u_R|^2 \, \big| |u_R|^2 - 1 \big| \, |x|^2 G \, dx \, dt \leq \frac{\lambda A}{4} + \frac{c_7}{\lambda}(1 + E_0 + \Psi^+(R))/R^\varepsilon \,, \quad \text{for } \lambda \in (0,1) \,.$$

Proof. We follow the same line of proof as that for Lemma 1.3. Multiplying (1.1) by $u_R\phi(|u_R|^2 - 1)|x|^2 G$, to obtain

$$\int_{T_1^+} kR^2 \, \big| |u_R|^2 - 1 \big| \, |u_R|^2 |x|^2 G = \int_{T_1^+ \cap \{||u_R|^2 - 1| \leq \frac{1}{k}\}} + \int_{T_1^+ \cap \{||u_R|^2 - 1| > \frac{1}{k}\}}$$
$$\leq \int_{T_1^+} R^2 |u_R|^2 |x|^2 G \, dx \, dt + \int_{T_1^+} kR^2 (|u_R|^2 - 1)\phi(|u_R|^2 - 1)|x|^2 G \, dx \, dt \,.$$

The first term is again bounded by $c(m)R^2 \leq c(m)$, and the second term is now given by

$$-\int_{T_1^+} \frac{R}{2t} V_R \cdot u_R \phi(|u_R|^2 - 1)|x|^2 G \, dx \, dt - \int_{T_1^+} |\nabla u_R|^2 \phi(|u_R|^2 - 1)|x|^2 G \, dx \, dt$$
$$-2\int_{T_1^+} |u_R \nabla u_R|^2 \phi'(|u_R|^2 - 1)|x|^2 G \, dx \, dt - 2\int_{T_1^+} x \cdot \nabla u_R \cdot u_R \phi(|u_R|^2 - 1) G \, dx \, dt$$
$$\leq \int_{T_1^+} 2|\nabla u_R|^2 |x|^2 G \, dx \, dt + \int_{T_1^+} |u_R|^2 G \, dx \, dt$$
$$+\frac{\lambda}{4} \int_{T_1^+} \frac{R}{-2t} V_R^2 \, G \, dx \, dt + \frac{1}{\lambda} \int_{T_1^+} \frac{R}{-2t} |u_R|^2 |x|^4 G \, dx \, dt$$
$$\leq \frac{\lambda}{4} A + \frac{c(m)}{\lambda} + 2\int_{T_1^+} |\nabla u_R|^2 |x|^2 G \, dx \, dt \,.$$

Finally we estimate the last term $\int_{T_1^+} |x|^2 |\nabla u_R|^2 G \, dx \, dt$ as follows:

$$\int_{T_1^+} |x|^2 |\nabla u_R|^2 G \, dx \, dt = \int_{-4}^{-1} \int_{|x| \leq \frac{1}{R^{\varepsilon/2}}, x_m > 0} |x|^2 |\nabla u_R|^2 G \, dx \, dt$$

$$+\int_{-4}^{-1}\int_{|x|\geq R^{-\varepsilon/2},x_m>0}|x|^2|\nabla u_R|^2 G\,dx\,dt\,.$$

The first term is clearly bounded by $R^{-\varepsilon}\Psi^+(R)$ (see the definition of $\Psi^+(R)$). The second term is bounded by

$$c(m)R^{-\varepsilon}e^{-\frac{1}{16R^\varepsilon}}\int_{-4}^{-1}\int_{\mathbb{R}^m_+}|\nabla u_R|^2\,dx\,dt\leq c(m)R^{-\varepsilon-m}e^{-\frac{R^{-\varepsilon}}{16}}\int_{-4R^2}^{-R^2}|\nabla u|^2\,dx\,dt$$
$$\leq c(m)R^{-\varepsilon-m+2}e^{-\frac{R^{-\varepsilon}}{16}}E(u_0)\,.$$

Since $0<R\leq 1$, the right-hand side of the above inequality is bounded by $c(m,\varepsilon)E(u_0)$. The conclusion (1.12) follows. □

Proof of (1.9).

$$\left|\int_{T_1^+}kR^2(|u_R|^2-1)u_R\left(\frac{x'}{R}\cdot\nabla_{x'}h_R\right)G\,dx\,dt\right|$$
$$\leq(\|\nabla u_0\|_{L^\infty(M)}+1)\int_{T_1^+}kR^2|u_R|\;\big||u_R|^2-1\big|\;|x|G\,dx\,dt$$
$$\leq(\|\nabla u_0\|_{L^\infty(M)}+1)\int_{T_1^+}kR^2\,\big||u_R|^2-1\big|\;\left[1+|u_R|^2|x|^2\right]G\,dx\,dt\,.$$

Applying Lemma 1.4 and Lemma 1.5, one has the right-hand side of the above inequality is bounded by

$$(\|\nabla u_0\|_{L^\infty(M)}+1)\left[\frac{\lambda}{2}A+\frac{c_6+c_7}{\lambda}(1+E_0+\Psi^+(R))/R^\varepsilon\right],\ \text{for all }\lambda\in(0,1),$$
$$\leq\ \frac{A}{4}+c_8(1+E_0+\Psi^+(R))/R^\varepsilon\,.$$

The latter inequality follows by letting $\lambda=\frac{1}{2}(1+\|\nabla u_0\|_{L^\infty(M)})^{-1}$, and $c_8=c_8(M,N,u_0)$. This completes the proof of (1.9) and hence the proof of Theorem 1.2. We note that one may take $\varepsilon=\frac{1}{4}$ in Theorem 1.2. □

2. Small Energy Regularity Theorem

Having established the monotonicity inequality for all points on $\overline{M}$ (for points inside M we refer to [S] and [CS]), we now want to prove the small energy regularity theorem for solutions of (1.1) on $\overline{M}$. We shall consider only those points at the boundary $\partial M \times R_+$. If a point p_0 is at the interior of M and the ball $B^M_{\rho_0}(p_0) = \{p \in \overline{M} : \mathrm{dist}_M(p, p_0) < \rho_0\}$ is cut by ∂M, the result can be proved in the same way as that for the boundary points. We shall also refer to [S] and [CS] for the case that the ball $B^M_{\rho_0}(p_0)$ is contained entirely in M.

Denote $P_R(x_0) = \{(x,t) : |x - x_0| < R, |t - t_0| < R^2\}, P_R(0)$, and $P_R^+ = P_R \cap \{x_m \geq 0\}$.

As in the previous section, various constants should depend only on M, ∂M, N and possibly E_0 and $u_0|_{\partial M}$. We have the following

Theorem 2.1. *Let $u = u^k : B^+_{\rho_0} \times [-T,T] \to N$ be a regular solution of* (1.1) *and assume that $T \leq \rho_0^2 \leq 1$. There exist constants $\varepsilon_0, \delta \in (0, \frac{1}{2})$ and c such that if for some $0 < R \leq \min(\varepsilon_0, \sqrt{T}/2)$, the inequality*

$$\Psi^+(R) \leq \varepsilon_0 \tag{2.1}$$

is satisfied, then there hold

$$\sup_{P^+_{\delta R}} e(u_k) \leq c\left[(\delta R)^{-2} + \|u_0\|_{c^2(\partial M)}\right] . \tag{2.2}$$

Proof. We prove (2.2) by a contradiction argument. Suppose Theorem 2.1 is not true, after various normalizations as those in [CS] and[C], one is lead to the existence of a sequence of solutions u_i of (1.1) in P_1^+ with the following properties:

(i) $\dfrac{\partial}{\partial t}u_i - \Delta u_i + k_i(|u_i|^2 - 1)u_i = 0$ in P_1^+,

(ii) $e_k(u_i) = \frac{1}{2}|\Delta u_i|^2 + \frac{k_i}{4}(|u_i|^2 - 1)^2 \leq 4$ in P_1^+,

(iii) $e_k(u_i)(x_i, 0) = 1$, with $x_i \to 0$ as $i \to \infty$,

(iv) $u_i\big|_{xm=0} = h_i(x')$ with $|\nabla^2 h_i| \leq \varepsilon_1^2 \|\nabla^2 u_0\|_{L^\infty(\partial M)} \to 0$, $|\nabla h_i| \leq \varepsilon_i \|\nabla u_0\|_{L^\infty(\partial M)}$,

(v) $\int_{P_1^+} e_k(u_i)\, dx\, dt \leq \varepsilon_i \to 0^+$ as $i \to \infty$,

(vi) $|h_i| = 1$ (cf. [CS] and [C]).

Moreover, via the calculation of [CS], we have the following Bochner-type inequality for $e_k(u_i)$:

$$\partial_t e_k(u_i) - \Delta e_k(u_i) \leq c_0 e_k(u_i) \in P_1^+ . \tag{2.3}$$

We now would like to obtain a contradiction from (i)–(vi) and (2.3).

To do so we may also assume that $k_i \geq 400$ in (ii). For, otherwise we would obtain from (i) and (iv) $W^{2,p}$-estimates for u_i, that is,

$$\|\nabla^2 u_i\|_{L^p(P^+_{1/2})} + \|\nabla u_i\|_{L^p(P^+_{1/2})} \leq c(p) \ , \tag{2.4}$$

for $1 < p < \infty$ (see [LSU]).

Moreover, by (v) and standard estimates for semilinear heat equations (cf. [LSU]), one has

$$\sup_{P^+_{1/2}} e_k(u_i) \leq c\left(\varepsilon_i + \|\nabla h_i\|^2_{c^1_{(\partial M)}}\right) \leq c\varepsilon_i \ . \tag{2.5}$$

The latter inequality contradicts to (iii).

Now since $k_i \geq 400$, (ii) implies in particular that

$$\big|\, |u_i|^2 - 1\big| \leq \frac{1}{5} \ .$$

We introduce a decomposition (polar decomposition) for $u_i = R_i W_i, R_i = |u_i|, W_i = \frac{u_i}{|u_i|}$ both are now well-defined. Moreover

$$|\nabla u_i|^2 = R_i^2 |\nabla W_i|^2 + |\nabla R_i|^2 \leq 4 \in P_1^+ \quad \text{and} \quad R_i \in \left[\frac{4}{5}, 1\right] \ . \tag{2.6}$$

From (i) we also derive the following equations

$$\frac{\partial W_i}{\partial t} - \Delta W_i - |\nabla W_i|^2 W_i - 2\frac{\nabla R_i}{R_i} \cdot \nabla W_i = 0 \ , \tag{2.7}$$

and

$$\frac{\partial R_i}{\partial t} - \Delta R_i + k_i(R_i^2 - 1)R_i + |\nabla W_i|^2 R_i = 0 \ . \tag{2.8}$$

Since $|\nabla W_i| \leq 7, |\frac{2\nabla R_i}{R_i}| \leq 5$ by (2.6), we obtain from (2.7) that (cf.[LSU]])

(2.9)

$$\sup_{P^+_{2/3}} |\nabla W_i|^2 \leq c_p \left[\int_{P+1} |\nabla W_i|^2 + \|\nabla W_i\|^2_{L^p(P_1^+)} + |\nabla h_i|^2_{c^1(\partial M)}\right] \ , \text{ for } p = 2m$$
$$\leq c\varepsilon_i^{\frac{1}{m}} \to 0 \quad \text{as } i \to \infty \ .$$

In particular, we have $\|\nabla W_i\|_{L^\infty(\partial M \times [-\frac{2}{3}, 0])} \leq c\varepsilon_i^{\frac{1}{m}} \to 0$.

Next we look at the equation for $\rho_i = 1 - R_i$:

$$\frac{\partial \rho_i}{\partial t} - \Delta\rho_i = R_i|\nabla W_i|^2 - k_i\rho_i R_i(1 + R_i) \ . \tag{2.10}$$

Since $0 \leq \rho_i \leq 1$ in P_1^+ and $\rho_i = 0$ on $\{x_m = 0\}$, we have

$$\begin{cases} \frac{\partial \rho_i}{\partial t} - \Delta\rho_i \leq c\varepsilon_i^{\frac{1}{m}} \in P_{2/3}^+ \\ \rho_i\Big|_{x_m=0} = 0 \ . \end{cases} \tag{2.11}$$

Hence, for $x \in P_{1/2}^+$,

$$\begin{aligned} \rho_i(x) &\leq c\Big(\int_{P_{2/3}^+} \rho_i + \varepsilon_i^{\frac{1}{m}}\Big)x_m \ , \quad \text{(cf. [LSU])} \\ &\leq c x_m \varepsilon_i^{\frac{1}{m}} \ , \quad \text{by (v).} \end{aligned}$$

Therefore, we also have

$$\|\nabla\rho_i\|_{L^\infty(\partial M\times[-\frac{1}{2},0])} = \|\nabla R_i\|_{L^\infty(\partial M\times[-\frac{1}{2},0])} \leq c\varepsilon_i^{\frac{1}{m}} \ . \tag{2.12}$$

Let $\tilde{e} = \max\{0, e_k(u_i) - 2c\varepsilon_i^{\frac{1}{m}}\}$, then (2.3) implies that

$$\partial_t\tilde{e} - \Delta\tilde{e} \leq c_0\tilde{e} \ , \quad \text{in } P_{1/2}^+ \ .$$

Moreover, above arguments show $\tilde{e}|_{x_m=0} = 0$. Thus the Moser's estimate for the linear heat equations implies that

$$\sup_{P_{1/4}^+} \tilde{e} \leq c\int_{P_{1/2}^+} \tilde{e} \leq c\int_{P_{1/2}^+} e_k(u_i) \leq c\varepsilon_i \ , \tag{2.13}$$

which goes to zero as $i \to \infty$.

(2.13) is an obvious contradiction to (iii), and thus we complete the proof of Theorem 2.1. □

Remark 2.1. The proof of the main theorem (stated in the introduction) is now identical to that in [S], [C], and [CS], and therefore we omit the details here.

3. General Target Manifolds

Here we shall consider the target manifold N being a compact smooth Reimannian submanifold of $\mathbb{R}^{n+\ell}$ without boundary. Instead of (1.1), we consider approximate solutions, $u = u^k, k = 1, 2, \ldots$, to following equations: (cf. [C] or[CS])

$$\partial_t u - \Delta u + k\chi'(\mathrm{dist}^2(u,N))\frac{d}{du}\left(\frac{\mathrm{dist}^2(u,N)}{2}\right) = 0 \;, \tag{3.1}$$

for $(x,t) \in M \times \mathbb{R}_+$, and

$$u(x,0) = u_0(x) \;, \quad u(\,\cdot\,,t)\Big|_{\partial M} = u_0(\,\cdot\,)\Big|_{\partial M} \;, \quad \text{for } t \in \mathbb{R}_+ \tag{3.2}$$

where χ is smooth monotone function on $\overline{\mathbb{R}}_+$ with $\chi(x) = s$ for $0 \le s \le \delta_N^2, \chi(x) \equiv 2\delta_N^2$, for $s \ge 2\delta_N^2$. Here $\delta_N \in (0, 1/2)$ is a positive constant so that the nearest neighbor projection $\pi_N : \mathbb{R}^{n+\ell} \to N$ is well-defined and smooth in a $2\delta_N$-neighborhood of N. Moreover, we may also assume that $\|D\pi_N(u) - P_N(u)\| \le 1/4$, for $u \in \mathbb{R}^{n+\ell}, \mathrm{dist}(u,N) \le 2\delta_N$. Here $P_N(u)$ is orthonormal projection of $\mathbb{R}^{n+\ell}$ onto $T_{\pi_N(u)}N$, the tangent space of N at $\pi_N(u)$.

For each fixed $k = 1, 2, \ldots$, it is again standard to show (cf. [LSU]) that there is a unique smooth solution of (3.1)–(3.2). Moreover, it satisfies the energy identity (1.2) (with the term $\frac{k}{2}\int_M (|u|^2 - 1)^2\,dM$ replacing by $\frac{k}{2}\int_M \chi(\mathrm{dist}^2(u,N))\,dM$).

As in [C] and [CS], we define

$$e_k(u) = \frac{1}{2}g^{\alpha\beta}u_{x_\alpha} \cdot u_{x_\beta} + \frac{k}{4}\chi(\mathrm{dist}^2(u,N))$$

and $\Psi^+(R)$ as before, etc... We claim $\Psi^+(R)$ satisfies the monotonicity inequality (1.3).

To see this, we follow the proof of Theorem 1.2. As in (1.4), we have (for $M = R_+^M$ case)

$$\begin{aligned} \frac{d}{dR}\Psi^+(R) \ge & \int_{T_1^+} \nabla V_R \cdot \nabla u_R G\,dx\,dt \\ & + \int_{T_1^+} \frac{1}{2}kR^2\chi'(\mathrm{dist}^2(u_R,N))\left(\frac{d}{du}\mathrm{dist}^2(u_R,N)\right) V_R G\,dx\,dt \;. \end{aligned} \tag{3.3}$$

Applying integration by parts as in (1.5) and (1.6), we then obtain

$$\frac{d}{dR}\Psi^+(R) \geq A + B + C + D \,, \tag{3.4}$$

where

$$A = -\int_{T_1^+} \frac{R}{2t} V_R^2 G\, dx\, dt \geq 0 \,,$$
$$B \geq \frac{A}{4} - c_1 \,,$$
$$D \geq -\Psi^+(R) - c_2 \,.$$

Here A, B, D are as in (1.6) before, and the absolute value of C is given by the left-hand side of (3.6) below.

Hence the issue is to verify

$$C \geq -\frac{A}{4} - \frac{c_3}{R^\varepsilon}(\Psi^+(R) + E_0 + 1) \,, \quad \varepsilon \in \left(0, \frac{1}{2}\right) \tag{3.5}$$

where c_1, c_2 and c_3 are constants as before.

(3.5) is equivalent to

$$\begin{aligned} &\left| \int_{T_1^+} kR^2 \chi'(\mathrm{dist}^2(u_R, N)) \left(\frac{d}{du}\mathrm{dist}^2(u_R, N) \right) \frac{x'}{R} \nabla_{x'} h_R G\, dx\, dt \right| \\ &\leq \|\nabla u_0\|_{L^\infty(\partial M)} \int_{T_1} kR^2 \chi'(\,\cdot\,)\mathrm{dist}(u_R, N)|x| G\, dx\, dt \\ &\leq \frac{A}{4} + \frac{c_3}{R^\varepsilon}(\Psi^+(R) + E_0 + 1) \,. \end{aligned} \tag{3.6}$$

For $\mathrm{dist}(u_R, N) < 2\delta_N$, we let $\frac{d}{du}\mathrm{dist}^2(u_R, N) = 2\nu(u_R)\mathrm{dist}(u_R, N)$. Then $\nu(u_R)$ is a well-defined unit vector as long as $\mathrm{dist}(u_R, N) > 0$. Moreover, $\nu(u_R)\mathrm{dist}(u_R, N)$ is a smooth function of u_R, for u_R in $2\delta_N$-neighborhood of N.

We let $\phi(s)$ be a monotone increasing, smooth function on R_+ with $\phi(s) \equiv 0$ for $s \leq \frac{1}{4k^2}$ and $\phi(s) \equiv 1$ for $s \geq \frac{1}{k^2}$. As in Section 2, we would like to multiply equation (3.1) by $\phi(\mathrm{dist}^2(u_R, N))\nu(u_R)\chi'(\mathrm{dist}^2(u_R, N))G$. Since this is a smooth function of u_R and since it is supported on $\{u_R \in \mathbb{R}^{n+\ell} : \mathrm{dist}(u_R, N) \in$

$[\frac{1}{2k}, \sqrt{2}\delta_N]\}$ we should find an equation for $\operatorname{dist}(u_R, N)$ in $\Omega = \{(x,t) \in T_1^+ : 0 < \operatorname{dist}(u_R, N) \le \sqrt{2}\delta_N\}$.

Let $u_R = v_R + (u_R - v_R), v_R = \pi_N(u_R)$. Then $(u_R - v_R) = \nu(u_R)\operatorname{dist}(u_R, N)$, for $(x,t) \in \Omega$. Denote $d = \operatorname{dist}(u_R, N)$, then d satisfies

(3.7)
$$d_t - \Delta d - d\langle \Delta\nu(u_R), \nu(u_R)\rangle + R^2 k\chi'(d^2)d - \langle \Delta(\pi_N(u_R)), \nu(u_R)\rangle = 0 \text{ in } \Omega.$$

(Note that (3.7) is simply the component of (3.1) in $\nu(u_R)$ direction.)

We note that $-\langle \Delta\nu(u_R), \nu(u_R)\rangle = |\nabla\nu(u_R)|^2$ and that

$$|\nabla v_R| \le \|D\pi_N\|_{L^\infty}|\nabla u_R| \le C|\nabla u_R|.$$

Hence $|\nabla(u_R - v_R)| \le C|\nabla u_R|$, and $|\langle \Delta\pi_N(u_R), \nu(u_R)\rangle| \le C|\nabla u_R|^2$. The last inequality follows from a direct computation, see e.g., (3.15) below.

We therefore have

$$(3.8)\quad d_t - \Delta d + \left(|\Delta\nu(u_R)|^2 + \chi'(d^2)kR^2\right) d \le C|\nabla u_R|^2\,, \quad \text{for } (x,t) \in \Omega\,.$$

Now let us estimate first the quantity

$$\begin{aligned}
&\int_{T_1^+} kR^2 \operatorname{dist}(u_R, N)\chi'\left(\operatorname{dist}^2(u_R, N)\right) G\,dx\,dt \\
\le &\int_{T_1^+\cap\{\operatorname{dist}(u_R,N)\le\frac{1}{k}\}} R^2\|\chi'\|_{L^\infty} G\,dx\,dt \\
&+ \int_{T_1^+\cap\{\frac{1}{k}<\operatorname{dist}(u_R,N)\le\delta_N\}} kR^2\operatorname{dist}(u_R,N)\phi(\operatorname{dist}^2(u_R,N))\,|\chi'(\operatorname{dist}^2(u_R,N))|^2 G\,dx\,dt \\
&+ \int_{T_1^+} kR^2\chi(\operatorname{dist}^2(u_R,N))G\,dx\,dt \cdot \frac{\|\chi'\|_{L^\infty}}{\delta_N} \\
\le &\ \frac{c_0}{\delta_N}\Psi^+(R) + c_0 + \int_{T_1^+} kR^2\operatorname{dist}(u_R,N)\phi(\operatorname{dist}^2(u_R,N))\left[\chi'(\operatorname{dist}^2(u_R,N))\right]^2 G\,dx\,dt \\
\triangleq &\ I + c_0\left(1 + \frac{\Psi^+(R)}{\delta_N}\right)\,.
\end{aligned}$$

Therefore, via $\chi' \ge 0$,

$$\int_{T_1^+} kR^2\operatorname{dist}(u_R, N)G\,dx \le I + 2c_0\left(1 + \frac{\Psi^+(R)}{\delta_N}\right)\,.$$

To estimate I, we multiply equation (3.8) by $\phi(d^2)\chi'(d^2)G$ to obtain

$$\int_{T_1^+} (d_t - \Delta d)\phi(\cdot)\chi'(\cdot)G\,dx\,dt + \int_{T_1^+} \phi(\cdot)(\chi'(\cdot))^2 kR^2 dG\,dx\,dt \le \int_{T_1^+} c|\nabla u|^2\phi(\cdot)\chi'(\cdot)G\,dx\,dt \le c\Psi^+(R) \tag{3.9}$$

But

$$\int_{T_1^+} -\Delta d\,\phi(\cdot)\chi'(\cdot)G\,dx\,dt = \int_{T_1^+} |\nabla d|^2(2\chi''(\cdot)\phi + 2\phi'(\cdot)\chi')\,d\,G\,dx\,dt + \int_{T_1} \frac{x}{2t}\cdot\nabla d\phi(\cdot)\chi'(\cdot)G\,.$$

Since $|\nabla d|^2 \le |\nabla(u - \pi_N u)|^2 \le c|\nabla u|^2$ we obtain, from (3.9), that

$$\begin{aligned} I &\le c\Psi^+(R) - \int_{T_1}\left(d_t + \frac{x}{2t}\nabla d\right)\phi(\cdot)\chi'(\cdot)G \\ &\le c\Psi^+(R) + \frac{\lambda}{4}A + \frac{c(m)}{\lambda}\,, \quad \forall\,\lambda\in(0,1)\,. \end{aligned} \tag{3.10}$$

Here we have used the fact that

$$u_R = d(u_R)\nu(u_R) + \pi_N(u_R) \quad \text{and} \quad \left|d_t + \frac{x}{2t}\nabla d\right|^2 \le \frac{R}{2|t|}\left|x\cdot\nabla u_R + 2t\,\partial_t u_R\right|^2/R^2.$$

Similarly, if we multiply the equation (3.1) and (3.8) by $|x|^2G\phi(d^2)\chi'(d^2)$, then we obtain, as in Section 1, that

(3.11)

$$\int kR^2\mathrm{dist}(u_R, N)|x|^2G\,dx \le \frac{\lambda}{4}A + \frac{c(m)}{\lambda} + \frac{c}{R^\varepsilon}\left(\Psi^+(R) + 1 + E_0\right)\,.$$

This completes the proof of the monotonicity inequality.

Finally, to the end of the paper, we outline the modification for the proof of Theorem 2.1 for general N. As in the proof of Theorem 2.1, it reduces to show the following is impossible (cf. also [C] and [CS]): there is a sequence of u^i solutions of (3.1) such that

(i) $\dfrac{\partial u^i}{\partial t} - \Delta u^i + k_i\chi'(\mathrm{dist}^2(u^i, N)\dfrac{d}{du}(\dfrac{\mathrm{dist}^2(u^i, N)}{2}) = 0$ in P_1^+,

(ii) $e_k(u^i) = \frac{1}{2}|\nabla u^i|^2 + \frac{k_i}{4}\chi(\mathrm{dist}^2(u^i, N)) \le 4$, in P_1^+,

(iii) $e_k(u^i)(x_i, 0) = 1$, with $(x_i, 0) \in P_1^+$ and $x_i \to 0$, as $i \to \infty$,

(iv) $u^i|_{x_m=0} = h_i(x')$, $\|\nabla h_i\|_{L^\infty(\partial M)} + \|\nabla^2 h_i\|_{L^\infty(\partial M)} \le \delta_i \to 0$,

(v) $\int_{P_1^+} e_k(u^i)\,dx\,dt = \varepsilon_i \to 0$ as $i \to +\infty$

(vi) $h_i(x') \in N$.

Moreover,

$$\frac{\partial}{\partial t} e_k(u^i) - \Delta e_k(u^i) \le c_0 e_k(u^i) \in P_1^+ . \tag{3.12}$$

Also $k_i \to +\infty$ as $i \to \infty$.

As (2.7) and (2.8), we consider the equations satisfied by $\pi_N(u^i)(x,t) \in N$ and $u^i - \pi_N(u^i) \perp T_{\pi_N(u^i)}N$. To do so we choose a point $(x_0, t_0) \in P_\delta^+$ and coordinate systems of $\mathbb{R}^{n+\ell}$ so that near $\pi_N u(x_0, t_0) = \underline{0} \in \mathbb{R}^{n+\ell}$, N can be represented by a graph $G : B_{4\delta}^n(0) \to \mathbb{R}^\ell$ where $\delta \in (0, \delta_{N/4})$ is a constant depending only on N. Moreover, G satisfies

$$N \cap (B_{4\delta}^n(0) \times [-4\delta, 4\delta]) = \mathrm{graph}(G) .$$
$$G(\underline{0}) = |\nabla G(\underline{0})| = 0 ,$$

and

$$|\nabla^2 G| + |\nabla^3 G| \le c_N , \quad \text{on } B_\delta^n(0) , \quad \text{with } c_N 4\delta < \frac{1}{10} .$$

Since (ii), we may assume that $u(P_\delta^+) \subset B_{3\delta}^n(0) \times [-3\delta, 3\delta]$.

We also choose a smooth orthonormal from $\{e_1, \dots, e_{n+\ell}\}$ along graph G so that $e_i(0) = (0, \dots, 1_{i^{th}}, \dots, 0)$, and that $\{e_1, \dots, e_n(p)\}$ span $\mathbb{T}_p N$.

Let us define a diffeomorphism $F : u \in \mathbb{R}^{n+\ell} \to V \in \mathbb{R}^{n+\ell}$ near $\underline{0} \in \mathbb{R}^{n+\ell}$ as follows:

$$\begin{cases} V_j = e_j(0) \cdot \pi_N(u) , & \text{for } j = 1, \dots, n , \quad \text{and} \\ V_j = e_j(\pi_N u) \cdot (u - \pi_N u) , & \text{for } j = n+1, \dots, n+\ell . \end{cases} \tag{3.13}$$

(Note that $\pi_N(u) = (V_1, \dots, V_n, G(V_1, \dots, V_n))$.)

Equivalently, one has

$$u = (V_1, \dots, V_n, G(V_1, \dots, V_n)) + \sum_{j=n+1}^{n+\ell} V_j e_j(V) , \tag{3.14}$$

here $e_j(V) = e_j(V_1, \dots, V_n, G(V_1, \dots, V_n))$.

Now we calculate the equation for $u^i - \pi_N u^i$ and $\pi_N u^i$ at the point $(x_0, t_0) \in P_\delta^+$. We may also assume

$$u^i(x_0, t_0) = (0, \dots, 0, d_i), d_i = |u^i(x_0, t_0) - \pi_N u^i(x_0, t_0)| \geq 0$$

for simplicity. At (x_0, t_0), (i) reduces to

$$\begin{aligned}(3.15)\quad (I_n + d_i M_1)\frac{\partial V^\top}{\partial t} - (I_n + d_i M_2)\Delta V^\top \\ = M_3\left(\nabla V^\top, \nabla V^\perp\right) + d_i M_4\left(\nabla V^\top, \nabla V^\top\right)\ ,\end{aligned}$$

and

(3.16)
$$\frac{\partial V^\perp}{\partial t} - \Delta V^\perp = M_5\langle \nabla V^\top, \nabla V^\top\rangle + d_i M_6\left(\nabla V^\top, \nabla V^\top\right) - k\chi'(d_i^2)\, d_i e_{n+\ell}(0)$$

where $V^\top = (V_i, \dots, V_n), V^\perp = (V_{n+\ell}, \dots, V_{n+\ell})$, $d_i M_1, d_i M_2, d_i M_4, d_i M_6$ are smooth matrix-valued functions of V and is bounded by cd_i at $(x_0, t_0), (d_i = |\pi_N u^i - u^i|), M_3$ and M_5 are also smooth matrix-valued functions of V. Note that all M_j's depend only on N, hence the function G definition N, and bounded by $\|\nabla^2 G\|_{L^\infty} + \|\nabla^3 G\|_{L^\infty}$, and also that $|\nabla V| \leq c|\nabla u^i|$.

From (3.15) one thus concludes that $\pi_N u^i$ satisfies an inequality of the form

$$\begin{aligned}(3.17)\quad &\left|A\frac{d(\pi_N u^i)}{dt} - B\Delta(\pi_N u^i)\right| \leq c_N|\nabla \pi_N u^i|\ , \quad (x,t) \in P_\delta^+ \quad \text{with} \\ &\|A - I_n\|_{L^\infty} + \|B - I_n\|_{L^\infty} \leq d_N|u^i - \pi_N u^i| \to 0 \quad \text{as } i \to \infty\ .\end{aligned}$$

Moreover, $\pi_N u^i|_{x_m=0} = h_i(x')$. Hence the $W^{2,p}$ estimate for linear parabolic equations and (3.17) imply that

$$\|\nabla \pi_N(u^i)\|^2_{L^\infty(P^+_{\delta/2})} \leq c_\delta\left(\int_{P_\delta^+} |\nabla u^i|^2 + \|\nabla h_i\|^2_{c^1}\right) \leq c_\delta(\varepsilon_i + \delta_i) \to 0 \quad \text{as } i \to \infty\ .$$

On the other hand, if we take the component of equation (3.16) in $e_{n+\ell}(0)$ direction, we obtain

$$(3.18)\qquad \frac{\partial d}{dt} - \Delta d \leq -k\chi'(d^2)d + c|\nabla u|^2$$

whenever $d(x,t) > 0$, (cf. also (2.10)) and $(x,t) \in P_\delta^+$ since $d|_{x_m=0} = 0$, we obtain from (3.18) that

$$\begin{aligned} d(x,t) &\le \tilde{d}(x,t) && \text{in } P_\delta^+ \quad \text{where} \\ \frac{\partial}{\partial t}\tilde{d} - \Delta \tilde{d} &= c|\nabla u|^2 && \text{in } P_\delta^+ \quad \text{and} \\ \tilde{d} &= d && \text{on } \partial_p P_\delta^+ - \{t=0\}\ . \end{aligned} \tag{3.19}$$

It is easy to see, from (ii) and (v) that

$$\tilde{d} \le c\varepsilon_i^{\frac{1}{m}} x_m \to 0\ , \quad \text{in } P_{\delta/2}^+\ , \quad \text{as } i \to \infty\ , \tag{3.20}$$

and therefore

$$e_k(u^i)\Big|_{x_m=0} \to 0\ , \quad \text{for } -\frac{\delta}{2} \le t \le 0\ .$$

The desired contradiction follows as before.

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