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Spacelike CMC surfaces near null infinity of the Schwarzschild spacetime

Luen-Fai Tam*

Motivated by a result of Treibergs, given a smooth function $f(\mathbf{y})$ on the standard sphere $\mathbb{S}^2$, $\mathbf{y} \in \mathbb{S}^2$, and any positive constant H_0, we construct a spacelike surface with constant mean curvature H_0 in the Schwarzschild spacetime, which is the graph of a function $u(\mathbf{y}, r)$ defined on $r > r_0$ for some $r_0 > 0$ in the standard coordinates exterior to the blackhole. Moreover, u has the following asymptotic behavior:

$$\left| u(\mathbf{y}, r) - r_* - \left(f(\mathbf{y}) + r^{-1}\phi(\mathbf{y}) + \frac{1}{2} r^{-2}\psi(\mathbf{y}) \right) \right| \le Cr^{-3}$$

for some $C > 0$, where $r_* = r + 2m\log(\frac{r}{2m} - 1)$. Here ϕ, ψ are functions on $\mathbb{S}^2$ given by

$$\begin{cases} \phi = \frac{1}{2}\left(H_0^{-2} + |\nabla_{\mathbb{S}^2} f|^2_{\mathbb{S}^2}\right); \\ \psi = \frac{1}{2}\left(H_0^{-2}\Delta_{\mathbb{S}^2} f + \langle \nabla_{\mathbb{S}^2} |\nabla_{\mathbb{S}^2} f|^2_{\mathbb{S}^2}, \nabla_{\mathbb{S}^2} f\rangle_{\mathbb{S}^2}\right). \end{cases}$$

In particular, the surface intersects the future null infinity with the cut given by the function f. In addition, we prove that the function $u - r_*$ is uniformly Lipschitz near the future null infinity.

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1. Introduction

In [12], Treibergs proved the following: Given a C^2 function $f(\mathbf{y})$ on the standard sphere $\mathbb{S}^{n-1}$ and a constant $H_0 > 0$ there exists an entire spacelike surface in the Minkowski space $\mathbb{R}^{n,1}$ with constant mean curvature H_0 which is the entire graph of a function u such that

$$\lim_{r\to\infty} (u(\mathbf{y}, r) - r) = f(\mathbf{y}).$$

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Here $(\mathbf{y}, r) \in \mathbb{S}^{n-1} \times (0, \infty)$ is the spherical coordinates of $\mathbb{R}^n$. The result implies that the surface will intersect the future null infinity at the cut given by $(\mathbf{y}, f(\mathbf{y}))$.

Motivated by this result, we want to study what one may obtain for Schwarzschild spacetime. Recall the standard Schwarzschild metric defined on $r > 2m > 0$ is:

$$g_{\text{Sch}} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\sigma, \tag{1.1}$$

$-\infty < t < \infty$, where $r = \sum_{i=1}^{3}(x^i)^2$ with $(x^1, x^2, x^3) \in \mathbb{R}^3$ and σ is the standard metric of the unit sphere $\mathbb{S}^2$. The future null infinity $\mathcal{I}^+$ of the Schwarzschild spacetime is of the form $\mathbb{S}^2 \times \mathbb{R}$ with $\mathbb{S}^2$ being the standard sphere, see §2 for more details. Given a cut $\mathcal{C}$ in $\mathcal{I}^+$ represented as $(\mathbf{y}, f(\mathbf{y}))$, $\mathbf{y} \in \mathbb{S}^2$ and f is a function of $\mathbf{y}$, we want to construct a spacelike constant mean curvature (CMC) surface with positive constant mean curvature which intersects $\mathcal{I}^+$ at this cut. To state our result, let

$$r_* = r + 2m \log(\frac{r}{2m} - 1). \tag{1.2}$$

We obtain the following:

Theorem 1.1. *Let f be a smooth function on $\mathbb{S}^2$. For any constant $H_0 > 0$, there exists $u(\mathbf{y}, r)$ defined for $\mathbf{y} \in \mathbb{S}^2$, $r > r_0$, for some $r_0 > 2m$ such that the graph of u in the Schwarzschild spacetime is a spacelike hypersurface of constant mean curvature $H_0 > 0$ with boundary value at the future null infinity given by f. More precisely, u satisfies:*

$$\lim_{r \to \infty} (u(\mathbf{y}, r) - r_*) = f(\mathbf{y}),$$

for all $\mathbf{y} \in \mathbb{S}^2$. In fact, there exists $C > 0$ such that

$$\left| u(\mathbf{y}, r) - r_* - \left(f(\mathbf{y}) + r^{-1}\phi(\mathbf{y}) + \frac{1}{2} r^{-2}\psi(\mathbf{y}) \right) \right| \le C r^{-3}$$

for all $\mathbf{y} \in \mathbb{S}^2$, $r > r_0$, where

$$\begin{cases} \phi = \frac{1}{2}\left(H_0^{-2} + |\widetilde{\nabla} f|^2_{\mathbb{S}^2}\right); \\ \psi = \frac{1}{2}\left(H_0^{-2}\widetilde{\Delta} f + \langle \widetilde{\nabla}|\widetilde{\nabla} f|^2_{\mathbb{S}^2}, \widetilde{\nabla} f\rangle_{\mathbb{S}^2}\right). \end{cases}$$

Here $\widetilde{\nabla}$ and $\widetilde{\Delta}$ are the covariant derivative and Laplacian of the standard $\mathbb{S}^2$ respectively. The inner product is taken with respect to the standard metric of $\mathbb{S}^2$.

We should emphasis that unlike [12], we can only construct a surface which is defined near the future null infinity.

In addition to the results on spacelike CMC surfaces in the Minkowski space by Treibergs [12], there is a well-known result by Bartnik [2] which states that there exists a complete spacelike maximal hypersurface asymptotic to the spatial infinity in an asymptotically flat spacetime satisfying a uniform *interior condition* (see [2, p.169] for the definition). In [1], Andersson and Iriondo proved the existence of a complete spacelike CMC surface with positive constant mean curvature on an *asymptotically Schwarzschild spacetime* (see [1, Definition 2.1]) which satisfies a uniform *future interior condition* (see [1, Definition 4.1]). The constructed surface intersects the future null infinity at $(\mathbf{y}, f(\mathbf{y}))$ with $f(\mathbf{y})$ =constant. In [3], Bartnik, Chruściel and Ó Murchada studied complete spacelike surfaces which are maximal outside a spatially compact set on certain asymptotically flat spacetimes. Recently, spacelike graph of a function which is asymptotically zero in the Minkowski spacetime $\mathbb{R}^{n,1}$ with prescribed mean curvature outside a compact set in $\mathbb{R}^n$ has been constructed in [4] by Bartolo, Caponio and Pomponio. On the other hand, spacelike CMC surfaces in the Schwarzschild spacetime have been studied by many people. In particular, In [7, 8] K-W Lee and Y-I Lee gave a complete description of spacelike spherical symmetric constant mean curvature surfaces in the Kruskal extension of Schwarzschild spacetime. See also the references therein.

In Theorem 1.1, the constructed surface is asymptotically to a cut in the null infinity. Some higher order rate of approximation is also obtained. The main idea is to construct a good foliation near the future null infinity as in [1] with good estimates so that one can obtain estimates of the so-called tilt factor of a spacelike surface, using a result in Bartnik [2]. We also need to construct suitable barrier. Our construction is to use the results by Li, Shi and the author in [9]. Without further assumptions on f one might not be able to construct a better barrier to obtain a better approximation. See Remark 4.1 for details.

A natural question is on the regularity of the function $u - r_*$. In [11], Stumbles constructed spacelike CMC surfaces in the Minkowski spacetime (or nearby spacetime) so that the surfaces are C^3 near and up to the future null infinity, provided the cut is represented by $(\mathbf{y}, f(\mathbf{y}))$ with f being close to a constant. One may not expect a C^4 regularity by the results in [9]. In our case, the foliation mentioned above in our construction is given by a time function. From the construction, the so-called tilt factor (see the definition in §3) of the constructed surface with respect to this time function is uniformly bounded. Using this fact, we have the following:

Corollary 1.1. *The function $Q(\mathbf{y}, s) = r_* - u(\mathbf{y}, r)$ with $r = s^{-1}$ is uniformly Lipschitz on $\mathbb{S}^2 \times (0, s_0)$ for some $s_0 > 0$.*

This is a corollary of a more general result. See Theorem 5.1 for details. This theorem might also be applied to the spacelike CMC surface in [1, Theorem 4.1].

The organization of the paper is as follows. In §2, we will recall the structure of future null infinity $\mathcal{I}^+$ in the Schwarzschild spacetime and will construct a suitable foliation near $\mathcal{I}^+$. In §3, we will give detailed estimation on the foliation which will be used later. In §4 we will prove Theorem 1.1. In §5 we will discuss a general Lipschitzian regularity property of spacelike surfaces near $\mathcal{I}^+$ and prove Corollary 1.1.

2. Future null infinity and a foliation

2.1. Future null infinity

Let us recall the future null infinity of the Schwarzschild spacetime. We always assume that $\frac{\partial}{\partial t}$ is future pointing. Consider the retarded null coordinate

$$v = t - r_*, \tag{2.1}$$

where r_* is given by (1.2). Let $s = r^{-1}$, then

$$\begin{aligned} g = g_{\mathrm{Sch}} &= -(1-2ms)dv^2 + 2s^{-2}dvds + s^{-2}\sigma \\ &= s^{-2}(-s^2(1-2ms)dv^2 + 2dvds + \sigma) \\ &=: s^{-2}\bar{g}, \end{aligned} \tag{2.2}$$

with $0 < s < \frac{1}{2m}$, $-\infty < v < \infty$. Here the unphysical metric $\bar{g}$ is the product metric:

$$\bar{g} = (\sigma_{AB}) \oplus \begin{pmatrix} 0 & 1 \\ 1 & -s^2(1-2ms) \end{pmatrix}, \tag{2.3}$$

where (σ_{AB}) is the standard metric for $\mathbb{S}^2$ in local coordinates y^1, y^2. So y^1, y^2, s, v are coordinates of the spacetime. We also write (y^1, y^2, s, v) as

(y^1, y^2, y^3, y^4). $\overline{g}$ can be extended as a smooth Lorentz metric defined on $\mathbf{y} \in \mathbb{S}^2$, $s \in [0, 1/2m)$, $v \in \mathbb{R}$. The future null infinity $\mathcal{I}^+$ is identified with the boundary $s = 0$, which is a null hypersurface. For later reference,

$$(\overline{g}^{ab}) = (\overline{g})^{-1} = (\sigma^{AB}) \oplus \begin{pmatrix} s^2(1-2ms) & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

where (σ^{AB}) is the inverse of (σ_{AB}). Hence for the physical metric, $g^{ab} = s^2\overline{g}^{ab}$.

Convention: In the following $a, b, c \ldots$ run from 1 to 4; $i, j, k, \ldots$ run from 1 to 3 and $A, B, C, \ldots$ run from 1 to 2. Einstein summation convention will be used.

2.2. Foliation

Given a smooth function $f(\mathbf{y})$ on $\mathbb{S}^2$. Consider the cut $\mathcal{C}$ given by $(\mathbf{y}, f(\mathbf{y})), \mathbf{y} \in \mathbb{S}^2$ in $\mathcal{I}^+$. We want to extend it to a spacelike CMC surface in the Schwarzschild spacetime. As in [1], we need to construct a suitable foliation near $\mathcal{I}^+$ related to f. For $\tau > 0$, let

$$P(\mathbf{y}, s, \tau) = f(\mathbf{y}) + s\phi(\tau, \mathbf{y}) + \frac{1}{2!}s^2\psi(\tau, \mathbf{y}), \tag{2.5}$$

where $\phi = P_s, \psi = P_{ss}$ at $s = 0$ are smooth functions in $\tau, \mathbf{y}$, given by

$$\begin{cases} \phi = -\frac{1}{2}\left(\tau^2 + |\widetilde{\nabla} f|^2_{\mathbb{S}^2}\right); \\ \psi = \frac{1}{2}\left(\tau^2\widetilde{\Delta} f + \langle \widetilde{\nabla}|\widetilde{\nabla} f|^2_{\mathbb{S}^2}, \widetilde{\nabla} f\rangle_{\mathbb{S}^2}\right). \end{cases} \tag{2.6}$$

The choice of ϕ, ψ is motivated by the result in [9, Theorem 3.1], so that if Σ_τ is the surface given by $(\mathbf{y}, s) \to (\mathbf{y}, s, -P)$ in the $\mathbf{y}, s, v$ coordinates, then Σ_τ is spacelike near $s = 0$ and its mean curvature H is such that $H = \tau^{-1}$ and $\partial_s H = 0$ at $s = 0$.

Direct computations give:

$$\begin{cases} P_\tau = -\tau s + \frac{1}{2}\tau s^2 \widetilde{\Delta} f = -\tau s\left(1 - \frac{1}{2}s\widetilde{\Delta} f\right); \\ P_s = \phi + s\psi; \\ P_A = f_A + s\phi_A + \frac{1}{2}s^2\psi_A, A = 1, 2. \end{cases} \tag{2.7}$$

Here for a smooth function θ in $\mathbf{y}, s, \tau$, the partial derivative of θ with respect to s is denoted by θ_s etc.

Let $0 < \tau_1 < \tau_2 < \infty$ be fixed. Let

$$M = \{\mathbf{y} \in \mathbb{S}^2, s \in (0, \frac{1}{2m}), \tau \in (\tau_1, \tau_2)\} = \mathbb{S}^2 \times (0, s_0) \times (\tau_1, \tau_2).$$

Consider the map Φ from M to the Schwarzschild spacetime in $\mathbf{y}, s, v$ coordinates defined by:

$$\Phi(\mathbf{y}, s, \tau) = (\mathbf{y}, s, v(\mathbf{y}, s, \tau)) \tag{2.8}$$

with $v(\mathbf{y}, s, \tau) = -P(\mathbf{y}, s, \tau)$.

Lemma 2.1. *There is $\frac{1}{2m} > s_0 > 0$ depending only on τ_1, τ_2, f such that Φ is a diffeomorphism onto its image. Hence $\Phi(M)$ is parametrized by $\mathbf{y}, s, \tau$. Moreover,*

$$\frac{\partial \tau}{\partial v} = -\frac{1}{P_\tau}; \frac{\partial \tau}{\partial s} = -\frac{P_s}{P_\tau}; \frac{\partial \tau}{\partial y^A} = -\frac{P_A}{P_\tau}, A = 1, 2. \tag{2.9}$$

Proof. It is easy to see that if $s_0 > 0$ is small enough, then $P_\tau < 0$. From this and some computations, it is easy to see the lemma is true. □

Let $s_0 > 0$ be as in the lemma, then

$$\Phi(M) = \{(\mathbf{y}, s, v) |\ P(\mathbf{y}, s, \tau_2) < v < P(\mathbf{y}, s, \tau_1)\}. \tag{2.10}$$

By the lemma, we can see that τ is a smooth function on $\Phi(M)$.

Given $\tau \in (\tau_1, \tau_2)$, let

$$\Sigma_\tau = \{v = -P(\mathbf{y}, s, \tau)\}$$

which is a level surface of τ. For fixed τ, let $F(\mathbf{y}, s, v) = v + P(\mathbf{y}, s, \tau)$. To simplify notation, define

$$L =: -\left(2P_s + s^2(1 - 2ms)P_s^2 + |\widetilde{\nabla} P|^2\right) = -\overline{g}(\overline{\nabla} F, \overline{\nabla} F) \tag{2.11}$$

where $\overline{\nabla}$ is the derivative with respect to the unphysical metric $\overline{g}$. Here and later, we simply write $|\widetilde{\nabla} P|$ instead of $||\widetilde{\nabla} P||_{\mathbb{S}^2}$ if this does not cause confusion.

Lemma 2.2. *There is $\frac{1}{2m} > s_0 > 0$ depending only on τ_1, τ_2 and f such that Σ_τ is spacelike in $(0, s_0)$ for $\tau \in (\tau_1, \tau_2)$. In fact,*

$$\nabla \tau = -P_\tau^{-1} \left(g^{va} + g^{ia} P_i\right) \partial_{y^a},$$

and

$$g(\nabla\tau,\nabla\tau) = -s^2 P_\tau^{-2} L.$$

Moreover, for all $\tau \in (\tau_1, \tau_2)$, Σ_τ is a smooth up to $\mathcal{I}^+$ in the sense that P is smooth up to $s = 0$, which intersects $\mathcal{I}^+$ at the cut $\mathcal{C}$ given by $\{(\mathbf{y}, f(\mathbf{y}))|\ \mathbf{y} \in \mathbb{S}^2\}$.

Proof. First let s_0 be as in Lemma 2.1 so that $P_\tau < 0$. Recall that

$$(y^1, y^2, y^3, y^4) = (y^1, y^2, s, v).$$

Denote the coordinate frame by ∂_a. For $\tau \in (\tau_1, \tau_2)$, by (2.9), we have

$$\begin{aligned}\nabla\tau =& g^{ab}\frac{\partial\tau}{\partial y^a}\partial_b\\ =& \left(g^{vb}\frac{\partial\tau}{\partial v} + g^{ib}\frac{\partial\tau}{\partial y^i}\right)\partial_b\\ =& -\frac{1}{P_\tau}\left(g^{vb} + g^{ib}P_i\right)\partial_b.\end{aligned}$$

On the other hand, direct computation shows

$$\begin{aligned}\langle\nabla\tau,\nabla\tau\rangle =& s^2\overline{g}^{ab}\frac{\partial\tau}{\partial y^a}\frac{\partial\tau}{\partial y^b}\\ =& -s^2 P_\tau^{-2} L.\end{aligned}$$

By (2.7), $P_\tau = -(\tau s + O(s))$. By (2.7) and (2.11),

$$\begin{aligned}L =& -\left(-\tau^2 - |\widetilde{\nabla} f|^2 + |\widetilde{\nabla} f|^2 + O(s)\right)\\ =& \tau^2 + O(s).\end{aligned} \tag{2.12}$$

It is easy to see that if $0 < s_0 < \frac{1}{2m}$ is small enough, depending only on τ_1, τ_2 and f, then Σ_τ is spacelike in $0 < s < s_0$. The last assertion is obvious. ☐

Let s_0 be as in the lemma. Since $\frac{\partial}{\partial t} = \partial_v$, we have

$$\begin{aligned}g(\nabla\tau, \frac{\partial}{\partial t}) =& -P_\tau^{-1} g((g^{va} + g^{ia}P_i)\partial_a, \partial_v)\\ =& -P_\tau^{-1}(g^{va} + g^{ia}P_i)g_{av}\\ =& -P_\tau^{-1}\\ >& 0.\end{aligned} \tag{2.13}$$

So τ is a time function on $\Phi(M)$ with $\nabla\tau$ being past directed.

3. Estimates on the foliation

Let s_0 be as in Lemma 2.2 so that Σ_τ is spacelike for $0 < s < s_0$. Let $M = \mathbb{S}^2 \times (0, s_0) \times (\tau_1, \tau_2)$. Then Φ is a parametrization of $\Phi(M)$, with τ being a time function. Recall that, if θ is a function in $\mathbf{y}, s, \tau$, then the partial derivatives will be denoted by $\theta_A, \theta_s, \theta_\tau$ etc. On the other hand, when consider τ as a function of $(y^1, y^2, y^3, y^4) = (y^1, y^2, s, v)$, the derivative of θ with respect to y^a will be denoted by $\partial_a \theta$. Hence

$$\partial_A\theta = \theta_\tau \tau_A + \theta_A, \ \partial_s\theta = \theta_\tau\tau_s + \theta_s, \quad \partial_v\theta = \theta_\tau\tau_v. \tag{3.1}$$

Let T be the unit future pointing timelike normal of Σ_τ so that

$$T = -\alpha\nabla\tau \tag{3.2}$$

where $\alpha > 0$ is the *lapse function* of τ given by

$$\alpha^2 = -(\langle\nabla\tau, \nabla\tau\rangle)^{-1} = s^{-2}P_\tau^2 L^{-1}. \tag{3.3}$$

For a spacelike hypersurface Σ with future directed unit normal $\mathbf{n}$, the *tilt factor* ν with respect to T is defined as $\nu = -g_{\mathrm{Sch}}(T, \mathbf{n})$.

We want to apply a result of Bartnik [2] to estimate the tilt factor for spacelike surfaces in $\Phi(M)$. First recall the following setting in the Bartnik's work. In $\Phi(M)$, introduce the Riemannian metric Θ:

$$\Theta = g_{\mathrm{Sch}} + 2\omega\otimes\omega \tag{3.4}$$

where ω is the dual of the unit normal T. For example, for a vector field V, $||V||_\Theta^2 = \sum_{i=1}^3 \langle V, w_i\rangle^2 + \langle V, T\rangle^2$, where w_1, w_2, w_3 form an orthonormal basis of Σ_τ with respect to metric induced by the Schwarzschild metric g. In order to apply [2, Theorem 3.1(iii)] (see also remarks on [2, p.162]) to a compact spacelike hypersurface Σ with smooth boundary $\partial\Sigma$ in $\Phi(M)$ so that τ =constant on $\partial\Sigma$, we need to estimate the following quantities:

$$\alpha, ||\alpha^{-1}\nabla\alpha||_\Theta, ||\mathcal{K}||_\Theta, ||\nabla T||_\Theta, ||\nabla\nabla T||_\Theta, ||\vec{H}_{\partial\Sigma}||_\Theta \tag{3.5}$$

where $\mathcal{K}$ is the second fundamental form of Σ_τ and ∇ is the connection of g_{Sch} and $\vec{H}_{\partial\Sigma}$ is the mean curvature vector of $\partial\Sigma$. We have used the fact that the g_{Sch} is Ricci flat. Our result will be summarized in Theorem 3.1 below. We proceed as in [1].

Since we may cover $\mathbb{S}^2$ with finitely many coordinate neighborhoods, we may work on a coordinate neighborhood first. Hence let us fix a coordinate neighborhood U with local coordinates y^1, y^2. The coordinate frame with respect to this coordinate is given by:

$$
\begin{cases}
e_A =: \Phi_*(\frac{\partial}{\partial y^A}) = -P_A\partial_v + \partial_A, A = 1, 2;\\
e_3 =: \Phi_*(\frac{\partial}{\partial s}) = -P_s\partial_v + \partial_s;\\
e_4 =: \Phi_*(\frac{\partial}{\partial \tau}) = -P_\tau\partial_v.
\end{cases}
\tag{3.6}
$$

Here ∂_a are coordinate frames with respect to $y^1, y^2, y^3 = s, y^4 = v$. Note that if θ is a smooth function in $\mathbf{y}, s, \tau$, then $e_A(\theta) = \theta_A$ etc. Note also that e_1, e_2, e_3 are tangential to Σ_τ, i.e. τ =constant. It is easy to see:

$$
\begin{cases}
\partial_v = -\frac{1}{P_\tau} e_4;\\
\partial_s = -\frac{P_s}{P_\tau} e_4 + e_3;\\
\partial_A = -\frac{P_A}{P_\tau} e_4 + e_A, \quad A = 1, 2.
\end{cases}
\tag{3.7}
$$

We may assume that σ_{AB} is smooth up to the boundary of U and that the eigenvalues of (σ_{AB}) is bounded below by some constant $\lambda > 0$.

Notation: In the following $c(s^\ell), c_{ab}(s^\ell), \ldots$ for integers ℓ will denote functions of the form $s^\ell\Lambda$ where Λ is a smooth function in $\mathbf{y}, s, \tau$ in $\overline{U} \times [0, s_0] \times [\tau_1, \tau_2]$. They may vary from line to line. For example, in (3.7), we have

$$\partial_v = c(s^{-1})e_4,$$

if s_0 is small enough.

Lemma 3.1. *In the above setting, for $\mathbf{y} \in U$, then the following are true:*

(i) The metric $\overline{g}$ in the frame e_a is given by

$$
\begin{cases}
\overline{g}(e_A, e_B) = \sigma_{AB} - s^2(1 - 2ms)P_A P_B, 1 \le A, B \le 2;\\
\overline{g}(e_A, e_3) = \overline{g}(e_3, e_A) = -P_A - s^2(1 - 2ms)P_A P_s, 1 \le A \le 2;\\
\overline{g}(e_3, e_3) = -2P_s - s^2(1 - 2ms)P_s^2\\
\overline{g}(e_A, e_4) = \overline{g}(e_4, e_A) = -s^2(1 - 2ms)P_\tau P_A, 1 \le A \le 2;\\
\overline{g}(e_3, e_4) = \overline{g}(e_4, e_3) = -s^2(1 - 2ms)P_\tau P_s;\\
\overline{g}(e_4, e_4) = -s^2(1 - 2ms)P_\tau^2.
\end{cases}
$$

(ii) Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ be an orthonormal basis for Σ_τ with respect to $\overline{g}$ obtained from e_1, e_2, e_3 using Gram-Schmidt process with respect to the metric induced by $\overline{g}$. Then $\varepsilon_i = c_{ik}(s^0)e_k$, $e_i = c^{ik}(s^0)\varepsilon_k$.

(iii) If $s_0 > 0$ is small enough depending only τ_1, τ_2 and f, then $\alpha = 1 + c(s)$ and

$$T = c_i(s)e_i + \alpha^{-1}e_4.$$

Proof. Using (3.6) and (2.3), direct computations give (i).

In the following, we always assume $s_0 > 0$ is small depending only on τ_1, τ_2 and f. Let $\overline{g}_{ab} = \overline{g}(e_a, e_b)$. Recall that $\overline{g}_{ab}$ can be extended smoothly up to $s = 0$. Moreover, at $s = 0$, $P_A = f_A$. Hence at $s = 0$, for any $(\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3$, let $f_A = \sigma_{AB} f^B$, for any $\varepsilon > 0$ we have:

$$\begin{aligned}
\overline{g}_{ij}\xi^i\xi^j =&\sigma_{AB}\xi^A\xi^B - 2f_A\xi^A\xi^3 + (\tau^2 + |\widetilde{\nabla} f|^2)(\xi^3)^2 \\
=&\sigma_{AB}\xi^B\xi^B - 2\sigma_{AB}f^B\xi^A\xi^3 + (\tau^2 + \sigma_{AB}f^Af^B)(\xi^3)^2 \\
\geq&\sigma_{AB}\xi^B\xi^B - \left(\varepsilon\sigma_{AB}\xi^A\xi^B + \varepsilon^{-1}\sigma_{AB}f^Af^B(\xi^3)^2\right) \\
&+ (\tau^2 + \sigma_{AB}f^Af^B)(\xi^3)^2 \\
=&(1-\varepsilon)\sigma_{AB}\xi^B\xi^B + (\tau^2 + (1 - \varepsilon^{-1}\sigma_{AB}f^Af^B)(\xi^3)^2 \\
\geq&C\left((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2\right),
\end{aligned}$$

for some $C > 0$ depending only on λ, τ_1, τ_2 and $|\widetilde{\nabla} f|$, if we choose $\varepsilon < 1$, ε close to 1 so that $\tau^2 + (1 - \varepsilon^{-1}\sigma_{AB}f^Af^B) \geq \tau^2/2$. On the other hand, away from $s = 0$, $(\overline{g}_{ij})$ is smooth and positive definite. Let $\varepsilon_i = c_{ik}e_k$ as in the lemma, one can see that c_{ik} are smooth function of y^a. On the other hand,

$$\delta_{ij} = c_{ik}c_{jl}\overline{g}_{kl}.$$

Hence $\overline{g}^{ij} = c_{ki}c_{kj}$. In particular, for each i, $\overline{g}^{ii} = \sum_k c_{ik}^2$. From this one can conclude that $c_{ik} = c(s^0)$. Similarly one can prove that $c^{ik} = c(s^0)$.

(iii) By (3.3), (2.7) and (2.12),

$$\begin{aligned}
\alpha =& - s^{-1}P_\tau L^{-\frac{1}{2}} \\
=&1 + c(s).
\end{aligned}$$

By Lemma 2.2

$$\begin{aligned}
T =& - \alpha\nabla\tau \\
=&\alpha P_\tau^{-1}\left(g^{vb}\partial_v + g^{ib}P_i\right)\partial_b \\
=&\alpha s^2 P_\tau^{-1}\left[\sigma^{BA}P_B\partial_A + (1 + s^2(1-2ms)P_s)\partial_s + P_s\partial_v\right] \\
=&\alpha s^2 P_\tau^{-1}\Big[\sigma^{BA}P_B(-\frac{P_A}{P_\tau}e_4 + e_A) + (1 + s^2(1-2ms)P_s)(-\frac{P_s}{P_\tau}e_4 + e_3) \\
&- \frac{P_s}{P_\tau}e_4\Big]
\end{aligned}$$

$$
\begin{aligned}
&=\alpha s^2 P_\tau^{-1}\left(\sigma^{BA}P_B e_A + (1+s^2(1-2ms)P_s)e_3\right) + \alpha s^2 P_\tau^{-2} L e_4\\
&=c(s)e_i + \alpha^{-1}e_4,
\end{aligned}
$$

by (iii), (2.7), (3.3) and (3.7). This completes the proof of the lemma. □

Let

$$
w_i = s\varepsilon_i,\ i=1,2,3. \tag{3.8}
$$

Then w_i form an orthonormal frame for Σ_τ with respect to the metric induced by the Schwarzschild metric g.

Lemma 3.2. *If s_0 is small enough, depending only on τ_1,τ_2 and f, then $\alpha,\alpha^{-1},||\nabla\alpha||_\Theta$ are uniformly bounded in $U\times(0,s_0)\times(\tau_1,\tau_2)$.*

Proof. The estimates of α,α^{-1} follow immediately from Lemma 3.1. Let us estimate the derivatives of α. By Lemma 3.1, (3.6) and

$$
\begin{aligned}
w_i(\alpha) =& s\varepsilon_i(\alpha)\\
=& sc_{ik}(s^0)e_k(1+c(s))\\
=& c_i(s).\\
T(\alpha) =& (c_i(s)e_i + \alpha^{-1}e_4)(\alpha)\\
=& c(s).
\end{aligned}
$$

Hence $||\nabla\alpha||_\Theta$ is uniformly bounded in $U\times(0,s_0)\times(\tau_1,\tau_2)$. □

Let $\mathcal{K}$ be the second fundamental form of Σ_τ. We want to estimate $||\mathcal{K}||_\Theta$. Since the metric $\overline{g}$ is a product metric, it is more easy to compute the second fundamental form with respect to $\overline{g}$. Let us recall the following fact:

Lemma 3.3. *Let Σ be a spacelike hypersurface in a spacetime (M,g). Suppose $g=e^{2\lambda}\overline{g}$. Let $\mathbf{n}$ be a unit normal of M with respect to g. Let $\overline{\mathbf{n}}=e^\lambda\mathbf{n}$, which is a unit normal with respect to $\overline{g}$. Let $\mathcal{K},\overline{\mathcal{K}}$ be the second fundamental forms of Σ with respect to $g,\mathbf{n}$ and $\overline{g},\overline{\mathbf{n}}$ respectively. Then for any tangential vector fields X,Y, we have*

$$
\mathcal{K}(X,Y) = e^\lambda\left(\overline{\mathcal{K}}(X,Y) + d\lambda(\overline{\mathbf{n}})\overline{g}(X,Y)\right).
$$

Proof. Let $\nabla,\overline{\nabla}$ be the connections of $g,\overline{g}$ respectively. Then any smooth vector fields X,Y, we have

$$
\nabla_X Y = \overline{\nabla}_X Y + \Gamma(X,Y),
$$

where Γ is given by

$$g(\Gamma(X,Y),Z) = X(\lambda)g(Y,Z) + Y(\lambda)g(X,Z) - Z(\lambda)g(X,Y).$$

Let X, Y be tangent to Σ. Then

$$\begin{aligned}\mathcal{K}(X,Y) =& - g(\nabla_X Y, \mathbf{n})\\ =& - g(\overline{\nabla}_X Y, \mathbf{n}) - X(\lambda)g(Y,\mathbf{n}) - Y(\lambda)g(X,\mathbf{n}) + \mathbf{n}(\lambda)g(X,Y)\\ =& - e^{\lambda}\overline{g}(\overline{\nabla}_X Y, \overline{\mathbf{n}}) + e^{\lambda}\overline{\mathbf{n}}(\lambda)\overline{g}(X,Y)\\ =& e^{\lambda}\left(\overline{\mathcal{K}}(X,Y) + d\lambda(\overline{\mathbf{n}})\overline{g}(X,Y)\right).\end{aligned}$$

□

In our case, $\mathbf{n} = T$, $\lambda = -\log s$. Let $\overline{\mathbf{n}} = e^{\lambda}T = s^{-1}T$. Then by Lemma 2.2,

$$d\lambda(\overline{\mathbf{n}}) = -s^{-2}T(s) = \alpha s^{-2}\nabla\tau(s) = -\alpha P_\tau^{-1}\left(1 + s^2(1-2ms)P_s)\right).$$

So the second fundamental forms $\mathcal{K}$, $\overline{\mathcal{K}}$ of Σ_τ with respect to $g, \overline{g}$ are related by:

$$\mathcal{K} = s^{-1}\left[\overline{\mathcal{K}} - \alpha P_\tau^{-1}\left(1 + s^2(1-2ms)P_s\right)\overline{g}\right]. \tag{3.9}$$

The following lemma basically is contained in [9].

Lemma 3.4. *Let $\mathcal{K}$ be the second fundamental form of Σ_τ. Then in $U \times (0, s_0) \times (\tau_1, \tau_2)$*

$$\mathcal{K}(w_i, w_j) = \tau^{-1}\delta_{ij} + c_{ij}(s),$$

where w_1, w_2, w_3 are given by (3.8) which form an orthonormal basis of Σ_τ with respect to g. In particular, $||\mathcal{K}||_\Theta$ is uniformly bounded.

Proof. Let e_i, ε_i be as in (3.6) and Lemma 3.1. By Lemma 3.5 below, we have

$$\overline{\mathcal{K}}(e_i, e_j) = c_{ij}(s^0).$$

Hence using Lemma 3.1, (2.7) and (3.9), we have

$$\begin{aligned}\mathcal{K}(w_i, w_j) =& s^2\mathcal{K}(\varepsilon_i, \varepsilon_j)\\ =& s\left(\overline{\mathcal{K}}(\varepsilon_i, \varepsilon_j) - \alpha P_\tau^{-1}\left(1 + s^2(1-2ms)P_s\right)\overline{g}(\varepsilon_i, \varepsilon_j)\right)\\ =& s\overline{\mathcal{K}}(\varepsilon_i, \varepsilon_j) - \alpha s P_\tau^{-1}\left(1 + s^2(1-2ms)P_s\right)\delta_{ij}\end{aligned}$$

$$
\begin{aligned}
&= s c_{ik}(s^0) c_{jl}(s^0) \overline{\mathcal{K}}(e_i, e_j) + \tau^{-1}\delta_{ij} + c(s) \\
&= \tau^{-1}\delta_{ij} + c(s).
\end{aligned}
$$

□

Lemma 3.5. *With the notation as in Lemma 3.4, we have* $\overline{\mathcal{K}}(e_i, e_j) = c_{ij}(s^0)$ *in* $U \times (0, s_0) \times (\tau_1, \tau_2)$

Proof. Using Lemma 2.3, direction computations show:

$$
(3.10) \qquad \begin{cases}
\overline{\nabla}_{\partial_A}\partial_B = \widetilde{\nabla}_{\partial_A}\partial_B, 1 \le A, B \le 2; \\
\overline{\nabla}_{\partial_A}\partial_a = \overline{\nabla}_{\partial_a}\partial_B = 0,\ 3 \le a \le 4, 1 \le A \le 2; \\
\overline{\nabla}_{\partial_3}\partial_3 = 0; \\
\overline{\nabla}_{\partial_4}\partial_4 = s^3(1 - 5ms + 6m^2s^2)\partial_3 + s(1 - 3ms)\partial_4 \\
\overline{\nabla}_{\partial_3}\partial_4 = \overline{\nabla}_{\partial_4}\partial_3 = -s(1 - 3ms)\partial_3.
\end{cases}
$$

On the other hand,

$$
(3.11) \qquad \begin{aligned}
\overline{\nabla}_{e_i} e_j =& \overline{\nabla}_{(-P_i\partial_4 + \partial_i)}(-P_j\partial_4 + \partial_j) \\
=& P_i\partial_4(P_j)\partial_4 + P_iP_j\overline{\nabla}_{\partial_4}\partial_4 - \partial_i(P_j)\partial_4 - P_i\overline{\nabla}_{\partial_4}\partial_j + \overline{\nabla}_{\partial_i}\partial_j \\
=& \left[P_i\partial_4(P_j) - ms^2P_iP_j - \partial_i(P_j)\right]\partial_4 - ms^4(1 - 2ms)P_iP_j\partial_3 \\
& - P_i\overline{\nabla}_{\partial_4}\partial_j + \overline{\nabla}_{\partial_i}\partial_j.
\end{aligned}
$$

We want to compute $\overline{g}(\overline{\nabla}_{e_i}e_j, \overline{\mathbf{n}})$ where $\overline{\mathbf{n}} = s^{-1}T$ is the unit normal of Σ_τ with respect to $\overline{g}$. By Lemma 3.1, $g(T, e_4) = -\alpha$. By (3.7), and the fact that $\overline{g}(T, e_i) = 0$, we have,

$$
(3.12) \qquad \begin{cases}
\overline{g}(\overline{\mathbf{n}}, \partial_i) = sg(T, \partial_i) = s\alpha P_\tau^{-1}P_i, \quad 1 \le i \le 3; \\
\overline{g}(\overline{\mathbf{n}}, \partial_4) = sg(T, \partial_4) = s\alpha P_\tau^{-1}.
\end{cases}
$$

Moreover,

$$
\begin{aligned}
P_A\partial_4(P_B) - \partial_A(P_B) =& -P_\tau^{-1}P_Ae_4(P_B) + P_\tau^{-1}P_Ae_4(P_B) - e_A(P_B) \\
=& -P_{AB}.
\end{aligned}
$$

Similarly, for $1 \le A \le 2$,

$$
P_3\partial_4(P_A) - \partial_3(P_A) = -P_{As}; P_3\partial_4(P_3) - \partial_3(P_3) = -P_{ss}.
$$

Combining these with (3.11), (3.12) and (2.7), the results follow. □

Next we want to estimate of $||\nabla T||_\Theta$ and $||\nabla^2 T||_\Theta$. First we have the following:

Lemma 3.6. *Let w_i be as in (3.8). Denote T by w_4. Then*

$$\begin{cases} [w_i, w_j] = c_{ijk}(s^0)w_k, 1 \le i, j, k \le 3; \\ [T, w_i] = \sum_{a=1}^4 c_{ia}(s^0)w_a, 1 \le i \le 3. \end{cases}$$

Proof. Observe that e_a in (3.6) are coordinate frames with respect to the coordinates y^1, y^2, s, τ. Hence $[e_a, e_b] = 0$. Now by Lemma 3.1

$$\begin{aligned} [w_i, w_j] =&[s\varepsilon_i, s\varepsilon_j] \\ =&[sc_{ik}(s^0)e_k, sc_{jl}(s^0)e_l] \\ =&sc_{ik}(s^0)e_k(sc_{jl}(s^0))e_l - sc_{jl}(s^0)e_l(sc_{ik}(s^0))e_k \\ =&c_{ijk}(s)e_k \\ =&c_{ijk}(s^0)w_k. \end{aligned}$$

By Lemma 3.1 again, we have

$$\begin{aligned} [T, w_i] =&[c_k(s)e_k + \alpha^{-1}e_4, sc_{ij}(s^0)e_j] \\ =&c_k(s)e_k + [\alpha^{-1}e_4, sc_{ij}(s^0)e_j] \\ =&c_{ia}(s^0)w_a \end{aligned}$$

where we have used the fact that $e_4(s) = 0$ and $e_4 = \alpha(T - c_k(s)e_k)$. □

Lemma 3.7. *$||\nabla T||_\Theta$ is uniformly bounded in $U \times (0, s_0) \times (\tau_1, \tau_2)$.*

Proof. Let w_i be as in (3.8). To estimate $||\nabla T||_\Theta$ it is sufficient to estimate $||\nabla_{w_i} T||_\Theta$ and $||\nabla_T T||_\Theta$. Now

$$g(\nabla_{w_i} T, T) = 0; \quad g(\nabla_{w_i} T, w_j) = \mathcal{K}(w_i, w_j).$$

By Lemma 3.4, $||\nabla_{w_i} T||_\Theta$ are uniformly bounded for $1 \le i \le 3$.

Next, we want to estimate $||\nabla_T T||_\Theta$. It is easy to see that $g(\nabla_T T, T) = 0$. Since $g(T, w_i) = 0$, we have

$$\begin{aligned} g(\nabla_T T, w_i) =& - g(T, \nabla_T w_i) \\ =& - g(T, [T, w_i]) + g(T, \nabla_{w_i} T) \\ =& - g(T, [T, w_i]). \end{aligned}$$

By Lemma 3.6, we conclude that $||\nabla_T T||_\Theta$ is uniformly bounded. This completes the proof of the lemma. □

For $\nabla\nabla T$, we have:

Lemma 3.8. $||\nabla\nabla T||_\Theta$ *is uniformly bounded in* $U \times (0, s_0) \times (\tau_1, \tau_2)$.

Proof. It is sufficient to prove that for all $1 \le a, b \le 4$, $||\nabla_{w_a}\nabla_{w_b}T||_\Theta$ is uniformly bounded. Here $w_4 = T$.

(i) To estimate $||\nabla_T\nabla_T T||_\Theta$:

$$g(\nabla_T\nabla_T T, T) = -g(\nabla_T T, \nabla_T T),$$

which is uniformly bounded by Lemma 3.7. On the other hand,

$$\begin{aligned} g(\nabla_T\nabla_T T, w_i\rangle =& T(g(\nabla_T T, w_i)) - g(\nabla_T T, \nabla_T w_i) \\ =& T(g(\nabla_T T, w_i)) - g(\nabla_T T, [T, w_i]) - g(\nabla_T T, \nabla_{w_i} T). \end{aligned}$$

By Lemmas 3.7, 3.6, the last two terms above are uniformly bounded. By Lemma 3.6

$$\begin{aligned} T(g(\nabla_T T, w_i)) =& - T(g(T, [T, w_i])) \\ =& T(c_i(s^0)) \\ =& c_i(s^0), \end{aligned} \tag{3.13}$$

by Lemma 3.1(iii). Hence $||\nabla_T\nabla_T T||_\Theta$ is uniformly bounded.

(ii) To estimate $||\nabla_{w_i}\nabla_T T||_\Theta$:

$$g(\nabla_{w_i}\nabla_T T, T) = -g(\nabla_T T, \nabla_{w_i} T),$$

which is uniformly bounded by Lemma 3.7. Next,

$$g(\nabla_{w_i}\nabla_T T, w_j) = w_i(g(\nabla_T T, w_j)) - g(\nabla_T T, \nabla_{w_i} w_j).$$

The first term on the RHS is uniformly bounded similar to (3.13). Consider the second term, we have

$$g(\nabla_T T, \nabla_{w_i} w_j) = g(\nabla_T T, w_k) \cdot g(\nabla_{w_i} w_j, w_k).$$

Now

$$g(\nabla_{w_i} w_j, w_k) = \frac{1}{2}\left(g([w_i, w_j], w_k) - g([w_i, w_k], w_j) - g([w_j, w_k], w_i)\right). \tag{3.14}$$

Hence by Lemma 3.7 and 3.6, the second term on the RHS is also uniformly bounded. So $||\nabla_{w_i}\nabla_T T||_\Theta$ is uniformly bounded.

(iii) To estimate $||\nabla_T\nabla_{w_i}T||_\Theta$:

$$\begin{aligned} g(\nabla_T\nabla_{w_i}T,T) =& -g(\nabla_{w_i}T,\nabla_T T)\\ =& -g(\nabla_{w_i}T,w_j)\cdot g(\nabla_T T,w_j)\\ =& -\mathcal{K}(w_i,w_j)g(\nabla_T T,w_j),\\ =&c(s^0)\end{aligned}$$

which is uniformly bounded by Lemmas 3.4 and 3.7. Next,

$$\begin{aligned} g(\nabla_T\nabla_{w_i}T,w_j) =&T(g(\nabla_{w_i}T,w_j))-g(\nabla_{w_i}T,\nabla_T w_j)\\ =&T(\mathcal{K}(w_i,w_j))-g(\nabla_{w_i}T,w_k))\cdot g(\nabla_T w_i,w_k)\\ =&T(\mathcal{K}(w_i,w_j))-\mathcal{K}(w_i,w_k)\left(g([T,w_i],w_k)-\mathcal{K}(w_i,w_k)\right),\end{aligned}$$

which is uniformly bounded by Lemmas 3.1, 3.4, and 3.6.

Hence $||\nabla_T\nabla_{w_i}T||_\Theta$ is uniformly bounded.

(iv) To estimate $||\nabla_{w_i}\nabla_{w_j}T||_\Theta$:

$$g(\nabla_{w_i}\nabla_{w_j}T,T)=-g(\nabla_{w_j}T,\nabla_{w_i}T),$$

which is uniformly bounded by Lemma 3.4.

$$\begin{aligned} g(\nabla_{w_i}\nabla_{w_j}T,w_k\rangle =&w_i(g(\nabla_{w_j}T,w_k))-g(\nabla_{w_j}T,\nabla_{w_i}w_k)\\ =&w_i(\mathcal{K}(w_j,w_k))-\mathcal{K}(w_j,w_l)\cdot g(\nabla_{w_i}w_k,w_l).\end{aligned}$$

As before, one can see that this is uniformly bounded. This completes the proof of the lemma. □

Finally, we want to estimate $||\mathbf{H}_{\tau,s}||_\Theta$, where $\mathbf{H}_{\tau,s}$ is the mean curvature vector of the two-surface given by $\tau=$constant, $s=$constant.

Lemma 3.9. *$||\mathbf{H}_{\tau,s}||_\Theta$ is uniformly bounded for $s\in(0,s_0)$, $\tau\in(\tau_1,\tau_2)$ and $\mathbf{y}\in U$.*

Proof. Let $N\subset\Sigma_\tau$ which is the level set of s. Let e_a, ε_i, w_i be as in (3.6), Lemma 3.1, and (3.8). Observe that e_1,e_2 form a basis for the tangent space of N, and $\varepsilon_1,\varepsilon_2,\varepsilon_3$ form an orthonormal basis for Σ_τ obtained by Gram-Schmidt process on e_1,e_2,e_3 with respect to $\overline{g}$. Hence w_1,w_2 form an orthonormal basis for the tangent space of N. w_3,T form an orthonormal basis for the normal bundle of N.

$$
\begin{aligned}
\mathbf{H}_{\tau,s} &= \left(\sum_{A=1}^{2} \nabla_{w_A} w_A\right)^{\perp} \\
&= -\sum_{A=1}^{2} g(\nabla_{w_A} w_A, T)T + \sum_{A=1}^{2} g(\nabla_{w_A} w_A, w_3)w_3 \\
&= \sum_{A=1}^{2} \mathcal{K}(w_A, w_A)T + \sum_{A=1}^{2} g(\nabla_{w_A} w_A, w_3)w_3.
\end{aligned}
$$

By Lemmas 3.4, 3.6 and (3.14), we conclude that the lemma is true. □

Since $\mathbb{S}^2$ can be covered by finitely many coordinate neighborhoods, by Lemmas 2.2, 3.2, 3.5, 3.7, 3.8 and 3.9, we have the following:

Theorem 3.1. *There is $s_0 > 0$ depending only on τ_1, τ_2, f such that for any $\tau \in (\tau_1, \tau_2)$ the level set Σ_τ is spacelike. Moreover, if α is the lapse function of the time function τ, T is the future pointing unit normal of Σ_τ and $\vec{H}_{\tau,s}$ is the mean curvature vector of the surface $\tau =$ constant, $s =$ constant, then the following are all uniformly bounded in $\mathbb{S}^2 \times (0, s_0) \times (\tau_1, \tau_2)$:*

$$
\alpha, \alpha^{-1}, ||\nabla\alpha||_\Theta, ||\nabla T||_\Theta, ||\nabla\nabla T||_\Theta, ||\vec{H}_{\tau,s}||_\Theta.
$$

Moreover, the mean curvature H of Σ_τ is given by

$$
H = \tau^{-1} + c(s).
$$

4. Construction of CMC surfaces

Using $t, \mathbf{x} = (x^1, x^2, x^3)$ as coordinates for the Schwarzschild metric in the form (1.1) with $r = |\mathbf{x}| = \left(\sum_{i=1}^{3}(x^i)^2\right)^{\frac{1}{2}}$,

$$
g_{\mathrm{Sch}} = -h dt^2 + g_{ij}(x) dx^i dx^j, \tag{4.1}
$$

where $h = 1 - \frac{2m}{r} = 1 - 2ms$ with $s = r^{-1}$ and

$$
g_{ij} = \delta_{ij} + (h^{-1} - 1) r^{-2} x^i x^j.
$$

Notation: In this section, we use g_{Sch} to denote the Schwarzschild metric and $g_{ij}(x)dx^i dx^j$ will be denoted by g. The inverse of (g_{ij}) is denoted by

g^{ij}. For a function v of $\mathbf{x} = (x^1, x^2, x^3)$, $Dv = g^{ij} v_i \frac{\partial}{\partial x^j}$ where $v_i = \frac{\partial v}{\partial x^i}$. $D^i v = g^{ij} v_j$. $|Dv|^2 = D^i v D_i v$. The Hessian of v with respect to g will be denoted by v_{ij}.

We will prove Theorem 1.1 for the case $H_0 = 1$. The other case is similar. Let $f(\mathbf{y})$ be a smooth function on $\mathbb{S}^2$. Let

$$M = \mathbb{S}^2 \times (0, s_0) \times (\frac{1}{2}, 2),$$

and define the map Φ as in § 2.2 given by $P(\mathbf{y}, s, \tau)$ in (2.5) with f replaced by $-f$. Let s_0 be as in Lemma 2.1. Let β be a constant, define

$$Q(\mathbf{y}, s; \beta) = -f(\mathbf{y}) + \phi(\mathbf{y}) s + \frac{1}{2}\psi(\mathbf{y}) s^2 + \beta s^3 = P(\mathbf{y}, s, 1) + \beta s^3,$$

where ϕ, ψ are defined as in (2.6) with f replaced by $-f$ and $\tau = 1$. For fixed β, if s_0 is small enough then $P(\mathbf{y}, s, 2) < Q(\mathbf{y}, s; \beta) < P(\mathbf{y}, s, \frac{1}{2})$. Hence the surface Σ given by $v = -Q(\mathbf{y}, s; \beta)$ will be in $\Phi(M)$, provided s_0 is small enough depending only on f and the bound of β.

We want to compute the mean curvature of Σ. All mean curvature will be computed with respect to future pointing unit normal.

Lemma 4.1. *There exist $\beta_1 < 0, \beta_2 > 0$ and $s_0 > 0$ depending only on f such that the surfaces Σ_1, Σ_2 which are the graphs of $v = -Q_1, v = -Q_2$ respectively, are in $\Phi(M)$. Here $Q_1(\mathbf{y}, s) =: Q(\mathbf{y}, s; \beta_1)$, $Q_2(\mathbf{y}, s) =: Q(\mathbf{y}, s; \beta_1)$. Moreover, the mean curvature of Σ_1 is smaller than 1 and the mean curvature of Σ_2 is larger than 1.*

Proof. The first part of the lemma is obvious. To simplify notation, in the following let us denote Q_1 by Q and Σ_1 by Σ. We may assume that Σ is spacelike. Let H be the mean curvature of Σ. By the computation in [9, Lemma 2.2],

(4.2)
$$\begin{aligned} -3HL^{\frac{3}{2}} =& sL\left(s^2(1-2ms)Q_{ss} + \widetilde{\Delta} Q\right) \\ &- \frac{1}{2}s\left(L_s + s^2(1-2ms)L_s Q_s + \langle \widetilde{\nabla} L, \widetilde{\nabla} Q\rangle\right) - s^2 L P_s - 3L \end{aligned}$$

where

$$L = -(2Q_s + s^2(1-2ms)Q_s^2 + |\widetilde{\nabla} Q|^2).$$

One can see that H is smooth up to $s = 0$, provided $s_0 > 0$ is small enough depending only on f. By the choice of ϕ, ψ and [9, Theorem 3.1], at $s = 0$ $H = 1, H_s = 0$.

In below, c, c_k will denote smooth functions in $\mathbf{y}, s$ up to $s = 0$, which are independent of β, it may vary from line to line. It is easy to see that at $s = 0$, $Q_s = c, Q_{ss} = c, Q_{sss} = 6\beta$. At $s = 0$, $L = 1$, $L_s = c$, $L_{ss} = -12\beta + c$. Therefore, at $s = 0$

$$\begin{aligned}(-3HL^{\frac{3}{2}})_{ss} =& - 3H_{ss}L^{\frac{3}{2}} - 3H(L^{\frac{3}{2}})_{ss} \\ =& - 3H_{ss} - \frac{9}{2}(L^{\frac{1}{2}}L_s)_s \\ =& - 3H_{ss} + 54\beta + c;\end{aligned}$$

$$\begin{aligned}\left[sL\left(s^2(1-2ms)Q_{ss} + \widetilde{\Delta}Q\right)\right]_{ss} =& 2\left[L\left(s^2(1-2ms)Q_{ss} + \widetilde{\Delta}Q\right)\right]_s \\ =& c;\end{aligned}$$

$$\begin{aligned}&-\frac{1}{2}\left[s\left(L_s + s^2(1-2ms)L_sQ_s + \langle\widetilde{\nabla}L, \widetilde{\nabla}Q\rangle\right)\right]_{ss} \\ &= - \left(L_s + s^2(1-2ms)L_sQ_s + \langle\widetilde{\nabla}L, \widetilde{\nabla}Q\rangle\right)_s \\ &= 12\beta + c;\end{aligned}$$

and

$$(-s^2LQ_s - 3L)_{ss} = 36\beta + c.$$

Hence we have

$$-3H_{ss} + 54\beta + c_1 = c_2 + 12\beta + c_3 + 36\beta + c_4.$$

Or

$$H_{ss} = 2\beta + c. \tag{4.3}$$

First choose $\beta = \beta_1 < 0$ so that $2\beta_1 + c < 0$. Then β_1 depends only on f.

$$H = 1 + \frac{1}{2}(2\beta_1 + c)s^2 + O(s^3). \tag{4.4}$$

In particular, $H < 1$ for $0 < s < s_0$ provided s_0 is small enough depending only on f. Similarly, one can choose $\beta_2 > 0$ so that $2\beta_2 + c > 0$. This completes the proof of the lemma. □

Remark 4.1. The construction in the above lemma does not work for higher order. Namely, suppose

$$Q(\mathbf{y}, s) = \sum_{i=0}^{k} \frac{1}{i!} f_i(\mathbf{y}) s^i + \beta s^{k+1}, \tag{4.5}$$

and suppose we can choose f_i so that the mean curvature H satisfies $H = 1$, and $\frac{\partial^i H}{\partial s^i} = 0$ for $1 \le i \le k-1$ at $s = 0$. Then at $s = 0$

$$3\partial_s^k H = (3-k)(k+1)!\beta + c$$

where c is a function of $\mathbf{y}, s$. Note that $(3-k)(k+1)! \le 0$ if $k \ge 3$ in contrast to (4.3). Another issue is that in general one cannot find f_i so that $\partial_s^i H = 0$ at $s = 0$ if $k \ge 4$, see [9, Theorem 3.1].

Let $t, \mathbf{x}$ be as in (4.1).

$$\nabla t = -h^{-1} \frac{\partial}{\partial t}.$$

The lapse function $\widetilde{\alpha}$ for the time function t is given by:

$$\widetilde{\alpha}^{-2} = -g_{\mathrm{Sch}}(\nabla t, \nabla t) = h^{-1}.$$

So $\widetilde{\alpha} = h^{\frac{1}{2}}$. The future pointing unit normal of $t =$ constant is:

$$\widetilde{T} = h^{-\frac{1}{2}} \frac{\partial}{\partial t}.$$

Lemma 4.2. *Let T be the future pointing unit normal of τ =constant. Then $g_{\mathrm{Sch}}(T, \widetilde{T}) = -s^{-1} L^{-\frac{1}{2}} h^{-\frac{1}{2}}$, where L is given by (2.11) with $s = r^{-1}$.*

Proof. By (3.3), the lapse function of τ is $\alpha = -s^{-1} P_\tau L^{-\frac{1}{2}}$. By (2.13),

$$g_{\mathrm{Sch}}(\nabla \tau, \frac{\partial}{\partial t}) = P_\tau^{-1}.$$

Hence

$$g_{\mathrm{Sch}}(T, \widetilde{T}) = -\alpha g_{\mathrm{Sch}}(\nabla \tau, h^{-\frac{1}{2}} \partial_t) = -s^{-1} L^{-\frac{1}{2}} h^{-\frac{1}{2}}.$$

□

Consider a surface given by the graph of $u(\mathbf{x})$, where $\mathbf{x} = (x^1, x^2, x^3)$, namely, it is given by $t = u(\mathbf{x})$. Then it is the level surface of $F(t, \mathbf{x}) = t - u(\mathbf{x}) = 0$. Normal is given by

$$\nabla F = -h^{-1}\frac{\partial}{\partial t} - D^i u \frac{\partial}{\partial x^i}.$$
$$g_{\mathrm{Sch}}(\nabla F, \nabla F) = g^{ab}F_a F_b = -h^{-1} + |Du|^2.$$

Hence the surface is spacelike if and only if

$$1 - h|Du|^2 > 0. \tag{4.6}$$

If u is spacelike, the future pointing unit normal is:

$$\widetilde{\mathbf{n}} = \left(h^{-1} - |Du|^2\right)^{-\frac{1}{2}} \nabla F = \left(h^{-1} - |Du|^2\right)^{-\frac{1}{2}} \left(h^{-1}\frac{\partial}{\partial t} + D^i u \frac{\partial}{\partial x^i}\right).$$

The tilt factor with respect to $\widetilde{T}$ is given by:

$$\widetilde{\nu} = -g_{\mathrm{Sch}}(\widetilde{T}, \widetilde{\mathbf{n}}) = h^{-\frac{1}{2}}\left(h^{-1} - |Du|^2\right)^{-\frac{1}{2}} = (1 - h|Du|^2)^{-\frac{1}{2}}. \tag{4.7}$$

Suppose the surface is spacelike, it is more easy to appeal to [2, p.160] to obtain the mean curvature equation of u. Namely, its graph has mean curvature H if and only if:

$$3H = \mathrm{Div}\left(\frac{U}{(1 - |U|^2)^{\frac{1}{2}}}\right) + 3\widetilde{\nu}H^o + \widetilde{\nu} g_{\mathrm{Sch}}(U, \nabla_{\widetilde{T}}\widetilde{T}) + \frac{1}{2}\widetilde{\nu}^3 \widetilde{T}(|U|^2).$$

Here Div is the divergence with respect to the metric (g_{ij}), and $U = \widetilde{\alpha} Du$. $|U|$ is the norm with respect to g so that $|U|^2 = \widetilde{\alpha}^2|Du|^2 = h|Du|^2$. H^o is the mean curvature of t =constant, which is zero. Note that $\widetilde{T}(|U|^2) = 0$, because $|U|^2$ does not depend on t.

$$g_{\mathrm{Sch}}(U, \nabla_{\widetilde{T}}\widetilde{T}) = -\frac{1}{2}D^i u D_i \log h.$$

Therefore, the graph of u has mean curvature H if and only if

$$\mathrm{Div}\left(\frac{h^{\frac{1}{2}} Du}{(1 - h|Du|^2)^{\frac{1}{2}}}\right) - \frac{1}{2}(1 - h|Du|^2)^{-\frac{1}{2}} D^i u D_i \log h = 3H. \tag{4.8}$$

Hence the mean curvature equation is of the form:

$$A^{ij}u_{ij} + B(x, Du) = 3h^{-\frac{1}{2}}(1 - h|Du|^2)^{\frac{1}{2}}H. \tag{4.9}$$

where

$$\begin{cases} A^{ij} = (1 - h|Du|^2)g^{lj} + hD^iuD^ju, \\ B(x, Du) = h^{-\frac{1}{2}}g(Du, D(h^{\frac{1}{2}})) + \frac{1}{2}\frac{|Du|^2 g(Du,Dh)}{(1-h|Du|^2)}. \end{cases}$$

Here u_{ij} is the Hessian of u with respect to g.

Lemma 4.3. *Assume the graph of u is spacelike, then any $\mathbf{a} = (a_1, a_2, a_3)$, we have*

$$|\mathbf{a}|^2 \ge A^{lj}a_l a_j \ge (1 - h|Du|^2)|\mathbf{a}|^2,$$

where $a^j = g^{ij}a_i$ and $|\mathbf{a}|^2 = a_i a^i$.

Proof.

$$\begin{aligned} A^{lj}a_l a_j =&(1 - h|Du|^2)|\mathbf{a}|^2 + hg^{ij}g^{kl}u_i u_k a_l a_j \\ =&(1 - h|Du|^2)|\mathbf{a}|^2 + h(\sum_i u_i a^i)^2. \end{aligned}$$

From this the lemma follows. □

Recall the following basic fact, see [2, Lemma 3.3]:

Lemma 4.4. *In a Lorentzian vector space with inner product $\langle\,,\,\rangle$, Let T_1, T_2, T_3 be future-directed unit timelike vectors. Then*

$$1 \le -\langle T_1, T_2\rangle \le 2\langle T_1, T_3\rangle\langle T_2, T_3\rangle.$$

We are now ready to prove Theorem 1.1:

Proof of Theorem 1.1. For simplicity, we prove the case that $H_0 = 1$. The other case is similar. Consider the foliation by $P(\mathbf{y}, s, \tau)$ at the beginning of the section with $M = \mathbb{S} \times (0, s_0) \times (\frac{1}{2}, 2)$. We assume that s_0 is chosen such that in the retarded null coordinate $v = t - r_*$, $\Phi(\mathbf{y}, s, \tau) = (\mathbf{y}, s, -P)$ is a diffeomorphism between M and $\Phi(M)$. Note that in terms standard coordinates as in (1.1),

(4.10)
$$\Phi(M) = \{(\mathbf{y}, r, t)|\ \ \mathbf{y} \in \mathbb{S}^2, r > \frac{1}{s_0}, r_* - P(\mathbf{y}, r^{-1}, 2) > t > r_* - P(\mathbf{y}, r^{-1}, \frac{1}{2})\}.$$

Choose $\beta_1 < 0, \beta_2 > 0$ as in Lemma 4.1 and let $w_1(\mathbf{y}, r) = r_* - Q_1(\mathbf{y}, r^{-1})$, $w_2(\mathbf{y}, r) = r_* - Q_2(\mathbf{y}, r^{-1})$ where Q_1, Q_2 are as in the lemma. Here s_0 is chosen so that the conclusion of the lemma is true and so that the conclusion of Theorem 3.1 is also true for $\tau_1 = \frac{1}{2}, \tau_2 = 2$. Also, let $w(\mathbf{y}, r) = r_* - P(\mathbf{y}, r^{-1}, 1)$.

Let $\frac{1}{2}r_0 = \frac{1}{s_0}$. For any $R > r_0$, consider the spacetime N_R given by $\mathbb{S}^2 \times (\frac{1}{2}r_0, 2R) \times \mathbb{R}$ with metric induced by the Schwarzschild metric. One can see that the surface given by $t = w(\mathbf{y}, r)$ is spacelike and acausal in N_R, see [10, Corollary 46]. That is: no two different points on the surface are causally related. By [5, Theorem 5.1], we can find smooth function u_R of $\mathbf{y}, r$, with $r_0 \le r \le R$ so that the graph of u_R is spacelike with constant mean curvature 1, so that u_R has the same boundary value as $w(\mathbf{y}, r)$. Since $\beta_1 < 0, \beta_2 > 0$, we have $Q_1(\mathbf{y}, s) < P(\mathbf{y}, s, 1) < Q_2(\mathbf{y}, s)$. We have $w_1 > w > w_2$. Moreover, the mean curvature of the graph of w_1 is less than 1, and the mean curvature of w_2 is larger than 1. By the form of (4.9) and the fact that the graphs of w_1, w_2, u_R are all spacelike up to the boundary, by (4.6) and Lemma 4.3, one can apply the comparison principle [6, Theorem 10.1] to conclude that

$$w_1(\mathbf{y}, r) \ge u_R(\mathbf{y}, r) \ge w_2(\mathbf{y}, r). \tag{4.11}$$

Hence the graph of u_R is in $\Phi(M)$. In the $(\mathbf{y}, s, \tau)$ coordinates, this graph is given by $(\mathbf{y}, s, \tau(\mathbf{y}, s))$ with $\mathbf{y} \in \mathbb{S}^2$, $\frac{1}{R} < s < \frac{1}{r_0}$ and

$$u_R(\mathbf{y}, r) = r_* - P(\mathbf{y}, s, \tau(\mathbf{y}, s)),$$

with $r = s^{-1}$. On the boundary $s = \frac{1}{R}, \frac{1}{r_0}$, we have $\tau(\mathbf{y}, s) = 1$.

The next step is to prove that u_R will subconverge to a solution of (4.9) with $H = 1$ as $R \to \infty$. In order to do this, by Lemma 4.3 we need to estimate the tilt factor of the surface with respect to the time function t. So let $\mathbf{n}_R$ be the future pointing unit normal of the surface and let $\nu_R = -g_{\mathrm{Sch}}(\mathbf{n}_R, T)$ be the tilt factor with respect to the time function τ where T is given by (3.2). By Theorem 3.1 and the fact that the Schwarzschild spacetime is Ricci flat, we can apply the Bartnik's gradient estimate [2, Theorem 3.1(iii)] to conclude that

$$\nu_R \le C_1 \tag{4.12}$$

for some constant C_1 independent of R. Let $\widetilde{T}$ be the future pointed unit normal of the surface $t =$constant, by Lemma 4.4 and (2.13), we have

$$\widetilde{\nu}_R =: -g_{\mathrm{Sch}}(\mathbf{n}_R, \widetilde{T}) \le 2g_{\mathrm{Sch}}(\mathbf{n}_R, T)g_{\mathrm{Sch}}(\widetilde{T}, T) \le C_2 s^{-1} = C_2 r \tag{4.13}$$

for some constant C_2 independent of R. Here we use the fact that the time function τ restricted on the graph is bounded between $\frac{3}{4}$ and $\frac{2}{3}$. Hence by (4.7), for any open set Ω with compact closure in the region $r > r_0$, there is a constant C_3 independent of R such that

$$(1 - h|Du_R|^2)^{-\frac{1}{2}} \le C_3.$$

In particular, $|Du_R| \le C_4$ in Ω for some constant C_4 independent of R. By (4.9) with $H = 1$, we may apply [6, Theorem 13.6] to obtain a uniform Hölder estimate for u_i. Using Schauder estimates, we conclude that there is a subsequence $R_k \to \infty$ such that u_{R_k} converge in C^∞_{loc} in $\{r > r_0\}$ to function u so that its graph is spacelike and has constant mean curvature 1. By (4.11), for any $R > r_0$, we have

$$w_2(\mathbf{y}, r) \le u(\mathbf{y}, r) \le w_1(\mathbf{y}, r).$$

where $s = r^{-1}$. Hence

$$|u(\mathbf{y}, r) - r_* - P(\mathbf{y}, r^{-1}, 1)| \le \max\{-\beta_1, \beta_2\} r^{-3}.$$

This completes the proof of the theorem. □

As a corollary of the proof, in particular by (4.12), we have:

Corollary 4.1. *Let u be the solution in Theorem 1.1. The tilt factor of the graph of u with respect to the time function τ is uniformly bounded by a constant.*

We should remark that Lemma 4.1 also implies the following:

Corollary 4.2. *Let f be a smooth function on $\mathbb{S}^2$. Suppose u is a function defined on $r > r_0$ such that the graph of u in the Schwarzschild spacetime is spacelike with constant mean curvature $H_0 > 0$ so that $u(r, \mathbf{y}) - r_* \to f(\mathbf{y})$ are $r \to \infty$. Suppose there is $C > 0$ such that*

$$u(\mathbf{y}, r_i) - r_* - \left(f(\mathbf{y}) + r_i^{-1}\phi(\mathbf{y}) + \frac{1}{2} r_i^{-2}\psi(\mathbf{y})\right) - C r_i^{-3} \le 0.$$

for some $r_i \to \infty$, where ϕ, ψ are as in Theorem 1.1. Then we have

$$\limsup_{r\to\infty} \left(u(\mathbf{y}, r) - r_* - \left(f(\mathbf{y}) + r^{-1}\phi(\mathbf{y}) + \frac{1}{2} r^{-2}\psi(\mathbf{y})\right) - C' r^{-3}\right) \le 0.$$

for some $C' > 0$. Similar result is true for the lower bound estimate.

Proof. In the proof of Lemma 4.1, we may choose $\beta_1 > 0$ large enough so that $\beta_1 + c < 0$ in the notation in the proof and so that $\beta_1 > C$. Then by the maximum principle, it is easy to see that the corollary is true. □

5. Lipschitzian regularity

We want to prove that the solution u given by Theorem 1.1 is Lipschitz near infinity in the sense that the function $r_* - u$ is Lipschitz up to $s = 0$ in the coordinates $\mathbf{y} \in \mathbb{S}^2, s$ and $v = t - r_*$. In fact, more general result can be obtained. Here is the setup. Let $\mathbf{y}, s, v$ be as in §2. Consider the metric given by

$$G = \omega^{-2}\overline{G} \tag{5.1}$$

where $\overline{G} = \overline{g} + p$ and $\omega = s(1 + c(s^3))$. Here $\overline{g}$ is the unphysical metric (2.3) and $p = p_{ab}dy^a dy^b$ with $p_{ab} = p_{ab}(s^3)$ in the coordinates $\mathbf{y} = (y^1, y^2), y^3 = s, y^4 = v$. Here $p_{ab}(s^3)$ means that $p_{ab} = s^3\Lambda_{ab}$ where Λ_{ab} is a smooth function on $\mathbb{S}^2 \times [0, s_0) \times \mathbb{R}$ for some $s_0 > 0$. Similar definition for $c(s^3)$. Hence for fixed $v_1 < v_2$, on $\mathbb{S}^2 \times (0, s_0) \times (v_1, v_2)$ we have $\overline{G}^{ab} = \overline{g}^{ab} + p^{ab}$, with $p^{ab} = p^{ab}(s^3)$, provided s_0 is small enough.

Let f be a smooth function on $\mathbb{S}^2$. Suppose $P(\mathbf{y}, s, \tau)$, $\tau > 0$ is such that

$$P(\mathbf{y}, s, \tau) = f(\mathbf{y}) - \frac{1}{2}\left(\tau^2 + |\widetilde{\nabla} f|^2\right) s + s^2 c(\mathbf{y}, s, \tau) \tag{5.2}$$

where c is smooth function on $\mathbb{S}^2 \times [0, s_0) \times (0, \infty)$. As before, one can see that for fixed $0 < \tau_1 < \tau_2$, $(\mathbf{y}, s, \tau) \to (\mathbf{y}, s, v)$ with $v = -P(\mathbf{y}, s, \tau)$ is a diffeomorphism from $M =: \mathbb{S}^2 \times (0, s_0) \times (\tau_1, \tau_2)$ onto its image $\mathcal{N}$, provided s_0 is small enough. Its image is:

$$\mathcal{N} = \{(\mathbf{y}, s, v) | \ \mathbf{y} \in \mathbb{S}^2, s \in (0, s_0), P(\mathbf{y}, s, \tau_1) < v < P(\mathbf{y}, s, \tau_2)\}.$$

Moreover, in terms of the metric G, $\nabla\tau$ is timelike. Here ∇ is the derivative with respect to G. Let $T = -\alpha\nabla\tau$ as before, where $\alpha^{-2} = -G(\nabla\tau, \nabla\tau)$. We have the following:

Theorem 5.1. *Suppose Σ is a spacelike surface inside $\mathcal{N}$ for some $0 < \tau_1 < \tau_2$, which is given by $v + Q(\mathbf{y}, s) = 0$, $(\mathbf{y}, s) \in \mathbb{S}^2 \times (0, s_0)$. Suppose the tilt factor of Σ with respect to T is bounded, that is suppose $-G(T, \mathbf{n}) \le C$ on Σ for some $C > 0$ where $\mathbf{n}$ is the future pointing unit normal of Σ. Then Q is uniformly Lipschitz on $\mathbb{S}^2 \times (0, s_1)$ for some $0 < s_1 < s_0$.*

By Corollary 4.1 and the proof of Theorem 1.1 we have:

Corollary 5.1. *Let u be the solution in Theorem 1.1. Let $Q(\mathbf{y}, s) = r_* - u(\mathbf{y}, r)$ with $s = r^{-1}$. Then $Q(\mathbf{y}, s)$ is uniformly Lipschitz in $\mathbb{S}^2 \times (0, s_0)$ for some $s_0 > 0$.*

Proof. Let u_R be as in the proof of Theorem 1.1. Since u_R converges to u in C^∞_{loc}, by (4.12), one can conclude that Q satisfies the conditions in the theorem. Hence the corollary is true. □

Remark 5.1. It seems likely that Theorem 5.1 can also be applied to the spacelike CMC surface constructed by Andersson and Iriondo in [1, Theorem 4.2].

Before we prove Theorem 5.1, we need to obtain some estimates. Consider the coordinates t, x^1, x^2, x^3 with $t = v + r_*$, $r = s^{-1}$ and $\mathbf{y}, r$ are the spherical coordinates of $\mathbb{R}^3$. In the following, we always assume that $\tau_1 < \tau < \tau_2$. Hence we are doing estimates in M or $\mathcal{N}$.

Lemma 5.1. *$\frac{\partial}{\partial t}$ is timelike with respect to G provided s_0 is small enough. Moreover, if $G_{ij} = G(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ is the induced metric on t =constant, and if $\widetilde{\alpha}$ is the lapse function and β^i is the shift vector, then $G_{ij} = \delta_{ij} + O(s)$, $\widetilde{\alpha} = 1 + O(s)$, $\beta_i = O(s)$. Here $\beta_i = G_{ij}\beta^j$.*

Proof. $\frac{\partial}{\partial t} = \partial_v$. By the assumption on p,

$$G(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \omega^{-2}(\overline{g}_{vv} + p_{vv}) < 0$$

if $s_0 > 0$ is small enough. Since $t = v + r_*$, $\partial_v(t) = 1, \partial_s t = -s^{-2}h^{-1}, \partial_{y^A} t = 0$, for $A = 1, 2$, where $h = 1 - 2ms$ as before. Let $\partial_a = \partial_{y^a}$. Here y^1, y^2 are local coordinates of $\mathbb{S}^2$, $y^3 = s, y^4 = v$. So

$$\nabla t = G^{ab}\partial_a t \partial_b = \left(G^{vb} - s^{-2}h^{-1}G^{sb}\right)\partial_b.$$

$$\begin{aligned} G(\nabla t, \nabla t) =& G^{ab}\partial_a t \partial_b t \\ =& G^{vv} - 2s^{-2}G^{vs}h^{-1} + G^{ss}s^{-4}h^{-2} \\ =& \omega^2(\overline{G}^{vv} - 2s^{-2}\overline{G}^{vs}h^{-1} + \overline{G}^{ss}s^{-4}h^{-2}) \\ =& \omega^2\left[(\overline{g}^{vv} + p^{vv}) - 2s^{-2}(\overline{g}^{vs} + p^{vs})h^{-1} + (\overline{g}^{ss} + p^{ss})s^{-4}h^{-2}\right] \\ =& -1 + O(s). \end{aligned}$$

Hence

$$\widetilde{\alpha} = 1 + O(s).$$

Comparing with g_{Sch}, we see that in the coordinates t, x^i,

$$G = s^2\omega^{-2}g_{\text{Sch}} + \omega^{-2}p.$$

On the other hand,

$$\frac{\partial}{\partial t} = \partial_v,$$

and

$$\frac{\partial}{\partial x^i} = \frac{\partial r}{\partial x^i}\frac{\partial}{\partial r} + \frac{\partial y^A}{\partial x^i}\frac{\partial}{\partial_{y^A}} = \frac{x^i}{r}\left(-h^{-1}\partial_v - s^2\partial_s\right) + \frac{\partial y^A}{\partial x^i}\partial_{y^A}.$$

Here we have used:

$$\frac{\partial}{\partial r} = \frac{\partial v}{\partial r}\partial_v + \frac{\partial s}{\partial r}\partial_s = -h^{-1}\partial_v - s^2\partial_s.$$

Hence

$$\begin{aligned}
\beta_i =& G(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}) \\
=& \omega^{-2}p(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}) \\
=& \omega^{-2}p(\partial_v, \frac{x^i}{r}\left(-h^{-1}\partial_v - s^2\partial_s\right)\partial_s + \frac{\partial y^A}{\partial x^i}\partial_{y^A}) \\
=& \omega^{-2}\left(\frac{x^i}{r}h^{-1}p_{vv} - \frac{x^i}{r}s^2 p_{vs} + \frac{\partial y^A}{\partial x^i}p_{vA}\right) \\
=& O(s).
\end{aligned}$$

Here we have used the fact that $\frac{\partial y^A}{\partial x^i} = O(s)$.

$$\begin{aligned}
G_{ij} =& s^2\omega^{-2}g_{\text{Sch}}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) + \omega^{-2}p(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) \\
=& \delta_{ij} + O(s) + \omega^{-2}p\Bigg(\frac{x^i}{r}\left(-h^{-1}\partial_v - s^2\partial_s\right)\partial_s + \frac{\partial y^A}{\partial x^i}\partial_{y^A}, \\
& \frac{x^j}{r}\left(-h^{-1}\partial_v - s^2\partial_s\right)\partial_s + \frac{\partial y^B}{\partial x^j}\partial_{y^B}\Bigg) \\
=& \delta_{ij} + O(s).
\end{aligned}$$

This completes the proof of the lemma. □

As before, we consider $\frac{\partial}{\partial t}$ as future pointing.

Lemma 5.2. *$\nabla\tau$ is future pointing. Let α be the lapse function of the time function τ, we have*

$$\alpha = 1 + O(s).$$

Moreover if $\widetilde{T} = -\widetilde{\alpha}\nabla t$, then

$$-G(T, \widetilde{T}) = (\tau s)^{-1} + O(1).$$

Proof. In the coordinates $y^1, y^2, y^3 = s, y^4 = v$,

$$\nabla\tau = G^{ab}\partial_{y^a}\tau\partial_{y^b}.$$

As in Lemma 2.1,

$$G(\frac{\partial}{\partial t}, \nabla\tau) = -\frac{1}{P_\tau} > 0,$$

provided s small enough. Hence $\nabla\tau$ is past directed. As in the proof of Lemma 2.1

$$\begin{aligned}
G(\nabla\tau, \nabla\tau) =& G^{ab}\partial_{y^a}\tau\partial_{y^b}\tau \\
=& \omega^2(\overline{g}^{ab}\partial_{y^a}\tau\partial_{y^b}\tau + p^{ab}\partial_{y^a}\tau\partial_{y^b}) \\
=& \omega^2(P_\tau^{-2}(2P_s + s^2hP_s^2 + |\widetilde{\nabla}P|^2 + O(s^{-2})) \\
=& s^2P_\tau^{-2}(2P_s + s^2hP_s^2 + |\widetilde{\nabla}P|^2) + O(s).
\end{aligned}$$

Hence $\alpha = 1 + O(s)$. Next,

$$\begin{aligned}
G(T, \widetilde{T}) =& \alpha\widetilde{\alpha}G(\nabla\tau, \nabla T) \\
=& \alpha\widetilde{\alpha}\omega^2(\overline{g}^{ab} + p^{ab})\partial_{y^a}\tau\partial_{y^b}t \\
=& \alpha\widetilde{\alpha}\omega^2(s^{-2}P_\tau^{-1} + O(1)) \\
=& -(\tau s)^{-1} + O(1).
\end{aligned}$$

□

Proof of Theorem 5.1. Let $F(\mathbf{y}, s, v) = v + Q(\mathbf{y}, s)$. Then the surface Σ given by $F = 0$ is spacelike. First, let us prove that $-G(\nabla F, \nabla F)$ is bounded on Σ. For $\tau_1 < \tau < \tau_2$, by (2.9) we have

(5.3)
$$\partial_v\tau = (\tau s + O(s^2))^{-1}, \partial_s\tau = (\tau s + O(s^2))^{-1}P_s, \partial_{y^A}\tau = (\tau s + O(s^2))^{-1}P_A,$$

$A = 1, 2$. Moreover, P_s, P_A are all bounded. We will work on a coordinate neighborhood U of $\mathbb{S}^2$, so that the standard metric σ_{AB} is bounded from above and the eigenvalues of (σ_{AB}) is bounded from below by a positive constant on U.

Let $T, \widetilde{T}$ be as in Lemma 5.2. Then by Lemma 4.4 and the assumption on $-G(T, \mathbf{n})$,

$$-G(T, \widetilde{T}) \leq 2G(T, \mathbf{n})G(\widetilde{T}, \mathbf{n}) \leq -C_1 G(\widetilde{T}, \mathbf{n})$$

for some $C_1 > 0$. By Lemma 5.2, we have

$$-G(\widetilde{T}, \mathbf{n}) \geq C_2 s^{-1} \tag{5.4}$$

for some $C_2 > 0$. Here and below, we implicitly assume that $0 < s < s_0$ with s_0 is small enough.

In the t, x^i coordinates, $F = t - r_* + Q =: t - u$. We have

$$-G(\nabla F, \nabla F) = \widetilde{\alpha}^{-2}\left((1 + \beta_i u^i)^2 - \widetilde{\alpha}^2 u^i u_i\right) > 0.$$

where $u_i = \frac{\partial u}{\partial x^i}$ and $u^i = G^{ij} u_j$. From this inequality, by Lemma 5.1, we conclude that $u^i u_i$ is uniformly bounded on Σ and hence $\frac{3}{2} \geq 1 + \beta_i u^i \geq \frac{1}{2} > 0$ provided s is small enough. We can write

$$-G(\nabla F, \nabla F) = \widetilde{\alpha}^{-2}(1 + \beta^i u_i)^2(1 - |U|^2),$$

where

$$U = \frac{\widetilde{\alpha} D u}{1 + \beta_i u^i}$$

and $Du = u^i \frac{\partial}{\partial x^i}$. On the other hand,

$$-G(\widetilde{T}, \mathbf{n}) = (1 - |U|^2)^{-\frac{1}{2}}.$$

By (5.4) we have

$$1 - |U|^2 \leq C_3 s^2$$

for some $C_3 > 0$. We conclude that by Lemma 5.1,

$$-G(\nabla F, \nabla F) \leq C_4 s^2. \tag{5.5}$$

for some $C_4 > 0$, because $1+\beta_i u^i$ is bounded. By the assumption on the tilt factor with respect to the time function τ, we have

$$
\begin{aligned}
C_5 \geq & -G(T,\mathbf{n}) \\
= & -\alpha(-G(\nabla F,\nabla F))^{-\frac{1}{2}}G(\nabla\tau,\nabla F) \\
= & -\alpha(-G(\nabla F,\nabla F))^{-\frac{1}{2}}\omega^2\left(\overline{G}^{ab}\partial_{y^a}\tau\partial_{y^b}F\right) \\
= & -\alpha(-G(\nabla F,\nabla F))^{-\frac{1}{2}}\omega^2\left((\overline{g}^{ab}+p^{ab})\partial_{y^a}\tau\partial_{y^b}F\right) \\
= & -\alpha(-G(\nabla F,\nabla F))^{-\frac{1}{2}}\omega^2\times \\
& \Big[\overline{g}^{vv}\partial_v\tau\partial_v F+\overline{g}^{vs}(\partial_s\tau\partial_v F+\partial_v\tau\partial_s F)+\overline{g}^{ss}\partial_s\tau\partial_s F \\
& +\overline{g}^{AB}\partial_{y^A}\tau\partial_{y^B}F+q^b\partial_{y^b}F\Big] \\
= & -\alpha(-G(\nabla F,\nabla F))^{-\frac{1}{2}}\omega^2\times \\
& \left[\frac{1}{P_\tau}\left(-(P_s+Q_s)-s^2(1-2ms)P_sQ_s-\langle\widetilde{\nabla}P,\widetilde{\nabla}Q\rangle\right)+q^b\partial_{y^b}F\right]
\end{aligned}
$$

where $q^b = O(s^2)$ and we have used (5.3). By (5.3) and (5.5) we conclude that

$$-Q_s - s^2(1-2ms)P_sQ_s - \langle\widetilde{\nabla}P,\widetilde{\nabla}Q\rangle + q^b\partial_{y^b}F \leq C_6$$

for some constant $C_6 > 0$. Since P_s, P_A are uniformly bounded and

$$\sigma^{AB}Q_AQ_B \geq C\sum_{A=1}^{2}Q_A^2$$

for some $C > 0$, we have for any $\varepsilon > 0$, we have

$$-(1+O(s^2))Q_s - (\varepsilon+O(s^2))|\widetilde{\nabla}Q|^2 \leq C_7(\varepsilon) \tag{5.6}$$

for some constant C_7 which also depends on ε.

Since Σ is spacelike, we have

$$G(\nabla F,\nabla F)\leq 0.$$

Computations similar to the above, we have

$$2Q_s + s^2(1-2ms)Q_s^2 + |\widetilde{\nabla}Q|^2 + O(s^3)\left(1+Q_s^2+|\widetilde{\nabla}Q|^2\right) \leq 0.$$

This implies that $Q_s \le Cs^3$ and

$$(5.7)\quad (2+O(s^3))Q_s + (s^2(1-2ms)+O(s^3))Q_s^2 + (1+O(s^3))|\widetilde{\nabla}Q|^2 \le C_8$$

for some $C_8 > 0$. Multiply (5.7) by $\delta > 0$ and add it to (5.6), if $s > 0$ is small enough, we have

$$\begin{aligned} &-[1+O(s^2)-\delta(2+O(s^3))]Q_s + [\delta(1+O(s^3)) - (\varepsilon + O(s^2))]|\widetilde{\nabla}Q|^2 \\ &\quad \le C_7 + \delta C_8 \end{aligned}$$

Let $\delta = 2\varepsilon$ and $\varepsilon = \frac{1}{8}$, we can conclude that

$$-Q_s \le C_9$$

for some $C_9 > 0$ provided s is small enough. Hence $-C_9 \le Q_s \le Cs^3$ if s is small enough. From this and (5.7), we conclude that $|\widetilde{\nabla}Q|^2$ is uniformly bounded provided s is small enough. This completes the proof of the theorem. □

References

[1] L. Andersson and M. S. Iriondo, Existence of constant mean curvature hypersurfaces in asymptotically flat spacetimes, *Ann. Global Anal. Geom.* **17** (1999), no. 6, 503–538. MR1728086

[2] R. Bartnik, Existence of maximal surfaces in asymptotically flat spacetimes, *Comm. Math. Phys.* **94** (1984), no. 2, 155–175. MR0761792

[3] R. Bartnik, P. T. Chruściel and N. Murchadha, On maximal surfaces in asymptotically flat space-times, *Comm. Math. Phys.* **130** (1990), no. 1, 95–109 MR1055687

[4] R. Bartolo, E. Caponio and A. Pomponio, Spacelike graphs with prescribed mean curvature on exterior domains in the Minkowski spacetime, *Proc. Amer. Math. Soc.* **149** (2021), no. 12, 5139–5151. MR4327421

[5] C. Gerhardt, *H*-surfaces in Lorentzian manifolds, *Comm. Math. Phys.* **89** (1983), no. 4, 523–553. MR0713684

[6] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., rev. 3rd printing. Berlin; New York: Springer, 1998. MR3793605

[7] K-W Lee and Y-I Lee, Spacelike spherically symmetric CMC foliation in the extended Schwarzschild spacetime, *Ann. Henri Poincaré* **17** (2016), no. 6, 1477–1503. MR3500222

[8] K-W Lee and Y-I Lee, Spacelike spherically symmetric CMC hypersurfaces in Schwarzschild spacetimes (I): Construction, arXiv:1111.2679v2, 2011.

[9] C. Li, Y.-G. Shi and L.-F. Tam, Boundary behaviors of spacelike constant mean curvature surfaces in Schwarzschild spacetime, Ann. Global Anal. Geom. **65** (2024), no.3, Paper No. 23, 33 pp. MR4728753

[10] B. O'Neill, *Semi-Riemannian geometry: with applications to relativity*, New York: Academic Press 1983. MR0719023

[11] S. Stumbles, Hypersurfaces of constant mean extrinsic curvature, *Annals of Physics* **133** (1980), 28–56. MR0626082

[12] A. E. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, *Invent. Math.* **66** (1982), no. 1, 39–56. MR0652645

Luen-Fai Tam
The Institute of Mathematical Sciences and the Department of Mathematics
The Chinese University of Hong Kong
Shatin
Hong Kong
China
E-mail address: lftam@math.cuhk.edu.hk

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