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The Bartnik quasi-local mass conjectures

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In Memory of Robert Bartnik

This paper is a tribute to Robert Bartnik and his work and conjectures on quasi-local mass. We present a framework in which to clearly analyse Bartnik's static vacuum extension conjecture. While we prove that this conjecture is not true in general, it remains a fundamental open problem to understand the realm of its validity.

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1. Introduction

This paper is a tribute to Robert Bartnik and his work on quasi-local mass in General Relativity. I met Robert in person only twice, at the Newton Institute in 2005 and then a few years later in Oberwolfach and my personal interactions with him were sadly somewhat limited. Robert's conjectures on the realization of the quasi-local mass, related to the structure of the space of static vacuum solutions of the Einstein equations have had a profound impact on a significant portion of my research over the last 20 years. Personally, I was not trained early in GR and came to the subject rather indirectly through a study of the geometrization problem for 3-manifolds (the Thurston conjecture). In that approach to geometrization, a surprising connection arose between the degeneration of metrics of controlled scalar curvature and the structure of static vacuum Einstein metrics in 3-dimensions.[1]

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[1]The singularity models of such degeneration are static vacuum Einstein metrics, much like the singularity models of Ricci flow are Ricci solitons. These two equations are superficially somewhat similar and it would be of interest to understand any deeper relations between them.

I hope this paper may contribute in a small way to a deeper understanding and appreciation of the Bartnik quasi-local mass conjectures.

The Bartnik mass of a bounded domain Ω with smooth boundary $\partial\Omega$ is a very natural and direct localization of the global ADM mass m_{ADM}. The original definition of Bartnik is the following, cf. [16], [17]. For simplicity, we restrict here to the time-symmetric or Riemannian case where $K = 0$.[2] Let Ω be a 3-dimensional domain with smooth connected boundary $\partial\Omega$ (we will usually assume $\partial\Omega \simeq S^2$) and let g_Ω be a smooth Riemannian metric on Ω, smooth up to $\partial\Omega$, with non-negative scalar curvature, $s_{g_\Omega} \geq 0$. Let (M, g) be an asymptotically flat (AF) extension of (Ω, g_Ω); thus $\partial M = \partial\Omega$ and the union $\Omega \cup M$ is a smooth, complete AF Riemannian manifold with non-negative and integrable scalar curvature. Assume in addition that (M, g) has no horizons, i.e. (M, g) has no minimal surfaces surrounding ∂M. Let $\mathcal{P}_\Omega$ denote the set of such AF extensions (M, g). The Bartnik mass of the smooth bounded domain (Ω, g_Ω) is then defined by

$$m_B(\Omega, g_\Omega) = \inf\{m_{ADM}(M, g) : (M, g) \in \mathcal{P}_\Omega\}, \tag{1.1}$$

where the infimum is taken over all $(M, g) \in \mathcal{P}_\Omega$.

Bartnik observed in [16], [17] that an AF extension (M, g) of $\partial\Omega$ with $\widehat{M} = \Omega \cup M$ which realizes the infimum in (1.1) will in general only be Lipschitz along the "seam" $\partial\Omega = \partial M$. By a simple and elegant argument using the 2nd variational formula for area, he showed that a minimizer should obey the boundary conditions

$$\gamma_{\partial\Omega} = \gamma_{\partial M}, \qquad H_{\partial\Omega} = H_{\partial M}, \tag{1.2}$$

where γ is the induced metric on the boundary and H is the mean curvature of ∂M with respect to the unit normal pointing into M, both with respect to g_Ω and g respectively. The relation (1.2) implies that the scalar curvature is defined as a non-negative distribution across the seam $\partial M = \partial\Omega$.[3] Standard minimal surface arguments show that if $H_{\partial M} \leq 0$ then any extension (M, g) has a horizon, so that it is common practice to assume

$$H = H_{\partial M} > 0. \tag{1.3}$$

[2]The case of general initial data on Ω is considerably more difficult, cf. [2] for recent progress.

[3]Actually this holds for $H_{\partial\Omega} \geq H_{\partial M}$ and this condition is sometimes used instead of (1.2), cf. [28].

As discussed in more detail in §2, there are several further reasons why the restriction (1.3) is essential. In the following, we will always assume (1.3), unless explicitly noted otherwise.

The quasi-local data (γ, H) of Ω on $\partial\Omega = \partial M$ are now called Bartnik boundary data. There are a number of modifications or variations of the definition of the Bartnik mass m_B; we refer to [28] and further references therein for a careful and detailed discussion.

By general principles, it is to be expected that an extension (M, g) realizing the infimum in (1.1) satisfies strong conditions. In [16], [17], Bartnik presented a natural physical argument that extensions (M, g) realizing the infimum in (1.1) should be solutions of the static vacuum Einstein equations. Briefly, any dynamical gravitational field carries energy and so mass and so an extension of minimal mass should have no gravitational dynamics, i.e. be time-independent. For similar reasons, a minimal-mass extension should have no mass coming from matter sources, and so be vacuum. A time-independent vacuum solution which is time-symmetric ($K = 0$) is static vacuum.

The static vacuum Einstein equations are the equations for a pair (g, u) on M where u is a potential function (forming the lapse function of the space-time $\mathcal{M} = I \times M$) given by

$$u\mathrm{Ric}_g = D^2 u, \quad \Delta u = 0, \tag{1.4}$$

where Ric_g is the Ricci curvature, $D^2 u$ is the Hessian of u and Δu is the Laplacian of u, all with respect to g. These equations are equivalent to the statement that the space-time metric

$$g_{\mathcal{M}} = -u^2 dt^2 + g \tag{1.5}$$

is Ricci-flat, i.e. a vacuum solution of the Einstein equations.[4] It is important both mathematically and physically to add the requirements that $u > 0$ in M and $u \to 1$ at infinity.

A clearer approach as to why the static vacuum equations (1.4) arise as minimizers was suggested by Bartnik in [18], following a proposal by Brill-Deser-Fadeev [20] regarding the positive mass theorem. Thus consider the Regge-Teitelboim Hamiltonian (with zero shift) given by

$$\mathcal{H}_{RT} = -\int_M u s_g dv_g + 16\pi m_{ADM}, \tag{1.6}$$

[4]This holds in both Lorentizian and Riemannian signature.

where $s_g = \mathrm{tr}_g \mathrm{Ric}_g$ is the scalar curvature. The first term corresponds to the Einstein-Hilbert action in 4-dimensions (modulo divergence terms) while the mass term is introduced to give a well-defined, and in particular differentiable variational problem on the space of fields. The configuration space of fields is here given by general AF static metrics as in (1.5), but not necessarily vacuum, i.e. general pairs (g, u) with $u > 0$; the AF conditions are described more precisely in §2. The Hamiltonian $\mathcal{H}_{RT}$ is thus a smooth function on such static pairs (g, u).

Let

$$\mathcal{C} = \{g : s_g = 0\}.$$

This is the vacuum constraint set (in the time-symmetric case $K = 0$). Let $\mathcal{C}_{(\gamma,H)} \subset \mathcal{C}$ be the subset where the boundary data (γ, H) induced by g is fixed. A necessary first step in the Bartnik program is to prove that $\mathcal{C}_{(\gamma,H)}$ is a smooth (Banach) manifold; this was proved to be the case in [13]. The Hamiltonian $\mathcal{H}_{RT}$ then restricts to be the smooth function

$$\mathcal{H}_{RT} = 16\pi m_{ADM} : \mathcal{C}_{(\gamma,H)} \to \mathbb{R}$$

on $\mathcal{C}_{(\gamma,H)}$. Thus, critical points of the ADM mass on $\mathcal{C}_{(\gamma,H)}$ are just the critical points of $\mathcal{H}_{RT}$ on $\mathcal{C}_{(\gamma,H)}$. It is then straightforward to verify that such critical points are exactly static vacuum Einstein metrics. Further, cf. [13], one has $u > 0$ everywhere and $u \to 1$ at infinity, at least for critical points realizing the infimum of m_{ADM} on $\mathcal{C}_{(\gamma,H)}$, i.e. the Bartnik mass.

There is a useful analogy, cf. [17], [19], [26], of the Bartnik mass with the gravitational capacity of a body $\Omega \subset \mathbb{R}^3$ in Newtonian gravity, or a charged body in electrostatics. Here one minimizes the Dirichlet energy,

$$E(v) = \int_M |dv|^2, \tag{1.7}$$

over $M = \mathbb{R}^3 \setminus \Omega$ with boundary conditions $v = 1$ at $\partial\Omega$ and $v \to 0$ at infinity.[5] (One could also set $v' = 1 - v$ with $v' \to 1$ at infinity). Classical results show that the infimum of (1.7) is realized by a unique harmonic function v_0, $\Delta v_0 = 0$ on M; v_0 represents the gravitational potential of the single layer $\partial\Omega$. The capacity of Ω, equal to the total mass or charge up to a constant, is then given by

$$E(\Omega) = \inf E(v) = -\int_{\partial M} N(v_0), \tag{1.8}$$

[5]In (1.7) and below the volume form will often be dropped from the notation when its choice is obvious.

where N is the unit normal into M at ∂M.

Partly based on the Newtonian analogy, Bartnik [16], [17], made the bold conjecture that a minimizer (M, g) in (1.1) also always exists and is unique.

Conjecture I (Bartnik Minimization Conjecture)**.** For any given smooth boundary data (γ, H), $H > 0$, of a domain Ω, $\partial\Omega \simeq S^2$, the infimum in (1.1) is realized by a unique (up to isometry) AF extension (M, g), i.e.

$$m_{ADM}(M, g) = m_B.$$

Moreover, there is a positive potential function $u > 0$ on $M \simeq \mathbb{R}^3 \setminus B$ with $u \to 1$ at infinity such that the triple (M, g, u) is a solution of the static vacuum Einstein equations (1.4) with boundary data (γ, H).

The validity of this conjecture would ensure that the Bartnik mass is well-behaved as a function of the boundary data (γ, H) coming from the solid body (Ω, g_Ω). For example, it would follow that the Bartnik mass is a smooth function of the boundary data (γ, H); in fact the mass would be effectively computable via the expression

$$m_B = \frac{1}{4\pi} \int_{\partial M} N(u),$$

where (M, g, u) is the unique static vacuum extension of (γ, H). Let Conjecture I^+ be the restriction of Conjecture I to boundary data (γ, H) which are boundary data of a fill-in (Ω, g_Ω) with $s_{g_\Omega} \geq 0$, cf. [16]. Only in this case can one guarantee that $m_B \geq 0$ via the positive mass theorem.

If true, Conjecture I suggests the following conjecture [16], [17], which thus serves as a test of the minimization conjecture but is also of independent interest in geometric PDE theory.

Conjecture II (Bartnik Static Extension Conjecture)**.** Given smooth boundary data (γ, H), $H > 0$, on $\partial M \simeq S^2$, there exists a unique (up to isometry) AF solution (M, g, u), $M \simeq \mathbb{R}^3 \setminus B$, $u > 0$, $u \to 1$ at infinity, of the static vacuum Einstein equations (1.4) which induces the data (γ, H) at ∂M.

As above, let Conjecture II^+ denote the restriction of Conjecture II to boundary data (γ, H) having a non-negative scalar curvature fill-in.

Conjecture II may be considered as a pure PDE problem: it concerns the unique global solvability of a non-linear elliptic boundary value problem. Conjecture I (I^+) does not logically imply Conjecture II (II^+) since a static vacuum solution may not minimize the Bartnik mass. Nevertheless, we

expect that any counterexample to Conjecture I leads also to counterexamples to Conjecture II. Of course counterexamples to existence in Conjecture II are counterexamples to Conjecture I.

The static vacuum Einstein equations are seemingly among the simplest set of geometric (diffeomorphism invariant) PDE for a general metric g, either on a Lorentzian or Riemannian 4-manifold. In the Lorentzian or General Relativity setting, the simplest vacuum Einstein solutions are static, i.e. time independent with vanishing initial momentum and so have a hypersurface-orthogonal time-like Killing field. Similarly, Einstein metrics on 4-manifolds are of fundametal interest in Riemannian geometry and the existence of a global non-vanishing Killing field reduces the Einstein PDE system to the simpler system (1.4) on a 3-manifold M. While the equations (1.4) lie at the crossroads or intersection of Lorentzian and Riemannian geometry, gaining an understanding of the space of solutions and of Conjectures I and II has remained very difficult. One could impose a further symmetry, i.e. Killing field to simplify the problem further to 2 dimensions. This leads to the class of axi-symmetric or Weyl solutions, discussed further in detail in §4: however, even in this case, Conjectures I and II remain very challenging.

For Conjecture II, the source of these difficulties is of course the nonlinear nature of the PDE (1.4) and the boundary data (γ, H). This is also a coupled system of PDE, adding significantly to the complexity. To gain some general perspective, it is worth comparing the global existence and uniqueness problem for static vacuum Einstein metrics with other natural geometric problems. In fact, global existence and uniqueness for solutions of nonlinear elliptic geometric PDE is rather rare and is much more the exception than the rule. The list of examples is huge, so we mention only the following illustrative examples:

Global existence and uniqueness:

• Kähler-Einstein metrics. Among the few situations where global existence and uniqueness hold, the most celebrated is Yau's solution of the Calabi problem for compact Kähler manifolds with $c_1 = 0$ and the Aubin-Yau theorem in the case $c_1 < 0$, as well as the more recent work of Chen-Donaldson-Sun in the more difficult positive case $c_1 > 0$. These existence proofs use a version of the continuity method, discussed briefly in §2.

• Weyl embedding problem. Any smooth metric of positive Gauss curvature on S^2 admits a smooth isometric embedding into $\mathbb{R}^3$ as the boundary of a convex body, unique up to rigid motion of $\mathbb{R}^3$. Again, the existence proof of this result by Nirenberg uses the continuity method.

Global existence, non-uniqueness:

• The Yamabe problem. Global existence was established by variational methods through the combined work of Yamabe, Trudinger, Aubin and Schoen. However, it is well-known that uniqueness, either of constant scalar curvature metrics or of Yamabe minimizers in a conformal class, fails in general.

• Minimal varieties. There is a large and very satisfactory theory for existence of minimal varieties spanning a given boundary in an ambient Riemannian manifold, using either parametric methods in low dimensions (Plateau problem) or non-parametric methods (geometric measure theory) in all dimensions. However, it is again well-known that uniqueness fails in many situations.

Failure of global existence and uniqueness:

• Nirenberg problem. A very simple sounding problem is the Nirenberg problem of determining the existence and uniqueness of metrics on S^2 conformal to the round metric with prescribed Gauss curvature. Here, both existence and uniqueness fail in general and the number of solutions depends in a very complicated way on the structure of the Gauss curvature function $K : S^2 \to \mathbb{R}$.

• Vacuum Einstein constraint equations in GR. The main method of solving these constraint equations is the conformal method of Lichnerowicz, Choquet-Bruhat and York. Here one has full existence and uniqueness results for constant mean curvature (CMC) data, but failure of both existence and uniqueness for data far from CMC data.

One could easily go on and on; but the lesson is that non-linear geometric problems usually have complicated global behavior.

We note that there is now also a large and growing literature of estimates of the Bartnik mass $m_B(\gamma, H)$ from below and above in terms of the boundary data (γ, H), as well as comparisons of m_B with other notions of quasi-local mass, such as the Hawking, Brown-York, or Wang-Yau masses. These estimates and comparisons may be viewed as analogs of the extensive and well-known estimates of the capacity of a domain $\Omega \subset \mathbb{R}^3$, or of the first eigenvalue λ_1 of a compact Riemannian manifold, in terms of other geometric data.

However, the theory behind the capacity of a domain or of the first eigenvalue of the Laplacian is simple and easy to establish; they are realized by a harmonic function or a first eigenfunction whose existence and uniqueness is easy to establish, since the Laplacian is a single scalar, linear PDE. This fundamental theory is still badly lacking for the definition of the Bartnik mass and is an important motivation for Conjectures I and II. In particular,

the Bartnik mass formally exists as a number, but one should understand its structure and global behavior as a function of the boundary data (γ, H).

In accord with most of the examples discussed above, it turns out that both Conjectures I and II (and their restricted versions I^+ and II^+) are false in full generality. For example, we prove:

Theorem 1.1. *There is no static vacuum Einstein metric (M, g, u) smooth up to ∂M with boundary data (γ, H), where γ is any smooth metric on S^2 for which the Gauss curvature K_γ is a Morse function on S^2 and*

$$0 < H < H_0,$$

where H_0 is a positive constant, sufficiently small depending on γ and an upper bound for $||H||_{C^\infty}$.

We refer to Theorem 3.5 below for a more precise statement. It is to be expected that this non-existence result leads naturally to local non-uniqueness for Conjecture II; this is discussed in detail in §3. Conjecture I was previously proved to be false in certain (quite different) situations in [13].

It remains then a fundamental challenge to identify conditions or restrictions on the boundary data (γ, H) which ensure the validity of the Bartnik Conjectures.

A brief summary of the contents of this paper is as follows. In §2, we discuss the basic framework in which to analyse Conjecture II and show that the local structures involved are very well behaved. In §3, we turn to global issues of existence and uniqueness and prove in particular Theorem 1.1. A large class of relatively explicit examples of static vacuum Einstein metrics are the Weyl metrics, discussed in detail in §4. Finally, in an Appendix, §5, we prove a regularity result for (local) minimal surfaces in static vacuum Einstein metrics needed for the proof of Theorem 3.5.

2. Local behavior

Let $\mathcal{B}$ be the set of Bartnik boundary data (γ, H) on S^2 with $H > 0$ as in (1.3). Let $\mathcal{E}$ be the set of isometry classes of static vacuum Einstein solutions with $H > 0$ at the boundary ∂M.[6] Thus $\mathcal{E}$ is the set of equivalence

[6]It is an open question whether $\mathcal{E}$ is connected or not. Thus $\mathcal{E}$ is understood to be the component containing the standard flat exterior metric $\mathbb{R}^3 \setminus B^3(1)$, with g the Euclidean metric and $u \equiv 1$.

classes of AF static vacuum Einstein metrics (M, g, u), modulo the equivalence relation

$$(g_1, u_1) \sim (g_2, u_2)$$

if there is a diffeomorphism $\psi : M \to M$ with $\psi = \mathrm{Id}$ on ∂M and $\psi \to \mathrm{Id}$ at infinity, (cf. (2.4) below), such that $\psi^* g_2 = \psi^* g_1$ and $\psi^* u_2 = u_1$. One has a natural Bartnik boundary map

$$\begin{aligned} \Pi_B : \mathcal{E} &\to \mathcal{B}, \\ (g, u) &\to (\gamma, H). \end{aligned} \tag{2.1}$$

Thus the static extension Conjecture II is equivalent to the statement that Π_B is a bijection. This means that the space of solutions (M, g, u) (up to isometry) should be set-theoretically parametrized by its boundary data (γ, H). However, it is natural to introduce topology at this point. Well-posedness in PDE also typically assumes stability properties; small perturbations of the data (boundary data in this situation, initial or initial + boundary data in the case of parabolic or hyperbolic PDE) implies small perturbations of the solution; this corresponds to continuity of the inverse map Π_B^{-1} in (2.1).

The set $\mathcal{B}$ of boundary data may be naturally topologized in several ways. For convenience, we choose here the Hölder space topology,

$$\mathcal{B} = \mathrm{Met}^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2).$$

Here $\mathrm{Met}^{m,\alpha}(S^2)$ is the space of $C^{m,\alpha}$ Riemannian metrics on S^2 and $C_+^{m-1,\alpha}(S^2)$ is the space of positive $C^{m-1,\alpha}$ functions on S^2. The space $\mathcal{B}$ has the structure of a Banach manifold (an open subset of a linear Banach space)[7].

Next, let $\mathrm{Met}_\delta^{m,\alpha}(M)$ be the space of $C^{m,\alpha}$ Riemannian metrics g on $M \simeq \mathbb{R}^3 \setminus B$ which are asymptotically flat of order δ, where $\delta \in (\frac{1}{2}, 1)$, i.e. for r sufficiently large,

$$|g - g_{Eucl}| \leq c r^{-\delta}, \quad |\partial^{k+\alpha} g| \leq c r^{-\delta-k-\alpha}, \tag{2.2}$$

[7]In place of $C^{m,a}$, it is actually better to choose the so-called little Hölder spaces $c^{m,\alpha}$, i.e. the closure of C^∞ in the $C^{m,\alpha}$ Hölder norm, which has the structure of a separable Banach space; however, this is not an essential distinction in the discussion to follow.

$1 \leq k \leq m$. Similarly, let $C_+^{m,\alpha}(M)$ denote the space of positive potential functions u on M which are asymptotically flat of order δ:

$$|u-1| \leq cr^{-\delta}, \quad |\partial^{k+\alpha}u| \leq cr^{-\delta-k-\alpha}. \tag{2.3}$$

The pair (g,u) assemble to form the static or time-independent metric on the 4-manifold $\mathcal{M}$, as in (1.5). A $C^{m+1,\alpha}$ diffeomorphism $\psi : M \to M$ is AF if

$$|\psi - \mathrm{Id}| \leq cr^{-\delta}, \quad |\partial^{k+\alpha}\psi| \leq cr^{-\delta-k-\alpha}, \tag{2.4}$$

for $1 \leq k \leq m+1$.

The product $\mathrm{Met}_\delta^{m,\alpha}(M) \times C_+^{m,\alpha}(M)$ then also has the structure of a Banach manifold. Let $\mathbb{E} \subset \mathrm{Met}_\delta^{m,\alpha}(M) \times C_+^{m,\alpha}(M)$ be the subset of solutions of the static vacuum Einstein equations (1.4) with $H > 0$ at ∂M. It is proved in [14] that $\mathbb{E}$ is a smooth Banach submanifold of $\mathrm{Met}_\delta^{m,\alpha}(M) \times C_+^{m,\alpha}(M)$. Moreover, the group $\mathrm{Diff}_1^{m+1,\alpha}(M)$ of $C^{m+1,\alpha}$ AF diffeomorphisms of M equal to the identity on the boundary ∂M and asymptotic to Id at infinity as in (2.4), acts smoothly, freely and properly on $\mathbb{E}$, with a smooth local slice. Consequently, the quotient

$$\mathcal{E} = \mathbb{E}/\mathrm{Diff}_1^{m+1,\alpha}(M), \tag{2.5}$$

representing isometry classes of solutions, is also a smooth Banach manifold.

In addition, the Bartnik boundary map Π_B in (2.1) is a smooth Fredholm map, of Fredholm index 0, cf. again [14]. Thus the linearization or derivative $D_{(g,u)}\Pi_B$ at any point (M,g,u) is a Fredholm map of Banach spaces. This means that the linear map $D\Pi_B$ has closed range and finite dimensional kernel and cokernel. In more detail, let K be the finite dimensional kernel of $D\Pi_B$ at some point $(M,g,u) \in \mathcal{E}$. The closed subspace K admits a closed complement D,

$$T_{(g,u)}\mathcal{E} = D \oplus K,$$

and similarly the image, $W = \mathrm{Im}D\Pi_B$ is a closed complemented subspace of $T_{(\gamma,H)}\mathcal{B}$. Let C denote the finite dimensional complement. Then $dimC = dimK = k < \infty$ since $D\Pi_B$ has Fredholm index 0. The restricted map

$$D_{(g,u)}\Pi|_D : D \to W,$$

is an isomorphism.

If $dimK = 0$, then $D\Pi_B$ itself is an isomorphism and so the inverse function theorem for Banach manifolds implies Π_B is a local diffeomorphism

near (M, g, u). Suppose $dimK = k > 0$. Then the implicit function theorem (also known as the constant rank theorem) for Banach manifolds states that there is a local submanifold $\mathcal{U}$ of codimension k in $\mathcal{E}$, with $T_{(g,u)}\mathcal{U} = D$ and a local submanifold $\mathcal{V}$ of codimension k in the boundary data space $\mathcal{B}$, with $T_{(\gamma,H)}\mathcal{V} = W$ such that

$$\Pi_B|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V} \tag{2.6}$$

is a diffeomorphism. Thus, near any given solution (M, g, u), one has local existence (and local uniqueness) of solutions, parametrized by boundary data (γ, H) in an open set of a local submanifold of codimension k in $\mathcal{B}$.

We see then that the local structure of the space of solutions $\mathcal{E}$ and the local behavior of the map Π_B relating solutions with boundary data are both very well behaved. However, this structure of the mapping Π_B breaks down at the locus $H = 0$ (minimal surface boundary) basically due to the black hole uniqueness theorem. To see this, let $\bar{\mathcal{E}}$ be the closure of $\mathcal{E}$ in the $C^{m,\alpha}$ norm. This consists of static vacuum Einstein metrics (M, g, u) with $H \geq 0$ and $u \geq 0$ on ∂M.[8] Similarly, let $\widetilde{\mathcal{B}}$ be the full space of boundary data, without the restriction that $H > 0$. One still has the Bartnik boundary map

$$\begin{aligned} \Pi_B : \bar{\mathcal{E}} &\to \widetilde{\mathcal{B}}, \\ (M, g, u) &\to (\gamma, H). \end{aligned} \tag{2.7}$$

Proposition 2.1. *Let (M, g, u) be an AF static vacuum Einstein metric in $\bar{\mathcal{E}}$ and let Σ be a minimal surface surrounding ∂M, (possibly $\Sigma = \partial M$). Then $\Sigma = \partial M$ and (M, g, u) is a Schwarzschild metric with $u = 0$ on the horizon ∂M.*

Proof. This is basically a consequence of the well-known black hole uniqueness theorem [27], [35], [21] and its extension by an elegant argument of Galloway [24]. We refer also to [31] for a different proof, (which relies however on the more difficult positive mass theorem).

To begin, without loss of generality, by standard minimal surface arguments we may assume Σ is outer-minimizing, cf. (3.4) below, and so in particular Σ is a stable minimal surface. By the 2nd variational formula for area

$$\int_\Sigma |df|^2 - (|A|^2 + \mathrm{Ric}(N, N))f^2 \geq 0, \tag{2.8}$$

[8] It does not really make sense in this framework to consider metrics where "$u < 0$" somewhere.

for all Lipschitz functions f with $f \geq 0$; clearly (2.8) then also holds for all Lipschitz functions f. By the static vacuum equations (1.4), we have

$$-\mathrm{Ric}(N,N) = -u^{-1}NN(u) = u^{-1}\Delta_\Sigma u,$$

since $H = 0$, cf. also the line preceding (5.3) in the Appendix. Choosing $f = u$ in (2.8) gives

$$\int_\Sigma |du|^2 + u\Delta_\Sigma u - u^2|A|^2 \geq 0,$$

and so by the divergence theorem

$$\int_\Sigma u^2|A|^2 = 0.$$

Again, the static vacuum Einstein equations imply that $A = 0$ at any locus where $u = 0$. Hence, it follows that

$$A = 0, \tag{2.9}$$

on Σ. If also $u = 0$ on Σ, then the result follows by the black hole uniqueness theorem, so assume u is not identically 0 on Σ. Since by the constraint equation (5.1), (cf. the Appendix), $-2\mathrm{Ric}(N,N) = s_{\gamma_\Sigma}$, (2.8) becomes

$$\int_\Sigma |df|^2 + K_\Sigma f^2 \geq 0.$$

Thus u is a lowest eigenfunction of the operator $-\Delta + K_\Sigma$ and hence, (since u is not identically 0), $u > 0$ on Σ.

Now consider the (so-called) optical metric $\widehat{g} = u^{-2}g$ on (M,g) exterior to Σ and the normal exponential map $\widehat{exp}_\Sigma$ from Σ in the metric $\widehat{g}$. It is proved in [24] that one has the monotonicity formula

$$\frac{d}{dt}(\frac{H}{u}) = -|A|^2, \tag{2.10}$$

where d/dt is the $\widehat{g}$-unit tangent to the flow lines of $\widehat{exp}_\Sigma$; all other metric quantities in (2.10) are with respect to g. (The sign for H used here is opposite to that in [24]). By (2.9), $\frac{d}{dt}(\frac{H}{u}) = 0$ at $t = 0$ and similarly $\frac{d^2}{dt^2}(\frac{H}{u}) = 0$ at $t = 0$. Thus

$$\frac{d^3}{dt^3}(\frac{H}{u}) = -2|\frac{d}{dt}A|^2. \tag{2.11}$$

This corresponds to the 4th variational formula for area. By the outer-minimizing property the left side of (2.11) must be non-negative when integrated over Σ. Hence, it follows that

$$\frac{d}{dt}A = 0,$$

at $t = 0$. By a standard computation, using the fact that $A = 0$,

$$\frac{d}{dt}A = -D^2u - R_N u = 0,$$

where $R_N(X, Y) = \langle R(N, X)Y, N\rangle$. In 3 dimensions, one has $-R_N = \mathrm{Ric} + \mathrm{Ric}(N, N)g$, and so by the static vacuum equations (1.4), it follows that $\mathrm{Ric}(N, N) = 0$. In turn, by the Hamiltonian constraint (5.1), this gives $K_\Sigma = 0$, which implies $\Delta_\Sigma u = 0$, cf. (5.3). Hence $u = const$ on Σ. Also $N(u) = const$ on Σ by the divergence constraint (5.4). Thus the Cauchy data for (M, g, u) at Σ are the trivial data ($\gamma = flat, u = const, A = 0, N(u) = const$). It then follows from unique continuation, or more simply just analyticity (static vacuum solutions with minimal boundary and $u > 0$ are analytic up to the boundary), that (M, g) is flat and u is an affine function. This again gives a contradiction; there are no compact minimal surfaces in $\mathbb{R}^3$. □

Remark 2.2. Proposition 2.1 is stable in the following sense: if (M, g, u) is a static vacuum Einstein metric with boundary data (γ, H) and H is sufficiently small, and if the $C^{m,\alpha}$ norm, $m \geq 4$, of (g, u) up to ∂M is uniformly bounded, then (M, g, u) is close to a Schwarzschild metric $g_{Sch}(m)$ in $\bar{\mathcal{E}}$. The proof of this is the same as that of Proposition 2.1, by taking a sequence of such metrics with boundary data $(\gamma_i, H_i) \to (\gamma, 0)$.

This statement will be significantly generalized in §3, cf. Theorem 3.5.

Proposition 2.1 implies

$$(\Pi_B)^{-1}(\mathrm{Met}^{m,\alpha}(S^2), 0) = g_{Sch}, \tag{2.12}$$

i.e. the inverse image of infinite dimensional space $\mathrm{Met}^{m,\alpha}(S^2) \times \{0\} \subset \widetilde{\mathcal{B}}$ is the 1-dimensional curve of Schwarzschild metrics. The linearized version of Proposition 2.1 implies that the image of $D_{g_{Sch}}\Pi_B$ has infinite codimension in $T\widetilde{\mathcal{B}}$ and so is not Fredholm. The Schwarzschild metrics lie at the boundary or edge of the space $\mathcal{E}$ but the relation (2.12) shows this boundary is just a

curve, not a hypersurface as one might expect. This indicates that the local behavior of Π_B must change significantly on approach to the locus $H = 0$ away from round metrics; this will be explored and discussed in more detail in §3.

Note also that the static vacuum equations (1.4) are no longer strictly elliptic (in any gauge) at regions in the locus $\{u = 0\}$; instead the equations become degenerate elliptic. Namely, the leading order symbol for (1.4) is $u\Delta g_{ij} = -2\partial_i\partial_j u$, $\Delta u = 0$ in a harmonic gauge for g, which is strictly elliptic only when $u > 0$. We note also that when $H > 0$ at the boundary ∂M, a static vacuum solution (M, g, u), C^2 smooth up to ∂M cannot have $u = 0$ at any point of M or ∂M. Namely, by (5.3)

$$\Delta u + HN(u) = \frac{u}{2}(|A|^2 - H^2 + s_\gamma).$$

If $u \geq 0$ on ∂M and $u(p) = 0$ at $p \in \partial M$, then at p, $\Delta u \geq 0$ and $N(u) > 0$ by the Hopf maximum principle. However, the right side vanishes at p giving a contradiction if $H(p) > 0$.

In sum, the map Π_B is Fredholm only where $u > 0$ up to the boundary, and the requirement $H > 0$ at ∂M ensures this condition. All of the above are reasons for enforcing (1.3).

Points $(g, u) \in \mathcal{E}$ where $D\Pi_B$ is an isomorphism are called regular or non-degenerate points of the map Π_B. Recall that boundary data $(\gamma, H) \in \mathcal{B}$ is a regular value of Π_B if every point in the inverse image $(\Pi_B)^{-1}(\gamma, H)$ is a regular point of Π_B in $\mathcal{E}$. By the Smale-Sard theorem [37], the regular values are generic (of second Baire category) in the target $\mathcal{B}$. (Note that, by fiat, the empty set is a regular value). A point (g, u) is singular (for Π_B) if it is not a regular point.

An important locus of singularities of a smooth index 0 Fredholm map $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ between Banach manifolds is a fold locus. This is given by a (local) hypersurface $\mathcal{H} \subset \mathcal{M}$ (a codimension 1 submanifold) where $\mathcal{H}$ consists of singular points of $\mathcal{F}$ with $dim\mathrm{Ker}\, D\mathcal{F} = 1$ and the map $\mathcal{F}$ is 2-1 in an open neighborhood of $\mathcal{H}$ in $\mathcal{M}$ off of $\mathcal{H}$; compare with (2.6), (2.13) and the discussion following (3.21) below. In the context of (Riemannian) Einstein metrics, the most well-known example of such fold behavior is the Hawking-Page phase transition for the AdS Schwarzschild metrics [25]. Similar fold behavior holds for the the interior Dirichlet boundary map for Ricci-flat Schwarzschild metrics [1]. Such behavior also occurs for the general interior Bartnik boundary map [9], i.e. for the analog of (2.1) where M is replaced by a compact (interior) manifold with boundary.

It is natural to conjecture that generically, i.e. at least on an open dense subset of $\mathcal{E}$, the linearization $D\Pi_B$ is an isomorphism, i.e. the regular or non-degenerate points in the domain $\mathcal{E}$ are generic. This issue has been widely considered and analysed in many other geometric PDE (and ODE) problems. For example, for geodesics in Riemannian manifolds, this is the statement that the regular points of the exponential map exp_p are open and dense in T_pM; generically, two points in M are not conjugate along a geodesic, i.e. there are no Jacobi fields vanishing at the end points. Similarly for minimal submanifolds, for a generic boundary in fixed Riemannian manifold, (or for a generic Riemannian metric), minimal surfaces have no Jacobi fields vanishing on the boundary.

In fact, based on an analogy with the monotonicity of the first eigenvalue of the Laplacian on bounded domains with Dirichlet boundary condition with respect to inclusion of domains, White [40] proved that the set of minimal immersions with non-trivial Jacobi fields vanishing on the boundary is of codimension 1 in the space of all minimal immersions $(M, \partial M) \to (N, g_N)$. This is of course a much stronger conclusion than generic. The proof in [40] uses a variational approach, together with a normal interior variation of the domain of the minimal surface and the Calderon unique continuation theorem for elliptic systems. A similar result was proved for the interior Bartnik problem in [9] using an analogous approach, but based on a unique continuation property for Einstein metrics and their linearization; this is considerably more difficult partly due to gauge issues.

Using essentially this same approach but with some modifications, An-Huang [3], [4] have recently shown that the regular points of Π_B are indeed generic in many cases.

As discussed above, while the spaces $\mathcal{E}$, $\mathcal{B}$ and the boundary map Π_B are well-behaved locally, this does not go very far towards resolving Conjecture II, the static vacuum extension conjecture. Consider for instance a very simple toy model. In place of $\mathcal{E}$, consider the circle $S = S^1(r) \subset \mathbb{R}^2$ of radius r about the origin. Similarly, replace $\mathcal{B}$ by the x-axis and the boundary Π_B by the projection π to the x-axis:

$$\begin{aligned} \pi &: S \to \mathbb{R}, \\ \pi(x, y) &= x. \end{aligned} \tag{2.13}$$

For r small, the image of π in $\mathbb{R}$ is small; for most values in $\mathbb{R}$ there is no solution. The space of boundary data realized by solutions increases as r increases, but for any $r < \infty$, there is still a large space of boundary data without solutions. Only in the limit $r \to \infty$ where $S^1(r)$ becomes $\mathbb{R}$ and π

becomes the identity (after suitable translations) does the situation improve. In all cases of course, the local structures involved are very well-behaved. There are only 2 singular or degenerate points, both fold points, at $x = \pm r$; all other points in S are regular.

In the example above, the degree of π is 0, so there are generically either 2 distinct solutions or no solutions. On the other hand, consider the following simple modification. Let T be the graph of the function $\tan(x)$ over the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and consider

$$\begin{aligned} \pi : T &\to \mathbb{R}, \\ \pi(x, y) &= x. \end{aligned} \tag{2.14}$$

Then either there is a unique solution y to $\pi(x, y) = x$ or there are no solutions. The degree of π here is not well-defined. This is because the map π is no longer proper; the issue of properness is fundamental in understanding the global behavior of non-linear mappings and will be discussed in more detail for the map Π_B in §3.

One of the most effective methods of proving global existence and uniqueness is the continuity method, as mentioned above in the works on the solution to the Calabi conjecture and the solution to the Weyl embedding problem. In the current context, this method can be described as follows. Choose boundary data for which a static vacuum solution is known to exist; for instance $(\gamma_0, H_0) = (\gamma_{+1}, 2)$, which is realized by the exterior flat metric $M = \mathbb{R}^3 \setminus B^3(1)$, $g = g_{Eucl}$ is the Euclidean metric and $u = 1$. Given any boundary data $(\gamma, H) \in \mathcal{B}$, choose a path from (γ_0, H_0) to (γ, H) in $\mathcal{B}$; for instance

$$(\gamma_s, H_s) = (1 - s)(\gamma_0, H_0) + s(\gamma, H). \tag{2.15}$$

To prove there is a solution in $\mathcal{E}$ with boundary data (γ, H), one shows that the set of s for which there is a solution g_s with boundary data (2.15) is both open and closed in $[0, 1]$. The openness property amounts to showing that solutions (M, g_s, u_s) with $\Pi_B(M, g_s, u_s) = (\gamma_s, H_s)$ are always non-degenerate. In case the degeneracy locus is non-empty, for example a collection of fold hypersurfaces in $\mathcal{E}$, then a very useful replacement or generalization of the method of continuity is the degree theory, e.g. the Leray-Schauder degree or more generally the Smale degree [37]. This does not require the non-degeneracy property, but to establish a well-defined degree does require the properness of the map Π_B. The properness of Π_B is essentially equivalent to the closedness property above.

3. Global behavior

The key issue in understanding the global behavior of Π_B is to understand how well the boundary data (γ, H) controls the behavior of any possible solution (M, g, u) with such boundary data. As always with elliptic problems, this requires apriori estimates on the geometry of solutions (M, g, u) in terms of (γ, H). Proving such estimates is equivalent to proving that Π_B is a proper map, i.e. the inverse image $(\Pi_B)^{-1}(K)$ of compact sets $K \subset \mathcal{B}$ are compact sets in the domain $\mathcal{E}$. We point out that Fredholm maps are always locally proper, cf. [37].

As a starting point of this analysis, we recall that one does have optimal apriori estimates of solutions (M, g, u) in the interior of M, away from ∂M. Let $t(x) = dist_g(x, \partial M)$ be the distance function to the boundary ∂M and let $\nu = \log u$. We recall that, by definition, $u > 0$ in M. By [5], there is a universal constant $K < \infty$ such that

$$|\mathrm{Rm}|(x) \leq \frac{K}{t^2(x)}, \quad |d\nu(x)| \leq \frac{K}{t(x)}, \tag{3.1}$$

for any static vacuum solution (M, g, u); here Rm is the Riemann curvature tensor of g. The estimates (3.1) are scale invariant and such scale-invariant estimates also hold for all higher derivatives. It follows (from the Cheeger-Gromov compactness theorem) that if one has weak uniform control on the global geometry of the locus $L_1 = \{t = 1\}$ in (M, g), namely a lower bound on its area and upper bound on its diameter,

$$\mathrm{area}_g L_1 \geq v > 0, \quad \mathrm{diam}_g L_1 \leq D < \infty, \tag{3.2}$$

then one obtains global uniform control on (M, g, u) away from ∂M. Under the bounds (3.2), any sequence of static vacuum solutions $(M, g_i, u_i) \in \mathcal{E}$ has a subsequence which converges, in C^∞_δ and away from the boundary ∂M and modulo AF diffeomorphisms, to a limit AF static vacuum solution (M, g, u), cf. [6], [14] for further details. In other words, one has good compactness properties away from the boundary – irrespective of the boundary conditions.

Thus, the issue is the behavior of solutions arbitrarily near the boundary ∂M.

The conjecture that Π_B in (2.1) is a bijection (Conjecture II) suggests the closely related conjecture that Π_B is a diffeomorphism. However, this turns out not to be the case. It was observed by the author in [14] that Π_B is not a proper map, and so in particular is not a homeomorphism.

The reason for this is simple: the structure of a manifold-with-boundary $(M, \partial M)$ may degenerate with uniform control on the boundary data (γ, H). In other words, the embedded boundary ∂M in M may "degenerate" from an embedding to an immersion.

As a simple illustration, consider for instance any fixed static vacuum metric (M, g, u) with boundary ∂M and boundary data $(\gamma, H) \in \mathcal{B}$. Let Σ be any embedded 2-sphere $S^2 \subset M$ homologous to the boundary ∂M with $H_\Sigma > 0$. One may then cut off M at Σ to obtain a new exterior solution M' with $\partial M' = \Sigma$ and corresponding boundary data $(\gamma_\Sigma, H_\Sigma) \in \mathcal{B}$ at Σ. Suppose however Σ is an *immersed* 2-sphere in (M, g) homologous to ∂M. This again has induced "boundary" data $(\gamma_\Sigma, H_\Sigma)$ from (M, g), but Σ is not the natural smooth boundary of an exterior manifold-with-boundary M'. If the immersed surface Σ is isotopic to the boundary ∂M in M with $H_\Sigma > 0$, then at least in many cases one may easily find curves of spheres Σ_s in (M, g) which smoothly deform an embedded sphere to an immersed sphere, with boundary data (γ_s, H_s) smoothly controlled for all s and with H_s remaining uniformly positive. In this process, the ambient metric g and potential u remain fixed up to ∂M; only the domain M_s on which they are defined is changing. Thus, control of the "boundary data" (γ_s, H_s) does not control the structure of the manifold-with-boundary M_s. On the other hand, when (M, g) is an extension of a smooth interior body (Ω, g_Ω), the corresponding interior geometry (Ω_s, g_{Ω_s}) is well-controlled and does not degenerate at all.

Remark 3.1. It was proved in [13] that this type of degeneration leads to counterexamples to the Bartnik minimization conjecture, Conjecture I. There exists a large family of immersed spheres S^2 in $\mathbb{R}^3$ whose boundary data $(\gamma, H) \in \mathcal{B}$ are not realized by a static vacuum extension of minimal mass; cf. also [11] for further discussion and conjectures.

Remark 3.2. One immediate consequence of this degeneration is a kind of symmetry breaking of the gauge group. As discussed in §2, the natural gauge group for the Bartnik boundary value problem is the group $\mathrm{Diff}_1(M)$ of AF diffeomorphisms of M equal to the identity on the boundary ∂M. This corresponds to isometric metrics, and preserves the boundary data (γ, H). Suppose then (γ, H) are realized as boundary data of a static vacuum solution (M, g, u), with ∂M embedded. Then clearly so are $(\varphi^*\gamma, \varphi^*H)$, for any diffeomorphism φ of $\partial M \simeq S^2$. Namely any such diffeomorphism extends to an AF diffeomorphism, also called φ, of M and the triple $(M, \varphi^*g, \varphi^*u)$ is a solution of the static vacuum Einstein equations with boundary data $(\varphi^*\gamma, \varphi^*H)$. If however Σ is merely an immersed "boundary" in a larger space M, then this may not be the case. It is no longer clear that if (γ, H)

are the boundary data of Σ, then $(\varphi^*\gamma, \varphi^*H)$ are realized as boundary values of a static vacuum solution, for general diffeomorphisms φ of $\Sigma \simeq S^2$. Thus the symmetry group of the problem has been reduced in the immersed case.

This breakdown of diffeomorphism invariance of the problem implies that one must work in a suitable fixed gauge. One can no longer deal with metrics up to isometry (i.e. moduli spaces of metrics or solutions to the equations) but only with metrics in a fixed gauge, i.e. (local) coordinate system. It is worth noting that the Weyl metrics, discussed in detail in §4, do have a preferred global gauge (coordinate system).

Let $(M, g_i, u_i) \in \mathcal{E}$ be a sequence of static vacuum solutions for which the boundary data $(\gamma_i, H_i) \in \mathcal{B}$ converge to a limit (γ, H), $H \geq 0$ in $\widetilde{\mathcal{B}}$. There are only two ways that the sequence can degenerate, i.e. fail to have a subsequence which converges to a limit $(M, g, u) \in \widetilde{\mathcal{E}}$, cf. (2.7)). First the structure of an exterior manifold with boundary may break down as above; this occurs only if there is no uniform lower bound on the distance to the cut locus of the g_i-normal exponential map of ∂M into (M, g_i). Second, the norm of the Riemann curvature Rm of g_i may tend to infinity at or arbitrarily near ∂M,

$$|\mathrm{Rm}_{g_i}(x_i)| \to \infty, \tag{3.3}$$

for some sequence such that $t(x_i) \to 0$, (cf. [14]). Of course it is possible that both of these behaviors occur simultaneously. We note that a special case of the analysis given in [14] shows that (3.3) does not occur if there are uniform lower and upper bounds on the potential u at ∂M, i.e. constants c, C such that

$$0 < c \leq u_i \leq C < \infty \quad \text{on} \quad \partial M.$$

There is one, seemingly natural, condition that can be used to rule out both of these possibilities of degeneration at once. Thus, recall that the boundary ∂M is called outer-minimizing in (M, g) if for any surface $S \subset M$ surrounding ∂M, i.e. S is homologous to ∂M in M, one has

$$\mathrm{area}_g\, S \geq \mathrm{area}_g\, \partial M. \tag{3.4}$$

The boundary ∂M is strictly outer-minimizing if strict inequality holds in (3.4), for all $S \neq \partial M$. Note that by the first variational formula for area, (3.4) implies (1.3), $H \geq 0$ at ∂M. It is obvious but important to note that the outer-minimizing condition depends on the global structure of the solution (M, g); it cannot be expressed only in terms of the boundary data (γ, H).

The following result was proved by the author in [14].

Theorem 3.3. *Let $(M, g_i, u_i) \in \bar{\mathcal{E}}$ be a sequence of static vacuum Einstein metrics with boundary data (γ_i, H_i) with $H_i \geq 0$, $m \geq 4$, such that*

$$(\gamma_i, H_i) \to (\gamma, H),$$

in $\widetilde{\mathcal{B}}$. Suppose also that ∂M is outer-minimizing in (M, g_i) for each i:

$$area_{g_i} S \geq area_{g_i} \partial M,$$

for all S surrounding ∂M. Then a subsequence converges in $\widetilde{\mathcal{E}}$ to a limit static vacuum Einstein metric $(M, g, u) \in \bar{\mathcal{E}}$ with boundary data (γ, H).

Thus, one has good compactness properties under an outer-minimizing condition.

In light of Theorem 3.3, it might seem beneficial to restrict to static vacuum solutions with (strictly) outer-minimizing boundary. Let then $\mathcal{E}^{out} \subset \mathcal{E}$ be the space of static vacuum solutions for which the boundary is strictly outer-minimizing:

$$\text{area}_g S > \text{area}_g \partial M, \tag{3.5}$$

for $S \neq \partial M$. This is clearly an open condition so that $\mathcal{E}^{out}$ remains a smooth Banach manifold. The map

$$\Pi_B : \mathcal{E}^{out} \to \mathcal{B}, \tag{3.6}$$

is still well-defined, smooth and Fredholm, of index 0.

In fact it is not unusual in the literature to modify the definition of the Bartnik mass by replacing the no horizon condition (no minimal surfaces surrounding the boundary) by an outer-minimizing condition. Thus define

$$m_B^{out}(\Omega, g_\Omega) = \inf\{m_{ADM}(M, g) : (M, g) \in \mathcal{P}_\Omega^{out}\}, \tag{3.7}$$

where $\mathcal{P}_\Omega^{out}$ is the space of strictly outer-minimizing extensions of (Ω, g_Ω) or equivalently of the boundary data (γ, H); cf. [28] for a detailed discussion. Similarly, the minimizing and static extension Conjectures I and II are then rephrased to assert the existence and uniqueness of static vacuum solutions with strictly outer-minimizing boundary. One advantage of this modified definition is the fundamental estimate of Huisken-Illmanen [26],

$$m_B^{out}(\gamma, H) \geq m_H(\gamma, H) = \sqrt{\frac{\text{area}_g \partial M}{16\pi}}(1 - \frac{1}{16\pi}\int_{\partial M} H^2),$$

comparing the outer-minimizing Bartnik mass with the Hawking mass of the boundary.

The problem with these reformulations say of Conjecture II is that one needs the strict inequality (3.5) to have a satisfactory manifold structure to the space $\mathcal{E}^{out}$ of solutions. Working instead with the (weak) outer-minimizing condition leads, at best, to the structure of a Banach manifold with boundary; this is not suitable to try to apply general existence methods, such as the method of continuity or the more general degree theory techniques discussed briefly in §2.

On the other hand, working with the strict outer-minimizing condition, the map Π_B in (3.6) cannot possibly be proper. One can easily find sequences $(M, g_i, u_i) \subset \mathcal{E}^{out}$ with $(\gamma_i, H_i) \to (\gamma, H) \in \mathcal{B}$ smoothly, but which have no limit (M, g, u) in $\mathcal{E}^{out}$. In fact it is easy to find such sequences of boundaries in a fixed background solution (M, g, u) with limit only weakly outer-minimizing

$$\operatorname{area}_g S \geq \operatorname{area}_g \partial M.$$

Thus, neither the weak or strict outer-minimizing condition seems satisfactory. We are not aware of any way out of this quandary.

Remark 3.4. One might raise the same objection with the no-horizon condition itself; if a static vacuum solution had a horizon, (i.e. a minimal surface surrounding ∂M), then it could not be an admissible extension and so could not realize the Bartnik mass as in Conjecture I. One would have to consider only static vacuum solutions without horizons, which would be very awkward. Fortunately, Proposition 2.1 rules out this situation.

Of course, the main trouble here is that the outer-minimizing property (strict or weak) cannot be phrased in terms of the boundary data (γ, H). There has been some speculation that suitable conditions on the boundary data (γ, H) would ensure that static vacuum solutions with such boundary data have strict outer-minimizing boundary; a common suggestion, based on the resolution of the Weyl embedding problem, is the condition

$$K_\gamma > 0 \quad \text{and} \quad H > 0.$$

However, the next result shows that this cannot possibly work, at least for H small.

Using Theorem 3.3, we prove the following non-existence result. This may be viewed as a stable version of the black hole uniqueness theorem, as in Proposition 2.1, with H small replacing $H = 0$.

Theorem 3.5. *Let γ be a smooth metric on S^2 such that the Gauss curvature K_γ is a Morse function in the domain $K_\gamma > 0$, i.e. K_γ has only non-degenerate critical points when $K_\gamma > 0$.*

Let $\mu(p) = \min |\lambda_i(p)|$, where $\lambda_i(p) \neq 0$ are the eigenvalues of the Hessian $D^2 K_\gamma$ at a critical point p of K_γ, $K_\gamma(p) > 0$. Let also $C = ||\gamma||_{C^{m,\alpha}} + ||H||_{C^{m-1,\alpha}}$, for some $m \geq 4$.

Then there is a constant $H_0 > 0$, depending only on an upper bound for C and a positive lower bound for $\mu_0 = \min_p \mu(p) > 0$, such that: for any smooth function H satisfying

$$0 < H < H_0, \tag{3.8}$$

pointwise, the Bartnik boundary data (γ, H) are not the boundary data of a static vacuum Einstein metric $(M, g, u) \in \mathcal{E}$. Thus for such $(\gamma, H) \in \mathcal{B}$,

$$\Pi_B^{-1}(\gamma, H) = \emptyset.$$

We note that at least a large class of boundary data satisfying (3.8) above do have positive scalar curvature fill-ins, i.e. are boundary data of compact domains (Ω, g_Ω) with positive scalar curvature, so that Conjecture II^+ does not hold. This is the case for instance if γ has positive Gauss curvature.

Proof. The proof is by contradiction. Thus, let $(\gamma_i, H_i) \in \mathcal{B}$, $(H_i > 0)$, be a sequence of boundary data converging in $\widetilde{\mathcal{B}}$ to $(\gamma, 0) \notin \mathcal{B}$ and satisfying the assumptions of Theorem 3.5. Suppose there is a sequence of static vacuum solutions (M, g_i, u_i), $u_i > 0$ with Bartnik boundary data (γ_i, H_i). We then need to derive a contradiction.

In case the boundary ∂M is (weakly) outer-minimizing in (M, g_i) for i sufficiently large, Theorem 3.5 is proved in [11], based on the discussion in [13]. We sketch the argument for completeness. Since ∂M is outer-minimizing for each g_i, Theorem 3.3 implies that a subsequence of (M, g_i, u_i) converges in $\widetilde{\mathcal{E}}$ to a limit solution $(M, g, u) \in \widetilde{\mathcal{E}}$ of the static vacuum Einstein equations with $u \geq 0$ and $u \to 1$ at infinity. The Bartnik boundary data of (M, g, u) is given by

$$\lim_{i \to \infty} (\gamma_i, H_i) = (\gamma, 0).$$

By the black hole uniqueness theorem as in Proposition 2.1, the only static vacuum solution (M, g, u) with $u \geq 0$, $u \to 1$ at infinity and smooth horizon boundary $H = 0$ is the Schwarzschild metric with boundary data $(\gamma_{2m}, 0)$. Since the Gauss curvature $K_{\gamma_{2m}} \equiv (2m)^{-2}$ is not a Morse function, this gives a contradiction. Namely the smooth convergence of (M, g_i, u_i) up

to the boundary implies that K_i is smoothly close to the constant $(2m)^{-2}$; however, the eigenvalues of the Hessian D^2K_i are bounded away from 0, say at $\max K_i$.

Now in general, consider a static vacuum solution (M, g, u) (say equal to some (M, g_i, u_i) above) with boundary data (γ, H) with $H > 0$ but close to 0. Let Σ be the outer-minimizing hull of ∂M in M, cf. [26]. This is the embedded surface surrounding ∂M in (M, g) of smallest area. Note first that by Proposition 2.1, $\Sigma \subset M$ cannot be disjoint from ∂M, so

$$\Sigma \cap \partial M \neq \emptyset.$$

Let $U \subset \partial M$ be the interior of $\partial M \cap \Sigma$ and let $V = \Sigma \backslash \bar{U} \subset \Sigma$. Then (again by Proposition 2.1), both U and V are non-empty open sets; of course, neither U nor V is assumed to be connected. By the outer-minimizing property, one has

$$H = 0 \quad \text{on} \quad V.$$

The surface Σ is $C^{1,1}$ at the seam or corner $\partial U \cap \partial V$, smooth away from $\partial U \cap \partial V$ and has non-negative distributional mean curvature H across $\partial U \cap \partial V$, cf. [26].

Now given (M, g_i, u_i) as above, consider the new sequence of static vacuum solutions $(\widehat{M}_i, g_i, u_i)$ where $\widehat{M}_i \subset M$ is the region of M exterior to Σ_i, so $\partial \widehat{M}_i = \Sigma_i$. The analysis to follow will be to show that that the same arguments as before where ∂M_i was outer-minimizing can be applied to the new sequence $(\widehat{M}_i, g_i, u_i)$ to again obtain a contradiction.

Let $\gamma_{U_i} = g_i|_{U_i}$ and $\gamma_{V_i} = g|_{V_i}$. The metric γ_{U_i} is uniformly controlled in $C^{m,\alpha}$ by hypothesis, since $U_i \subset \partial M_i$. Now assume first, for simplicity, that the sequence (V_i, γ_{V_i}) is uniformly bounded in $C^{4,\alpha}$ (modulo diffeomorphisms), so that the Gauss curvature K_i of (V_i, γ_{V_i}) is uniformly bounded in $C^{2,\alpha}$. The same then holds for the induced metric $\widehat{\gamma}_i$ on $\partial \widehat{M}_i$, away from the seam or corner $\partial V_i \cap \partial U_i \subset \Sigma_i$. Thus away from the corners, one has $H_i \to 0$ in $C^{3,\alpha}$ with $\widehat{\gamma}_i$ bounded in $C^{4,\alpha}$. One may then still apply the proof of Theorem 3.3 in this setting to conclude that the curvature Rm_{g_i} and its derivatives up to order $2+\alpha$ are uniformly bounded in $\widehat{M}_i$, up to the boundary ∂M_i; we refer to [14] for further details here. Thus, in a subsequence, $(\widehat{M}_i, \widehat{g}_i, \widehat{u}_i)$ converges in $C^{4,\alpha}$ away from corners to a round Schwarzschild metric, cf. also the proof of Proposition 2.1 and Remark 2.2. In particular, this convergence holds in $U_i \subset \partial M_i$ and so gives the same contradiction as above – even if components of U_i shrink to points for instance.

Now the surfaces (V_i, γ_{V_i}) are stable minimal surfaces in (M, g_i). In the Appendix, we prove a regularity result for such stable minimal surfaces in static vacuum Einstein manifolds which implies that if both the scalar curvature s_{V_i} of (V_i, γ_{V_i}) and the full Riemann curvature Rm of (M, g_i) are uniformly bounded along V_i,

$$|s_{V_i}| + |\mathrm{Rm}_{g_i}| \leq C \tag{3.9}$$

on V_i, then in fact (V_i, γ_{V_i}) is uniformly bounded in C^∞ modulo diffeomorphisms. Note this statement is standard for the trivial static vacuum Einstein metric $\mathbb{R}^3$. It also follows easily in regions of V_i where $t_i = dist_{g_i}(\partial M_i, \cdot)$ is bounded away from 0, by the estimates (3.1). Thus, the proof follows just as above if (3.9) holds, and so it suffices to prove (3.9).

We prove (3.9) by a blow-up argument. The discussion below applies to each $(\widehat{M}_i, g_i, u_i)$ but we drop the index i from the notation; the estimates below are then understood to hold uniformly, for all i large.

We begin with some basic uniform integral estimates at the boundary Σ. First, the outer-minimizing property and 2nd variation of area gives

$$\int_\Sigma (|df|^2 + (N(H) + H^2)f^2) dv_{\gamma_\Sigma} \geq 0. \tag{3.10}$$

The normal variation of H is given by $N(H) = -|A|^2 - \mathrm{Ric}(N, N)$. By the Hamiltonian constraint (Gauss) equation (5.1), $|A|^2 - H^2 + s_\gamma = -2\mathrm{Ric}(N, N)$, so that

$$N(H) = -\tfrac{1}{2}(|A|^2 + H^2 - s_{\gamma_\Sigma}).$$

Choosing $f = 1$ in (3.10) thus gives the bound

$$\int_\Sigma |A|^2 \leq \int_\Sigma s_{\gamma_\Sigma} + H^2 = 4\pi\chi(\Sigma) + \int_\Sigma H^2.$$

Since $H^2 \to 0$ uniformly, it follows that one has the uniform bound

$$\int_\Sigma |A|^2 \leq 8\pi + 1. \tag{3.11}$$

Note this bound is scale invariant. Next, again from the Gauss equation, cf. (5.3), one has

$$|A|^2 - H^2 + s_\gamma = 2u^{-1}(\Delta u + HN(u)).$$

Integrating over Σ and applying divergence formula gives

$$\int_\Sigma |d^T\nu|^2 \le 8\pi - \int_\Sigma HN(\nu),$$

where $d^T\nu$ is the tangential gradient of ν on Σ. On the domain $V \subset \Sigma$, $H = 0$ while on the locally outer-minimizing complement $U \subset \partial M$, $HN(\nu)$ converges to a well-defined limit by Theorem 3.3, cf. [14] for details. It follows that

$$\int_\Sigma |d^T\nu|^2 \le C, \tag{3.12}$$

for some uniform C, independent of i.

Next, for the blow-up argument, choose points $p_i \in \Sigma_i$ realizing the maximum of

$$Q_i(x) = |s_{\gamma_{V_i}}(x)| + |\mathrm{Rm}_{g_i}(x)|, \tag{3.13}$$

$x \in \Sigma_i$. Without loss of generality, assuming $Q_i(p_i) \to \infty$, we rescale the metrics g_i (and correspondingly γ_{V_i}) by

$$\bar{g}_i = Q_i(p_i) g_i,$$

so that

$$\bar{Q}_i \le 1 \quad \text{and} \quad \bar{Q}_i(p_i) = 1, \tag{3.14}$$

on Σ_i. Clearly one has $p_i \in V_i$. We also renormalize u to $\bar{u}_i = u_i/u(p_i)$. Again by (the proof of) Theorem 3.3, $|\mathrm{Rm}_{\bar{g}_i}|$ is uniformly bounded on $(\widehat{M}_i, \bar{g}_i)$.

To prove that a subsequence $(\widehat{M}_i, \bar{g}_i, p_i)$ converges modulo diffeomorphisms, (i.e. in the Cheeger-Gromov sense), in $C^{1,\alpha}$, we need to rule out the possibility of collapse of the metrics, i.e. the collapse of the volume of unit balls at p_i.

Lemma 3.6 (Non-collapse)**.** *For the rescaled sequence $(\widehat{M}_i, \bar{g}_i, p_i)$, there is a uniform lower bound*

$$\mathrm{vol}_{\bar{g}_i} B_{p_i}(1) \ge v_0 > 0. \tag{3.15}$$

Similarly, the boundary geometry $(\Sigma_i, \bar{\gamma}_i, p_i)$ does not collapse:

$$\mathrm{area}_{\bar{\gamma}_i} D_{p_i}(1) \ge a_0 > 0. \tag{3.16}$$

Proof. First, as discussed in the Appendix, cf. (5.5), for any $(M, g, u) \in \mathcal{E}$, the conformally equivalent metric $\widetilde{g} = u^2 g$ has positive Ricci curvature. The metric $\widetilde{g}$ is AF and hence by the Bishop-Gromov volume comparison theorem, there is a uniform scale-invariant lower bound on the volume of r-balls in $(M, \widetilde{g})$:

$$\frac{\mathrm{vol}_{\widetilde{g}} B(r)}{r^3} \geq v_0 > 0.$$

Now apply this to the rescaled metrics $(\widehat{M}_i, \bar{g}_i, \bar{u}_i)$ based at p_i. Since $\bar{u}_i$ is uniformly bounded above and below in $(B_{p_i}(1), \bar{g}_i)$, (cf. (5.8)), the metrics $\bar{g}_i$ and $\widetilde{g}_i$ are uniformly quasi-isometric in this region. This proves (3.15).

To see that the boundary geometry also cannot collapse, recall that $(\Sigma_i, \bar{\gamma}_i)$ has uniformly bounded (Gauss) curvature. If the geometry is collapsing at p_i, then the geometry of a large geodesic disc $D_{p_i}(R)$ about p_i in $(\Sigma_i, \bar{\gamma}_i)$ is that of a long cylinder which has a foliation by short (geodesic) loops. Since the ambient curvature is uniformly bounded, if ℓ denotes the length of the short loops in $(\Sigma_i, \bar{\gamma}_i)$ near p_i, then $|A| \sim \ell^{-1}$ and hence

$$\int_I |A|^2 dv_\gamma \sim \int_I \ell \ell^{-2} \sim \ell^{-1}.$$

Since A is bounded in L^2 by (3.11) and since (3.11) is scale-invariant, this gives a uniform lower bound on ℓ, proving (3.16). □

The discussion above proves that, after passing to a subsequence, the sequence $(\widehat{M}_i, \bar{g}_i, \bar{u}_i, p_i)$ converges in the pointed $C^{1,\alpha}$ topology, modulo diffeomorphisms, to a complete non-compact limit $(M_\infty, \bar{g}_\infty, \bar{u}_\infty, p)$, with complete, non-compact boundary $(S, \gamma_S, \bar{u}_\infty, p)$. The rest of the argument to follow is to prove that the limit is flat, with flat boundary S. By results in the Appendix, cf. Proposition 5.2, the convergence to the limit is in the strong C^2 topology, giving a contradiction to the continuity of Q in (3.14) under C^2 convergence. This will complete the proof of Theorem 3.5.

All the discussion to follow takes place on the limit M_∞ and S so we drop the bar and ∞ below to simplify the notation.

On the blow-up limit boundary S, we have A and $d\nu$ are in $L^2(S, \gamma_S)$, by (3.11) and (3.12). Moreover, the ambient curvature $|\mathrm{Rm}|$ is uniformly bounded along S. As above, $D(r)$ denotes a geodesic r-disc in (S, γ_S) about p.

Lemma 3.7. *The limit (S, γ_S) has at most quadratic area growth, i.e.*

$$\mathrm{area}_{\gamma_S} D(r) \leq V r^2, \tag{3.17}$$

for some $V < \infty$.

Proof. On (S, γ_S), we have

$$|A|^2 + s_{\gamma_S} = 2u^{-1}\Delta u. \tag{3.18}$$

Let $v(r)$ be the length of the boundary $S(r) = \partial D(r)$. Integrating (3.18) over $D(r)$ and applying the Gauss-Bonnet and divergence theorems gives

$$4\pi - 2v'(r) = -\int_{D(r)} |A|^2 + 2\int_{D(r)} |d^T\nu|^2 + 2\int_{S(r)} \partial_r \nu.$$

By (3.11), this gives

$$v'(r) \leq C + \int_{S(r)} |d^T\nu|.$$

On intervals where $v'(r) \leq C$, (3.17) holds (by integration), so assume only

$$v'(r) \leq C\int_{S(r)} |d^T\nu|.$$

By the Hölder inequality, $\int_{S(r)} |d^T\nu| \leq (\int_{S(r)} |d^T\nu|^2)^{1/2} v(r)^{1/2}$, so that $((\sqrt{v'(r)})')^2 \leq C\int_{S(r)} |d^T\nu|^2$. Let $q(r) = \sqrt{v'(r)}$, so that via (3.12) we obtain

$$\int_0^r (q')^2 \leq C.$$

Then $q(r) \leq \int_0^r q'(r) \leq (\int_0^r (q'(r)^2)^{1/2} r^{1/2} \leq Cr^{1/2}$, so that again $v(r) \leq Cr$. This proves the result. □

Lemma 3.8. *The surface $(S, \gamma_S) \subset (M, g)$ is AF in the weak sense that*

$$Q(x_k) \to 0,$$

on any sequence $x_k \to \infty$ in S.

Proof. Let x_k be any divergent sequence in S and consider the geometry of pointed manifolds (S, γ_S, x_k) in regions about x_k. By Lemma 3.6, this sequence does not collapse, and so has a $C^{1,\alpha}$ limit (in a subsequence). Also, $A \to 0$ and $d\nu \to 0$ in L^2_{loc}, by (3.11) and (3.12). By the regularity estimates in the Appendix, A is uniformly bounded in L^∞, as is $u \in L^{2,p}_{loc}$ on

the boundary S and in M, cf. (5.7), (5.8). Hence $N(u)$ is also uniformly bounded. By the constraint equation (5.4),

$$\delta A - A(d\nu) = u^{-1} dN(u).$$

The term $A(d\nu) \to 0$ in L^2_{loc}, while the term $\delta A \to 0$ in $L^{-1,2}_{loc}$. It follows that $dN(u) \to 0$ in $L^{-1,2}_{loc}$ and hence $N(u) \to const$ weakly in L^2_{loc}. Since $N(u)$ is also uniformly bounded in L^∞, $N(u) \to const$ strongly in L^2_{loc}. It follows that on the limit (which is smooth by regularity results from Appendix), one has the limit data

$$A = 0, \quad u = const, \quad N(u) = const.$$

Similarly, $NN(u) = 0$ on the limit, etc. Since $\Delta_g u = 0$ on M, unique continuation for the Laplacian implies that $D^2 u = 0$ on (M, g) so that u is an affine function and the metric g is flat. Since $A = 0$, the boundary metric γ is also flat.

It follows that $Q \equiv 0$ on the limit. By the strong convergence (Proposition 5.2) in the Appendix, it follows that $Q(x_k) \to 0$, as claimed. □

It follows, by repeated use of Lemma 3.8 at various blow-down scales that all tangent cones at infinity of (S, γ_S) are flat, with flat ambient geometry, i.e. $(D(r), \frac{1}{r^2}\gamma_S)$ converges to a flat metric away from the origin p and similarly with g_S in place of γ_S. Now apply the stability inequality (3.10) on the complete (S, γ_S). By the well-known log-cutoff trick, (cf. [23] for example) one has

$$\int_S |A|^2 - s_{\gamma_S} \leq 0,$$

so that by the Gauss-Bonnet theorem,

$$\int_S |A|^2 \leq \int_S s_\gamma = 2\pi - \lim_{r\to\infty} v'(r). \tag{3.19}$$

This implies first that $\lim_{r\to\infty} \frac{v(r)}{r} \leq 2\pi$. Suppose strict inequality holds, $\lim_{r\to\infty} \frac{v(r)}{r} < 2\pi$. The tangent cone at infinity C is S is then a flat cone, of cone angle $\alpha < 2\pi$, which bounds a flat static vacuum solution. By the scale-invariant apriori estimates (3.1), the limit static vacuum solution is smooth away from the boundary cone C, so cannot be of the form $C \times \mathbb{R}^+$, which is singular in the interior: any singularity of the cone C cannot propagate into the exterior region. On the other hand, if C is a singular cone,

$\alpha < 2\pi$, in an ambient smooth flat geometry, then one has $\int_C |A|^2 = \infty$, again a contradiction. (Alternately, such cones are not outer-minimizing in the ambient geometry).

It follows that

$$\lim_{r\to\infty} \frac{v(r)}{r} = 2\pi, \tag{3.20}$$

and hence by (3.19),

$$\int_S |A|^2 = 0,$$

so that $A = 0$ on S. The potential equation (3.18) then becomes

$$K = u^{-1}\Delta u.$$

Integrating this over $D(r)$ and taking the limit gives

$$\lim_{r\to\infty} [\int_{D(r)} |d^T\nu|^2 + \int_{S(r)} \partial_r \nu] = 0,$$

since $\lim_{r\to\infty} \int_{D(r)} K = 0$. As in the proof of Lemma 3.7, this implies

$$\int_S |d^T\nu|^2 = 0,$$

so that $u = const$ on S. Thus the Cauchy data $(\gamma, A, u, N(u))$ for the static vacuum Einstein equations equal that of a flat solution with affine potential. By unique continuation as in the proof of Lemma 3.8, it follows that the limit is flat, with flat boundary, so that $Q = 0$ on S. This again contradicts the strong convergence to the limit, cf. Proposition 5.2.

This completes the proof of Theorem 3.5. □

The condition that K_γ is a Morse function is a sufficient but we don't believe a necessary condition. It is natural to conjecture that Theorem 3.5 holds for all γ, with then H sufficiently small depending only on lower and upper bounds for the distance of γ to a round metric in say $C^{4,\alpha}$, (and an upper bound on $|H|_{C^{3,\alpha}}$).

It follows in particular that the mass m_B cannot be realized by boundary data (γ, H) with H sufficiently small and γ not a round metric, i.e. Conjecture I and Conjecture I$^+$ also fail in this situation.

Remark 3.9. Theorem 3.5 should be contrasted with the following result, proved in [10]; the proof also strongly uses the outer-minimizing property: for any given smooth boundary data $(\gamma, H) \in \mathcal{B}$, there is a $\lambda > 0$ such that

$$(\gamma, \lambda H) \in \operatorname{Im} \Pi_B,$$

i.e. there is a static vacuum solution (M, g, u) with boundary data $(\gamma, \lambda H)$. Moreover, ∂M is outer-minimizing in (M, g). Thus, for any γ, if one increases H sufficiently by a positive multiplicative factor, then there is a solution to the static vacuum equations realising the given Bartnik boundary data.

For boundary data $(\gamma, H) \notin \operatorname{Im} \Pi_B$ as in Theorem 3.5, a minimizing sequence for $m_B(\gamma, H)$ cannot converge, so it must degenerate. How? Consider first the boundary data $(\gamma, 0) \in \widetilde{\mathcal{B}}$, with γ not a round metric on S^2, compare with (2.12). The black hole uniqueness theorem (Proposition 2.1) implies that a minimizing sequence for this boundary data must degenerate. The work of Mantoulidis-Schoen [30] indicates how this occurs – when the first eigenvalue $\lambda_1(-\Delta_\gamma + K_\gamma) > 0$ – for minimizing sequences with respect to the outer-minimizing mass $m_B^{out}(\gamma, 0)$ in (3.7). Briefly, one portion of the minimizing sequence (containing the AF end) converges to a Schwarzschild metric $g_{Sch(m)}$ up to or arbitrarily near the horizon, where m is given by

$$m = \sqrt{\frac{\operatorname{area}_\gamma \partial M}{16\pi}}.$$

The remaining portion of the sequence forms an (arbitrarily) long cylinder of non-negative scalar curvature, connecting the nearly round metric near the horizon at one end to the metric γ at the other end. The cylindrical region is'hidden behind' the Schwarzschild horizon; since this region does not contribute to the ADM mass, one has no effective control on the behavior of a minimizing sequence in this region. We refer also to the work of Miao-Xie [32] for related results.

We conjecture similar behavior occurs for minimizing sequences for $m_B(\gamma, H)$ when $H > 0$ is sufficiently small, depending on γ as in Theorem 3.5.

The non-existence region above strongly suggests there is a large region of fold behavior for the map Π_B, which, in turn, implies a large region of non-uniqueness for Conjecture II (compare with the discussion in and preceding the toy model (2.13)).

We describe the situation on a concrete example. Choose a fixed Schwarzschild metric $g_{Sch}(m)$ and choose a round sphere $S(t)$ at fixed distance $t > 0$ to the horizon. Let (γ_0, H_0) be the induced Bartnik boundary

data on $S(t)$, so γ_0 is a round metric and $H_0 > 0$ is a small constant. It is proved in [14] that $\operatorname{Ker} D\Pi_B = 0$ at the Schwarzschild metric exterior to the horizon $H = 0$ (even though $D\Pi_B$ is not Fredholm there); hence $\operatorname{Ker} D\Pi_B = 0$ at the exterior region to $S(t)$ for t small, i.e. this exterior metric is a regular point of Π_B. Now choose any fixed metric γ satisfying the hypotheses of Theorem 3.5, i.e. K_γ is a Morse function in the region $K_\gamma > 0$. Consider the curve of boundary data

$$L(s) = (1-s)(\gamma_0, H_0) + s(\gamma, H_0) \subset \mathcal{B},$$

for $s \in [0,1]$ and the corresponding inverse image

$$\Pi_B^{-1}(L(s)).$$

By the Smale-Sard theorem [37], if necessary one may perturb $L : I \to \mathcal{E}$ slightly to $\widetilde{L} : I \to \mathcal{E}$, keeping the endpoints fixed, so that $\widetilde{L} : I \to \mathcal{E}$ is transverse to Π_B. The inverse image

$$\Pi_B^{-1}(\widetilde{L}),$$

is then a collection of 1-dimensional curves $\{\sigma_i(s)\}$ in $\mathcal{E}$. If there is more than one component of $\Pi_B^{-1}(\widetilde{L})$ then of course one already has non-uniqueness for Conjecture II, so suppose then that $\sigma = \Pi_B^{-1}(\widetilde{L})$ is a connected curve. By Theorem 3.5, for H_0 sufficiently small depending on γ, there is $s_0 < 1$ (possibly close to 1) such that either

$$\Pi_B^{-1}(\widetilde{L}[s_0,1]) = \emptyset \quad \text{or} \quad \Pi_B^{-1}(\widetilde{L}(s_0,1]) = \emptyset. \tag{3.21}$$

Without loss of generality, assume $\Pi_B^{-1}(\widetilde{L}(s)) \neq \emptyset$ for $s < s_0$. We discuss these two possibilities in more detail.

I. In the first case above, as $s \to s_0$ with $s < s_0$, the curve $\sigma(s)$ diverges to infinity in $\mathcal{E}$, i.e. Π_B is not a proper map over $\widetilde{L}$. Since at $s = 0$ the boundary $S(t)$ is strictly outer-minimizing in the Schwarzschild metric, the boundary ∂M of $\sigma(s)$ remains outer-minimizing for s sufficiently small. However, by Theorem 3.3, the boundary cannot remain outer-minimizing for all $s \in [0, s_0)$. The divergence of the curve $\sigma(s)$ as $s \to s_0$ is due either to a loss of the manifold-with-boundary structure or the curvature blows up on approach to the boundary (or both).

II. In the second case, the map Π_B is proper at least over $\widetilde{L}[0, s_1)$ for some $s_1 > s_0$. In this case, Π_B exhibits fold behavior (like $x \to x^2$) over $\widetilde{L}$.

Such fold points $\sigma(s_0)$ are critical points of Π_B and the map Π_B is locally 2-1 over $\widetilde{L}$ near $\widetilde{L}(s_0)$. This again gives non-uniqueness for Conjecture II.

The same analysis as above applies to general curves in $\mathcal{B}$, which start at data (γ_0, H_0) for which $(\Pi_B)^{-1}(\gamma_0, H_0) \neq \emptyset$ and which end at points (γ, H) satisfying the conditions of Theorem 3.5, for which $(\Pi_B)^{-1}(\gamma, H) = \emptyset$. We conjecture that, at least for a large class of curves, Case I does not occur, so the fold behavior of Case II holds.

4. Weyl metrics

It is of course important to have a large class of examples on which one can test the Bartnik conjectures in various regions of boundary data. The most interesting class of (relatively) explicit solutions are the Weyl solutions [39], [15], which have an additional axial symmetry. These metrics have a hypersurface-orthogonal isometric S^1 action, so that the metric $g = g_M$ on M has the form

$$g_M = f^2 d\varphi^2 + g_V, \tag{4.1}$$

where the orbit space (V, g_V) is a Riemannian surface with $V \simeq (\mathbb{R}^3 \setminus B)/S^1 \simeq (\mathbb{R}^2)^+ \setminus D$. Thus the Ricci flat 4-metric $(\mathcal{M}, g_{\mathcal{M}})$ (as in (1.5)) has the form

$$g_{\mathcal{M}} = \pm u^2 dt^2 + f^2 d\varphi^2 + g_V, \tag{4.2}$$

with u, f positive functions on V. This gives the 4-manifold $\mathcal{M}$ the structure of a toric Einstein 4-manifold (when the t-factor is compactified to S^1 in the Riemannian setting). The static vacuum equations (1.4) are then reduced to equations on (V, g_V).

As in (2.5), let

$$\mathcal{E}^{m,\alpha}_{S^1} = \mathbb{E}^{m,\alpha}_{S^1}/\mathrm{Diff}^{m+1,\alpha}_1(M), \tag{4.3}$$

denote the (abstract) space of isometry classes of such AF Weyl metrics of the form (4.1), with $H > 0$ at ∂M. Similarly, the space of S^1-invariant boundary data is given by

$$\mathcal{B}^{m,\alpha}_{S^1} = \mathrm{Met}^{m,\alpha}_{S^1}(S^2) \times C^{m-1,\alpha}_{S^1,+}(S^2).$$

Here $\mathrm{Met}^{m,\alpha}_{S^1}(S^2)$ denotes the space of $C^{m,\alpha}$ metrics on S^2 of the form $\alpha^2 d\theta^2 + \beta^2 d\varphi^2$, $\alpha = \alpha(\theta) > 0, \beta = \beta(\theta)$ where (θ, φ) are standard spherical coordinates on $S^2(1)$; there is no $d\varphi d\theta$ cross term in (4.1). At the poles

$\theta = 0, \pi$, one has $\beta = 0$ and $\beta' = \pm\alpha$ with similar higher order conditions on β at 0, π for $C^{m,\alpha}$ smoothness. The functions in $C^{m-1,\alpha}_{S^1,+}(S^2)$ are positive functions H of the form $H = H(\theta)$ with similar smoothness conditions at the poles.

The associated Bartnik boundary map

$$\begin{aligned} \Pi_B : \mathcal{E}_{S^1} &\to \mathcal{B}_{S^1}, \\ (M, g, u) &\to (\gamma, H), \end{aligned} \tag{4.4}$$

is still a smooth Fredholm map of Fredholm index 0. One expects or at least hopes that it should be considerably simpler to understand the validity of Conjectures I and II in the context of Weyl metrics compared with the general case.

An important feature is that Weyl metrics have a canonical choice of coordinates, Weyl cylindrical coordinates and it is useful to derive this in some detail, cf. also, [6] [29] for background material on these metrics. First, note that both $M_u = V \times_f S^1$ with metric $g_V + f^2 d\theta^2$ and $M_f = V \times_u S^1$ with metric $g_V + u^2 d\theta^2$ are static vacuum solutions on the respective 3-manifolds, with potentials u and f. Thus one has a natural dual pairing $u \leftrightarrow f$, cf. [6] for further discussion. On M_u,

$$0 = \Delta_{M_u} u = \Delta_V u + \langle d \log f, du \rangle,$$

while on M_f,

$$0 = \Delta_{M_f} f = \Delta_V f + \langle d \log u, df \rangle.$$

It follows that the function

$$r = fu \tag{4.5}$$

representing the area of the toral fibers in (4.2), is harmonic on V, $\Delta_V r = 0$. Noting that V is simply connected, let z be the harmonic conjugate of r on V. Then (r, z) give isothermal coordinates for the metric g_V. Adding the coordinate $\varphi \in [0, 2\pi]$ gives the Weyl canonical cylindrical coordinates (r, z, φ) on $M = V \times_f S^1$. In these coordinates, the metric g on M has the form

$$g = g_M = u^{-2}[e^{2\lambda}(dr^2 + dz^2) + r^2 d\varphi^2], \tag{4.6}$$

for some function λ. Let $\nu = \log u$. Further analysis using the static vacuum equations (1.4) shows that λ satisfies

$$\lambda_r = r(\nu_r^2 - \nu_z^2), \quad \lambda_z = 2r\nu_r\nu_z, \tag{4.7}$$

so that the 1-form $d\lambda$ is given by $d\lambda = r(\nu_r^2 - \nu_z^2)dr + 2r\nu_r\nu_z dz$. Thus, λ is completely determined by u, up to a constant. On the z-axis A where $r = 0$ (so $\theta = 0, \pi$) one has $\lambda = const$. The metric (4.6) is regular (i.e. smooth) at A only when $\lambda = 0$; otherwise there is a cone singularity along A. Thus, λ is uniquely determined by ν. Note that r in (4.5) is uniquely determined by (M, g, u)[9] while z is uniquely determined up to a constant; this constant may be fixed by imposing the normalization condition

$$\int_{\partial M} z\, dv_\gamma = 0. \tag{4.8}$$

We note that (r, z) normalized as in (4.8) vary smoothly with (g, u).

The coordinates (r, z, φ) represent standard cylindrical coordinates on $\mathbb{R}^3$. Most importantly, the potential function $\nu = \log u$ is an axi-symmetric (i.e. φ-invariant) harmonic function with respect to the *Euclidean* Laplacian:

$$\Delta_{Eucl}\nu = 0. \tag{4.9}$$

Observe that the full Weyl solution (4.6) is determined by the single potential function ν in these coordinates.

One main point here is that given any (abstract) Weyl metric, there is a canonical choice of coordinates, i.e. gauge, read off from the geometry of S^1 fibers, compare with Remark 3.2. In this chart, the Weyl solution (M, g, u) is completely determined by the axi-symmetric Euclidean harmonic function ν and the location of the boundary ∂M. This is the remarkable Weyl reduction of a non-linear system of equations to a linear equation.

Working in these coordinates, a boundary is specified by a, say $C^{m+1,\alpha}$, embedded curve

$$\sigma : [0, \pi] \to (\mathbb{R}^2)^+ \tag{4.10}$$

in $(\mathbb{R}^2)^+$, $\sigma(\theta) = (r(\theta), z(\theta))$ meeting the axis A smoothly and orthogonally only at $\theta = 0, \pi$. The rotation of σ about the axis A generates an embedding

$$F : S^2 \to \mathbb{R}^3$$

of a sphere as a surface of revolution in $\mathbb{R}^3$. The image $\Sigma = \operatorname{Im} F$ divides $\mathbb{R}^3$ into a compact interior region diffeomorphic to a 3-ball B and a non-compact exterior region diffeomorphic to $\mathbb{R}^3 \setminus B$. Here we focus exclusively

[9] Since $u > 0$ on M and g is smooth on M, note that (4.6) also shows that r is a well-defined coordinate, i.e. $dr \neq 0$ on M.

on the exterior manifold-with-boundary M with $\partial M = \Sigma$. We will assume that σ is oriented[10] in that $\theta = 0$ corresponds to the north pole of S^2 while $\theta = \pi$ corresponds to the south pole and the normal vector ∂_z at $\sigma(0)$ points into the exterior region M.

The induced metric γ on the boundary is given by

$$\gamma = F^* g = u^{-2}[e^{2\lambda}((r'(\theta))^2 + (z'(\theta))^2)d\theta^2 + r^2 d\varphi^2]. \tag{4.11}$$

By computation, one has

$$e^{\lambda-\nu} H = H_{Eucl} + N_E(\lambda - 2\nu), \tag{4.12}$$

where N_E is the Euclidean unit normal pointing into M and H_{Eucl} is the mean curvature of ∂M with respect to the Euclidean metric; this is given by

$$H_{Eucl} = \frac{N_E(r)}{r} + \kappa_E,$$

corresponding to the standard formula for the mean curvature of surfaces of revolution in $\mathbb{R}^3$; here κ_E is the Euclidean geodesic curvature of σ, given as $\langle \nabla_T N, T\rangle$ in the Euclidean metric.

Examples. A very useful and large class of examples of Weyl solutions, closely related to Newtonian gravity, are given as follows. Let $d\mu$ be any positive Radon measure, compactly supported on the axis A. Then the Newtonian potential of ν, i.e.

$$\nu(x) = -\int_A \frac{1}{|x-y|} d\mu_y, \tag{4.13}$$

gives an axisymmetric harmonic function on $\mathbb{R}^3 \setminus \operatorname{supp} d\mu$ and so generates an AF Weyl solution. For example, taking

$$d\mu = \frac{1}{2} d\ell_{[-m,m]},$$

where $d\ell$ is the Lebesque measure on the interval $[-m, m]$ gives the Schwarzschild metric of mass m. On the other hand, taking

$$d\mu = \kappa d\ell_{[-m,m]},$$

[10]This will be dropped later.

for any $\kappa \neq \frac{1}{2}$ generates a metric g which is singular, i.e. not smooth up to the boundary $\operatorname{supp} d\mu$.

Taking $d\mu = m\delta_0$ to be a multiple of the Dirac delta function supported at the origin of A, gives

$$\nu = -\frac{m}{R},$$

a multiple of the Green's function on $\mathbb{R}^3$, with $R^2 = r^2 + z^2$. The resulting solution, called the Curzon solution, is given explicitly as

$$g_C(m) = e^{2m/R}[e^{-m^2r^2/R^4}(dr^2 + dz^2) + r^2 d\varphi^2].$$

This metric becomes highly singular on approach to the origin $0 \in \mathbb{R}^3$.

It is not difficult to see (cf. [6]) that for all solutions as in (4.13), $\nu \to -\infty$, so $u \to 0$, on approach to at least a dense subset of $\operatorname{supp} d\mu$. Thus, such static vacuum solutions are maximal, in the sense that they cannot be extended to any larger domain.[11]

One may then take a boundary $\partial M = \operatorname{Im} F$ as above to obtain a Weyl solution (M, g, u) with induced Bartnik boundary data (γ, H) on ∂M. These solutions correspond to varying domains in a fixed ambient maximal Weyl solution generated by ν in (4.13).

Not all Weyl solutions (M, g, u) are of this form however. The most general solutions are given by solving the Dirichlet problem for ν.[12]

Proposition 4.1. *Let $\sigma : I \to \mathbb{R}^2$ be a $C^{m+1,\alpha}$ embedding as in (4.10) and let ν be a $C^{m,\alpha}$ function on $\operatorname{Im}\sigma$. Then ν extends uniquely to an axisymmetric harmonic function on the exterior domain $M = \mathbb{R}^3 \setminus B$ with $\nu \to 0$ at infinity, and generates an AF Weyl metric (M, g, u) which is $C^{m,\alpha}$ up to ∂M.*

Conversely, up to isometry in $\operatorname{Diff}_1^{m+1,\alpha}(M)$, any AF Weyl metric (M, g, u) which is $C^{m,\alpha}$ up to ∂M is uniquely given by such a pair (σ, ν) satisfying (4.8).

Proof. The first statement is standard from elliptic regularity theory, given the discussion above. For the second statement, given a specific representative $(M, g, u) \in \mathbb{E}_{S^1}^{m,\alpha}$ representing a class in $\mathcal{E}_{S^1}^{m,\alpha}$, construct the Weyl

[11] One could also take potentials ν as in (4.13) with more general signed measures or even distributions supported in A, generating Weyl metrics in the same way. However, when $d\mu$ is a positive measure, boundaries have fill-ins (Ω, g_Ω) with non-negative scalar curvature and the ADM mass m_g of the solution is always non-negative.

[12] One could also solve the Neumann or a suitable Robin boundary value problem instead.

canonical coordinates (r, z, φ), normalized as in (4.8). These coordinates restrict to ∂M to give an axi-symmetric embedding $F : \partial M \simeq S^2 \to \mathbb{R}^3$ and hence the embedding $\sigma : I \to (\mathbb{R}^2)^+$. A different representative (M', g', u') of (M, g, u) in $\mathcal{E}^{m,\alpha}_{S^1}$ gives the same map F, since the diffeomorphisms in $\mathrm{Diff}^{m+1,\alpha}_1(M)$ fix the boundary pointwise.

To see that $\sigma \in C^{m+1,\alpha}(\theta)$, by hypothesis ν, and hence λ, is $C^{m,\alpha}$ up to ∂M, while $H \in C^{m-1,\alpha}(\theta)$. Boundary regularity for harmonic functions implies that $N(\nu)$ is $C^{m-1,\alpha}$ up to ∂M and hence the same for $N(\lambda)$. Thus, by (4.12), $H_{Eucl} \in C^{m-1,\alpha}(\theta)$. This together with the fact that the Euclidean arclength parameter $\sqrt{(r')^2 + (z')^2} \in C^{m,\alpha}$ implies $\sigma \in C^{m+1,\alpha}(\theta)$. □

Remark 4.2. Generic boundary data ν on ∂M lead to harmonic functions on M which do not extend to any larger region $M' \supset M$ containing ∂M in the interior. Thus the examples above, although simple and useful, are not generic. Note that when the boundary data ∂M and ν are analytic, then ν and so g does extend past ∂M to a slightly larger domain M'.

Remark 4.3. Proposition 4.1 suggests that the simplest boundary data for a Weyl metric are the Dirichlet boundary data. This corresponds to the isometric embedding of a prescribed metric $u^2 dt^2 + \gamma$ on the boundary $S^1 \times S^2$ into an ambient 4-dimensional Weyl metric. However, it is proved in [7] that Dirichlet boundary data are not elliptic boundary data for (general) Einstein metrics.

We see then that the space $\mathbb{E}^{m,\alpha}_W$ of $C^{m,\alpha}$ AF Weyl metrics in Weyl canonical coordinates has a very explicit (and simple) form:

$$\begin{gathered}\mathbb{E}^{m,\alpha}_W \simeq [\mathrm{Emb}^{m+1,\alpha}(I) \times C^{m,\alpha}(I)]/T, \\ (M, g, u) \leftrightarrow (\sigma, \nu).\end{gathered}$$

where T is the action of translations along the z-axis A on the first factor. The Bartnik boundary map is thus given by

$$\begin{gathered}\Pi_B : \mathbb{E}^{m,\alpha}_W \to \mathcal{B}^{m,\alpha}_{S^1}, \\ \Pi_B(\nu, \sigma) = (F^* g, H_{F,g}).\end{gathered} \tag{4.14}$$

The three (arbitrary) functions $(\sigma, \nu) = (r(\theta), z(\theta), \nu(\theta))$ describing the points in $\mathbb{E}_W$ correspond to the three (arbitrary) functions $(\alpha(\theta), \beta(\theta), H(\theta))$ in $\mathcal{B}_{S^1}$.

However, in contrast to the (abstract) map Π_B in (4.4), the map in (4.14) is no longer smooth; although it is C^0, it is not even C^1. Namely, let F_s be

a smooth curve of $C^{m+1,\alpha}$ axi-symmetric embeddings of S^2 into $\mathbb{R}^3$, for example $F_s = F + sX$ where X is a $C^{m+1,\alpha}$ axi-symmetric vector field on M. Then the derivative $D\Pi_B(X)$ in (4.14) involves the terms $X(\lambda - \nu)$ and $X(\nu)$ (cf. (4.11)) which are only $C^{m-1,\alpha}$ and not $C^{m,\alpha}$ on M up to ∂M.[13]

Thus, the natural identification provided by Proposition 4.1

$$\Phi : \mathcal{E}^{m,\alpha}_{S^1} \simeq \mathbb{E}^{m,\alpha}_W,$$

is a homeomorphism, but is not a C^1 identification. In other words, for a C^∞ curve of $C^{m+1,\alpha}$ diffeomorphisms $\varphi_s : M \to M$ with $\varphi_s \neq \mathrm{Id}$ on ∂M, the curve $(\varphi_s^* g, \varphi_s^* u)$ is not in general a smooth curve in the abstract space $\mathcal{E}_{S^1}$ or $\mathcal{E}$. In the abstract setting, the "location" of the boundary ∂M is not specified and the Bartnik boundary map should be viewed as a free boundary value problem. Nevertheless, the lack of smoothness of the identification Φ is not an essential issue for most purposes.

Despite the concrete setting of the Bartnik boundary map in (4.14), it does not appear to be significantly easier to understand the validity of Conjectures I and II for Weyl metrics. For instance, we are not aware of any substantially simpler proofs of Proposition 2.1 or Theorem 3.5 in this setting.

To make some progress, let us consider a much simpler setting. Namely, fix the (global) potential ν as in (4.13). Then ν, λ are globally defined, independent of the location of the boundary. Let

$$\mathbb{E}^{m,\alpha}_\nu \subset \mathbb{E}^{m,\alpha}_W \tag{4.15}$$

be the subspace with potential fixed in this way. Clearly, $\mathbb{E}_\nu$ is a Banach submanifold. For the restriction of the Bartnik boundary map to $\mathbb{E}_\nu$, it is natural to drop the map to the mean curvature (freezing a scalar field in the domain corresponds to freezing a scalar field in the target).[14] This leads to consideration of the Dirichlet boundary data map

$$\begin{aligned} \Pi_D : \mathbb{E}^{m,\alpha}_\nu &\to \widehat{\mathrm{Met}}^{m,\alpha}_{S^1}(S^2), \\ \sigma &\to F^* g. \end{aligned} \tag{4.16}$$

[13]This loss of derivative for the action of diffeomorphisms on the space of metrics is well-known. Note that Π_B is smooth when ν is, say, C^∞ up to ∂M, so that Π_B is smooth in the context of Frechet manifolds. If (M, g, u) extends to a slightly larger domain M', as in Remark 4.2, then ν is automatically as smooth as ∂M in M', since static vacuum Einstein pairs (g, u) are analytic in the interior.

[14]Another natural choice would be to keep the mean curvature but take only the conformal class of the metric; cf. [8] for further discussion.

Here $\widehat{\mathrm{Met}}^{m,\alpha}_{S^1}(S^2)$ is the modification of $\mathrm{Met}^{m,\alpha}_{S^1}(S^2)$ consisting of α, β as before with $\alpha \in C^{m,\alpha}(\theta)$ but $\beta \in C^{m+1,\alpha}(\theta)$. This modification is due to the different levels of differentiability of α and β in (4.11).[15] Also $g = g(\nu)$. Clearly (4.16) corresponds to the isometric embedding problem: given an axially symmetric metric on S^2, does there exist a (unique) isometric embedding into a Weyl metric (M, g, u) with fixed u?

We consider in detail the case $u = 1$ so $\nu = 0$ and $g = g_{Eucl}$ corresponding to the Euclidean isometric embedding problem. First, it turns out that the map Π_D is not onto, i.e. not every axi-symmetric metric on S^1 is realized by an isometric embedding or immersion as a surface of revolution in $\mathbb{R}^3$. The following is a well-known necessary condition.

Lemma 4.4. *Let $\sigma(\theta) = (r(\theta), z(\theta))$ be a $C^{m+1,\alpha}$ immersion into $(\mathbb{R}^2)^+$ generating the surface of revolution F with induced metric $\alpha^2 d\theta^2 + \beta^2 d\varphi^2$. Then for all θ,*

$$\alpha(\theta) \geq |\beta'(\theta)|. \tag{4.17}$$

Further equality holds at the endpoints $\theta = 0, \pi$, i.e. at the poles of the surface.

Proof. For such an immersion, one clearly has

$$\alpha^2 = (r')^2 + (z')^2 \tag{4.18}$$

$$\beta = r, \tag{4.19}$$

so that

$$(z')^2 = \alpha^2 - (\beta')^2 \geq 0. \tag{4.20}$$

Since necessarily $\alpha > 0$, this proves (4.17). Further, as noted above, smoothness at the poles implies $z' = 0$, proving the second statement. □

Although it is obvious, it is worth pointing out explicitly that (4.19) shows that the function $r(\theta)$ of $\sigma(\theta) = (r(\theta), z(\theta))$ is uniquely determined by the metric γ. Thus we need only consider the behavior of z in terms of the boundary data (α, β).

Let $\mathcal{V}_0 \subset \widehat{\mathrm{Met}}^{m,\alpha}_{S^1}(S^2)$ be the open set such that

$$\alpha(\theta) > |\beta'(\theta)|, \quad \forall \theta \in (0, \pi), \tag{4.21}$$

[15] The regularizing property of the mean curvature is no longer present in this situation.

with $\beta'(\theta) = \alpha(\theta)$ for $\theta = 0, \pi$. Let $\mathcal{U}_0 = \Pi_D^{-1}(\mathcal{V}_0) \subset \mathbb{E}^{m,\alpha}_{\nu=0}$.

Lemma 4.5. *The map*

$$\Pi_D : \mathcal{U}_0 \to \mathcal{V}_0$$

is a diffeomorphism.

Proof. By (4.20), one has

$$z' = \pm\sqrt{\alpha^2 - (\beta')^2}. \tag{4.22}$$

Since by hypothesis, $z' \neq 0$ on $(0, \pi)$, there is a unique choice of sign for z'. As noted following (4.10), the choice of orientation of σ then gives

$$z' = -\sqrt{\alpha^2 - (\beta')^2},$$

on $[0, \pi]$. By integration, this then uniquely defines z under the normalization condition (4.8). As noted above, the function $r(\theta)$ is already defined by β. Clearly, the curve σ is a graph over the z-axis (except at the poles), so that σ is an embedding.

The remarks above prove that Π_D is a bijection on $\mathcal{U}_0$. It is also smooth, with derivative given by

$$D\Pi_D(\rho, \zeta) = (\frac{1}{\alpha}(r'\rho + z'\zeta), \rho). \tag{4.23}$$

Since $z' \neq 0$, $\mathrm{Ker} D\Pi_D = 0$. Similarly, it is easy to see that $D\Pi_D$ is surjective, so that Π_D is a diffeomorphism. □

However this result breaks down at curves where (4.21) fails. In fact, the map $D_\sigma \Pi_D$ is no longer a Fredholm map at σ if

$$(\alpha - |\beta'|)(\theta_0) = 0, \tag{4.24}$$

i.e. $z'(\theta_0) = 0$, for some $\theta_0 \in (0, \pi)$. Namely, to solve $D\Pi_D(\rho, \zeta) = (q, \rho)$ as in (4.23) requires $r'\rho + z'\zeta = \alpha q$, so that

$$\zeta = \frac{\alpha q - r'\rho}{z'},$$

where α, ρ, r' and z' are given. This is solvable only if $\alpha q - r'\rho$ vanishes on the zero-set of z', and the quotient ζ is then only $C^{m-1,\alpha}$. Thus $D\Pi_D$ maps

at most only onto a dense subset of a codimension one subspace of $C^{m,\alpha}$ so that Π_D is not Fredholm.[16]

Next let $\mathcal{V}_1 \subset \widehat{\mathrm{Met}}_{S^1}(S^2)$ be the subset such that

$$\chi(t) = \alpha(\theta) - |\beta'(t)| \geq 0 \tag{4.25}$$

with $\chi = 0$ at only finitely many points $\theta_i \in [0, \pi]$, each of which is a non-degenerate minimum of χ. As noted above, points where $\chi = 0$ correspond to points where $z' = 0$, i.e. the tangent line to σ is horizontal. As simple examples show, one can no longer expect that curves having points where $z' = 0$ remain embedded in general.[17] For this and related reasons, we modify the definition of $\mathbb{E}_{\nu=0}$ (without changing its notation) to

$$\mathbb{E}^m_{\nu=0} = \mathrm{Imm}_0{}^{m+1}(I)/\mathrm{Isom}(\mathbb{R}).$$

Here $\mathrm{Imm}_0{}^{m+1}(I)$ is the space of unoriented immersions isotopic to embeddings relative to the endpoints. We have changed (m, α) to m (since there are no longer any elliptic regularity issues) and taken the quotient by action of the rotation invariant subgroup $\mathrm{Isom}(\mathbb{R}) \subset \mathrm{Isom}(\mathbb{R}^3)$, consisting of translations along the z-axis A and reflections in planes $z = const$. The group $\mathrm{Isom}(\mathbb{R})$ is the relevant group of Euclidean congruences.

As before, we have the Dirichlet (boundary) map

$$\Pi_D : \mathbb{E}^m_{\nu=0} \to \widehat{\mathrm{Met}}^m_{S^1}(S^2), \quad \sigma \to F^*(g_{Eucl}).$$

As above, set $\mathcal{U}_1 = (\Pi_D)^{-1}(\mathcal{V}_1) \subset \mathbb{E}^m_{\nu=0}$. It is worth noting that neither $\mathcal{V}_1$ nor $\mathcal{U}_1$ is connected; each space has many components. For instance, passing from points in $\mathcal{U}_0$ to points in $\mathcal{U}_1$ requires passing through curves where $\chi(t)$ has degenerate minima.

Lemma 4.6. *The smooth (but not Fredholm and so not proper) map*

$$\Pi_D : \mathcal{U}_1 \to \mathcal{V}_1,$$

is a bijection. Further $\mathrm{Ker} D\Pi_D = 0$ *everywhere.*

[16]To avoid this loss of derivative, one could pass to C^∞ maps and so Frechet spaces. While $D_\sigma \Pi_D$ would be Fredholm if σ has isolated points where (4.24) holds to finite order, Π_D would not be a tame Fredholm map, due to the factor $(z')^{-1}$.

[17]Compare with the discussion at the beginning of §3.

Proof. Let $(\alpha, \beta) \in \mathcal{V}_1^m$ be arbitrary. As with the proof of Lemma 4.5, the issue here is the determination of the sign of z' in terms of α, β. Once this is determined, z and hence $\sigma = (r, z)$ is determined as above.

We first note that the order of vanishing of z' is completely determined by α, β. Namely, by (4.20),

$$z'z'' = \alpha\alpha' - \beta'\beta'', \tag{4.26}$$

and

$$(z'')^2 + z'z''' = (\alpha\alpha')' - (\beta'\beta'')'. \tag{4.27}$$

Thus at θ_i where $z' = 0$, whether $z'' \neq 0$ or $z'' = 0$ is completely determined by whether the minimum of $\alpha - |\beta'|$ is a non-degenerate minimum or not.

Now starting at the pole $\theta = 0$, we have either $z'' > 0$ or $z'' < 0$. Suppose $z'' > 0$. Then the sign of z' and hence as before z and so σ, is uniquely determined up to first zero θ_1 of χ. Since $z''(\theta_1) \neq 0$, by continuity the sign of z' is uniquely determined slightly past θ_1, so that σ is then uniquely determined up to the second zero θ_2 of χ. Proceeding in this way uniquely determines a curve σ_1 on $[0, \pi]$ with $\Pi_D(\sigma_1) = (\alpha, \beta)$.

If $z''(0) < 0$, one may perform the same process, obtaining a curve σ_2 also with $\Pi_D(\sigma_2) = (\alpha, \beta)$. However, up to translations along the z-axis, the $\mathbb{Z}_2$ reflection through the plane $z = z(0)$ maps σ_1 to σ_2, so that σ_1 and σ_2 are congruent and represent the same point in $\mathcal{U}_1$. This proves Π_D is a bijection on $\mathcal{U}_1$. The second statement follows easily from (4.23) as above. □

One may proceed inductively in this way to higher order degeneracies of z'. Thus, let $\mathcal{V}_2 \subset \widehat{\mathrm{Met}}^m_{S^1}(S^2)$ be the subset such that the minima $\theta_i \in (0, \pi)$ of χ are non-degenerate at order either 2 or 4; note that $\mathcal{V}_1 \subset \mathcal{V}_2$. As before, set $\mathcal{U}_2 = (\Pi_D)^{-1}(\mathcal{V}_2) \subset \mathbb{E}^m_{\nu=0}$. Differentiating (4.27) twice gives

$$3(z''')^2 + 4z''z^{(4)} + z'z^{(5)} = (\alpha\alpha')''' - (\beta'\beta'')''' = \tfrac{1}{2}((\alpha^2)^{(4)} - (\beta^2)^{(5)}).$$

Thus, for $\sigma \in \mathcal{U}_2$ (with $m \geq 4$) if $z'(\theta_i) = z''(\theta_i) = 0$ for some θ_i, then $z'''(\theta_i) \neq 0$.

Note that smoothness requires $z''' = 0$ at the poles $0, \pi$. In any case, choose a generic point $\bar{\theta}$ sufficiently near 0 so that $(\alpha\alpha' - \beta'\beta'')(\bar{\theta}) \neq 0$. By (4.26), $z'(\bar{\theta}) \neq 0$ and $z''(\bar{\theta}) \neq 0$. Suppose $z''(\bar{\theta}) > 0$. Then the sign of z' is uniquely determined by α, β and so by (4.22), z is uniquely determined by α, β near $\bar{\theta}$. The same arguments as above then show that $(\alpha, \beta) \in \mathcal{V}_2$

uniquely determine a curve $\sigma_1 \in \mathcal{U}_2$. Again if $z''(\bar{\theta}) < 0$, then the same construction produces another curve $\sigma_2 \in \mathcal{U}_2$ which is congruent in $\mathrm{Isom}(\mathbb{R})$ to σ_1.

One may construct in the same way spaces $\mathcal{V}_k \subset \mathcal{V}_{k+1} \subset \cdots$, where $\mathcal{V}_k$ consists of metrics for which χ has isolated zeros θ_i each of which is non-degenerate at some order $\leq 2k$. This of course requires $m \geq 2k$. In the same way, given sufficient smoothness, one has the inclusion of Π_D-inverse images $\mathcal{U}_k \subset \mathcal{U}_{k+1} \cdots$. The same proof of Lemmas 4.5 – 4.6 as above shows that

$$\Pi_D : \mathcal{U}_k \to \mathcal{V}_k,$$

is a bijection. Similarly, $\mathrm{Ker} D\Pi_D = 0$ on $\mathcal{U}_k$.

Suppose however $\sigma \in \mathcal{U}_k$, $m \geq 2k$ is a curve for which there exist zeros θ_i and θ_j of χ (not necessarily consecutive) at which χ vanishes at all orders $\leq 2(k-1)$ and is non-vanishing at order $2k$. Then the derivatives $z^{(j)} = 0$ for all $j \leq k$ at θ_i and θ_j but $z^{(k+1)} \neq 0$ at θ_i and θ_j. Suppose also

$$z(\theta_i) = z(\theta_j).$$

One may perform a $\mathbb{Z}_2$-reflection of σ through the plane $z = z(\theta_i)$ over the interval $[\theta_i, \theta_j]$ to obtain another curve $\widetilde{\sigma}$ which agrees with σ outside $[\theta_i, \theta_j]$. These two curves differ only by a local, not global, $\mathbb{Z}_2$-reflection and so are non-congruent curves. Note however that while $\sigma \in \mathbb{E}^m_{\nu=0}$, with $m \geq 2k$, $\widetilde{\sigma}$ is only in $\mathbb{E}^k_{\nu=0}$, so that these curves are not in the same space. This loss of derivatives can be cured (only) by going to C^∞, and when z' vanishes to infinite order at θ_i and θ_j. Thus there are regions where Π_D becomes (at least) a $2-1$ map in certain regions in the C^∞ context.

Finally, to conclude this discussion suppose that $\chi = \alpha - |\beta'|$ on an open θ-interval (θ_0, θ_1). The surface $\Sigma = \mathrm{Im} F$ is then a flat annulus in this region. Let f be any smooth function of compact support in (θ_0, θ_1). Then the deformation

$$\sigma_s = \sigma + sfN,$$

$N = \partial_z$, induces an infinitesimal isometric deformation of Σ, $\mathcal{L}_{fN}\gamma = 0$, Thus $\mathrm{Ker} D\Pi_D$ is now infinite dimensional. Also, for any s and f as above, the surfaces $\Sigma_\pm$ generated by

$$\sigma_\pm = \sigma \pm sfN$$

are isometric but not congruent. Thus, one has here the $2-1$ fold behavior discussed in §2 and §3.

Summarizing, the discussion above proves:

Proposition 4.7. *For any* $\gamma = (\alpha, \beta) \in \mathrm{Met}^{\infty}_{S^1}(S^2)$ *satisfying* (4.17), *there is a* C^∞ *axi-symmetric isometric immersion* $F : S^2 \to \mathbb{R}^3$, *so that*

$$F^*(g_{Eucl}) = \gamma.$$

The same result holds for C^m *smoothness, for any* $1 \leq m < \infty$. *However, in general the map*

$$\Pi_D : \mathrm{Imm}^{\infty}_{S^1}(S^2) \to \mathrm{Met}^{\infty}_{S^1}(S^2),$$

may be finite-to-one or ∞*-to-one.*

Remark 4.8. All of the results above hold for general $\mathbb{E}_\nu$ as in (4.15), when the condition (4.17) is replaced by

$$e^\lambda \alpha \geq |\beta'|,$$

where λ is determined from ν as in (4.7).

We also note that the infinitesimal rigidity (i.e. $\mathrm{Ker} D\Pi_D = 0$) on the spaces $\mathcal{U}_k$ discussed above fails badly when one considers non-axisymmetric embeddings of axi-symmetric metrics $S^2 \to \mathbb{R}^3$.[18] There is a large classical literature on this, starting from the remarkable examples of Cohn-Vossen [22], see also [34], [38], [36].

In conclusion, it would be interesting to understand if some of the complicated behavior of the isometric embedding problem for surfaces of revolution in $\mathbb{R}^3$ exhibited above carries over to general Weyl metrics and the corresponding Bartnik boundary map in (4.4) or more generally in (2.1).

5. Appendix

In this Appendix, we first collect several standard facts regarding static vacuum Einstein metrics. Following this, we prove a regularity result for minimal surfaces in static vacuum spaces used in the proof of Theorem 3.5.

On a Riemannian manifold (M, g) with (local) boundary $(\partial M, \gamma)$ the Gauss and Gauss-Codazzi equations (also known as the Hamiltonian and momentum constraint equations) are:

$$|A|^2 - H^2 + s_\gamma = s_g - 2\mathrm{Ric}(N, N), \tag{5.1}$$

[18] Thus the symmetry of the metric is not assumed apriori to extend to a symmetry of $\mathbb{R}^3$.

$$\delta A + dH = -\mathrm{Ric}(N, \cdot), \tag{5.2}$$

where N is the unit normal to ∂M and δ is the divergence operator on ∂M. For static vacuum Einstein metrics, $s_g = 0$ and $\mathrm{Ric}(N, N) = u^{-1}NN(u)$. Using the standard relation $\Delta_g v = NN(v) + HN(v) + \Delta_{\partial M} v$, (5.1) gives

$$2u^{-1}(\Delta_{\partial M} u + HN(u)) = |A|^2 - H^2 + s_\gamma. \tag{5.3}$$

Similarly, the static vacuum equations give $\mathrm{Ric}(N, \cdot) = u^{-1}D^2u(N, \cdot) = u^{-1}(dN(u) - A(du))$. It follows that

$$\delta(uA) + udH = -dN(u). \tag{5.4}$$

or equivalently

$$\delta(uA - N(u)\gamma) = -udH.$$

Next, consider the conformally related metric $\widetilde{g} = u^2 g$. Standard formulas for conformal change of metric show that the static vacuum Einstein equations (1.4) are equivalent to the system

$$\mathrm{Ric}_{\widetilde{g}} = 2d\nu \cdot d\nu, \quad \Delta_{\widetilde{g}}\nu = 0, \tag{5.5}$$

where $\nu = \log u$. Two advantages of these equations are that the Ricci curvature $\mathrm{Ric}_{\widetilde{g}}$ is positive, and second, the Ricci curvature is lower order (1st order) in ν, which is not the case for (1.4).

Next we turn to the regularity properties of local minimal surfaces (V, γ) in static vacuum Einstein metrics (M, g, u). To begin, as in (3.9), assume a bound on Q in (3.13), so

$$|s_\gamma| + |\mathrm{Rm}_g| \leq C, \tag{5.6}$$

on (V, γ).

We first discuss what uniform control the bound (5.6) gives on the geometry of (M, g) and (V, γ). It follows from the Cheeger-Gromov compactness theorem (or more simply from the uniformization theorem in 2 dimensions) that γ is controlled, in harmonic coordinates, in $C^{1,\alpha}$. Similarly, g is also controlled in $C^{1,\alpha}$, cf. also [12]. (In case the geometry is (arbitrarily) collapsed, one may unwrap the collapse in universal covers and obtain the same conclusion).

Since $H = 0$, it follows first from the Hamiltonian constraint (5.1) that A is uniformly bounded in L^∞,

$$|A| \leq C, \quad \text{on} \quad V.$$

Next, assume (as in §3) that u is normalized at a base point $p \in V$ so that $u(p) = 1$. It then follows from (5.3) and elliptic regularity for the Laplacian on (V, γ) that u is uniformly bounded in $L^{2,p}_{loc}$, for any $p < \infty$:

$$|u|_{L^{2,p}_{loc}(V)} \leq C. \tag{5.7}$$

Since $\Delta_g u = 0$ on (M, g), elliptic regularity for the Laplacian on (M, g) implies that u is $L^{2,p}$ and so $C^{1,\alpha}$ up to ∂M:

$$|u|_{L^{2,p}_{loc}(M)} \leq C, \tag{5.8}$$

locally near V. Hence $N(u)$ is bounded in $L^{1,p}_{loc}$ and so in C^α on V. Finally, it then follows from the momentum constraint (5.4) and elliptic regularity for the (δ, tr) elliptic system on (V, γ) that A is bounded in $L^{1,p}_{loc}$ and so C^α_{loc} on V:

$$|A|_{C^\alpha_{loc}(V)} \leq C. \tag{5.9}$$

The main step in proving regularity is the following result. Recall $\widetilde{g} = u^2 g$.

Lemma 5.1. *The system*

$$\mathrm{Ric}_{\widetilde{g}} = 2d\nu \cdot d\nu, \quad \Delta_{\widetilde{g}}\nu = 0, \tag{5.10}$$

on $(M, \widetilde{g}, \nu)$ *with boundary conditions*

$$(\widetilde{\gamma}, H) \tag{5.11}$$

where $H = u\widetilde{H} - 2N(\nu)$ *on* ∂M*, is an elliptic boundary value problem in Bianchi gauge.*

We note that H in (5.11) is the mean curvature of ∂M with respect to g.

Proof. It suffices to prove the result for the linearized system and for deformations h of $g_{\mathcal{M}}$ satisfying the Bianchi gauge condtion $\beta(h) = \delta h + \frac{1}{2} d\mathrm{tr} h = 0$, where the divergence and trace are with respect to the 4-metric $g_{\mathcal{M}}$ as in (1.5). In the interior, the leading order term is just $\mathrm{Ric}'(h) \sim -\frac{1}{2}\Delta_{\widetilde{g}} h$, which is elliptic with principal symbol $-|\xi|^2 I$. The proof that the boundary conditions (5.11) are elliptic is exactly the same as that given in [7], [14] for the boundary conditions $(\widetilde{\gamma}, \widetilde{H})$; thus we refer to [7] or [14] for details. $\square$

Proposition 5.2. *Suppose (V, γ) is a (local) minimal surface in a static vacuum Einstein manifold (M, g, u) satisfying the bound (5.6). Then the geometry of (V, γ) is bounded in the C^∞ norm.*

Proof. The proof follows the usual bootstrap method in elliptic PDE. To begin, by (5.8) we have uniform $C^{1,\alpha}$ control on ν. By (5.10) this gives uniform C^α control on $\widetilde{\mathrm{Ric}}$, and hence uniform $C^{2,\alpha}$ control on $\widetilde{g}$ in the interior of M. Next, the Hamiltonian constraint (5.1) in the $\widetilde{g}$ metric gives

$$|\widetilde{A}|^2 - \widetilde{H}^2 + s_{\widetilde{\gamma}} = s_{\widetilde{g}} - 2\mathrm{Ric}_{\widetilde{g}}(\widetilde{N}, \widetilde{N}).$$

The right side is 1st order in ν, and $\widetilde{A}$, $\widetilde{H}$ are uniformly controlled in C^α by (5.9) and the control on ν. Hence $s_{\widetilde{\gamma}}$ is uniformly controlled in C^α. As above, by the uniformization theorem (or Cheeger-Gromov compacness theorem) it follows that $\widetilde{\gamma}$ is uniformly controlled in $C^{2,\alpha}$ modulo diffeomorphisms, i.e. in harmonic coordinates. It then follows from boundary regularity for elliptic systems, cf. [12] that $\widetilde{g}$ is uniformly controlled in $C^{2,\alpha}$ up to the boundary V. This in turn gives $C^{1,\alpha}$ control on $\widetilde{H} = \mathcal{L}_{\widetilde{N}} dv_{\widetilde{g}}/dv_{\widetilde{g}}$ and so, since $H = 0$, $C^{1,\alpha}$ control on $N(\nu)$. Via the elliptic equation $\widetilde{\Delta}\nu = 0$, this gives uniform $C^{2,\alpha}$ control on ν up to the boundary V and hence uniform control of g in $C^{2,\alpha}$ up to the boundary V. This then also gives uniform control of A in $C^{1,\alpha}$ on V.

This shows that the initial regularity of (M, g) up to the boundary (V, γ) has been increased by one derivative. Iterating this process then proves the result. □

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