ADE transform

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Dedicated to Cumrun Vafa on the occasion of his 60th birthday

There is a beautiful correspondence between configurations of lines
on a rational surface and tautological bundles over that surface. We
extend this correspondence to families, by means of a generalized
Fourier-Mukai transform that relates spectral data to bundles over
a rational surface fibration.

1. General comments

It has long been appreciated that there are remarkable relations between
del Pezzo surfaces and exceptional Lie groups. One way this manifests itself
is through a correspondence between configurations of lines on a del Pezzo
surface, and tautological $\widetilde{E}_n$ bundles over the surface, where $\widetilde{E}_n = E_n \times_{\mathcal{Z}} \mathbf{C}^*$
and $\mathcal{Z}$ is the center of E_n.

For example, the homology classes of the twenty-seven lines l_i on a cubic
surface are associated to the twenty-seven dimensional representation of E_6.
The corresponding bundle

$$(1.1) \qquad\qquad V = \bigoplus_{i=1}^{27} \mathcal{O}(l_i)$$

is an $\widetilde{E}_6$ vector bundle of rank twenty-seven on the cubic surface, where
$\widetilde{E}_6 = E_6 \times_{\mathbf{Z}_3} \mathbf{C}^*$. When restricted to a fixed elliptic curve embedded as the
anti-canonical divisor of the del Pezzo, and twisted to get a degree zero
bundle, the structure group of the bundle reduces to E_n. By varying the
surface, this correspondence yields all possible semi-stable E_n bundles over
an elliptic curve [1–3].

This construction has played an important role in heterotic/F-theory
duality [3–5]. Moreover the correspondence is not restricted to exceptional
groups, and can be extended to other Lie groups by considering more general
rational surfaces [6, 7], for essentially all one needs in the construction is a
rational surface whose homology lattice contains an ADE sublattice.

Our purpose in this work is to extend this correspondence to families of rational surfaces. In order to accomplish this, we will introduce an integral transform for rational surface fibrations. It plays a role somewhat analogous to that of the Fourier-Mukai transform for bundles over elliptically fibered manifolds.

For convenience let us briefly recall some aspects of the Fourier-Mukai transform. Given an elliptic fibration

$$(1.2) \qquad \pi_Z : Z \to B,$$

the Fourier-Mukai transform [8, 9] takes a sheaf $\mathcal{F}$ on Z, or a complex of sheaves, to another such complex:

$$(1.3) \qquad \mathbf{FM}(\mathcal{F}) = Rp_{1*}(p_2^* \mathcal{F} \otimes \mathcal{P}),$$

where $\mathcal{P}$ is the (relative) Poincare sheaf on the fiber product $Z \times_B Z$, and p_1 and p_2 are the projections to the first and second factor of $Z \times_B Z$ respectively. A particularly interesting case is when the sheaf $\mathcal{F}$ is a stable vector bundle on Z and the transform $\mathbf{FM}(\mathcal{F})$ is a line bundle L on a spectral cover $C \subset Z$. Conversely, stable vector bundles $\mathcal{F}$ on Z can be constructed as the Fourier-Mukai transform of spectral data (C, L), consisting of a line bundle L on a spectral cover $C \subset Z$. This powerful algebraic tool is useful also for understanding many aspects of heterotic string compactifications [3, 10] and F-theory [11, 12].

We extend this correspondence in the following way. Given an elliptic fibration $\pi_Z : Z \to B$ and a spectral cover $C \subset Z$, we construct a fibration

$$(1.4) \qquad \pi_Y : Y_C \to B$$

whose fibers are (essentially) Leung's rational surfaces, with 'boundary' $Z \subset Y_C$. Here Y_C depends on the spectral cover C, but not on the spectral sheaf L. Our transform $\mathbf{T}$ then takes a line bundle L supported on the spectral cover $C \subset Z$ to a sheaf $\mathbf{T}(L)$ (or sometimes complex) supported on Y_C. Furthermore, its restriction to $Z \subset Y_C$ is the usual Fourier-Mukai transform $\mathbf{FM}(L)$ of L.

As for all integral transforms, our transform is given by a variant of formula (1.3). In our description the analogue of the Poincaré sheaf is asymmetric: it lives on the fiber product

$$(1.5) \qquad \widehat{Y}_C = Y_C \times_B C,$$

which is a rational surface fibration over C.

The points in the fiber of $C \to B$ parametrize lines in the corresponding rational surface. As a result there is a 'universal line', which is a divisor $D \subset \widehat{Y}_C$. The Poincaré sheaf is given, in (3.22), as a twisted version of $\mathcal{O}(D)$. Its restriction to $Z \times_B C$ is also the restriction of the usual Poincaré sheaf, on $Z \times_B Z$. The identification of $\mathbf{FM}(L)$ with the restriction of $\mathbf{T}(L)$ follows from this. We work mostly with the A_n case. A few remarks are made about D_n and E_n, but detailed analysis of those cases is left for a future work.

This construction is clearly 'right' at points of B over which the cover $C \to B$ is unramified. It is more interesting to figure out what happens over the branch locus. In this direction we give two results. In (3.40) we show that near a generic branch point our $\mathbf{T}(L)$ is still locally free, i.e. it is an honest vector bundle. We further show in (3.55) that at a generic branch point, $\mathbf{T}(L)$ is in fact the regular representative, as defined in [13].

The construction has some rather interesting string-theoretic implications, which we develop in a separate paper [14]. The basic point is that the traditional geometric formulation of M-theory and F-theory compactifications is not able to capture certain non-abelian Hodge theoretic structures that are present in dual formulations. The ADE transform shows that such structures can in fact be described, provided we extend the traditional formulation of an M-theory or F-theory compactification to one with bundles or sheaves. Although at first sight this might seem like a radical proposal, as a result we can resolve a number of previously problematic issues involving missing branches of the moduli space, Yukawa couplings and stability conditions.

2. ADE surfaces and canonical ADE bundles

2.1. Lines and bundles

In this section we give a brief review of configurations of lines on a rational surface, and bundles over that surface. In particular for G a simple Lie group of type ADE, we review the relation between lines on G-surface S_G as defined in [6, 7], canonical G-bundles on S_G, and spectral data on a 'boundary' elliptic curve $E \subset S_G$.

The idea is as follows. Suppose we are given a surface S for which the group of line bundles $\mathrm{Pic}(S)$ is discrete, i.e. $\mathrm{Pic}(S) \cong H^2(S, \mathbf{Z})$, like for a rational surface. Suppose we are also given a lattice N, and define

$$(2.1) \qquad\qquad T = N \otimes \mathbf{C}^*$$

Then up to isomorphism, T-bundles on S are classified by

$$(2.2) \qquad \text{Hom}(N^\vee, \text{Pic}(S))$$

where

$$(2.3) \qquad N^\vee = \text{Hom}(N, \mathbf{Z}) = \text{Hom}(T, \mathbf{C}^*)$$

is the dual of N.

Now suppose that $H^2(S, \mathbf{Z})$ has a sublattice which is isomorphic to a root lattice Λ_{rt} for a simple ADE Lie group G. We will take $N = \Lambda_{wt}$ so that $N^\vee = \Lambda_{rt}$. Then, we get a canonical element of $\text{Hom}(N^\vee, \text{Pic}(S))$, namely the inclusion of Λ_{rt} into $H^2(S, \mathbf{Z})$, and therefore (up to isomorphism) we get a canonical T-bundle on S. More precisely, the map (and hence the bundle) is canonical up to the action by the Weyl group. But T-bundles related by an action of the Weyl group determine the same G-bundle, where we identify T with a maximal torus of G. So we get a canonical G-bundle on S.

Although this correspondence is quite general, we are going to consider some particular rational surfaces which one may informally think of as a compactification of ADE type ALE spaces by an elliptic curve E at infinity. The elliptic curve will arise as an anti-canonical divisor, so the pair (S, E) will be a log Calabi-Yau manifold, a Kähler surface whose log canonical bundle

$$(2.4) \qquad K_{(S,E)} \equiv K_S + E$$

is trivial. A log Calabi-Yau (S, E) admits a holomorphic volume form with a simple pole along E, which is interpreted as a holomorphic version of the boundary of S. Such manifolds give a large and mostly overlooked class of string compactifications where we get an interesting interplay between bulk and boundary. The boundary behaves as a kind of holographic screen in the sense that holomorphic constructions over S can be translated to constructions over E.

Thus we can restrict our canonical G-bundle on S to get a G-bundle on E. If furthermore $[E] \in \Lambda_{rt}^\perp \subset \text{Pic}(S)$ in the sense that $\alpha(E) = 0$ for any $\alpha \in \Lambda_{rt}$, then we get a flat G-bundle on E. As discussed in [6, 7, 15], all flat G-bundles on E may be recovered in this way, and we can pick an essentially unique rational surface S_G such that the moduli space of the flat G-bundle on E equals the complex structure moduli space of pairs (S, E') with an isomorphism $E' \to E$. The surface S_G is rational and can be constructed very explicitly by a sequence of blow-ups. We will refer to S_G as a rational

surface of type G. For $G = E_k$ these are just the del Pezzo surfaces, with one extra blow-up as defined in [6, 7]. In string theory applications one would typically add a few more blow-ups, in order to embed the correspondence in heterotic/F-theory duality.

The ADE lattice can also be obtained from the homology classes of lines living on the rational surface. Then we can try to construct associated vector bundles $\mathcal{V}_\rho$ for each representation ρ of G by considering associated configurations of lines. By restriction, they would yield associated bundles on E. The homology classes of the lines however generate a lattice that is slightly larger than the ADE root lattice. As a result, one typically has to deal with vector bundles that have a slightly larger structure group of the form $G_{ADE} \times_{\mathcal{Z}} \mathbf{C}^*$, where $\mathcal{Z}$ is the center of G_{ADE}.

Let us discuss the D_n and A_{n-1} cases in more detail. We take a Hirzebruch surface $\mathbf{F}^1$ and blow up in n points to get the D_n surface S_{D_n}. The homology lattice is spanned by b, f and the exceptional curves l_i, and the canonical divisor is given by $K = -2b - 3f + \sum l_i$. To get the A_{n-1} surface we can contract b to get $S_{A_{n-1}}$ (or equivalently start with $\mathbf{P}^2$ and blow up n points on that). However in practice we will keep b around and also denote it as l_0. The root lattice is given by

$$(2.5) \qquad \Lambda_{rt} = \{x \in \mathrm{Pic}(S) | x \cdot K = x \cdot f = x \cdot b = 0\}$$

Since $x \cdot b = 0$, this descends to $S_{A_{n-1}}$. The lattice Λ_{rt} is a root lattice of A_{n-1} type, and the simple roots may be taken as

$$(2.6) \qquad \alpha_1 = l_1 - l_2, \ldots, \alpha_{n-1} = l_{n-1} - l_n$$

The elliptic curve E is obtained by embedding into $\mathbf{P}^2$ using the linear system $|3p_0|$, where p_0 is the identity on E. Then we blow up $\mathbf{P}^2$ at p_0 and generic points p_i on E to get l_i. For $SU(n)$ we also want to impose that $\sum_{i=1}^n p_i = np_0$ in the group law on E. Then the tautological A_{n-1} bundle on S for the adjoint representation is given by

$$(2.7) \qquad \mathcal{O}^{\oplus n-1} \oplus \sum_{\alpha \in \Phi} \mathcal{O}_S(\alpha)$$

The adjoint representation is not faithful, and the bundle above may be equally regarded as an $SU(n)$ bundle as well as a $U(n) = SU(n) \times_{\mathbf{Z}_n} U(1)$

bundle. The tautological bundle associated to the fundamental representation on the other hand is given by

$$(2.8) \qquad \mathcal{W} = \sum_{i=1}^{n} \mathcal{O}_S(l_i)$$

which is merely a $U(n)$ bundle. It is convenient to twist this bundle by tensoring with $\mathcal{O}_S(-l_0)$:

$$(2.9) \qquad \widetilde{\mathcal{W}} = \sum_{i=1}^{n} \mathcal{O}_S(l_i - l_0)$$

Although still a $U(n)$ bundle on our rational surface S, we have $(l_i - l_0) \cdot E = 0$ so each of the factors restricts to a degree zero bundle on E. The restricted bundle

$$(2.10) \qquad \mathcal{V} = \mathcal{O}_E(p_1 - p_0) \oplus \cdots \oplus \mathcal{O}_E(p_n - p_0)$$

on E does naturally allow a reduction of the structure group from $U(n)$ to $SU(n)$.

For D_n, the natural faithful representation to consider is the $2n$-dimensional vector representation. The associated vector bundle can be constructed from the lines as follows:

$$(2.11) \qquad \mathcal{W} = \bigoplus_{i=1}^{n} \mathcal{O}_S(l_i) \oplus \mathcal{O}_S(f - l_i)$$

Its first Chern class is non-vanishing, so the structure group is actually a $U(1)$ extension of $SO(2n)$. Considering the twisted version

$$(2.12) \qquad \widetilde{\mathcal{W}} = \bigoplus_{i=1}^{n} \mathcal{O}_S(l_i - l_0) \oplus \mathcal{O}_S(f - l_i - l_0)$$

we get the following bundle when restricted to E:

$$(2.13) \qquad \mathcal{V} = \bigoplus_{i=1}^{n} \mathcal{O}_E(p_i - p_0) \oplus \mathcal{O}_E(-p_i + p_0)$$

The bundle $\mathcal{V}$ does admit the structure of a flat $SO(2n)$ bundle.

2.2. Singular surfaces and the regular representative

The correspondence reviewed above works for generic, smooth rational surfaces. Now we would like to ask what happens to this correspondence when the surface is allowed to degenerate. This is a very common situation and the singularities often have interesting physics associated to them. Remarkably, the construction above extends to singular rational surfaces with singularities of ADE type. Physically such singularities are usually associated to enhanced gauge symmetries. Moreover such singularities are generically unavoidable when we study families of rational surfaces, so understanding what happens will be an important aspect of our construction.

Let us first consider this from the point of view of the boundary elliptic curve E. For simplicity we consider the rank two vector bundle $\mathcal{O}_E(p_1) \oplus \mathcal{O}_E(p_2)$. Suppose that we take the limit as $p_2 \to p_1$. By the classification of bundles on an elliptic curve by Atiyah [16], there are two possibilities for the resulting rank two bundle. The first is $\mathcal{O}_E(p_1) \oplus \mathcal{O}_E(p_1)$, and the second is the non-decomposable rank two bundle F_2 defined by the extension sequence

$$(2.14) \qquad\qquad 0 \to \mathcal{O}_E(p_1) \to F_2 \to \mathcal{O}_E(p_1) \to 0$$

These two bundles define the same S-equivalence class, but they are not isomorphic. If we consider a generic family, then we end up with F_2. It is called the regular representative of the S-equivalence class, meaning it is the representative with the smallest automorphism group, or the maximal Jordan block structure. Its Fourier- Mukai transform, i.e. its T-dual, is $\mathcal{O}_{2p_1}$, the basic example of a degenerate brane with a gluing VEV.

Now we want to understand the corresponding bundles on the rational surface. This was studied by [17] and [18]. Let us denote the line on S_G corresponding to p_1 by l_1 and the line corresponding to p_2 by l_2. Then the corresponding rank two bundle on S_G is $\mathcal{O}_S(l_1) \oplus \mathcal{O}_S(l_2)$. Now we let $p_1 \to p_2$. As l_1 approaches l_2 and we get a surface with an A_1 singularity. We can blow up the surface to get an exceptional -2-curve C, in the class $[l_2 - l_1]$. Let us denote the blow up by $\widetilde{S}_G$.

To extend the correspondence, we would like a tautological bundle on S_G (or $\widetilde{S}_G$) that naturally arises as a limit of $\mathcal{O}_S(l_1) \oplus \mathcal{O}_S(l_2)$, has some kind of interpretation as a regular representative, and restricts to F_2 on E. One might think that one candidate for such a rank two bundle is $V = \mathcal{O}_S(l_1) \oplus \mathcal{O}_S(l_1 + C)$. However this clearly restricts to $\mathcal{O}_E(p_1) \oplus \mathcal{O}_E(p_1)$ on the boundary elliptic curve, not the regular representative. So what is the tautological bundle that restricts to F_2?

Let us first consider V restricted to the exceptional curve C. Then we get that

$$(2.15) \qquad V|_C = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(1).$$

This bundle is 'unbalanced'. It admits a unique deformation to a more generic rank two bundle on C with the same topological data, namely $\mathcal{O}(0) \oplus \mathcal{O}(0)$. We can construct it explicitly as an extension

$$(2.16) \qquad 0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C(0) \oplus \mathcal{O}_C(0) \to \mathcal{O}_C(1) \to 0$$

On a rational surface this deformation can be uniquely lifted to an irreducible deformation of the direct sum bundle V. To see this, we start with the exact sequence

$$(2.17) \qquad 0 \to \mathcal{O}_{\widetilde{S}} \to \mathcal{O}_{\widetilde{S}}(C) \to \mathcal{O}_C(C) \to 0$$

The associated long exact sequence in cohomology gives us

$$(2.18) \qquad \cdots \to 0 \to H^1(\mathcal{O}_{\widetilde{S}}(C)) \to H^1(\mathcal{O}_C(C)) \to 0 \to \cdots$$

where we used that $h^{1,0}(\widetilde{S}) = h^{2,0}(\widetilde{S}) = 0$. But we have $\mathrm{Ext}^1(\mathcal{O}(l_1), \mathcal{O}(l_2)) \simeq H^1(\mathcal{O}_{\widetilde{S}}(C))$ and $\mathrm{Ext}^1_C(\mathcal{O}(1), \mathcal{O}(-1)) \simeq H^1(\mathcal{O}_C(C)) \simeq \mathbf{C}$, so our deformation on C lifts to $\widetilde{S}$, in fact we get the unique non-trivial extension

$$(2.19) \qquad 0 \to \mathcal{O}(l_2) \to W \to \mathcal{O}(l_1) \to 0$$

Note also that $\mathrm{Ext}^1(\mathcal{O}(l_1), \mathcal{O}(l_1)) = \mathrm{Ext}^1(\mathcal{O}(l_2), \mathcal{O}(l_2)) = 0$ since $h^{1,0}(\widetilde{S}) = 0$, so this is the only natural candidate for a bundle that restricts to F_2 on the boundary and arises as a limit as $l_1 \to l_2$. Therefore the deformed bundle W is the tautological bundle on $\widetilde{S}_G$ that corresponds to the regular representative. Since it is trivial when restricted to C, it may also be thought of as the pull-back of a rank two vector bundle on S_G itself, as opposed to a more general sheaf.

For later purposes, we would like to note that $\mathrm{Ext}^1_{\widetilde{S}}(\mathcal{O}(l_1), \mathcal{O}(l_2)) \cong H^1(\mathcal{O}(l_2 - l_1))$ vanishes when $l_2 - l_1$ is not effective. This is somewhat analogous to the orthogonality of Fourier modes.

To see this, we can look at the index

$$\text{(2.20)} \qquad\qquad \sum_i (-1)^i \dim \text{Ext}^i_{\widetilde{S}}(\mathcal{O}(l_1), \mathcal{O}(l_2))$$

which is invariant under deformations. Further,

$$\text{Ext}^2_{\widetilde{S}}(\mathcal{O}(l_1), \mathcal{O}(l_2)) \cong H^0(\mathcal{O}(l_1 - l_2) \otimes K_{\widetilde{S}})$$

vanishes on a rational surface, since $K_{\widetilde{S}}$ does not have sections and tensoring with $\mathcal{O}(l_2 - l_1)$ doesn't make it any better. Therefore if we vary l_1 and l_2, Ext^1 and Ext^0 must appear or disappear in pairs. But $\text{Ext}^0_{\widetilde{S}}(\mathcal{O}(l_1), \mathcal{O}(l_2)) = H^0(\mathcal{O}(C))$ is one-dimensional when $l_2 - l_1$ is effective, and zero dimensional when $l_2 - l_1$ is not effective. Therefore the unique Ext^1 we found when $l_2 - l_1$ is effective disappears when $l_2 - l_1$ ceases to be effective. An alternative proof which does not rely on deformations can be found in [17].

3. Fibering the correspondence

The correspondence discussed so far holds for individual rational surfaces. Now we want to extend it to families, so that we can fiber this correspondence over a base B.

3.1. Constructing Y_C

The first step is to construct a Kähler manifold $Y \to B$ whose fibers are ADE rational surfaces. We will assume that we are given a smooth elliptically fibered Kähler manifold Z with projection $\pi_Z : Z \to B$ and section σ. We further assume we are given a spectral cover $C \subset Z$ which is a finite cover $C \to B$. We could presumably generalize this set-up by allowing the fibers of $Y \to B$ to be more general surfaces, as long as their homology lattice contains an ADE root lattice, and allowing for abstract unembedded covers $C \to B$. In fact this would conceptually be a cleaner way to describe the correspondence. But working with embedded covers tends to make the set-up a little more concrete as we can write explicit equations.

To construct a bundle we further need a spectral sheaf $\mathcal{L}$ on C, or a holomorphic bundle $\mathcal{V}$ on Z whose Fourier-Mukai yields such a spectral sheaf supported on C. However $\mathcal{L}$ plays no role in constructing Y itself, so it will be ignored in this subsection.

Let us first consider fibrations by surfaces of type A_n. Recall that we constructed such surfaces by embedding an elliptic curve into $\mathbf{P}^2$ using

the linear system $|3p_0|$ and then blowing up $n+1$ points $p_0, \ldots, p_n$. The global analogue of this is as follows. Consider the line bundle $\mathcal{O}_Z(3\sigma)$ and its direct image $\pi_{Z*}\mathcal{O}_Z(3\sigma)$. This is a rank three bundle over B with fibers $H^0(E_b, \mathcal{O}_{E_b}(3p_0))$ over a point $b \in B$. Given a point $p \in E_b$ and a section $s \in H^0(E_b, \mathcal{O}_{E_b}(3p_0))$, we have the natural evaluation map $(p, s) \to s(p)$, so dually we get the embedding $E_b \hookrightarrow \mathbf{P}(H^0(E_b, \mathcal{O}_{E_b}(3p_0))^\vee) \cong \mathbf{P}^2$. Globally this means we get an embedding

$$(3.1) \qquad Z \hookrightarrow \mathbf{P}(\pi_{Z*}\mathcal{O}_Z(3\sigma)^\vee)$$

where the right-hand-side is a $\mathbf{P}^2$-bundle over B.

Now C sits in Z, and therefore by the above embedding it also sits in $\mathbf{P}(\pi_{Z*}\mathcal{O}_Z(3\sigma)^\vee)$. Then Y_C is obtained by blowing up $\mathbf{P}(\pi_{Z*}\mathcal{O}_Z(3\sigma)^\vee)$ along $C \cup \sigma$:

$$(3.2) \qquad Y_C = \mathrm{Bl}_{C\cup\sigma}[\mathbf{P}(\pi_{Z*}\mathcal{O}_Z(3\sigma)^\vee)]$$

This space is actually singular when C and σ intersect. The reason is that along $C \cap \sigma$ we are effectively blowing up along a length two scheme and such a blow-up yields a singular space. To resolve this we can do a further blow-up of Y_C, or alternatively we can blow-up along C and σ successively, which will do the extra blow-up for us (with exchanging the order being related by a flop). The extra blow-up is familiar from 'matter curves' in F-theory compactifications.

We note that it is straightforward to do the blow-up explicitly. Our elliptic Calabi-Yau Z is given by a cubic equation $c = 0$ in the $\mathbf{P}^2$-bundle $\mathbf{P}(\mathcal{O} \oplus L^2 \oplus L^3)$, where $L = K_B^{-1}$ and

$$(3.3) \qquad c = -y^2 z + x^3 + fxz^2 + gz^3$$

The spectral cover C in Z is specified by an equation $s = 0$ where

$$(3.4) \qquad s = p_0 + p_2 x + p_3 y + \cdots + p_{n+1} x^{(n+1)/2}$$

for n odd, with a similar equation for n even. Here the p_i are sections of L^{n+1-i}. To perform the blow-up, we only need to introduce additional projective coordinates (u, v) such that $uc = vs$. The sections p_i allow us to vary the rational surface fibration while keeping the elliptic fibration fixed.

The Picard lattice of our surface fiber over a point $b \in B$ is of the form

$$(3.5) \qquad H^2(S_b, \mathbf{Z}) = \Lambda_{A_{n-1}} \oplus \mathbf{Z}E_b \oplus \mathbf{Z}l_0 \oplus \mathbf{Z}f$$

and furthermore by construction these pieces do not mix as we vary the point b.

The construction for other Lie algebras is a little more involved. The cases E_6, E_7 and E_8 are effectively discussed in [3]. For D_n we can give a fairly explicit description using recent results on the Sen limit [5]. Let $F \to B$ be $\mathbf{P}^1$-fibration over B. Consider $\mathbf{P}(\mathcal{O}_F \oplus K_F \oplus L)$, which is a $\mathbf{P}^2$-fibration over F, and let (y, u, v) denote fiber coordinates of this $\mathbf{P}^2$-bundle. We take our rational surface fibration Y to be specified by an equation

$$(3.6) \qquad y^2 = b_2 u^2 + 2b_4 uv + b_6 v^2$$

where the b_i are sections of K_F^{-2}, $K_F^{-1} \otimes L^{-1}$ and L^{-2} respectively. The line bundle $L = K_F^3$ is what appears in the usual Sen limit. Geometrically the above equation cuts out a conic in each $\mathbf{P}^2$-fiber, and the fiber of $Y \to B$ is a rational surface which is fibered by conics. For $v = 0$ we recover the Calabi-Yau Z, given by $y^2/u^2 = b_2$ (a double cover of F). The spectral cover C is given by the intersection of Z with $R = \pi^{-1}(\Delta)$, where $\Delta = b_2 b_6 - b_4^2$ is a section of $K_F^{-2} \otimes L^{-2}$. If L has degree k on the fiber of $F \to B$, then this cover has degree $4k + 8$ over B.

In each case, the boundary of a rational surface fiber is the elliptic curve E_b, so Y_C contains an elliptic fibration over B which is our original Z. The collection of lines on each rational surface fit together in the cylinder $R \to B$, and the intersection of R (after embedding in Y) with Z gives the spectral cover $C \to B$.

Now given a spectral sheaf $\mathcal{L}$ supported on $C \subset Z$, associated to a bundle $\mathcal{V}$ on Z by the usual Fourier-Mukai transform, we want to construct a natural bundle $\mathcal{W}$ on Y. This bundle will have the property that restricting to the boundary gives us back $\mathcal{V}$, i.e. $\mathcal{W}|_Z = \mathcal{V}$. The structure group of $\mathcal{W}$ will be a $\mathbf{C}^*$-extension of the ADE structure group of $\mathcal{V}$.

In concrete applications we may encounter variants of the above situation. In one common situation, some part of the gauge symmetry remains unbroken. Then the lattice Λ_{rt} may further split up, eg. $\Lambda_{rt} = \Lambda_{rt,1} \oplus \Lambda_{rt,2}$, and these pieces don't mix as we vary over a base. Then of course we could construct bundles associated to the sublattices.

3.2. General comments on the construction

In order to understand the construction, it is useful to keep the conventional Fourier-Mukai transform in mind, so let us briefly review this. Our discussion is somewhat heuristic, in particular more attention should be paid to the branch locus of the spectral cover. For a more detailed treatment see eg. [8, 9].

Let $Z \to B$ be a smooth elliptically fibered Kähler manifold with section σ. Let us consider two copies of Z, denoted by Z_1 and Z_2. We want to think of the elliptic fibers of Z_1 as being the dual of the elliptic fibers of Z_2. More formally we can write

$$(3.7) \qquad\qquad Z_2 = \mathrm{Pic}^0(Z_1/B)$$

where $\mathrm{Pic}^0(Z/B)$ is the relative Picard group of degree zero divisors. Since $\mathrm{Pic}_0(E) \cong E^\vee \cong E$ and Z_1 has a section, Z_2 is isomorphic to Z_1. Now we construct the following double elliptic fibration

$$(3.8) \qquad\qquad Z_1 \times_B Z_2 = Z_1 \times_B \mathrm{Pic}_0(Z_1/B)$$

There are natural maps $\pi_1 : Z_1 \times_B Z_2 \to Z_1$ and $\pi_2 : Z_1 \times_B Z_2 \to Z_2$ that project on the two factors respectively.

On $Z_1 \times_B Z_2$ we can define a certain universal line bundle, the Poincaré line bundle $\mathcal{P}$, as follows: we label a point in $Z_1 \times_B Z_2$ as (p, q), where $p \in Z_1, q \in Z_2$, and $\pi_1(p) = \pi_2(q)$. We will also use $\bar{p}$ and $\bar{q}$ to denote the corresponding points on the elliptic fibers $E = \pi_1^{-1}\pi_1(p) \cong \pi_2^{-1}\pi_2(q)$. Since our elliptic fibrations have a section, there is a distinguished point $\bar{q}_0$ on every elliptic fiber. Then the fiber of $\mathcal{P}$ at (p, q) is given by

$$(3.9) \qquad\qquad \mathcal{P}_{(p,q)} = \mathcal{O}_E(\bar{q} - \bar{q}_0)|_{\bar{p}}.$$

The Poincaré sheaf is the sheaf of sections of $\mathcal{P}$, which we denote by the same name.

Now given a spectral sheaf $\mathcal{L}$ on Z_2, the dual bundle $\mathcal{V}$ on Z_1 is given by

$$(3.10) \qquad\qquad \mathcal{V} = \pi_{1*}(\pi_2^*\mathcal{L} \otimes \mathcal{P})$$

Note that if the spectral cover has degree n over B, then $\pi_2^{-1}C = Z \times_B C$ has degree n over Z_1, and so π_{1*} gives a rank n bundle on Z_1.

We would like to make two comments. The first comment is that the Poincaré sheaf is defined on all of $Z_1 \times_B Z_2$, but in formula (3.10) we actually

only need the restriction of $\mathcal{P}$ to $Z \times_B C \subset Z_1 \times_B Z_2$, since we are tensoring with the sheaf $\pi_2^* \mathcal{L}$ which is only supported on $Z \times_B C$. Therefore we may also formulate the Fourier-Mukai transform as follows. Let us define

$$(3.11) \qquad \widehat{Z} = Z \times_B C$$

Due to the inclusion $Z \times_B C \to Z \times_B Z$ we may pull back the Poincaré sheaf to a sheaf $\mathcal{P}_{\widehat{Z}}$ on $\widehat{Z}$. Since the spectral sheaf is supported on C, we may then also express $\mathcal{V}$ as

$$(3.12) \qquad \mathcal{V} = \mathbf{FM}(\mathcal{L}) = \pi_{1*}(\pi_2^* \mathcal{L} \otimes \mathcal{P}_{\widehat{Z}})$$

We will use this observation in the new construction.

The second comment we would like to make is that the Poincaré sheaf often fails to be a line bundle, but this is not a cause for concern. Indeed the cross product $Z_1 \times_B Z_2$ has singularities over points in the base B where we take the cross product of nodal elliptic curves, and the Poincaré sheaf fails to be a line bundle at these singular points. We will see a very similar phenomenon in our new construction.

Now we would like to construct natural ADE bundles on a complex manifold Y that admits a fibration $Y \to B$ by rational surfaces. The idea of our construction will be to proceed along the same lines as above: we define a suitable 'tautological' sheaf and then do an analogue of the Fourier-Mukai transform. Let us explain why such a tautological sheaf should exist. We are grateful to T. Pantev for the argument below.

Consider the space $S \times \mathrm{Pic}(S)$. There is a universal sheaf $\mathcal{P}$ on this space: given a point $y \in \mathrm{Pic}(S)$, y determines a sheaf $\mathcal{L}_y$ on S, and we take

$$(3.13) \qquad \mathcal{P}|_{S \times y} = \mathcal{L}_y.$$

We can easily extend this to families. Given a fibration $Y \to B$, the relative Picard group $\mathrm{Pic}(Y/B)$ is defined as $R^1 \pi_* \mathcal{O}_Y^*$. On a generic fiber S_b this yields $H^1(S_b, \mathcal{O}^*)$ which is the usual Picard group of S_b. We define a new space by crossing with Y:

$$(3.14) \qquad \mathcal{Y} = Y_C \times_B \mathrm{Pic}(Y_C/B)$$

There is a universal sheaf $\mathcal{P}$ on $\mathcal{Y}$, as follows: given a point $(x, y) \in \mathcal{Y}$ with $\pi(x) = \pi(y) = b$, y determines a sheaf on S_b, so we use this as the fibers of $\mathcal{P}$.

Now we define

$$(3.15) \qquad \widehat{Y} = Y_C \times_B C$$

It will be convenient to again use π_1 and π_2 for projection on the first and second factor respectively. There is a natural map from C to $\mathrm{Pic}(Y_C/B)$, since by construction of Y_C a point p on C with $\pi_C(p) = b$ determines a line l in S_b and a generator $\mathcal{O}_{S_b}(l)$ of $\mathrm{Pic}(S_b)$. Then by pulling back the universal sheaf $\mathcal{P}$ we get a sheaf $\mathcal{P}_{\widehat{Y}}$ on $\widehat{Y} = Y_C \times_B C$.

Once we have a Poicaré sheaf, we can proceed as in the usual Fourier-Mukai transform, and define a bundle $\mathcal{W}$ on Y_C as the transform

$$(3.16) \qquad \mathcal{W} = \mathbf{T}(\mathcal{L}) = \pi_{1*}(\pi_2^* \mathcal{L} \otimes \mathcal{P}_{\widehat{Y}})$$

Furthermore by construction, when we restrict this to $Z \times_B C \subset Y_C \times_B C$, we reproduce the usual Fourier-Mukai transform for elliptically fibered manifolds reviewed above.

3.3. Explicit description of the Poincaré sheaf

The construction thus boils down to defining a suitable generalized Poincaré sheaf $\mathcal{P}_{\widehat{Y}}$. In this section we describe this Poincaré sheaf more explicitly.

Let us introduce the cylinder R, the collection of lines on each rational surface fibered over B. R has the structure of a $\mathbf{P}^1$-fibration over C. After embedding in Y, it can be identified with the exceptional divisor of the blow-up (3.2). We also define $\widehat{R} = R \times_B C$. Since the lines are naturally embedded in the surface fibers, we have an embedding $\widehat{R} \hookrightarrow \widehat{Y}$. Furthermore since $R \to C$ is a $\mathbf{P}^1$-fibration, we also get the $\mathbf{P}^1$-fibration

$$(3.17) \qquad \widehat{R} \to C \times_B C$$

Now we consider the diagonal $\Delta \in C \times_B C$. Pulling Δ back to $\widehat{R}$ and embedding in $\widehat{Y}$ gives a natural divisor $D \subset \widehat{Y}$. The relations are summarized in the following diagram:

$$(3.18) \qquad \begin{array}{ccccc} D & \hookrightarrow & R \times_B C & \hookrightarrow & \widehat{Y} = Y \times_B C \\ \downarrow & & \downarrow & & \downarrow \qquad \searrow \\ \Delta & \hookrightarrow & C \times_B C & & Y \qquad\quad C \end{array}$$

Now our tentative definition of the Poincaré sheaf is simply

$$(3.19) \qquad \mathcal{P}_{\widehat{Y}} = \mathcal{O}_{\widehat{Y}}(D)$$

In practice one typically wants to consider a small modification of this, in order for the restriction of the bundle $\mathcal{W}$ to Z to have an ADE structure group instead of a $\mathbf{C}^*$-extension of this. Let us do this explicitly for the A_{n-1} case. Fiber by fiber the correspondence is essentially unchanged if we tensor $\mathcal{O}_{\widehat{Y}}(D)$ by a line bundle pulled back from C or from Y. The first modification we want to make is for the Poincaré sheaf to be flat when restricted to the elliptic fibers of Z. We saw that this is achieved on the rational surface fibers by tensoring with the line bundle $\mathcal{O}(-l_0)$, which produces the collection $\{\mathcal{O}(l_1 - l_0), \ldots, \mathcal{O}(l_n - l_0)\}$ instead of $\{\mathcal{O}(l_1), \ldots, \mathcal{O}(l_n)\}$. Fibering the line l_0 over B gives the divisor R_σ in Y_C. Then we modify $\mathcal{P}$ as:

$$(3.20) \qquad \mathcal{P}_{\widehat{Y}} = \mathcal{O}_{\widehat{Y}}(D - R_\sigma \times_B C)$$

Secondly, in order to produce the usual Poincaré sheaf on $Z \times_B C$, we want to make this a little more symmetric. We can do this by modifying $\mathcal{P}$ by the pull-back of $\mathcal{O}_C(\Sigma)$, where $\Sigma = \sigma \cap C$: to

$$(3.21) \qquad \mathcal{P}_{\widehat{Y}} = \mathcal{O}_{\widehat{Y}}(D - R_\sigma \times_B C - \pi_2^{-1}(\Sigma))$$

Note that we can think of $\pi_2^{-1}(\Sigma)$ as $(Y \times_B C) \cap (Y \times_B \sigma)$ in $Y \times_B Z$. In this way we reproduce the more conventional expression $\mathcal{O}(\Delta - \sigma \times_B C - \pi_2^{-1}\Sigma)$ when restricted to $Z \times_B C$. Finally, although our twist ensured that the restriction $\mathcal{W}|_Z$ has vanishing first Chern class along the elliptic fibers, it still has non-vanishing first Chern class along the base B. To fix this we need to further twist by a factor of $\pi_B^* K_B^{-1}$, where π_B is the natural projection $Y \times_B C \to B$. Thus the final expression that we settle on is given by

$$(3.22) \qquad \mathcal{P}_{\widehat{Y}} = \mathcal{O}_{\widehat{Y}}(D - R_\sigma \times_B C - \pi_2^{-1}\Sigma) \otimes \pi_B^* K_B^{-1}$$

Most of the interesting behaviour however has to do with the factor $\mathcal{O}_{\widehat{Y}}(D)$.

3.4. Behaviour of the Poincaré sheaf at the branch locus

There wasn't much choice in defining our Poincaré sheaf, the whole construction being essentially tautological. It is not hard to see that our transform reproduces the correspondence discussed in section 2 for generic fibers. In that sense, the whole point now is to check that we get the expected behaviour also in more degenerate situations. In particular if we have a non-trivial fibration $Y_C \to B$, then some of the fibers are going to be singular even in the generic case. This corresponds to the branch locus of the spectral cover C, or equivalently the locus where two lines l_1 and l_2 on the surface

fibers of Y_C come together and get exchanged (which we may refer to as the branch locus of Y_C).

We want to take a closer look at the divisor D in such a situation. It turns out that $\widehat{Y}$ is singular in codimension three even for generic spectral covers, and D is a Weil but not a Cartier divisor on $\widehat{Y}$. Thus our $\mathcal{P}_{\widehat{Y}}$ is a rank one sheaf and not a line bundle, although it looks like a line bundle away from the singular locus. As we observed previously, a similar phenomenon occurs for the conventional Poincaré sheaf on $Z_1 \times_B Z_2$.

To see this more explicitly, let's look at some local models. Let us denote by t a local parameter on the base B. Then we have the following local models for C near the branch locus and for Y when two lines come together:

$$(3.23) \qquad C : \mathbf{C}[t,s]/(t-s^2), \qquad Y : \mathbf{C}[t,x,y,z]/(t-x^2+y^2-z^2)$$

We then find the following local model for $\widehat{Y} = Y \times_B C$:

$$(3.24) \qquad \begin{aligned} \widehat{Y} &: \mathbf{C}[t,s,x,y,z]/(t-s^2, t-x^2+y^2-z^2) \\ &\cong \mathbf{C}[s,x,y,z]/(s^2-x^2+y^2-z^2) \end{aligned}$$

We see that $\widehat{Y}$ has conifold singularities in codimension three over the branch locus in B.

Now we can write D explicitly in this local model for $\widehat{Y}$. We first write the corresponding local models for $\widehat{R}$ and $C \times_B C$:

$$(3.25) \qquad \widehat{R} : \mathbf{C}[v,s_1,s_2]/(s_1^2-s_2^2), \qquad C \times_B C : \mathbf{C}[s_1,s_2]/(s_1^2-s_2^2)$$

Here we used the local model $\mathbf{C}[v,s_1,t]/(s_1^2-t)$ for R, with v denoting a coordinate along the $\mathbf{P}^1$ fiber of $R \to C$. For a fixed t, this describes the lines l_1 and l_2 that we met previously in section 2.

Now the diagonal $\Delta \subset C \times_B C$ is given by

$$(3.26) \qquad \Delta : \mathbf{C}[s_1,s_2]/(s_1-s_2)$$

This lifts to the subset

$$(3.27) \qquad D : \mathbf{C}[v,s_1,s_2]/(s_1-s_2) \subset \widehat{R}$$

Then finally we want to embed D into $\widehat{Y}$. Equivalently we find a map from the coordinate ring of $\widehat{Y}$ to the coordinate ring of D, and we can take this

to be

(3.28)
$$(x, y, z, s) \to (v, v, s_1, s_2)$$

Note that even locally we need not one but two equations to specify D in $\widehat{Y}$, namely $x = y$ and $z = s$, so D is not a Cartier divisor. Another way to see this is that D itself is clearly smooth, even when it intersects the singular locus of $\widehat{Y}$. This means that D cannot be a Cartier divisor, since a Cartier divisor would have to be singular at this locus. Indeed the above is pretty much the standard example of a Weil divisor that is not Cartier.

The conifold singularity of $\widehat{Y}$ can be removed by a small resolution. It is perhaps interesting to note that the Poincaré sheaf lifts to an actual line bundle if we pick one of the two small resolutions.

3.5. Structure of ADE transform near the branch locus

The next point we want to understand is the behaviour of the bundle we constructed near the branch locus. In particular we would like to establish two properties. We would like to show that over the branch locus, $\mathbf{T}(\mathcal{L})$ reproduces the tautological bundle corresponding to the regular representative discussed in section 2.2. We would further like to show that $\mathbf{T}(\mathcal{L})$ is locally free, so that it defines a smooth vector bundle on Y. In this subsection we will establish both of these properties. This is slightly tricky since we saw that $\widehat{Y}$ is actually singular and $\mathcal{P}_{\widehat{Y}}$ fails to be a line bundle at the singularities.

The story is (mostly) local, and so we first want to compute $(\pi_{\widehat{Y}/Y})_* \mathcal{O}_{\widehat{Y}}(D)$ in the local models given above.

To simplify the discussion, let us also consider the divisor D' defined as follows:

(3.29)
$$D' : \mathbf{C}[x, y, z, s]/(x - y, z + s) \quad \subset \quad \widehat{Y}$$

It is also a Weil divisor, but the sum $D + D'$ is defined by a single equation (namely $x = y$) and so corresponds to a Cartier divisor. Thus although neither $\mathcal{O}(D)$ or $\mathcal{O}(D')$ is a line bundle, $\mathcal{O}(D + D')$ actually is a line bundle on $\widehat{Y}$. Its sections are local meromorphic functions that are allowed to have a pole at $x = y$, in other words expressions of type

(3.30)
$$f(x, y, z, s)/(x - y)$$

where f is holomorphic. Furthermore, $\mathcal{O}(D + D')$ is actually the pull-back of the line bundle $\mathcal{O}_Y(R)$ downstairs, since $\pi^{-1}_{\widehat{Y}/Y}(R) = D + D'$. Indeed the equation of R in Y is given simply by $x = y$, describing a pair of lines l_1 and l_2 for each t.

Now we can start computing the push-forwards. We write

$$(3.31) \qquad \mathcal{O}_{\widehat{Y}}(D) = \mathcal{O}_{\widehat{Y}}(D + D') \otimes \mathcal{O}_{\widehat{Y}}(-D')$$

so that

$$(3.32) \qquad \pi_{\widehat{Y}/Y*}\mathcal{O}_{\widehat{Y}}(D) = \mathcal{O}_Y(R) \otimes \pi_{\widehat{Y}/Y*}\mathcal{O}(-D')$$

The advantage of writing it this way is that $\mathcal{O}_{\widehat{Y}}(-D')$ is an ideal sheaf, and the push-forward of an ideal sheaf is a bit easier to understand.

We will use the standard short exact sequence associated to a divisor:

$$(3.33) \qquad 0 \to \mathcal{O}_{\widehat{Y}}(-D') \to \mathcal{O}_{\widehat{Y}} \to \mathcal{O}_{D'} \to 0$$

In order to save some notation, we are going to denote the map $\pi_{\widehat{Y}/Y}$ simply by π in the remainder of this subsection. Applying the push-forward we get

$$(3.34) \qquad 0 \to \pi_*\mathcal{O}_{\widehat{Y}}(-D') \to \pi_*\mathcal{O}_{\widehat{Y}} \to \pi_*\mathcal{O}_{D'} \to 0$$

The sequence terminates because π is a finite map, hence $R^1\pi_* = 0$. Let us first understand $\pi_*\mathcal{O}_{\widehat{Y}}$. Sections of $\mathcal{O}_{\widehat{Y}}$ are of the form

$$(3.35) \qquad 1 \cdot f(x, y, z, s)$$

where f is any element of the ring $\mathbf{C}[x, y, z, s]/(x^2 - y^2 + z^2 - s^2)$. Applying π_* means that we should regard any such section as a generator over the ring $\mathbf{C}[x, y, z, t]/(x^2 - y^2 + z^2 - t)$ where $t = s^2$. Clearly we can write any such section as a combination of two generators, 1 and s. Furthermore any appearance of t can be eliminated using the ring relation (equivalently any occurrence of s^2 can be eliminated using the ring relation even before pushing forward). Thus we can write sections of $\pi_*\mathcal{O}_{\widehat{Y}}$ as

$$(3.36) \qquad f_1(x, y, z) + s f_2(x, y, z)$$

The map $\mathcal{O}_{\widehat{Y}} \to \mathcal{O}_{D'}$ simply imposes the relations $x - y = z + s = 0$, so we may use this to eliminate y and s altogether, i.e our section maps as

$$(3.37) \qquad f_1(x, y, z) + s f_2(x, y, z) \to (f_1(x, y, z) - z f_2(x, y, z))|_{y=x}$$

The kernel of this map consists of sections $f_1(x, y, z) + s f_2(x, y, z)$ where

$$(3.38) \qquad f_1(x, y, z) = z f_2(x, y, z) + (x - y) f_3(x, y, z)$$

Rearranging as

$$(3.39) \qquad f_1 + s f_2 = (z + s) f_2 + (x - y) f_3$$

we see that $\pi_* \mathcal{O}_{\widehat{Y}}(-D')$ is freely generated over $\mathbf{C}[x, y, z]$ or $\mathbf{C}[x, y, z, t]$ by $x - y$ and $z + s$ (recall t can always be eliminated using the ring relation). This is probably not so surprising given that D' is defined on $\widehat{Y}$ as $x - y = z + s = 0$, although we have to be careful to say which ring we are working over. At any rate, we see that in our local model $\pi_* \mathcal{O}_{\widehat{Y}}(-D')$ is a locally free sheaf (a vector bundle) of rank two. In fact from (3.39) and the fact that the push-forward of the Poincaré sheaf is given by tensoring $\pi_* \mathcal{O}_{\widehat{Y}}(-D')$ with an honest line bundle, we see that locally we have the isomorphism

$$(3.40) \qquad (\pi_{\widehat{Y}/Y})_* \mathcal{P}_{\widehat{Y}} \cong \mathcal{O}_Y \oplus \mathcal{O}_Y$$

This is the first property we set out to prove in this subsection.

In order to establish the second property, let us restrict our bundle to the surface fibers of the fibration $Y_C \to B$. It is straightforward to see that away from the branch locus, the restriction yields a direct sum of line bundles. We need to show that when the bundle we constructed is restricted to a surface fiber over the branch locus (here $t = 0$ in the local model), it can be expressed as the non-trivial extension in equation (2.19). The argument has a global and a local component. We start the local one.

Let us take a closer look at the bundle $\pi_{\widehat{Y}/Y *} \mathcal{O}(-D')$ restricted to the surface fiber over $t = 0$. In order to save some notation, let us simply denote $\pi_{\widehat{Y}/Y *} \mathcal{O}(-D')|_{t=0}$ by F_2. We also denote the fiber over $t = 0$ (with equation $x^2 - y^2 + z^2 = 0$) by S_0, and its resolution by $\nu : \widetilde{S}_0 \to S_0$. We now claim that when we lift F_2 to $\widetilde{S}_0$, it can be expressed as the extension of two specific line bundles $\mathcal{O}_{\widetilde{S}_0}(-l_1)$ and $\mathcal{O}_{\widetilde{S}_0}(-l_2)$, thereby recovering the 'regular representative' discussed in section 2.2.

To see this, let us take general sections of F_2 as expressed in (3.39), and consider the restriction mapping $s \to 0$. Note this is consistent with $s^2 = 0$ at $t = 0$. We are left with

$$(3.41) \qquad z f_2 + (x - y) f_3$$

which are precisely the sections of the ideal sheaf $\mathcal{I}_{(x-y,z)}$ on $x^2 - y^2 + z^2 = 0$. Thus the map $s \to 0$ describes an onto map from F_2 to $\mathcal{I}_{(x-y,z)}$. We can express this as a sequence:

$$(3.42) \qquad F_2 \xrightarrow{r} \mathcal{I}_{(x-y,z)} \to 0.$$

We would like to complete this to a short exact sequence:

$$(3.43) \qquad 0 \to \mathcal{K} \to F_2 \xrightarrow{r} \mathcal{I}_{(x-y,z)} \to 0.$$

which we will eventually claim is essentially the extension sequence (2.19) we are trying to recover. To do this we need to look at the kernel of the map r. Again we identify this sheaf by its sections. The kernel is a sheaf with sections of the form (3.39) which are in the kernel of the map $s \to 0$. Such sections are of the form

$$(3.44) \qquad s\, f_2(x, y, z)$$

where $z\, f_2 + (x - y) f_3 = 0$. This defines $\mathcal{K}$, albeit somewhat implicitly.

In order to get a more recognizable description of the kernel sheaf, we will lift to the blow-up $\widetilde{S}_0$, where we can solve the equation $z\, f_2 + (x - y) f_3 = 0$ more easily. To do this we introduce projective coordinates (λ_1, λ_2), and instead of $x^2 - y^2 + z^2 = 0$ we impose the pair of equations

$$(3.45) \qquad \begin{pmatrix} x - y & z \\ -z & x + y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0$$

on $\mathbf{C}^3 \times \mathbf{P}^1$. On the blow-up we can solve $z\, f_2 + (x - y) f_3 = 0$ by writing

$$(3.46) \qquad f_3 = \lambda_1\, q(x, y, z, \lambda), \qquad f_2 = \lambda_2\, q(x, y, z, \lambda),$$

where q is an arbitrary polynomial. Then the kernel is isomorphic to a sheaf with sections

$$(3.47) \qquad \lambda_2\, q(x, y, z, \lambda)$$

which defines the ideal sheaf $\mathcal{I}_{(\lambda_2)}$, and the map from $\mathcal{I}_{(\lambda_2)}$ to F_2 is given by multiplying such an expression by s, i.e. it is the inclusion

$$(3.48) \qquad i : \lambda_2 q \to s(\lambda_2 q)$$

Thus we have derived the short exact sequence

$$(3.49) \qquad 0 \to \mathcal{I}_{(\lambda_2)} \xrightarrow{i} \nu^* F_2 \xrightarrow{r} \nu^* \mathcal{I}_{(x-y,z)} \to 0$$

But as one may check from equation (3.45), $\lambda_2 = 0$ also implies that $x - y = z = 0$, so $\lambda_2 = 0$ is the equation for the line for the line l_1 on $\widetilde{S}_0$ (the proper transform of the line $x - y = z = 0$ downstairs on S_0). Thus the ideal sheaf $\mathcal{I}_{(\lambda_2)}$ is in fact $\mathcal{O}(-l_1)$. Therefore when pulled back to the resolution of $x^2 - y^2 + z^2 = 0$, we found the following short exact sequence:

$$(3.50) \qquad 0 \to \mathcal{O}(-l_1) \xrightarrow{i} \nu^* F_2 \xrightarrow{r} \nu^* \mathcal{I}_{(x-y,z)} \to 0$$

Furthermore, when we pull back $\mathcal{I}_{(x-y,z)}$ to the resolution, we are looking at $\mathcal{O}(-l_2)$, where l_2 is the locus $x - y = z = 0$ on $\widetilde{S}_0$, equivalently the pull-back of the line $x - y = z = 0$ downstairs on S_0. Thus l_2 corresponds to $l_1 \cup C$, where C is the exceptional $\mathbf{P}^1$ of the blow-up. Putting this together, we see that after lifting to the blow-up, we have found the short exact sequence

$$(3.51) \qquad 0 \to \mathcal{O}_{\widetilde{S}_0}(-l_1) \to \nu^* F_2 \to \mathcal{O}_{\widetilde{S}_0}(-l_2) \to 0$$

with $l_2 = l_1 + C$.

We now want to argue that the extension is non-trivial, i.e. $\nu^* F_2$ is not isomorphic to the direct sum $\mathcal{O}(-l_1) \oplus \mathcal{O}(-l_2)$. For this we really need to compactify to a rational surface, since higher cohomology vanishes on affine varieties and the sheaf would just be isomorphic to $\mathcal{O} \oplus \mathcal{O}$ in the local model. The precise compactification doesn't matter much, as long as l_1 and l_2 are promoted to -1-curves on a rational surface, because then we can use the argument in section 2.2, which says that it is enough to check the extension over the exceptional curve C. For example a simple way to compactify the surface fibers of our local model is as follows:

$$(3.52) \qquad x^2 - y^2 + w^2(z_1^2 - tz_0^2) = 0$$

Here (z_0, z_1) are projective coordinates on $\mathbf{P}^1$ and (x, y, w) are projective coordinates on $\mathbf{P}^2$. For each t the equation (3.52) defines a conic bundle over $\mathbf{P}^1$, and in the patch $z_0 = w = 1$ we recover our previous local model. This can be viewed as a Hirzebruch surface blown up at two points, so it fits with the A_n and D_n rational surfaces discussed previously. The fiber over a generic point (z_0, z_1) is a smooth conic, which degenerates to a pair of lines over $z = \pm\sqrt{t}$. Denoting the generic fiber by f, we may denote the

two lines over $z = \sqrt{t}$ by l_1 and $f - l_1$ respectively, since their sum deforms to a generic fiber. It follows that $0 = f^2 = ((f - l_1) + l_1)^2$, and using the symmetry between l_1 and $f - l_1$ and the fact that they intersect at a single point, we deduce that $l_1^2 = -1$. The same argument applies at $z = -\sqrt{t}$, so the lines $x = y$, $z = \pm\sqrt{t}$ are promoted to -1-curves. They collide on the fiber over $t = 0$, creating the A_1 singularity.

To argue that the extension is non-trivial over C is straightforward. Using the fact that l_1 and l_2 have self-intersection -1, we have $\mathcal{O}(-l_1)|_C = \mathcal{O}_C(-1)$ and $\mathcal{O}(-l_2)|_C = \mathcal{O}_C(1)$. Furthermore we have already shown, $\nu^* F_2$ is the pull-back of a vector bundle downstairs (a locally free sheaf) and so it restrict to the trivial rank two bundle on C. Therefore on C we get the short exact sequence

$$(3.53) \qquad 0 \to \mathcal{O}_C(-1) \to \mathcal{O}_C(0) \oplus \mathcal{O}_C(0) \to \mathcal{O}_C(1) \to 0$$

which indeed describes the non-trivial extension of $\mathcal{O}_C(1)$ by $\mathcal{O}_C(-1)$, as required.

We are now essentially done. We only need to remember that $\nu^* F_2$ corresponds to $\pi_{\widehat{Y}/Y *}\mathcal{O}(-D')$ at $t = 0$, but the push-forward of our Poincaré sheaf is $\pi_{\widehat{Y}/Y *}\mathcal{O}(D)$. To save on notation, we denote the latter by $\widetilde{F}_2$. The difference is given by tensoring with the line bundle $\mathcal{O}(R)$, i.e. we have

$$(3.54) \qquad (\pi_{\widehat{Y}/Y})_* \mathcal{P}_{\widehat{Y}} = \widetilde{F}_2 = \mathcal{O}_{S_0}(R) \otimes F_2.$$

Here R is the divisor defined by the equation $x = y$, which describes the locus $x - y = z^2 = 0$ in the fiber over $t = 0$, which in turn describes the divisor $2l_1 + C = l_1 + l_2$ on the resolution $\widetilde{S}_0$. Therefore after tensoring with $\mathcal{O}(l_1 + l_2)$ we have

$$(3.55) \qquad 0 \to \mathcal{O}_{\widetilde{S}_0}(l_2) \to \nu^* \widetilde{F}_2 \to \mathcal{O}_{\widetilde{S}_0}(l_1) \to 0$$

This is precisely the non-trivial extension sequence (2.19) that we wished to recover, so our transform has the expected behaviour at the branch locus. This concludes the second property that we wanted to show.

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