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Characteristic gluing with Λ: 1. Linearised Einstein equations on four-dimensional spacetimes

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We establish a gluing theorem for linearised vacuum gravitational fields in Bondi gauge on a class of characteristic hypersurfaces in static vacuum four-dimensional backgrounds with cosmological constant $\Lambda \in \mathbb{R}$ and arbitrary topology of the compact cross-sections of the null hypersurface. This generalises and complements, in the linearised case, the pioneering analysis of Aretakis, Czimek and Rodnianski, carried-out on light-cones in Minkowski spacetime.

1. Introduction

In their pioneering work [1, 2, 3], Aretakis, Czimek and Rodnianski presented a gluing construction, along a null hypersurface, of characteristic Cauchy data for the vacuum Einstein equations, for a class of asymptotically Minkowskian data. We wish to generalise their construction to null hypersurfaces with non-spherical sections, and to allow for a cosmological constant, in spacetimes of dimension four or higher.

As a first step towards this, in this paper we consider four-dimensional vacuum Einstein equations, with a cosmological constant $\Lambda \in \mathbb{R}$, linearised at Birmingham-Kottler metrics. Recall, now, that the analysis in [1, 2, 3] is based on the Christodoulou-Klainerman version of the Newman-Penrose formalism. Recall also that there exists a well-studied Bondi-type parameterisation of the metric (see [4] and references therein), which generalises readily to any dimensions. So it appears of interest to analyse the problem in Bondi coordinates, to be able to exploit results which have already been established in the Bondi setting, with an eye-out both for possible advantages of the alternative setting and for higher dimensions. Indeed, while we are concerned with four-dimensional spacetimes in this work, we carry-out the higher dimensional construction in a companion paper [5].

Interestingly enough, some more work needs to be done in other topologies and dimensions because of different properties of the differential operators involved. In fact, the analysis on null three-dimensional hypersurfaces with spherical cross-sections turns out to be somewhat simpler than the general case. One of key new aspects of other topologies or dimensions, when

compared to null hypersurfaces with two-dimensional spherical sections, is the existence of non-trivial transverse-traceless two-covariant tensors. Their existence leads to new difficulties which need to be addressed. While the collection of TT-tensors is finite-dimensional on two dimensional manifolds, these tensors carry the bulk of information about the geometry in higher dimensions.

To make things precise, we consider the linearisation of the vacuum Einstein equations at a metric

$$g = -\big(\varepsilon - \alpha^2 r^2 - \frac{2m}{r}\big) du^2 - 2du\, dr + r^2 \mathring{\gamma}_{AB} dx^A dx^B \,, \tag{1}$$

with

$$\alpha \in \{0, \sqrt{\Lambda/3}\} \subset \mathbb{R} \cup \sqrt{-1}\mathbb{R}\,, \quad m \in \mathbb{R}\,,$$

where $\mathring{\gamma}_{AB} dx^A dx^B$ is a u and r-independent metric with scalar curvature 2ε, with $\varepsilon \in \{0, \pm 1\}$. Roughly speaking, the question addressed here is the following: given two smooth linearised solutions of the vacuum Einstein equations defined near the null hypersurfaces $\{u = 0\,,\ r < r_1\}$ and $\{u = 0\,,\ r > r_2\}$, where $r_2 > r_1$, can we find characteristic initial data on the missing region $\{u = 0\,,\ r_1 \le r \le r_2\}$ which, when evolved to a solution of the linearised Einstein equations, provide a linearised metric perturbation which coincides on $\{u = 0\}$, together with u-derivatives up to order k, with the original data. We refer to this construction as the $C^k_u\, C^\infty_{(r,x^A)}$-gluing. The resolution of this problem is presented in Theorem 4.1, p. 727 below, which is the main result of this paper. This theorem is the key step towards a nonlinear gluing [6], where a suitable implicit function theorem is used.

An equivalent way of formulating the gluing problem, advocated in [1], is that of connecting two sets of "sphere data" using null-hypersurface data. This perspective can also be taken in our setting, with "sphere data" replaced by suitable linearised data on codimension-two spacelike manifolds, viewed as cross-sections of a null hypersurface.

It was found by Aretakis et al., in the case $\Lambda = 0$ and $\varepsilon = 1$, that there exists a ten-parameter family of obstructions to do such a gluing-up-to-gauge, when requiring continuity of two u-derivatives of the metric components along the null-hypersurface. Our analysis shows that the analysis is affected both by the dimension, by the cosmological constant, by the topology of sections of the level sets of u (which we assume to be compact), by the mass, and by the number of transverse derivatives which are required to be continuous. In the spherical four-dimensional case with $m = 0$ we provide an alternative proof of the corresponding result in [2] for $C^2_u\, C^\infty_{(r,x^A)}$-gluing, with

Table 1.1: The dimension of the space of obstructions for $C^2_u C^\infty_{(r,x^A)}$-gluing. The radial charges $\overset{[a]}{Q}$, a=1,2, are defined in (59), p. 705 and (72), p. 708; the radially-conserved tensor fields q_{AB}, $\overset{[2]}{q}_{AB}$, and $\overset{[3,i]}{Q}$ are defined in (100), p. 714, (159), p. 725, and (115), p. 717; $\mathfrak{g}$ is the genus of the cross-sections of the characteristic initial data hypersurface; the superscripts [H], respectively [TT], denote the L^2-orthogonal projection on the set of harmonic 1-forms, respectively on transverse-traceless tensors. On S^2 the four obstructions associated with $\overset{[2]}{Q}$ correspond to spacetime translations, the three obstructions associated with $\overset{[1]}{Q}$ when $m \neq 0$ correspond to rotations of S^2, with the further three obstructions arising when $m = 0$ corresponding to boosts

	S^2	$\mathbb{T}^2$	higher genus
$\overset{[1]}{Q}$: $m = 0$	6	2	0
$m \neq 0$	3	2	0
$\overset{[2]}{Q}$: $m = 0$	4	1	1
$m \neq 0$	1	1	1
$\overset{[3,1]}{Q}{}^{[\mathrm{H}]}$: $m = 0$	0	coincides with $\overset{[2]}{Q}$	$2\mathfrak{g}$
$m \neq 0$	0	0	0
$q^{[\mathrm{TT}]}_{AB}$: $m = 0$, $\alpha = 0$	0	2	$6(\mathfrak{g} - 1)$
$m = 0$, $\alpha \neq 0$	0	0	0
$m \neq 0$	0	0	0
$\overset{[3,2]}{Q}{}^{[\mathrm{H}]}$: $m = 0$	0	0	$2\mathfrak{g}$
$m \neq 0$	0	0	0
$\overset{[2]}{q}{}^{[\mathrm{TT}]}_{AB}$: $m = 0$	0	2	$6(\mathfrak{g} - 1)$
$m \neq 0$	0	0	0
together: $m = 0$, $\alpha = 0$	10	7	$16\mathfrak{g} - 11$
$m = 0$, $\alpha \neq 0$	10	5	$10\mathfrak{g} - 5$
$m \neq 0$	4	3	1

the same number of obstructions. Table 1.1 lists the obstructions which arise in the linearised gluing depending upon the geometry of the cross-sections of the initial data hypersurface and the mass parameter m. A key role in our construction is played by the radially-constant function χ (cf. (80), p. 710

below), the existence of which has already been pointed-out in [7], and the radially constant fields q_{AB} and $\overset{[3,1]}{Q}{}_A$ (cf. (100), p. 714 and (106), p. 715), which do not seem to have been noticed so far in the Bondi gauge, and which seem to be related to the radially constant fields $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$ discovered in [1]. We point out a slightly different interpretation of the result, namely that the gluing can be performed without obstructions after adding fields, which carry the missing radial charges and which we describe explicitly, to the data on $\{r > r_2\}$. We describe the additional obstructions that arise for $C^k_u\, C^\infty_{(r,x^A)}$-gluing, $k \geq 3$, when linearising on a background with $m = 0$, see Tables 4.1, p. 732, and 4.2, p. 733. We show, with a considerable amount of work, that the higher-order obstructions disappear on backgrounds with $m \neq 0$.

In their introduction, the authors of [1] discuss several applications of their construction. The results presented here lead immediately to corresponding results for the linearised fields in our setting.

This work is organised as follows: In Section 2 we introduce some of our notations. In Section 3 we analyse the linearised Einstein equations in the Bondi gauge, following [7]. As already observed in [1, 2, 3], a key part of the gluing is played by the residual gauges, discussed in Section 3.2. The main new element, as compared to [7], is Section 3.7, where inductive formulae for higher-order transverse derivatives are presented. The gluing construction is carried-out in Section 4. We present our strategy in Section 4.1, with further details provided in the remaining sections there. In Section 6 we reformulate our gluing result as an unobstructed gluing-with-perturbation problem for the data on $\{r > r_2\}$. Various technical results are presented in the appendices.

We note that the first arxiv version of this work established solvability of the linearised gluing problem through an argument which was inadequate for the full gluing problem, addressed in [6], because of insufficient differentiability of solutions of some of the equations used. The current, alternative version resolves these issues.

2. Notation

Let $\mathring{\gamma} = \mathring{\gamma}_{AB} dx^A dx^B$ be a metric on a 2-dimensional, compact, orientable manifold $\mathbf{S}$, with covariant derivative $\mathring{D}$. We let $\mathring{\mathrm{div}}_{(1)}$, respectively $\mathring{\mathrm{div}}_{(2)}$, denote the divergence operator on vector fields ξ, respectively on two-index tensor fields h:

$$\mathring{\mathrm{div}}_{(1)}\, \xi := \mathring{D}_A \xi^A\,, \qquad (\mathring{\mathrm{div}}_{(2)}\, h)_A := \mathring{D}_B h^B{}_A\,. \tag{2}$$

Given a function f we denote by $f^{[1]}$ the L^2-orthogonal projection of f on the constants:

$$f^{[1]} := \frac{1}{|\mathbf{S}|_{\mathring{\gamma}}} \int_{\mathbf{S}} f d\mu_{\mathring{\gamma}}\,, \quad \text{where} \quad |\mathbf{S}|_{\mathring{\gamma}} = \int_{\mathbf{S}} d\mu_{\mathring{\gamma}}\,. \tag{3}$$

Note that $f^{[1]}$ should not be confused with $f^{[=1]}$, which we use when decomposing a function or a tensor field in spherical harmonics on S^2. We set

$$f^{[1^\perp]} := f - f^{[1]}\,. \tag{4}$$

Let CKV, respectively KV, denote the space of conformal Killing vector fields on $\mathbf{S}$, respectively Killing vector fields. Thus (cf. Appendix C.2), CKV is six-dimensional on S^2, consists of covariantly constant vectors on $\mathbb{T}^2$, and is trivial on manifolds of higher genus. Given a vector field ξ on $\mathbf{S}$ we denote by $(\xi^A)^{[\mathrm{CKV}]}$ the L^2-orthogonal projection on the space CKV, with

$$(\xi^A)^{[\mathrm{CKV}^\perp]} := \xi^A - (\xi^A)^{[\mathrm{CKV}]}\,,$$

with a similar notation for $(\xi^A)^{[\mathrm{KV}]}$ and $(\xi^A)^{[\mathrm{KV}^\perp]}$.

We will denote by H the space of harmonic 1-forms:

$$\mathrm{H} = \{\xi_A \,|\, \mathring{D}^A \xi_A = 0 = \epsilon^{AB} \mathring{D}_A \xi_B\}\,. \tag{5}$$

By standard results (cf., e.g., [8, Theorems 19.11 and 19.14] or [9, Theorem 18.7]), the space H has dimension $2\mathfrak{g}$ on cross-sections $\mathbf{S}$ with genus $\mathfrak{g}$, in particular it is trivial on spherical sections. We will denote by $\xi_A^{[\mathrm{H}]}$ the L^2-orthogonal projection of ξ_A on H, and by $\xi_A^{[\mathrm{H}^\perp]}$ the projection on the L^2-orthogonal to H.

Let TT denote the space of transverse-traceless symmetric two tensors:

$$\mathrm{TT} = \{h_{AB} \,|\, h_{[AB]} = 0 = \mathring{\gamma}^{CD} h_{CD} = \mathring{D}^E h_{EF}\}\,. \tag{6}$$

The space TT is trivial on S^2, consists of covariantly constant tensors on $\mathbb{T}^2$, and is $6(\mathfrak{g}-1)$-dimensional on two-dimensional manifolds of genus $\mathfrak{g} \geq 2$ (cf., e.g., [10] Theorem 8.2 and the paragraph that follows, or [11, Theorem 6.1 and Corollary 6.1]).

Given a tensor field $h = h_{AB} dx^A dx^B$ we denote by $h_{AB}^{[\mathrm{TT}]}$ the L^2-orthogonal projection of h on TT, and set

$$h_{AB}^{[\mathrm{TT}^\perp]} := h_{AB} - h_{AB}^{[\mathrm{TT}]}\,. \tag{7}$$

Clearly, for two-covariant traceless symmetric tensors on S^2 it holds that $h_{AB}^{[\mathrm{TT}^\perp]} = h_{AB}$, but this is not true anymore for the remaining two-dimensional compact manifolds.

We will often follow terminology and notation from [1]. In particular, scalar functions, vector fields, and traceless two-covariant symmetric tensors on S^2 will be decomposed into spherical harmonics, see Appendix C.1 for a summary. The notation $t^{[=\ell]}$ will denote the L^2-orthogonal projection of a tensor t on the space of ℓ-spherical harmonics. Then

$$t^{[\le\ell]} = \sum_{i=0}^{\ell} t^{[=i]}\,, \qquad t^{[>\ell]} = t - t^{[\le\ell]}\,, \tag{8}$$

with obvious similar definition of $t^{[<\ell]}$, etc.

One of the operators appearing below is the operator

$$\mathring{L} := \Delta_{\mathring{\gamma}}(\Delta_{\mathring{\gamma}} + 2\varepsilon) \tag{9}$$

acting on functions. Its kernel consists of linear combinations of $\ell = 0$ and $\ell = 1$ spherical harmonics on S^2, and of constants on the remaining compact orientable two-dimensional manifolds. Given a function f we will write $f^{[\ker \mathring{L}]}$ for the L^2-orthogonal projection on this kernel, and $f^{[(\ker \mathring{L})^\perp]}$ for the projection on its orthogonal.

3. Linearised characteristic constraint equations in Bondi coordinates

Let $(\mathscr{M}, g)$ be a $(3+1)$-dimensional spacetime. Locally, near a null hypersurface for which the optical divergence scalar is non-vanishing, we can use Bondi-type coordinates (u, r, x^A) in which the metric takes the form

$$\begin{aligned} g_{\alpha\beta}dx^\alpha dx^\beta = &-\frac{V}{r}e^{2\beta}du^2 - 2e^{2\beta}dudr \\ &+ r^2\gamma_{AB}\Big(dx^A - U^A du\Big)\Big(dx^B - U^B du\Big)\,, \end{aligned} \tag{10}$$

where

$$\det[\gamma_{AB}] = \det[\mathring{\gamma}_{AB}]\,, \tag{11}$$

with $\mathring{\gamma}_{AB}(x^C)$ being a metric of constant scalar curvature 2ε. In particular, $\det[\gamma_{AB}]$ is r and u-independent, which implies

$$\gamma^{AB}\partial_r\gamma_{AB} = 0\,, \qquad \gamma^{AB}\partial_u\gamma_{AB} = 0\,. \tag{12}$$

As such, the inverse metric reads

$$g^\sharp = e^{-2\beta}\frac{V}{r}\,\partial_r^2 - 2e^{-2\beta}\,\partial_u\partial_r - 2e^{-2\beta}U^A\,\partial_r\partial_A + \frac{1}{r^2}\gamma^{AB}\,\partial_A\partial_B\,. \tag{13}$$

Note that each surface $\{u = \text{constant}\}$ is a null hypersurface with null normal proportional to ∂_r, and r is a parameter which varies along the null generators. Finally, the x^C's are local coordinates on the codimension-two surfaces of constant (u, r) which, as r varies, foliate each null hypersurface of constant u.

The restriction of the Einstein equations (E.E.) to a null hypersurface gives a set of null constraint equations for the metric functions $(V, \beta, U^A, \gamma_{AB})$, which leads to obstructions to the gluing of characteristic data. In this work we will study the linearised problem around a null hypersurface in a Birmingham-Kottler background, which includes a Minkowski, anti-de Sitter or de Sitter background. In Bondi coordinates the background metrics can be written as

$$g \equiv g_{\alpha\beta}dx^\alpha dx^\beta = g_{uu}du^2 - 2du\,dr + r^2\mathring{\gamma}_{AB}dx^A dx^B\,, \tag{14}$$

with

$$g_{uu} := -\big(\varepsilon - \alpha^2 r^2 - \frac{2m}{r}\big)\,, \quad \varepsilon \in \{0, \pm 1\}\,, \quad \alpha \in \{0, \sqrt{\Lambda/3}\}\,, \quad m \in \mathbb{R}\,,$$

where $\mathring{\gamma}_{AB}dx^A dx^B$ is a u- and radially constant metric of scalar curvature 2ε, and note that $\alpha \in \mathbb{R} \cup i\mathbb{R}$: a purely imaginary value of α is allowed to accommodate for a cosmological constant $\Lambda < 0$. It holds that

$$g^{\alpha\beta}\partial_\alpha\partial_\beta = -2\partial_u\partial_r - g_{uu}(\partial_r)^2 + r^{-2}\mathring{\gamma}^{AB}\partial_A\partial_B\,.$$

Consider now a perturbation of the metric of the form

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon h_{\mu\nu}\,, \tag{15}$$

where ϵ should be thought as being very small. The conditions on the linearised fields such that the perturbed metric is still in the Bondi form to $O(\epsilon)$ are,

$$h_{rA} = h_{rr} = \mathring{\gamma}^{AB}h_{AB} = 0\,. \tag{16}$$

In what follows for perturbations around a Birmingham-Kottler background, we shall sometimes find it convenient to use fields $\{\delta V, \delta\beta, \delta U_A := \mathring{\gamma}_{AB}\delta U^B\}$

to denote metric perturbations. These correspond respectively to

$$\{\delta V + 2V\delta\beta, \delta\beta, \delta U_A\} \equiv \{-rh_{uu}, -h_{ur}/2, -h_{uA}/r^2\}\,. \tag{17}$$

We will also use the notation

$$\check{h}_{\mu\nu} := h_{\mu\nu}/r^2\,. \tag{18}$$

3.1. The linearised $C^k_u C^\infty_{(r,x^A)}$-gluing problem

One of the key objects that arise in the characteristic gluing construction of [1] are the "sphere data". Roughly speaking, these are data that are needed on a cross-section of a characteristic surface for the integration of the transport equations (see below).

Using a Bondi parameterisation of the metric, these data can be defined as follows. Let $\mathscr{N}_I$ be a null hypersurface $\{u = u_0, r \in I\}$, where I is an interval in $\mathbb{R}$, and let $\mathbf{S}$ be a cross-section of $\mathscr{N}$, i.e. a two-dimensional submanifold of $\mathscr{N}$ meeting each null generator of $\mathbf{S}$ precisely once. Let $2 \le k \in \mathbb{N}$ be the number of derivatives of the metric that we want to control at $\mathbf{S}$. Using the Bondi parameterisation of the metric, we define linearised Bondi *cross-section data* of order k as the collection of fields

$$\mathfrak{d}_{\mathbf{S}} = (\partial_u^\ell \partial_r^j h_{AB}|_{\mathbf{S}},\, \partial_u^\ell \partial_r^j \delta\beta|_{\mathbf{S}},\, \partial_u^\ell \partial_r^j \delta U^A|_{\mathbf{S}},\, \partial_u^\ell \partial_r^j \delta V|_{\mathbf{S}})\,, \tag{19}$$

for integers ℓ, j such that $\ell + j \le k$.[1]

For simplicity we assume that all the fields in (19) are smooth, though a finite sufficiently large degree of differentiability would suffice for our purposes, as can be verified by chasing the number of derivatives in the relevant equations; compare Section 3.6 below.

A natural threshold for the gluing is $k = 2$, as then one expects existence of an associated space-time solving the vacuum Einstein equations when the fields are sufficiently differentiable in directions tangent to $\mathbf{S}$ (cf. [13] for a small data result in a different gauge; see [14, 15, 16] for existence without smallness restrictions under more stringent differentiability conditions). In the linearised $C^k_u C^\infty_{(r,x^A)}$-gluing problem we start with two sections $\mathbf{S}_1$ and $\mathbf{S}_2 \subset J^+(\mathbf{S}_1)$ of a null hypersurface $\{u = 0\}$ equipped with Bondi coordinates as in (10), each with constant r, and their linearised Bondi cross-section data of order k, $\mathfrak{d}_{\mathbf{S}_1}$ and $\mathfrak{d}_{\mathbf{S}_2}$. The goal is to interpolate between $\mathfrak{d}_{\mathbf{S}_1}$

[1] The data $\mathfrak{d}_{\mathbf{S}}$ are closely related to the data $\Psi_{\mathrm{Bo}}[\mathbf{S}, k]$ of [12, Section 5]. As discussed in more detail there, some of the fields in (19) are not independent, but this is irrelevant for our purposes.

and $\mathfrak{d}_{\mathbf{S}_2}$ along a null hypersurface $\mathscr{N}_{[r_1,r_2]}$ such that (i) $\mathfrak{d}_{\mathbf{S}_1}$ agrees with the restriction to r_1 of the interpolating field along $\mathscr{N}_{[r_1,r_2]}$; (ii) $\mathfrak{d}_{\mathbf{S}_2}$ agrees with the restriction to $r = r_2$ of the interpolating field; and (iii) the constructed field satisfies the linearised null constraint equations. We shall see in Section 3.3 how the linearised null constraint equations lead to obstructions to the gluing.

Since linearised Bondi data are defined up to linearised gauge transformations, we shall use these transformations to help us with the gluing.

3.2. Gauge freedom

Recall that linearised gravitational fields are defined up to a gauge transformation

$$h \mapsto h + \mathcal{L}_\zeta g \tag{20}$$

determined by a vector field ζ. Once the metric perturbation has been put into Bondi gauge, there remains the freedom to make gauge transformations which preserve this gauge:

$$\mathcal{L}_\zeta g_{rr} = 0\,, \tag{21}$$

$$\mathcal{L}_\zeta g_{rA} = 0\,, \tag{22}$$

$$g^{AB}\mathcal{L}_\zeta g_{AB} = 0\,, \tag{23}$$

For the metric (14) this is solved by (cf., e.g., [7])

$$\zeta^u(u,r,x^A) = \xi^u(u,x^A)\,, \tag{24}$$

$$\zeta^B(u,r,x^A) = \xi^B(u,x^A) - \frac{1}{r}\mathring{D}^B\xi^u(u,x^A)\,, \tag{25}$$

$$\zeta^r(u,r,x^A) = -\frac{1}{2}r\mathring{D}_B\xi^B(u,x^A) + \frac{1}{2}\Delta_{\mathring{\gamma}}\xi^u(u,x^A)\,, \tag{26}$$

for some fields $\xi^u(u,x^A)$, $\xi^B(u,x^A)$, and where $\mathring{D}_A$ and $\Delta_{\mathring{\gamma}}$ are respectively the covariant derivative and the Laplacian operator associated with the two-dimensional metric $\mathring{\gamma}_{AB}$ appearing in (14).

We define

$$\mathring{\mathcal{L}}_\zeta$$

to be the Lie-derivation in the x^A-variables with respect to the vector field $\zeta^A\partial_A$.

The transformation (20) can be viewed as a result of linearised coordinate transformation to new coordinates $\tilde{x}^\mu$ such that

$$x^\mu = \tilde{x}^\mu + \epsilon\zeta^\mu(\tilde{x}^\mu)\,, \tag{27}$$

where ϵ is as in (15). Writing g_{uu} as

$$g_{uu} = -\varepsilon + \alpha^2 r^2 + \frac{2m}{r} =: \varepsilon N^2\,, \text{ where } \varepsilon \in \{\pm 1\},$$

under (27), the linearised metric perturbation transforms as

$$\begin{aligned} h_{uA} \to \tilde{h}_{uA} &= h_{uA} + \mathscr{L}_\zeta g_{uA} \\ &= h_{uA} + \partial_A(\epsilon N^2\zeta^u - \zeta^r) + r^2\mathring{\gamma}_{AB}\partial_u\zeta^B \\ &= h_{uA} - \frac{1}{2}\partial_A\left[(\Delta_{\mathring{\gamma}}\xi^u + 2\varepsilon\xi^u) - r(\mathring{D}_B\xi^B - 2\partial_u\xi^u)\right] \\ &\quad + r^2\big(\mathring{\gamma}_{AB}\partial_u\xi^B + \big(\alpha^2 + \frac{2m}{r^3}\big)\partial_A\xi^u\big)\,, \end{aligned} \tag{28}$$

$$\begin{aligned} h_{ur} \to \tilde{h}_{ur} &= h_{ur} + \mathscr{L}_\zeta g_{ur} = h_{ur} - \partial_u\zeta^u + \epsilon N^2\partial_r\zeta^u - \partial_r\zeta^r \\ &= h_{ur} - \partial_u\xi^u + \frac{1}{2}\mathring{D}^A\xi_A\,, \end{aligned} \tag{29}$$

$$\begin{aligned} h_{uu} \to \tilde{h}_{uu} &= h_{uu} + \mathscr{L}_\zeta g_{uu} = h_{uu} + \epsilon\zeta^r\partial_r N^2 + 2\partial_u(\epsilon N^2\zeta^u - \zeta^r) \\ &= h_{uu} - (2\varepsilon + \Delta_{\mathring{\gamma}})\partial_u\xi^u + r\big(\mathring{D}_B\partial_u\xi^B + \big(\alpha^2 - \frac{m}{r^3}\big)\Delta_{\mathring{\gamma}}\xi^u\big) \\ &\quad + \big(\alpha^2 r^2 + \frac{2m}{r}\big)(2\partial_u\xi^u) - \big(\alpha^2 r^2 - \frac{m}{r}\big)\mathring{D}_B\xi^B\,, \end{aligned} \tag{30}$$

$$h_{AB} \to \tilde{h}_{AB} = h_{AB} + \mathscr{L}_\zeta g_{AB} = h_{AB} + 2r\zeta^r\mathring{\gamma}_{AB} + r^2\mathring{\mathcal{L}}_\zeta\mathring{\gamma}_{AB}\,, \tag{31}$$

with

$$\mathrm{TS}[X_{AB}] := \frac{1}{2}(X_{AB} + X_{BA} - \mathring{\gamma}^{CD}X_{CD}\mathring{\gamma}_{AB})$$

denoting the traceless symmetric part of a tensor on a section $\mathbf{S}$.

Given $\mathbf{S}_{u_0,r_0}$ corresponding to a $\{u = u_0, r = r_0\}$-section of some $\mathscr{N}$, equations (28)-(31) together with all their u- and r-derivatives up to order k define a new set of order-k cross-section data on

$$\begin{aligned} \tilde{\mathbf{S}}_{u_0,r_0} &:= \{\tilde{u} = u_0, \tilde{r} = r_0\} \\ &= \{u = u_0 + \epsilon\zeta^u(u_0, r_0, x^A), r = r_0 + \epsilon\zeta^r(u_0, r_0, x^A)\}\,, \end{aligned}$$

a section lying close to the original $\mathbf{S}_{u_0,r_0}$, in terms of the gauge fields

$$\{\partial_u^i \xi^B|_{u=u_0}, \partial_u^i \xi^u|_{u=u_0}\}_{0\leq i\leq 3}$$

as well as the original metric perturbations evaluated on $\tilde{\mathbf{S}}_{u_0,r_0}$.

Equation (29) shows that we can always choose ζ so that

$$\tilde{h}_{ur} = 0\,. \tag{32}$$

After having done this, we are left with a residual set of gauge transformations, defined by a u-parameterised family of vector fields $\xi^A(u,\cdot)$, and $\xi^u(u,\cdot)$, with the condition

$$\partial_u \xi^u(u,x^A) = \frac{\mathring{D}_B \xi^B(u,x^A)}{2} \tag{33}$$

needed to preserve the gauge $\tilde{h}_{ur} = 0$.

Under the residual gauge transformations with (33), the transformed fields take the form

$$\begin{aligned}\tilde{h}_{uA} &= h_{uA} - \frac{1}{2}\mathring{D}_A \Delta_{\mathring{\gamma}} \xi^u + \epsilon N^2 \mathring{D}_A \xi^u + r^2 \partial_u \xi_A \\ &= h_{uA} - \frac{1}{2}\mathring{D}_A \left[\,(\Delta_{\mathring{\gamma}} \xi^u + 2\varepsilon \xi^u)\right] \\ &\quad + r^2 \big[\mathring{\gamma}_{AB} \partial_u \xi^B + \big(\alpha^2 + \frac{2m}{r^3}\big)\mathring{D}_A \xi^u\big]\,,\end{aligned} \tag{34}$$

$$\begin{aligned}\tilde{h}_{uu} &= h_{uu} + r\big[\big(\alpha^2 - \frac{m}{r^3}\big)\Delta_{\mathring{\gamma}} \xi^u + \mathring{D}_B \partial_u \xi^B\big] \\ &\quad - \big(\varepsilon + \frac{1}{2}\Delta_{\mathring{\gamma}} - \frac{3m}{r}\big)\mathring{D}_B \xi^B\,,\end{aligned} \tag{35}$$

$$\tilde{h}_{AB} = h_{AB} + 2r^2\,\mathrm{TS}[\mathring{D}_A \xi_B] - 2r\,\mathrm{TS}[\mathring{D}_A \mathring{D}_B \zeta^u]\,. \tag{36}$$

Let $\mathfrak{d}_{\mathbf{S}_1}$ and $\mathfrak{d}_{\mathbf{S}_2}$ be linearised Bondi cross-section data of order k on $\mathbf{S}_1$ and $\mathbf{S}_2$ respectively. Given gauge fields

$$\{\partial_u^i \xi^B|_{\tilde{\mathbf{S}}_a}, \partial_u^i \xi^u|_{\tilde{\mathbf{S}}_a}\}_{0\leq i\leq k+1\,,1\leq a\leq 2}\,,$$

the associated transformed Bondi cross-section data are given by (28)-(31) and their ∂_u and ∂_r derivatives. In the linearised gluing problem, we shall allow for such gauge transformations to the data; that is, we consider gluing along a null hypersurface of the transformed data $\tilde{\mathfrak{d}}_{\tilde{\mathbf{S}}_1}$ and $\tilde{\mathfrak{d}}_{\tilde{\mathbf{S}}_2}$ with the

freedom of choosing gauge fields to achieve the gluing. We shall call this gluing-up-to-gauge.

To simplify notation we will write

$$L_1(\xi^u)_A := -\frac{1}{2}\mathring{D}_A\,[\,\Delta_{\mathring{\gamma}}\xi^u + 2\varepsilon\xi^u] = -\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\mathring{D}_B\xi^u]\,, \tag{37}$$

$$C(\zeta)_{AB} := \mathrm{TS}[\mathring{D}_A\zeta_B]\,, \tag{38}$$

$$L_2(\xi) := -\left(\varepsilon + \frac{1}{2}\Delta_{\mathring{\gamma}}\right)\mathring{D}_B\xi^B\,. \tag{39}$$

For further convenience we note the transformation laws, in this notation and for $i \geq 1$,

$$\tilde{h}_{uA} = h_{uA} + L_1(\xi^u)_A + r^2\big(\partial_u\xi_A + \big(\alpha^2 + \frac{2m}{r^3}\big)\mathring{D}_A\xi^u\big)\,, \tag{40}$$

$$\begin{aligned}\partial_u^i\tilde{h}_{uA} &= \partial_u^i h_{uA} + \frac{1}{2}L_1(\mathring{D}_B\partial_u^{i-1}\xi^B)_A \\ &\quad + r^2\big[\partial_u^{i+1}\xi_A + \frac{1}{2}\big(\alpha^2 + \frac{2m}{r^3}\big)\mathring{D}_A\mathring{D}_B\partial_u^{i-1}\xi^B\big]\,,\end{aligned} \tag{41}$$

$$\begin{aligned}\tilde{h}_{uu} &= h_{uu} + r\big[\big(\alpha^2 - \frac{m}{r^3}\big)\Delta_{\mathring{\gamma}}\xi^u + \mathring{D}_B\partial_u\xi^B\big] + L_2(\xi) \\ &\quad + \frac{3m}{r}\mathring{D}_B\xi^B\,,\end{aligned} \tag{42}$$

$$\begin{aligned}\tilde{h}_{AB} &= h_{AB} + 2r^2C(\zeta)_{AB} \\ &= h_{AB} + 2r^2C(\xi)_{AB} - 2r\,\mathrm{TS}[\mathring{D}_A\mathring{D}_B\xi^u]\,,\end{aligned} \tag{43}$$

$$\begin{aligned}\partial_u^i\tilde{h}_{AB} &= \partial_u^i h_{AB} + 2r^2C(\partial_u^i\xi)_{AB} \\ &\quad - r\,\mathrm{TS}[\mathring{D}_A\mathring{D}_B\mathring{D}_C\partial_u^{i-1}\xi^C]\,,\end{aligned} \tag{44}$$

$$\begin{aligned}\mathring{D}^A\tilde{h}_{uA} &= \mathring{D}^A h_{uA} - \frac{1}{2}\Delta_{\mathring{\gamma}}\,(\Delta_{\mathring{\gamma}} + 2\varepsilon)\xi^u \\ &\quad + r^2\big[\mathring{D}_A\partial_u\xi^A + \big(\alpha^2 + \frac{2m}{r^3}\big)\Delta_{\mathring{\gamma}}\xi^u\big]\,,\end{aligned} \tag{45}$$

$$\begin{aligned}\mathring{D}^B\tilde{h}_{AB} &= \mathring{D}^B h_{AB} + r^2(\Delta_{\mathring{\gamma}} + \varepsilon)\xi_B - r\mathring{D}_A(\Delta_{\mathring{\gamma}} + 2\varepsilon)\xi^u \\ &= \mathring{D}^B h_{AB} + r^2(\Delta_{\mathring{\gamma}} + \varepsilon)\xi_A + 2rL_1(\xi^u)_A\,,\end{aligned} \tag{46}$$

$$\begin{aligned}\mathring{D}^A\mathring{D}^B\tilde{h}_{AB} &= \mathring{D}^A\mathring{D}^B h_{AB} + r^2(\Delta_{\mathring{\gamma}} + 2\varepsilon)\mathring{D}_A\xi^A \\ &\quad - r\Delta_{\mathring{\gamma}}(\Delta_{\mathring{\gamma}} + 2\varepsilon)\xi^u \\ &= \mathring{D}^A\mathring{D}^B h_{AB} - 2r^2L_2(\xi) - r\Delta_{\mathring{\gamma}}(\Delta_{\mathring{\gamma}} + 2\varepsilon)\xi^u\,.\end{aligned} \tag{47}$$

3.3. Null constraint equations

We now turn our attention to Einstein equations,

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{48}$$

and their linearisation in Bondi coordinates.

3.3.1. h_{ur}. The G_{rr} component of the Einstein tensor, which we reproduce from [4], reads:

$$\frac{r}{4} G_{rr} = \partial_r \beta - \frac{r}{16} \gamma^{AC} \gamma^{BD} (\partial_r \gamma_{AB})(\partial_r \gamma_{CD}) \,. \tag{49}$$

Since the right-hand side of (49) is quadratic in $\partial_r \gamma_{AB}$, after linearising in vacuum we find

$$\partial_r \delta\beta = 0 \quad \Longleftrightarrow \quad \delta\beta = \delta\beta(u, x^A) \,. \tag{50}$$

Using a terminology somewhat similar to that of [1], we thus obtain a pointwise radial conservation law for $\delta\beta$, and an apparent obstruction to gluing: two linearised fields can be glued together if and only if their Bondi functions $\delta\beta$ coincide.

However, it follows from (32) that we can always choose a gauge so that $\delta\beta \equiv 0$. Thus, (50) does not lead to an obstruction for gluing-up-to-gauge. Hence, when gluing, we will always use the gauge where $\delta\beta = 0$. As such, in the current section we will *not* assume $\delta\beta = 0$ unless explicitly indicated otherwise.

3.3.2. h_{uA}. From the G_{rA}-component of the Einstein equations one has

$$\begin{aligned} \partial_r \left[r^4 e^{-2\beta} \gamma_{AB} (\partial_r U^B) \right] &= 2r^4 \partial_r \Big(\frac{1}{r^2} D_A \beta \Big) \\ &\quad - r^2 \gamma^{EF} D_E (\partial_r \gamma_{AF}) + 16\pi r^2 T_{rA} \,. \end{aligned} \tag{51}$$

The linearisation of G_{rA} at a Birmingham-Kottler metric reads

$$\begin{aligned} 2r^2 \delta G_{rA} &= \partial_r \left[r^4 \mathring{\gamma}_{AB} (\partial_r \delta U^B) \right] - 2r^4 \partial_r \Big(\frac{1}{r^2} \mathring{D}_A \delta\beta \Big) \\ &\quad + r^2 \partial_r \left(r^{-2} \mathring{D}^B h_{AB} \right) \,. \end{aligned} \tag{52}$$

The linearised vacuum Einstein equation thus gives

$$\partial_r \left[r^4 \partial_r (r^{-2} h_{uA}) + 2r^2 \mathring{D}_A \delta\beta \right] = 8r \mathring{D}_A \delta\beta + \mathring{D}^B r^2 \partial_r \left(r^{-2} h_{AB} \right) . \tag{53}$$

Integration of this transport equation gives us a representation formula for $\partial_r h_{uA}$:

$$\left[s^4 \partial_s (s^{-2} h_{uA}) + 2s^2 \mathring{D}_A \delta\beta \right]_{s=r_1}^{r} = \int_{r_1}^{r} 8s \mathring{D}_A \delta\beta + \mathring{D}^B s^2 \partial_s \left(s^{-2} h_{AB} \right) ds . \tag{54}$$

In the gauge $\delta\beta = 0$, and after performing an integration by parts on the right-hand side, this can be written as,

$$r^4 \partial_r \check{h}_{uA}|_r = r_1^4 \partial_r \check{h}_{uA}|_{r_1} + \left[\mathring{D}^B h_{AB} \right]_{r_1}^{r} - 2 \int_{r_1}^{r} \hat{\kappa}_1(s) \mathring{D}^B h_{AB} \, ds \tag{55}$$

where we have defined,

$$\hat{\kappa}_1(s) := \frac{1}{s} . \tag{56}$$

Given $\mathfrak{d}_{\mathbf{S}_1}$ and $\mathfrak{d}_{\mathbf{S}_2}$, equation (55) evaluated at $r = r_2$ gives a condition for the field $h_{AB}(r)$, where $r \in (r_1, r_2)$ which has to hold when constructing the solution to the gluing problem on $\mathscr{N}_{[r_1,r_2]}$.

Now, the cokernel of the operator $\mathring{\mathrm{div}}_{(2)}$

$$\mathring{\mathrm{div}}_{(2)} : \varphi_{AB} \mapsto \mathring{D}^B \varphi_{AB}$$

acting on traceless symmetric tensors φ_{AB}, and which appears in (53) in front of h_{AB}, is spanned by solutions of the system

$$\mathrm{TS}[\mathring{D}_A \pi_B] = 0 , \tag{57}$$

with $\pi_A = \pi_A(u, x^B)$. The space of solutions of (57) is the space of conformal Killing vector fields, which we denote by CKV. This space is six-dimensional on S^2, and is isomorphic to the Lie algebra of the Lorentz group. On a two-dimensional torus $\mathbb{T}^2$, solutions of (57) belong to the two-dimensional space of covariantly constant vectors. Finally, the space of solutions of (57) on a two-dimensional negatively curved compact manifold is trivial; cf . Appendix C.2.

The projection of (53) onto π_A in the gauge $\delta\beta = 0$ gives

$$\partial_r \int_{\mathbf{S}} \pi^A \left[r^4 \partial_r (r^{-2} h_{uA})\right] \, d\mu_{\mathring{\gamma}} = \int_{\mathbf{S}} \pi^A \mathring{D}^B \left[r^2 \partial_r \left(r^{-2} h_{AB}\right)\right] d\mu_{\mathring{\gamma}} = \int_{\mathbf{S}} \mathrm{TS}[\mathring{D}^B \pi^A] \left(r^2 \partial_r \left(r^{-2} h_{AB}\right)\right) d\mu_{\mathring{\gamma}} = 0 \,, \tag{58}$$

and thus the integrals

$$\overset{[1]}{Q}(\pi^A)[\mathbf{S}] := \int_{\mathbf{S}} \pi^A \left[r^4 \partial_r (r^{-2} h_{uA})\right] \, d\mu_{\mathring{\gamma}} \tag{59}$$

form a family of *radially conserved charges*, with

$$\partial_r \overset{[1]}{Q} = 0$$

along any $u =$ constant null hypersurfaces with the gauge choice $\delta\beta = 0$.

This leads to a six-dimensional family of obstructions to gluing on S^2, two-dimensional on $\mathbb{T}^2$, and no obstructions on null surfaces with sections of higher-genus.

We shall denote the dependence of $\overset{[1]}{Q}$ on $\mathfrak{d}_{\mathbf{S}}$ as $\overset{[1]}{Q} = \overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}}]$. Thus in the gauge $\delta\beta = 0$, *to achieve gluing of* $\mathfrak{d}_{\mathbf{S}_1}$ *and* $\mathfrak{d}_{\mathbf{S}_2}$, *it must hold that*

$$\overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_1}] = \overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_2}] \,. \tag{60}$$

Indeed, it follows from Appendix C.3, p. 780 that (60) is a necessary and sufficient condition for $r_2^4 \partial_r \check{h}_{uA}|_{\mathbf{S}_2} - r_2^4 \partial_r \check{h}_{uA}|_{\mathbf{S}_1}$ to lie in the image of the operator $\mathring{\mathrm{div}}_{(2)}$ acting on traceless symmetric tensors, or equivalently, for the existence of a solution $\tilde{\varphi}_{AB}(x^C)$ to the equation

$$r_2^4 \partial_r \check{h}_{uA}|_{\mathbf{S}_2} = r_1^4 \partial_r \check{h}_{uA}|_{\mathbf{S}_1} - \mathring{D}^B h_{AB}|_{\mathbf{S}_1} - \mathring{D}^B \tilde{\varphi}_{AB} \,. \tag{61}$$

The gluing condition (55) evaluated at $r = r_2$ can thus be achieved by interpolating h_{AB} on $\mathscr{N}_{(r_1,r_2)}$ so that

$$\langle h_{AB}, \hat{\kappa}_1 \rangle = \tilde{\varphi}_{AB} \,, \tag{62}$$

where $\tilde{\varphi}_{AB}$ is the solution to (61), and where we write, for $f, h : (r_1, r_2) \to \mathbb{R}$,

$$\langle f, h\rangle := \int_{r_1}^{r_2} f(s)h(s)ds\,.$$

Under the gauge transformation (34), $\overset{[1]}{Q}$ transforms as

$$\begin{aligned}
(63)\qquad & \int_{\mathbf{S}} \pi^A \left(r^4\partial_r \check{h}_{uA}\right)\, d\mu_{\mathring{\gamma}} \\
& \to \int_{\mathbf{S}} \pi^A r^4 \partial_r \Bigg(\check{h}_{uA} + \frac{1}{r^2} L_1(\xi^u)_A \\
& \qquad\qquad + (\mathring{\gamma}_{AB}\partial_u \xi^B + \big(\alpha^2 + \frac{2m}{r^3}\big)\partial_A \xi^u)\Bigg)\, d\mu_{\mathring{\gamma}} \\
& = \int_{\mathbf{S}} \pi^A \Bigg(r^4 \partial_r \check{h}_{uA} + 2r\mathring{D}^B\ \mathrm{TS}[\mathring{D}_A \mathring{D}_B \xi^u] - 6m\partial_A \xi^u\Bigg)\, d\mu_{\mathring{\gamma}} \\
& = \int_{\mathbf{S}} \left(\pi^A r^4 \partial_r \check{h}_{uA} + 6m\xi^u \mathring{D}_A \pi^A\right) d\mu_{\mathring{\gamma}}\,.
\end{aligned}$$

So on S^2, if $m = 0$ we see that $\overset{[1]}{Q}$ is gauge invariant, hence

$$\overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_1}] = \overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_2}] \quad \Longleftrightarrow \quad \overset{[1]}{Q}[\mathfrak{d}_{\tilde{\mathbf{S}}_1}] = \overset{[1]}{Q}[\mathfrak{d}_{\tilde{\mathbf{S}}_2}]\,.$$

If $m \neq 0$, $\overset{[1]}{Q}$ is invariant under gauge transformations for which $\mathring{D}_A \pi^A$ vanishes; these generate rotations of S^2.

On the remaining topologies we have $\mathring{D}_A \pi^A = 0$, so that the charges $\overset{[1]}{Q}$ are gauge-invariant independently of whether or not the mass parameter m vanishes.

Now, let ψ_A denote (compare (52)),

$$(64)\qquad \psi_A := -2r^4 \partial_r \left(\tfrac{1}{r^2} D_A \delta\beta\right) + r^2 \partial_r \left(r^{-2} \mathring{D}^B h_{AB}\right)\,.$$

Integrating (52) in r twice one obtains a representation formula for h_{uA}:

$$\begin{aligned}
(65)\qquad h_{uA}(u, r, x^B) \;=\; & r^2 \mu_A(u, x^B) + \frac{\lambda_A(u, x^B)}{r} \\
& - r^2 \int_{r_1}^{r} \psi_A(u, s, x^B) \left(\frac{1}{3r^3} - \frac{1}{3s^3}\right) ds\,,
\end{aligned}$$

with μ_A and λ_A determined by $h_{uA}(u, r_1, x^B)$ and $\partial_r h_{uA}(u, r_1, x^B)$.

The part of (65) involving h_{AB} can be viewed as the following map:

$$
\begin{aligned}
h_{AB} \quad &\mapsto \quad -r^2 \int_{r_1}^{r} s^2 \partial_s \left(s^{-2} \mathring{D}^B h_{AB} \right) \left(\frac{1}{3r^3} - \frac{1}{3s^3} \right) ds \\
&= \quad -\frac{r^2}{3} \mathring{D}^B \left[\int_{r_1}^{r} \partial_s \left(s^{-2} h_{AB} \right) \left(\frac{s^2}{r^3} - \frac{1}{s} \right) ds \right] \\
&= \quad -\frac{r^2}{3} \mathring{D}^B \Big[h_{AB}(u, s, x^A) \left(\frac{1}{r^3} - \frac{1}{s^3} \right) \Big|_{r_1}^{r} \\
&\qquad - \int_{r_1}^{r} h_{AB} \left(\frac{2}{sr^3} + \frac{1}{s^4} \right) ds \Big] .
\end{aligned} \tag{66}
$$

When $\delta\beta \equiv 0$ we thus obtain

$$
\begin{aligned}
h_{uA}(u, r, x^B) = r^2 \mu_A(u, x^B) &+ \frac{\lambda_A(u, x^B)}{r} \\
&+ \mathring{D}^B h_{AB}(u, r_1, x^A) \left(\frac{r}{3} - \frac{r^2}{3r_1^3} \right) \\
&+ \frac{r^2}{3} \int_{r_1}^{r} \mathring{D}^B h_{AB} \left(\frac{2}{sr^3} + \frac{1}{s^4} \right) ds .
\end{aligned} \tag{67}
$$

For future use we will track the differentiability orders of the fields involved. Denoting the Sobolev spaces over $\mathbf{S}$ as H_{k_U} for $\delta U^A \equiv -r^{-2} \mathring{\gamma}^{AB} h_{uB}$, and H_{k_γ} for $\delta\gamma_{AB} \equiv \check{h}_{AB} = r^{-2} h_{AB}$, Equation (67) implies

$$
k_\gamma \geq k_U + 1 . \tag{68}
$$

We emphasise that these spaces keep only track of the differentiability in directions tangent to $\mathbf{S}$ at given r, with no information concerning the behaviour in the r-direction. But note that in our setup all fields on the interpolating initial data hypersurface are smooth in the r-variable.

3.3.3. h_{uu}. To obtain the transport equation for the function V occurring in the Bondi form of the metric, it turns out to be convenient to consider the expression for $2G_{ur} + 2U^A G_{rA} - V/r\, G_{rr}$:

$$
\begin{aligned}
r^2 e^{-2\beta} (2G_{ur} + 2U^A G_{rA} - V/r\, G_{rr}) = R[\gamma] - 2\gamma^{AB} \Big[D_A D_B \beta \\
+ (D_A \beta)(D_B \beta) \Big] + \frac{e^{-2\beta}}{r^2} D_A \Big[\partial_r (r^4 U^A) \Big]
\end{aligned} \tag{69}
$$

$$-\frac{1}{2}r^4 e^{-4\beta}\gamma_{AB}(\partial_r U^A)(\partial_r U^B) - 2e^{-2\beta}\partial_r V\,,$$

(It follows directly from the definition of $G_{\mu\nu}$ and the Bondi parametrisation of the metric that $r^2 e^{-2\beta}(2G_{ur} + 2U^A G_{rA} - V/r\, G_{rr})$ can equivalently be written as $r^2 g^{AB} R_{AB}$; compare Appendix D).

Let $\mathring{R}_{AB} = \varepsilon\mathring{\gamma}_{AB}$ denote the Ricci tensor of the metric $\mathring{\gamma}_{AB}$. As h_{AB} is $\mathring{\gamma}$-traceless we have

$$\begin{aligned} r^2\delta(R[\gamma])|_{\gamma=\mathring{\gamma}} &= -\mathring{D}^A\mathring{D}_A(\mathring{\gamma}^{BC}h_{BC}) + \mathring{D}^A\mathring{D}^B h_{AB} - \mathring{R}^{AB}h_{AB} \\ &= \mathring{D}^A\mathring{D}^B h_{AB}\,. \end{aligned} \tag{70}$$

Linearising (69) around a Birmingham-Kottler background thus gives

$$\begin{aligned} \partial_r(\delta V - \frac{r^2}{2}\mathring{D}_A\delta U^A) = \frac{1}{2}\Big\{\mathring{D}^A\mathring{D}^B\check{h}_{AB} - 2\gamma^{AB}\mathring{D}_A\mathring{D}_B\delta\beta\Big\} \\ + r\mathring{D}_A\delta U^A + 2(\varepsilon - r^2\Lambda)\delta\beta\,. \end{aligned} \tag{71}$$

We note that since $\delta(G_{ur} + U^A G_{rA}) = \delta G_{ur}$, (71) is equivalent to the equation $r^2\delta G_{ur} = r^2\Lambda h_{ur}$.

In the $\delta\beta = 0$ gauge, Equation (71) provides another family of radially conserved charges:

$$\overset{[2]}{Q}(\lambda) := \int_{\mathbf{S}} \lambda\left[\delta V - \frac{r}{2}\partial_r\left(r^2\mathring{D}^A\delta U_A\right)\right]d\mu_{\mathring{\gamma}}\,, \tag{72}$$

where the functions $\lambda(x^A)$ are solutions of the equation

$$\mathrm{TS}[\mathring{D}_A\mathring{D}_B\lambda] = 0\,. \tag{73}$$

The only solutions of this equation on a torus or on a higher genus manifold are constants. On S^2 such λ's are linear combinations of $\ell = 0$ or $\ell = 1$ spherical harmonics [17]. We thus obtain another four-dimensional family of obstructions on S^2, and a one-dimensional family of obstructions in the remaining topologies.

The conservation equation $\partial_r\overset{[2]}{Q} = 0$ is the consequence of an identity, already observed in [7], of the form

$$\delta G_{ur} - \frac{1}{r}\mathring{D}^A\delta G_{rA} = \partial_r(....)\,, \tag{74}$$

which can be derived as follows:

$$
\begin{aligned}
\partial_r\left[\delta V - \frac{r}{2}\partial_r\left(r^2\mathring{D}^A\delta U_A\right)\right] &= \partial_r\delta V - \partial_r\left(\frac{r}{2}\partial_r\left(r^2\mathring{D}^A\delta U_A\right)\right) \\
&= \underbrace{\partial_r\delta V - 2r\mathring{D}_A\delta U^A - \frac{r^2}{2}\mathring{D}_A\partial_r\delta U^A}_{=1/2\mathring{D}^A\mathring{D}^B\check{h}_{AB}} \\
&\quad \underbrace{-\frac{1}{2}(r^3\partial_r^2\mathring{D}_A\delta U^A + 4r^2\partial_r\mathring{D}_A\delta U^A)}_{=r/2\partial_r\left(\mathring{D}^A\mathring{D}^B\check{h}_{AB}\right)} \\
&= \mathring{D}^A\mathring{D}^B\left[\frac{1}{2}\check{h}_{AB} + \frac{r}{2}\partial_r\check{h}_{AB}\right] \\
&= \frac{1}{2}\partial_r\left(r\mathring{D}^A\mathring{D}^B\check{h}_{AB}\right).
\end{aligned}
\tag{75}
$$

Hence

$$
\partial_r\overset{[2]}{Q} = \frac{1}{2}\int_{\mathbf{S}}\lambda\partial_r\big(r\mathring{D}^A\mathring{D}^B\check{h}_{AB}\big)\,d\mu_{\mathring{\gamma}} = 0. \tag{76}
$$

Under a gauge transformation the radial charge $\overset{[2]}{Q}$ transforms as

$$
\begin{aligned}
&\int_{\mathbf{S}}\lambda\Big[\delta V + \frac{r}{2}\mathring{D}^A\partial_r h_{uA}\Big]\,d\mu_{\mathring{\gamma}} \\
&\to \int_{\mathbf{S}}\lambda\Big[\delta V + r\left(\varepsilon + \frac{1}{2}\Delta_{\mathring{\gamma}} - \frac{3m}{r}\right)\big(\mathring{D}_B\xi^B\big) \\
&\qquad - r^2\big[\mathring{D}_B\partial_u\xi^B + \big(\alpha^2 - \frac{m}{r^3}\big)\Delta_{\mathring{\gamma}}\xi^u\big] \\
&\qquad + \frac{r}{2}\Big(\mathring{D}^A\partial_r h_{uA} + 2r\big[\mathring{D}_B\partial_u\xi^B + \big(\alpha^2 - \frac{m}{r^3}\big)\Delta_{\mathring{\gamma}}\xi^u\big]\Big)\Big]\,d\mu_{\mathring{\gamma}} \\
&= \int_{\mathbf{S}}\lambda\Big[\delta V + \frac{r}{2}\partial_r\mathring{D}^A h_{uA}\Big]\,d\mu_{\mathring{\gamma}} \\
&\qquad + \int_{\mathbf{S}}\lambda\Big[r\left(\varepsilon + \frac{1}{2}\Delta_{\mathring{\gamma}} - \frac{3m}{r}\right)\big(\mathring{D}_B\xi^B\big)\Big]\,d\mu_{\mathring{\gamma}}\,.
\end{aligned}
\tag{77}
$$

Taking $\mathring{D}^A$ of (73) gives,

$$
\mathring{D}_B\Delta_{\mathring{\gamma}}\lambda = -2\mathring{R}_{AB}\mathring{D}^A\lambda\,, \tag{78}
$$

where $\mathring{R}_{AB}$ is the Ricci tensor of the metric $\mathring{\gamma}$. Inserting this into (77) gives

$$\overset{[2]}{Q} \mapsto \int_{\mathbf{S}} \lambda\Big[\delta V + \frac{r}{2}\mathring{D}^A\partial_r h_{uA}\Big] d\mu_{\mathring{\gamma}} - 3m\int_{\mathbf{S}} \lambda \mathring{D}_B\xi^B\,. \tag{79}$$

Therefore, a) $\overset{[2]}{Q}(\lambda)$ is gauge invariant <u>when $m=0$</u> for all λ's satisfying (73), while b) <u>when $m\neq 0$</u>, only $\overset{[2]}{Q}(\lambda^{[1]})$ is gauge invariant.

As already pointed out, (75) takes the form of a pointwise radial conservation law:

$$\partial_r\chi = 0\,,$$

where

$$\begin{aligned}\chi &:= -\delta V + \frac{r}{2}\partial_r\Big(r^2\mathring{D}^A\delta U_A\Big) + \frac{1}{2}r\mathring{D}^A\mathring{D}^B\check{h}_{AB}\\ &= -\delta V - \frac{r}{2}\partial_r\mathring{D}^A h_{uA} + \frac{1}{2r}\mathring{D}^A\mathring{D}^B h_{AB}\,.\end{aligned} \tag{80}$$

We note that (71) can be used to rewrite χ as (cf. [7, Equation (D.4)])

$$\chi = -r^2\partial_r h_{uu} + \mathring{D}^A h_{uA}\,. \tag{81}$$

Under gauge transformations χ transforms as

$$\chi \mapsto \chi - \frac{1}{2}\underbrace{(\Delta_{\mathring{\gamma}} + 2\varepsilon)\Delta_{\mathring{\gamma}}\xi^u}_{=\mathring{L}(\xi^u)} + 3m\mathring{D}^B\xi_B\,. \tag{82}$$

This shows that, on S^2, $\chi^{[(\ker\mathring{L})^\perp]} = \chi^{[\geq 2]}$ can be made to achieve any desired value by a suitable choice of $(\xi^u)^{[\geq 2]}$. For the remaining topologies this is the case for $\chi^{[(\ker\mathring{L})^\perp]} = \chi^{[1^\perp]}$ using $(\xi^u)^{[1^\perp]}$.

It further follows from (82) that the projection $\chi^{[1]}$ is gauge invariant for all topologies regardless of m. Moreover, when $m=0$, the projection $\chi^{[\leq 1]}$ is also gauge-invariant on S^2. These projections are determined by the radial charge (72), i.e.

$$\begin{aligned}\overset{[2]}{Q}(\lambda) &= r\int_{\mathbf{S}} \lambda\big[-h_{uu} + \frac{1}{2}\partial_r\mathring{D}^A h_{uA}\big] d\mu_{\mathring{\gamma}}\\ &= -r\int_{\mathbf{S}} \lambda h_{uu} d\mu_{\mathring{\gamma}} + \frac{r}{2}\partial_r\int_{\mathbf{S}} \lambda\mathring{D}^A h_{uA} d\mu_{\mathring{\gamma}}\,,\end{aligned} \tag{83}$$

where λ is a linear combination of $\ell = 0$ and $\ell = 1$ spherical harmonics on S^2, and is a constant in the remaining cases, as follows: Recall that

$$\begin{aligned}\overset{[1]}{Q}(\mathring{D}\lambda) &= \int_{\mathbf{S}} \mathring{D}^A\lambda \left[r^4\partial_r(r^{-2}h_{uA})\right]\, d\mu_{\mathring{\gamma}} \\ &= -\int_{\mathbf{S}} \lambda \left[r^4\partial_r(r^{-2}\mathring{D}^A h_{uA})\right]\, d\mu_{\mathring{\gamma}} \\ &= -r^4\partial_r\big(r^{-2}\int_{\mathbf{S}} \lambda\mathring{D}^A h_{uA}\, d\mu_{\mathring{\gamma}}\big)\,,\end{aligned} \tag{84}$$

see (59). Integrating (84) over $\mathbf{S}$ shows that there exists a function $\overset{[1]}{C}(u, x^A)$ such that

$$\int_{\mathbf{S}} \lambda\mathring{D}^A h_{uA}\, d\mu_{\mathring{\gamma}} = \frac{\overset{[1]}{Q}(\mathring{D}\lambda)}{3r} + \overset{[1]}{C}r^2\,, \tag{85}$$

which is non-zero on S^2 only. It then follows from (83) that

$$\begin{aligned}\int_{\mathbf{S}} \lambda h_{uu} d\mu_{\mathring{\gamma}} &= -\frac{\overset{[2]}{Q}(\lambda)}{r} + \frac{1}{2}\partial_r \int_{\mathbf{S}} \lambda\mathring{D}^A h_{uA} d\mu_{\mathring{\gamma}} \\ &= -\frac{\overset{[2]}{Q}(\lambda)}{r} - \frac{\overset{[1]}{Q}(\mathring{D}\lambda)}{6r^2} + \overset{[1]}{C}r\,.\end{aligned} \tag{86}$$

Hence, whatever the topology,

$$\int_{\mathbf{S}} \lambda\chi d\mu_{\mathring{\gamma}} \;=\; \int_{\mathbf{S}} \lambda\mathring{D}^A h_{uA} d\mu_{\mathring{\gamma}} - r^2\partial_r \int_{\mathbf{S}} \lambda h_{uu} d\mu_{\mathring{\gamma}} = -\overset{[2]}{Q}(\lambda)\,. \tag{87}$$

Writing H_{k_V} for the Sobolev space of the δV's, (71) above implies

$$k_\gamma \geq k_V + 2 \text{ and } k_U \geq k_V + 1. \tag{88}$$

3.3.4. $\partial_u h_{AB}$. We continue with $\partial_u h_{AB}$, as determined from [4, Equation (32)]:

$$\begin{aligned}&\partial_r\Big[r\partial_u\gamma_{AB} - \frac{1}{2}V\partial_r\gamma_{AB} - \frac{1}{2r}V\gamma_{AB}\Big] \\ &\quad = -\frac{1}{2}\partial_r(V/r)\gamma_{AB} - \frac{1}{r}\,\mathrm{TS}\,\Big[e^{2\beta}R[\gamma]_{AB} - 2e^{\beta}D_AD_Be^{\beta}\end{aligned} \tag{89}$$

$$
\begin{aligned}
&+ \gamma_{CA} D_B[\partial_r(r^2 U^C)] - \frac{1}{2} r^4 e^{-2\beta} \gamma_{AC}\gamma_{BD}(\partial_r U^C)(\partial_r U^D) \\
&+ \frac{r^2}{2}(\partial_r \gamma_{AB})(D_C U^C) + r^2 U^C D_C(\partial_r \gamma_{AB}) \\
&- r^2(\partial_r \gamma_{AC})\gamma_{BE}(D^C U^E - D^E U^C) + \Lambda e^{2\beta} g_{AB} - 8\pi e^{2\beta} T_{AB}\Big] .
\end{aligned}
$$

The linearisation of (89) around a Birmingham-Kottler background in vacuum reads, keeping in mind that $\mathrm{TS}[R[\gamma]_{AB}] = 0$ in dimension two,

$$
\begin{aligned}
0 &= \frac{1}{r}\,\mathrm{TS}[\delta G_{AB}] \\
&= \partial_r\Big[r\partial_u \check{h}_{AB} - \frac{1}{2} V \partial_r \check{h}_{AB} - \frac{1}{2r} V \check{h}_{AB} - r\,\mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right]\Big] \\
&\quad \underbrace{+\frac{1}{2}\partial_r(V/r)}_{mr^{-2}-\alpha^2 r}\check{h}_{AB} - r^{-1}\Big(2\mathring{D}_A\mathring{D}_B\delta\beta + r\,\mathrm{TS}\left[\mathring{D}_A\check{h}_{uB}\right]\Big) .
\end{aligned} \tag{90}
$$

Integrating this equation gives

$$
\begin{aligned}
s\partial_u \check{h}_{AB}\Big|_{r_1}^{r} &= \Big[\frac{1}{2} V \partial_s \check{h}_{AB} + \frac{V}{2s}\check{h}_{AB} + s\,\mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right]\Big]_{r_1}^{r} \\
&\quad + \int_{r_1}^{r}\Big[-\frac{1}{2}\partial_s(V/s)\check{h}_{AB} \\
&\qquad + 1/s\Big(2\mathring{D}_A\mathring{D}_B\delta\beta + s\,\mathrm{TS}\left[\mathring{D}_A\check{h}_{uB})\right]\Big)\Big] ds .
\end{aligned} \tag{91}
$$

Denoting $H_{k_{\partial_u\gamma}} \ni \partial_u \check{h}_{AB}$, (91) implies

$$
k_\beta \geq k_{\partial_u\gamma} + 2 \text{ and } k_U \geq k_{\partial_u\gamma} + 1. \tag{92}
$$

When $\delta\beta = 0$, using (67) in the last term of (91) leads to

$$
\begin{aligned}
\partial_u h_{AB} &= r\frac{\partial_u h_{AB}|_{r_1}}{r_1} - \frac{r}{2}\Big[V\partial_r \check{h}_{AB} + \frac{1}{s} V \check{h}_{AB} \\
&\qquad - \Big(\frac{1}{s^2} - \frac{1}{r^2}\Big) TS\Big[\mathring{D}_A \mathring{D}^C h_{BC}\Big]\Big]\Big|_{s=r_1} \\
&\quad + 2r\,\mathrm{TS}\Big\{\frac{r_1^2(r_1^2 - r^2)}{4r^2}\partial_s \mathring{D}_A \check{h}_{uB}|_{r_1} - r_1 \mathring{D}_A \check{h}_{uB}|_{r_1} \\
&\quad + r\mathring{D}_A\Big[r^2 \mu_B(u, x^C) + \frac{\lambda_B(u, x^C)}{r} \\
&\qquad - \frac{r^2}{3}\mathring{D}^C h_{BC}(u, r_1, x^A)\left(\frac{1}{r_1^3} - \frac{1}{r^3}\right)\Big]\Big\}
\end{aligned} \tag{93}
$$

$$+ r\left[\frac{V}{2r}\Big[r\partial_r\check{h}_{AB}+\check{h}_{AB}\Big]\right.$$
$$\left.+\int_{r_1}^{r}\Big[(\frac{\alpha^2}{s}-\frac{m}{s^4})h_{AB}+\Big(\frac{1}{3sr^2}+\frac{2r}{3s^4}\Big)\,\mathrm{TS}\,\big[\mathring{D}_A\mathring{D}^C h_{BC}\big]\Big]ds\right].$$

Recall that

$$V = r\varepsilon - \alpha^2 r^3 - 2m\,.$$

Let us write $\text{b.d.}|_{r_1}$ for terms known from "boundary data at r_1". We rewrite (93) as

$$\partial_u h_{AB} = \text{b.d.}|_{r_1} + r\left[\frac{V}{2r}\Big[r\partial_r\check{h}_{AB}+\check{h}_{AB}\Big]+\int_{r_1}^{r}\Big[(\frac{\alpha^2}{s}-\frac{m}{s^4})h_{AB}\right.$$
$$\left.+(\frac{1}{3sr^2}+\frac{2r}{3s^4})\underbrace{\mathrm{TS}\,\big[\mathring{D}_A\mathring{D}^C h_{BC}\big]}_{=:Ph_{AB}}\Big]ds\right]$$
$$= \frac{r(\varepsilon r-\alpha^2r^3-2m)}{2}\Big[\partial_r(r^{-2}h_{AB})+\frac{1}{r^3}h_{AB}\Big]$$
$$+\int_{r_1}^{r}\Big[\underbrace{(\frac{\alpha^2 r}{s}-\frac{mr}{s^4})}_{\overset{(1,0)}{\psi}}h_{AB}+\underbrace{(\frac{1}{3sr}+\frac{2r^2}{3s^4})}_{\overset{(1,1)}{\psi}(s,r)}Ph_{AB}\Big]ds+\text{b.d.}|_{r_1}\,. \tag{94}$$

For further use it is convenient to separate the terms involving α and m from the remaining ones:

(95)
$$\partial_u h_{AB} = \frac{\varepsilon}{2}\Big[\partial_r h_{AB}-\frac{1}{r}h_{AB}\Big]+\int_{r_1}^{r}\Big(\frac{1}{3sr}+\frac{2r^2}{3s^4}\Big)Ph_{AB}\,ds$$
$$-(\frac{\alpha^2r^2}{2}+\frac{m}{r})\Big[\partial_r h_{AB}-\frac{1}{r}h_{AB}\Big]+\int_{r_1}^{r}(\frac{\alpha^2 r}{s}-\frac{mr}{s^4})h_{AB}\,ds+\text{b.d.}|_{r_1}\,.$$

3.4. A pointwise radial conservation law

In this section we show that the equation

$$\mathrm{TS}\,\big(\frac{1}{r}\delta G_{AB}+\mathring{D}_A\delta G_{rB}\big)=0 \tag{96}$$

can be written as a radial conservation law, $\partial_r(....) = 0$ when $m = 0 = \alpha = \delta\beta$, where P is as in (94):

$$Ph_{AB} := \mathrm{TS}[\mathring{D}_A \mathring{D}^C h_{BC}]\,. \tag{97}$$

We further show that the equation obtained by taking $\mathring{\mathrm{div}}_{(2)}$ of (96),

$$\mathring{D}^A\big[\,\mathrm{TS}\,\big(\frac{1}{r}\delta G_{AB} + \mathring{D}_A \delta G_{rB}\big)\big] = 0 \tag{98}$$

can likewise be written as a radial conservation law when $m = 0 = \delta\beta$, for any α. This is likely to be related to the contracted Bianchi identity discussed in Section 3.5 below, but if and how is not clear.

Indeed, when $\delta\beta = 0$, taking $\frac{1}{2r^2} \times C$ of (53) gives

$$\frac{1}{2r^2}\partial_r\left[r^4\partial_r(r^{-2}\,\mathrm{TS}[\mathring{D}_B h_{uA}])\right] - \frac{1}{2}\partial_r\left(r^{-2}Ph_{AB}\right) = 0\,. \tag{99}$$

Subtracting (99) from (90) leads to

$$\begin{aligned}\partial_r\Big[\underbrace{r\partial_u\check{h}_{AB} - \frac{V}{2}\partial_r\check{h}_{AB} - \frac{V}{2r}\check{h}_{AB} - \frac{\partial_r\big(r^2\,\mathrm{TS}[\mathring{D}_A h_{uB}]\big)}{2r^2} + \frac{P\check{h}_{AB}}{2}}_{:=q_{AB}}\Big] \\ = \big(\frac{\alpha^2}{r} - \frac{m}{r^4}\big)h_{AB}\,.\end{aligned} \tag{100}$$

Hence q_{AB} is radially conserved when $\alpha = m = 0$.

Under a gauge transformation q_{AB} transforms as

$$\begin{aligned}q_{AB} \mapsto q_{AB} - \Big[\,\mathrm{TS}[\mathring{D}_A\mathring{D}_B\mathring{D}_C\xi^C] - \big(P - \varepsilon + \alpha^2 r^2 - \frac{2m}{r}\big)C(\xi)_{AB}\big) \\ - \big(2\alpha r + \frac{m}{r^2}\big)\,\mathrm{TS}[\mathring{D}_A\mathring{D}_B\xi^u]\Big]\,.\end{aligned} \tag{101}$$

Since $C(X)^{[\mathrm{TT}]} = 0$ for any vector field X^A (cf. Proposition C.3, Appendix C.2 below), the field $q_{AB}^{[\mathrm{TT}]}$ is gauge-independent and, when $\alpha = 0 = m$, gives a 2-dimensional family of radially conserved charges on $\mathbb{T}^2$, and a $6(\mathfrak{g}-1)$-dimensional family of such charges on sections with genus $\mathfrak{g} \geq 2$.

Next, taking the divergence of (100) and using (53) we find

$$\partial_r(\mathring{D}^B q_{AB}) = -\frac{\alpha^2}{2}\partial_r\left[r^4\partial_r(r^{-2}h_{uA}) - \mathring{D}^B h_{AB}\right] - \frac{m}{r^4}\mathring{D}^B h_{AB}\,. \tag{102}$$

Setting $\delta\beta = 0$ in (53) we have

$$\partial_r\left(3\check{h}_{uA} + r\partial_r\check{h}_{uA} - r^{-3}\mathring{D}^B h_{AB}\right) = r^{-4}\mathring{D}^B h_{AB}\,. \tag{103}$$

Equation (103) allows us to rewrite (102) as:

$$\begin{aligned}\partial_r\Big[\mathring{D}^B q_{AB} + \frac{\alpha^2}{2}\left(r^4\partial_r(\check{h}_{uA}) - \mathring{D}^B h_{AB}\right)\\ + m\big(3\check{h}_{uA} + r\partial_r\check{h}_{uA} - r^{-3}\mathring{D}^B h_{AB}\big)\Big] = 0\,.\end{aligned} \tag{104}$$

We define

$$\begin{aligned}\frac{\overset{[3,1]}{Q}{}_A}{2} :=& r\mathring{D}^B\partial_u\check{h}_{AB} - \frac{1}{2}V\partial_r\mathring{D}^B\check{h}_{AB} - \frac{1}{2r}V\mathring{D}^B\check{h}_{AB}\\ &- \frac{1}{2r^2}\partial_r\big(r^2\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A h_{uB}]\big)\\ &+ \frac{1}{2}\mathring{D}^B P\check{h}_{AB} + \frac{\alpha^2}{2}\big(r^4\partial_r(r^{-2}h_{uA}) - \mathring{D}^B h_{AB}\big)\\ &+ m(3\check{h}_{uA} + r\partial_r\check{h}_{uA} - r^{-3}\mathring{D}^B h_{AB})\,,\end{aligned} \tag{105}$$

with $\overset{[3,1]}{Q}{}_A$ being $r-$ independent by (104), where the notation $\overset{[3,1]}{Q}{}_A$ should be clear from (115) below. Equivalently, the field

$$\begin{aligned}\overset{[3,1]}{Q}{}_A =& \mathring{D}^B\Big[2r\partial_u\check{h}_{AB} - V\partial_r\check{h}_{AB} - \frac{1}{r^2}\partial_r\big(r^4\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\big)\\ &+ \big(P-\varepsilon\big)\check{h}_{AB}\Big] + \alpha^2 r^4\partial_r\check{h}_{uA} + 2m(3\check{h}_{uA} + r\partial_r\check{h}_{uA})\end{aligned} \tag{106}$$

is radially conserved.

To continue, for $\pi^A \in \mathrm{CKV}$ and $i \geq 0$ we set

$$\overset{[4,i]}{Q}(\pi^A) := \int_{\mathbf{S}} \pi^A\left(3\partial_u^i\check{h}_{uA} + r\partial_r\partial_u^i\check{h}_{uA}\right)d\mu_{\mathring{\gamma}}\,. \tag{107}$$

It follows from the u-derivatives of (103) that these are radially conserved charges. Furthermore it holds that

$$\int_{\mathbf{S}} \pi^A\overset{[3,1]}{Q}{}_A\,d\mu_{\mathring{\gamma}} = \int_{\mathbf{S}} \pi^A(\alpha^2 r^4\partial_r\check{h}_{uA} + 2m(3\check{h}_{uA} + r\partial_r\check{h}_{uA}))\,d\mu_{\mathring{\gamma}} \tag{108}$$

$$= \alpha^2 \overset{[1]}{Q}(\pi^A) + 2m \overset{[4,0]}{Q}(\pi^A)\,;$$

recall that $\overset{[1]}{Q}(\pi^A)$ has been defined in (59). Thus $\overset{[3,1]}{Q}{}^{[\mathrm{CKV}]}_A$ is zero if $m = 0 = \alpha$, and is determined by $\overset{[1]}{Q}$ and $\overset{[4,0]}{Q}$ otherwise.

Under a gauge transformation $\overset{[3,1]}{Q}{}_A$ transforms as

$$\overset{[3,1]}{Q}{}_A \mapsto \overset{[3,1]}{Q}{}_A + 2\underbrace{\mathring{D}^B\Big\{ - \mathrm{TS}[\mathring{D}_A\mathring{D}_B\mathring{D}_C\xi^C] + \big(P-\varepsilon\big)\,\mathrm{TS}[\mathring{D}_A\xi_B]\Big\}}_{=(\widehat{\mathrm{L}}\xi)_A} + 6m\partial_u\xi_A\,, \tag{109}$$

where the operator $\widehat{\mathrm{L}}$ can be written as

$$\widehat{\mathrm{L}} = -\mathring{\mathrm{div}}_{(2)}\, C\, \mathrm{L}\,, \quad \mathrm{L} := \mathring{D}\,\mathring{\mathrm{div}}_{(1)} - \mathring{\mathrm{div}}_{(2)}\, C + \varepsilon\,. \tag{110}$$

It follows from Proposition C.6, Appendix C.4, that the gauge transformations (109) associated with ξ_A act transitively on $\overset{[3,1]}{Q}{}^{[(\mathrm{CKV}+\mathrm{H})^\perp]}_A$. <u>In the case $m \neq 0$</u>, the gauge-transformations associated with $\partial_u\xi_A$ clearly act transitively on the collection of all radial charges $\overset{[3,1]}{Q}{}_A$.

On S^2 we have $\mathrm{H} = \{0\}$, and <u>when $m = 0$</u> but $\alpha \neq 0$ we conclude that there the integrals

$$\overset{[3,1]}{Q}{}^{[\mathrm{CKV}]}_A = \overset{[3,1]}{Q}{}^{[<2]}_A = \overset{[3,1]}{Q}{}^{[=1]}_A\,,$$

provide a 6-dimensional family of gauge-invariant radially-conserved charges.

On $\mathbb{T}^2$ we have (compare (327)-(328) below)

$$\widehat{\mathrm{L}}(\xi)_A = -\frac{1}{2}\Delta_{\mathring{\gamma}}(\mathring{D}_A\mathring{D}^C\xi_C - \frac{1}{2}\Delta_{\mathring{\gamma}}\xi_A)\,, \tag{111}$$

$$\mathrm{L}(\xi)_A = \mathring{D}_A\mathring{D}^C\xi_C - \frac{1}{2}\Delta_{\mathring{\gamma}}\xi_A\,, \tag{112}$$

with kernels and cokernels spanned by covariantly constant vectors. So $\mathrm{CKV} = \mathrm{KV} = \mathrm{H}$, and <u>when $m = 0$</u> it follows that on a torus the gauge transformations (109) act transitively on $\overset{[3,1]}{Q}{}^{[\mathrm{KV}^\perp]}_A$, and that $\overset{[3,1]}{Q}{}^{[\mathrm{KV}]}_A$ gives a 2-dimensional family of gauge-invariant radially-conserved charges.

On negatively curved two dimensional manifolds with genus $\mathfrak{g}$ we have $\mathrm{CKV} = \{0\}$ so that $\mathrm{CKV} + \mathrm{H} = \mathrm{H}$ and, again <u>when $m = 0$</u>, $\overset{[3,1]}{Q}{}_A$ leads to a $2\mathfrak{g}$-dimensional family of gauge-invariant radially-conserved charges $\overset{[3,1]}{Q}{}^{[\mathrm{H}]}$.

Summarising: we can always choose $\overset{(2)}{\xi}{}_A$ so that

$$\overset{[3,1]}{Q}{}[\mathfrak{d}_{\mathbf{S}_1}]^{[(\mathrm{CKV}+\mathrm{H})^\perp]} = \overset{[3,1]}{Q}{}[\mathfrak{d}_{\mathbf{S}_2}]^{[(\mathrm{CKV}+\mathrm{H})^\perp]} \tag{113}$$

holds. <u>When $m = 0$</u> the equality

$$\overset{[3,1]}{Q}{}[\mathfrak{d}_{\mathbf{S}_1}]^{[\mathrm{CKV}+\mathrm{H}]} = \overset{[3,1]}{Q}{}[\mathfrak{d}_{\mathbf{S}_2}]^{[\mathrm{CKV}+\mathrm{H}]} \tag{114}$$

provides an obstruction to gluing; <u>when $m \neq 0$</u> it can be enforced by an appropriate choice of $\partial_u \xi_A^{[\mathrm{CKV}+\mathrm{H}]}$. On S^2 and on $\mathbb{T}^2$ the condition (114) is trivially satisfied when $m = \alpha = 0$, and reduces to the requirement of conservation of $\overset{[1]}{Q}$ and $\overset{[4,0]}{Q}$ if $m\alpha \neq 0$.

It should be clear from the above that if we set, for $i \geq 1$,

$$\begin{aligned}\overset{[3,i+1]}{Q}{}_A :=& \mathring{D}^B\Big[2r\partial_u^{i+1}\check{h}_{AB} - V\partial_r(\partial_u^i\check{h}_{AB}) - \frac{1}{r^2}\partial_r\big(r^4\,\mathrm{TS}[\mathring{D}_A\partial_u^i\check{h}_{uB}]\big)\\ &+ \big(P-\varepsilon\big)\partial_u^i\check{h}_{AB}\Big] + \alpha^2 r^4\partial_r\partial_u^i\check{h}_{uA}\\ &+ 2m(3\partial_u^i\check{h}_{uA} + r\partial_r\partial_u^i\check{h}_{uA})\,,\end{aligned} \tag{115}$$

then we have:

Lemma 3.1. *Suppose that for $i \geq 0$ the i'th u-derivative of (53) and (90) with $\delta\beta \equiv 0$ hold. Then*

$$\partial_r \overset{[3,i+1]}{Q}{}_A = 0\,. \qquad \square$$

Similarly to (108), for conformal Killing vectors π^A we have

$$\begin{aligned}(116)\qquad &\int_{\mathbf{S}} \pi^A \overset{[3,i]}{Q}{}_A \, d\mu_{\mathring{\gamma}} \\ &= \int_{\mathbf{S}} \pi^A \big(\alpha^2 r^4 \partial_r \partial_u^{i-1} \check{h}_{uA} + 2m(3\partial_u^{i-1}\check{h}_{uA} + r\partial_r\partial_u^{i-1}\check{h}_{uA})\big)\, d\mu_{\mathring{\gamma}} \\ &=: \alpha^2 \overset{[1,i-i]}{Q}(\pi^A) + 2m \overset{[4,i-1]}{Q}(\pi^A)\,,\end{aligned}$$

so that the left-hand side obviously vanishes if $\alpha = 0 = m$. In fact, for $i \geq 2$ there is no obstruction regardless of α and m, as our arguments below show that the right-hand side of the last equation is continuous at r_2 when the Einstein equations together with a sufficient number of their u-derivatives hold on $\mathscr{N}$.

Under gauge transformations, it follows from (109) that

$$(117)\qquad \overset{[3,i+1]}{Q}{}_A \mapsto \overset{[3,i+1]}{Q}{}_A + 2(\widehat{\mathrm{L}}\partial_u^i \xi)_A + 6m\partial_u^{i+1}\xi_A\,.$$

3.5. The remaining Einstein equations

Let us start by recalling that the Einstein equations

$$\mathscr{E}_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi T_{\mu\nu}$$

can be split as

$$(118)\qquad \mathscr{E}^u{}_\mu = 0\,, \qquad \mathscr{E}_{AB} - \tfrac{1}{2} g^{CD}\mathscr{E}_{CD} g_{AB} = 0\,,$$

$$(119)\qquad g^{CD}\mathscr{E}_{CD} = 0\,,$$

$$(120)\qquad \partial_r(r^2 e^{2\beta}\mathscr{E}^r{}_u) = 0\,, \qquad \partial_r(r^2 e^{2\beta}\mathscr{E}^r{}_A) = 0\,,$$

and the following holds (cf., e.g., [4, Section 3]): Suppose that (118) holds on a null hypersurface $\mathscr{N}$ and that

$$(121)\qquad \partial_u \mathscr{E}^u{}_\mu|_{\mathscr{N}} = 0\,.$$

Then a) (119) is satisfied automatically on $\mathscr{N}$, and b) the equations $\mathscr{E}^r{}_u|_{\mathscr{N}} = \mathscr{E}^r{}_A|_{\mathscr{N}} = 0$ will hold if they are satisfied at one single value of r. This follows from the observation that, in Bondi coordinates, we have the identity

$$(122)\qquad \nabla_\mu \mathscr{E}^\mu{}_\nu = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}\mathscr{E}^\mu{}_\nu) + \frac{1}{2}\mathscr{E}_{\mu\sigma}\partial_\nu g^{\mu\sigma}\,.$$

In the current context this implies, using $\partial_\nu g^{u\mu} = 0 = \partial_u g^{\mu\sigma}$ and the divergence identity,

$$0 = \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}\delta\mathscr{E}^\mu{}_\nu) + \frac{1}{2}\sum_{\mu,\sigma\neq u}\delta\mathscr{E}_{\mu\sigma}\underbrace{\partial_\nu g^{\mu\sigma}}_{0 \text{ if } \nu=u}\,. \tag{123}$$

Since $\delta\mathscr{E}^u{}_\mu = -\delta\mathscr{E}_{r\mu}$, when the main equations (118) are satisfied (123) becomes

$$\begin{aligned} 0 &= \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}\delta\mathscr{E}^\mu{}_\nu) + \frac{1}{2}\sum_{\mu,\sigma\notin\{u,r\}}\delta\mathscr{E}_{\mu\sigma}\partial_\nu g^{\mu\sigma} \\ &= \frac{1}{\sqrt{|\det g|}}\partial_\mu(\sqrt{|\det g|}\delta\mathscr{E}^\mu{}_\nu) + \frac{1}{2}\delta\mathscr{E}_{AB}\underbrace{\partial_\nu g^{AB}}_{0 \text{ if } \nu=u}\,. \end{aligned} \tag{124}$$

In what follows we assume

$$\delta\mathscr{E}^u{}_\mu|_{\mathscr{N}} = 0 = \partial_u\delta\mathscr{E}^u{}_\mu|_{\mathscr{N}}\,. \tag{125}$$

We review the standard argument, which is a somewhat simplified version of what needs to be done in our gluing. Setting $\nu = r$ in (124) one obtains immediately

$$0 = -\frac{1}{r}g^{AB}\delta\mathscr{E}_{AB}|_{\mathscr{N}}\,, \tag{126}$$

hence the linearisation of (119) holds on $\mathscr{S}$. So the linearised version of the second equation in (118) is equivalent to $\delta\mathscr{E}_{AB}|_{\mathscr{N}} = 0$. Then $\delta\mathscr{E}^A{}_B|_{\mathscr{N}} = g^{AC}\delta\mathscr{E}_{CB}|_{\mathscr{N}} = 0$, and (124) with $\nu = A$ becomes

$$0 = \frac{1}{r^2}\partial_r(r^2\delta\mathscr{E}^r{}_A)|_{\mathscr{N}}\,, \tag{127}$$

as desired. So, if $\mathscr{E}^r{}_A$ vanishes for some r on $\mathscr{N}$, it will vanish throughout $\mathscr{N}$. Now, (124) with $\nu = u$ reduces to

$$0 = \frac{1}{r^2}\partial_r(r^2\delta\mathscr{E}^r{}_u)|_{\mathscr{N}} + \frac{1}{r^2}\partial_A(r^2\delta\mathscr{E}^A{}_u)|_{\mathscr{N}}\,. \tag{128}$$

and what has been said about $\delta\mathscr{E}^A{}_u|_{\mathscr{N}}$ gives the result.

The above means that there is no need to integrate in r these Einstein equations which have not been discussed so far, namely $g^{AB}\mathscr{E}_{AB} = 0$, $\mathscr{E}_{uA} = 0$ and $\mathscr{E}_{uu} = 0$, when (125) holds. Indeed, once the already analysed

equations (118) are solved, together with their first u-derivatives, the whole set of Einstein equations will be solved by ensuring that $\mathscr{E}^r{}_A = 0 = \mathscr{E}^r{}_u$ holds at one value of r; this is equivalent to ensuring $\mathscr{E}_{uA} = 0 = \mathscr{E}_{uu}$ at one value of r.

The same scheme applies to the set of equations obtained by further differentiating the Einstein equation in u an arbitrary number of times.

3.5.1. $\partial_u \partial_r h_{uA}$. The equations $\mathscr{E}_{uA} = 0$ are too long to be usefully displayed here. Their linearisation $\delta\mathscr{E}_{uA} \equiv -\delta\mathscr{E}^r{}_A + (\epsilon - \alpha^2 r^2 - \frac{2m}{r})\delta\mathscr{E}_{rA}$ in vacuum reads

$$
\begin{aligned}
(129)\quad 0 = 2\delta\mathscr{E}_{uA} = \frac{1}{r^2}\Bigg[& \mathring{D}^B \mathring{D}_A h_{uB} - \mathring{D}^B \mathring{D}_B h_{uA} + \partial_u \mathring{D}^B h_{AB} \\
& - r^2\Bigg(\left(\varepsilon - r^2\alpha^2 - \frac{2m}{r}\right)\partial_r^2 h_{uA} + (2\alpha^2 + \frac{4m}{r^3})h_{uA} \\
& - r^2 \partial_r \partial_u \left(\frac{h_{uA}}{r^2}\right) + \partial_r \mathring{D}_A h_{uu}\Bigg)\Bigg] .
\end{aligned}
$$

This equation is satisfied both by $\mathfrak{d}_{\mathbf{S}_1}$ and $\mathfrak{d}_{\mathbf{S}_2}$ in vacuum.

Assuming $\delta G_{rA} = 0$, using the transport equation (53) to eliminate $\partial_r^2 \check{h}_{uA}$ and the identity (81) to eliminate $\partial_r h_{uu}$, we can rewrite (129) as

$$
\begin{aligned}
(130)\quad -r^4 \partial_r \partial_u \left(\frac{h_{uA}}{r^2}\right) &= \mathring{D}^B \mathring{D}_A h_{uB} - \mathring{D}^B \mathring{D}_B h_{uA} + \partial_u \mathring{D}^B h_{AB} \\
&\quad - r^2\Bigg(\left(\varepsilon - r^2\alpha^2 - \frac{2m}{r}\right)\partial_r^2 h_{uA} \\
&\qquad + (2\alpha^2 + \frac{4m}{r^3})h_{uA} + \partial_r \mathring{D}_A h_{uu}\Bigg) \\
&= \mathring{D}^B \Big[-2\,\mathrm{TS}[\mathring{D}_B h_{uA}] + \partial_u h_{AB} \\
&\qquad + (\alpha^2 r^2 - \varepsilon + \frac{2m}{r}) r^2 \partial_r \left(r^{-2} h_{AB}\right)\Big] + \mathring{D}_A \chi .
\end{aligned}
$$

Using the fact that $\partial_r \chi = 0$ we obtain, for any $\pi^A(x^B)$ satisfying $\mathrm{TS}[\mathring{D}_A \pi_B] = 0$,

$$
(131)\qquad \partial_r \int_{\mathbf{S}} \pi^A r^4 \partial_u \partial_r \check{h}_{uA} d\mu_{\mathring{\gamma}} \equiv \partial_r \overset{[1,1]}{Q}(\pi^A) = 0 ,
$$

where we recall from (116) that for $0 \leq i \in \mathbb{N}$,

$$\overset{[1,i]}{Q}(\pi^A) := \int_{\mathbf{S}} \pi^A r^4 \partial_u^i \partial_r \check{h}_{uA} d\mu_{\mathring{\gamma}} \,. \tag{132}$$

Clearly, by u-differentiating (130), we conclude that $\partial_u^i \delta\mathscr{E}_{uA} = 0$ implies

$$\partial_r \overset{[1,i+1]}{Q}(\pi^A) = 0 \,. \tag{133}$$

for $i \geq 0$.

Denoting $H_{k_{\partial_u U}} \ni \partial_u \delta U^A$, (129) implies

$$k_U \geq k_{\partial_u U} + 2,\ k_{\partial_u \gamma} \geq k_{\partial_u U} + 1 \text{ and } k_V \geq k_{\partial_u U} + 1. \tag{134}$$

3.5.2. $\partial_u h_{uu}$. The equation $\mathscr{E}_{uu} = 0$ is likewise too long to be usefully displayed here. Its linearised version is shorter and, in vacuum, can be rewritten as an equation for the transverse derivative $\partial_u(rh_{uu} - \mathring{D}^A h_{uA})$:

$$\begin{aligned} 0 &= 2\delta\mathscr{E}_{uu} \\ &= \frac{1}{r^2}\Big[2\Big(\partial_u + \big(\alpha^2 r^2 - \varepsilon + \frac{2m}{r}\big)\partial_r + \frac{3m}{r^2} - \frac{\varepsilon}{r}\Big)\mathring{D}^A h_{uA} \\ &\quad - \mathring{D}^A \mathring{D}_A h_{uu} - \big(\alpha^2 r^2 - \varepsilon + \frac{2m}{r}\big)\Big(\frac{\mathring{D}^A \mathring{D}^B h_{AB}}{r^2}\Big) \\ &\quad - 2r\partial_u h_{uu} - 2\big(\alpha^2 r^2 - \varepsilon + \frac{2m}{r}\big)\partial_r(rh_{uu})\Big] \end{aligned} \tag{135}$$

This must be satisfied by $\mathfrak{d}_{\mathbf{S}_1}$ and $\mathfrak{d}_{\mathbf{S}_2}$ when the linearised vacuum Einstein equations hold.

Denoting $H_{k_{\partial_u V}} \ni \partial_u \delta V$, (135) implies

$$k_U \geq k_{\partial_u V} + 1,\ k_V \geq k_{\partial_u V} + 2 \text{ and } k_\gamma \geq k_{\partial_u V} + 2. \tag{136}$$

3.6. Regularity

The regularity analysis carried-out so far is summarised by the following inequalities for the regularity of the metric components:

$$h_{uA} \text{ equation}: \quad k_\beta \geq k_U + 1\,, \quad k_\gamma \geq k_U + 1\,, \tag{137}$$

$$h_{uu} \text{ equation}: \quad k_\gamma \geq k_V + 2\,, \quad k_U \geq k_V + 1\,, \tag{138}$$

$$\partial_u h_{AB} \text{ equation}: \quad k_\beta \geq k_{\partial_u \gamma} + 2\,, \quad k_U \geq k_{\partial_u \gamma} + 1\,, \tag{139}$$

$$\partial_u\partial_r h_{uA} \text{ equation}: \quad k_U \geq k_{\partial_u U}+2\,, \quad k_V \geq k_{\partial_u U}+1\,, \quad k_{\partial_u\gamma} \geq k_{\partial_u U}+1\,, \tag{140}$$

$$\partial_u h_{uu} \text{ equation}: \quad k_U \geq k_{\partial_u V}+1\,, \quad k_V \geq k_{\partial_u V}+2\,, \quad k_\gamma \geq k_{\partial_u V}+2\,. \tag{141}$$

A consistent scheme for the linearised equations will thus be obtained if we choose any field h_{AB} such that $h_{AB}(r,\cdot) \in H_{k_\gamma}(\mathbf{S})$, for all $r \in [r_1, r_2]$, with $k_\gamma \geq 4$ and

$$k_\beta = k_\gamma\,, \quad k_U = k_\gamma - 1\,, \quad k_V = k_\gamma - 2\,, \quad k_{\partial_u U} = k_\gamma - 3\,, \tag{142}$$

$$k_{\partial_u V} = k_\gamma - 4\,, \quad k_{\partial_u\gamma} = k_\gamma - 2\,. \tag{143}$$

Note that the question of regularity of r-derivatives of γ has been swept under the rug using integration by parts. This question will need to be addressed when dealing with the nonlinear problem.

The regularity properties of the metric will be compatible with gauge transformations (28)-(31) if we assume, using obvious notation,

$$h_{uA} \text{ equation}: \quad k_{\xi^u} \geq k_U + 3\,, \quad k_{\partial_u\xi^u} \geq k_U + 1\,, \quad k_{\xi^A} \geq k_U + 2\,, \tag{144}$$

$$h_{ur} \text{ equation}: \quad k_{\partial_u\xi^u} \geq k_\beta\,, \quad k_{\xi^A} \geq k_\beta + 1\,, \tag{145}$$

$$h_{uu} \text{ equation}: \quad k_{\partial_u\xi^u} \geq k_V + 2\,, \quad k_{\partial_u\xi^B} \geq k_V + 1\,, \quad k_{\xi^u} \geq k_V + 2\,, \quad k_{\xi^B} \geq k_V + 1\,, \tag{146}$$

$$h_{AB} \text{ equation}: \quad k_{\xi^A} \geq k_\gamma + 1\,, \quad k_{\xi^u} \geq k_\gamma + 2\,, \tag{147}$$

$$\partial_u h_{AB} \text{ equation}: \quad k_{\partial_u\xi^A} \geq k_{\partial_u\gamma} + 1\,, \quad k_{\partial_u\xi^u} \geq k_{\partial_u\gamma} + 2\,. \tag{148}$$

A scheme consistent with (142)-(148) results by choosing

$$k_{\xi^u} = k_\gamma + 2\,, \quad k_{\partial_u\xi^u} = k_\gamma\,, \quad k_{\xi^A} = k_\gamma + 1\,, \quad k_{\partial_u\xi^A} = k_\gamma - 1\,. \tag{149}$$

3.7. Further u-derivatives

The representation formula for higher u-derivatives of the linearised metric components can be obtained by taking the u-derivatives of the existing equations. This gives, for $i \geq 0$, representation formulae of the form

$$\partial_u^i h_{AB} = \overset{(i)}{\Psi}_{AB}(u,r,x^A) + \sum_{0\leq j+k\leq i, k\neq i} \overset{(i,j,k)}{\psi}(r)\partial_r^j P^k h_{AB} \tag{150}$$

$$+\int_{r_1}^{r}\sum_{j=0}^{i}\overset{(i,j)}{\psi}(s,r)P^j h_{AB}\,ds\,,$$

$$\partial_u^i \check{h}_{uA} = \overset{(i)}{X}_A(u,r,x^A) + \mathring{D}^B\Big[\sum_{0\le j+k\le i, k\neq i}\overset{(i,j,k)}{\chi}(r)\partial_r^j P^k h_{AB} + \int_{r_1}^{r}\sum_{j=0}^{i}\overset{(i,j)}{\chi}(s,r)P^j h_{AB}\,ds\Big]\,, \tag{151}$$

where $\overset{(i)}{X}$ and $\overset{(i)}{\Psi}$ depend only on data at r_1; recall that P denotes the operator

$$Ph_{AB} = \mathrm{TS}[D_A D^C h_{BC}]\,. \tag{152}$$

The above is proved by induction (see Appendix B), which is initialised with $i=0$ as follows:

1. Order zero for (150) is trivial, with

$$\overset{(0,0,0)}{\psi}(r) = 1\,,\quad \overset{(0)}{\Psi}_{AB}(u,r,x^A) = 0 = \overset{(0,0)}{\psi}(s,r)\,. \tag{153}$$

   We note that order one for (150) is obtained from (94), with

$$\begin{aligned}
\overset{(1,0,0)}{\psi}(r) &= -\frac{\varepsilon}{2r} + \frac{\alpha^2 r}{2} + \frac{m}{r^2}\,,\\
\overset{(1,1,0)}{\psi}(r) &= \frac{1}{2}\left(\varepsilon - \alpha^2 r^2 - \frac{2m}{r}\right)\,,\\
\overset{(1,0,1)}{\psi}(r) &= 0\,,\quad \overset{(1,1)}{\psi}(s,r) = \frac{2r^2}{3s^4} + \frac{1}{3sr}\,,\\
\overset{(1,0)}{\psi}(s,r) &= \frac{\alpha^2 r}{s} - \frac{mr^2}{s^4}\,.
\end{aligned}$$

2. Order zero for (151) follows from (67), where μ and λ are determined from $h_{uA}|_{r_1}$ and $\partial_r h_{uA}|_{r_1}$, with

$$\overset{(0,0,0)}{\chi}(r) = 0\,,\quad \overset{(0,0)}{\chi}(s,r) = \frac{1}{3}\left(\frac{2}{sr^3} + \frac{1}{s^4}\right)\,. \tag{154}$$

We note that the terms involving $\overset{(i,j,k)}{\psi}$ and $\overset{(i,j,k)}{\chi}$ are innocuous at $r=r_2$, as they are determined by known boundary data at r_2. However, they are

essential for the induction procedure for $r \neq r_2$, as they contribute to the key terms $\overset{(i,j)}{\psi}$ and $\overset{(i,j)}{\chi}$ in the iteration. This implies in particular that the explicit form of $\overset{(i,j,k)}{\psi}$ etc. with the highest index $i = \ell$ is *not* needed when gluing at order ℓ.

Again by induction (cf. Appendix B), one shows the following:

1. All the integral kernels in (150)-(151), depending upon r and s, are polynomials in s^{-1} with coefficients depending upon r;
2. <u>when $m = 0$</u>, $\overset{(i,0)}{\psi}$ is proportional to $\alpha^2 s^{-1}$.
3. The highest power of $1/s$ in $\overset{(1,j)}{\psi}$ is s^{-4}.
4. The highest power of $1/s$ in $\overset{(i,j)}{\psi}$ with $1 \leq j \leq i$ is $s^{-(i+3)}$ <u>when $m = 0$</u>, and this power is not larger than $s^{-2i+j-3}$ when $m \neq 0$; cf. Lemma B.1, Appendix B.2.
5. It holds that
$$\overset{(i+1,i+1)}{\psi}(s,r) = \int_s^r \overset{(i,i)}{\psi}(y,r)\,\overset{(1,1)}{\psi}(s,y)\,dy\,,$$
with
$$\overset{(1,1)}{\psi}(s,r) = \frac{2r^2}{3s^4} + \frac{1}{3rs}\,,$$
independently of m.
6. The highest power of $1/s$ in $\overset{(i,j)}{\chi}$ with $0 \leq j \leq i$ is $s^{-(i+4)}$ <u>when $m = 0$</u>, and this power is not larger than $s^{-2i+j-4}$ <u>when $m \neq 0$</u>.

In what follows we will often use the notation

$$\hat{\kappa}_i(s) := \frac{1}{s^i}\,. \tag{155}$$

We have collected the explicit formulae for all the integral kernels appearing in (150)-(151), and needed for $C^2_u\, C^\infty_{(r,x^A)}$-gluing, in Appendix B.

3.7.1. The transverse-traceless part. For most of our further purposes, the essential role is played by the integral kernels $\overset{(i,j)}{\chi}$ and $\overset{(i,j)}{\psi}$ appearing in (150)-(151). However, it turns out that when $m = 0$ the TT-part of $\partial_u^i h_{AB}$ leads to obstructions to gluing, in which case the boundary terms in (150) become significant. This forces us to revisit the induction, as follows:

We first consider the L^2-projection of (93) on TT, with $m = 0$:

$$\partial_u h^{[\mathrm{TT}]}_{AB} = r \underbrace{\left[\frac{\partial_u h^{[\mathrm{TT}]}_{AB}\big|_{r_1}}{r_1} - \frac{1}{2}(\varepsilon - \alpha^2 r_1^2)\left(\frac{1}{r_1}\partial_r h^{[\mathrm{TT}]}_{AB}\big|_{r_1} - \frac{1}{r_1^2} h^{[\mathrm{TT}]}_{AB}\big|_{r_1} \right) \right]}_{q^{[\mathrm{TT}]}_{AB}\big|_{r_1}} + \frac{r}{2}(\varepsilon - \alpha^2 r^2)\left(\frac{1}{r}\partial_r h^{[\mathrm{TT}]}_{AB} - \frac{1}{r^2} h^{[\mathrm{TT}]}_{AB} \right) + \alpha^2 r \int_{r_1}^{r} \frac{1}{s} h^{[\mathrm{TT}]}_{AB}\, ds\,; \tag{156}$$

equivalently,

$$\underbrace{\frac{1}{r}\partial_u h^{[\mathrm{TT}]}_{AB} - \frac{1}{2}(\varepsilon - \alpha^2 r^2)\left(\frac{1}{r}\partial_r h^{[\mathrm{TT}]}_{AB} - \frac{1}{r^2} h^{[\mathrm{TT}]}_{AB} \right)}_{q^{[\mathrm{TT}]}_{AB}\big|_{r}} = \underbrace{\left[\frac{\partial_u h^{[\mathrm{TT}]}_{AB}\big|_{r_1}}{r_1} - \frac{1}{2}(\varepsilon - \alpha^2 r_1^2)\left(\frac{1}{r_1}\partial_r h^{[\mathrm{TT}]}_{AB}\big|_{r_1} - \frac{1}{r_1^2} h^{[\mathrm{TT}]}_{AB}\big|_{r_1} \right) \right]}_{q^{[\mathrm{TT}]}_{AB}\big|_{r_1}} + \alpha^2 \int_{r_1}^{r} \frac{1}{s} h^{[\mathrm{TT}]}_{AB}\, ds\,. \tag{157}$$

This can of course also be derived directly from (100), but note that this calculation makes it clear how the tensor field q_{AB} appears in the formalism.

It follows that when $\alpha = 0 = m$, the field $q^{[\mathrm{TT}]}_{AB}$ provides a 2-dimensional family of gauge-independent radially conserved charges on $\mathbb{T}^2$, and a $6(\mathfrak{g}-1)$-dimensional family of such charges on sections with genus $\mathfrak{g} \geq 2$.

When $\alpha \neq 0$ but m remains zero, taking u-derivatives of (157) leads to

$$\overset{[p+1]}{q}{}^{[\mathrm{TT}]}_{AB}\Big|_{r_1}^{r} = \alpha^2 \int_{r_1}^{r} \hat{\kappa}_1(s)\partial_u^p h^{[\mathrm{TT}]}_{AB}\, ds\,, \tag{158}$$

where, for $i \geq 1$,

$$\overset{[i]}{q}_{AB} := \frac{1}{r}\partial_u^i h_{AB} - \frac{1}{2}(\varepsilon - \alpha^2 r^2)\big(\frac{1}{r}\partial_r \partial_u^{i-1} h_{AB} - \frac{1}{r^2}\partial_u^{i-1} h_{AB}\big)\,. \tag{159}$$

Making use again of (156) we find

$$\int_{r_1}^{r} \hat{\kappa}_1(s)\partial_u h^{[\mathrm{TT}]}_{AB}\, ds \tag{160}$$

$$
\begin{aligned}
&= \int_{r_1}^{r} \left[q^{[\mathrm{TT}]}_{AB}\big|_{r_1} + \frac{1}{2}(\varepsilon - \alpha^2 s^2)\big(\frac{1}{s}\partial_s h^{[\mathrm{TT}]}_{AB}\big|_s - \frac{1}{s^2} h^{[\mathrm{TT}]}_{AB}\big|_s\big)\right] ds \\
&\quad + \alpha^2 \int_{r_1}^{r}\int_{r_1}^{s} \left(\frac{1}{y} h^{[\mathrm{TT}]}_{AB}\big|_y\right) dy\, ds \\
&= \left[s\; q^{[\mathrm{TT}]}_{AB}\big|_{r_1} + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) h^{[\mathrm{TT}]}_{AB}\big|_s \right]_{r_1}^{r} + \alpha^2 r \int_{r_1}^{r} \hat{\kappa}_1(s) h^{[\mathrm{TT}]}_{AB}\; ds\,.
\end{aligned}
$$

It follows by induction that

$$
\begin{aligned}
(161)\qquad & \partial_u^p \int_{r_1}^{r} \hat{\kappa}_1(s) h^{[\mathrm{TT}]}_{AB}\, ds \\
&= \sum_{k=0}^{p-1} (\alpha^2 r)^k \partial_u^{p-1-k} \left[s\; q^{[\mathrm{TT}]}_{AB}\big|_{r_1} + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) h^{[\mathrm{TT}]}_{AB}\big|_s \right]_{r_1}^{r} \\
&\qquad + (\alpha^2 r)^p \int_{r_1}^{r} \hat{\kappa}_1(s) h^{[\mathrm{TT}]}_{AB}\, ds \\
&= \sum_{k=0}^{p-1} (\alpha^2 r)^k \left[s\; \overset{[p-k]}{q}{}^{[\mathrm{TT}]}_{AB}\big|_{r_1} + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) \partial_u^{p-1-k} h^{[\mathrm{TT}]}_{AB}\big|_s \right]_{r_1}^{r} \\
&\qquad + (\alpha^2 r)^p \int_{r_1}^{r} \hat{\kappa}_1(s) h^{[\mathrm{TT}]}_{AB}\, ds\,.
\end{aligned}
$$

This allows us to rewrite (158) as

$$
\begin{aligned}
(162)\qquad & \overset{[p+1]}{q}{}^{[\mathrm{TT}]}_{AB}\Big|_{r_1}^{r} \\
&= \alpha^2 \sum_{k=0}^{p-1} (\alpha^2 r)^k \left[s\; \overset{[p-k]}{q}{}^{[\mathrm{TT}]}_{AB}\big|_{r_1} + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) \partial_u^{p-1-k} h^{[\mathrm{TT}]}_{AB}\big|_s \right]_{r_1}^{r} \\
&\quad + \alpha^{2(p+1)} r^p \int_{r_1}^{r} \hat{\kappa}_1(s) h^{[\mathrm{TT}]}_{AB}\, ds\,.
\end{aligned}
$$

4. Gluing up to gauge

We now present a scheme for matching, up-to residual gauge, the linearised fields

$$
(163)\qquad \{h_{\mu\nu}, \partial_u h_{\mu\nu} \ldots \partial_u^k h_{\mu\nu}\}
$$

in Bondi gauge, with $2 \le k < \infty$. We will assume, for simplicity, that each of the fields $\partial_u^i h_{\mu\nu}\big|_{\{u=0\}}$, $0 \le i \le k$, is smooth. The collection of fields of

this differentiability class will be denoted by $C^k_u C^\infty_{(r,x^A)}$.

Let $0 \le r_0 < r_1 < r_2 < r_3 \in \mathbb{R}$. Consider two sets of vacuum linearised gravitational fields in Bondi gauge, of $C^k_u C^\infty_{(r,x^A)}$-differentiability class, defined in spacetime neighbourhoods of $\mathscr{N}_{(r_0,r_1]}$ and $\mathscr{N}_{[r_2,r_3)}$. Let us denote by $\mathbf{S}_1$ the section of $\mathscr{N}_{(r_0,r_1]}$ at $r = r_1$. The linearised gravitational field near $\mathscr{N}_{(r_0,r_1]}$ induces a set of Bondi cross-section data on $\mathbf{S}_1$, which we denote as $\mathfrak{d}_{\mathbf{S}_1}$. Similarly, we denote by $\mathbf{S}_2$ the section of $\mathscr{N}_{[r_2,r_3)}$ at $r = r_2$ and the induced gluing data by $\mathfrak{d}_{\mathbf{S}_2}$. Let us also denote by $\tilde{\mathbf{S}}_1$ (resp. $\tilde{\mathbf{S}}_2$) the codimension-two section obtained by gauge-transforming $\mathbf{S}_1$ (resp. $\mathbf{S}_2$) using arbitrary gauge fields $\overset{(1)}{\xi}{}^\mu$ (resp. $\overset{(2)}{\xi}{}^\mu$), the associated gluing data by $\tilde{\mathfrak{d}}_{\tilde{\mathbf{S}}_1}$ (resp. $\tilde{\mathfrak{d}}_{\tilde{\mathbf{S}}_2}$) and the outgoing null hypersurface on which it lies by $\widetilde{\mathscr{N}}_{(r_0,r_1]}$ (resp. $\widetilde{\mathscr{N}}_{[r_2,r_3)}$).

Of course, in the linearised gluing the initial hypersurface $\mathscr{N}_{(r_0,r_3)}$ does not change, thus $\widetilde{\mathscr{N}}_{(r_0,r_3)} = \mathscr{N}_{(r_0,r_3)}$ as a set, but the Bondi coordinates on either $\mathscr{N}_{(r_0,r_1]}$ or on $\mathscr{N}_{[r_2,r_3)}$ need to be "infinitesimally deformed" both in transverse and in tangential directions. We use the symbol $\widetilde{\mathscr{N}}$ to emphasise the infinitesimal adjustment of Bondi coordinates, as an adjustment of $\mathscr{N}_{(r_0,r_1]}$ or $\mathscr{N}_{[r_2,r_3)}$ is generically needed when passing to the nonlinear gluing both in our case and in [3].

The goal is to glue $\tilde{\mathfrak{d}}_{\tilde{\mathbf{S}}_1}$ and $\tilde{\mathfrak{d}}_{\tilde{\mathbf{S}}_2}$ along $\widetilde{\mathscr{N}}_{[r_1,r_2]}$ so that the resulting linearised field on $\widetilde{\mathscr{N}}_{(r_0,r_3)}$ provide smooth characteristic data for Einstein equations together with a matching of k transverse derivatives. Indeed, we claim:

Theorem 4.1. *A $C^k_u C^\infty_{(r,x^A)}$-linearised vacuum data set on $\mathscr{N}_{(r_0,r_1]}$ can be smoothly glued to another such set on $\mathscr{N}_{[r_2,r_3)}$ if and only if the obstructions listed in Tables 4.1-4.2 are satisfied.*

The rest of this section is devoted to the proof of this theorem.

Let v_{AB} be any symmetric traceless tensor field defined on a neighbourhood of $\mathscr{N}_{[r_1,r_2]}$ which interpolates between the original fields $h_{AB}|_{\mathscr{N}_{(r_0,r_1]}}$ and $h_{AB}|_{\mathscr{N}_{[r_2,r_3)}}$, so that the resulting field on $\mathscr{N}_{(r_0,r_3)}$ is as differentiable as the original fields. When attempting a $C^k_u C^\infty_{(r,x^A)}$-gluing, we can add to v_{AB} a field $w_{AB}|_{[r_1,r_2]}$ which vanishes smoothly (i.e. together with r-derivatives of all orders) at the end cross-sections $\{r_1\} \times \mathbf{S}$ and $\{r_2\} \times \mathbf{S}$ without affecting the gluing of h_{AB}. To take into account the gauge freedom, let $\phi(r) \ge 0$ be a smooth function which equals 1 near $r = r_1$ and equals 0 near $r = r_2$. Let $\overset{(1)}{\xi}{}^u$ and $\overset{(1)}{\xi}_A$ be gauge fields used to gauge the metric around $\mathscr{N}_{(r_0,r_1]}$, and

let $\overset{(2)}{\xi}{}^u$ and $\overset{(2)}{\xi}{}_A$ be gauge fields used to gauge the metric around $\mathscr{N}_{[r_2,r_3)}$. For $r_1 \le r \le r_2$ we set

$$\tilde{h}_{AB} = v_{AB} + w_{AB} + \phi r^2 \,\mathrm{TS}[\mathring{\mathcal{L}}_{\underset{\zeta}{(1)}} \mathring{\gamma}_{AB}] + (1-\phi) r^2 \,\mathrm{TS}[\mathring{\mathcal{L}}_{\underset{\zeta}{(2)}} \mathring{\gamma}_{AB}] \,. \tag{164}$$

(Recall that $\zeta^A = \xi^A - \mathring{D}^A \xi^u / r$, cf. (25).)

In the gluing problem, the gauge fields evaluated on $\tilde{\mathbf{S}}_{1,2}$ and the field w_{AB} on $\widetilde{\mathscr{N}}_{(r_1,r_2)}$ are *free fields* which can be chosen arbitrarily. Our aim in what follows is to show how to choose these fields to solve the transport equations of Section 3.3-3.7 to achieve gluing-up-to-gauge. When extending fields across r_1 by solving the transport equations, we will always choose initial data at r_1 which guarantee smoothness of the fields there.

For the $C^k_u C^\infty_{(r,x^A)}$-gluing we will need smooth functions

$$\kappa_i : (r_1, r_2) \to \mathbb{R} \,, \quad i \in \{0, \dots, k_{[m]} + 4\} \,,$$

where $k_{[m]} = k$ <u>when $m = 0$</u> and $k_{[m]} = 2k$ <u>when $m \neq 0$</u>, satisfying

$$\langle \kappa_i, \hat{\kappa}_j \rangle \equiv \int_{r_1}^{r_2} \kappa_i(s) \hat{\kappa}_j(s)\, ds = 0 \quad \text{for } i > j \,, \tag{165}$$

$$\langle \kappa_i, \hat{\kappa}_i \rangle = 1 \,, \tag{166}$$

and vanishing near the end points $r \in \{r_1, r_2\}$, which is possible since the $\hat{\kappa}_i$'s are linearly independent; see Appendix A.

The fields w_{AB} of (164) will be taken of the following form: for $s \in [r_1, r_2]$,

$$w_{AB}(s) = \sum_{i=1}^{k_{[m]}+4} \kappa_i(s) \overset{[i]}{\varphi}_{AB} \,. \tag{167}$$

We define

$$\overset{[j]}{\hat{\varphi}}_{AB} := \langle w_{AB}, \hat{\kappa}_j \rangle \,. \tag{168}$$

We will show how to construct the fields $\{\overset{[j]}{\hat{\varphi}}_{AB}\}_{j=1}^{k_{[m]}+4}$. In view of (165)-(166)

we have

$$\begin{pmatrix} \overset{[1]}{\hat{\varphi}}_{AB} \\ \overset{[2]}{\hat{\varphi}}_{AB} \\ \vdots \\ \overset{[k_{[m]}+4]}{\hat{\varphi}}_{AB} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \langle \kappa_1, \hat{\kappa}_2 \rangle & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \langle \kappa_1, \hat{\kappa}_{k_{[m]}+4} \rangle & \langle \kappa_2, \hat{\kappa}_{k_{[m]}+4} \rangle & \cdots & 1 \end{pmatrix} \begin{pmatrix} \overset{[1]}{\varphi}_{AB} \\ \overset{[2]}{\varphi}_{AB} \\ \vdots \\ \overset{[k_{[m]}+4]}{\varphi}_{AB} \end{pmatrix}, \tag{169}$$

which allows one to determine $\{\overset{[j]}{\varphi}_{AB}\}_{j=1}^{k_{[m]}+4}$ in terms of $\{\overset{[j]}{\hat{\varphi}}_{AB}\}_{j=1}^{k_{[m]}+4}$ in the obvious way.

4.1. Strategy

A collection of fields $\{\partial_u^i h_{\mu\nu}\}_{0\leq i\leq k}$ on a null hypersurface $\mathscr{N}$ will be called *characteristic $C_u^k\, C^\infty_{(r,x^A)}$-data for linearised vacuum Einstein equations on $\mathscr{N}$*, or simply *$C_u^k\, C^\infty_{(r,x^A)}$-data*, if the fields $\partial_u^i h_{\mu\nu}$ are smooth on $\mathscr{N}$ and satisfy on $\mathscr{N}$ the equations which are obtained by differentiating the linearised vacuum Einstein equations in u up to k-times, and in which no-more than k derivatives of the $h_{\mu\nu}$'s with respect to u occur. In Bondi gauge this means that the equations $\partial_u^i \mathscr{E}_{\mu\nu} = 0$ should hold with $0 \leq i \leq k-1$, and that in addition we also have $\partial_u^k \mathscr{E}_{rA} = 0 = \partial_u^k \mathscr{E}_{rr} = \partial_u^k \mathscr{E}_{ur}$.

We will say that $C_u^k\, C^\infty_{(r,x^A)}$-data are smooth if the $\partial_u^i h_{\mu\nu}$'s are smooth on $\mathscr{N}$.

We note that the linearised Einstein equations are invariant under linearised gauge transformations. In our scheme we will perform gauge transformations which will be needed to ensure the continuity of the fields, but which will have no influence on the question whether or not the linearised Einstein equations hold.

A set of $C_u^k\, C^\infty_{(r,x^A)}$-data can be obtained by restricting a smooth solution of linearised vacuum equations, and its transverse derivatives, to a null hypersurface. The converse is also true for null hypersurfaces with boundary, e.g. $\mathscr{N}_{[r_0,r_1)}$ or $\mathscr{N}_{[r_0,r_1]}$, in the following sense: any such data set arises by restriction of (many) solutions of vacuum Einstein equations to $\mathscr{N}$. This can be realised by solving a characteristic Cauchy problem with two null hypersurfaces intersecting transversally at $\{r = r_0\}$, and requires providing data on both hypersurfaces. We note that losses of differentiability are unavoidable in the characteristic Cauchy problem when the data are not smooth: solutions constructed from characteristic initial data which are of C^k-differentiability class will typically be of differentiability class C^{k-k_0}, for some $k_0 \in \mathbb{N}$ which typically depends upon k. Compare [13, 14].

Our gluing procedure for such fields rests on the following elementary result. Let $a < b < c$, and let us for simplicity assume that all fields $\partial_u^i h_{\mu\nu}$, $i \in \mathbb{N}$, on $\mathscr{N}_{(a,b]}$ and $\mathscr{N}_{[b,c)}$ are smooth in all variables, up-to-and-including the common boundary at b; a similar result for finitely-differentiable fields, with distinct finite losses of differentiability for distinct fields, can be established using the results of Section 3.6, and is left as an exercise to a concerned reader.

Lemma 4.2. *Let $k \in \mathbb{N}$. Two $C_u^k C^\infty_{(r,x^A)}$ data sets in Bondi gauge on $\mathscr{N}_{(a,b]}$ and $\mathscr{N}_{[b,c)}$, with h_{AB} extending smoothly across $\{r = b\}$, extend to smooth $C_u^k C^\infty_{(r,x^A)}$ data on $\mathscr{N}_{(a,c)}$ if and only if the fields*

1. *$\partial_u^i h_{ur}$, $\partial_u^i h_{uA}$, $\partial_u^i h_{AB}$, with $0 \le i \le k$, as well as*
2. *$\partial_r h_{uA}$ and h_{uu}*

extend by continuity at $\{r = b\}$ to continuous fields.

Proof. The necessity is obvious. The sufficiency follows from the equations in Sections 3.3-3.5, together with their u-derivatives, as follows:

Suppose that $\delta\beta$ extends by continuity at b, then (50) shows that $\delta\beta$ extends to a smooth function. Next, (53) shows that continuity of $\partial_r(r^{-2}h_{uA})$ at b guarantees a smooth extension of $\partial_r(r^{-2}h_{uA})$. But then, by another integration, continuity of h_{uA} at b guarantees smooth extendability. One can now use (71) and (90) to similarly show that continuity, at b, of δV and $\partial_u h_{AB}$ leads to smooth extensions of these fields. In particular $\partial_u h_{AB}$ is now smooth on $\mathscr{N}_{(a,c)}$, and one can apply the argument just given to the equations obtained by u-differentiating the vacuum Einstein equations to obtain smoothness on $\mathscr{N}_{(a,c)}$ of $\partial_u h_{\mu\nu}$ and $\partial_u^2 h_{AB}$.

Iterating this argument a finite number of times establishes the result. □

As such, Lemma (4.2) will apply directly at $r = r_2$, once we have shown that all desired equations hold for $r \in (r_0, r_2)$. However, the argument that we are about to present is more complicated because, within our construction, for $r \in [r_1, r_2]$ we can only solve some of the Einstein equations. Fortunately the conditions of the Lemma are not independent, and the crux of the argument is to isolate and enforce the independent ones in a hierarchical way, proving as we progress both the continuity of the fields listed in the Lemma, and the satisfaction of the linearised Einstein equations, as well as their u-derivatives, on $\mathscr{N}_{(r_0,r_2]}$.

Given $k \in \mathbb{N}$, $k \geq 2$, in order to carry out a $C^k_u C^\infty_{(r,x^A)}$-gluing the smooth solution on $\mathscr{N}_{(r_0,r_1]}$ is extended to one on $\mathscr{N}_{(r_0,r_2]}$ using a smooth interpolating field v_{AB} as in (167) and smooth gauge fields $\overset{(1)}{\zeta}$ and $\overset{(2)}{\zeta}$ as in (164), with the u-derivatives extended using the equations in Section (3.7). This guarantees that some of the Einstein equations are satisfied. It now remains to show that we can choose v, $\overset{(1)}{\zeta}$ and $\overset{(2)}{\zeta}$ to satisfy the remaining conditions of Lemma 4.2 together with the Einstein equations on $\mathscr{N}_{(r_0,r_2]}$. This can be done in three steps:

1. The requirement of continuity of the fields $\partial_u^p \tilde{h}_{\mu\nu}$ for $0 \leq p \leq k$ at $\tilde{\mathbf{S}}_2$ imposes conditions on the given data $\mathfrak{d}_{\mathbf{S}_1}$ and $\mathfrak{d}_{\mathbf{S}_2}$, as well as on the gauge fields $\overset{(2)}{\xi}_A$ and $\overset{(2)}{\xi}{}^u$ and the gluing fields $\overset{[p]}{\hat{\varphi}}_{AB}$. We summarise these conditions here (cf. Tables 4.1-4.2), with further details presented in the next section:
   i. $\tilde{h}_{uu}$: Continuity of $\tilde{h}_{uu}$ at $\tilde{\mathbf{S}}_2$ requires
   $$\chi[\mathfrak{d}_{\tilde{\mathbf{S}}_1}] = \chi[\mathfrak{d}_{\tilde{\mathbf{S}}_2}]\,. \tag{170}$$
   The continuity of $\chi^{[\ker \mathring{L}]}$ at r_2 requires the radial-charge matching-condition
   $$\overset{[2]}{Q}[\mathfrak{d}_{\mathbf{S}_1}](\lambda) = \overset{[2]}{Q}[\mathfrak{d}_{\mathbf{S}_2}](\lambda)\,, \tag{171}$$
   for all λ with vanishing Hessian, where $\mathring{L}$ has been defined in (9). <u>When $m = 0$</u>, this condition is a gauge-invariant obstruction to gluing. The continuity of the remaining part $\chi^{[(\ker \mathring{L})^\perp]}$ can be achieved using the gauge field $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ (see (82)).
   <u>When $m \neq 0$</u>, the gauge-invariant obstruction is (171) with constant λ. The remaining projection of χ can be made continuous at r_2 using the same gauge field as when $m = 0$, together with *proper* conformal Killing vectors $\overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$ (which are zero unless we are on S^2).
   ii. $\partial_r \tilde{h}_{uA}$: Continuity of $\partial_r \tilde{h}_{uA}$ at $\tilde{\mathbf{S}}_2$ requires the radial-charge matching condition
   $$\overset{[1]}{Q}[\mathfrak{d}_{\tilde{\mathbf{S}}_1}] = \overset{[1]}{Q}[\mathfrak{d}_{\tilde{\mathbf{S}}_2}]\,, \tag{172}$$

Table 4.1: Fields used to ensure the continuity at r_2 when $m = 0$. The continuity for the fields in the last two lines follows from Bianchi identities. The fields $\tilde{h}_{\mu\nu}$ are the gauge-transformed fields $h_{\mu\nu}$ using the gauge fields $\overset{(1)}{\xi}$ for $r \le r_1$ and $\overset{(2)}{\xi}$ for $r \ge r_2$, cf. Section 3.2; the operator $\mathring{L}$ has been defined in (9); the fields v_{AB} and $\overset{[k]}{\hat{\varphi}}_{AB}$ are defined in (164) and (167)-(168); projections such as $(\cdot)^{[\mathrm{TT}]}$ and $(\cdot)^{[\mathrm{H}]}$ (both trivial on S^2), or $(\cdot)^{[\mathrm{CKV}^\perp]}$ (identity on higher genus) are defined in Section 2; the radial charges $\overset{[a]}{Q}$, a=1,2, are defined in (59) and (72); the radially-conserved fields $\overset{[i]}{q}_{AB}$ and $\overset{[3,i]}{Q}$ are defined in (159) and (115); the operator P has been defined in (94); the coefficients $\overset{(p,j)}{\chi}_{p+4}$ are defined inductively in (151) and (219).

	Gluing field	Gauge field	Obstruction
h_{AB}	v_{AB}		
$\partial_u^i \tilde{h}_{ur}$, $i \ge 0$		$\partial_u^{i+1} \overset{(1)}{\xi}{}^u$ and $\partial_u^{i+1} \overset{(2)}{\xi}{}^u$	
$\tilde{h}_{uu}$		$(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$	$\overset{[2]}{Q}(\lambda)$
$\partial_r \tilde{h}_{uA}$	$\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$		$\overset{[1]}{Q}(\pi)$
$\tilde{h}_{uA}$	$\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$	$\partial_u \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$	
$\partial_u \tilde{h}^{[\mathrm{TT}^\perp]}_{AB}$: $\mathfrak{g} \le 1$ $\mathfrak{g} \ge 2$		$\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A$ $\overset{(2)}{\xi}{}^{[\mathrm{H}^\perp]}_A$	$\overset{[1]}{Q}(\pi)$ if $\alpha \ne 0$ $\overset{[3,1]}{Q}{}^{[\mathrm{H}]}$
$\partial_u \tilde{h}^{[\mathrm{TT}]}_{AB}$, $\alpha \ne 0$	$\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$		
$\partial_u \tilde{h}^{[\mathrm{TT}]}_{AB}$, $\alpha = 0$			$q^{[\mathrm{TT}]}_{AB}$ (trivial on S^2)
$\partial_u^p \tilde{h}^{[\mathrm{TT}^\perp]}_{AB}$, $2 \le p \le k$		$\partial_u^{p-1} \overset{(2)}{\xi}{}^{[(\mathrm{CKV}+\mathrm{H})^\perp]}_A$	$\overset{[3,p]}{Q}{}^{[\mathrm{H}]}$ if $\mathfrak{g} \ge 2$
$\partial_u^p \tilde{h}^{[\mathrm{TT}]}_{AB}$, $\alpha = 0$ $\alpha \ne 0$ $2 \le p \le k$			$\overset{[p]}{q}{}^{[\mathrm{TT}]}_{AB}$ see (239), involves $\overset{[p]}{q}{}^{[\mathrm{TT}]}_{AB}[\mathfrak{d}_{\mathbf{S}_a}]$
$\partial_u^p \tilde{h}_{uA}$, $1 \le p \le k-2$	$\overset{[p+4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$	$\partial_u^{p+1} \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$	$\ker\left(\sum_{j=0}^p \overset{(p,j)}{\chi}_{p+4} P^j\right)$ (trivial if $\mathfrak{g} \ge 2$)
$\partial_u^p \tilde{h}_{uA}$, $k-1 \le p \le k$	$\overset{[p+4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$	only S^2 and $\mathbb{T}^2$ $\mathbb{T}^2$: $\partial_u^{p+1} \overset{(2)}{\xi}{}^{[1]}_A$ S^2: $\partial_u^{p+1} \overset{(2)}{\xi}{}^{[1 \le \ell \le p+1]}_A$	
$\partial_u^p \tilde{h}_{uu}$, $1 \le p \le k$			
$\partial_u^p \partial_r \tilde{h}_{uA}$, $1 \le p \le k$			

Table 4.2: Fields used to ensure the continuity at r_2 when $m \neq 0$. The notation, and the last two lines, are as in Table 4.1. "KV of $\mathbf{S}$" stands for "Killing vector of $(\mathbf{S}, \mathring{\gamma})$"

	Gluing field	Gauge field	Obstruction
h_{AB}	v_{AB}		
$\partial_u^i \tilde{h}_{ur}\,,\ i \geq 0$		$\partial_u^{i+1} \overset{(1)}{\xi}{}^u$ and $\partial_u^{i+1} \overset{(2)}{\xi}{}^u$	
$\tilde{h}_{uu}$		$(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ and $\overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$, with $\mathring{D}^A \overset{(2)}{\xi}_A \neq 0$	$\overset{[2]}{Q}(\lambda^{[1]})$
$\partial_r \tilde{h}_{uA}$	$\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$	only on S^2: $(\overset{(2)}{\xi}{}^u)^{[=1]}$	$\overset{[1]}{Q}(\pi^A)$ π^A – KV of $\mathbf{S}$
$\tilde{h}_{uA}$	$\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$	$\partial_u \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$ only for S^2 and $\mathbb{T}^2$: $\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A$	
$\partial_u \tilde{h}^{[\mathrm{TT}]}_{AB}$	$\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$ and $\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$		
$\{\partial_u^p \tilde{h}^{[\mathrm{TT}^\perp]}_{AB}\}_{1\leq p\leq k}$		$\{\partial_u^p \overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]}_A\}_{0\leq p\leq k}$, $\{\partial_u^p \overset{(2)}{\xi}{}^{[\mathrm{H}]}_A\}_{0\leq p\leq k}$	
$\partial_u^p \tilde{h}^{[\mathrm{TT}]}_{AB}$, $2 \leq p \leq k$ (cf. (242))	$\overset{[2p+2]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$ and $\overset{[2p+1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$		
$\{\partial_u^p \tilde{h}_{uA},\ 1 \leq p \leq k\}$	$\{\overset{[j]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}\}_{5\leq j\leq 2k+4}$	$\partial_u^{p+1} \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$	
$\partial_u^p \tilde{h}_{uu},\ 1 \leq p \leq k$			
$\partial_u^p \partial_r \tilde{h}_{uA},\ 1 \leq p \leq k$			

as well as a suitable choice of the field $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$. This condition is equivalent to the obstruction

$$\overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_1}] = \overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_2}]\,, \tag{173}$$

except if $\mathbf{S} = S^2$ and $m \neq 0$. In this last case we use the gauge field $(\overset{(2)}{\xi}{}^u)^{[=1]}$ to get rid of the obstructions associated with proper conformal Killing vectors in (173).

iii. $\tilde{h}_{uA}$: Continuity of $\tilde{h}_{uA}$ at $\tilde{\mathbf{S}}_2$ is achieved by a suitable choice of

$\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ and of $\partial_u \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$.

iv. $\partial_u^p \tilde{h}_{AB}$ for $1 \le p \le k$: This is addressed in Sections 4.2.2, 4.2.6 and 4.3.2. The continuity of $\partial_u^p \tilde{h}^{[\mathrm{TT}^\perp]}_{AB}$ requires

$$\overset{[3,p]}{Q}{}_A[\mathfrak{d}_{\tilde{\mathbf{S}}_1}] = \overset{[3,p]}{Q}{}_A[\mathfrak{d}_{\tilde{\mathbf{S}}_2}]\,. \tag{174}$$

<u>In the case $m = 0$</u>, the gauge fields $\partial_u^{p-1} \overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]}_A$ can be used to achieve the matching of $(\overset{[3,p]}{Q})^{[(\mathrm{CKV+H})^\perp]}$. $(\overset{[3,p]}{Q})^{[\mathrm{H}]}$ provide obstructions for gluing on negatively curved manifolds of $\mathfrak{g} \ge 2$. The requirement of continuity of $\partial_u^p \tilde{h}^{[\mathrm{TT}]}_{AB}$, which is non-trivial only on $\mathbb{T}^2$ and manifolds of higher genus, is subjected to the obstruction (239), p. 754, namely

$$\begin{aligned}\partial_u^p q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_2} - \partial_u^p q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_1} = \alpha^2 \sum_{k=0}^{p-1} (\alpha^2 r_2)^k \Big[& s\, \overset{[p-k]}{q}{}^{[\mathrm{TT}]}_{AB}\Big|_{r_1} \\ & + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) \partial_u^{p-1-k} h^{[\mathrm{TT}]}_{AB}\Big|_s \Big]_{r_1}^{r_2} \\ & + (\alpha^2 r_2)^p (q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_2} - q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_1})\,,\end{aligned} \tag{175}$$

which simplifies considerably when $\alpha = 0$.

<u>In the case $m \neq 0$</u>, the collection $\{\partial_u^p \overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]}_A\}_{p=0}^k$ of gauge fields can be used to match $\{\overset{[3,p]}{Q}{}^{[(\mathrm{CKV+H})^\perp]}\}_{p=1}^k$. The gauge fields $\partial_u^p \overset{(2)}{\xi}{}^{[\mathrm{H}]}_A$ can be used to achieve the matching of $\overset{[3,p]}{Q}{}^{[\mathrm{H}]}$. The requirement of continuity of $\partial_u^p \tilde{h}^{[\mathrm{TT}]}_{AB}$ is ensured by a suitable choice of

$$\overset{(p,0)}{\psi}_{2p+2}(r_2)\, \overset{[2p+2]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB} + \overset{(p,0)}{\psi}_{2p+1}(r_2)\, \overset{[2p+1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}\,. \tag{176}$$

v. $\partial_u^p \tilde{h}_{uA}$ for $1 \le p \le k$: In the case $m = 0$, the continuity of $\partial_u^p \tilde{h}_{uA}$ at $\tilde{\mathbf{S}}_2$ determines $\overset{[p+4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ and $\partial_u^{p+1} \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$, with additional obstructions coming from the kernel of the operator $\sum_{j=0}^p \overset{(p,j)}{\chi}_{p+4} P^j$. We provide an analysis of this kernel in Appendix C.1.

In the case $m \neq 0$, the continuity of $\{\partial_u^p \tilde{h}_{uA}\}_{p=1}^k$ at $\tilde{\mathbf{S}}_2$ is obtained by solving a system of equations for the collections $\{\overset{[p]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}\}_{p=5}^{2k+4}$ and $\{\partial_u^{p+1} \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A\}_{p=1}^k$.

2. Once the gauge fields and the fields $\overset{[p]}{\hat{\varphi}}_{AB}$ with $1 \leq p \leq k+4$ in the case $m = 0$, and with $1 \leq p \leq 2k+4$ in the case $m \neq 0$, have been determined, we construct the fields $\partial_u^p \tilde{h}_{\mu\nu}$ on $\widetilde{\mathscr{N}}_{[r_1,r_2)}$ by setting $\tilde{h}_{AB}$ according to (164) and using this to solve the transport equations of Section 3.3-3.7:
   i. $\partial_u^p \tilde{h}_{ur}$ for $0 \leq p \leq k$: We set $\partial_u^p \tilde{h}_{ur}|_{\widetilde{\mathscr{N}}} \equiv 0$, which guarantees both smoothness of $\tilde{h}_{ur}$ and the validity of the equations, for all i,

$$0 = \partial_u^i \delta\mathscr{E}_{rr}|_{\widetilde{\mathscr{N}}} \equiv -\partial_u^i \delta\mathscr{E}^u{}_r|_{\widetilde{\mathscr{N}}} \equiv \partial_u^i \delta\mathscr{E}^{uu}|_{\widetilde{\mathscr{N}}} \,. \tag{177}$$

   ii. $\partial_u^p \tilde{h}_{uA}$ for $0 \leq p \leq k$: Using the representation formulae (151), with all $h_{\mu\nu}$'s there replaced by $\tilde{h}_{\mu\nu}$'s. This guarantees that on $\widetilde{\mathscr{N}}_{[r_0,r_2)}$ we have

$$\partial_u^p \delta\mathscr{E}_{rA}|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} \equiv -\partial_u^p \delta\mathscr{E}^u{}_A|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} = 0 \,. \tag{178}$$

   It follows that

$$\partial_u^p \delta\mathscr{E}^A{}_B|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} = g^{AC} \partial_u^p \delta\mathscr{E}_{CB}|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} \,. \tag{179}$$

   The divergence identity

$$\begin{aligned} 0 &\equiv \nabla_\mu \delta\mathscr{E}^\mu{}_A \\ &= r^{-2}\partial_r(r^2 \delta\mathscr{E}^r{}_A) + \partial_u \delta\mathscr{E}^u{}_A + \mathring{D}_B \delta\mathscr{E}^B{}_A \,, \end{aligned} \tag{180}$$

   together with its u-derivatives, shows that we also have $\forall\, 0 \leq i \leq k-1$,

$$\left(r^{-2}\partial_r(r^2 \partial_u^i \delta\mathscr{E}^r{}_A) + \mathring{D}_B \partial_u^i \delta\mathscr{E}^B{}_A\right)\Big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} = 0 \,. \tag{181}$$

   iii. $\partial_u^p \tilde{h}_{uu}$ for $0 \leq p \leq k$: We impose $\partial_r \overset{[p]}{\chi}|_{\widetilde{\mathscr{N}}_{[r_1,r_2)}} = 0$ with the initial conditions $\overset{[p]}{\chi}|_{r_1} = \overset{[p]}{\chi}[\mathfrak{d}_{\tilde{\mathbf{S}}_1}]$, together with the value of

$\partial_u^{p-1}\tilde{h}_{uA}\big|_{\tilde{\mathscr{N}}_{[r_1,r_2)}}$ determined in (ii) above. This ensures

$$\partial_u^p\delta\mathscr{E}_{ru}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} - \frac{1}{2r}\mathring{D}^A\partial_u^p\delta\mathscr{E}_{rA}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} = 0\,. \tag{182}$$

Together with (178), Equation (182) ensures

$$\partial_u^p\delta\mathscr{E}_{ru}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} \equiv -\partial_u^p\delta\mathscr{E}^u{}_u\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} = 0\,. \tag{183}$$

iv. $\partial_u^p\tilde{h}_{AB}$ for $1\le p\le k$: We use the representation formulae (150), with all $h_{\mu\nu}$'s replaced by $\tilde{h}_{\mu\nu}$'s. This ensures that

$$\mathrm{TS}\left(\delta\partial_u^{p-1}\mathscr{E}_{AB}\right)\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} = 0\,. \tag{184}$$

The u differentiated divergence identity (123) with $\nu = r$ reads

$$\begin{aligned} 0 \equiv{}& \partial_u^p\delta\mathscr{E}^u{}_r + \frac{1}{r^2}\partial_r(r^2\delta\partial_u^{p-1}\mathscr{E}^r{}_r) \\ &+ \frac{1}{\sqrt{|\det\mathring{\gamma}|}}\partial_A(\sqrt{|\det\mathring{\gamma}|}\delta\partial_u^{p-1}\mathscr{E}^A{}_r) \\ &- \frac{1}{r}g^{AB}\delta\partial_u^{p-1}\mathscr{E}_{AB}\,, \end{aligned} \tag{185}$$

so that, in view of (178) and (183), we have now

$$\forall\ 0\le i\le k \qquad 0 = \frac{1}{r}g^{AB}\partial_u^i\delta\mathscr{E}_{AB}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}}\,. \tag{186}$$

Together with (184), it follows that

$$\forall\ 0\le i\le k-1 \qquad \partial_u^i\delta\mathscr{E}_{AB}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} = 0\,. \tag{187}$$

Equation (181) then gives, $\forall\ 0\le i\le k-1$,

$$\begin{aligned} 0 &= r^{-2}\partial_r(r^2\partial_u^i\delta\mathscr{E}^r{}_A)\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} \\ &= -r^{-2}\partial_r(r^2\partial_u^i\delta\mathscr{E}_{uA})\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}}\,, \end{aligned} \tag{188}$$

where we have used

$$\begin{aligned} \partial_u^i\delta\mathscr{E}^r{}_A\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} &= -g_{uu}\partial_u^i\delta\mathscr{E}_{rA}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} - \partial_u^i\delta\mathscr{E}_{uA}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}} \\ &= -\partial_u^i\delta\mathscr{E}_{uA}\big|_{\tilde{\mathscr{N}}_{[r_0,r_2)}}\,; \end{aligned}$$

note that the last equality is justified by (178). Continuity at r_1, where all the fields $\partial_u^i \mathscr{E}_{\mu\nu}$, $i \in \mathbb{N}$, vanish when the data there arise from a smooth solution of linearised Einstein equations, together with (188) implies that

$$\forall\ 0 \leq i \leq k-1 \quad \partial_u^i \delta\mathscr{E}^r{}_A\big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} = 0 = \partial_u^i \delta\mathscr{E}_{uA}\big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} . \tag{189}$$

Meanwhile, the divergence identity for the Einstein tensor with a free lower index u now reduces to $\forall\ 0 \leq i \leq k-1$,

$$0 \equiv \partial_u^i \nabla_\mu \delta\mathscr{E}^\mu{}_u\big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} = r^{-2}\partial_r(r^2 \partial_u^i \delta\mathscr{E}^r{}_u)\big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} . \tag{190}$$

Continuity and vanishing at r_1 together with (177) and (183) implies that $\forall\ 0 \leq i \leq k-1$

$$\begin{aligned} 0 = \partial_u^i \delta\mathscr{E}_{uu}\big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} &= -\partial_u^i \delta\mathscr{E}^r{}_u\big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} \\ &= \partial_u^i \delta\mathscr{E}^{rr}\big|_{\widetilde{\mathscr{N}}_{[r_0,r_2)}} . \end{aligned} \tag{191}$$

3. The construction above guarantees the continuity of $\tilde{h}_{uu}$, $\partial_u \tilde{h}_{AB}$, $\partial_u^p \tilde{h}_{uA}$ with $0 \leq p \leq k$, and $\partial_u^i \tilde{h}_{AB}^{[\mathrm{TT}]}$ with $2 \leq i \leq k$ at r_2. Continuity of the fields $\partial_r \partial_u^p \tilde{h}_{uA}$ and $\partial_u^p \tilde{h}_{uu}$ for $1 \leq p \leq k$ and $\partial_u^i \tilde{h}_{AB}^{[\mathrm{TT}^\perp]}$ for $2 \leq i \leq k$ at r_2 follows now by induction: The explicit form (129) of the equation $\delta\mathscr{E}_{uA} = 0$ together with the continuity of $\tilde{h}_{uu}$, $\tilde{h}_{uA}$ and $\partial_u \tilde{h}_{AB}$ at r_2 ensures the continuity of $\partial_r \partial_u \tilde{h}_{uA}$ at r_2. This guarantees continuity of $\overset{[1,1]}{Q}$ and $\overset{[4,1]}{Q}$, hence of $\overset{[3,2]}{Q}{}^{[\mathrm{CKV}]}$ in view of (116), namely

$$\int_{\mathbf{S}} \pi^A \overset{[3,k]}{Q}{}_A \, d\mu_{\hat{\gamma}} = \alpha^2 \overset{[1,k-1]}{Q}(\pi^A) + 2m \overset{[4,k-1]}{Q}(\pi^A) , \tag{192}$$

with $k = 2$. This, together with point 1.iv ensures continuity of $\overset{[3,2]}{Q}{}_A$ at r_2, which in turn renders $\partial_u^2 \tilde{h}_{AB}^{[\mathrm{TT}^\perp]}$ continuous there.
Meanwhile the explicit form (135) of $\delta\mathscr{E}_{uu} = 0$ together with smoothness at r_2 of $\tilde{h}_{uu}$, $\tilde{h}_{uA}$ and $\partial_u \tilde{h}_{AB}$, leads to continuity of $\partial_u \tilde{h}_{uu}$ at r_2.
Now, suppose that the continuity of the fields $\partial_u^p \tilde{h}_{uu}$, $\partial_r \partial_u^p \tilde{h}_{uA}$ and $\partial_u^p \tilde{h}_{AB}^{[\mathrm{TT}^\perp]}$ has been achieved up to $p = k-1$. It follows that we have $\partial_u^{k-2} \delta\mathscr{E}_{uA}\big|_{\widetilde{\mathscr{N}}} = 0$ and thus $\partial_r \overset{[1,k-1]}{Q}(\pi^A)\big|_{\widetilde{\mathscr{N}}} = 0 = \partial_r \overset{[4,k-1]}{Q}(\pi^A)\big|_{\widetilde{\mathscr{N}}}$

(compare (133)). Further, (192) implies that the $(\overset{[3,k]}{Q})^{[\mathrm{CKV+H}]}$-part of the radial charges on S^2 and $\mathbb{T}^2$ are continuous. Meanwhile, recall that continuity of the radial charge $(\overset{[3,k]}{Q})^{[(\mathrm{CKV+H})^\perp]}$ was ensured using the gauge field $\partial_u^{k-1}\overset{(2)}{\xi}_A$, while on higher-genus sections continuity of the charge $(\overset{[3,k]}{Q})^{[\mathrm{CKV+H}]} = (\overset{[3,k]}{Q})^{[\mathrm{H}]}$ is achieved by $\partial_u^k \overset{(2)}{\xi}{}_A^{[\mathrm{H}]}$ when $m \neq 0$ and is an obstruction whose continuity has to be assumed when $m = 0$. We have now the continuity of $(\overset{[3,k]}{Q}) = (\overset{[3,k]}{Q})^{[(\mathrm{CKV+H})^\perp]} + (\overset{[3,k]}{Q})^{[(\mathrm{CKV+H})]}$, which ensures the continuity of $\partial_u^k \tilde{h}_{AB}^{[\mathrm{TT}^\perp]}$ at r_2.
Next, by differentiation of (129) we obtain the explicit form of (189) with $i = k-1$

$$\begin{aligned} r^2\partial_r\left(\frac{\partial_u^k h_{uA}}{r^2}\right) = &-\frac{1}{r^2}\Big[\mathring{D}^B\mathring{D}_A\partial_u^{k-1}h_{uB} - \mathring{D}^B\mathring{D}_B\partial_u^{k-1}h_{uA} \\ &\qquad + \partial_u\mathring{D}^B\partial_u^{k-1}h_{AB}\Big] \\ &+\left(\varepsilon - r^2\alpha^2 - \frac{2m}{r}\right)\partial_r^2\partial_u^{k-1}h_{uA} \\ &+ (2\alpha^2 + \frac{4m}{r^3})\partial_u^{k-1}h_{uA} + \partial_r\mathring{D}_A\partial_u^{k-1}h_{uu}\,. \end{aligned}$$

This equation, together with the continuity of $\partial_u^{k-1}\tilde{h}_{uu}$, $\partial_u^{k-1}\tilde{h}_{uA}$ and $\partial_u^k\tilde{h}_{AB}$, ensures the continuity of $\partial_r\partial_u^k\tilde{h}_{uA}$ at r_2.
Finally, the explicit form of (191) with $i = k-1$, i.e.

$$\begin{aligned} 0 &= \partial_u^{k-1}\delta\mathscr{E}_{uu}\big|_{\mathscr{N}_{[r_1,r_2)}} \\ &= \frac{1}{r^2}\Big\{2\Big[\partial_u^k + \big(\alpha^2r^2 - \varepsilon + \frac{2m}{r}\big)\partial_r + \frac{3m}{r^2} - \frac{\varepsilon}{r}\Big]\mathring{D}^A h_{uA} \\ &\quad - \mathring{D}^A\mathring{D}_A\partial_u^{k-1}h_{uu} - (\alpha^2r^2 - \varepsilon + \frac{2m}{r})\big(\frac{\mathring{D}^A\mathring{D}^B\partial_u^{k-1}h_{AB}}{r^2}\big) \\ &\quad - 2r\partial_u^k h_{uu} - 2(\alpha^2r^2 - \varepsilon + \frac{2m}{r})\partial_r(r\partial_u^{k-1}h_{uu})\Big\}\,, \end{aligned}$$

together with smoothness at r_2 of $\partial_u^{k-1}\tilde{h}_{uu}$, $\partial_u^{k-1}\tilde{h}_{uA}$ and $\partial_u^{k-1}\tilde{h}_{AB}$, ensures the continuity of $\partial_u^k\tilde{h}_{uu}$ at r_2.

We now pass to a more detailed presentation of some of the arguments above.

4.2. Continuity at r_2

4.2.1. Gluing of $\delta\beta$. The sets of gauge functions $\partial_u^i \overset{(1)}{\xi}{}^u|_{\tilde{\mathbf{S}}_1}$ and $\partial_u^i \overset{(2)}{\xi}{}^u|_{\tilde{\mathbf{S}}_2}$ with $i \leq k+1$ allow us to transform $\partial_u^j \delta\tilde{\beta}$ for $j \leq k$ to zero on $\tilde{\mathbf{S}}_1$ and $\tilde{\mathbf{S}}_2$, and hence, by invoking the ur-component of the linearised Einstein equation (50), on the whole $\widetilde{\mathscr{N}}_{[r_1,r_2]}$. In what follows, we assume that this gauge choice has been made, and set $\partial_u^j \delta\tilde{\beta} = 0$ for $j \leq k$ everywhere.

Furthermore, to simplify notation we omit the "$|_{\tilde{\mathbf{S}}_j}$" on all gauge fields, with the understanding that all $\overset{(1)}{\xi}$ fields, and their u-derivatives, are evaluated on $\tilde{\mathbf{S}}_1$, while all $\overset{(2)}{\xi}$ fields, and their u-derivatives, are evaluated on $\tilde{\mathbf{S}}_2$, unless indicated otherwise.

4.2.2. Freezing part of the gauge. First, recall that the radial charge $\overset{[1]}{Q}$ is gauge invariant except in the case $m \neq 0$ on S^2. In this case, we can use the gauge field $(\overset{(2)}{\xi}{}^u)^{[=1]}$ for the matching of $\overset{[1]}{Q}(\pi_A)$ when the conformal Killing field π_A is such that $\mathring{D}^A \pi_A \neq 0$, i.e. a proper conformal Killing vector field. According to (63), this is achieved by choosing $(\overset{(2)}{\xi}{}^u)^{[=1]}$ so that

$$\int_{\mathbf{S}_2} \pi^A (r_2^4 \partial_r \check{h}_{uA}\big|_{r_2} - 6m\mathring{D}_A \overset{(2)}{\xi}{}^u) d\mu_{\mathring{\gamma}} = \int_{\mathbf{S}_1} \pi^A (r_1^4 \partial_r \check{h}_{uA}|_{r_1}) d\mu_{\mathring{\gamma}} \,. \tag{193}$$

However, for Killing vector fields the terms explicitly involving m integrate-out to zero, and we obtain an obstruction to gluing.

Next, we determine the gauge fields needed to ensure continuity of χ. For this, we evaluate the function χ of (80) at $\mathbf{S}_1$:

$$\chi[\mathfrak{d}_{\mathbf{S}_1}] = \Big(-\delta V + \frac{r}{2}\partial_r \big(r^2 \mathring{D}^A \delta U_A\big) + \frac{1}{2} r \mathring{D}^A \mathring{D}^B \check{h}_{AB} \Big)\Big|_{\mathbf{S}_1} \,, \tag{194}$$

and use the transformation law (82), i.e.

$$\chi \mapsto \chi - \frac{1}{2}(\Delta_{\mathring{\gamma}} + 2\varepsilon)\Delta_{\mathring{\gamma}} \xi^u + 3m\mathring{D}^B \xi_B \,, \tag{195}$$

to do the gluing. As already pointed out, and clearly seen from this equation, the integral average of χ over $\mathbf{S}$, i.e. $\overset{[2]}{Q}(\lambda = 1)$, is gauge-invariant regardless

of topology or of the value m. Now, <u>when $m=0$</u>, we can find a function $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ so that

$$-\frac{1}{2}\mathring{L}(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]} \equiv -\frac{1}{2}(\Delta_{\hat\gamma}+2\varepsilon)\Delta_{\hat\gamma}(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]} = \chi^{[(\ker \mathring{L})^\perp]}[\mathfrak{d}_{\mathbf{S}_1}] - \chi^{[(\ker \mathring{L})^\perp]}[\mathfrak{d}_{\mathbf{S}_2}]\,. \tag{196}$$

The smooth fields $(\overset{(2)}{\xi}{}^u)^{[\ker \mathring{L}]}$ are left arbitrary at this stage.

<u>When $m\neq 0$,</u> to ensure the desired gluing we use the field $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ determined in (196) together with the equation

$$(3m\mathring{D}_A\overset{(2)}{\xi}{}^A)^{[\ker \mathring{L}]} = \chi^{[\ker \mathring{L}]}[\mathfrak{d}_{\mathbf{S}_1}] - \chi^{[\ker \mathring{L}]}[\mathfrak{d}_{\mathbf{S}_2}]\,. \tag{197}$$

On S^2 this determines the proper-conformal-Killing-vectors part of $\overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$, but does not provide a useful equation for the remaining topologies as there are no proper conformal Killing vectors there.

Next, we determine the gauge fields $\partial_u^i\overset{(2)}{\xi}{}_A$ for $0\le i\le k$ as follows: We evaluate the radially-constant covector field $\overset{[3,i+1]}{Q}{}_A$ of (115) at $\mathbf{S}_1$:

$$\begin{aligned}\overset{[3,i+1]}{Q}{}_A[\mathfrak{d}_{\mathbf{S}_1}] =& \mathring{D}^B\Big(2r\partial_u^{i+1}\check{h}_{AB} - V\partial_r(\partial_u^i\check{h}_{AB}) \\ &- \frac{1}{r^2}\partial_r\big(r^4\,\mathrm{TS}[\mathring{D}_A\partial_u^i\check{h}_{uB}]\big) + (P-\varepsilon)\partial_u^i\check{h}_{AB}\Big)\Big|_{\mathbf{S}_1} \\ &+ \big(\alpha^2r^4\partial_r\partial_u^i\check{h}_{uA} + 2m(3\partial_u^i\check{h}_{uA} + r\partial_r\partial_u^i\check{h}_{uA})\big)\big|_{\mathbf{S}_1}\,.\end{aligned} \tag{198}$$

The idea now is to use the transformation law (117) to find gauge-vector fields such that

$$\begin{aligned}&2(\widehat{\mathrm{L}}\partial_u^i\overset{(2)}{\xi}{}^{[(\mathrm{CKV}+\mathrm{H})^\perp]})_A + 6m\partial_u^{i+1}\overset{(2)}{\xi}{}_A \\ &\quad= \underbrace{(\overset{[3,i+1]}{Q}{}_A)[\mathfrak{d}_{\mathbf{S}_1}] - (\overset{[3,i+1]}{Q}{}_A)[\mathfrak{d}_{\mathbf{S}_2}]}_{=:\Delta\overset{[3,i+1]}{Q}{}_A}\,.\end{aligned} \tag{199}$$

We start by noting that the image of $\widehat{\mathrm{L}}$ is orthogonal to CKV + H; (cf. Proposition C.6, p. 782). (The spaces H and CKV can be viewed as

being orthogonal to each other, in that H a) is trivial on S^2; b) can be ignored on $\mathbb{T}^2$ being a subset of CKV; c) is non-trivial on the remaining topologies but CKV is trivial there.)

Note next that for $\pi^A \in$ CKV we have

$$\begin{aligned}(200)\qquad &\int_{\mathbf{S}} \pi^A \overset{[3,i]}{Q}{}_A \, d\mu_{\hat{\gamma}} \\ &= \int_{\mathbf{S}} \pi^A (\alpha^2 r^4 \partial_r \partial_u^{i-1} \check{h}_{uA} + 2m(3\partial_u^{i-1}\check{h}_{uA} + r\partial_r\partial_u^{i-1}\check{h}_{uA}))\, d\mu_{\hat{\gamma}} \\ &= \alpha^2 \overset{[1,i-1]}{Q}(\pi^A) + 2m \overset{[4,i-1]}{Q}(\pi^A)\,.\end{aligned}$$

Thus the continuity of $\overset{[3,i]}{Q}{}^{[\mathrm{CKV}]}_A$ will be guaranteed by the continuity of $\overset{[1,i-1]}{Q}$ and $\overset{[4,i-1]}{Q}$, with the latter two being addressed in point 3, p. 737. Hence we ignore for the moment the condition $\Delta \overset{[3,i]}{Q}{}^{[\mathrm{CKV}]}_A = 0$ and consider the system of equations

$$(201)\qquad 2(\widehat{\mathrm{L}}\overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]})_A + 6m\partial_u \overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]}_A = \big(\Delta \overset{[3,1]}{Q}{}_A\big)^{[(\mathrm{CKV+H})^\perp]}\,,$$

$$(202)\qquad 6m\partial_u \overset{(2)}{\xi}{}^{[\mathrm{H}]}_A = \big(\Delta \overset{[3,1]}{Q}{}_A\big)^{[\mathrm{H}]}\,,$$

$$\vdots$$

$$(203)\qquad 2(\widehat{\mathrm{L}}\partial_u^{k-2}\overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]})_A + 6m\partial_u^{k-1}\overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]}_A = \big(\Delta \overset{[3,k-1]}{Q}{}_A\big)^{[(\mathrm{CKV+H})^\perp]}\,,$$

$$(204)\qquad 6m\partial_u^{k-1}\overset{(2)}{\xi}{}^{[\mathrm{H}]}_A = \big(\Delta \overset{[3,k-1]}{Q}{}_A\big)^{[\mathrm{H}]}\,,$$

$$(205)\qquad 2(\widehat{\mathrm{L}}\partial_u^{k-1}\overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]})_A + 6m\partial_u^{k}\overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]}_A = \big(\Delta \overset{[3,k]}{Q}{}_A\big)^{[(\mathrm{CKV+H})^\perp]}\,,$$

$$(206)\qquad 6m\partial_u^{k}\overset{(2)}{\xi}{}^{[\mathrm{H}]}_A = \big(\Delta \overset{[3,k]}{Q}{}_A\big)^{[\mathrm{H}]}\,.$$

<u>In the case $m \neq 0$</u> we set, e.g.,

$$(207)\qquad \partial_u^k \overset{(2)}{\xi}{}^{[(\mathrm{CKV+H})^\perp]}_A = 0\,,$$

and we can solve (205)-(206) to guarantee continuity of $\overset{[3,k]}{Q}{}^{[\mathrm{CKV}^\perp]}_A$ on $\mathbb{T}^2$ and

S^2, and $\overset{[3,k]}{Q}{}^{[\mathrm{H}]}_A$ on other topologies. The solution of (205) can then be inserted into (203) to obtain a solution of (203)-(204). Continuing in this way, after a finite number of steps we obtain a solution of the system (201)-(206).

Clearly the argument breaks down <u>when $m = 0$</u>, in which case we can still solve (199) for $\partial_u^i \overset{(2)}{\xi}{}^{[(\mathrm{CKV}+\mathrm{H})^\perp]}_A$, but the equations

$$\big(\Delta \overset{[3,i]}{Q}{}_A\big)^{[\mathrm{H}]} = 0\,, \quad i = 0, \dots, k\,,$$

provide obstructions to gluing.

4.2.3. Continuity of $\tilde{h}_{uu}$. It follows from the pointwise radial conservation of the function χ defined in (80) that the gluing of $\tilde{h}_{uu}$ requires

$$\chi[\mathfrak{d}_{\tilde{\mathbf{S}}_1}] = \chi[\mathfrak{d}_{\tilde{\mathbf{S}}_2}]\,. \tag{208}$$

This is achieved by the condition $\overset{[2]}{Q}[\mathfrak{d}_{\mathbf{S}_1}] = \overset{[2]}{Q}[\mathfrak{d}_{\mathbf{S}_2}]$ together with formulae (196)-(197) for the projected gauge fields $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ and $\overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$.

4.2.4. Continuity of $\partial_r \tilde{h}_{uA}$. Taking into account the allowed gauge perturbations to Bondi data, the gluing of $\partial_r \tilde{h}_{uA}$ requires $\tilde{h}_{AB}$ to satisfy on $\widetilde{\mathscr{N}}_{(r_1,r_2)}$,

$$\begin{aligned} r_2^4 \partial_r \check{h}_{uA}|_{\tilde{\mathbf{S}}_2} = {} & 2r_2 L_1(\overset{(2)}{\xi}{}^u)_A + 2r_2^2 \mathring{D}^B C(\overset{(2)}{\zeta})_{AB} - 6m \mathring{D}_A \overset{(2)}{\xi}{}^u \\ & + \Phi_A(x^C) + \mathring{D}^B h_{AB}|_{\tilde{\mathbf{S}}_2} - 2\int_{r_1}^{r_2} \hat{\kappa}_1(s) \mathring{D}^B \tilde{h}_{AB}\, ds\,. \end{aligned} \tag{209}$$

We have

$$\mathring{D}_A(\overset{(2)}{\xi}{}^u)^{[1]} = 0\,, \qquad L_1\big((\overset{(2)}{\xi}{}^u)^{[\ker \mathring{L}]}\big) = 0\,, \qquad C(\overset{(2)}{\xi}{}^{[\mathrm{CKV}]})_{AB} = 0\,,$$

so that the gauge-part of the right-hand side of (209) involving $\overset{(2)}{\xi}{}^u$, except for the term explicitly involving m, depends only on $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ and has already been determined in terms of the given data by (196). When $m \neq 0$ and $\mathbf{S} \approx S^2$ the term $-6m\mathring{D}_A \overset{(2)}{\xi}{}^u$ depends only on $(\overset{(2)}{\xi}{}^u)^{[=1]}$, which is already

known from (193). When $m \neq 0$ and $\mathbf{S} \approx S^2$ the remaining part, which contributes to $-6m\mathring{D}_A \overset{[2]}{\xi}{}^u$, is already known from (193). For the remaining topologies the kernel of $\mathring{L}$ consists of constants, which do not contribute to the right-hand side. Thus in all cases, the terms in (209) involving $\overset{(2)}{\xi}{}^u$ are either vanishing or already determined.

To clarify the freedom left, let us rewrite (209) as an equation for $\overset{[1]}{\hat{\varphi}}_{AB} \equiv \langle \hat{\kappa}_1, w_{AB} \rangle$, where w_{AB} is as in (164):

$$\mathring{D}^B \overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB} = \tilde{\Phi}_A(x^C) + \mathring{D}^B \Big[r_2^2 C(\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]})_{AB} - 2\int_{r_1}^{r_2} \hat{\kappa}_1(s)(1-\phi)s^2 C(\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}(s))_{AB}\, ds \Big], \tag{210}$$

where the already known fields such as $\check{h}_{uA}|_{\tilde{\mathbf{S}}_2}$, v_{AB} and $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$, as well as the gauge fields $\overset{(1)}{\xi}{}^A$ and $\overset{(1)}{\xi}{}^u$ have been collected into the term $\tilde{\Phi}_A$.

Now, the divergence operator on traceless symmetric two-tensors in two dimensions is elliptic; it has a cokernel spanned on conformal Killing vectors; on S^2 it has no kernel (see Appendix C.3). It follows that (210) determines a unique tensor field $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ on S^2 provided that the source term $\tilde{\Phi}$ is L^2-orthogonal to the cokernel. This orthogonality is guaranteed by the condition $\overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_1}] = \overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_2}]$ and either the gauge invariance of $\overset{[1]}{Q}$ in the case $m = 0$, or by a suitable choice of the gauge field $(\overset{(2)}{\xi}{}^u)^{[=1]}$ if $m \neq 0$. In other words, if the radial charge $\overset{[1]}{Q}$ of the linearised field on $\mathscr{N}|_{(r_0,r_1]}$ coincides with that of the linearised field on $\mathscr{N}|_{[r_2,r_3)}$, the field $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ satisfying (210) exists, and is uniquely determined in terms of the given data and the gauge field $\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A$.

By a similar analysis for the remaining topologies, (210) determines $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ uniquely in terms of the given data and the gauge field $\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A$ provided that the radial charges $\overset{[1]}{Q}$ at $r = r_1$ and $r = r_2$ coincide.

4.2.5. Continuity of $\tilde{h}_{uA}$. Taking into account the allowed gauge perturbations of Bondi data, it follows from (34) and (67) that the continuity

of $\tilde{h}_{uA}$ at r_2 can be achieved by choosing $\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ so that

$$
\begin{aligned}
(211)\qquad & h_{uA}|_{\tilde{\mathbf{S}}_2} + L_1(\overset{(2)}{\xi}{}^u)_A + r_2^2\big[\partial_u \overset{(2)}{\xi}{}_A + (\alpha^2 + \frac{2m}{r_2^3})\mathring{D}_A \overset{(2)}{\xi}{}^u\big] \\
& = X_A(x^C) + \frac{1}{3}\mathring{D}^B \int_{r_1}^{r_2} \tilde{h}_{AB}\left(\frac{2\hat{\kappa}_1(s)}{r_2} + \hat{\kappa}_4(s) r_2^2\right) ds\,,
\end{aligned}
$$

where X_A depends only on data at r_1. More explicitly:

$$
\begin{aligned}
(212)\qquad & \frac{r_2^2}{3}\mathring{D}^B \overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB} \equiv \frac{r_2^2}{3}\mathring{D}^B \langle \hat{\kappa}_4, w_{AB}\rangle \\
& = r_2^2 \partial_u \overset{(2)}{\xi}{}_A - \tilde{X}_A(x^C) - \frac{2}{3r_2}\mathring{D}^B \overset{[1]}{\hat{\varphi}}_{AB} + (\alpha^2 + \frac{2m}{r^3}) r_2^2 \mathring{D}_A \overset{(2)}{\xi}{}^u \\
& \quad - \frac{2}{3}\int_{r_1}^{r_2}\left(\frac{2\hat{\kappa}_1(s)}{r_2} + \hat{\kappa}_4(s) r_2^2\right)(1-\phi) s^2 \mathring{D}^B C(\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]})_{AB}\, ds \\
& = r_2^2 \partial_u \overset{(2)}{\xi}{}_A - \tilde{X}_A(x^C) - \frac{2r_2}{3}\mathring{D}^B C(\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]})_{AB} \\
& \quad - \frac{2}{3r_2}\tilde{\Phi}_A + (\alpha^2 + \frac{2m}{r^3}) r_2^2 \mathring{D}_A (\overset{(2)}{\xi}{}^u)^{[\ker \mathring{L}]} \\
& \quad - \frac{2r_2^2}{3}\int_{r_1}^{r_2} \hat{\kappa}_4(s)(1-\phi) s^2 \mathring{D}^B C(\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]})_{AB}\, ds\,,
\end{aligned}
$$

where once again the already known fields such as

$$
h_{uA}|_{\tilde{\mathbf{S}}_2},\ v_{AB} \text{ and } (\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}
$$

as well as the gauge fields $\overset{(1)}{\xi}{}^A$ and $\overset{(1)}{\xi}{}^u$ have been collected into the term $\tilde{X}_A$. Since $\ker(C) = \mathrm{im}(\mathring{\mathrm{div}}_{(2)})^\perp$, we can use the freedom in choosing the smooth fields

$$
(213)\qquad \big(\partial_u \overset{(2)}{\xi}{}_A + (\alpha^2 + \frac{2m}{r^3})\mathring{D}_A \overset{(2)}{\xi}{}^u\big)^{[\mathrm{CKV}]}
$$

to arrange that the right-hand side of (212) lies in the image of $\mathring{\mathrm{div}}_{(2)}$. (Note that the first term in (213) has not been determined so far, while the remaining ones have only been determined if $m \neq 0$.) It follows that (212) can be solved uniquely for both $\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ and $\partial_u \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$ in terms of $\partial_u \overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A$

and $\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A$ when $\alpha = 0 = m$. For $\alpha, m \neq 0$ there remains some freedom in the gauge field $(\overset{(2)}{\xi}{}^u)^{[\ker \mathring{L}]}$, made clear by (213). On sections with higher genus, it follows from the surjectivity of $\mathring{\mathrm{div}}_{(2)}$ (Lemma C.4, Appendix C.3) that (212) determines $\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ uniquely in terms of $\partial_u \overset{(2)}{\xi}{}_A$.

A conserved, gauge dependent, radial charge involving h_{uA} and $\partial_r h_{uA}$ is derived in Appendix E.1.

4.2.6. Continuity of $\partial_u \tilde{h}_{AB}$.

4.2.6.1. <u>*The case $m = 0$*</u>. It follows from the pointwise radial conservation of $\overset{[3,1]}{Q}{}_A$ that the gluing of $\partial_u \tilde{h}^{[\mathrm{TT}^\perp]}_{AB}$ requires

$$\overset{[3,1]}{Q}{}_A[\mathfrak{d}_{\tilde{\mathbf{S}}_1}] = \overset{[3,1]}{Q}{}_A[\mathfrak{d}_{\tilde{\mathbf{S}}_2}]\,. \tag{214}$$

This is achieved on S^2 by the condition $\overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_1}] = \overset{[1]}{Q}[\mathfrak{d}_{\mathbf{S}_2}]$ together with the expressions (199) with $i = 0$ for the gauge field $\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A$.

For the remaining topologies we use (100) to obtain an equation for $q^{[\mathrm{TT}]}_{AB}$:

$$\partial_r q^{[\mathrm{TT}]}_{AB} = \partial_r \Big[r\partial_u \check{h}^{[\mathrm{TT}]}_{AB} - \frac{1}{2} V \partial_r \check{h}^{[\mathrm{TT}]}_{AB} - \frac{1}{2r} V \check{h}^{[\mathrm{TT}]}_{AB} \Big] = \frac{\alpha^2}{r} h^{[\mathrm{TT}]}_{AB}\,. \tag{215}$$

Integrating, we obtain

$$\begin{aligned} q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_2} - q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_1} &= \alpha^2 \int_{r_1}^{r_2} \hat{\kappa}_1(s) \tilde{h}^{[\mathrm{TT}]}_{AB}\, ds \\ &= \alpha^2 \overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB} + \alpha^2 \int_{r_1}^{r_2} \hat{\kappa}_1(s) v^{[\mathrm{TT}]}_{AB}\, ds\,. \end{aligned}$$

This provides an equation for $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$ when $\alpha \neq 0$:

$$\alpha^2 \overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB} = q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_2} - q^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_1} - \alpha^2 \int_{r_1}^{r_2} \hat{\kappa}_1(s) v^{[\mathrm{TT}]}_{AB}\, ds\,. \tag{216}$$

When $\alpha = 0$, $\partial_u h^{[\mathrm{TT}]}_{AB}$ is part of the radially conserved charge $q^{[\mathrm{TT}]}_{AB}$ of (100). In this case, the continuity of $\partial_u \tilde{h}^{[\mathrm{TT}]}_{AB}$ at r_2 imposes the radial

conservation law

$$q^{[\mathrm{TT}]}_{AB}[\mathfrak{d}_{\mathbf{S}_1}] = q^{[\mathrm{TT}]}_{AB}[\mathfrak{d}_{\mathbf{S}_2}]\,. \tag{217}$$

4.2.6.2. The case $m \neq 0$. Taking into account the allowed gauge perturbations of Bondi data, it follows from (94) that we need to satisfy the equation

$$\begin{aligned}
\partial_u h_{AB}|_{\tilde{\mathbf{S}}_2} = {} & \overset{(1)}{\tilde{\Psi}}_{AB}(r_2, x^A) - 2r_2^2 C(\partial_u \overset{(2)}{\zeta})_{AB} \\
& + r_2\left(\varepsilon - \alpha^2 r_2^2 - \frac{2m}{r_2}\right) C(\overset{(2)}{\xi})_{AB} \\
& + (\alpha^2 r_2 + \frac{1}{3r_2}P)\overset{[1]}{\hat{\varphi}}_{AB} - (mr_2 - \frac{2r^2}{3}P)\overset{[4]}{\hat{\varphi}}_{AB} \\
& + 2\int_{r_1}^{r_2} (\alpha^2 r_2 + \frac{1}{3r_2}P)(1-\phi)\kappa_1(s)s^2 C(\overset{(2)}{\xi})_{AB} \\
& - 2\int_{r_1}^{r_2} (mr_2 - \frac{2r^2}{3}P)\kappa_4(s)s^2 C(\overset{(2)}{\xi})_{AB}\,.
\end{aligned}$$

Since $C(\xi)^{[\mathrm{TT}]} = 0$ for any vector field ξ, and similarly the image of P is orthogonal to TT, continuity of $\partial_u \tilde{h}^{[\mathrm{TT}]}_{AB}$ at $\tilde{\mathbf{S}}_2$ requires $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$ and $\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$ to satisfy

$$\partial_u h^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_2} = \overset{(1)}{\tilde{\Psi}}{}^{[\mathrm{TT}]}{}_{AB}(r_2, x^A) + \alpha^2 r_2 \overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB} - mr_2 \overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}\,, \tag{218}$$

which can be achieved by setting, for example, $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB} = 0$ and solving (218) for $\overset{[4]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$.

4.3. Higher derivatives

Recall from Section 3.7 that the terms $\overset{(i,j)}{\chi}(s,r)$ are linear combinations of $\hat{\kappa}_j(s)$'s with $1 \le j \le i_{[m]}$, $j \neq 2,3$, with $i_{[m]} = i+4$ when $m = 0$ and $i_{[m]} = 2i+4$ when $m \neq 0$, where $i_{[m]}$ is not necessarily optimal unless $i = 0$; see Appendix B. We shall henceforth write them as

$$\overset{(i,j)}{\chi}(s,r) = \sum_{\ell=1}^{i_{[m]}} \overset{(i,j)}{\chi}_{\ell}(r)\,\hat{\kappa}_\ell(s)\,, \quad \text{with } \overset{(i,j)}{\chi}_2 = 0 = \overset{(i,j)}{\chi}_3\,. \tag{219}$$

Similarly we write, for $i \geq 1$,

$$\overset{(i,j)}{\psi}(s,r) = \sum_{\ell=1}^{i_{[m]}-1} \overset{(i,j)}{\psi}_{\ell}(r)\,\hat{\kappa}_\ell(s)\,, \quad \text{with } \overset{(i,j)}{\psi}_2 = 0 = \overset{(i,j)}{\psi}_3\,, \tag{220}$$

where again the upper bound $i_{[m]}-1$ is not necessarily optimal unless $i=1$.

4.3.1. Continuity of $\partial_u^i \tilde{h}_{uA}$. While this is not needed for the current argument, we note that a conserved, gauge dependent, radial charge involving $\partial_u h_{uA}$ is derived in Appendix E.2.

To continue, let k be the order at which we want to perform the gluing, i.e. the number of u-derivatives of $h_{\mu\nu}$ which we want to be continuous, and let $1 \leq p \leq k$.

4.3.1.1. The case $m=0$. After performing a gauge transformation, Equation (151) at order $i=p$ together with (41) provides a gluing equation of the form,

$$\begin{aligned}
\partial_u^p \check{h}_{uA}|_{\check{\mathbf{S}}_2} = &-\frac{1}{2r_2^2} L_1(\mathring{D}^C \partial_u^{p-1} \overset{(2)}{\xi}_C) - \Big(\partial_u^{p+1} \overset{(2)}{\xi}_A + \frac{\alpha^2}{2} \mathring{D}_A \mathring{D}^C \partial_u^{p-1} \overset{(2)}{\xi}_C\Big) \\
&+ \overset{(p)}{X}_A + \sum_{0\leq j+\ell\leq p, \ell\neq p} \overset{(p,j,\ell)}{\chi}(r) \partial_r^j \mathring{D}^B P^\ell h_{AB} \\
&+ \sum_{0\leq j+\ell\leq p, \ell\neq p} \overset{(p,j,\ell)}{\chi}(r) \partial_r^j \mathring{D}^B P^\ell \Big(2r^2 C(\overset{(2)}{\zeta})_{AB}\Big) \\
&\quad + \sum_{j=0}^{p} \mathring{D}^B \int_{r_1}^{r_2} \overset{(p,j)}{\chi}(s,r_2) P^j \tilde{h}_{AB}(s)\, ds\,,
\end{aligned}$$

which equals

$$\begin{cases}
-\partial_u^{p+1} \overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A + \overset{(p)}{\tilde{X}}_A + \sum_{j=0}^{p} \sum_{\ell=1}^{p+4} \mathring{D}^B \int_{r_1}^{r_2} \overset{(p,j)}{\chi}_\ell(r_2) \hat{\kappa}_\ell(s) P^j w_{AB}(s)\, ds\,; \\
-\partial_u^{p+1} \overset{(2)}{\xi}_A + \overset{(p)}{\tilde{X}}_A + \sum_{j=0}^{p} \sum_{\ell=1}^{p+4} \mathring{D}^B \int_{r_1}^{r_2} \overset{(p,j)}{\chi}_\ell(r_2) \hat{\kappa}_\ell(s) P^j w_{AB}(s)\, ds,
\end{cases}$$

with the first case arising when $p \le k-2$ and the second for $p \in \{k-1, k\}$, and which we rewrite as

$$\begin{cases} -\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]} + \overset{(p)}{\tilde{X}}{}_A + \sum_{j=0}^{p} \sum_{\ell=1}^{p+4} \overset{(p,j)}{\chi}{}_\ell(r_2) \mathring{D}^B P^j \overset{[\ell]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]}, & p \le k-2; \\ -\partial_u^{p+1} \overset{(2)}{\xi}{}_A + \overset{(p)}{\tilde{X}}{}_A + \sum_{j=0}^{p} \sum_{\ell=1}^{p+4} \overset{(p,j)}{\chi}{}_\ell(r_2) \mathring{D}^B P^j \overset{[\ell]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]}, & p = k-1, k. \end{cases}$$

Here we used the fact that $P|_{\mathrm{TT}} = 0$, that the fields $\partial_u^\ell \overset{(2)}{\xi}{}_C^{[\mathrm{CKV}^\perp]}$ with $\ell \le k-1$ are already known from Section 4.2.2, and we included them, together with all other already known fields, in $\overset{(p)}{\tilde{X}}{}_A = \overset{(p)}{\tilde{X}}{}_A(r, x^A)$. Recall that $\overset{[1]}{\hat{\varphi}}{}_{AB}$ has been determined in Section 4.2.4, $\overset{[4]}{\hat{\varphi}}{}_{AB}$ in (212), and we further set

$$\overset{[2]}{\hat{\varphi}}{}_{AB} = \overset{[3]}{\hat{\varphi}}{}_{AB} = 0\,.$$

For the sake of induction, suppose that the fields $\overset{[\ell]}{\hat{\varphi}}{}_{AB}$ with $4 \le \ell \le p+3$ are known. Together with $\overset{(p)}{\tilde{X}}{}_A$ and $\partial_u^p \check{h}_{uA}|_{\tilde{\mathbf{S}}_2}$ we collect them into a term $\overset{(p)}{\hat{\tilde{X}}}{}_A$, so that the requirement that $\partial_u^p \tilde{h}_{uA}$ be continuous at r_2 results in an equation of the form

(221)

$$\mathring{D}^B \Big(\sum_{j=0}^{p} \overset{(p,j)}{\chi}{}_{p+4}(r_2) P^j \overset{[p+4]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]} \Big) = \begin{cases} \partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]} - \overset{(p)}{\hat{\tilde{X}}}{}_A, & p \le k-2; \\ \partial_u^{p+1} \overset{(2)}{\xi}{}_A - \overset{(p)}{\hat{\tilde{X}}}{}_A, & p = k-1, k. \end{cases}$$

Now, the operator at the left-hand side of this equation, namely

$$\overset{(p)}{\chi} := \mathring{\mathrm{div}}_{(2)} \circ \sum_{j=0}^{p} \overset{(p,j)}{\chi}{}_{p+4}(r_2) P^j \overset{[p+4]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]}\,, \tag{222}$$

is elliptic, of order $2p+1$, and has a non-trivial cokernel on S^2 and $\mathbb{T}^2$. (For example, the cokernel on S^2 is the space of spherical harmonic vectors with index $1 \le \ell \le p+1$, see Appendix C.5.2; it is 30-dimensional when $p = 2$

and 48-dimensional when $p = 3$.) Recall that the fields $\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]}$, $i \geq 2$, have not been determined so far. We can use this freedom to arrange that $\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]} - \overset{(p)}{\hat{\tilde{X}}}_A$ is in the image of $\mathring{\mathrm{div}}_{(2)}$. For all $p \leq k$ we can therefore find a unique $\overset{[p+4]}{\hat{\varphi}}{}_{AB}^{[(\ker \overset{(p)}{\chi})^\perp]}$ and $\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]}$ satisfying

$$\overset{(p)}{\chi}\big(\overset{[p+4]}{\hat{\varphi}}{}^{[(\ker \overset{(p)}{\chi})^\perp]}\big) = \partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]} - \overset{(p)}{\hat{\tilde{X}}}{}^{[\mathrm{im}\overset{(p)}{\chi}]}. \tag{223}$$

For $p \leq k-1$ the $(\mathrm{im}\overset{(p)}{\chi})^\perp$-part of $\overset{(p)}{\hat{\tilde{X}}}_A$ constitutes thus a further obstruction to the solvability of (221), as it is not clear whether or not the right-hand side is orthogonal to the cokernel.

However, if $p = k$ or $k+1$, since the fields $\partial_u^k \overset{(2)}{\xi}{}_A$ and $\partial_u^{k+1} \overset{(2)}{\xi}{}_A$ are unconstrained so far, we can use (221) to choose these fields so that continuity of $\partial_u^{k-1}\tilde{h}_{uA}$ and $\partial_u^k \tilde{h}_{uA}$ at r_2 holds:

$$(\mathrm{im}\overset{(p)}{\chi})^\perp \ni \partial_u^{p+1} \overset{(2)}{\xi}{}_A := \overset{(p)}{\hat{\tilde{X}}}{}_A^{[(\mathrm{im}\overset{(p)}{\chi})^\perp]}. \tag{224}$$

4.3.1.2. The case $m \neq 0$. After performing a gauge transformation, Equation (151) at order $i = p$ together with (41) provides a gluing equation of the form,

$$\begin{aligned}
\partial_u^p \check{h}_{uA}|_{\tilde{\mathbf{S}}_2} &= -\frac{1}{2r_2^2} L_1(\mathring{D}^C \partial_u^{p-1} \overset{(2)}{\xi}{}_C)_A - \partial_u^{p+1} \overset{(2)}{\xi}{}_A \\
&\quad - \Big(\frac{\alpha^2}{2} + \frac{m}{r_2^3}\Big) \mathring{D}_A \mathring{D}^C \partial_u^{p-1} \overset{(2)}{\xi}{}_C + \overset{(p)}{X}_A \\
&\quad + \sum_{0 \leq j+\ell \leq p, \ell \neq p} \overset{(p,j,\ell)}{\chi}(r_2) \partial_r^j \mathring{D}^B P^\ell h_{AB} \\
&\quad + \sum_{0 \leq j+\ell \leq p, \ell \neq p} \overset{(p,j,\ell)}{\chi}(r_2) \partial_r^j \mathring{D}^B P^\ell \Big(2r_2^2 C(\overset{(2)}{\zeta})_{AB}\Big) \\
&\quad + \sum_{j=0}^{p} \mathring{D}^B \int_{r_1}^{r_2} \overset{(p,j)}{\chi}(s, r_2) P^j \tilde{h}_{AB}(s)\, ds \\
&= -\partial_u^{p+1} \overset{(2)}{\xi}{}_A + \overset{(p)}{\tilde{X}}_A
\end{aligned} \tag{225}$$

$$
\begin{aligned}
&+\sum_{j=0}^{p}\sum_{\ell=1}^{2p+4}\mathring{D}^B\int_{r_1}^{r_2}\overset{(p,j)}{\chi}_\ell(r_2)\hat\kappa_\ell(s)P^j w_{AB}(s)\,ds\\
&=-\partial_u^{p+1}\overset{(2)}{\xi}_A+\overset{(p)}{\tilde X}_A+\sum_{\ell=1}^{2p+4}\underbrace{\sum_{j=0}^{p}\overset{(p,j)}{\chi}_\ell(r_2)\mathring{D}^BP^j}_{=:L_{p,\ell}}\overset{[\ell]}{\hat\varphi}_{AB}\,,
\end{aligned}
$$

where we have collected the fields $\partial_u^j\overset{(2)}{\xi}_A$ for $j\le p$, together with all other already known fields, into $\overset{(p)}{\tilde X}_A$.

For $k\ge 1$ we let

$$
\Phi_k:=\begin{pmatrix}\overset{[5]}{\hat\varphi}\\ \vdots\\ \overset{[4+k]}{\hat\varphi}\end{pmatrix},\qquad \Psi_k:=\begin{pmatrix}\overset{[5+k]}{\hat\varphi}\\ \vdots\\ \overset{[4+2k]}{\hat\varphi}\end{pmatrix},
$$

and

$$
X_k:=\begin{pmatrix}\partial_u\check h_{uA}|_{\tilde{\mathbf{S}}_2}+\partial_u^2\overset{(2)}{\xi}_A-\overset{(1)}{\tilde X}_A\\ \vdots\\ \partial_u^k\check h_{uA}|_{\tilde{\mathbf{S}}_2}+\partial_u^{k+1}\overset{(2)}{\xi}_A-\overset{(k)}{\tilde X}_A\end{pmatrix}.
$$

The system of equations (225) with $1\le p\le k$ takes the form

$$
\mathring{\mathrm{div}}_{(2)}\circ L_k\Phi_k+\mathring{\mathrm{div}}_{(2)}\circ M_k\Psi_k=X_k\,, \tag{226}
$$

where

$$
L_k=\begin{pmatrix}L_{1,5}&\dots&L_{1,k+4}\\ \vdots&\ddots&\vdots\\ L_{k,5}&\dots&L_{k,k+4}\end{pmatrix},\quad M_k=\begin{pmatrix}L_{1,k+5}&\dots&L_{1,2k+4}\\ \vdots&\ddots&\vdots\\ L_{k,k+5}&\dots&L_{k,2k+4}\end{pmatrix}. \tag{227}
$$

Recall that the fields $\partial_u^i\overset{(2)}{\xi}{}^{[\mathrm{CKV}]}_A$, $i\ge 2$, have not been determined so far. We can use this freedom to arrange that $\partial_u\check h_{uA}|_{\tilde{\mathbf{S}}_2}+\partial_u^2\overset{(2)}{\xi}_A-\overset{(1)}{\tilde X}_A$ is in the image

of $\mathring{\mathrm{div}}_{(2)}$. Inserting the result into $\overset{(2)}{\tilde{X}}_A$ we can arrange that

$$\partial_u^2 \check{h}_{uA}|_{\tilde{\mathbf{S}}_2} + \partial_u^3 \overset{(2)}{\xi}_A - \overset{(2)}{\tilde{X}}_A$$

also lies in the image of $\mathring{\mathrm{div}}_{(2)}$. Continuing in this way we obtain that all the entries in X_k are in the image of $\mathring{\mathrm{div}}_{(2)}$.

We continue by noting that the operator L_k is elliptic in the sense of Agmon, Douglis and Nirenberg (cf., e.g., [18], in particular see the estimates of Theorem C there). Indeed, in each row of L_k the operators $L_{i,j}$ are of order less than $2i$, except for the operator $L_{i,i}$ lying on the diagonal, which is elliptic and precisely of order $2i$. This shows that the Agmon-Douglis-Nirenberg condition on the orders $s_i + t_j$ of $L_{i,j}$ holds by setting $s_i = 2i$ and $t_j = 0$; ellipticity readily follows.

It is instructive to consider explicitly the case $k = 2$:

$$L_2 = \begin{pmatrix} \frac{1}{4}(P+2\varepsilon) & -\frac{3m}{2} \\ \frac{15m\alpha^2}{8} - \frac{3m}{8r_2^2}(P+2\varepsilon) & \frac{9m^2}{4r_2^2} + \frac{1}{10}(P+2\varepsilon)(P+5\varepsilon) \end{pmatrix}, \tag{228}$$

$$M_2 = \begin{pmatrix} 0 & 0 \\ -\frac{7m}{4}(P + \frac{11\varepsilon}{3}) & \frac{75m^2}{8} \end{pmatrix},$$

so that (226) can be rewritten as

$$\begin{pmatrix} L_{1,5}(P) & -a \\ L_{2,5}(P) & L_{2,6}(P) \end{pmatrix} \begin{pmatrix} \overset{[5]}{\hat{\varphi}} \\ \overset{[6]}{\hat{\varphi}} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ L_{2,7}(P) & c \end{pmatrix} \begin{pmatrix} \overset{[7]}{\hat{\varphi}} \\ \overset{[8]}{\hat{\varphi}} \end{pmatrix} = (\mathring{\mathrm{div}}_{(2)})^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{229}$$

with $ac \neq 0$, where the $L_{i,j}(P)$'s are polynomials in P, and where $(\mathring{\mathrm{div}}_{(2)})^{-1}$ is the inverse of $\mathring{\mathrm{div}}_{(2)}$ viewed as a map from $(\ker \mathring{\mathrm{div}}_{(2)})^\perp$ to $\mathrm{im}\, \mathring{\mathrm{div}}_{(2)} = (\ker C)^\perp$. It should be clear from this equation that for generic m the operator L_2 is invertible from $(\ker \mathring{\mathrm{div}}_{(2)})^\perp$ to $(\ker C)^\perp$, so that for such m's we can solve the problem by setting $\overset{[7]}{\hat{\varphi}} = \overset{[8]}{\hat{\varphi}} \equiv 0$ and solving (229) for $\overset{[5]}{\hat{\varphi}}$ and $\overset{[6]}{\hat{\varphi}}$. However, this genericity condition is not needed; indeed, for any $m \neq 0$

we can proceed as follows: From the first equation of (229) we find

$$\overset{[6]}{\hat\varphi} = a^{-1}\big(L_{1,5}(P)\overset{[5]}{\hat\varphi} - (\mathring{\mathrm{div}}_{(2)})^{-1}x_1\big)\,. \tag{230}$$

Inserting into the second equation we obtain

$$\underbrace{L_{2,6}(P)L_{1,5}(P)}_{=:\mathfrak{L}_2}\overset{[5]}{\hat\varphi} + aL_{1,2}(P)\overset{[5]}{\hat\varphi} = a\big((\mathring{\mathrm{div}}_{(2)})^{-1}x_2 - L_{2,7}(P)\overset{[7]}{\hat\varphi} - c\overset{[8]}{\hat\varphi}\big)\,. \tag{231}$$

We can choose smooth fields $\overset{[7]}{\hat\varphi}$ and $\overset{[8]}{\hat\varphi}$ so that

$$L_{2,7}(P)\overset{[7]}{\hat\varphi} + c\overset{[8]}{\hat\varphi} = \big((\mathring{\mathrm{div}}_{(2)})^{-1}x_2\big)^{[\mathrm{im}(\mathfrak{L}_2)^\perp]} \in (\mathrm{im}\mathfrak{L}_2)^\perp$$

(e.g., by setting $\overset{[7]}{\hat\varphi} = 0$, but other choices might be more convenient); note that $(\mathrm{im}\mathfrak{L}_2)^\perp$ is finite dimensional and is spanned by smooth functions by ellipticity of $\mathfrak{L}_2^\dagger$. We then solve the elliptic equation

$$\mathfrak{L}_2\overset{[5]}{\hat\varphi} + aL_{1,2}(P)\overset{[5]}{\hat\varphi} = a\big((\mathring{\mathrm{div}}_{(2)})^{-1}x_2\big)^{[\mathrm{im}\mathfrak{L}_2]}\,, \tag{232}$$

with a unique $\overset{[5]}{\hat\varphi} \in (\ker\mathfrak{L}_2)^\perp$. Inserting into (230) we have thus obtained a desired solution of

$$L_k\Phi_k + M_k\Psi_k = (\mathring{\mathrm{div}}_{(2)})^{-1}X_k\,, \tag{233}$$

with $k = 2$.

Let us pass now to general k's. We note first that ellipticity and self-adjointness of P implies existence of a complete set of smooth, pairwise L^2-orthogonal, eigenfunctions ϕ_ℓ with a corresponding discrete set of eigenvalues $\lambda_\ell \to_{\ell\to\infty} \infty$. We can therefore write

$$\Phi_k = \sum_\ell \Phi_{k,\ell}\phi_\ell\,,\quad \Psi_k = \sum_\ell \Psi_{k,\ell}\phi_\ell\,,\quad (\mathring{\mathrm{div}}_{(2)})^{-1}X_k = \sum_\ell X_{k,\ell}\phi_\ell\,. \tag{234}$$

Equation (233) can be solved mode-by-mode:

$$L_k\Phi_{k,\ell}+M_k\Psi_{k,\ell} = X_{k,\ell} \quad\Longleftrightarrow\quad L_k|_{P\mapsto\lambda_\ell}\Phi_{k,\ell}+M_k|_{P\mapsto\lambda_\ell}\Psi_{k,\ell} = X_{k,\ell}\,, \tag{235}$$

where $P \mapsto \lambda_\ell$ means that every occurrence of P should be replaced by λ_ℓ.

Now, $\det L_k|_{P\mapsto\lambda_\ell}$ is a polynomial in λ_ℓ. Keeping in mind that in each line of the matrix L_k the highest power of P is on the diagonal, we see that $\det L_k|_{P\mapsto\lambda_\ell}$ is non-zero for ℓ large enough, and therefore there exists $N(k)$ such that we can find a unique solution of (235) with $\Psi_{k,\ell}=0$ for all $\ell > N(k)$. It remains to show that (235) can be solved in the finite dimensional space of Φ_k's and Ψ_k's of the form

(236)
$$\Phi_k = \sum_{\ell\le N(k)} \Phi_{k,\ell}\phi_\ell\,, \quad \Psi_k = \sum_{\ell\le N(k)} \Psi_{k,\ell}\phi_\ell\,, \quad (\mathring{\mathrm{div}}_{(2)})^{-1}X_k = \sum_{\ell\le N(k)} X_{k,\ell}\phi_\ell\,.$$

This is equivalent to the requirement that all the linear maps obtained by juxtaposing $L_k|_{P\mapsto\lambda_\ell}$ and $M_k|_{P\mapsto\lambda_\ell}$ with $\ell < N(k)$ are surjective. (Note, by the way, that we have already established surjectivity for $\ell \ge N(k)$.) This, in turn, is equivalent to the fact that the adjoints of these linear maps have no kernel.

Let us denote by $(L_k\, M_k)$ the relevant matrices. For simplicity write $L_{i,j}$ for $L_{i,j}(P)|_{P\mapsto\lambda_\ell}$. It follows from Appendix B.2 that $(L_k\, M_k)$ is of the form

$$\left(\begin{array}{cc} L_{1,4+1} & L_{1,4+2} \\ L_{2,4+1} & L_{2,4+2} \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ L_{k,4+1} & L_{k,4+2} \end{array} \cdots \begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ L_{k-j,3+2(k-j)} & L_{k-j,4+2(k-j)} \\ \vdots & \vdots \\ L_{k,3+2(k-j)} & L_{k,4+2(k-j)} \end{array} \cdots \begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ \cdot & \cdot \\ \vdots & \vdots \\ 0 & 0 \\ L_{k,3+2k} & L_{k,4+2k} \end{array}\right),$$

where the (i,j)-th entry is $L_{i,4+j}$. Note that each of the two-columns pairs, as grouped above, has a specific vanishing-block structure. This gives the following adjoint matrix:

(237)
$$\left(\begin{array}{ccccccccccc} L_{1,4+1} & L_{2,4+1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & L_{k,4+1} \\ \boxed{L_{1,4+2}} & L_{2,4+2} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & L_{k,4+2} \\ & & & & & \vdots & & & & & \\ 0 & 0\dots & 0 & L_{k-j,4+2(k-j)-1} & \dots & \dots & \dots & L_{k,4+2(k-j)-1} & & & \\ 0 & 0\dots & 0 & \boxed{L_{k-j,4+2(k-j)}} & \dots & \dots & \dots & L_{k,4+2(k-j)} & & & \\ & & & & & \vdots & & & & & \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots 0 & L_{k-1,4+2k-3} & L_{k,4+2k-3} & & \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots 0 & \boxed{L_{k-1,4+2k}} & L_{k,4+2k-2} & & \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots 0 & 0 & L_{k,4+2k-1} & & \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots 0 & 0 & \boxed{L_{k,4+2k}} & & \end{array}\right).$$

Equation (300) shows that the entries (boxed in the above) $L_{i,4+2i}$ are non-zero when $m \neq 0$. This easily implies that that the matrix (237) has maximal rank, and thus has trivial kernel.

We have therefore proved:

THEOREM 4.3. *The system (225) with $p \in \{1, \dots, k\}$ can be solved by a choice of gluing fields and gauge fields for any finite k. Its solutions are determined by an elliptic system, uniquely up to a finite number of eigenfunctions of P.*

4.3.2. Continuity of $\partial_u^p \tilde{h}_{AB}$, $p \geq 2$.

4.3.2.1. The case $m = 0$. It follows from the pointwise radial conservation law of $\overset{[3,p]}{Q}$ (cf. (115)) that the continuity of $\partial_u^p \tilde{h}_{AB}^{[\mathrm{TT}^\perp]}$ at r_2 requires

$$\overset{[3,p]}{Q}[\mathfrak{d}_{\tilde{\mathbf{S}}_1}] = \overset{[3,p]}{Q}[\mathfrak{d}_{\tilde{\mathbf{S}}_2}]\,. \tag{238}$$

The gauge field $\partial_u^{p-1} \overset{(2)}{\xi}{}_A^{[(\mathrm{CKV+H})^\perp]}$ is used to achieve the matching of $(\overset{[3,p]}{Q})^{[(\mathrm{CKV+H})^\perp]}$ according to (199).

On S^2, this ensures the continuity of $\partial_u^p \tilde{h}_{AB}$ at r_2 since then $\partial_u^p \tilde{h}_{AB}^{[\mathrm{TT}^\perp]} = \partial_u^p \tilde{h}_{AB}$.

For the remaining topologies we return to (162). Taking into account the gauge invariance of $h_{AB}^{[\mathrm{TT}]}$, Equation (162) provides a necessary and sufficient condition for the continuity of $\partial_u^p \tilde{h}_{AB}^{[\mathrm{TT}]}$ at r_2 according to:

$$\begin{aligned}
&\partial_u^p q_{AB}^{[\mathrm{TT}]}|_{\tilde{\mathbf{S}}_2} - \partial_u^p q_{AB}^{[\mathrm{TT}]}|_{\tilde{\mathbf{S}}_1} \\
&= \alpha^2 \sum_{k=0}^{p-1} (\alpha^2 r_2)^k \left[s \overset{[p-k]}{q}{}_{AB}^{[\mathrm{TT}]}\Big|_{r_1} + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) \partial_u^{p-1-k} h_{AB}^{[\mathrm{TT}]}\Big|_s \right]_{r_1}^{r_2} \\
&\quad + \alpha^{2(p+1)} r_2^p \left[\overset{[1]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}]} + \int_{r_1}^{r_2} \hat{\kappa}_1(s) v_{AB}^{[\mathrm{TT}]}\, ds \right] \\
&= \alpha^2 \sum_{k=0}^{p-1} (\alpha^2 r_2)^k \left[s \overset{[p-k]}{q}{}_{AB}^{[\mathrm{TT}]}\Big|_{r_1} + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) \partial_u^{p-1-k} h_{AB}^{[\mathrm{TT}]}\Big|_s \right]_{r_1}^{r_2} \\
&\quad + (\alpha^2 r_2)^p (q_{AB}^{[\mathrm{TT}]}|_{\tilde{\mathbf{S}}_2} - q_{AB}^{[\mathrm{TT}]}|_{\tilde{\mathbf{S}}_1})\,,
\end{aligned} \tag{239}$$

where we used the formula (216) for $\overset{[1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$ in the last step. Equation (239) provides a further obstruction to be satisfied by the data. When $\alpha = 0$, the condition reduces to

$$\forall\, 0 \le p \le k-1 \qquad \overset{[p+1]}{q}{}^{[\mathrm{TT}]}_{AB}[\mathfrak{d}_{\mathbf{S}_1}] = \overset{[p+1]}{q}{}^{[\mathrm{TT}]}_{AB}[\mathfrak{d}_{\mathbf{S}_2}]\,. \tag{240}$$

4.3.2.2. The case $m \neq 0$. The continuity at r_2 of the $\mathrm{TT}^{\perp}$-part of $\partial_u^p h_{AB}$ has already been addressed in Section 4.2.2. Taking into account the allowed gauge perturbations of the linearised gravitational field, it follows from (150) that it remains to satisfy the equation

$$\begin{aligned}\partial_u^p h^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_2} = {}& \Big(\overset{(p)}{\tilde{\Psi}}{}_{AB}(r_2, x^A) - 2r_2^2\,\mathrm{TS}[\mathring{D}_A \partial_u^p \overset{(2)}{\xi}{}_B]\Big)^{[\mathrm{TT}]} \\ &+ \Big(\sum_{j=0}^{2}\sum_{\ell=2p+1}^{2p+2} \overset{(p,j)}{\psi}{}_{\ell}(r_2) P^j \overset{[\ell]}{\hat{\varphi}}_{AB}\Big)^{[\mathrm{TT}]}\,.\end{aligned} \tag{241}$$

Here, for the sake of induction, we treated the fields $\partial_u^j \overset{(2)}{\xi}{}_A$ for $0 \le j \le p-1$ and $\overset{[\ell]}{\hat{\varphi}}_{AB}$ for $1 \le \ell \le 2p$ as known, and collected them together with the remaining known fields into the term $\overset{(p)}{\tilde{\Psi}}{}_{AB}(r_2, x^A)$. This equation is non-trivial only for $\mathbb{T}^2$ and for cross-sections $\mathbf{S}$ of higher genus, can be solved using a linear combination of $\overset{[2p+2]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$ and $\overset{[2p+1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}$:

(242)

$$\partial_u^p h^{[\mathrm{TT}]}_{AB}|_{\tilde{\mathbf{S}}_2} = \overset{(p)}{\tilde{\Psi}}{}^{[\mathrm{TT}]}_{AB}(r_2, x^A) + \overset{(p,0)}{\psi}{}_{2p+2}(r_2)\,\overset{[2p+2]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB} + \overset{(p,0)}{\psi}{}_{2p+1}(r_2)\,\overset{[2p+1]}{\hat{\varphi}}{}^{[\mathrm{TT}]}_{AB}\,.$$

5. An isomorphism theorem

In Section 3.6 we have verified that a consistent scheme for the linearised equations is obtained

$$\boxed{\text{if we assume that } h_{AB}(r,\cdot) \in H_{k_\gamma}(\mathbf{S}), \text{ for all } r \in [r_1, r_2], \text{ with } k_\gamma \ge 4}$$

and

$$\begin{aligned} &k_\beta = k_\gamma\,, \quad k_U = k_\gamma - 1\,, \quad k_V = k_\gamma - 2\,, \quad k_{\partial_u U} = k_\gamma - 3\,, \\ &k_{\partial_u V} = k_\gamma - 4\,, \quad k_{\partial_u \gamma} = k_\gamma - 2\,. \end{aligned} \tag{243}$$

as well as

$$k_{\xi^u} = k_\gamma + 2\,, \quad k_{\partial_u \xi^u} = k_\gamma\,, \quad k_{\xi^A} = k_\gamma + 1\,, \quad k_{\partial_u \xi^A} = k_\gamma - 1\,. \tag{244}$$

We wish to check that this is consistent with the equations satisfied by the gluing functions, and that the gluing equations provide surjections in the relevant spaces, with splitting kernels, so that the implicit function theorem can be applied to the full nonlinear problem.

More precisely, we assume that the gluing fields take the form as in (164) and (167). Assuming that the fields at r_1 and r_2 satisfy (243), and that all r-derivatives of the interpolating field v_{AB} are in $H_{k_\gamma} \equiv H_{k_\gamma}(\mathbf{S})$, we want to show that the gluing fields $\overset{[i]}{\hat\varphi}_{AB}$ will be in H_{k_γ}, and that the gauge functions will satisfy (244).

Note that within the scheme presented above, the r-derivatives of all fields preserve their Sobolev class.

Whether for the sake of induction when $m = 0$, or for the analysis of the elliptic system (226) when $m \neq 0$, we assume that each u-derivative of all fields obtained in a previous induction step has a loss of no more than two degrees of differentiability:

$$k_{\partial_u^i U} = k_\gamma - 1 - 2i\,, \quad k_{\partial_u^i V} = k_\gamma - 2i\,, \quad k_{\partial_u^i \gamma} = k_\gamma - 2i\,, \tag{245}$$

while for the gauge fields it holds that

$$k_{\partial_u^i \xi^u} = k_\gamma + 2 - 2i\,, \quad k_{\partial_u^i \xi^A} = k_\gamma + 1 - 2i \tag{246}$$

(we have listed the condition on $\partial_u^i \overset{(2)}{\xi}{}^u$ with $i \geq 1$ for clarity of the arguments below, but in fact it follows from the one on $\partial_u^i \overset{(2)}{\xi}{}^A$ since we impose (33), namely

$$\partial_u \overset{(2)}{\xi}{}^u(u, x^A) = \frac{\mathring{D}_B \overset{(2)}{\xi}{}^B(u, x^A)}{2}\,.) \tag{247}$$

We start with the β-equation, the linearised version of which, namely (49), is clearly consistent with the above.

Next, the field $\chi[\mathfrak{d}_{\mathbf{S}_1}]$ given by (194), p. 739 is in $H_{k_\gamma - 2}$. <u>When $m = 0$</u>, it follows by standard elliptic estimates that the solution $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ of (196)

is in $H_{k_\gamma+2}$. Since the field $(\overset{(2)}{\xi}{}^u)^{[\ker \mathring{L}]}$, which is left undetermined at that stage of the argument, is smooth, we conclude that

$$\overset{(2)}{\xi}{}^u \in H_{k_\gamma+2}\,,$$

as desired. <u>When $m \neq 0$</u>, we use again the field $(\overset{(2)}{\xi}{}^u)^{[(\ker \mathring{L})^\perp]}$ determined by (196) and, on S^2, the field $(\mathring{D}_A \overset{(2)}{\xi}{}^A)^{[=1]}$, determined by (197); this is implemented by a smooth field $\overset{(2)}{\xi} \in$ CKV.

We pass now to the covector fields $\overset{[3,i+1]}{Q}{}_A$ of (198), which are in $H_{k_\gamma-2i-3}$. <u>When $m = 0$</u> the vector fields $\partial_u^i \overset{(2)}{\xi}{}_A^{[(\mathrm{CKV+H})^\perp]}$, $i \geq 0$, solve the fourth-order elliptic equation (199) (cf. Proposition C.6, p. 782), hence we have

$$\partial_u^i \overset{(2)}{\xi}{}_A \in H_{k_\gamma-2i+1}\,, \quad i \geq 0\,,$$

as desired. <u>When $m \neq 0$</u>, the solution of the set of equations (201)-(206), as obtained in Section 4.2.2, is again of the desired differentiability class when the condition (207) is imposed.

As the next step, consider the field $\overset{[1]}{\hat{\varphi}}_{AB} \equiv \langle \hat{\kappa}_1, w_{AB} \rangle$ solving (210), cf. also (209). Here a note on the differentiability class of the boundary term appearing in (210), or in the equations that will be referred to in what follows, is in order. We have already checked that the differentiability classes in (245)-(246) are consistent with all the equations. The boundary terms in all integrated equations are obtained by integration by parts in the r-variable, which does not change the differentiability in the **S**-directions. The boundary terms for the gauge equations only involve the known boundary data, which have the needed regularity by hypothesis. The boundary terms for the gluing fields are either determined by the known boundary data, or from the gauge fields. <u>When $m = 0$</u>, the hierarchical structure of our proof implies that the boundary terms arising at each further step of our analysis will have the right differentiability properties for the induction argument. When $m \neq 0$ the solutions are obtained by a global system of equations, with boundary terms determined by the known boundary data.

In any case, one can directly chase through the derivatives of the already known fields which appear in (209)-(210), and use that on two-dimensional

manifolds the operator $\mathring{\mathrm{div}}_{(2)}$ is elliptic, and that the field $\overset{[1]}{\hat\varphi}{}^{[\mathrm{TT}]}_{AB}$ solving (216) is smooth, to obtain that

$$\overset{[1]}{\hat\varphi}{}_{AB} \in H_{k_\gamma}\,.$$

as needed.

When $m=0$, we continue with (221), where p is an induction parameter. The operator appearing at the left-hand side is elliptic, of order $2p+1$. By the arguments just given, or by a direct inductive calculation, the boundary terms $\overset{(p)}{\hat{\hat X}}_A$ are in $H_{k_\gamma-2p-3}$. For $p\le k-2$, where k is the number of derivatives that we wish to glue, the operator recovers the number of derivatives lost by the right-hand side, which results in a field

$$\overset{[p+4]}{\hat\varphi}{}^{[\mathrm{TT}^\perp]}_{AB} \in H_{k_\gamma}\,. \tag{248}$$

The case $m\neq 0$ is taken care of by Theorem 4.3.

Keeping in mind that all the operators considered in the analysis of the linearised equations have splitting kernels, we have proved:

THEOREM 5.1. *The linearised gluing map, assigning to the collection*

$$\{\textit{data at } r_1\textit{, the gluing fields, and the gauge fields}\}$$

the data at r_2, as described above, is continuous and surjective in the Sobolev spaces defined in (245)-(246)*, with splitting kernel, modulo the obstructions listed in Tables 4.1, p. 732 and 4.2, p. 733.*

6. Unobstructed gluing to perturbed data

Given that there exist obstructions to glue two arbitrary characteristic data sets of order k, the question arises whether something can be done about that. Since we are dealing with linear equations, the simplest solution is to add to the data another data set with charges chosen to compensate for the obstructions. This requires families of data sets with a sufficient number of radial charges to cover all obstructions.

Now, a static family of such data sets can be obtained by differentiating the Birmingham-Kottler metrics with respect to mass:

$$\frac{d}{dm}\Big[\big(\varepsilon-\alpha^2r^2-\frac{2m}{r}\big)du^2-2du\,dr+r^2\mathring\gamma_{AB}dx^Adx^B\Big]=-\frac{2}{r}du^2\,. \tag{249}$$

These metric perturbations can be used to compensate for the missing charge $\overset{[2]}{Q}(\lambda)$ with $\lambda = 1$.

Another such family is obtained by differentiating (1) with respect to a parameter along a curve of metrics $\lambda \mapsto \mathring{\gamma}_{AB}(\lambda)$ with constant scalar curvature:

$$\frac{d}{d\lambda}\Big[\big(\varepsilon-\alpha^2 r^2-\frac{2m}{r}\big)du^2-2du\,dr+r^2\mathring{\gamma}_{AB}dx^A dx^B\Big] = r^2\frac{d\mathring{\gamma}_{AB}}{d\lambda}dx^A dx^B\,. \tag{250}$$

By [19, Theorem 8.15] every TT-tensor, say $\mathring{m}_{AB}$, is tangent to such a curve, and thus metric perturbations of the form

$$r^2\mathring{m}_{AB}dx^A dx^B\,,\ \text{with } \mathring{D}_A\mathring{m}^{AB} = 0 = \mathring{\gamma}_{AB}\mathring{m}^{AB} \tag{251}$$

provide the missing radial charges $\overset{[i]}{q}{}^{[\mathrm{TT}]}_{AB}$.

Yet another, time-independent, family is provided by differentiating the Kerr-(Anti) de Sitter metrics with respect to the angular-momentum. (Since there is no explicit formula for these metrics in Bondi coordinates, the associated linearised metrics can only be obtained by an indirect calculation.) When calculated at the (A)dS solution (with $m \neq 0$), this leads to the metric perturbation (252) below, where $\mathring{\lambda}^A\partial_A$ is a u-independent Killing vector on S^2, with the remaining fields there vanishing.[2]

It turns out that we can obtain a family of metric perturbations compensating for all radial charges needed for $C^2_u C^\infty_{(r,x^A)}$-gluing by setting

$$\begin{aligned}\mathring{h} \;=\;& \Big(\frac{\mathring{\mu}(u,x^C)}{r} - \frac{\mathring{D}^A\mathring{\lambda}_A(u,x^C)}{2r^2} + \frac{1}{2r}\mathring{D}^A\mathring{D}^B\mathring{s}_{AB}(u,x^C)\Big)du^2\\ &+\Big(\frac{\mathring{\lambda}_A(u,x^C)}{r} + \frac{1}{2}\mathring{D}^B\mathring{s}_{AB}(u,x^C)\Big)dx^A du\\ &+(r\mathring{s}_{AB}(u,x^C) + r^2\mathring{m}_{AB}(u,x^C))dx^A dx^B\,,\end{aligned} \tag{252}$$

with symmetric $\mathring{\gamma}$-traceless tensors $\mathring{s}_{AB}$ and $\mathring{m}_{AB}$. In addition, anticipating the fact that $\mathring{D}^B\mathring{s}_{AB}$ plays a role in adjusting $(\overset{[3,i]}{Q})^{[\mathrm{H}]}$, we impose

$$\mathring{D}^A\mathring{D}^B\mathring{s}_{AB}(u,x^A) = 0\,. \tag{253}$$

After using

$$\mathring{D}_A\Delta_{\mathring{\gamma}}\psi^A = (\Delta_{\mathring{\gamma}} + \varepsilon)\mathring{D}_A\psi^A\,, \tag{254}$$

[2]We are grateful to Finn Gray for checking this.

the linearised Einstein equations will hold if and only if

$$\mathring{D}^A\mathring{D}^B\mathring{m}_{AB}=0\,,\quad \alpha^2\mathring{s}_{AB}-\partial_u\mathring{m}_{AB}=0\,,\quad 3m\mathring{D}^A\mathring{\lambda}_A=0\,, \tag{255}$$

$$\mathrm{TS}\left[\mathring{D}_A\mathring{\lambda}_B\right]+m\mathring{s}_{AB}=0\,, \tag{256}$$

$$3\alpha^2\mathring{D}^A\mathring{\lambda}_A+2\partial_u\mathring{\mu}=0\,,\quad 3\partial_u\mathring{\lambda}_A-\mathring{D}_A\mathring{\mu}+\frac{1}{2}(\Delta_{\mathring{\gamma}}-\varepsilon)\mathring{D}^B\mathring{s}_{AB}=0\,, \tag{257}$$

$$2\varepsilon\mathring{\lambda}_A+\mathring{D}_A\mathring{D}^B\mathring{\lambda}_B-\mathring{D}^B\mathring{D}_A\mathring{\lambda}_B+\Delta_{\mathring{\gamma}}\mathring{\lambda}_A+2m\mathring{D}^B\mathring{s}_{AB}=0\,, \tag{258}$$

$$\varepsilon\mathring{D}^A\mathring{\lambda}_A+\frac{1}{2}\Delta_{\mathring{\gamma}}\mathring{D}^A\mathring{\lambda}_A=0\,. \tag{259}$$

For completeness we listed above all conditions obtained from the linearised Einstein equations, cf. Sections 3.3 and 3.5, but we note that (255)-(257) suffice. Indeed, taking $2\times\mathring{\mathrm{div}}_{(2)}$ of (256) gives equation (258), while equation (259) can be obtained by taking $\mathring{\mathrm{div}}_{(1)}$ of (258) and by making use of (255).

Equation (255) implies that $\mathring{\mathrm{div}}_{(1)}\,\mathring{\lambda}$ has to vanish when $m\neq 0$, and $\partial_u\mathring{m}_{AB}$ has to vanish when $\alpha=0$ or when we are on S^2. In addition, it follows from (256) that $\mathring{\lambda}_A$ has to vanish when $m=0$ and $\varepsilon=-1$.

Equations (257) together with their u-differentiated versions show that

$$2\partial_u^2\mathring{\mu}=-3\alpha^2\Delta_{\mathring{\gamma}}\mathring{\mu}\,, \tag{260}$$

$$\partial_u^2\mathring{\lambda}_A=-\frac{\alpha^2}{2}\mathring{D}_A\mathring{D}^B\mathring{\lambda}_B-\frac{1}{6}(\Delta_{\mathring{\gamma}}-\varepsilon)\mathring{D}^B\partial_u\mathring{s}_{AB}\,. \tag{261}$$

When $m\neq 0$ we can use (256) to rewrite the last equation as

$$\partial_u^2\mathring{\lambda}_A=-\frac{\alpha^2}{2}\mathring{D}_A\mathring{D}^B\mathring{\lambda}_B+\frac{1}{6m}(\Delta_{\mathring{\gamma}}-\varepsilon)\mathring{D}^B\partial_u\,\mathrm{TS}\left[\mathring{D}_A\mathring{\lambda}_B\right]\,. \tag{262}$$

So, when $m\neq 0$, Equations (260) and (262) provide evolution equations for $\mathring{\mu}$ and $\mathring{\lambda}_A$, solutions of which determine the time-evolution of the remaining fields.

To continue, we note that equation (258) can be rewritten as,

$$\frac{1}{2}(\Delta_{\mathring{\gamma}}+\varepsilon)\mathring{\lambda}_A+m\mathring{D}^B\mathring{s}_{AB}=0\,. \tag{263}$$

Next, the second equation in (257), together with (263), implies that

$$(\Delta_{\mathring{\gamma}}+\varepsilon)\mathring{D}_A\mathring{\mu}=3(\Delta_{\mathring{\gamma}}+\varepsilon)\partial_u\mathring{\lambda}_A+\frac{1}{2}(\Delta_{\mathring{\gamma}}+\varepsilon)(\Delta_{\mathring{\gamma}}-\varepsilon)\mathring{D}^B\mathring{s}_{AB}$$

$$= -6m(\Delta_{\mathring{\gamma}} + \varepsilon)\mathring{D}^B \partial_u \mathring{s}_{AB} + \frac{1}{2}(\Delta_{\mathring{\gamma}} + \varepsilon)(\Delta_{\mathring{\gamma}} - \varepsilon)\mathring{D}^B \mathring{s}_{AB}\,.$$

Taking $\mathring{\mathrm{div}}_{(1)}$ of this and making use of $\mathring{D}^A\mathring{D}^B\mathring{s}_{AB} = 0 = \partial_u \mathring{D}^A\mathring{D}^B\mathring{s}_{AB}$ gives

$$\mathring{D}^A(\Delta_{\mathring{\gamma}} + \varepsilon)\mathring{D}_A\mathring{\mu} = \Delta_{\mathring{\gamma}}(\Delta_{\mathring{\gamma}} + 2\varepsilon)\mathring{\mu} = 0\,.$$

In particular, when we are not on S^2, the "mass aspect function" $\mathring{\mu}$ must be x^A-independent, while on S^2 it is a linear combination of $\ell = 0$ and $\ell = 1$ spherical harmonic. It follows that

$$\partial_u^2\mathring{\mu} = \begin{cases} 3\alpha^2\mathring{\mu}, & \mathbf{S} = S^2 \text{ and } \mathring{\mu} \text{ has no } \ell = 0 \text{ harmonics;} \\ 0, & \mathbf{S} \neq S^2, \text{ or } \mathbf{S} = S^2 \text{ and } \mathring{\mu} \text{ has no } \ell = 1 \text{ harmonics.} \end{cases}$$

Next, when $m = 0$ the space of $\mathring{\lambda}$'s satisfying (256) is six-dimensional on S^2 and two-dimensional on $\mathbb{T}^2$; for negatively curved $\mathbf{S}$ one finds $\mathring{\lambda}_A \equiv 0$.

The tensor field $\mathring{h}_{AB}$ carries the full set of conserved radial charges needed for $C^2_u C^\infty_{(r,x^A)}$-gluing when $\mathring{\mu}$, $\mathring{\lambda}_A$, $\mathring{s}_{AB}$ and $\mathring{m}_{AB}$ run over the set of solutions of (255)-(257):

$$\overset{[1]}{Q}(\pi) = -3\int_{\mathbf{S}} \pi^A \mathring{\lambda}_A \, d\mu_{\mathring{\gamma}}\,, \qquad \overset{[2]}{Q}(\lambda) = -\int_{\mathbf{S}} \lambda\mathring{\mu}\, d\mu_{\mathring{\gamma}}\,, \tag{264}$$

$$q^{[\mathrm{TT}]}_{AB} = -\frac{V}{2r}\mathring{m}^{[\mathrm{TT}]}_{AB} + \alpha^2 r\mathring{s}^{[\mathrm{TT}]}_{AB} + \partial_u \mathring{s}^{[\mathrm{TT}]}_{AB}\,, \tag{265}$$

$$\overset{[2]}{q}{}^{[\mathrm{TT}]}_{AB} = -\frac{\alpha^2 V}{2r}\mathring{s}^{[\mathrm{TT}]}_{AB} + \alpha^2 r\partial_u \mathring{s}^{[\mathrm{TT}]}_{AB} + \partial_u^2 \mathring{s}^{[\mathrm{TT}]}_{AB}\,, \tag{266}$$

$$(\overset{[3,1]}{Q})^{[\mathrm{H}]}_A = -3\alpha^2\mathring{\lambda}^{[\mathrm{H}]}_A + \frac{2m}{r}(\mathring{D}^B\mathring{m}_{AB})^{[\mathrm{H}]} - \frac{m}{r^2}(\mathring{D}^B\mathring{s}_{AB})^{[\mathrm{H}]} + 2(\mathring{D}^B\partial_u\mathring{s}_{AB})^{[\mathrm{H}]}\,, \tag{267}$$

$$(\overset{[3,2]}{Q})^{[\mathrm{H}]}_A = -3\alpha^2\partial_u\mathring{\lambda}^{[\mathrm{H}]}_A + \frac{2m\alpha^2}{r}(\mathring{D}^B\mathring{s}_{AB})^{[\mathrm{H}]} - \frac{m}{r^2}(\mathring{D}^B\partial_u\mathring{s}_{AB})^{[\mathrm{H}]} + 2(\mathring{D}^B\partial_u^2\mathring{s}_{AB})^{[\mathrm{H}]}\,, \tag{268}$$

where π and λ satisfy, respectively, (57) and (73). Note that when $\alpha = 0 = m$, we have $V/r = \varepsilon$, in which case all expressions in (264)-(268) are r-independent, as they should be in this case.

Keeping in mind that $C^2_u C^\infty_{(r,x^A)}$-gluing with $m \neq 0$ needs only the matching of $\overset{[1]}{Q}$ and $\overset{[2]}{Q}$, we have proved:

THEOREM 6.1. *Any $C^2_u C^\infty_{(r,x^A)}$ linearised vacuum data on $\mathscr{N}_{(r_0,r_1]}$ can be $C^2_u C^\infty_{(r,x^A)}$-glued to any $C^2_u C^\infty_{(r,x^A)}$ linearised vacuum data on $\mathscr{N}_{[r_2,r_3)}$ after adding to one of them a suitable field of the form* (252).

Proof. Indeed, when $m \neq 0$ we only need (cf. Table 1.1, p. 693)

$$\mathring{h} = \frac{\mathring{\mu}}{r} du^2 + \frac{\mathring{\lambda}_A(x^C)}{r} dx^A du \,, \tag{269}$$

with $\mathring{\mu}$ being a constant and $\mathring{\lambda}_A$ being a combination of $\ell = 1$ vector harmonics satisfying $\mathring{D}^A \mathring{\lambda}_A = 0$ on S^2; with constant $\mathring{\mu}$ and covariantly constant $\mathring{\lambda}_A$ on $\mathbb{T}^2$; with constant $\mathring{\mu}$ and vanishing $\mathring{\lambda}_A$ on higher genus manifolds. In all cases the fields are chosen so that the radial charges

$$\overset{[1]}{Q}(\pi) = -3 \int_{\mathbf{S}} \pi^A \mathring{\lambda}_A \, d\mu_{\mathring{\gamma}} \,, \qquad \overset{[2]}{Q}(\lambda) = -\int_{\mathbf{S}} \lambda \mathring{\mu} \, d\mu_{\mathring{\gamma}} \,, \tag{270}$$

compensate for the difference of radial charges calculated from the fields at r_1 and at r_2.

When $m = 0$ we obtain the desired fields by choosing $\mathring{\mu}$ and $\mathring{\lambda}_A$ so that the radial charges in (270) compensate for the difference of the respective radial charges at r_1 and r_2 at $u = 0$, and by choosing

$$\mathring{m}_{AB}\big|_{u=0} = 0 = \mathring{s}_{AB}\big|_{u=0} \,. \tag{271}$$

The remaining fields vanish on S^2, in which case we are done.

Otherwise recall the obstruction (239) with $p = 1$:

$$\begin{aligned} \overset{[2]}{q}{}^{[\mathrm{TT}]}_{AB}\big|_{\tilde{\mathbf{S}}_2}\Big|_{r_1}^{r_2} &= \alpha^2 \left[s q^{[\mathrm{TT}]}_{AB}\big|_{r_1} + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) h^{[\mathrm{TT}]}_{AB}\big|_s \right]_{r_1}^{r_2} + \alpha^2 r_2 q^{[\mathrm{TT}]}_{AB}\Big|_{r_1}^{r_2} \\ &= \alpha^2 \left[s q^{[\mathrm{TT}]}_{AB}\big|_s + \frac{1}{2s}(\varepsilon - \alpha^2 s^2) h^{[\mathrm{TT}]}_{AB}\big|_s \right]_{r_1}^{r_2} . \end{aligned}$$

So at, say, $r = r_2$ we can compensate all radial charge deficits by choosing the remaining fields as

$$\partial_u \mathring{s}^{[\mathrm{TT}]}_{AB}\big|_{u=0} = \begin{cases} -q^{[\mathrm{TT}]}_{AB}\Big|_{r_1}^{r_2} \\ 0 \,, \end{cases}$$

$$\partial_u^2 \mathring{s}^{[\mathrm{TT}]}_{AB}\big|_{u=0} = \begin{cases} -\overset{[2]}{q}{}^{[\mathrm{TT}]}_{AB}\Big|_{r_1}^{r_2} \\ -\overset{[2]}{q}{}^{[\mathrm{TT}]}_{AB}\Big|_{r_1}^{r_2} + \alpha^2\left[sq^{[\mathrm{TT}]}_{AB}\big|_s + \frac{1}{2s}(\varepsilon - \alpha^2 s^2)h^{[\mathrm{TT}]}_{AB}\big|_s\right]_{r_1}^{r_2} , \end{cases}$$

with the upper case being for $\alpha = 0$ and the lower case for $\alpha \neq 0$, and

$$(\mathring{D}^B \partial_u \mathring{s}_{AB})^{[\mathrm{H}]}\big|_{u=0} = \begin{cases} 0 , & \mathfrak{g} = 1 , \\ -\frac{1}{2}\overset{[3,1]}{Q}{}^{[\mathrm{H}]}_{A}\Big|_{r_1}^{r_2} , & \mathfrak{g} \geq 2 , \end{cases}$$

$$(\mathring{D}^B \partial_u^2 \mathring{s}_{AB})^{[\mathrm{H}]}\big|_{u=0} = \begin{cases} 0 , & \mathfrak{g} = 1 , \\ -\frac{1}{2}\overset{[3,2]}{Q}{}^{[\mathrm{H}]}_{A}\Big|_{r_1}^{r_2} , & \mathfrak{g} \geq 2 , \end{cases}$$

where $\big[f(r)\big]_{r_1}^{r_2} \equiv \big[f(s)\big]_{r_1}^{r_2} \equiv f(s)\big|_{r_1}^{r_2} := f(r_2) - f(r_1)$, and where we have used that $\mathring{\lambda}_A$ vanishes if $\mathfrak{g} \geq 2$. □

Appendix A. Constructing the κ_i's

Recall that $\hat{\kappa}_i(s) = s^{-i}$. We wish to construct a sequence of smooth functions κ_i compactly supported in (r_1, r_2) satisfying

$$\langle \kappa_i, \hat{\kappa}_j \rangle \equiv \int_{r_1}^{r_2} \kappa_i(r)\hat{\kappa}_j(r)\, dr = 0 \quad \text{for } j < i , \tag{272}$$

$$\langle \kappa_i, \hat{\kappa}_i \rangle = 1 . \tag{273}$$

This can be done as follows: Let χ be any smooth non-negative function supported away from neighborhoods of r_1 and r_2, with integral 1. Let

$$\kappa_i = c_i \chi f_i ,$$

where the f_i's are constructed by a Gram-Schmidt orthonormalisation procedure from the family of monomials in $1/r$, namely $\{1, r^{-1}, r^{-2}, \ldots\}$, in the space $\mathbb{H} := L^2([r_1, r_2], \chi dr)$, so that the scalar product is

$$\langle \phi, \psi \rangle_{\mathbb{H}} = \int_{r_1}^{r_2} (\phi\psi\chi)(r)\, dr ,$$

and the c_i's are constants chosen so that (273) holds; the possibility of doing so will be justified shortly. Then, by construction, f_i is a polynomial

of order i in $1/r$ which is $\mathbb{H}$-orthogonal to any such polynomial of order $j < i$; this is (272). As for (273), we note that each of the functions r^{-i} can be decomposed in the basis $\{f_j\}_{j\in\mathbb{N}}$ as $r^{-i} = \sum_{j=0}^{i} a_{ij} f_j(r)$, with $a_{ii} \neq 0$ since otherwise the right-hand side would be a polynomial in $1/r$ of order less than or equal to $i-1$. This shows that

$$\int_{r_1}^{r_2} r^{-i} f_i(r)\chi(r)\, dr = a_{ii} \neq 0\,,$$

so that we can indeed choose $c_i = 1/a_{ii}$ to fulfill (273).

Appendix B. Recursion formulae

For ease of further reference we collect here all the integral kernels appearing in (150)-(151), as needed for $C^2_u\, C^\infty_{(r,x^A)}$-gluing and for various induction arguments in the rest of this Appendix:

$$\overset{(1,0,0)}{\psi}(r) = -\frac{1}{2r}\left(\varepsilon - \alpha^2 r^2 - \frac{2m}{r}\right)\,, \tag{274}$$

$$\overset{(1,1,0)}{\psi}(r) = \frac{1}{2}\left(\varepsilon - \alpha^2 r^2 - \frac{2m}{r}\right)\,, \qquad \overset{(1,0,1)}{\psi}(r) = 0\,, \tag{275}$$

$$\overset{(1,0)}{\psi}(s,r) = \alpha^2 r\hat\kappa_1(s) - mr\hat\kappa_4(s)\,, \qquad \overset{(1,1)}{\psi}(s,r) = \frac{2r^2\hat\kappa_4(s)}{3} + \frac{\hat\kappa_1(s)}{3r}\,, \tag{276}$$

$$\begin{aligned}\overset{(2,0)}{\psi}(s,r) = {}& \frac{9m^2 r}{2}\hat\kappa_6(s) + \frac{\alpha^2(2m+4r^3\alpha^2)}{4r}\hat\kappa_1(s)\\ & - \frac{8m^2+16mr^3\alpha^2}{16r}\hat\kappa_4(s) - \frac{3mr\varepsilon}{2}\hat\kappa_5(s)\,,\end{aligned} \tag{277}$$

$$\begin{aligned}\overset{(2,1)}{\psi}(s,r) = {}& (r^2\varepsilon - \frac{3mr}{4})\hat\kappa_5(s) + \frac{(9m-4r\varepsilon)\hat\kappa_1(s)}{12r^3} + \frac{\varepsilon r\hat\kappa_4(s)}{3}\\ & - 3mr^2\hat\kappa_6(s)\,,\end{aligned} \tag{278}$$

$$\overset{(2,2)}{\psi}(s,r) = \frac{r^2\hat\kappa_5(s)}{2} - \frac{\hat\kappa_1(s)}{6r^2} - \frac{r\hat\kappa_4(s)}{3}\,, \tag{279}$$

$$\overset{(0,0,0)}{\chi}(r) = 0\,, \quad \overset{(1,1,0)}{\chi}(r) = \overset{(1,0,1)}{\chi}(r) = 0\,, \quad \overset{(1,0,0)}{\chi}(r) = \frac{\varepsilon}{2r^4} - \frac{\alpha^2}{2r^2}\,, \tag{280}$$

$$\overset{(0,0)}{\chi}(s,r) = \frac{1}{3}\left(\frac{2\hat\kappa_1(s)}{r^3} + \hat\kappa_4(s)\right)\,, \tag{281}$$

$$\overset{(1,0)}{\chi}(s,r) = -\frac{3m\hat\kappa_6(s)}{2} - \frac{m\hat\kappa_4(s)}{2r^2} + \frac{\alpha^2\hat\kappa_1(s)}{2r^2} + \frac{\varepsilon\hat\kappa_5(s)}{2}\,, \tag{282}$$

$$\overset{(1,1)}{\chi}(s,r) = \frac{\hat\kappa_5(s)}{4} - \frac{\hat\kappa_1(s)}{4r^4}\,, \tag{283}$$

$$
\begin{aligned}
(284)\quad \overset{(2,0)}{\chi}(s,r) &= m\Big(\frac{75m}{8}\hat\kappa_8(s) - \frac{77\varepsilon}{12}\hat\kappa_7(s) + \frac{3}{8}(5\alpha^2 - \frac{2\varepsilon}{r^2})\hat\kappa_5(s)\Big)\\
&\quad + \frac{\alpha^2(15m+8r\varepsilon)}{24r^4}\hat\kappa_1(s) - \frac{15m^2+8r(m+r^3\alpha^2)\varepsilon}{24r^4}\hat\kappa_4(s)\\
&\quad + (\frac{9m^2}{4r^2}+\varepsilon^2)\hat\kappa_6(s)\,,\\
(285)\quad \overset{(2,1)}{\chi}(s,r) &= -\frac{7m}{4}\hat\kappa_7(s) - \frac{3m}{8r^2}\hat\kappa_5(s) + \frac{7\varepsilon}{10}\hat\kappa_6(s)\\
&\quad + \frac{6m-2r^3\alpha^2+r\varepsilon}{6r^3}\hat\kappa_4(s) + \frac{15m-80r^3\alpha^2+16r\varepsilon}{120r^6}\hat\kappa_1(s)\,,\\
(286)\quad \overset{(2,2)}{\chi}(s,r) &= \frac{\hat\kappa_1(s)}{15r^5} - \frac{\hat\kappa_4(s)}{6r^2} + \frac{\hat\kappa_6(s)}{10}\,.
\end{aligned}
$$

These are all linear combinations of the $\hat\kappa_i$'s with $0 \le i \le 8$, $i \notin \{2,3\}$, with coefficients which might depend upon r.

Next, recall (95):

$$
\begin{aligned}
(287)\quad \partial_u h_{AB} &= \frac{\varepsilon}{2}\Big[\partial_r h_{AB} - \frac{1}{r}h_{AB}\Big] + \int_{r_1}^{r}\Big(\frac{1}{3sr} + \frac{2r^2}{3s^4}\Big)Ph_{AB}\,ds\\
&\quad - (\frac{\alpha^2r^2}{2} + \frac{2m}{r})\Big[\partial_r h_{AB} - \frac{1}{r}h_{AB}\Big] + \int_{r_1}^{r}(\frac{\alpha^2 r}{s} - \frac{mr}{s^4})h_{AB}\,ds\\
&\quad + \text{b.d.}|_{r_1}\,,
\end{aligned}
$$

where $\text{b.d.}|_{r_1}$ stands for terms known from data at r_1.

B.1. $\alpha = m = 0$

When $\alpha = m = 0$, inserting (287) into the u-derivative of (150) leads to

(288)

$$
\begin{aligned}
\partial_u^{i+1}h_{AB} &= \sum_{0\le j+k\le i, k\ne i} \overset{(i,j,k)}{\psi}(r)\partial_r^j P^k \partial_u h_{AB}\\
&\quad + \sum_{j=0}^{i}\int_{r_1}^{r}\overset{(i,j)}{\psi}(s,r)P^j\partial_u h_{AB}\,ds + \text{b.d.}|_{r_1}\\
&= \sum_{0\le j+k\le i, k\ne i} \overset{(i,j,k)}{\psi}(r)\partial_r^j P^k\Big[\frac{\varepsilon}{2}\Big[\partial_r h_{AB} - \frac{1}{r}h_{AB}\Big]\\
&\qquad\qquad + \int_{r_1}^{r}\Big(\frac{1}{3sr} + \frac{2r^2}{3s^4}\Big)Ph_{AB}\,ds\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=0}^{i}\int_{r_1}^{r}\underbrace{\overset{(i,j)}{\psi}(s,r)P^j\left[\frac{\varepsilon}{2}\Big[\partial_s h_{AB}-\frac{1}{s}h_{AB}\Big]_s\right.}_{\text{integrate by parts}}\\
&\qquad\qquad\left.+\int_{r_1}^{s}\Big(\frac{1}{3ys}+\frac{2s^2}{3y^4}\Big)Ph_{AB}|_y dy\right]ds+\text{b.d.}|_{r_1}\\
&=\sum_{0\le j+k\le i,k\neq i}\overset{(i,j,k)}{\psi}(r)\partial_r^j P^k\Big[\frac{\varepsilon}{2}\Big[\partial_r h_{AB}-\frac{1}{r}h_{AB}\Big]\\
&\qquad\qquad\qquad\qquad+\int_{r_1}^{r}\Big(\frac{1}{3sr}+\frac{2r^2}{3s^4}\Big)Ph_{AB}\,ds\Big]\\
&+\frac{\varepsilon}{2}\sum_{j=0}^{i}\overset{(i,j)}{\psi}(s,r)\big|_{s=r}P^j h_{AB}\\
&+\sum_{j=0}^{i}\int_{r_1}^{r}\big(-\frac{\varepsilon}{2}\partial_s\overset{(i,j)}{\psi}(s,r)-\frac{\varepsilon}{2s}\overset{(i,j)}{\psi}(s,r)\big)P^j h_{AB}\big|_s ds\\
&+\sum_{j=0}^{i}\underbrace{\int_{r_1}^{r}\overset{(i,j)}{\psi}(s,r)\int_{r_1}^{s}\Big(\frac{1}{3ys}+\frac{2s^2}{3y^4}\Big)P^{j+1}h_{AB}|_y dy\,ds}_{\int_{r_1}^{r}\Big(\int_s^r\Big(\frac{1}{3ys}+\frac{2y^2}{3s^4}\Big)\overset{(i,j)}{\psi}(y,r)dy\Big)P^{j+1}h_{AB}|_s ds}+\text{b.d.}|_{r_1}\,.
\end{aligned}
$$

One finds that a term

$$
a_{ki\ell}s^{-\ell}\ \text{ in }\ \overset{(k,i)}{\psi} \tag{289}
$$

with $\ell\notin\{0,3\}$ induces terms s^{-1}, s^{-4} and

$$
a_{ki\ell}\varepsilon\frac{\ell-1}{2}s^{-(\ell+1)}\ \text{ in }\ \overset{(k+1,i)}{\psi}\ \text{ and }\ a_{ki\ell}\frac{\ell-1}{\ell(\ell-3)}s^{-(\ell+1)}\ \text{ in }\ \overset{(k+1,i+1)}{\psi}\ ; \tag{290}
$$

see Figure B.1, where we have anticipated the fact that the highest powers of s^{-1} are not affected by α.

We thus find

$$
\overset{(k,k)}{\psi}(s,r)=\underbrace{\frac{2r^2}{(k-1)!(k+2)}}_{=:\overset{(k,k)}{\psi}_{k+3}(r)}\frac{1}{s^{k+3}}+\dots\,, \tag{291}
$$

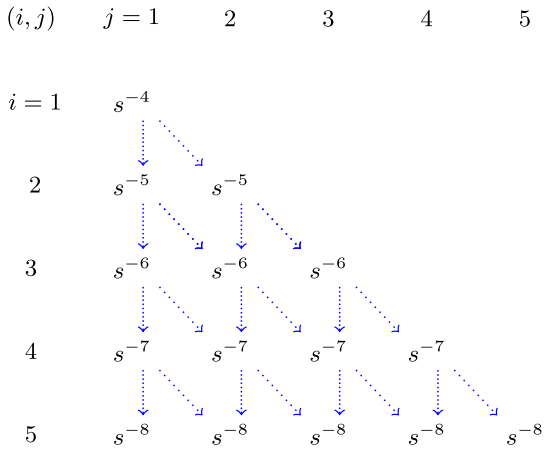

Figure B.1: Highest powers of s^{-1} in $\overset{(i,j)}{\psi}$ when $m=0$, $\alpha\in\mathbb{R}$. The structure of the tree for $\overset{(i,j)}{\chi}$ is identical after replacing (i,j) in the table by $(i-1,j-1)$, thus $(1,1)$ becomes $(0,0)$, etc.

where $\ldots$ denotes a sum of lower-order powers of s^{-1}.

An identical calculation applies to the $\overset{(k,i)}{\chi}$'s, since (151) has an identical structure as (150) from the point of view of induction. In particular the recurrence relation (289)-(290) remains unchanged. After taking into account the initialisation of the recurrence, which is different for the χ's and ψ's, one obtains

$$\overset{(k,k)}{\chi} = \underbrace{\frac{1}{(k)!(k+3)}}_{=:\overset{(k,k)}{\chi}_{k+4}} \frac{1}{s^{k+4}} + \ldots . \tag{292}$$

Let us write

$$\overset{(i,j)}{\chi}(s,r) = \sum_{\ell=1}^{i+4} \overset{(i,j)}{\chi}_{\ell}(r)\, s^{-\ell}\,, \qquad \overset{(i,j)}{\psi}(s,r) = \sum_{\ell=1}^{i+4} \overset{(i,j)}{\psi}_{\ell}(r)\, s^{-\ell}\,. \tag{293}$$

Since (cf. (276)-(283) with $m=0$, and regardless of α)

$$\overset{(2,0)}{\psi}_{5}(r) = 0 =: 2r^2 \overset{(0,-1)}{\chi}_{5}(r)\,, \qquad \overset{(2,1)}{\psi}_{5}(r) = 2r^2 \overset{(1,0)}{\chi}_{5}(r)\,,$$
$$\overset{(2,2)}{\psi}_{5}(r) = 2r^2 \overset{(1,1)}{\chi}_{5}(r)\,,$$

it follows by induction from (290) that

$$\overset{(i,j)}{\psi}_{i+3}(s,r) = 2r^2 \overset{(i-1,j-1)}{\chi}_{i+3}(s,r)\,. \tag{294}$$

Next, using

$$\overset{(1,0)}{\chi}(s,r) = \frac{\alpha^2\hat\kappa_1(s)}{2r^2} + \frac{\varepsilon\hat\kappa_5(s)}{2}\,, \quad \overset{(1,1)}{\chi}(s,r) = \frac{\hat\kappa_5(s)}{4} - \frac{\hat\kappa_1(s)}{4r^4}\,, \tag{295}$$

(cf. (282)-(283)) it follows by induction that

$$\overset{(i,j)}{\chi}_{i+3}(s,r) = 0\,. \tag{296}$$

B.2. The general case

When $\alpha \neq 0$ and $m \neq 0$, we will have instead

$$\begin{aligned}
&\partial_u^{i+1} h_{AB} \\
&= \text{right-hand side of (288)} \\
&\quad + \sum_{0\le j+k\le i, k\neq i} \overset{(i,j,k)}{\psi}(r)\partial_r^j P^k \Bigg[-(\frac{\alpha^2 r^2}{2} + \frac{m}{r})\Big[\partial_r h_{AB} - \frac{1}{r}h_{AB}\Big] \\
&\qquad\qquad + \int_{r_1}^r (\frac{\alpha^2 r}{s} - \frac{mr}{s^4}) h_{AB}\, ds \Bigg] \\
&\quad + \sum_{j=0}^{i} \underbrace{\int_{r_1}^r \overset{(i,j)}{\psi}(s,r) P^j \Bigg[-(\frac{\alpha^2 s^2}{2} + \frac{m}{s})\Big[\partial_s h_{AB}\big|_s - \frac{1}{s}h_{AB}\big|_s\Big]}_{\text{integrate by parts}} \\
&\qquad\qquad + \int_{r_1}^s (\frac{\alpha^2 s}{y} - \frac{ms}{y^4}) h_{AB}\big|_y dy \Bigg] ds \\
&= \text{right-hand side of (288)} \\
&\quad + \sum_{0\le j+k\le i, k\neq i} \overset{(i,j,k)}{\psi}(r)\partial_r^j P^k \Bigg[-(\frac{\alpha^2 r^2}{2} + \frac{m}{r})\Big[\partial_r h_{AB} - \frac{1}{r}h_{AB}\Big] \\
&\qquad\qquad + \int_{r_1}^r (\frac{\alpha^2 r}{s} - \frac{mr}{s^4}) h_{AB}\, ds \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=0}^{i}\Big(\big(\frac{\alpha^2 r^2}{2}+\frac{m}{r}\big)\overset{(i,j)}{\psi}(s,r)\Big)\Big|_{s=r}P^j h_{AB}\Big|_r \\
&+\sum_{j=0}^{i}\int_{r_1}^{r}\Bigg(\underbrace{\partial_s\big[\big(\frac{\alpha^2 s^2}{2}+\frac{m}{s}\big)\overset{(i,j)}{\psi}(s,r)\big]+\big(\frac{\alpha^2 s}{2}+\frac{m}{s^2}\big)\overset{(i,j)}{\psi}(s,r)}_{=(\frac{\alpha^2 s^2}{2}+\frac{m}{s})\partial_s\overset{(i,j)}{\psi}(s,r)+\frac{3\alpha^2 s}{2}\overset{(i,j)}{\psi}(s,r)}\Bigg)P^j h_{AB}\Big|_s\,ds \\
&+\sum_{j=0}^{i}\underbrace{\int_{r_1}^{r}\overset{(i,j)}{\psi}(s,r)\int_{r_1}^{r}\big(\frac{\alpha^2 s}{y}-\frac{ms}{y^4}\big)P^j h_{AB}\Big|_y dy\,ds}_{=\int_{r_1}^{r}\left(\int_s^r(\frac{\alpha^2 y}{s}-\frac{my}{s^4})\overset{(i,j)}{\psi}(y,r)dy\right)P^j h_{AB}|_s\,ds} \\
&+\text{b.d.}|_{r_1}\,,
\end{aligned}
$$

It follows that, in addition to (290), a term

$$a_{ki\ell}s^{-\ell}\ \text{ in }\ \overset{(k,i)}{\psi}\,,$$

with $k\geq 1$ and $0\leq\ell\neq 2$, induces terms involving $1/s$, $1/s^4$, and a term

$$a_{ki\ell}\frac{(1-\ell)}{2(2-\ell)}\Big(\alpha^2(4-\ell)s^{-\ell+1}+2m(1-\ell)s^{-\ell-2}\Big)\ \text{ in }\ \overset{(k+1,i)}{\psi}\,; \tag{297}$$

cf. Figures B.2 and B.3. This shows in particular that the recursion formulae (291) and (292), established with $\alpha=0$, remain valid for $\alpha,m\in\mathbb{R}$; but e.g. (294) does not hold anymore when $m\neq 0$.

To continue, it is convenient to set

$$\overset{(k,-1)}{\psi}_{\ell}=0\,. \tag{298}$$

Using this notation, putting together (290) with (297) we find the recursion formula, for $k\geq i\geq 0$ and $k\geq 1$,

$$
\begin{aligned}
\overset{(k+1,i)}{\psi}(s,r)=\ &\overset{(k+1,i)}{\psi}_{0}(r)+\frac{\overset{(k+1,i)}{\psi}_{1}(r)}{s}+\frac{\overset{(k+1,i)}{\psi}_{4}(r)}{s^4} \\
&+\sum_{\ell=4}^{k+3}\Big[\frac{(1-\ell)}{2(2-\ell)s^{\ell}}\Big(\alpha^2(4-\ell)s-\frac{(\ell-2)\varepsilon}{s}+\frac{2m(1-\ell)}{s^2}\Big)\overset{(k,i)}{\psi}_{\ell}(r) \\
&\qquad+\frac{(\ell-1)}{\ell(\ell-3)s^{\ell+1}}\overset{(k,i-1)}{\psi}_{\ell}(r)\Big]
\end{aligned}
$$

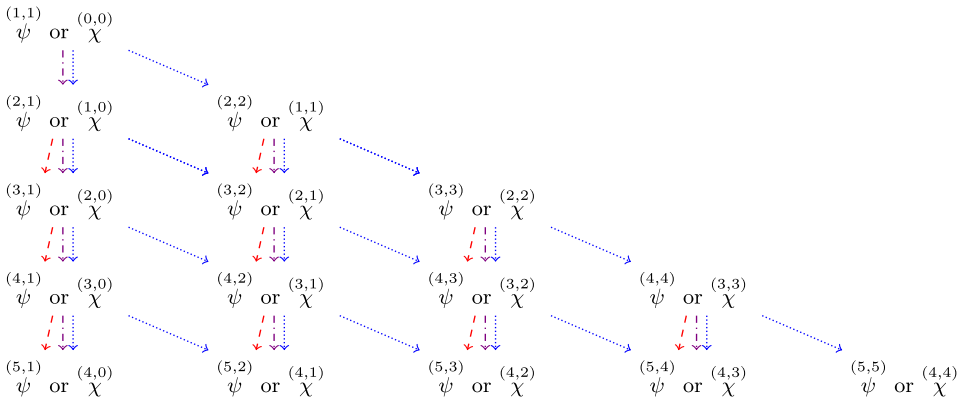

Figure B.2: Recursion tree for the integral kernels. The dash-dotted lines describe the contributions from the mass parameter m, increasing each power by 2. The dotted lines describe the contributions from the Gauss curvature ε, increasing each power by 1. The dashed lines arise from the cosmological constant, and are slanted to the left to visualise the fact that they decrease powers by 1.

$$
\begin{aligned}
&= \overset{(k+1,i)}{\psi}{}_0(r) + \frac{\overset{(k+1,i)}{\psi}{}_1(r)}{s} + \frac{\overset{(k+1,i)}{\psi}{}_4(r)}{s^4} \\
&\quad + \sum_{\ell=4}^{k+3}(\ell-1)\Bigg[\frac{\alpha^2(4-\ell)}{2(\ell-2)}\frac{\overset{(k,i)}{\psi}{}_\ell(r)}{s^{\ell-1}} - \frac{m(\ell-1)}{\ell-2}\frac{\overset{(k,i)}{\psi}{}_\ell(r)}{s^{\ell+2}} \\
&\qquad + \Big(\frac{\varepsilon}{2}\overset{(k,i)}{\psi}{}_\ell(r) + \frac{1}{\ell(\ell-3)}\overset{(k,i-1)}{\psi}{}_\ell(r)\Big)\frac{1}{s^{\ell+1}}\Bigg],
\end{aligned}
$$

An identical formula holds for χ with $k \geq i \geq 0$ after setting

$$
\overset{(k,-1)}{\chi}{}_\ell = 0\,. \tag{299}
$$

One is led to:

Lemma B.1. *The integral kernels* $\overset{(i,j)}{\psi}$ *and* $\overset{(i,j)}{\chi}$ *are polynomials in* $1/s$ *with coefficients depending upon* r*, with no terms* $1/s^2$ *and* $1/s^3$*. Moreover*

a) When $m = 0$*, the integral kernels* $\overset{(i,j)}{\psi}$*,* $i \geq 1$*,* $1 \leq j \leq i$*, are polynomials in* $1/s$ *of order* $i+3$*, with* $\overset{(i,0)}{\psi}$ *of order 1.*

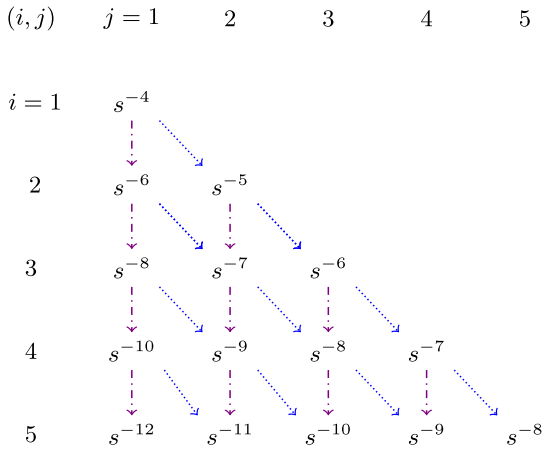

Figure B.3: Highest powers of s^{-1} in $\overset{(i,j)}{\psi}$ when $m \neq 0$. The dash-dotted lines describe the contributions from the mass parameter m, corresponding to an increase of the highest power by 2. The dotted lines describe the contributions from (290), increasing the power by 1. The tree for $\overset{(i,j)}{\chi}$ is identical after replacing (i,j) in the table by $(i-1, j-1)$.

b) When $m = 0$, the integral kernels $\overset{(i,j)}{\chi}$, $0 \leq j \leq i$, are polynomials in $1/s$ of order $i+4$.

c) When $m \neq 0$, the integral kernels $\overset{(i,j)}{\psi}$, $i \geq 1$, $0 \leq j \leq i$, are polynomials in $1/s$ of order not larger than $2i+3-j$, with $\overset{(i,0)}{\psi}$ and $\overset{(i,1)}{\psi}$ of order $2i+2$, and $\overset{(i,i)}{\psi}$ of order $i+3$.

d) When $m \neq 0$, the integral kernels $\overset{(i,j)}{\chi}$, $0 \leq j \leq i$, are polynomials in $1/s$ of order not larger than $2i+4-j$, with $\overset{(i,0)}{\chi}$ of order $2i+4$, and $\overset{(i,i)}{\chi}$ of order $i+4$.

Proof. We summarise the arguments so far, and add some details:

1. The functions that initialise the induction for $\partial_u h_{AB}$ involve only $1/s$ and $1/s^4$ terms, and the functions that initialise the induction for $\partial_u h_{uA}$ involve only $1/s$ and $1/s^5$ terms.
2. One then applies the recursion formulae (290) and (297); cf. Figures B.1 and B.3. We note that $\ln r$- and/or $\ln s$-terms could *a priori* arise in the induction from $1/s$ terms in some integrals, but the multi-

plicative coefficients $(\ell-1)$ which appear in the second and third lines in (290) and (297) guarantee that there will be no s^{-2} terms in any of the integral kernels, which in turns guarantees that no logarithmic terms will occur.

3. When $m=0$, the fact that $\overset{(i,j)}{\psi}$ is of order $i+3$ in s^{-1} follows from (346).
4. Point a) together with the equality $\overset{(i,j)}{\psi}_{i+3}(s,r) = 2r^2\,\overset{(i-1,j-1)}{\chi}_{i+3}(s,r)$ (cf. (294)) establishes b).
5. It follows from (291) that $\overset{(i,i)}{\psi}$ is of order s^{-i-3} when $m=0$, and Figure B.3 makes it clear that this is not affected by the non-vanishing of m.
6. By following the dashed-dotted arrows in Figure B.3 starting from the $(1,1)$ entry makes it clear that $\overset{(i,1)}{\psi}$ is of order $2i+2$ in s^{-1} when $m\neq 0$. The same holds for $\overset{(i,0)}{\psi}$ since the recursion formulae do not depend upon the index j of $\overset{(i,j)}{\psi}$, and both initialising polynomials $\overset{(1,0)}{\psi}$ and $\overset{(1,1)}{\psi}$ are of order 4. In fact one checks that

$$\overset{(k,0)}{\chi}_{2k+4} = \frac{(-m)^k}{3}\frac{\big((2k+1)!\big)^2}{2^{3k}(k!)^3}\,, \tag{300}$$

$$\overset{(k,0)}{\psi}_{2k+2} = \frac{r(-m)^k}{2^{3k-1}(k-1)!}\left(\frac{(2k)!}{k!}\right)^2 = -\frac{3m}{2r}\overset{(k,1)}{\psi}_{2k+2}\,, \tag{301}$$

which further implies

$$\overset{(k,0)}{\chi}_{2k+4} = -\frac{1}{3mr}\overset{(k+1,0)}{\psi}_{2k+4} = \frac{1}{2r^2}\overset{(k+1,1)}{\psi}_{2k+4}\,. \tag{302}$$

□

We finish this section with the following relations, needed for (370):

Lemma B.2. *For $k\geq 2$ we have*

$$\overset{(k-1,0)}{\chi}_{2k+2}\overset{(k,0)}{\psi}_{2k+1} = \overset{(k-1,0)}{\chi}_{2k+1}\overset{(k,0)}{\psi}_{2k+2}\,, \tag{303}$$

$$\overset{(k-1,0)}{\chi}_{2k+2}\overset{(k,1)}{\psi}_{2k+1} = \overset{(k-1,1)}{\chi}_{2k+1}\overset{(k,0)}{\psi}_{2k+2} + \overset{(k-1,0)}{\chi}_{2k+1}\overset{(k,1)}{\psi}_{2k+2}\,, \tag{304}$$

$$\overset{(k-1,0)}{\chi}_{2k+2}\overset{(k,2)}{\psi}_{2k+1} = \overset{(k-1,1)}{\chi}_{2k+1}\overset{(k,1)}{\psi}_{2k+2}\,. \tag{305}$$

Proof. We start by noting the following recursion formulae, which can be read off (298)-(299), for $k \geq 1$, $k \geq i \geq 0$, and $n \geq 5$:

$$\overset{(k,i)}{\chi}_n = m_n \overset{(k-1,i)}{\chi}_{n-2} + \varepsilon_n \overset{(k-1,i)}{\chi}_{n-1} + \alpha_n \overset{(k-1,i)}{\chi}_{n+1} + \iota_n \overset{(k-1,i-1)}{\chi}_{n-1} ,$$
$$\overset{(k,i)}{\psi}_n = m_n \overset{(k-1,i)}{\psi}_{n-2} + \varepsilon_n \overset{(k-1,i)}{\psi}_{n-1} + \alpha_n \overset{(k-1,i)}{\psi}_{n+1} + \iota_n \overset{(k-1,i-1)}{\psi}_{n-1} ,$$

where m_n arises from the mass m, ε_n from the Gauss curvature ε of $\mathring{\gamma}$, α_n from the cosmological constant encoded in α, and ι_n is associated with the term containing a shift in i:

$$m_n = -m\frac{(n-3)^2}{n-4} , \quad \varepsilon_n = \varepsilon\frac{n}{2} , \quad \alpha_n = -\alpha^2\frac{n(n-3)}{2(n-1)} ,$$
$$\iota_n = \frac{n-2}{(n-1)(n-4)} .$$

By Lemma B.1, the coefficients $\overset{(i,j)}{\chi}_\ell$ vanish for $\ell + j > 2i + 4$, and the coefficients $\overset{(i,j)}{\psi}_\ell$ vanish for $\ell + j > 2i + 3$. Thus, for $k \geq 2$ we can write

$$\overset{(k,0)}{\psi}_{2k+2} = m_{2k+2} \overset{(k-1,0)}{\psi}_{2k} , \qquad \overset{(k,1)}{\psi}_{2k+2} = m_{2k+2} \overset{(k-1,1)}{\psi}_{2k} ,$$
$$\overset{(k,i)}{\psi}_{2k+1} = m_{2k+1} \overset{(k-1,i)}{\psi}_{2k-1} + \varepsilon_{2k+1} \overset{(k-1,i)}{\psi}_{2k} + \alpha_{2k+1} \overset{(k-1,i)}{\psi}_{2k+2} + \iota_{2k+1} \overset{(k-1,i-1)}{\psi}_{2k} ,$$
$$\overset{(k,2)}{\psi}_{2k+1} = m_{2k+1} \overset{(k-1,2)}{\psi}_{2k-1} + \underbrace{\varepsilon_{2k+1} \overset{(k-1,2)}{\psi}_{2k} + \alpha_{2k+1} \overset{(k-1,2)}{\psi}_{2k+2}}_{=0} + \iota_{2k+1} \overset{(k-1,1)}{\psi}_{2k} ,$$
$$\overset{(k,1)}{\psi}_{2k+1} = m_{2k+1} \overset{(k-1,1)}{\psi}_{2k-1} + \varepsilon_{2k+1} \overset{(k-1,1)}{\psi}_{2k} + \alpha_{2k+1} \underbrace{\overset{(k-1,1)}{\psi}_{2k+2}}_{=0} + \iota_{2k+1} \overset{(k-1,0)}{\psi}_{2k} ,$$
$$\overset{(k,0)}{\psi}_{2k+1} = m_{2k+1} \overset{(k-1,0)}{\psi}_{2k-1} + \varepsilon_{2k+1} \overset{(k-1,0)}{\psi}_{2k} + \underbrace{\alpha_{2k+1} \overset{(k-1,0)}{\psi}_{2k+2} + \iota_{2k+1} \overset{(k-1,-1)}{\psi}_{2k}}_{=0} .$$

Similarly for $k \geq 3$ we have

$$
\begin{aligned}
\overset{(k-1,0)}{\chi}{}_{2k+2} &= m_{2k+2}\overset{(k-2,0)}{\chi}{}_{2k}\,,\\
\overset{(k-1,1)}{\chi}{}_{2k+1} &= m_{2k+1}\overset{(k-2,1)}{\chi}{}_{2k-1} + \underbrace{\varepsilon_{2k+1}\overset{(k-2,1)}{\chi}{}_{2k} + \alpha_{2k+1}\overset{(k-2,1)}{\chi}{}_{2k+2}}_{=0}\\
&\quad + \iota_{2k+1}\overset{(k-2,0)}{\chi}{}_{2k}\,,\\
\overset{(k-1,0)}{\chi}{}_{2k+1} &= m_{2k+1}\overset{(k-2,0)}{\chi}{}_{2k-1} + \varepsilon_{2k+1}\overset{(k-2,0)}{\chi}{}_{2k}\\
&\quad + \underbrace{\alpha_{2k+1}\overset{(k-2,0)}{\chi}{}_{2k+2} + \iota_{2k+1}\overset{(k-2,-1)}{\chi}{}_{2k}}_{=0}\,.
\end{aligned}
$$

We now check that (303)-(305) hold with $k = 2$:

$$
\underbrace{\overset{(1,0)}{\chi}{}_{6}}_{-\frac{3m}{2}\text{ by (282)};\ -\frac{3mr\varepsilon}{2},\text{ by (277)};}\ \underbrace{\overset{(2,0)}{\psi}{}_{5}}_{} = \underbrace{\overset{(1,0)}{\chi}{}_{5}}_{\frac{\varepsilon}{2}\text{ by (282)};\ \frac{9m^2r}{2}\text{ by (277)};}\ \underbrace{\overset{(2,0)}{\psi}{}_{6}}_{}\,,
$$

$$
\begin{aligned}
\underbrace{\overset{(1,0)}{\chi}{}_{6}}_{-\frac{3m}{2}\text{ by (282)};\ r^2\varepsilon-\frac{3mr}{4}\text{ by (278)};}\ \underbrace{\overset{(2,1)}{\psi}{}_{5}}_{} &= \underbrace{\overset{(1,1)}{\chi}{}_{5}}_{\frac{1}{4}\text{ by (283)};\ \frac{9m^2r}{2}\text{ by (277)};}\ \underbrace{\overset{(2,0)}{\psi}{}_{6}}_{}\\
&\quad + \underbrace{\overset{(1,0)}{\chi}{}_{5}}_{\frac{\varepsilon}{2}\text{ by (282)};\ -3mr^2\text{ by (278)};}\ \underbrace{\overset{(2,1)}{\psi}{}_{6}}_{}\,,
\end{aligned}
$$

$$
\underbrace{\overset{(1,0)}{\chi}{}_{6}}_{-\frac{3m}{2}\text{ by (282)};\ \frac{r^2}{2}\text{ by (279)};}\ \underbrace{\overset{(2,2)}{\psi}{}_{5}}_{} = \underbrace{\overset{(1,1)}{\chi}{}_{5}}_{\frac{1}{4}\text{ by (283)};\ -3mr^2\text{ by (278)};}\ \underbrace{\overset{(2,1)}{\psi}{}_{6}}_{}\,.
$$

To continue, let $k \geq 3$ and assume that (303) holds with k replaced by $k-1$, then:

$$
\begin{aligned}
&\overset{(k-1,0)}{\chi}{}_{2k+2}\overset{(k,0)}{\psi}{}_{2k+1} - \overset{(k-1,0)}{\chi}{}_{2k+1}\overset{(k,0)}{\psi}{}_{2k+2}\\
&= m_{2k+2}\overset{(k-2,0)}{\chi}{}_{2k}\big(m_{2k+1}\overset{(k-1,0)}{\psi}{}_{2k-1} + \cancel{\varepsilon_{2k+1}\overset{(k-1,0)}{\psi}{}_{2k}}\big)\\
&\quad -\big(m_{2k+1}\overset{(k-2,0)}{\chi}{}_{2k-1} + \cancel{\varepsilon_{2k+1}\overset{(k-2,0)}{\chi}{}_{2k}}\big)m_{2k+2}\overset{(k-1,0)}{\psi}{}_{2k}\\
&= m_{2k+2}\,m_{2k+1}\big(\overset{(k-2,0)}{\chi}{}_{2k}\overset{(k-1,0)}{\psi}{}_{2k-1} - \overset{(k-2,0)}{\chi}{}_{2k-1}\overset{(k-1,0)}{\psi}{}_{2k}\big)\\
&\quad = 0\,.
\end{aligned}
$$

Next, we assume that (304) holds with k replaced by $k-1$. Then

$$
\begin{aligned}
&\overset{(k-1,0)}{\chi}_{2k+2}\overset{(k,1)}{\psi}_{2k+1}-\overset{(k-1,1)}{\chi}_{2k+1}\overset{(k,0)}{\psi}_{2k+2}-\overset{(k-1,0)}{\chi}_{2k+1}\overset{(k,1)}{\psi}_{2k+2}\\
&=m_{2k+2}\overset{(k-2,0)}{\chi}_{2k}\overset{(k,1)}{\psi}_{2k+1}-\overset{(k-1,1)}{\chi}_{2k+1}m_{2k+2}\overset{(k-1,0)}{\psi}_{2k}\\
&\quad-\overset{(k-1,0)}{\chi}_{2k+1}m_{2k+2}\overset{(k-1,1)}{\psi}_{2k}\\
&=m_{2k+2}\Big[\overset{(k-2,0)}{\chi}_{2k}\Big(m_{2k+1}\overset{(k-1,1)}{\psi}_{2k-1}+\cancel{\varepsilon_{2k+1}\overset{(k-1,1)}{\psi}_{2k}}\\
&\qquad\qquad+\cancel{\iota_{2k+1}\overset{(k-1,0)}{\psi}_{2k}}\Big)\\
&\qquad-\Big(m_{2k+1}\overset{(k-2,1)}{\chi}_{2k-1}+\cancel{\iota_{2k+1}\overset{(k-2,0)}{\chi}_{2k}}\Big)\overset{(k-1,0)}{\psi}_{2k}\\
&\qquad-\Big(m_{2k+1}\overset{(k-2,0)}{\chi}_{2k-1}+\cancel{\varepsilon_{2k+1}\overset{(k-2,0)}{\chi}_{2k}}\Big)\overset{(k-1,1)}{\psi}_{2k}\Big]\\
&=m_{2k+2}m_{2k+1}\Big[\overset{(k-2,0)}{\chi}_{2k}\overset{(k-1,1)}{\psi}_{2k-1}-\overset{(k-2,1)}{\chi}_{2k-1}\overset{(k-1,0)}{\psi}_{2k}\\
&\qquad\qquad-\overset{(k-2,0)}{\chi}_{2k-1}\overset{(k-1,1)}{\psi}_{2k}\Big]\\
&=0\,.
\end{aligned}
$$

Finally, suppose that (305) holds with k replaced by $k-1$. Then

$$
\begin{aligned}
&\overset{(k-1,0)}{\chi}_{2k+2}\overset{(k,2)}{\psi}_{2k+1}-\overset{(k-1,1)}{\chi}_{2k+1}\overset{(k,1)}{\psi}_{2k+2}\\
&=\quad m_{2k+2}\overset{(k-2,0)}{\chi}_{2k}\Big(m_{2k+1}\overset{(k-1,2)}{\psi}_{2k-1}+\cancel{\iota_{2k+1}\overset{(k-1,1)}{\psi}_{2k}}\Big)\\
&\qquad-\Big(m_{2k+1}\overset{(k-2,1)}{\chi}_{2k-1}+\cancel{m_{2k+2}\overset{(k-2,0)}{\chi}_{2k}}\Big)\iota_{2k+1}\overset{(k-1,1)}{\psi}_{2k}\\
&=\quad m_{2k+2}m_{2k+1}\Big(\overset{(k-2,0)}{\chi}_{2k}\overset{(k-1,2)}{\psi}_{2k-1}-\overset{(k-2,1)}{\chi}_{2k-1}\overset{(k-1,1)}{\psi}_{2k}\Big)\\
&=\quad 0\,.
\end{aligned}
$$

The validity of (303)-(305) follows thus by induction. □

Appendix C. Operators on S

The aim of this appendix is to analyse the mapping properties of several operators acting on tensor fields defined on a compact orientable two-

dimensional manifold $({}^2M \equiv \mathbf{S}, \mathring{\gamma})$ with constant Gauss curvature $\varepsilon \in \{0, \pm 1\}$.

C.1. Vector and tensor spherical harmonics

For integers $\ell \geq 1$, $-\ell \leq m \leq \ell$, let $Y^{(\ell m)}$ be the standard spherical harmonics on the unit sphere. Following the notations and conventions of [2, 20], we define the *vector spherical harmonics*, as well as trace-free symmetric 2-*tensor spherical harmonics* on S^2 as:

1. For $\ell \geq 1$, $-\ell \leq m \leq \ell$, define the vector fields

(306)
$$E_A^{(\ell m)} := -\frac{1}{\sqrt{\ell(\ell+1)}} \mathring{D}_A Y^{(\ell m)}\,, \quad H_A^{(\ell m)} := \frac{1}{\sqrt{\ell(\ell+1)}} \epsilon_{AB} \mathring{D}^B Y^{(\ell m)}\,,$$

where ϵ_{AB} denote the volume two-form of S^2.

2. For $\ell \geq 2$, $-\ell \leq m \leq \ell$, define the trace-free symmetric 2-tensors

$$\psi_{AB}^{(\ell m)} := -\frac{1}{\sqrt{\frac{1}{2}\ell(\ell+1) - 1}} C(E^{(\ell m)})_{AB}\,, \tag{307}$$

$$\phi_{AB}^{(\ell m)} := -\frac{1}{\sqrt{\frac{1}{2}\ell(\ell+1) - 1}} C(H^{(\ell m)})_{AB}\,, \tag{308}$$

where the operator $C(\xi)_{AB} = TS(\mathring{D}_A \xi_B)$ of (38) corresponds to the operator $-\not{D}_2^*$ of [2, 20].

Let us summarise the properties of these tensor harmonics, as needed in the main text. More details and proofs can be found in [20], see also [17].

Lemma C.1. *The following holds.*

1. *On S^2, L^2-integrable functions f, vector fields ξ and trace-free symmetric 2-tensors φ can be decomposed as*

$$f = \sum_{\ell \geq 0} \sum_{-\ell \leq m \leq \ell} f^{\ell m} Y^{(\ell m)}\,,$$
$$\xi_A = \sum_{\ell \geq 1} \sum_{-\ell \leq m \leq \ell} \xi_E^{(\ell m)} E_A^{(\ell m)} + \xi_H^{(\ell m)} H_A^{(\ell m)}\,,$$
$$\varphi_{AB} = \sum_{\ell \geq 2} \sum_{-\ell \leq m \leq \ell} \varphi_\psi^{(\ell m)} \psi_{AB}^{(\ell m)} + \varphi_\phi^{(\ell m)} \phi_{AB}^{(\ell m)}\,,$$

where

$$f^{(\ell m)} := \int_{S^2} f Y^{(\ell m)} d\mu_{\mathring{\gamma}}\,,$$

$$\xi_E^{(\ell m)} := \int_{S^2} \xi^A E_A^{(\ell m)}\,, \qquad \xi_H^{(\ell m)} := \int_{S^2} \xi^A H_A^{(\ell m)} d\mu_{\mathring{\gamma}}\,,$$

$$\varphi_\psi^{(\ell m)} := \int_{S^2} \varphi^{AB} \psi_{AB}^{(\ell m)}\,, \qquad \varphi_\phi^{(\ell m)} := \int_{S^2} \varphi^{AB} \phi_{AB}^{(\ell m)} d\mu_{\mathring{\gamma}}\,.$$

2. *It holds that for* $\ell \geq 2$,

$$\mathring{D}^A \psi_{AB}^{(\ell m)} = \sqrt{\frac{1}{2}\ell(\ell+1) - 1}\, E_B^{(\ell m)}\,, \tag{309}$$

$$\mathring{D}^A \phi_{AB}^{(\ell m)} = \sqrt{\frac{1}{2}\ell(\ell+1) - 1}\, H_B^{(\ell m)}\,. \tag{310}$$

3. *The space of conformal Killing vector fields on* S^2 *is spanned by* $E_A^{(1m)}$ *and* $H_A^{(1m)}$.

C.2. The conformal Killing operator

Consider the conformal Killing operator on a closed 2-dimensional Riemannian manifold $({}^2M, \mathring{\gamma})$:

$$\xi^A \mapsto \mathring{D}_A \xi_B + \mathring{D}_B \xi_A - \mathring{D}^C \xi_C \mathring{\gamma}_{AB} \equiv 2C(\xi)_{AB}\,. \tag{311}$$

We have

Proposition C.2. *The conformal Killing operator on two dimensional manifolds is elliptic, with*

1. *six dimensional kernel and no cokernel on* S^2*;*
2. *two dimensional kernel and cokernel on* $\mathbb{T}^2$*;*
3. *no kernel and* $6(\mathfrak{g}-1)$ *dimensional cokernel on manifolds of genus* $\mathfrak{g} \geq 2$.

Proof. We first show that C is elliptic. For this, let $0 \neq k \in T^*({}^2M)$ and let $\sigma(k)$ be the symbol of C, with kernel determined by the equation

$$\big(\sigma(k)\big)_{AB} \equiv \frac{1}{2}\big(k_A \xi_B + k_B \xi_A - k^C \xi_C \mathring{\gamma}_{AB}\big) = 0\,. \tag{312}$$

Contracting with $k^A k^B$ one obtains

$$k^A k_A k^C \xi_C = 0 \qquad \Longrightarrow \qquad k^C \xi_C = 0\,. \tag{313}$$

Equation (312) becomes now

$$k_A \xi_B + k_B \xi_A = 0\,. \tag{314}$$

Contracting with k^A one concludes that

$$k^A k_A \xi_B = 0\,. \tag{315}$$

Hence $\xi_B = 0$, and ellipticity of C follows.

Concerning the kernel in point 1., we start by noting that the equation

$$\mathring{D}_A \xi_B + \mathring{D}_B \xi_A - \mathring{D}^C \xi_C \mathring{\gamma}_{AB} = 0 \tag{316}$$

is conformally invariant. Hence it suffices to analyse it on the unit round sphere. Therefore, by Lemma C.1, its solution are of the form

$$\xi_A = \mathring{D}_A \varphi + \epsilon_{AB} \mathring{D}^B \psi\,,$$

where φ and ψ are linear combinations of $\ell = 1$ spherical harmonics. The φ-solutions are in one-to-one correspondence with the three generators of boosts of four-dimensional Minkowski space-time, while the ψ-solutions correspond to rotations.

The statements about the kernel in points 2. and 3. follow from Proposition C.3 which we are about to prove.

The statements about the cokernels follow from

$$C^\dagger = -\mathring{\mathrm{div}}_{(2)}$$

where $\mathring{\mathrm{div}}_{(2)}$ is the divergence operator on two-symmetric trace-free tensors,

$$(\mathring{\mathrm{div}}_{(2)}\, h)_A := \mathring{D}^B h_{AB}\,, \tag{317}$$

together with the results in Section C.3 below. □

Recall that we use the symbol CKV to denote the space of conformal Killing vectors, while TT denotes the space of trace-free divergence-free symmetric two-tensors, and orthogonality is defined in L^2. Then:

PROPOSITION C.3. *1. On $\mathbb{T}^2$ all conformal Killing vectors are covariantly constant, hence Killing.*

2. There are no nontrivial Killing vectors or conformal Killing vectors on higher genus two dimensional manifolds.

3. $\mathrm{im}(\mathring{\mathrm{div}}_{(2)} C) = \mathrm{CKV}^{\perp}$.

4. For any vector field ξ we have $C(\xi)^{[TT]} = 0$.

Proof. 1. and 2.: Taking the divergence of (316) and commuting derivatives leads to

$$\mathring{D}^A \mathring{D}_A \xi_B + \mathring{R}_{BC}\xi^C = 0\,. \tag{318}$$

Multiplying by ξ^B and integrating over ${}^2M$ one finds

$$\int (|\mathring{D}\xi|^2 - \mathring{R}_{BC}\xi^B\xi^C) = 0\,. \tag{319}$$

If $\mathring{R}_{BC} \leq 0$ we find that ξ is covariantly constant, vanishing if $\mathring{R}_{BC} < 0$.

3. Let η be L^2-orthogonal to the image of $\mathring{\mathrm{div}}_{(2)} C$, thus for any vector field ξ we have

$$\begin{aligned} 0 &= \int_{\mathbf{S}} \eta^A \mathring{D}_B(\mathring{D}_A\xi_B + \mathring{D}_B\xi_A - \mathring{D}^C\xi_C\mathring{\gamma}_{AB})d\mu_{\mathring{\gamma}} \\ &= 2\int_{\mathbf{S}} \eta^A \mathring{D}^B\big(\mathrm{TS}(\mathring{D}_A\xi_B)\big)d\mu_{\mathring{\gamma}} \\ &= -2\int_{\mathbf{S}} \mathring{D}^B\eta^A\,\mathrm{TS}(\mathring{D}_A\xi_B)d\mu_{\mathring{\gamma}} = -2\int_{\mathbf{S}} \mathrm{TS}(\mathring{D}^B\eta^A)\,\mathrm{TS}(\mathring{D}_A\xi_B)d\mu_{\mathring{\gamma}}\,. \end{aligned}$$

Letting $\xi = \eta$ we conclude that η is a conformal Killing vector.

4. The field $C(\xi)^{[\mathrm{TT}]}$ is obtained by L^2-projecting $C(\xi)$ on TT. As such, for any $h \in \mathrm{TT}$ we have

$$\begin{aligned} \int_{\mathbf{S}} h^{AB}C(\xi)_{AB}\,d\mu_{\mathring{\gamma}} &= \int_{\mathbf{S}} h^{AB}\big(\mathrm{TS}(\mathring{D}_A\xi_B)\big)\,d\mu_{\mathring{\gamma}} \\ &= \int_{\mathbf{S}} \mathrm{TS}(h^{AB})\mathring{D}_A\xi_B\,d\mu_{\mathring{\gamma}} = \int_{\mathbf{S}} h^{AB}\mathring{D}_A\xi_B\,d\mu_{\mathring{\gamma}} \\ &= -\int_{\mathbf{S}} \underbrace{\mathring{D}_A h^{AB}}_{0}\,\xi_B\,d\mu_{\mathring{\gamma}} = 0\,. \end{aligned}$$

Hence $C(\xi)^{[\mathrm{TT}]} = 0$. □

C.3. $\mathring{\mathrm{div}}_{(2)}$

We denote by $\mathring{\mathrm{div}}_{(1)}$ the divergence operator on vector fields:

$$\mathring{\mathrm{div}}_{(1)}\,\xi := \mathring{D}_A\xi^A\,. \tag{320}$$

and by $\mathring{\mathrm{div}}_{(2)}$ that on two-symmetric trace-free tensors,

$$(\mathring{\mathrm{div}}_{(2)}\,h)_A := \mathring{D}^B h_{AB}\,. \tag{321}$$

As is well-known, $\mathring{\mathrm{div}}_{(2)}$ is conformally covariant in all dimensions. In particular, in dimension two if $g_{AB} = e^{\varphi}\bar{g}_{AB}$ then

$$D_A h^{AB} = e^{-2\varphi}\bar{D}_A(e^{2\varphi}h^{AB})\,, \tag{322}$$

where D is the Levi-Civita connection of g and $\bar{D}$ that of $\bar{g}$. It follows that it suffices to understand the kernel for metrics of constant Gauss curvature.

As already pointed out, on a two-dimensional closed negatively curved manifold of genus $\mathfrak{g} \geq 2$, the operator $\mathring{\mathrm{div}}_{(2)}$ has a $6(\mathfrak{g}-1)$-dimensional kernel; it has no kernel on S^2; on a flat torus $\mathring{\mathrm{div}}_{(2)}$ has a two-dimensional kernel consisting of covariantly constant fields (cf., e.g., [10] Theorem 8.2 and the paragraph that follows or [11, Theorem 6.1 and Corollary 6.1]).

We claim that:

Lemma C.4. *Consider a two-dimensional Riemannian manifold* $({}^2M, \mathring{\gamma})$. *Then the operator* $\mathring{\mathrm{div}}_{(2)}$ *acting on symmetric traceless tensors is elliptic, and it holds that*

$$\mathrm{im}\,\mathring{\mathrm{div}}_{(2)} = \mathrm{CKV}^{\perp}\,.$$

In particular if $\mathring{R}_{BC} < 0$, *the operator* $\mathring{\mathrm{div}}_{(2)}$ *is surjective.*

Proof. We start with ellipticity. For this, let $0 \neq k \in T^*({}^2M)$ and let $\sigma(k)$ be the symbol of $\mathring{\mathrm{div}}_{(2)}$, with kernel determined by the equation

$$\big(\sigma(k)h\big)_A \equiv k^C h_{AC} = 0\,. \tag{323}$$

In an orthonormal frame in which $k^2 = 0$ this is equivalent to

$$h_{11} = h_{12} = 0\,. \tag{324}$$

For symmetric and traceless tensors h_{AB} this is the same as $h_{AB} = 0$. So $\sigma(k)$ has trivial kernel for $k \neq 0$, which is the definition of ellipticity.

Next, let ξ be L^2-orthogonal to the image of $\mathring{\mathrm{div}}_{(2)}$, then for all smooth symmetric traceless tensors h we have

$$0 = \int \xi^A \mathring{D}^B h_{AB} = -\int \mathring{D}^B \xi^A h_{AB} = -\int \mathrm{TS}(\mathring{D}^B \xi^A) h_{AB}\,. \tag{325}$$

This shows that $\mathrm{TS}(\mathring{D}^B \xi^A) = 0$, hence ξ^A is a conformal vector field.

Since no such fields exist when the Ricci tensor is negative by Proposition C.3, surjectivity for such metrics follows. □

C.4. $\widehat{\mathrm{L}}$ and L

To continue, we wish to analyse the operators

$$\widehat{\mathrm{L}} = -\mathring{\mathrm{div}}_{(2)}\, C\, \mathrm{L}\,, \quad \mathrm{L} = (\mathring{D}\mathring{\mathrm{div}}_{(1)} - \mathring{\mathrm{div}}_{(2)}\, C + \varepsilon)\,; \tag{326}$$

recall that $\mathring{\mathrm{div}}_{(1)}\, \xi = \mathring{D}_A \xi^A$, $(\mathring{\mathrm{div}}_{(2)}\, h)_A = \mathring{D}^B h_{AB}$, and that $\varepsilon \in \{0, \pm 1\}$ is the Gauss curvature of $\mathring{\gamma}$.

We consider first the operator $\xi \mapsto \mathring{\mathrm{div}}_{(2)}\, C(\xi)$. One finds

$$\big(\mathring{\mathrm{div}}_{(2)}\, C(\xi)\big)_A = \frac{1}{2}(\Delta_{\mathring{\gamma}} + \varepsilon)\xi_A\,, \tag{327}$$

which is elliptic, self-adjoint, with kernel and cokernel spanned by conformal Killing vectors.

Next, we turn our attention to L:

$$\begin{aligned} \mathrm{L}(\xi)_A &= \mathring{D}_A \mathring{D}^C \xi_C + \frac{1}{2}\big(\underbrace{\mathring{D}_A \mathring{D}^C \xi_C - \mathring{D}^C(\mathring{D}_A \xi_C}_{-\mathring{R}^C{}_A \xi_C} + \mathring{D}_C \xi_A)\big) + \varepsilon \xi_A \\ &= \mathring{D}_A \mathring{D}^C \xi_C + \frac{1}{2}\big(-\Delta_{\mathring{\gamma}} + \varepsilon\big)\xi_A\,. \end{aligned} \tag{328}$$

One readily checks that L is also elliptic and self-adjoint.

Applying $\mathring{D}^A$ to (328), commuting derivatives, and using

$$\mathring{R}_{AB} = \varepsilon \mathring{\gamma}_{AB} \tag{329}$$

one finds that the kernel of L consists of vector fields satisfying

$$\frac{1}{2}\Delta_{\mathring{\gamma}} \mathring{D}_A \xi^A = 0\,, \tag{330}$$

hence $\mathring{D}_A\xi^A = c$ for some constant c. Integrating this last equality over ${}^2M$ shows that $c = 0$. It now follows that the kernel of $\widehat{\mathrm{L}}$ consists of vector fields satisfying

$$\big(-\Delta_{\mathring{\gamma}} + \varepsilon\big)\xi_A = 0\,, \qquad \mathring{D}_A\xi^A = 0\,. \tag{331}$$

Recall the Hodge decomposition: on a compact two dimensional oriented manifold every one-form can be decomposed as

$$\xi_A = \mathring{D}_A\psi + \epsilon_{AB}\mathring{D}^B\phi + r_A\,, \tag{332}$$

where r_A is a harmonic one-form, i.e. a covector field satisfying

$$\mathring{D}^A r_A = 0 = \epsilon^{AB}\mathring{D}_A r_B = (-\Delta_{\mathring{\gamma}} + \varepsilon)r_A\,. \tag{333}$$

On S^2 the forms r_A vanish identically, and on manifolds with genus $\mathfrak{g}$ the space of r_A's is $2\mathfrak{g}$-dimensional; cf., e.g., [8, Theorems 19.11 and 19.14] or [9, Theorem 18.7].

From the second equation in (331) together with (332)-(333) we find that the Laplacian of ψ vanishes, hence ψ is constant, and the first equation in (331) gives

$$\epsilon^{AB}\mathring{D}_B\Delta_{\mathring{\gamma}}\phi = 0\,. \tag{334}$$

It readily follows that ϕ is also constant, hence $\xi_A = r_A$, and we conclude that:

Lemma C.5. *The operator* L *is elliptic, self-adjoint, with kernel and cokernel consisting of one-forms* r_A *satisfying* (333)*, hence of dimension equal to twice the genus of the compact, oriented, two-dimensional manifold.*

We are ready now to pass to the proof of:

Proposition C.6. *The operator* $\widehat{\mathrm{L}}$ *is elliptic, self-adjoint, with*

$$\ker\widehat{\mathrm{L}} = \operatorname{coker}\widehat{\mathrm{L}} = \mathrm{CKV} + \mathrm{H}\,.$$

In particular:

1. *on* S^2 *and on* T^2 *we have* $\ker\widehat{\mathrm{L}} = \operatorname{coker}\widehat{\mathrm{L}} =$ *CKV;*
2. *on two-dimensional compact orientable manifolds of genus* $\mathfrak{g} \geq 2$ *both the kernel and cokernel of* $\widehat{\mathrm{L}}$ *are spanned by the* $2\mathfrak{g}$*-dimensional space of harmonic 1-forms.*

Proof. We first check that L and $-\mathring{\mathrm{div}}_{(2)}\, C$ commute. In view of (327)-(328) it suffices to check the identity

$$(\Delta_{\mathring{\gamma}} + \varepsilon)\mathring{D}_A \mathring{D}^C \xi_C = \mathring{D}_A \mathring{D}^C (\Delta_{\mathring{\gamma}} + \varepsilon)\xi_C \,, \tag{335}$$

which follows from a straightforward commutation of derivatives. This shows that $\widehat{\mathrm{L}}$ is the composition of two commuting self-adjoint elliptic operators, hence elliptic and self-adjoint.

On S^2 the operator L is an isomorphism by Lemma C.5, hence the cokernel of $\widehat{\mathrm{L}}$ is determined by that of $-\mathring{\mathrm{div}}_{(2)}\, C$. The claim on the kernel follows by duality.

It should be clear that in manifestly flat coordinates on $\mathbb{T}^2$ the kernels of both L and $-\mathring{\mathrm{div}}_{(2)}\, C$ consist of covectors ξ_A with constant entries, which span the space of conformal Killing vectors on $\mathbb{T}^2$. Self-adjointness implies the result for the cokernel.

In the higher genus case the operator $-\mathring{\mathrm{div}}_{(2)}\, C$ is an isomorphism, so that the kernel of $\widehat{\mathrm{L}}$ coincides with the kernel of L, as given by Lemma C.5. One concludes as before. □

C.5. *P*

Consider the operator

$$P h_{AB} := \mathrm{TS}[\mathring{D}_A \mathring{D}^C h_{BC}] \,. \tag{336}$$

of (97). where h is symmetric and $\mathring{\gamma}$-traceless.

We have:

Proposition C.7. *The operator P is elliptic, self-adjoint and negative, with*

1. *six-dimensional cokernel and kernel on S^2;*
2. *two-dimensional kernel and cokernel on $\mathbb{T}^2$;*
3. *$6(\mathfrak{g}-1)$-dimensional cokernel and kernel on manifolds of genus $\mathfrak{g} \geq 2$.*

Proof. Note that

$$P = C \circ \mathring{\mathrm{div}}_{(2)} \tag{337}$$

is a composition of elliptic operators, hence is elliptic. Using

$$\mathring{\mathrm{div}}_{(2)}{}^{\dagger} = -C \,, \quad C^{\dagger} = -\mathring{\mathrm{div}}_{(2)} \,, \tag{338}$$

we have

$$P = -\mathring{\mathrm{div}}_{(2)}{}^{\dagger} \circ \mathring{\mathrm{div}}_{(2)} \ , \tag{339}$$

from which self-adjointness follows.

Finally, we have

$$\int h^{AB} P h_{AB} = -\int h\, \mathring{\mathrm{div}}_{(2)}{}^{\dagger} \circ \mathring{\mathrm{div}}_{(2)}\, h = -\int |\mathring{\mathrm{div}}_{(2)}\, h|^2 \le 0 \, , \tag{340}$$

hence all eigenvalues of P are negative, and $Ph = 0$ implies $\mathring{\mathrm{div}}_{(2)}\, h = 0$. □

C.5.1. S^2. As already discussed in Section C.1, it follows from [2, 20] that on S^2 we can write symmetric trace-free 2-tensors φ_{AB} as

$$\varphi_{AB} = \sum_{\ell\ge 2} \sum_{-\ell\le m\le \ell} \varphi_\psi^{(\ell m)} \psi_{AB}^{(\ell m)} + \varphi_\phi^{(\ell m)} \phi_{AB}^{(\ell m)} \, . \tag{341}$$

It follows from (307)-(308) and (309)-(310) that the operator P of (152), namely

$$P\varphi_{AB} = TS[\mathring{D}_A \mathring{D}^C \varphi_{BC}] \equiv C(\mathring{D}^C \varphi_{CD})_{AB} \, , \tag{342}$$

acts on φ_{AB} as

(343)

$$\begin{aligned} P\varphi_{AB} &= \sum_{\ell\ge 2} \sum_{-\ell\le m\le \ell} \varphi_\psi^{(\ell m)} C(\mathring{D}^B \psi_{AB}^{(\ell m)}) + \varphi_\phi^{(\ell m)} C(\mathring{D}^B \phi_{AB}^{(\ell m)}) \\ &= \sum_{\ell\ge 2} \sum_{-\ell\le m\le \ell} \sqrt{\frac{1}{2}\ell(\ell+1) - 1} \left(\varphi_\psi^{(\ell m)} C(E^{(\ell m)})_{AB} + \varphi_\phi^{(\ell m)} C(H_{AB}^{(\ell m)}) \right) \\ &= -\sum_{\ell\ge 2} \sum_{-\ell\le m\le \ell} \underbrace{\left(\frac{1}{2}\ell(\ell+1) - 1 \right)}_{>0 \text{ for } \ell\ge 2} \left(\varphi_\psi^{(\ell m)} \psi_{AB}^{(\ell m)} + \varphi_\phi^{(\ell m)} \phi_{AB}^{(\ell m)} \right) \, . \end{aligned}$$

In particular the operator P is self-adjoint and has trivial kernel on S^2. On the other hand the operator $\mathring{\mathrm{div}}_{(2)}\,(P+2)$, which appears in (221) with $p = 1$ and $m = 0$, acts according to

(344)

$$\mathring{\mathrm{div}}_{(2)}\,(P+2)(\varphi)_B := \mathring{D}^A (P+2)\varphi_{AB}$$

$$= - \sum_{\substack{\ell\geq 2\\ |m|\leq\ell}} \left(\tfrac{\ell(\ell+1)}{2} - 1 - 2\right) \mathring{D}^A \left(\varphi_\psi^{(\ell m)} \psi_{AB}^{(\ell m)} + \varphi_\phi^{(\ell m)} \phi_{AB}^{(\ell m)}\right)$$

$$= - \sum_{\substack{\ell\geq 2\\ |m|\leq\ell}} \underbrace{\left(\tfrac{\ell(\ell+1)}{2} - 3\right) \sqrt{\tfrac{\ell(\ell+1)}{2} - 1}}_{\substack{=0 \text{ for } \ell=1,2\\ >0 \text{ for } \ell>2}} \left(\varphi_\psi^{(\ell m)} E_B^{(\ell m)} + \varphi_\phi^{(\ell m)} H_B^{(\ell m)}\right) .$$

It follows that the L^2-orthogonal $\left(\mathrm{im}(\mathring{\mathrm{div}}_{(2)}\,(P+2))\right)^\perp$ of $\mathrm{im}(\mathring{\mathrm{div}}_{(2)}\,(P+2))$ is spanned by conformal Killing vectors together with spherical harmonic vector fields with $\ell = 2$. Subsequently, for any covector field $X_A \in L^2$ the equation

$$\mathring{D}^B\,(P+2)\,\varphi_{AB} - \xi_A^{[\leq 2]} = X_A \tag{345}$$

admits a unique solution with a symmetric traceless 2-tensor φ_{AB} and a covector field $\xi_A^{[\leq 2]}$.

For a $C_u^2\,C^\infty_{(r,x^A)}$ gluing we need the operator

$$\mathring{\mathrm{div}}_{(2)}\,(P^2 + 7\varepsilon P + 10\varepsilon)\,,$$

as determined from the coefficients of $\hat{\kappa}_6$ in the formulae (286) for $\overset{(2,i)}{\chi}$. On S^2, a calculation similar to that in (344) shows that its kernel consists of spherical harmonic tensors with $\ell = 1, 2, 3$, which results in a cokernel spanned on spherical harmonic vectors with $\ell = 1, 2, 3$.

C.5.2. Polynomials in P. In this section we assume that $m = 0$.

For C^k-gluing, the operator $\sum_{i=0}^k \overset{(k,i)}{\chi}_{k+4} P^i$ appearing in (221) is of the form

$$\sum_{i=0}^k \overset{(k,i)}{\chi}_{k+4}(r_2) P^i = \hat{c}_k \prod_{i=1}^k (P + \varepsilon a_i)\,, \qquad a_i = \frac{1}{2} i(3+i)\,, \tag{346}$$

where

$$\hat{c}_k = \frac{1}{k!(k+3)}\,.$$

This can be verified by induction.

Indeed, when $k = 1$ this follows from (282)-(283) with $\hat{c}_1 = 1/4$. Using the recursion formula (290), a straightforward calculation shows that the

$\hat{\kappa}_{k+5}$ component of the $(k+1)$-order coefficient are given by

$$\overset{(k+1,i)}{\chi}_{k+5} = c_k \times \begin{cases} \varepsilon a_{k+1} \overset{(k,0)}{\chi}_{k+4}\,, & i=0\,, \\ \varepsilon a_{k+1} \overset{(k,i)}{\chi}_{k+4} + \overset{(k,i-1)}{\chi}_{k+4}\,, & 1 \le i \le k\,, \\ \overset{(k,k)}{\chi}_{k+4}\,, & i = k+1\,, \end{cases} \tag{347}$$

with $c_k = \frac{k+3}{(k+4)(k+1)}$. Therefore, assuming (346), the operator at order $k+1$ is actually r_2-independent and reads,

$$\begin{aligned} \sum_{i=0}^{k+1} \overset{(k,i)}{\chi}_{k+5}(r_2)P^i &= c_k \sum_{i=0}^{k} \left(\overset{(k,i)}{\chi}_{k+4} P^{i+1} + \varepsilon a_{k+1} \overset{(k,i)}{\chi}_{k+4} P^i \right) \\ &= c_k \sum_{i=0}^{k} \left(\overset{(k,i)}{\chi}_{k+4} P^i \right) (P + \varepsilon a_{k+1}) \\ &= c_k \hat{c}_k (P + \varepsilon a_{k+1}) \prod_{i=1}^{k} (P + \varepsilon a_i) \\ &= \hat{c}_{k+1} \prod_{i=1}^{k+1} (P + \varepsilon a_i)\,, \qquad \text{with } \hat{c}_{k+1} = c_k \hat{c}_k\,. \end{aligned}$$

It thus follows from (343) that, on S^2, spherical harmonic vector fields with mode $\ell \ge 0$ satisfying

$$0 = \prod_{i=1}^{k} \left(-\frac{1}{2}\ell(\ell+1) + 1 + a_i \right) = \frac{1}{2} \prod_{i=1}^{k} (1+i-\ell)(2+i+\ell) \tag{348}$$

belong to $\ker\left(\sum_{i=0}^{k} \overset{(k,i)}{\chi}_{k+4} P^i \right)$. The corresponding values of ℓ are $\ell = 2, ..., k+1$.

For the remaining topologies, each of the operators

$$P + \varepsilon a_i$$

appearing in (346) is negative. On $\mathbb{T}^2$ its kernel, when acting on traceless tensors, is two-dimensional, consisting of covariantly constant tensors. Hence, in the toroidal case, the kernel of the left-hand side of (346) is also two-dimensional, which can be seen e.g. by a Fourier-series decomposition.

On higher genus manifolds $P+\varepsilon a_i$ is strictly negative and therefore has no kernel. Hence so does the left-hand side of (346).

Appendix D. A trace identity

The aim of this appendix is to prove the following curious consequence of Bianchi identities:

$$r^{-1}\gamma^{AB}\delta G_{AB} = -\frac{1}{2}\gamma^{AB}\partial_r\delta R_{AB} + \mathring{D}^A\delta G_{rA}\,, \tag{349}$$

when $\partial_u^i\delta\beta = 0$ (i.e., $\partial_u^i\delta G_{rr} = 0$) for $i = 0, 1$.

For this, we start by noting that the operator $g^{AB}R_{AB}$ is related to that appearing in (69), which can be seen as follows: From the definition (48) of the Einstein tensor $G_{\mu\nu}$ and the Bondi parametrisation of the metric (10) we have

$$G_{ur} = \frac{1}{2}e^{2\beta}g^{rr}G_{rr} - U^A G_{rA} + \frac{1}{2}e^{2\beta}g^{AB}R_{AB}\,. \tag{350}$$

Now, from the linearisation of (350), when $\delta\beta = 0$, $G_{rr} = 0$, and $\partial_u^i\delta G_{rr} = 0$, we have

$$\frac{1}{2}\mathring{\gamma}^{AB}\delta R_{AB} = r^2\delta G_{ur} \qquad\Longrightarrow\qquad \frac{1}{2}\mathring{\gamma}^{AB}\partial_r\delta R_{AB} = 2r\delta G_{ur} + r^2\partial_r\delta G_{ur}\,, \tag{351}$$

and hence the identity (349) is equivalent to

$$\mathring{D}^A\delta G_{rA} - \frac{1}{r}\mathring{\gamma}^{AB}\delta G_{AB} = 2r\delta G_{ur} + r^2\partial_r\delta G_{ur}\,. \tag{352}$$

Meanwhile, it follows from the divergence identity (122) with $\nu = r$ that

$$0 = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\mathscr{E}^\mu{}_r) + \frac{1}{2}\partial_r(g^{\mu\rho})\mathscr{E}_{\mu\rho}\,. \tag{353}$$

The linearisation of (353) with $\partial_u\delta G_{rr} = 0$ gives,

$$0 = -\frac{1}{r^2}\partial_r(r^2\delta G_{ur}) + \frac{1}{r^2}\mathring{D}^A\delta G_{rA} + \frac{1}{2}\partial_r\left(\frac{1}{r^2}\mathring{\gamma}^{AB}\right)\delta G_{AB}\,, \tag{354}$$

and hence,

$$2r\delta G_{ur} + r^2\partial_r\delta G_{ur} = \mathring{D}^A\delta G_{rA} - \frac{1}{r}\mathring{\gamma}^{AB}\delta G_{AB}\,, \tag{355}$$

which agrees with (352).

Appendix E. Conserved charges for $\partial_u^i \check{h}_{uA}$

The aim of this appendix is to present an alternative way to obtain the obstructions for gluing of $\partial_u^i h_{uA}$, $i = 0, 1$, in terms of gauge-dependent radial charges, as suggested to us by S. Czimek. We assume in what follows that

$$m = 0\,.$$

Similar calculations can be done for $m \neq 0$ and for higher derivatives, but we have not pursued these ideas any further.

E.1. $\check{h}_{uA}$

Recall (103):

$$\partial_r\Big(3\check{h}_{uA} + r\partial_r\check{h}_{uA} - r^{-3}\mathring{D}^B h_{AB}\Big) = r^{-4}\mathring{D}^B h_{AB}\,. \tag{356}$$

In particular we obtain a collection of radially-conserved charges

$$\partial_r \underbrace{\int_{\mathbf{S}} \pi^A\Big(3\check{h}_{uA} + r\partial_r\check{h}_{uA}\Big)d\mu_{\mathring{\gamma}}}_{=:\overset{[4,0]}{Q}(\pi^A)} = 0\,, \tag{357}$$

where the π^A's are conformal Killing vectors of $(\mathbf{S}, \mathring{\gamma})$. Keeping in mind our assumption that $m = 0$, under gauge transformations we have

$$3\check{h}_{uA} + r\partial_r\check{h}_{uA} \mapsto 3\check{h}_{uA} + r\partial_r\check{h}_{uA} + \frac{1}{r^2}L_1(\xi^u)_A + 3(\partial_u\xi_A + \alpha^2\mathring{D}_A\xi^u)\,. \tag{358}$$

Hence the gauge-dependent radial charge $\overset{[4,0]}{Q}$ transforms as,

$$\overset{[4,0]}{Q}(\pi^A) \mapsto \overset{[4,0]}{Q}(\pi^A) + 3\int \pi^A\big(\partial_u\xi_A + \alpha^2\mathring{D}_A\xi^u\big)d\mu_{\mathring{\gamma}}\,, \tag{359}$$

since the L_1-gauge term in (358) has zero projection onto the π^A's.

E.2. $\partial_u \check{h}_{uA}$

Now, we hope to obtain a conserved radial charge involving $\partial_u \check{h}_{uA}$. For this, take ∂_u of (356):

$$\partial_r\left(3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA}\right) = \frac{1}{r}\mathring{D}^B\partial_r\left(\partial_u \check{h}_{AB}\right) . \tag{360}$$

We can then achieve the goal by rewriting the RHS into terms of the form $\partial_r(\cdot) +$ (terms involving only $\partial_r^i h_{AB}$). For this, recall (90) with $\delta\beta = 0$:

$$\begin{aligned} 0 =& \partial_r\left[r\partial_u \check{h}_{AB} - \frac{1}{2}V\partial_r \check{h}_{AB} - \frac{1}{2r}V\check{h}_{AB} - r\,\mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right]\right] \\ &+ \frac{1}{2}\partial_r(V/r)\check{h}_{AB} - \mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right] . \end{aligned} \tag{361}$$

We wish to combine the above two equations by taking RHS of (360) $+\ Ar^c \times \mathring{D}^B$ (361) for some constants A, c such that all the $\partial_u \mathring{D}^B \check{h}_{AB}$ terms collect into the form $B\partial_r(r^a \mathring{D}^B \partial_u \check{h}_{AB})$, for some constants B, a. By a straightforward comparison of coefficients, we find that this is possible with $A = -\frac{1}{2} = -B$ and $c = -2$, $a = -1$. Indeed, we have,

$$\begin{aligned} \partial_r\left(3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA}\right) &= \frac{1}{r}\mathring{D}^B\partial_r\left(\partial_u \check{h}_{AB}\right) - \frac{1}{2r^2}\partial_r\left(r\partial_u \mathring{D}^B \check{h}_{AB}\right) \\ &\quad + \frac{1}{2r^2}\mathring{D}^B\partial_r\left[\frac{1}{2}V\partial_r \check{h}_{AB} + \frac{1}{2r}V\check{h}_{AB} + r\,\mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right]\right] \\ &\quad - \frac{1}{4r^2}\partial_r(V/r)\mathring{D}^B \check{h}_{AB} + \frac{1}{2r^2}\mathring{D}^B\,\mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right] \\ &= \frac{1}{2}\partial_r\left(\frac{1}{r}\partial_u \mathring{D}^B \check{h}_{AB}\right) \\ &\quad + \frac{1}{2r^2}\mathring{D}^B\partial_r\left[\frac{1}{2}V\partial_r \check{h}_{AB} + \frac{1}{2r}V\check{h}_{AB} + r\,\mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right]\right] \\ &\quad - \frac{1}{4r^2}\partial_r(V/r)\mathring{D}^B \check{h}_{AB} + \frac{1}{2r^2}\mathring{D}^B\,\mathrm{TS}\left[\mathring{D}_A \check{h}_{uB}\right] . \end{aligned}$$

We rewrite this as

$$\begin{aligned} &\partial_r\left(3\partial_u \check{h}_{uA} + r\partial_r\partial_u \check{h}_{uA} - \frac{1}{2r}\partial_u \mathring{D}^B \check{h}_{AB}\right) \\ &\quad = \frac{1}{2r^2}\mathring{D}^B\partial_r\left[\frac{V}{2}\partial_r \check{h}_{AB} + \frac{V}{2r}\check{h}_{AB}\right] - \frac{1}{4r^2}\partial_r\left(\frac{V}{r}\right)\mathring{D}^B \check{h}_{AB} \end{aligned} \tag{362}$$

$$+\frac{1}{2r^2}\partial_r\Big(r\mathring{D}^B\,\mathrm{TS}\left[\mathring{D}_A\check{h}_{uB}\right]\Big)+\frac{1}{2r^2}\mathring{D}^B\,\mathrm{TS}\left[\mathring{D}_A\check{h}_{uB}\right].$$

Next, we perform a similar trick, by taking RHS of (362) $+D\,r^d\times\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A[\text{LHS of (356)}]]$ to collect all the $\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uA}]$ terms into the form

$$\tilde{A}\partial_r\big(r^{\tilde{a}}\partial_r(r^{\tilde{b}}\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uA}])\big)$$

for some constants $D,d,\tilde{A},\tilde{a},\tilde{b}$. By power matching and comparing coefficients, we get $d=-1$ and $D=-3/8=\tilde{A}=-1/\tilde{b}=1/\tilde{a}$. Explicitly, as a first step we write

$$\begin{aligned}
&\frac{1}{2r^2}\partial_r\Big(r\mathring{D}^B\,\mathrm{TS}\left[\mathring{D}_A\check{h}_{uB}\right]\Big)+\frac{1}{2r^2}\mathring{D}^B\,\mathrm{TS}\left[\mathring{D}_A\check{h}_{uB}\right]\\
&\qquad-\frac{3}{8r}\underbrace{\partial_r\Big(3\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]+r\partial_r\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\Big)}_{=\frac{1}{r}\partial_r\left(\mathring{D}^BP\check{h}_{AB}\right)\text{ by (356)}}\\
&\qquad=-\frac{3}{8}\partial_r\big(r^{-8/3}\partial_r(r^{8/3}\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}])\big)\,.
\end{aligned}$$

Substituting this into (362) then gives,

$$\begin{aligned}
&\partial_r\left(3\partial_u\check{h}_{uA}+r\partial_r\partial_u\check{h}_{uA}-\frac{1}{2r}\partial_u\mathring{D}^B\check{h}_{AB}\right)\\
&\qquad=\quad\frac{1}{2r^2}\mathring{D}^B\partial_r\Big[\frac{V}{2}\partial_r\check{h}_{AB}+\frac{V}{2r}\check{h}_{AB}\Big]-\frac{1}{4r^2}\partial_r\left(\frac{V}{r}\right)\mathring{D}^B\check{h}_{AB}\\
&\qquad\quad-\frac{3}{8}\partial_r\big(r^{-\frac{8}{3}}\partial_r(r^{\frac{8}{3}}\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}])\big)+\frac{3}{8r^2}\partial_r\left(\mathring{D}^BP\check{h}_{AB}\right)\,,
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
&\partial_r\left(3\partial_u\check{h}_{uA}+r\partial_r\partial_u\check{h}_{uA}-\frac{1}{2r}\partial_u\mathring{D}^B\check{h}_{AB}+\frac{3}{8}r^{-\frac{8}{3}}\partial_r\big(r^{\frac{8}{3}}\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\big)\right)\\
&\qquad=\frac{1}{2r^2}\mathring{D}^B\partial_r\Big[\frac{V}{2}\partial_r\check{h}_{AB}+\frac{V}{2r}\check{h}_{AB}\Big]-\frac{1}{4r^2}\partial_r\left(\frac{V}{r}\right)\mathring{D}^B\check{h}_{AB}\\
&\qquad\quad+\frac{3}{8r^2}\partial_r\left(\mathring{D}^BP\check{h}_{AB}\right)\\
&\qquad=\partial_r\left(\frac{V}{4r^5}\mathring{D}^Bh_{AB}+\frac{V}{4r^4}\mathring{D}^B\partial_rh_{AB}+\frac{3}{8r^4}Ph_{AB}\right)\\
&\qquad\quad+\left(\frac{9V}{4r^6}-\frac{3\partial_rV}{4r^5}\right)\mathring{D}^Bh_{AB}+\frac{3}{4r^5}\mathring{D}^BPh_{AB}\,,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
&\partial_r\Big(3\partial_u\check{h}_{uA}+r\partial_r\partial_u\check{h}_{uA}-\frac{1}{2r}\partial_u\mathring{D}^B\check{h}_{AB}+\frac{3}{8}r^{-\frac{8}{3}}\partial_r\big(r^{\frac{8}{3}}\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\big)\\
&\quad-\frac{V}{4r^5}\mathring{D}^Bh_{AB}-\frac{V}{4r^4}\mathring{D}^B\partial_rh_{AB}-\frac{3}{8r^4}Ph_{AB}\Big)\\
&=\frac{3}{4r^5}\mathring{D}^B\Big(\underbrace{\frac{3V}{r}-\partial_rV}_{=2\varepsilon}+P\Big)h_{AB}\,.
\end{aligned}\tag{363}$$

Let us denote by,

$$\begin{aligned}
\overset{[1]}{q}_A :=&3\partial_u\check{h}_{uA}+r\partial_r\partial_u\check{h}_{uA}-\frac{1}{2r}\partial_u\mathring{D}^B\check{h}_{AB}+\frac{3}{8}r^{-\frac{8}{3}}\partial_r\big(r^{\frac{8}{3}}\mathring{D}^B\,\mathrm{TS}[\mathring{D}_A\check{h}_{uB}]\big)\\
&-\frac{V}{4r^5}\mathring{D}^Bh_{AB}-\frac{V}{4r^4}\mathring{D}^B\partial_rh_{AB}-\frac{3}{8r^4}Ph_{AB}\,.
\end{aligned}\tag{364}$$

Let $\mu_A\in\mathrm{coker}(\mathring{\mathrm{div}}_{(2)}\,(P+2\varepsilon))$. Equation (363) gives

$$\partial_r\overset{[5]}{Q}(\mu^A)=0,\tag{365}$$

where

$$\overset{[5]}{Q}(\mu^A):=\int_{\mathbf{S}}\mu^A\overset{[1]}{q}_Ad\mu_{\hat{\gamma}}\,.\tag{366}$$

We have thus found a gauge-dependent radial charge involving $\partial_u\check{h}_{uA}$. From the analysis in Appendix C.5.2, on S^2, $\mathrm{coker}(\mathring{\mathrm{div}}_{(2)}\,(P+2\varepsilon))$ is the 16-dimensional space of $\ell=1$ and $\ell=2$ spherical harmonic vectors; on $\mathbb{T}^2$, $\mathrm{coker}(\mathring{\mathrm{div}}_{(2)}\,(P+2\varepsilon))$ is the two dimensional space of covariantly constant vectors; on negatively curved manifolds of higher genus, $\mathrm{coker}(\mathring{\mathrm{div}}_{(2)}\,(P+2\varepsilon))$ is trivial. Note that $\mathrm{coker}(\mathring{\mathrm{div}}_{(2)}\,(P+2\varepsilon))$ corresponds exactly to the space of obstructions for solving (221) with $p=1$ (compare also (344)).

Under gauge transformations, $\overset{[1]}{q}_A$ transforms as

$$\begin{aligned}
\overset{[1]}{q}_A\mapsto\overset{[1]}{q}_A&-\frac{\varepsilon}{r^3}L_1(\xi^u)_A+\frac{3\alpha^2}{2}(\mathring{D}_A\mathring{D}_B\xi^B+\mathring{D}^BC(\xi)_{AB})\\
&+3\partial_u^2\xi_A-\frac{3}{4r^2}\mathring{D}^B(2\varepsilon+P)C(\xi)_{AB}\,.
\end{aligned}$$

Note that the r-dependent gauge terms in this equation vanish upon projection onto μ^A. Hence, the charge $\overset{[5]}{Q}$ transforms as,

(367)
$$\overset{[5]}{Q}(\mu^A) \mapsto \overset{[5]}{Q}(\mu^A) + 3\int_{\mathbf{S}} \mu^A \left(\frac{\alpha^2}{2}(\mathring{D}_A \mathring{D}_B \xi^B + \mathring{D}^B C(\xi)_{AB}) + \partial_u^2 \xi_A \right) d\mu_{\tilde{\gamma}} \,.$$

Appendix F. An identity

One way of ensuring continuity of $\partial_u^p \check{h}_{uA}$ and $\partial_u^p h_{AB}^{[\mathrm{TT}^\perp]}$ at r_2 is through the fields $\{\overset{[2p+4]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]}, \partial_u^{p+1} \overset{(2)}{\xi}{}_C^{[\mathrm{CKV}]}\}$ and the gauge field $\partial_u^p \overset{(2)}{\xi}{}_C^{[\mathrm{CKV}^\perp]}$ respectively. This turns out not to be convenient from the perspective of nonlinear theory, as it involves losses of derivatives, which is avoided by the argument presented in the main body of the paper. We present this alternative calculation here as it involves some unexpected identities which might be useful for further applications.

We thus revisit (225). For the sake of the induction here we assume that the fields $\overset{[\ell]}{\hat{\varphi}}_{AB}$ for $\ell \leq 2p+2$ are known and collect them, together with $-\partial_u^p \check{h}_{uA}|_{\tilde{\mathbf{S}}_2}$, into a new term $\overset{(p)}{\hat{\tilde{X}}}_A$, allowing us to rewrite the L^2-projections on CKV and CKV$^\perp$ of (225) respectively as

$$\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]} = \overset{(p)}{\hat{\tilde{X}}}{}_A^{[\mathrm{CKV}]}, \tag{368}$$

$$\begin{aligned} -\overset{(p,0)}{\chi}{}_{2p+4}(r_2) \mathring{D}^B \overset{[2p+4]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]} &= -\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}^\perp]} + \overset{(p)}{\hat{\tilde{X}}}{}_A^{[\mathrm{CKV}^\perp]} \\ &\quad + \mathring{D}^B (\overset{(p,0)}{\chi}{}_{2p+3}(r_2) + \overset{(p,1)}{\chi}{}_{2p+3}(r_2) P) \overset{[2p+3]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]} \,. \end{aligned} \tag{369}$$

An argument identical to that below (213) shows that, both on S^2 and $\mathbb{T}^2$, (368) determines $\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}]}$ uniquely in terms of $\overset{(p)}{\hat{\tilde{X}}}{}_A^{[\mathrm{CKV}]}$ while (369) determines $\overset{[2p+4]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]}$ uniquely in terms of $\overset{(p)}{\hat{\tilde{X}}}{}_A^{[\mathrm{CKV}^\perp]}$, $\overset{[2p+3]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]}$ and $\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}^\perp]}$; the fields $\overset{[2p+3]}{\hat{\varphi}}{}_{AB}^{[\mathrm{TT}^\perp]}$ and $\partial_u^{p+1} \overset{(2)}{\xi}{}_A^{[\mathrm{CKV}^\perp]}$ remain free to use for other gluing equations. On negatively curved sections with higher genus, (369)

determines $\overset{[2p+4]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ in terms of $\overset{(p)}{\hat{\tilde{X}}}_A$, $\overset{[2p+3]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ and $\partial_u^{p+1}\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_A = \partial_u^{p+1}\overset{(2)}{\xi}_A$, with the fields $\overset{[2p+3]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ and $\partial_u^{p+1}\overset{(2)}{\xi}_A$ remaining free.

Next, in order to take into account the dependence of $\overset{[2p+2]}{\hat{\varphi}}$ and $\overset{[2p+1]}{\hat{\varphi}}$ upon $\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_C$, we consider the equation obtained by acting with $\mathring{\mathrm{div}}_{(2)}$ on (241). There occur some miraculous cancellations, which are likely to have some simple origin:

(370)

$$\begin{aligned}
\mathring{D}^A\partial_u^p h^{[\mathrm{TT}^\perp]}_{AB}|_{\tilde{\mathbf{S}}_2} &= \mathring{D}^A\overset{(p)}{\tilde{\Psi}}{}^{[\mathrm{TT}^\perp]}_{AB}(r_2,x^A) - 2r_2^2\mathring{D}^A\,\mathrm{TS}[\mathring{D}_A\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_B] \\
&\quad + \mathring{D}^A(\overset{(p,0)}{\psi}{}_{2p+2}(r_2) + \overset{(p,1)}{\psi}{}_{2p+2}(r_2)P)\overset{[2p+2]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB} \\
&\quad + \mathring{D}^A(\overset{(p,0)}{\psi}{}_{2p+1}(r_2) + \overset{(p,1)}{\psi}{}_{2p+1}(r_2)P + \overset{(p,2)}{\psi}{}_{2p+1}(r_2)P^2)\overset{[2p+1]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB} \\
&= \mathring{D}^A\overset{(p)}{\tilde{\Psi}}{}^{[\mathrm{TT}^\perp]}_{AB}(r_2,x^A) - 2r_2^2\mathring{D}^A\,\mathrm{TS}[\mathring{D}_A\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_B] \\
&\quad + \overset{(p,0)}{\psi}{}_{2p+2}/\overset{(p-1,0)}{\chi}{}_{2p+2}\,\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_B \\
&\quad + \overset{(p,1)}{\psi}{}_{2p+2}/\overset{(p-1,0)}{\chi}{}_{2p+2}\,\mathring{D}^A C(\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]})_{AB} \\
&= \mathring{D}^A\overset{(p)}{\tilde{\Psi}}{}^{[\mathrm{TT}^\perp]}_{AB}(r_2,x^A) - 3mr_2\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_B\,,
\end{aligned}$$

where in the second equality we made use of the expression for $\overset{[2p+2]}{\hat{\varphi}}{}^{[\mathrm{TT}^\perp]}_{AB}$ from (369) at order $(p-1)$, while the last equality uses (302)-(305), Appendix B. Thus, continuity of $\partial_u^p h^{[\mathrm{TT}^\perp]}_{AB}$ can be achieved by solving (370) for $\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_C$:

$$3mr_2\partial_u^p\overset{(2)}{\xi}{}^{[\mathrm{CKV}^\perp]}_B = -\mathring{D}^A\partial_u^p h^{[\mathrm{TT}^\perp]}_{AB}|_{\tilde{\mathbf{S}}_2} + \mathring{D}^A\overset{(p)}{\tilde{\Psi}}{}^{[\mathrm{TT}^\perp]}_{AB}(r_2,x^A)\,. \tag{371}$$

In fact, Lemma B.2, p. 772 shows that we have the factorisation

$$\begin{aligned}
&\overset{(p-1,0)}{\chi}{}_{2p+2}(r_2)\big(\overset{(p,0)}{\psi}{}_{2p+1}(r_2) + \overset{(p,1)}{\psi}{}_{2p+1}(r_2)P + \overset{(p,2)}{\psi}{}_{2p+1}(r_2)P^2\big) \\
&\qquad = \big(\overset{(p,0)}{\psi}{}_{2p+2}(r_2) + \overset{(p,1)}{\psi}{}_{2p+2}(r_2)P\big)\big(\overset{(p-1,0)}{\chi}{}_{2p+1}(r_2) + \overset{(p-1,1)}{\chi}{}_{2p+1}(r_2)P\big)\,.
\end{aligned}$$

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