The Conway Moonshine Module is a reflected K3 theory

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Recently, Duncan and Mack-Crane established an isomorphism, as Virasoro modules at central charges $c = 12$, between the space of states of the Conway Moonshine Module and the space of states of a special K3 theory that was extensively studied some time ago by Gaberdiel, Volpato and the two authors. In the present work, we lift this result to the level of modules of the extensions of these Virasoro algebras to $N = 4$ super Virasoro algebras. Moreover, we relate the super vertex operator algebra and module structure of the Conway Moonshine Module to the operator product expansion of this special K3 theory by a procedure we call *reflection*. This procedure can be applied to certain superconformal field theories, transforming all fields to holomorphic ones. It also allows to describe certain superconformal field theories within the language of super vertex operator algebras. We discuss reflection and its limitations in general, and we argue that through reflection, the Conway Moonshine Module inherits from the K3 theory a richer structure than anticipated so far. The comparison between the Conway Moonshine Module and the K3 theory is considerably facilitated by exploiting the free fermion description as well as the lattice vertex operator algebra description of both theories. We include an explicit construction of cocycles for the relevant charge lattices, which are half integral. The transition from the K3 theory to the Conway Moonshine Module via reflection promotes the latter to the role of a medium that collects the symmetries of K3 theories from distinct points of the moduli space, thus uncovering a version of symmetry surfing in this context.

Introduction

Very few would have anticipated in 1978 that the innocuous decomposition $196884 = 196883 + 1$ would spark so much interest in the mathematics community, culminating with the award of a Fields medal to Borcherds twenty years later. The defining moment was John McKay's observation that 196883 is the dimension of the smallest non trivial irreducible representation of the Monster group $\mathbb{M}$, and that 196884 is the coefficient of the linear term in the Fourier expansion of the j-function, a Hauptmodul for $\mathrm{SL}(2, \mathbb{Z})$. What had appeared to be a mere coincidence at first turned out to be part of an intriguing pattern. Not only did all the Fourier coefficients of the j-function, bar the constant term, coincide with dimensions of representations of $\mathbb{M}$, exhibiting j as a graded dimension of some $\mathbb{M}$-module $V^\natural$, but *all* the so-called McKay-Thompson series [Tho79], which are graded characters for arbitrary elements of $\mathbb{M}$, are Hauptmoduln of genus zero subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$. This phenomenon was coined *Monstrous Moonshine* by Conway and Norton. The connections between the largest sporadic group and modular functions have deep roots and manifest themselves in string theory for example. To this day though, and despite the beautiful construction of the Monster Module by Frenkel, Lepowsky and Meurman [FLM84] using conformal field theory techniques, as well as its realization in terms of superstrings [DGH88], there has been little use of these connections in string theory.

A game changing event took place in 2010, when Eguchi, Ooguri and Tachikawa [EOT11] noticed an intriguing connection between the elliptic genus $\mathcal{E}_{\mathrm{K3}}$ of K3, and the sporadic group M_{24}. The arena is that of closed

superstrings propagating on K3 surfaces, where the existence of a world-sheet $N = (4, 4)$ superconformal symmetry is well established and intimately related to the hyperkähler structure of K3. One of us, in collaboration with Eguchi, Ooguri and Yang, used techniques pioneered by Witten to calculate the elliptic genus of K3 exploiting the structure of string partition functions based on orbifolds and Gepner models in [EOTY89]. Such partition functions are expressed in terms of massless and massive, unitary irreducible characters of the $N = 4$ superconformal algebra at central charge $c = 6$ [ET87, ET88a, ET88b], and it follows that $\mathcal{E}_{\text{K3}}$ may be decomposed into an infinite sum of such characters. The coefficients of all massive $N = 4$ characters appearing in that sum were conjectured in [EOT11] to be the dimensions of representations of M_{24}. In fact, an imprint of this relation had been anticipated previously by Govindarajan and Krishna [GK10, Gov11] in their studies of Borcherds-Kac-Moody Lie superalgebras obtained from dyon spectra in $\mathbb{Z}_N$-CHL orbifolds. The conjecture became more specific after the work of Cheng [Che10], Gaberdiel, Hohenegger and Volpato [GHV10b, GHV10a], and Eguchi and Hikami [EH11], who determined the expected twining genera. This *Mathieu Moonshine* was mathematically proven by Gannon [Gan16], but his approach does not shed much light on the role of M_{24} in string theory. This remains an open question, which in our eyes, is well worth studying. Indeed, investigations so far suggest that a full understanding of an M_{24} action in this context requires new conceptual thinking.

There have been several lines of attack to probe Mathieu Moonshine. As mentioned above, shortly after the observation by Eguchi, Ooguri and Tachikawa, and building on Thompson's idea of twist [Tho79], several groups constructed twining elliptic genera that were proven to be graded characters of an infinite-dimensional M_{24}-module [Gan16]. In another development, a new family of moonshines was discovered, of which Mathieu Moonshine is a member. Dubbed *Umbral Moonshine*, this family of connections between certain mock modular forms and automorphism groups of Niemeier lattices has opened the door to some fruitful collaborations that bridge mathematics and theoretical physics [CDH14a, CDH14b, DGO15]. In the meantime, we investigated the geometric symmetries of strings propagating on K3 surfaces of Kummer type. Using lattice techniques to introduce the concept of symmetry surfing the moduli space of K3 theories, we showed how the overarching group $\mathbb{Z}_2^4 : A_8$ emerges from *symmetry surfing* [TW13, TW15a]. The basic idea, at this level, is an application of Kondo's beautiful strategy of proof [Kon98] of Mukai's seminal classification result [Muk88] for symplectic automorphisms of K3 surfaces: the lattice of integral cohomology of

K3, which enters crucially in the construction of the moduli space of K3 theories [AM94, NW01], is replaced by an even, self-dual lattice of the same rank, which thereby serves as a medium to collect symmetries from distinct points of the moduli space. To move beyond generating the relevant groups, in order to construct the expected representations, one needs to leave the comfort zone of lattice techniques. In the context of $\mathbb{Z}_2$-orbifolds of toroidal conformal field theories, this has been achieved in [TW15b, GKP17], providing further evidence in favour of the idea of symmetry surfing.

Our present investigation picks up more recent efforts to establish connections between certain conformal field theories and certain super vertex operator algebras. Indeed, we aim at clarifying the relationship between the K3 theory studied in [GTVW14] and the Conway Moonshine Module of Duncan and Mack-Crane [Dun07, DMC16]. As Virasoro modules at central charge $c = 12$, Duncan and Mack-Crane showed that the spaces of states of the two models agree. However, while the former is an $N = (4, 4)$ superconformal field theory at central charges $c = \bar{c} = 6$, the latter is a super vertex operator algebra at central charge $c = 12$, together with its (unique, up to isomorphism, irreducible) canonically twisted supermodule. We show that the two are related[1] by a procedure which we call *reflection*. Only very special superconformal field theories allow reflection, which transforms all fields to holomorphic ones. The operator product expansion (OPE) of the K3 theory thereby induces the super vertex operator algebra and admissible module structure on the Conway Moonshine Module, and more, since in the K3 theory, an OPE is defined between *any* pair of fields. The reflection procedure provides a bridge between conformal field theory and vertex operator algebra techniques. This bridge may be used in both directions, hopefully allowing some of the experts in vertex operator algebras to enter the world of K3 theories, or even more general superconformal field theories.

By our interpretation of the Conway Moonshine Module as the image of a K3 theory under reflection, the modular properties of the partition function and its building blocks receive a natural explanation from superstring theory. It would be interesting to know whether all the genus zero properties of Conway Moonshine can be traced back to K3. On the level of lattices, reflection is an implementation of the techniques mentioned above, where the K3 lattice with signature $(4, 20)$ is replaced by an even, self-dual

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[1]up to exchanging the roles of bosons and fermions in the Ramond sector of the Conway Moonshine Module, while accounting for an extra factor of (-1) introduced by hand in the Ramond sector partition functions between [DMC16, (8.7)] and [DMC16, (9.10), (9.14)]

positive definite lattice of the same rank. This allows us to reveal the proposal of [DMC16], that is, the realisation of all symmetries of K3 theories as automorphisms of the Conway Moonshine Module, as an incarnation of symmetry surfing by means of lattice techniques [TW13, TW15a]. It also means that we do not expect a construction of an M_{24} vertex operator algebra that explains Mathieu Moonshine from the Conway Moonshine Module, since the latter disregards the twist of [TW15b, GKP17].

The work [CDR18] by Creutzig, Duncan and Riedler complements ours, with some overlap with our results. To clarify the relation between the Conway Moonshine Module and the K3 theory studied in [GTVW14], they introduce the notion of a *potential bulk SCFT*, which in our language amounts to the image of a SCFT under reflection, viewed as a module of its chiral-antichiral algebra. They find sufficient conditions for such a potential bulk SCFT to agree with (adequately) *nice* super vertex operator algebras, and they provide examples where these conditions hold. Given that the potential bulk SCFT obtained from the above-mentioned K3 theory is among these examples, they have in particular, independently from us, extended the identification with the Conway Moonshine Module to the level of modules of a supersymmetric extension of the previously studied two copies of the Virasoro algebra.

The present work is organised as follows. In Section 1, we start by revisiting the K3 theory based on the $\mathbb{Z}_2$-orbifold of the D_4-torus theory, whose symmetry group $\mathbb{Z}_2^8 \colon \mathbb{M}_{20}$ is one of the largest symmetry groups of K3 theories preserving $N = (4, 4)$ supersymmetry [GTVW14]. As a K3 sigma model, this theory is built on the tetrahedral Kummer surface studied in depth in [TW13, TW15b, TW15a]. With a view to compare this theory to the Conway Moonshine Module later on, we pay particular attention to its description in terms of twenty-four free Majorana fermions, twelve left- and twelve right-movers, and highlight its underlying affine current algebra[2] $(\widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R})^3 \subset \widehat{\mathfrak{so}}(8)_1^3$. One of the three summands $\widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R}$ is the affine algebra arising from the fermionic superpartners of the four left and four right-moving $U(1)$ currents of the bosonic D_4-torus theory. We make a choice of a left- (resp. right-)moving $U(1)$ current whose zero mode generates the Cartan subalgebra of an affine $\widehat{\mathfrak{su}}(2)_{1,L} \subset \widehat{\mathfrak{so}}(4)_{1,L}$ (resp. $\widehat{\mathfrak{su}}(2)_{1,R} \subset \widehat{\mathfrak{so}}(4)_{1,R}$), which is determined by the left- (resp.

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[2]We use the notation $\widehat{\mathfrak{g}}_1^n = \widehat{\mathfrak{g}}_1^{\oplus n}$ throughout.

right)-moving $N = 4$ superconformal algebra[3] at central charge $c = 6$ (resp. $\bar{c} = 6$). We use standard conformal field theory techniques to identify the spectrum of our special K3 theory in terms of the vacuum, vector and two spinor representations of $\widehat{\mathfrak{so}}(8)_1$ and to give an elegant description in terms of lattice vertex operator algebras. In addition, we provide the full partition function, including its explicit dependence on the charges measured by the zero modes of the $U(1)$ currents described above. The underlying charge lattice Γ of our K3 theory is half integral. Our presentation includes the classification of all equivalence classes of cocycles for a certain type of half integral lattices, in particular Γ. Moreover, in each such equivalence class of cocycles we explicitly construct a representative which obeys all required compatibility conditions with the real structure on our space of states.

Section 2 is devoted to a recapitulation of the Conway Moonshine Module presented in [Dun07, DMC15, DMC16]. We also offer a description in terms of lattice vertex operator algebras. In addition, we explain how to obtain $U(1)$ currents that allow an interpretation as images of the $U(1)$ currents in the K3 theory, under reflection, and we present the partition function for the Conway Moonshine Module with its dependence on the corresponding charges.

In Section 3, we determine some necessary and sufficient conditions for non-holomorphic superconformal field theories to allow *reflection*, that is, a mathematically consistent transformation of all fields to holomorphic ones. We show that reflection amounts to a complex conjugation for the anti-holomorphic parameters of the OPE when restricted to pairs of fields that create *real* states.[4] The real structure on the space of states of the original superconformal field theory is thus found to play a crucial role[5]. We show that our reflection procedure, if applicable, yields the structure of a super

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[3]In fact, this is the *small $N = 4$* superconformal algebra of [ABD+76]. We simply call it *the $N = 4$ superconformal algebra* to untangle the terminology.

[4]This procedure is reminiscent of that used by Harvey and Moore in their definition of BPS algebras [HM96]. However, we differ in the following crucial point: throughout our work, we parametrize fields $\phi(z, \bar{z})$ in the complex plane rather than on the cylinder. Therefore, on a formal level complex conjugation of the right-moving degrees of freedom amounts to restricting to $z = \bar{z}$. On the other hand, the prescription given in [HM96, §9], which also changes the right-moving fields to left-moving fields, amounts to enforcing $z = \bar{z}^{-1}$, i.e. using a complex conjugation on the cylinder. Although the latter may be a natural choice, our construction, in the context of lattice vertex operator algebras, entails the change in signature of the charge lattice required to make contact with the super vertex operator algebras.

[5]As is customary, we denote the real structure on $\mathbb{C}$ by $z \mapsto \bar{z}$ for $z \in \mathbb{C}$, and we write $i = \exp\left(\frac{i\pi}{2}\right)$ for our choice of $\sqrt{-1}$ throughout.

vertex operator algebra on the Neveu-Schwarz sector $\mathbb{H}^{NS}$ and that of an admissible $\mathbb{H}^{NS}$-module on the Ramond sector of the theory. We furthermore discuss how reflection induces an additional structure on the resulting super vertex operator algebra and admissible module.

In Section 4, we show that the Conway Moonshine Module emerges via reflection of the K3 theory with $\mathbb{Z}_2^8 \colon \mathbb{M}_{20}$ symmetry. In this case, we show that reflection amounts to replacing the lattice of signature $(6, 6)$, which governs the lattice vertex operator algebra description of the K3 theory, by a lattice of signature $(12, 0)$. Thereby, we interpret the ideas of [DMC16], namely to realize all symmetries of K3 theories as automorphisms of the Conway Moonshine Module, in terms of symmetry surfing. Although this yields a natural action of M_{24} on the Conway Moonshine Module that extends the action of the geometric symmetry group of the K3 theory, this cannot explain Mathieu Moonshine. Indeed, the twist that had already been observed in [TW15b] is not implemented in the Conway Moonshine Module. Three appendices summarise, respectively, our approach to superconformal field theory, technical background concerning cocycles on half integral lattices, and some useful identities for Jacobi theta functions.

1. A K3 theory with $\mathbb{Z}_2^8 \colon \mathbb{M}_{20}$ symmetry

In this section, we present the basic ingredients of the K3 theory which is central to this work[6], namely the *K3 theory based on the $\mathbb{Z}_2$-orbifold of the D_4-torus theory*. This model accounts for one of the largest possible discrete symmetry groups of K3 theories preserving $N = (4, 4)$ supersymmetry, namely $\mathbb{Z}_2^8 \colon \mathbb{M}_{20}$. That some K3 theory would possess this symmetry was predicted in the very interesting classification paper [GHV12]. As a consequence, it was a sound enterprise to determine and study such a model further, constructing its symmetries explicitly in order to gain further insight in relation to the M_{24} Moonshine phenomenon. The model was first investigated in [NW01], where it was denoted $(\widetilde{2})^4$ in reference to the fact that it can be constructed as a Gepner type model. It was studied extensively in [Wen02, GTVW14] from several perspectives, one of which involves a description[7] in terms of 12 left-moving Majorana fermions $\psi_j(z)$ and 12 right-moving Majorana fermions $\overline{\psi}_j(\overline{z})$, $j \in \{1, \dots, 12\}$.

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[6]For a summary of relevant notions from (super-)conformal field theory, see Appendix A.

[7]The authors of [DMC16] seem unaware of the fact that the free fermion description of the K3 theory predates their own account.

The description of this model in terms of free fermions, as given in [GTVW14], is summarized below since it is central in making contact with the recent works of Duncan and Mack-Crane [DMC15, DMC16]. For later convenience, we include a detailed discussion of the OPEs in this model, fixing in particular the delicate choices of phase factors.

1.1. Bosonic D_4-torus model

The *bosonic* D_4-torus model, which we consider as a starting point to the construction of our K3 theory, is a toroidal theory based on the 4-dimensional torus $\mathbb{T} = \mathbb{R}^4/L$, where L is the D_4-lattice $L_{D_4} \subset \mathbb{R}^4$ and where the B-field and the metric are chosen in such a way that the generic left-moving affine $\widehat{\mathfrak{u}}(1)^4$ algebra is enhanced to the affine algebra $\widehat{\mathfrak{so}}(8)_1$, and analogously for the right-movers (see [GTVW14, §2] for details). The lattice of $\widehat{\mathfrak{u}}(1)^4_L \oplus \widehat{\mathfrak{u}}(1)^4_R$ charges, which completely determines the D_4-torus model, equals

$$(1.1) \quad \Gamma_{d,d} := \left\{ (\boldsymbol{Q}; \overline{\boldsymbol{Q}}) \in \left(\mathbb{Z}^d \oplus \mathbb{Z}^d \right) \cup \left(\left(\tfrac{1}{2} + \mathbb{Z} \right)^d \times \left(\tfrac{1}{2} + \mathbb{Z} \right)^d \right) \right.$$
$$\left. \Big| \sum_{k=1}^{d} \left(Q_k + \overline{Q}_k \right) \equiv 0 \mod 2 \right\}$$

with $d = 4$, c.f. [GTVW14, (2.11)]. The lattice $\Gamma_{d,d} \subset \mathbb{R}^{d,d} = \mathbb{R}^d \oplus \mathbb{R}^d$ is equipped with the symmetric bilinear form

$$(1.2) \quad \forall (\boldsymbol{Q}; \overline{\boldsymbol{Q}}), (\boldsymbol{Q}'; \overline{\boldsymbol{Q}}') \in \Gamma_{d,d}: \quad (\boldsymbol{Q}; \overline{\boldsymbol{Q}}) \bullet (\boldsymbol{Q}'; \overline{\boldsymbol{Q}}') := \boldsymbol{Q} \cdot \boldsymbol{Q}' - \overline{\boldsymbol{Q}} \cdot \overline{\boldsymbol{Q}}',$$

where here and in the following, $\boldsymbol{Q} \cdot \boldsymbol{Q}' \in \mathbb{R}$ denotes the standard scalar product of $\boldsymbol{Q}, \boldsymbol{Q}' \in \mathbb{R}^d$. This yields an even, self-dual lattice $\Gamma_{d,d}$ in general, and the space of states of the bosonic D_4-torus model is

$$(1.3) \qquad \mathcal{H}_{D_4\text{-torus}} = \bigoplus_{\gamma \in \Gamma_{4,4}} \mathcal{H}_\gamma$$

with $\mathcal{H}_\gamma$, $\gamma = (\boldsymbol{Q}; \overline{\boldsymbol{Q}}) \in \Gamma_{4,4}$, the Fock space representation of 4 left-moving and 4 right-moving free bosons, built on a ground state υ_γ of $\widehat{\mathfrak{u}}(1)^4_L \oplus \widehat{\mathfrak{u}}(1)^4_R$ with charge γ and conformal dimensions $(h; \overline{h}) = \frac{1}{2}(\boldsymbol{Q} \cdot \boldsymbol{Q}; \overline{\boldsymbol{Q}} \cdot \overline{\boldsymbol{Q}})$. We generally choose such $\upsilon_\gamma \in \mathcal{H}_\gamma$ with $\upsilon_\gamma^* = \upsilon_{-\gamma}$ as unit vectors with respect to the scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_{D_4\text{-torus}}$. Moreover, we assume that $\upsilon_0 = \Omega$ is the vacuum.

By construction, the space $\mathcal{H}_{D_4\text{-torus}}$ decomposes into four sectors, since

$$(1.4) \quad \Gamma_{4,4} = \bigcup_{a=0}^{3} \left(\gamma_{D_4}^{(a)} + \Gamma_{D_4}^{(0)} \right), \quad \text{with}$$

$$\Gamma_{D_4}^{(0)} := \left\{ (\boldsymbol{Q}; \overline{\boldsymbol{Q}}) \in \mathbb{Z}^4 \oplus \mathbb{Z}^4 \; \Big| \; \sum_{k=1}^{4} Q_k \equiv \sum_{k=1}^{4} \overline{Q}_k \equiv 0 \mod 2 \right\},$$

$$\gamma_{D_4}^{(0)} := 0, \gamma_{D_4}^{(1)} := (\boldsymbol{e}_4; \boldsymbol{e}_4), \gamma_{D_4}^{(2)} := \frac{1}{2} \sum_{k=1}^{4} (\boldsymbol{e}_k; \boldsymbol{e}_k), \gamma_{D_4}^{(3)} := \gamma_{D_4}^{(1)} + \gamma_{D_4}^{(2)},$$

where here and in the following, $(\boldsymbol{e}_k)_{k \in \{1,\ldots,d\}}$ denotes the standard basis of $\mathbb{R}^d$.

1.1.1. Holomorphic and anti-holomorphic currents and fermionization. The four holomorphic currents of the model in the Cartan subalgebra of $\widehat{\mathfrak{so}}(8)_1$, $j_k(z)$ with $k \in \{1, \ldots, 4\}$, obey the OPEs

$$(1.5) \qquad j_k(z) j_\ell(w) \sim \frac{\delta_{k\ell}}{(z-w)^2} \quad \forall k, \ell \in \{1, \ldots, 4\}.$$

Analogously, in the right-moving sector, one has four anti-holomorphic $U(1)$ currents $\overline{\jmath}_k(\overline{z})$, $k \in \{1, \ldots, 4\}$.

One may fermionize the theory by the *Frenkel-Kac-Segal* construction [FK81, Seg81, GO84]. To do so, one introduces eight free left-moving Majorana fermions $\psi_i(z)$, $i \in \{5, \ldots 12\}$ and eight free right-moving Majorana fermions $\overline{\psi}_i(\overline{z})$, $i \in \{5, \ldots 12\}$, with OPEs

$$\psi_i(z) \psi_j(w) \sim \frac{\delta_{ij}}{z - w}, \quad \overline{\psi}_i(\overline{z}) \overline{\psi}_j(\overline{w}) \sim \frac{\delta_{ij}}{\overline{z} - \overline{w}}, \quad \forall i, j \in \{5, \ldots, 12\},$$

all with coupled spin structures. In terms of the free holomorphic Dirac fermions

$$(1.6) \qquad x_k := \frac{1}{\sqrt{2}} (\psi_{k+4} + i \psi_{k+8}), \quad x_k^* := \frac{1}{\sqrt{2}} (\psi_{k+4} - i \psi_{k+8}),$$
$$k \in \{1, \ldots, 4\},$$

which satisfy the OPEs

$$(1.7) \qquad x_k(z) \, x_k^*(w) \sim \frac{1}{z - w} \sim x_k^*(z) \, x_k(w), \quad k \in \{1, \ldots, 4\},$$

the four left-moving $U(1)$ currents are given by

$$(1.8) \qquad j_k(z) = {:}x_k(z)x_k^*(z){:} = -i{:}\psi_{k+4}(z)\psi_{k+8}(z){:},$$

as can be checked by calculating their OPEs (1.5) with the help of (1.7). Introducing $j_k(z) = i\partial\phi_k(z)$, one may identify

$$(1.9) \qquad x_k(z) = {:}\exp(i\phi_k(z)){:}\, c_k, \quad x_k^*(z) = {:}\exp(-i\phi_k(z)){:}\, c_{-k},$$
$$k \in \{1, \dots, 4\},$$

where c_k and c_{-k} are cocycle factors that ensure that the fermions of different species anticommute, as we shall discuss in greater detail below.

The analogous construction holds for the right-moving sector, through the introduction of four right-moving Dirac fermions

$$(1.10) \qquad \overline{x}_k := \frac{1}{\sqrt{2}}(\overline{\psi}_{k+4} + i\overline{\psi}_{k+8}), \quad \overline{x}_k^* := \frac{1}{\sqrt{2}}(\overline{\psi}_{k+4} - i\overline{\psi}_{k+8}),$$
$$k \in \{1, \dots, 4\},$$

as well as $\overline{j}_k(\overline{z}) := {:}\overline{x}_k(\overline{z})\overline{x}_k^*(\overline{z}){:} = i\overline{\partial}\overline{\phi}_k(\overline{z})$ with

$$i\overline{x}_k(\overline{z}) = {:}\exp(i\overline{\phi}_k(\overline{z})){:}\, \overline{c}_k, \quad i\overline{x}_k^*(\overline{z}) = {:}\exp(-i\overline{\phi}_k(\overline{z})){:}\, \overline{c}_{-k}, \quad k \in \{1, \dots, 4\}.$$

It is then straightforward to express the 24 currents of $\widehat{\mathfrak{so}}(8)_1$ associated with the roots of D_4, namely the vectors $\pm\boldsymbol{e}_j \pm \boldsymbol{e}_k$, $1 \leq j < k \leq 4$, in terms of the Dirac fermions (1.9): possibly up to cocycle factors, these 24 conformal weight $(h; \overline{h}) = (1; 0)$-fields may be realized as

$$(1.11) \qquad \begin{aligned} V_{\boldsymbol{e}_j+\boldsymbol{e}_k}(z) &= {:}x_j(z)x_k(z){:}, & V_{-\boldsymbol{e}_j-\boldsymbol{e}_k}(z) &= {:}x_j^*(z)x_k^*(z){:}, \\ V_{\boldsymbol{e}_j-\boldsymbol{e}_k}(z) &= {:}x_j(z)x_k^*(z){:}, & V_{-\boldsymbol{e}_j+\boldsymbol{e}_k}(z) &= {:}x_j^*(z)x_k(z){:}. \end{aligned}$$

In other words, $\Gamma_{D_4}^{(0)}$ contains the charge vectors $(\pm\boldsymbol{e}_j \pm \boldsymbol{e}_k; 0)$ that are responsible for the extended $\widehat{\mathfrak{so}}(8)_1$ symmetry of the model.

1.1.2. General momentum-winding fields. The currents (1.11) are special momentum-winding fields with left and right $\widehat{\mathfrak{u}}(1)^4$ charges $(\boldsymbol{Q}; \overline{\boldsymbol{Q}})$. The latter, a priori, are vectors of the charge lattice $\Gamma_{4,4} \subset \mathbb{R}^{4,4}$ of the D_4-torus model as in (1.1). The momentum-winding field for any $(\boldsymbol{Q}; \overline{\boldsymbol{Q}}) \in \Gamma_{4,4}$

may be written as

$$(1.12) \qquad V_{(\boldsymbol{Q};\overline{\boldsymbol{Q}})}(z,\overline{z}) := \; :\exp\left[i \sum_{k=1}^{4} Q_k \phi_k(z) + i \sum_{k=1}^{4} \overline{Q}_k \overline{\phi}_k(\overline{z}) \right]: \; c_{(\boldsymbol{Q};\overline{\boldsymbol{Q}})},$$

with $c_{(\boldsymbol{Q};\overline{\boldsymbol{Q}})}$ denoting appropriate *cocycle factors* [FK81, Seg81, GO84]. This means that for every $\gamma \in \Gamma_{4,4}$, we have a linear operator c_γ on $\mathbb{H}$, where $(c_\gamma)_{|\mathcal{H}_{\gamma'}} = \varepsilon(\gamma, \gamma') \cdot \mathrm{id}_{\mathcal{H}_{\gamma'}}$ for all charge vectors $\gamma, \gamma' \in \Gamma_{4,4}$, with *cocycles*

$$\varepsilon \colon \Gamma_{4,4} \times \Gamma_{4,4} \longrightarrow \{\pm 1\}.$$

Here, as is common in the physics literature, the term *cocycles* more precisely refers to 2-cocycles on $\Gamma_{4,4}$ with values in $\{\pm 1\}$ that obey the additional symmetry requirement (B.5) with respect to the bilinear form (1.2) on $\Gamma_{4,4}$. In Appendix B we review the definition of such cocycles. Since $\Gamma_{4,4}$ is an integral lattice, their explicit construction, also given in Appendix B, is well-known.

In the notations of (1.12), the bosonic $(h;\overline{h}) = (1;0)$-fields (1.11) have $\boldsymbol{Q} = \pm \boldsymbol{e}_j \pm \boldsymbol{e}_k$ and $\overline{\boldsymbol{Q}} = 0$, and we write

$$V_{\pm \boldsymbol{e}_j \pm \boldsymbol{e}_k}(z) := V_{(\pm \boldsymbol{e}_j \pm \boldsymbol{e}_k; \boldsymbol{0})}(z,\overline{z}).$$

According to (1.4), the four cosets in $\Gamma_{4,4}/\Gamma_{D_4}^{(0)}$, namely $\gamma_{D_4}^{(a)} + \Gamma_{D_4}^{(0)}$ for $a \in \{0,\dots,3\}$, induce the decomposition of the space of states of the bosonic D_4-torus model into representations of the left and right-moving $\widehat{\mathfrak{so}}(8)_1$-algebras as

$$\mathcal{H}_{D_4\text{-torus}} = (\mathcal{H}_{L,0} \otimes \mathcal{H}_{R,0}) \oplus (\mathcal{H}_{L,v} \otimes \mathcal{H}_{R,v})$$
$$\oplus (\mathcal{H}_{L,s} \otimes \mathcal{H}_{R,s}) \oplus (\mathcal{H}_{L,c} \otimes \mathcal{H}_{R,c}),$$

where $\mathcal{H}_{L,0}$ is the left-moving $\widehat{\mathfrak{so}}(8)_1$ vacuum representation, while $\mathcal{H}_{L,v}, \mathcal{H}_{L,s}$ and $\mathcal{H}_{L,c}$ are the vector and the two spinor representations, respectively. The $\mathcal{H}_{R,\bullet}$ denote the corresponding right-moving representations. Hence

$$\mathcal{H}_{L,0} \otimes \mathcal{H}_{R,0} = \bigoplus_{\gamma \in \Gamma_{D_4}^{(0)}} \mathcal{H}_\gamma$$

with the notations of (1.4) and (1.3). The vector representation

$$\mathcal{H}_{L,v} \otimes \mathcal{H}_{R,v} = \bigoplus_{\gamma \in \gamma_{D_4}^{(1)} + \Gamma_{D_4}^{(0)}} \mathcal{H}_\gamma$$

is generated by OPEs of the left and right-moving currents with the *vector* $(h; \overline{h}) := (\frac{1}{2}; \frac{1}{2})$ winding-momentum fields $V_{(Q; \overline{Q})}(z, \overline{z})$, where

$$\mathbf{Q} = \pm e_i , \quad \overline{\mathbf{Q}} = \pm e_j.$$

The spinor representations $\mathcal{H}_{L,s} \otimes \mathcal{H}_{R,s}$, $\mathcal{H}_{L,c} \otimes \mathcal{H}_{R,c}$ are analogously generated by the OPEs of the left and right-moving currents with the *spin* $(h; \overline{h}) := (\frac{1}{2}; \frac{1}{2})$ winding-momentum fields $V_{Q; \overline{Q}}(z, \overline{z})$ for

$$\mathbf{Q} = \frac{1}{2} \sum_{j=1}^{4} \varepsilon_j e_j , \quad \overline{\mathbf{Q}} = \frac{1}{2} \sum_{k=1}^{4} \delta_k e_k ,$$

$$\text{where } \varepsilon_j,\, \delta_k \in \{\pm 1\} \text{ and } \sum_{k=1}^{4} (Q_k + \overline{Q}_k) \equiv 0 \mod 2 .$$

In fact,

$$\mathcal{H}_{L,s} \otimes \mathcal{H}_{R,s} = \bigoplus_{\gamma \in \gamma_{D_4}^{(2)} + \Gamma_{D_4}^{(0)}} \mathcal{H}_\gamma , \quad \mathcal{H}_{L,c} \otimes \mathcal{H}_{R,c} = \bigoplus_{\gamma \in \gamma_{D_4}^{(3)} + \Gamma_{D_4}^{(0)}} \mathcal{H}_\gamma ,$$

i.e. $\sum_{k=1}^{4} Q_k$ and $\sum_{k=1}^{4} \overline{Q}_k$ are both even for $\mathcal{H}_{L,s} \otimes \mathcal{H}_{R,s}$ and both odd for $\mathcal{H}_{L,c} \otimes \mathcal{H}_{R,c}$.

Fermionizing the bosonic D_4-torus theory as mentioned earlier allows us to extend the definition of the momentum-winding fields in (1.12) to include *fermionic fields* $V_\gamma(z, \overline{z})$ with $\gamma \in (\mathbb{Z}^4 \oplus \mathbb{Z}^4) \setminus \Gamma_{4,4} \subset (\Gamma_0^{D_4})^*$, where

$$\Gamma_0^{D_4} := (\mathbb{Z}^4 \oplus \mathbb{Z}^4) \cap \Gamma_{4,4}.$$

Indeed[8], in (1.9) and (1.10) we have already presented special cases of such fermionic fields, namely

$$(1.13) \quad \begin{array}{l} x_k(z) = V_{e_k}(z), \quad x_k^*(z) = V_{-e_k}(z); \\ i\overline{x}_k(\overline{z}) = V_{(0; e_k)}(z, \overline{z}), \quad i\overline{x}_k^*(\overline{z}) = V_{(0; -e_k)}(z, \overline{z}), \quad k \in \{1, \ldots, 4\}, \end{array}$$

where $c_{(\pm e_k; 0)} := c_{\pm k}$ and $c_{(0; \pm e_k)} := \overline{c}_{\pm k}$. Following [GO86], [GNOS86], [GNO+87], we may actually extend further to $\mathcal{H}_{\gamma'}$ with $\gamma' \in \Gamma_{D_4}$,

$$\Gamma_{D_4} := \Gamma_{4,4} \cup ((0; e_4) + \Gamma_{4,4}) = (\Gamma_0^{D_4})^*,$$

---

[8]For later convenience, we use slightly different normalizations than the ones given in [GTVW14].

a half integral lattice. The lattice $\Gamma_0^{D_4} = (\Gamma_{D_4})^*$ is a sublattice of Γ_{D_4} of index 4, and as such, $(\Gamma_0^{D_4}, \Gamma_{D_4})$ form a $\mathbb{Z}_2$ lattice pair (Γ_0, Γ) of the type used in Appendix B. In particular, Appendix B includes an explicit construction of cocycles ε for this half integral lattice. These cocycles are bimultiplicative in the sense of (B.11), and they are in the special gauge (B.12). They take values in the group of eighth roots of unity in $\mathbb{C}^*$. To clear notations, we write the charge lattice, extended to include fermions, as

(1.14)
$$\Gamma_{D_4} = \bigcup_{a=0}^{3} \left(\widetilde{\gamma}_{D_4}^{(a)} + \Gamma_0^{D_4} \right), \quad \text{with} \quad \Delta := \left\{ \widetilde{\gamma}_{D_4}^{(1)}, \widetilde{\gamma}_{D_4}^{(2)}, \widetilde{\gamma}_{D_4}^{(3)} \right\},$$

$$\widetilde{\gamma}_{D_4}^{(0)} := 0, \ \widetilde{\gamma}_{D_4}^{(1)} := (0; \boldsymbol{e}_4), \ \widetilde{\gamma}_{D_4}^{(2)} := \gamma_{D_4}^{(2)} = \frac{1}{2} \sum_{k=1}^{4} (\boldsymbol{e}_k; \boldsymbol{e}_k), \ \widetilde{\gamma}_{D_4}^{(3)} := \widetilde{\gamma}_{D_4}^{(1)} + \widetilde{\gamma}_{D_4}^{(2)}.$$

For any $\gamma, \gamma' \in \Gamma_{D_4}$, the OPEs between momentum-winding fields $V_\gamma, V_{\gamma'}$ are given by

$$(1.15) \quad V_\gamma(z, \overline{z}) V_{\gamma'}(w, \overline{w}) \sim (z - w)^{\boldsymbol{Q} \cdot \boldsymbol{Q}'} (\overline{z} - \overline{w})^{\overline{\boldsymbol{Q}} \cdot \overline{\boldsymbol{Q}}'} \varepsilon\left(\gamma, \gamma'\right) V_{\gamma + \gamma'}(w, \overline{w})$$

$$\times \left\{ 1 + (z - w) \sum_{k=1}^{4} Q_k j_k(w) + (\overline{z} - \overline{w}) \sum_{k=1}^{4} \overline{Q}_k \overline{j}_k(\overline{w}) + \cdots \right\}.$$

Note that in these OPEs, apart from integral powers of $(z - w)$, $(\overline{z} - \overline{w})$ and $|\overline{z} - \overline{w}|$, odd integral powers of $(z - w)^{\pm \frac{1}{2}}$ and $(\overline{z} - \overline{w})^{\pm \frac{1}{2}}$ occur iff

$$\gamma \in \delta + \Gamma_0^{D_4}, \quad \gamma' \in \delta' + \Gamma_0^{D_4} \quad \text{and} \quad \delta, \delta' \in \Delta, \quad \delta \neq \delta',$$

or equivalently, $\gamma \bullet \gamma' \in \frac{1}{2} + \mathbb{Z}$. Then, implementation of (1.15) in an n-point function affords the restriction of the domain of definition to some contractible open $U \subset \mathbb{C}^n \setminus \cup_{i \neq j} \{z \in \mathbb{C}^n | z_i = z_j\}$. An unambiguous formulation of such an extension of (1.15) states for all $\gamma, \gamma' \in \Gamma_{D_4}$:

$$(1.16) \quad V_\gamma(z, \overline{z}) v_{\gamma'} \sim z^{\boldsymbol{Q} \cdot \boldsymbol{Q}'} \overline{z}^{\overline{\boldsymbol{Q}} \cdot \overline{\boldsymbol{Q}}'} \varepsilon\left(\gamma, \gamma'\right)$$

$$\times \left\{ 1 + z \sum_{k=1}^{4} Q_k a_{-1}^{(k)} + \overline{z} \sum_{k=1}^{4} \overline{Q}_k \overline{a}_{-1}^{(k)} + \cdots \right\} v_{\gamma + \gamma'},$$

where $z \in U$ with $U \subset \mathbb{C}^*$ a contractible open subset, and where $a_n^{(k)}$, $\overline{a}_n^{(k)}$ with $k \in \{1, \ldots, 4\}$, $n \in \mathbb{Z}$, denote the modes of $j_k(z)$, $\overline{j}_k(\overline{z})$.

Using (1.16), one checks that the coboundary condition (B.1) ensures *associativity* of the OPE,

$$\forall \alpha,\, \beta,\, \gamma \in \Gamma_{D_4}: \quad V_\alpha(z,\overline{z})\,(V_\beta(w,\overline{w})v_\gamma) \sim (V_\alpha(z,\overline{z})V_\beta(w,\overline{w}))\,v_\gamma.$$

The additional symmetry condition (B.5) ensures *semilocality*

$$\forall a \in \{1,\, 2,\, 3\},\ \forall \alpha,\, \beta \in \Gamma_0^{D_4} \cup \left(\widetilde{\gamma}_{D_4}^{(a)} + \Gamma_0^{D_4}\right):$$
$$V_\alpha(z,\overline{z})V_\beta(w,\overline{w}) \sim (-1)^{(\alpha \bullet \alpha)\cdot(\beta \bullet \beta)} V_\beta(w,\overline{w})V_\alpha(z,\overline{z}).$$

In other words, semilocality is only required to hold between $V_\alpha(z,\overline{z})$ and $V_\beta(w,\overline{w})$ if $\alpha \bullet \beta \in \mathbb{Z}$. Indeed, this condition cannot be imposed on all pairs $\alpha,\, \beta \in \Gamma_{D_4}$, since the above-mentioned square root cuts obstruct semilocality.

The choice of special gauge (B.12) for the cocycles ensures that the OPE (1.15) is compatible with the real structure on the space of states according to (A.6). Indeed, for $\alpha = (\boldsymbol{Q}; \overline{\boldsymbol{Q}}) \in \Gamma_{D_4}$ we see from (1.16) that the condition $\varepsilon(\alpha, 0) = 1$ ensures that the field $V_\alpha(z,\overline{z})$ creates the state v_α from the vacuum. Furthermore, $\varepsilon(-\alpha, \alpha) = 1$ amounts to the hermiticity condition that ensures that $(V_\alpha(z,\overline{z}))^\dagger = \overline{z}^{-\boldsymbol{Q}\cdot\boldsymbol{Q}}z^{-\overline{\boldsymbol{Q}}\cdot\overline{\boldsymbol{Q}}}V_{-\alpha}(\overline{z}^{-1}, z^{-1})$ is compatible with our requirements $v_\alpha^* = v_{-\alpha}$ and $v_0 = \Omega$. More generally, by (A.7) we have

$$\forall \alpha,\, \beta \in \Gamma_{D_4}: \quad \overline{< V_{\alpha+\beta}(w,\overline{w})V_{-\alpha}(x,\overline{x})V_{-\beta}(z,\overline{z}) >}$$
$$= < (V_{-\beta}(z,\overline{z}))^\dagger \, (V_{-\alpha}(x,\overline{x}))^\dagger \, (V_{\alpha+\beta}(w,\overline{w}))^\dagger >,$$

such that the above requirement for the Hermitian conjugate fields together with (1.15) yield the last equation in (B.12).

1.2. Supersymmetric D_4-torus model

The *supersymmetric D_4-torus model* is obtained by adjoining $d = 4$ free Majorana fermions $(\psi_k(z),\, \overline{\psi}_k(\overline{z}))$, $k \in \{1, \ldots, 4\}$, related to the U(1) currents $j_k(z)$ and their right-moving counterparts by world-sheet supersymmetry. Similarly to (1.6), it is more convenient to work with the Dirac fermions

$$(1.17) \quad \chi_j := \tfrac{1}{\sqrt{2}}(\psi_{2j-1} + i\psi_{2j}), \quad \chi_j^* := \tfrac{1}{\sqrt{2}}(\psi_{2j-1} - i\psi_{2j}), \quad j \in \{1, 2\},$$

and their right-moving counterparts, all of which have coupled spin struc-
tures. Hence these Dirac fermions give rise to the affine symmetry

$$\text{(1.18)} \qquad \widehat{\mathfrak{so}}(8)_1 \supset \widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R} \cong \widehat{\mathfrak{su}}(2)_1^4 \,.$$

Details of the construction of the corresponding currents in terms of the four
Majorana fermions may be found in [GTVW14, §2]. This model enjoys ex-
tended left- and right-moving worldsheet supersymmetry. We choose a par-
ticular left- (resp. right-) moving $N = 4$ superconformal algebra at central
charge $c = 6$ (resp. $\bar{c} = 6$), which comes with an affine $\widehat{\mathfrak{su}}(2)_{1,L} \subset \widehat{\mathfrak{so}}(4)_{1,L}$
(resp. $\widehat{\mathfrak{su}}(2)_{1,R} \subset \widehat{\mathfrak{so}}(4)_{1,R}$) for $\widehat{\mathfrak{so}}(4)_{1,L}$ and $\widehat{\mathfrak{so}}(4)_{1,R}$ in (1.18). Our choice of
$U(1)$ currents

$$\text{(1.19)} \qquad J := {:}\chi_1\chi_1^*{:} + {:}\chi_2\chi_2^*{:}, \quad \overline{J} := {:}\overline{\chi}_1^*\overline{\chi}_1{:} + {:}\overline{\chi}_2^*\overline{\chi}_2{:},$$

whose zero modes generate the Cartan subalgebras of the above-mentioned
$\widehat{\mathfrak{su}}(2)_{1,L}$ and $\widehat{\mathfrak{su}}(2)_{1,R}$, is of particular importance in what follows.

Altogether, the total affine symmetry of the supersymmetric D_4-torus
model is

$$\widehat{\mathfrak{so}}(8)_1 \oplus \widehat{\mathfrak{so}}(16)_1 \supset (\widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(8)_{1,L}) \oplus (\widehat{\mathfrak{so}}(4)_{1,R} \oplus \widehat{\mathfrak{so}}(8)_{1,R}) \,.$$

The pair $\chi_k, \chi_k^*, \overline{\chi}_k, \overline{\chi}_k^*$, $k \in \{1, 2\}$, of two left- and two right-moving
Dirac fermions, all with coupled spin structures, gives rise to a *fermionic
CFT* at central charges $c = 2$, $\bar{c} = 2$, three copies of which suffice to give a
complete description of the supersymmetric D_4-torus model, as was done
in [GTVW14, §3] and shall be recalled shortly. As a preparation, we first
give a description of this fermionic CFT by means of toroidal momentum-
winding fields as in Section 1.1, along the lines of [GTVW14, Appendix D],
including the fermionic contributions. Though the fermionic CFT at central
charges $c = 2$, $\bar{c} = 2$ possesses neither worldsheet nor spacetime supersym-
metry, Neveu-Schwarz and Ramond sectors are well-defined by means of the
fermion boundary conditions. By the above, the supersymmetric D_4-torus
model is the tensor product of the bosonic D_4-torus model of Section 1.1
and this fermionic CFT.

For each $U(1)$ current $\mathfrak{j}$ in the $\widehat{\mathfrak{so}}(8)_1$ current algebra of (1.18), similarly
to (1.8), (1.10), we may introduce $\mathfrak{j} = i\partial\varphi$. Thus we *bosonize* by writing

$$\text{(1.20)} \qquad \mathfrak{j}_k := -i\,{:}\psi_{2k-1}\psi_{2k}{:} = i\partial\varphi_k, \quad \overline{\mathfrak{j}}_k := -i\,{:}\overline{\psi}_{2k-1}\overline{\psi}_{2k}{:} = i\overline{\partial}\overline{\varphi}_k$$
$$\text{for}\ \ k \in \{1, 2\},$$

and we recover

$$\chi_k(z) = :\exp(i\varphi_k(z)): c_k, \quad \chi_k^*(z) = :\exp(-i\varphi_k(z)): c_{-k},$$
$$i\overline{\chi}_k(\overline{z}) = :\exp(i\overline{\varphi}_k(\overline{z})): c_{k+2}, \quad i\overline{\chi}_k^*(\overline{z}) = :\exp(-i\overline{\varphi}_k(\overline{z})): c_{-(k+2)}.$$

Analogously to (1.13), all contributions from the free fermions χ_k, χ_k^*, $\overline{\chi}_k$, $\overline{\chi}_k^*$, $k \in \{1,2\}$, are now generated by fields $V_{(\boldsymbol{Q};\overline{\boldsymbol{Q}})}(z;\overline{z})$ as in (1.12), where $(\boldsymbol{Q};\overline{\boldsymbol{Q}}) \in \widetilde{\Gamma}_{2,2} \subset \mathbb{R}^{2,2}$, a lattice equipped with the symmetric bilinear form
• that was introduced in (1.2). This half integral lattice, which extends the charge lattice $\Gamma_{2,2}$ given by (1.1) for $d = 2$, is needed to accommodate fermionic fields in the same way as was presented in Subsection 1.1.2. One has

$$(1.21) \quad \widetilde{\Gamma}_{2,2} := \Gamma_{2,2} \cup ((0;\boldsymbol{e}_2) + \Gamma_{2,2})$$

$$= (\mathbb{Z}^2 \oplus \mathbb{Z}^2) \cup \left(\frac{1}{2} + \mathbb{Z}\right)^2 \times \left(\frac{1}{2} + \mathbb{Z}\right)^2 = \bigcup_{i=0}^{3}\left(\widetilde{\gamma}^{(i)} + \widetilde{\Gamma}_0\right) \text{ with}$$

$$\widetilde{\Gamma}_0 := \left\{(\boldsymbol{Q};\overline{\boldsymbol{Q}}) \in \mathbb{Z}^2 \oplus \mathbb{Z}^2 \;\Big|\; \sum_{k=1}^{2}(Q_k + \overline{Q}_k) \equiv 0 \mod 2\right\},$$

$$\widetilde{\gamma}^{(0)} := 0, \; \widetilde{\gamma}^{(1)} := (0;\boldsymbol{e}_2), \; \widetilde{\gamma}^{(2)} := \frac{1}{2}\sum_{k=1}^{2}(\boldsymbol{e}_k;\boldsymbol{e}_k), \; \widetilde{\gamma}^{(3)} := \widetilde{\gamma}^{(1)} + \widetilde{\gamma}^{(2)}.$$

The above charge lattice $\widetilde{\Gamma}_{2,2}$ with its sublattice $\widetilde{\Gamma}_0$ yields another example of a $\mathbb{Z}_2$ lattice pair $(\widetilde{\Gamma}_0, \widetilde{\Gamma}_{2,2})$ of the type used in Appendix B. Hence analogously to the discussion in Section 1.1.2, a general winding-momentum field creating a ground state $v_\gamma \in \mathcal{H}_\gamma$, $\gamma = (\boldsymbol{Q};\overline{\boldsymbol{Q}}) \in \widetilde{\Gamma}_{2,2}$, has the form (1.12),

$$V_\gamma(z,\overline{z}) := :\exp\left[i\sum_{k=1}^{2}Q_k\varphi_k(z) + i\sum_{k=1}^{2}\overline{Q}_k\overline{\varphi}_k(\overline{z})\right]: c_\gamma.$$

Consistent cocycles governing the cocycle factors c_γ, $\gamma \in \widetilde{\Gamma}_{2,2}$, with the additional symmetry and gauge requirements (B.5), (B.11), (B.12), are constructed in our Appendix B. We refer to the end of Section 1.1.2 for the justification of these requirements.

Since all spin structures in our pair χ_k, χ_k^*, $\overline{\chi}_k$, $\overline{\chi}_k^*$, $k \in \{1,2\}$, of two left- and two right-moving Dirac fermions are coupled, the space of states arising from the standard Fock space representations of these fermions decomposes into the contributions from the vacuum, vector, spinor and antispinor representations of $\widehat{\mathfrak{so}}(8)_1 \supset \widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R}$ labelled 0, v, s, c, above. Similarly

to the discussion in Section 1.1 and with notations as in (1.3), we collect these contributions in sectors $\mathcal{H}_\mathcal{S}$ with $\mathcal{S} \in \{0, v, s, c\}$ and find

$$
\begin{aligned}
\mathcal{H}_0 = \bigoplus_{\gamma \in \widetilde{\Gamma}_0} \mathcal{H}_\gamma, \quad &\mathcal{H}_v = \bigoplus_{\gamma \in \widetilde{\gamma}^{(1)} + \widetilde{\Gamma}_0} \mathcal{H}_\gamma, \\
\mathcal{H}_s = \bigoplus_{\gamma \in \widetilde{\gamma}^{(2)} + \widetilde{\Gamma}_0} \mathcal{H}_\gamma, \quad &\mathcal{H}_c = \bigoplus_{\gamma \in \widetilde{\gamma}^{(3)} + \widetilde{\Gamma}_0} \mathcal{H}_\gamma.
\end{aligned}
$$

(1.22)

The bosonic sector of this model is $\mathcal{H}_0 \oplus \mathcal{H}_s$, while $\mathcal{H}_v \oplus \mathcal{H}_c$ yields the fermions. The Neveu-Schwarz sector is $\mathcal{H}_0 \oplus \mathcal{H}_v$, while the Ramond sector is $\mathcal{H}_s \oplus \mathcal{H}_c$.

Generalizing the definition (1.17) of the Dirac fermions $\chi_\ell, \overline{\chi}_\ell$, to include $\ell \in \{3, \ldots, 6\}$, the sectors of the bosonic D_4-torus theory of Section 1.1 arise as

$$(1.23) \qquad \mathcal{H}_{L,\mathcal{S}} \otimes \mathcal{H}_{R,\mathcal{S}} \cong \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{S}, \quad \mathcal{S} \in \{0, v, s, c\},$$

where $\otimes$ denotes a fermionic tensor product, whenever needed. The sector $\mathcal{H}_{L,\mathcal{S}} \otimes \mathcal{H}_{R,\mathcal{S}}$ is governed by the lattice Γ_{D_4}, which yields the charge lattice with respect to the zero modes of $(j_1, \ldots, j_4; \overline{j}_1, \ldots, \overline{j}_4)$ where as in (1.8), $j_k = -i : \psi_{k+4} \psi_{k+8} :, k \in \{1, \ldots, 4\}$, and similarly for $\overline{j}_k$. Our choice of $U(1)$ currents on $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{S}$ is

$$
\begin{aligned}
(\mathsf{j}_3, \ldots, \mathsf{j}_6; \overline{\mathsf{j}}_3, \ldots, \overline{\mathsf{j}}_6) = (&-i : \psi_5 \psi_6 :, \ldots, -i : \psi_{11} \psi_{12} : ; \\
&-i : \overline{\psi}_5 \overline{\psi}_6 :, \ldots, -i : \overline{\psi}_{11} \overline{\psi}_{12} :),
\end{aligned}
$$

as in (1.20).

1.3. $\mathbb{Z}_2$-orbifold of the supersymmetric D_4-torus model

In order to obtain a K3 theory, we now consider a $\mathbb{Z}_2$-*orbifold* of the supersymmetric D_4-torus model. The group $\mathbb{Z}_2$ acts in the usual manner on the fields of the bosonic D_4-model, i.e. it maps $j_k(z) \mapsto -j_k(z)$, $\overline{j}_k(\overline{z}) \mapsto -\overline{j}_k(\overline{z})$, $k \in \{1, \ldots, 4\}$, and $V_\gamma \mapsto V_{-\gamma}$ for all $\gamma \in \Gamma_{4,4}$. This action is induced by the transformation that leaves $\psi_5(z), \ldots, \psi_8(z)$ invariant, while mapping $\psi_i(z) \mapsto -\psi_i(z)$ where $i \in \{9, \ldots, 12\}$, as can be checked by inspection of (1.8). In other words, we have $x_k(z) \leftrightarrow x_k^*(z)$, and analogously for the right-moving fermions. Note that the $\mathbb{Z}_2$-orbifold action on the eight Majorana fermions $\psi_i(z)$, $i \in \{5, \ldots 12\}$, and their anti-holomorphic counterparts, which before orbifolding had coupled spin structures as demanded by the

$\widehat{\mathfrak{so}}(16)_1$ symmetry, decouples the boundary conditions of the first four Majorana spinors from the last four. Therefore, the $\mathbb{Z}_2$-orbifold action breaks the $\widehat{\mathfrak{so}}(16)_1$ symmetry of the supersymmetric D_4-torus model to $\widehat{\mathfrak{so}}(8)_1 \oplus \widehat{\mathfrak{so}}(8)_1$.

On the fermions $\psi_k(z)$ and $\overline{\psi}_k(\overline{z})$ for $k \in \{1, \ldots, 4\}$, which are the supersymmetric partners of the $U(1)$ currents $j_k(z)$ and $\overline{\jmath}_k(\overline{z})$, the group $\mathbb{Z}_2$ acts as $\psi_k \mapsto -\psi_k$ and likewise for the right-movers, $\overline{\psi}_k \mapsto -\overline{\psi}_k$. In particular, the orbifold leaves the $\widehat{\mathfrak{so}}(8)_1$ algebra in (1.18) invariant, since it is generated by all bilinear fermion combinations, whose $\widehat{\mathfrak{so}}(4)_{1,L}$ currents are given in [GTVW14, (2.13)–(2.16)].

Altogether, the orbifold thus has an affine current algebra of type

$$\left(\widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R}\right)^3 \subset \widehat{\mathfrak{so}}(8)_1^3.$$

The untwisted sector of the $\mathbb{Z}_2$-orbifold is generated by the $\mathbb{Z}_2$-invariant $(h; \overline{h}) = (1; 0)$-fields with $\mathbb{C}$-basis

for $j < k$, $V_{(\boldsymbol{e}_j + \boldsymbol{e}_k; 0)}(z) + V_{(-\boldsymbol{e}_j - \boldsymbol{e}_k; 0)}(z)$, $V_{(\boldsymbol{e}_j - \boldsymbol{e}_k; 0)}(z) + V_{(-\boldsymbol{e}_j + \boldsymbol{e}_k; 0)}(z)$,

along with the $\mathbb{Z}_2$-invariant $(h; \overline{h}) = (\frac{1}{2}; \frac{1}{2})$-fields which are of the form $V_\gamma(z, \overline{z}) + V_{-\gamma}(z, \overline{z})$.

In the twisted sector, the twisted ground states of our $\mathbb{Z}_2$-orbifold amount to the Ramond ground states for pairs of free Dirac fermions χ_k, χ_k^*, $\overline{\chi}_k$, $\overline{\chi}_k^*$ with $k \in \{1, 2\}$, $k \in \{3, 4\}$ and $k \in \{5, 6\}$, respectively. Hence our K3 theory allows an elegant free fermion description with respect to the $(\widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R})^3$ current algebra (c.f. [GTVW14, §3.2 and Appendix D]) introduced above: the spin structures of left- and right-movers within each of the three summands $\widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R}$ are coupled; the contributions of each of these summands to the Neveu-Schwarz sector, according to [GTVW14, (C.3), (C.4)], are

(1.24) $(\mathrm{NS}, \mathrm{NS}, \mathrm{NS})$ $(\mathrm{NS}, \mathrm{R}, \mathrm{R})$ $(\mathrm{R}, \mathrm{NS}, \mathrm{R})$ $(\mathrm{R}, \mathrm{R}, \mathrm{NS})$,

and those to the Ramond-sector come from

(1.25) $(\mathrm{R}, \mathrm{NS}, \mathrm{NS})$ $(\mathrm{R}, \mathrm{R}, \mathrm{R})$ $(\mathrm{NS}, \mathrm{NS}, \mathrm{R})$ $(\mathrm{NS}, \mathrm{R}, \mathrm{NS})$.

In terms of the vacuum, vector, spinor and antispinor representations of $\widehat{\mathfrak{so}}(8)_1 \supset \widehat{\mathfrak{so}}(4)_{1,L} \oplus \widehat{\mathfrak{so}}(4)_{1,R}$ let us denote by $\mathcal{H}_{\mathcal{S}_1 \mathcal{S}_2 \mathcal{S}_3}$ with $\mathcal{S}_k \in \{0, v, s, c\}$ the threefold (fermionic) tensor product of the respective $\mathcal{H}_{S_k}$ of (1.22),

according to the three entries in each triplet of (1.24), (1.25). Then (1.24) means that the Neveu-Schwarz sector of the theory has the following bosonic and fermionic spaces of states:

$$(1.26) \qquad \begin{aligned} \mathcal{H}_{\text{bos}}^{\text{NS}} &= \mathcal{H}_{000} \oplus \mathcal{H}_{0ss} \oplus \mathcal{H}_{s0s} \oplus \mathcal{H}_{ss0}, \\ \mathcal{H}_{\text{ferm}}^{\text{NS}} &= \mathcal{H}_{vvv} \oplus \mathcal{H}_{vcc} \oplus \mathcal{H}_{cvc} \oplus \mathcal{H}_{ccv}. \end{aligned}$$

Analogously, by (1.25), the Ramond sector of the theory has the following bosonic and fermionic spaces of states:

$$(1.27) \qquad \begin{aligned} \mathcal{H}_{\text{bos}}^{\text{R}} &= \mathcal{H}_{s00} \oplus \mathcal{H}_{sss} \oplus \mathcal{H}_{00s} \oplus \mathcal{H}_{0s0}, \\ \mathcal{H}_{\text{ferm}}^{\text{R}} &= \mathcal{H}_{cvv} \oplus \mathcal{H}_{ccc} \oplus \mathcal{H}_{vvc} \oplus \mathcal{H}_{vcv}. \end{aligned}$$

This is in accord with [DMC16, (11.15), (11.16)].

The explicit form (1.15) of the OPE now confirms that OPEs are well-defined without square root cuts between any two fields corresponding to states in $\mathcal{H}_{\text{bos}}^{\text{NS}} \oplus \mathcal{H}_{\text{bos}}^{\text{R}}$, and without such cuts also between any two fields corresponding to states in $\mathcal{H}_{\text{bos}}^{\text{NS}} \oplus \mathcal{H}_{\text{ferm}}^{\text{NS}}$, as they should.

The charge lattice Γ governing this theory is most conveniently described as a sublattice

$$(1.28) \qquad \Gamma \subset \widetilde{\Gamma}_{2,2} \oplus \widetilde{\Gamma}_{2,2} \oplus \widetilde{\Gamma}_{2,2} \subset \mathbb{R}^{6,6},$$

with $\widetilde{\Gamma}_{2,2}$ as in (1.21), equipped with the symmetric bilinear form $\bullet$ that was introduced in (1.2). Here, each of the three identical summands $\widetilde{\Gamma}_{2,2}$ in the overlattice governs the charges of one of the sectors $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ in $\mathcal{H}_{\mathcal{S}_1 \mathcal{S}_2 \mathcal{S}_3}$.

Now from (1.26), (1.27) and recalling (1.22), one reads that the bosonic sector $\mathcal{H}_{\text{bos}}^{\text{NS}} \oplus \mathcal{H}_{\text{bos}}^{\text{R}}$ is governed by the even self-dual lattice

$$\begin{aligned} \Gamma_{\text{bos}} &:= \left(\widetilde{\Gamma}_0 \cup (\widetilde{\gamma}^{(2)} + \widetilde{\Gamma}_0) \right) \oplus \left(\widetilde{\Gamma}_0 \cup (\widetilde{\gamma}^{(2)} + \widetilde{\Gamma}_0) \right) \oplus \left(\widetilde{\Gamma}_0 \cup (\widetilde{\gamma}^{(2)} + \widetilde{\Gamma}_0) \right) \\ &\cong \Gamma_{2,2} \oplus \Gamma_{4,4} = \Gamma_{6,6} \end{aligned}$$

and thus agrees, as a bosonic conformal field theory, with the bosonic sector of the toroidal superconformal field theory on the standard torus $\mathbb{R}^4/\mathbb{Z}^4$ with vanishing B-field. This was in fact already shown in [NW01, Rem. 3.8]. Using the notations of (1.21) and in keeping with the decomposition (1.28) into contributions from the three summands $\widetilde{\Gamma}_{2,2}$, we set

$$\begin{aligned} \gamma^{(0)} &:= 0, \quad \gamma^{(1)} := \left(\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(1)} \right), \\ \gamma^{(2)} &:= \left(\widetilde{\gamma}^{(2)}, \widetilde{\gamma}^{(2)}, \widetilde{\gamma}^{(2)} \right), \quad \gamma^{(3)} := \gamma^{(1)} + \gamma^{(2)}, \end{aligned}$$

and find that the charge lattice of our K3 theory is half-integral,

$$\Gamma = \Gamma_{\text{bos}} \cup \left(\gamma^{(1)} + \Gamma_{\text{bos}} \right).$$

The lattice Γ meets all the assumptions on the lattice Γ of Appendix B, with $\Gamma_0 = \Gamma^*$. By our construction in Appendix B, we thus obtain well-defined cocycles obeying the additional symmetry and gauge requirements (B.5), (B.11), (B.12).

1.4. Partition function

The free fermion description given above is convenient in order to determine the partition function of the theory and – by means of the elliptic genus – to confirm that it is a K3 theory. In fact, by the results of [EOTY89], the usual $\mathbb{Z}_2$-orbifold of every supersymmetric $(d = 4)$-dimensional torus model has the elliptic genus of K3 and thus is indeed a *K3 theory* by definition, see [Wen15].

In the following, we calculate the various contributions to the partition function that can be read from (1.26), (1.27). We use the standard notations for Jacobi theta functions, which we also summarize in Appendix C for the reader's convenience.

By (1.26), the contributions to the partition function

$$Z_{\widetilde{\text{NS}}}(\tau, z) = \text{tr}_{\mathcal{H}_{\text{bos}}^{\text{NS}}} \left(y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - 1/4} \overline{q}^{\overline{L}_0 - 1/4} \right) - \text{tr}_{\mathcal{H}_{\text{ferm}}^{\text{NS}}} \left(y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - 1/4} \overline{q}^{\overline{L}_0 - 1/4} \right)$$

from the Neveu-Schwarz sector as defined in (A.3), in terms of the different ingredients to (1.24), are given by

$$(1.29) \qquad (\text{NS}, \text{NS}, \text{NS}): \quad \frac{1}{4} \left(\left| \frac{\vartheta_3(\tau)}{\eta(\tau)} \right|^8 + \left| \frac{\vartheta_4(\tau)}{\eta(\tau)} \right|^8 \right) \cdot \left| \frac{\vartheta_4(\tau, z)}{\eta(\tau)} \right|^4$$

$$+ \frac{1}{2} \left| \frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)} \right|^4 \cdot \left| \frac{\vartheta_3(\tau, z)}{\eta(\tau)} \right|^4$$

$$(1.30) \qquad (\text{NS}, \text{R}, \text{R}): \quad \frac{1}{4} \left| \frac{\vartheta_2(\tau)}{\eta(\tau)} \right|^8 \cdot \left| \frac{\vartheta_4(\tau, z)}{\eta(\tau)} \right|^4,$$

$$(1.31) \quad (\text{R}, \text{NS}, \text{R}) + (\text{R}, \text{R}, \text{NS}): \quad \frac{1}{2} \left| \frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)} \right|^4 \cdot \left| \frac{\vartheta_1(\tau, z)}{\eta(\tau)} \right|^4$$

$$+ \frac{1}{2} \left| \frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)} \right|^4 \cdot \left| \frac{\vartheta_2(\tau, z)}{\eta(\tau)} \right|^4.$$

Analogously, by (1.27), (1.25), the contributions to the partition function

$$Z_{\widetilde{R}}(\tau, z) = \mathrm{tr}_{\mathcal{H}^{\mathrm{R}}_{\mathrm{bos}}}\left(y^{J_0}\overline{y}^{\overline{J}_0}q^{L_0-1/4}\overline{q}^{\overline{L}_0-1/4}\right) - \mathrm{tr}_{\mathcal{H}^{\mathrm{R}}_{\mathrm{ferm}}}\left(y^{J_0}\overline{y}^{\overline{J}_0}q^{L_0-1/4}\overline{q}^{\overline{L}_0-1/4}\right)$$

from the Ramond sector are

$$(1.32) \qquad (\mathrm{R,NS,NS}): \quad \frac{1}{4}\left(\left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right|^8 + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right|^8\right)\cdot\left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4$$
$$+ \frac{1}{2}\left|\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_2(\tau,z)}{\eta(\tau)}\right|^4,$$

$$(1.33) \qquad (\mathrm{R,R,R}): \quad \frac{1}{4}\left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right|^8\cdot\left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4,$$

$$(1.34) \quad (\mathrm{NS,NS,R})+(\mathrm{NS,R,NS}): \quad \frac{1}{2}\left|\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_4(\tau,z)}{\eta(\tau)}\right|^4$$
$$+ \frac{1}{2}\left|\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^4.$$

Altogether, the four parts of the partition function of (A.3) are given by

$$Z_{\mathrm{NS}}(\tau, z) = \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^4\left|\frac{\vartheta_k(\tau)}{\eta(\tau)}\right|^8\cdot\left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_4(\tau,z)}{\eta(\tau)}\right|^4\right.$$
$$\left. + \left|\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_2(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4\right),$$

$$Z_{\widetilde{\mathrm{NS}}}(\tau, z) = \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^4\left|\frac{\vartheta_k(\tau)}{\eta(\tau)}\right|^8\cdot\left|\frac{\vartheta_4(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^4\right.$$
$$\left. + \left|\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_2(\tau,z)}{\eta(\tau)}\right|^4\right),$$

$$Z_{\mathrm{R}}(\tau, z) = \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^4\left|\frac{\vartheta_k(\tau)}{\eta(\tau)}\right|^8\cdot\left|\frac{\vartheta_2(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4\right.$$
$$\left. + \left|\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_4(\tau,z)}{\eta(\tau)}\right|^4\right),$$

$$Z_{\widetilde{\mathrm{R}}}(\tau, z) = \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^4\left|\frac{\vartheta_k(\tau)}{\eta(\tau)}\right|^8\cdot\left|\frac{\vartheta_1(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_2(\tau,z)}{\eta(\tau)}\right|^4\right.$$
$$\left. + \left|\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_4(\tau,z)}{\eta(\tau)}\right|^4 + \left|\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right|^4\cdot\left|\frac{\vartheta_3(\tau,z)}{\eta(\tau)}\right|^4\right).$$

2. The Conway Moonshine Module

In this section, we summarize Duncan's construction of the *Conway Moonshine Module*[9] $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ [Dun07, DMC15]. Section 2.1 closely follows the exposition in [DMC16, §6] but accompanies it by a description in terms of a lattice theory, while in Section 2.2, we include some additional structure that we need for comparison to the K3 theory of Section 1.

By [Dun07, Thm. 5.15], $V^{s\natural}$ is the unique self-dual, C_2-cofinite super vertex operator algebra of CFT type with central charge $c = 12$, such that for the Virasoro zero mode $L_{(0)}$ on $V^{s\natural}$, the kernel of $L_{(0)} - \frac{1}{2}\mathrm{id}_{V^{s\natural}}$ is trivial. Moreover, $V^{s\natural}_{\mathrm{tw}}$ is an irreducible canonically twisted $V^{s\natural}$-supermodule, and as such, it is unique according to [DMC15, §4].

2.1. The construction of $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$

Both $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ are obtained using a standard construction [FFR91, DMC15] that attaches a super vertex operator algebra $A(\mathfrak{a})$ and a canonically twisted module $A(\mathfrak{a})_{\mathrm{tw}}$ for it to any finite dimensional complex vector space $\mathfrak{a}$ equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$. For later convenience, we will always assume that the dimension of $\mathfrak{a}$ is even. Moreover, by a slight abuse of terminology, we will call a family $(v_1, \ldots, v_k)$ of elements of $\mathfrak{a}$ *orthonormal*, iff for all $i, j \in \{1, \ldots, k\}$, we have $(v_i, v_j) = \delta_{ij}$.

For every $n \in \mathbb{Z}$, one now introduces a copy $\mathfrak{a}_{(n+1/2)} \cong \mathfrak{a}$ and sets

$$\widehat{\mathfrak{a}}^- := \bigoplus_{n<0} \mathfrak{a}_{(n+1/2)}, \quad A(\mathfrak{a}) := \bigwedge(\widehat{\mathfrak{a}}^-)\Omega \cong \bigwedge(\widehat{\mathfrak{a}}^-),$$

where Ω denotes a choice of a *vacuum state*, such that in particular, for $w \in \bigwedge(\widehat{\mathfrak{a}}^-)$, $w(\Omega) := w\Omega$ yields the isomorphism $A(\mathfrak{a}) \cong \bigwedge(\widehat{\mathfrak{a}}^-)$. The construction of the standard super vertex algebra on the vector space $A(\mathfrak{a})$ involves a choice of isomorphism $\mathfrak{a} \longrightarrow \mathfrak{a}_{(n+1/2)}$ for every $n \in \mathbb{Z}$, denoted[10]

---

[9]For the relevant definitions concerning super vertex operator algebras and their properties, we refer the reader to the literature, see e.g. [Kac98, FBZ04, LL04], as well as the very accessible summary in [DMC16, §5].

[10]As a warning to the bilingual reader we remark that our $v_{(\nu)}$ are denoted $v(\nu)$ in the vertex algebra literature, while their $v_{(\nu)}$ relate to our $v_{(\nu)}$ by a weight-dependent shift of ν.

$v \mapsto v_{(n+1/2)}$. The $v_{(n+1/2)}$ are regarded as linear maps on $A(\mathfrak{a})$, acting by left multiplication if $n < 0$, and obeying $v_{(n+1/2)}\Omega = 0$ for all $v \in \mathfrak{a}$ and $n \in \mathbb{N}$ as well as the Clifford algebra[11]

(2.1) $\forall v, \varphi \in \mathfrak{a}, \ \forall m, n \in \mathbb{Z}$:
$$v_{(n+1/2)}\varphi_{(m+1/2)} + \varphi_{(m+1/2)}v_{(n+1/2)} = \delta_{m+n+1,0} \cdot (v, \varphi).$$

This uniquely fixes the action of each $v_{(n+1/2)}$ on $A(\mathfrak{a})$ and by [FBZ04, Thm. 4.4.1] extends uniquely to a super vertex algebra structure on $A(\mathfrak{a})$. In physics terminology, the field associated to $v \in \mathfrak{a}$ is a free Majorana fermion. By the standard Sugawara construction [Sug68, Som68], applied to a maximal set of $U(1)$ currents with pairwise trivial OPEs (c.f. [FK81, Seg81, GO86], for example), $A(\mathfrak{a})$ enjoys the action of a Virasoro algebra at central charge $c = \frac{1}{2} \dim \mathfrak{a}$, promoting $A(\mathfrak{a})$ to a super vertex *operator* algebra. The standard modes generating this Virasoro algebra are denoted $L_{(n)}$, $n \in \mathbb{Z}$, in the following.

The canonically twisted module of $A(\mathfrak{a})$ is similarly obtained by introducing a copy $\mathfrak{a}_{(n)} \cong \mathfrak{a}$ and a choice of $\mathbb{C}$ vector space isomorphism $\mathfrak{a} \longrightarrow \mathfrak{a}_{(n)}$, $v \mapsto v_{(n)}$ for every $n \in \mathbb{Z}$. In addition, one chooses a polarization $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ for $\mathfrak{a}$ with respect to $(\cdot, \cdot)$. Let $(\mathfrak{a}^-)_{(0)}$ denote the image of $\mathfrak{a}^-$ under $\mathfrak{a} \longrightarrow \mathfrak{a}_{(0)}$, $v \mapsto v_{(0)}$. One then sets

$$\widehat{\mathfrak{a}}_{\mathrm{tw}}^- := (\mathfrak{a}^-)_{(0)} \oplus \bigoplus_{n<0} \mathfrak{a}_{(n)}, \quad A(\mathfrak{a})_{\mathrm{tw}} := \bigwedge(\widehat{\mathfrak{a}}_{\mathrm{tw}}^-)\Omega_{\mathrm{tw}} \cong \bigwedge(\widehat{\mathfrak{a}}_{\mathrm{tw}}^-),$$

where Ω_{tw} is a choice of a *twisted ground state*, and similarly to the above, for $w \in \bigwedge(\widehat{\mathfrak{a}}_{\mathrm{tw}}^-)$, $w(\Omega_{\mathrm{tw}}) := w\Omega_{\mathrm{tw}}$. As above, the $v_{(n)}$ are regarded as linear maps on $A(\mathfrak{a})_{\mathrm{tw}}$, acting by left multiplication if $n < 0$, and obeying $v_{(n)}\Omega_{\mathrm{tw}} = 0$ if $n > 0$, or $n = 0$ and $v \in \mathfrak{a}^+$, as well as the Clifford algebra (2.1) in its incarnation

$$\forall v, \varphi \in \mathfrak{a}, \ \forall m, n \in \mathbb{Z}: \quad v_{(n)}\varphi_{(m)} + \varphi_{(m)}v_{(n)} = \delta_{m+n,0} \cdot (v, \varphi).$$

As for $A(\mathfrak{a})$, this uniquely fixes the action of each $v_{(n)}$ on $A(\mathfrak{a})_{\mathrm{tw}}$. According to [FS04, §2.2], this extends uniquely to a canonically twisted $A(\mathfrak{a})$-module structure on $A(\mathfrak{a})_{\mathrm{tw}}$.

---

[11]Note that the normalization chosen by Duncan and Mack-Crane in [DMC15, DMC16] differs from ours and [FFR91, (2.41)] by a factor of -2 on the right hand side of (2.1).

The above construction ensures a natural action of the standard Clifford algebra $\text{Cliff}(\mathfrak{a})$ associated to $(\mathfrak{a}, (\cdot, \cdot))$ on $A(\mathfrak{a})_{\text{tw}}$, such that an element represented within the tensor algebra of $\mathfrak{a}$ by $v_1 \otimes \cdots \otimes v_k$ with $v_1, \ldots, v_k \in \mathfrak{a}$, $k \in \mathbb{N}$, acts by $(v_1)_{(0)} \circ \cdots \circ (v_k)_{(0)}$. The $\text{Cliff}(\mathfrak{a})$-submodule $\boldsymbol{CM}$ of $A(\mathfrak{a})_{\text{tw}}$ generated by Ω_{tw} is the unique (up to isomorphism) non-trivial irreducible representation of $\text{Cliff}(\mathfrak{a})$ [DMC16, (6.16)]. In addition, one chooses a *fermion number operator* $(-1)^F$ on $A(\mathfrak{a})$ and on $A(\mathfrak{a})_{\text{tw}}$, where

$$(-1)^F v_{(k)} + v_{(k)}(-1)^F = 0 \quad \forall v \in \mathfrak{a}, \ k \in \frac{1}{2}\mathbb{Z},$$
$$(-1)^F \Omega = \Omega, \quad (-1)^F \Omega_{\text{tw}} = \Omega_{\text{tw}}.$$

The algebra generated by the $v_{(0)}$ with $v \in \mathfrak{a}$ together with $(-1)^F$, in the physics literature is known as the *fermionic zero mode algebra*. In [DMC16], $(-1)^F$ is obtained by choosing a lift of $-\text{id}_{\mathfrak{a}} \in \text{SO}(\mathfrak{a})$ to $\text{Spin}(\mathfrak{a})$ which is compatible with the polarization $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ of $\mathfrak{a}$. The fermion number operator $(-1)^F$ induces a $\mathbb{Z}_2$-grading on $A(\mathfrak{a})$ and on $A(\mathfrak{a})_{\text{tw}}$, such that

$$A(\mathfrak{a}) = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1, \quad A(\mathfrak{a})_{\text{tw}} = A(\mathfrak{a})^0_{\text{tw}} \oplus A(\mathfrak{a})^1_{\text{tw}},$$

where $A(\mathfrak{a})^j$ and $A(\mathfrak{a})^j_{\text{tw}}$, with $j \in \{0, 1\}$, are the $(-1)^j$ eigenspaces of $(-1)^F$ on $A(\mathfrak{a})$, $A(\mathfrak{a})_{\text{tw}}$.

The super vertex operator algebra $V^{s\natural}$ and its canonically twisted module $V^{s\natural}_{\text{tw}}$ are now obtained from $\mathfrak{a} \cong \mathbb{C}^{24}$ with the standard bilinear form $(\cdot, \cdot)$ as

$$V^{s\natural} := A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{\text{tw}}, \quad V^{s\natural}_{\text{tw}} := A(\mathfrak{a})^0_{\text{tw}} \oplus A(\mathfrak{a})^1,$$

where according to [DMC15] (c.f. [DMC16, Prop. 8.1]), the $A(\mathfrak{a})^0$-module structure of $V^{s\natural}$ extends uniquely to a super vertex operator algebra structure on $V^{s\natural}$, and the $A(\mathfrak{a})^0$-module structure of $V^{s\natural}_{\text{tw}}$ extends uniquely to a canonically twisted $V^{s\natural}$-module structure. As mentioned above, by [Dun07, Thm. 5.15], $V^{s\natural}$ is the unique self-dual, C_2-cofinite super vertex operator algebra with central charge $c = 12$ and trivial $\ker\left(L_{(0)} - \frac{1}{2}\text{id}_{V^{s\natural}}\right)$. We call the subspace $V^{s\natural}$, equipped with its structure as a super vertex operator algebra, the *Neveu-Schwarz sector* of the Conway Moonshine Module, and $V^{s\natural}_{\text{tw}}$ its *Ramond sector*.

In physics terminology, every $v \in A(\mathfrak{a})^j$ or $v \in A(\mathfrak{a})^j_{\text{tw}}$, $j \in \{0, 1\}$, is a state in a free fermion theory (see, e.g., [DW09] for a systematic description of free fermion theories in the context of heterotic strings on Calabi-Yau three-folds), obtained from 24 free Majorana fermions with coupled spin structures. Analogously to the analysis of Section 1.3, any decomposition

$\mathfrak{a} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3$ with $\dim \mathfrak{b}_k = 8$ for each $k \in \{1, 2, 3\}$ allows a description of the contributions to $A(\mathfrak{a})^j$ and $A(\mathfrak{a})^j_{\text{tw}}$ in terms of threefold (fermionic) tensor products $U_{\mathcal{S}_1 \mathcal{S}_2 \mathcal{S}_3}$, where $\mathcal{S}_k \in \{0, v, s, c\}$ for $k \in \{1, 2, 3\}$ labels the vacuum, vector, spinor or antispinor representation of the affine algebra $\widehat{\mathfrak{so}}(8)_1$ corresponding to $\mathfrak{b}_k$:

$$
\begin{aligned}
V_{\text{bos}}^{s\natural} &:= A(\mathfrak{a})^0 \\
&= (A(\mathfrak{b}_1) \wedge A(\mathfrak{b}_2) \wedge A(\mathfrak{b}_3))^0 \\
&= U_{000} \oplus U_{0vv} \oplus U_{v0v} \oplus U_{vv0}, \\
V_{\text{ferm}}^{s\natural} &:= A(\mathfrak{a})^1_{\text{tw}} \\
&= (A(\mathfrak{b}_1)_{\text{tw}} \wedge A(\mathfrak{b}_2)_{\text{tw}} \wedge A(\mathfrak{b}_3)_{\text{tw}})^1 \\
&= U_{ccc} \oplus U_{css} \oplus U_{scs} \oplus U_{ssc}, \\
V_{\text{tw,bos}}^{s\natural} &:= A(\mathfrak{a})^0_{\text{tw}} \\
&= (A(\mathfrak{b}_1)_{\text{tw}} \wedge A(\mathfrak{b}_2)_{\text{tw}} \wedge A(\mathfrak{b}_3)_{\text{tw}})^0 \\
&= U_{scc} \oplus U_{sss} \oplus U_{ccs} \oplus U_{csc}, \\
V_{\text{tw,ferm}}^{s\natural} &:= A(\mathfrak{a})^1 \\
&= (A(\mathfrak{b}_1) \wedge A(\mathfrak{b}_2) \wedge A(\mathfrak{b}_3))^1 \\
&= U_{v00} \oplus U_{vvv} \oplus U_{00v} \oplus U_{0v0},
\end{aligned}
$$

(2.2)

c.f. [DMC16, (11.20),(11.21)].

This also shows that the structure of $V^{s\natural} \oplus V_{\text{tw}}^{s\natural}$ can be conveniently encoded in terms of a lattice vertex operator algebra, by bosonization. Indeed, analogously to the discussion in Section 1.3, and using the notations introduced there, we find

$$
V^{s\natural} \oplus V_{\text{tw}}^{s\natural} = \bigoplus_{\gamma \in \Gamma^{\text{refl}}} \mathcal{H}_\gamma,
$$

where the relevant charge lattice is

$$
\Gamma^{\text{refl}} := \mathbb{Z}^{12} \cup (\tfrac{1}{2} + \mathbb{Z})^{12} \subset \mathbb{R}^{12},
$$

equipped with the Euclidean scalar product. For later convenience we remark that

$$
\Gamma^{\text{refl}} \subset \widetilde{\Gamma}_{2,2}^{\text{refl}} \oplus \widetilde{\Gamma}_{2,2}^{\text{refl}} \oplus \widetilde{\Gamma}_{2,2}^{\text{refl}} \subset \mathbb{R}^{12},
$$

where

$$
\widetilde{\Gamma}_{2,2}^{\text{refl}} := \mathbb{Z}^4 \cup (\tfrac{1}{2} + \mathbb{Z})^4 \subset \mathbb{R}^4.
$$

More precisely, with $\widetilde{\Gamma}_0^{\text{refl}} := \{\mathbf{Q} \in \mathbb{Z}^4 | \sum_{k=1}^4 Q_k \equiv 0 \bmod 2\}$, an index 4 sublattice of $\widetilde{\Gamma}_{2,2}^{\text{refl}} \subset \mathbb{R}^4$, and the 4-vectors

$$(2.3) \quad \widetilde{\gamma}^{(0)} := 0, \quad \widetilde{\gamma}^{(1)} := \boldsymbol{e}_4, \quad \widetilde{\gamma}^{(2)} := \tfrac{1}{2} \sum_{k=1}^4 (\boldsymbol{e}_k, \boldsymbol{e}_k), \quad \widetilde{\gamma}^{(3)} := \widetilde{\gamma}^{(1)} + \widetilde{\gamma}^{(2)},$$

the cosets are

$$\forall a \in \{0, \ldots, 3\}: \quad \widetilde{\Gamma}_a^{\text{refl}} := \widetilde{\gamma}^{(a)} + \widetilde{\Gamma}_0^{\text{refl}}, \quad \text{hence} \quad \widetilde{\Gamma}_{2,2}^{\text{refl}} = \bigcup_{a=0}^3 \widetilde{\Gamma}_a^{\text{refl}}.$$

One has

$$\Gamma^{\text{refl}} = \Gamma_{\text{bos}}^{\text{refl}} \cup \Gamma_{\text{ferm}}^{\text{refl}}$$

with

$$(2.4) \quad \begin{aligned} \Gamma_{\text{bos}}^{\text{refl}} &:= \left(\widetilde{\Gamma}_0^{\text{refl}} \cup \widetilde{\Gamma}_1^{\text{refl}}\right) \oplus \left(\widetilde{\Gamma}_0^{\text{refl}} \cup \widetilde{\Gamma}_1^{\text{refl}}\right) \oplus \left(\widetilde{\Gamma}_0^{\text{refl}} \cup \widetilde{\Gamma}_1^{\text{refl}}\right) = \mathbb{Z}^{12}, \\ \Gamma_{\text{ferm}}^{\text{refl}} &:= \left(\widetilde{\Gamma}_2^{\text{refl}} \cup \widetilde{\Gamma}_3^{\text{refl}}\right) \times \left(\widetilde{\Gamma}_2^{\text{refl}} \cup \widetilde{\Gamma}_3^{\text{refl}}\right) \times \left(\widetilde{\Gamma}_2^{\text{refl}} \cup \widetilde{\Gamma}_3^{\text{refl}}\right) = (\tfrac{1}{2} + \mathbb{Z})^{12}. \end{aligned}$$

Note that

$$V_{\text{bos}}^{s\natural} \oplus V_{\text{tw,ferm}}^{s\natural} = \bigoplus_{\gamma \in \Gamma_{\text{bos}}^{\text{refl}}} \mathcal{H}_\gamma, \quad V_{\text{ferm}}^{s\natural} \oplus V_{\text{tw,bos}}^{s\natural} = \bigoplus_{\gamma \in \Gamma_{\text{ferm}}^{\text{refl}}} \mathcal{H}_\gamma;$$

our counterintuitive choice of notations will be justified in Section 4.1.

To introduce cocycles, we may again invoke the results of Appendix B, since $\Gamma_0^{\text{refl}} := \left(\Gamma^{\text{refl}}\right)^*$ is an even sublattice of index 4 in the half integral lattice Γ^{refl}. It is given by

$$\Gamma_0^{\text{refl}} = \left\{ \mathbf{Q} \in \mathbb{Z}^{12} \,\bigg|\, \sum_{k=1}^{12} Q_k \equiv 0 \mod 2 \right\}.$$

With

$$\begin{aligned} \gamma^{\text{refl}(0)} &:= 0, & \gamma^{\text{refl}(1)} &:= (\widetilde{\gamma}^{(3)}, \widetilde{\gamma}^{(3)}, \widetilde{\gamma}^{(3)}), \\ \gamma^{\text{refl}(2)} &:= (\widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(1)}, \widetilde{\gamma}^{(1)}) & \text{and} \quad \gamma^{\text{refl}(3)} &:= \gamma^{\text{refl}(1)} + \gamma^{\text{refl}(2)}, \end{aligned}$$

the four cosets are

$$\Gamma_a^{\text{refl}} = \gamma^{\text{refl}(a)} + \Gamma_0^{\text{refl}} \quad \forall a \in \{0, 1, 2, 3\}.$$

The two lattices $(\Gamma_0^{\mathrm{refl}}, \Gamma^{\mathrm{refl}})$ form a $\mathbb{Z}_2$ lattice pair in the terminology of [GNO$^+$87]. In this context one may associate the two Lie algebras D_{12} and B_{12} to this lattice pair. Indeed, the set of 264 vectors of length square 2 in Γ^{refl} form a root system of type D_{12}, which together with the set of 24 vectors of length square 1 in Γ^{refl} form a root system of type B_{12}. In fact, Γ_0^{refl} is a root lattice of type D_{12}, while Γ^{refl} is a root lattice of type B_{12}. We obtain well-defined cocycles $\varepsilon^{\mathrm{refl}}$ on Γ^{refl} from the construction summarized in Appendix B, and they obey the additional symmetry and gauge requirements (B.5), (B.11), (B.12).

By construction, $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$ enjoys a natural action of $\mathrm{Spin}(\mathfrak{a})$ which respects the super vertex operator algebra and twisted module structures and which on $A(\mathfrak{a})$ factors over $\mathrm{SO}(\mathfrak{a})$. Now let Λ denote the Leech lattice and $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{24}$ with the standard bilinear form $(\cdot, \cdot)$, such that $\mathrm{Co}_0 = \mathrm{Aut}(\Lambda) \subset \mathrm{SO}(\mathfrak{a})$. Then by [DMC15] (c.f. [DMC16, Prop. 7.1]), there is a unique lift $\widehat{\mathrm{Co}}_0$, i.e. a subgroup $\widehat{\mathrm{Co}}_0 \subset \mathrm{Spin}(\mathfrak{a})$ such that the natural map $\mathrm{Spin}(\mathfrak{a}) \to \mathrm{SO}(\mathfrak{a})$ induces an isomorphism $\widehat{\mathrm{Co}}_0 \cong \mathrm{Co}_0$, thus yielding a Co_0-action on the super vertex operator algebra $V^{s\natural}$ along with its canonically twisted module $V_{\mathrm{tw}}^{s\natural}$. Without loss of generality, one assumes $(-1)^F$ to yield the non-trivial central element of $\widehat{\mathrm{Co}}_0$ by modifying the polarization of $\mathfrak{a}$ accordingly if need be.

2.2. Choosing $U(1)$ currents

According to [DMC16], the choice of an appropriate $U(1)$ current for the Conway Moonshine Module allows to attach a weak Jacobi form to any symplectic derived equivalence of a K3 surface that fixes a suitable stability condition on K3. Following [DMC16, (9.5)], for $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$ we choose a $U(1)$ current J with zero mode J_0, by distinguishing four free Majorana fermions which are associated to an orthonormal basis of a four-dimensional subspace $\mathfrak{x}$ of $\mathfrak{a}$. Analogously to (1.6), these are combined to form Dirac fermions $\mathbf{a}_X^{\pm}$ and $\mathbf{a}_Z^{\pm}$, such that insertion into the bilinear form $(\cdot, \cdot)$ on $\mathfrak{a}$ yields $(\mathbf{a}_X^{\pm}, \mathbf{a}_X^{\mp}) = (\mathbf{a}_Z^{\pm}, \mathbf{a}_Z^{\mp}) = 1$, while inserting any other combination of $\mathbf{a}_X^{\pm}, \mathbf{a}_Z^{\pm}$ in $(\cdot, \cdot)$ yields zero. Then[12]

$$(2.5) \qquad\qquad J := \, :\mathbf{a}_X^+ \mathbf{a}_X^-: + :\mathbf{a}_Z^+ \mathbf{a}_Z^-: \, .$$

---

[12]Note that our formula differs from that given in [DMC16, (9.5)] by a factor of -2 due to the difference in the normalization of the Clifford algebra (2.1).

This introduces charges ± 1 for $\mathbf{a}_X^\pm$ and $\mathbf{a}_Z^\pm$ with respect to the zero mode J_0 of the associated field $J(z)$, while all fermions corresponding to states in $\mathfrak{a}$ that are perpendicular to $\mathfrak{r}$ remain uncharged. Below, we will see that this choice is compatible with the identification of $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ with the space of states underlying the K3 theory of Section 1, such that J is mapped to our choice (1.19) of $U(1)$ current in the left-moving $N = 4$ superconformal algebra of [GTVW14]. Since in that model, we also naturally have a right-moving $N = 4$ superconformal algebra, in addition to the data given in [DMC16], we need to determine another $U(1)$ current, to serve as the image of a choice of right-moving $U(1)$ current under the reflection procedure to be described in Section 3. The most natural candidate $\widehat{J}$ seems to arise by choosing another, disjoint set of four free fermions associated to an orthonormal basis of a four-dimensional subspace $\widehat{\mathfrak{r}}$ of $\mathfrak{a}$, which is perpendicular to the four-dimensional subspace $\mathfrak{r}$. Indeed at first sight, the structure of $(J, \widehat{J})$ is analogous to the one observed in (1.19) for the left- and right-moving $U(1)$ currents in the K3 theory. However, that we should have to make additional choices is counterintuitive. Below we will see that consistency with the identifications of [DMC16] actually requires to introduce instead, alongside the $U(1)$ current J, a second $U(1)$ current $\overline{J}$ with zero mode $\overline{J}_0$, by setting

$$(2.6) \qquad \overline{J} := :\mathbf{a}_X^+ \mathbf{a}_X^-: - :\mathbf{a}_Z^+ \mathbf{a}_Z^-: .$$

This introduces charges ± 1 for $\mathbf{a}_X^\pm$ and $\mathbf{a}_Z^\mp$ with respect to $\overline{J}_0$, while all other fermions remain uncharged.

In [DMC16], the choice of the four-dimensional subspace $\mathfrak{r}$ is interpreted in terms of the choice of a complex structure along with a stability condition on an algebraic K3 surface following [Huy06, Huy14, Huy16]. However, in all of these references this structure is only used to attach a new label to the refined geometric interpretations (c.f. [Wen06]) of the points in the moduli space of SCFTs on K3, following [AM94, NW01]. Indeed, even the subdivision of the four-dimensional space into two two-dimensional subspaces is never relevant in the work of [DMC16], other than yielding the interpretation in terms of stability conditions. The latter introduces the very unnatural restriction to algebraic K3 surfaces, which is unnecessary in the original interpretation of the moduli space and its refinements [AM94, NW01].

The above-mentioned observation that the natural $U(1)$ charges for a choice of orthonormal basis of the four-dimensional subspace $\mathfrak{r} \subset \mathfrak{a}$ turn out to be $\pm(1, 1)$ and $\pm(1, -1)$ is in accord with the observation in [NW01] that this four-dimensional subspace corresponds to a choice of *four charged Ramond ground states* of the K3 theory, with $U(1)$ charges $\pm(1, 1)$, $\pm(1, -1)$.

Indeed, under the state-field correspondence, the fields associated to $\mathbf{a}_X^-$, $\mathbf{a}_X^+ \in \mathfrak{a}$ create the states $(\mathbf{a}_X^-)_{(-1/2)}\Omega$ and $(\mathbf{a}_X^+)_{(-1/2)}\Omega$, both in $A(\mathfrak{a})^1 \subset V_{\mathrm{tw}}^{s\natural}$, which according to [DMC16, §8] are *Ramond states* in the Conway Moonshine Module. In fact, since the Ramond sector of the Conway Moonshine Module is $V_{\mathrm{tw}}^{s\natural} = A(\mathfrak{a})_{\mathrm{tw}}^0 \oplus A(\mathfrak{a})^1$, these states actually are Ramond *ground* states.

While by the above, the choice of the $U(1)$ current J is crucial to the main results of [DMC16], the charges with respect to its zero mode are only given for the twining elliptic genera, there. Let us determine this information, along with the charges with respect to $\overline{J}$, for all parts of the partition function. For notational convenience, we introduce

$$
\begin{aligned}
B(\tau, z, \zeta) &:= \frac{1}{2}\left(\frac{\vartheta_3(\tau, z)^2\vartheta_3(\tau, \zeta)^2}{\eta(\tau)^4} + \frac{\vartheta_4(\tau, z)^2\vartheta_4(\tau, \zeta)^2}{\eta(\tau)^4}\right) \\
&\overset{(\mathrm{C.13})}{=} \frac{1}{2}\left(\frac{\vartheta_3(\tau, z+\zeta)\vartheta_3(\tau, z-\zeta)\vartheta_3(\tau)^2}{\eta(\tau)^4} \right. \\
&\qquad\quad \left. + \frac{\vartheta_4(\tau, z+\zeta)\vartheta_4(\tau, z-\zeta)\vartheta_4(\tau)^2}{\eta(\tau)^4}\right), \\
F(\tau, z, \zeta) &:= \frac{1}{2}\left(\frac{\vartheta_3(\tau, z)^2\vartheta_3(\tau, \zeta)^2}{\eta(\tau)^4} - \frac{\vartheta_4(\tau, z)^2\vartheta_4(\tau, \zeta)^2}{\eta(\tau)^4}\right), \\
\widehat{F}(\tau, z, \zeta) &:= \frac{1}{2}\left(\frac{\vartheta_3(\tau, z+\zeta)\vartheta_3(\tau, z-\zeta)\vartheta_3(\tau)^2}{\eta(\tau)^4} \right. \\
&\qquad\quad \left. - \frac{\vartheta_4(\tau, z+\zeta)\vartheta_4(\tau, z-\zeta)\vartheta_4(\tau)^2}{\eta(\tau)^4}\right),
\end{aligned}
$$

(2.7)

with shorthand notations $B(\tau) := B(\tau, 0, 0)$, $F(\tau) := F(\tau, 0, 0)$. We also set $\widetilde{y} := e^{2\pi i \zeta}$ for $\zeta \in \mathbb{C}$. From (2.2), we then find that the bosons in $V^{s\natural}$ are counted by

$$
\begin{aligned}
(2.8) \quad Z_{\mathrm{NS}^0}^{\mathrm{DM-C}}(\tau, z, \zeta) &:= \mathrm{tr}_{A(\mathfrak{a})^0}\left(y^{J_0}\widetilde{y}^{\overline{J}_0}q^{L_{(0)}-1/2}\right) \\
&= \frac{1}{2}\left(\frac{\vartheta_3(\tau, z+\zeta)\vartheta_3(\tau, z-\zeta)}{\eta(\tau)^2} \cdot \frac{\vartheta_3(\tau)^{10}}{\eta(\tau)^{10}}\right. \\
&\qquad\quad \left. + \frac{\vartheta_4(\tau, z+\zeta)\vartheta_4(\tau, z-\zeta)}{\eta(\tau)^2} \cdot \frac{\vartheta_4(\tau)^{10}}{\eta(\tau)^{10}}\right) \\
&= B(\tau, z, \zeta)\left(B(\tau)^2 + F(\tau)^2\right) + \widehat{F}(\tau, z, \zeta) \cdot 2B(\tau)F(\tau).
\end{aligned}
$$

The fermions in $V^{s\natural}$ are counted by

$$(2.9) \qquad Z_{\mathrm{NS}^1}^{\mathrm{DM-C}}(\tau, z, \zeta) := \mathrm{tr}_{A(\mathfrak{a})_{\mathrm{tw}}^1} \left(y^{J_0} \widetilde{y}^{\overline{J}_0} q^{L_{(0)} - 1/2} \right)$$

$$= \frac{1}{2} \cdot \frac{\vartheta_2(\tau, z+\zeta)\vartheta_2(\tau, z-\zeta)}{\eta(\tau)^2} \cdot \left(\frac{\vartheta_2(\tau)}{\eta(\tau)} \right)^{10}$$

$$\overset{\mathrm{(C.3),(C.15)}}{=} F(\tau, z, \zeta) \cdot 4F(\tau)^2.$$

Similarly, the bosons in $V_{\mathrm{tw}}^{s\natural}$ are counted by

$$(2.10) \qquad Z_{\mathrm{R}^1}^{\mathrm{DM-C}}(\tau, z, \zeta) := \mathrm{tr}_{A(\mathfrak{a})_{tw}^0} \left(y^{J_0} \widetilde{y}^{\overline{J}_0} q^{L_{(0)} - 1/2} \right)$$

$$= \frac{1}{2} \cdot \frac{\vartheta_2(\tau, z+\zeta)\vartheta_2(\tau, z-\zeta)}{\eta(\tau)^2} \cdot \left(\frac{\vartheta_2(\tau)}{\eta(\tau)} \right)^{10}$$

$$\overset{\mathrm{(C.3),(C.15)}}{=} F(\tau, z, \zeta) \cdot 4F(\tau)^2.$$

The fermions in $V_{\mathrm{tw}}^{s\natural}$ are counted by

$$(2.11) \; Z_{\mathrm{R}^0}^{\mathrm{DM-C}}(\tau, z, \zeta) := \mathrm{tr}_{A(\mathfrak{a})^1} \left(y^{J_0} \widetilde{y}^{\overline{J}_0} q^{L_{(0)} - 1/2} \right)$$

$$= \frac{1}{2} \left(\frac{\vartheta_3(\tau, z+\zeta)\vartheta_3(\tau, z-\zeta)}{\eta(\tau)^2} \cdot \frac{\vartheta_3(\tau)^{10}}{\eta(\tau)^{10}} \right.$$

$$\left. - \frac{\vartheta_4(\tau, z+\zeta)\vartheta_4(\tau, z-\zeta)}{\eta(\tau)^2} \cdot \frac{\vartheta_4(\tau)^{10}}{\eta(\tau)^{10}} \right)$$

$$= B(\tau, z, \zeta) \cdot 2B(\tau)F(\tau) + \widehat{F}(\tau, z, \zeta) \left(B(\tau)^2 + F(\tau)^2 \right).$$

3. Reflecting right-moving degrees of freedom

Since it enjoys an action of a left- and a right moving Virasoro algebra at central charges $c = \bar{c} = 6$, the space of states

$$\mathcal{H}^{\mathrm{GTVW}} := \mathcal{H}_{\mathrm{bos}}^{\mathrm{NS}} \oplus \mathcal{H}_{\mathrm{ferm}}^{\mathrm{NS}} \oplus \mathcal{H}_{\mathrm{bos}}^{\mathrm{R}} \oplus \mathcal{H}_{\mathrm{ferm}}^{\mathrm{R}}$$

of the K3 theory of [GTVW14], described in Section 1, can be regarded as a representation of the diagonal Virasoro algebra generated (as a Lie algebra) by the $L_{(n)} := L_n + \overline{L}_n$, $n \in \mathbb{Z}$. As such, according to [DMC16, Prop. 11.1], $\mathcal{H}^{\mathrm{GTVW}}$ is isomorphic to the space of states $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$ of the Conway Moonshine Module [Dun07] of Section 2. The aim of the present

work is to compare the known additional structures on these two spaces of states in greater depth.

The most apparent difference between the structures on $\mathcal{H}^{\mathrm{GTVW}}$ and $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ is the lack of right-movers in the Conway Moonshine Module, while $\mathcal{H}^{\mathrm{GTVW}}$ is the space of states of a *left-right symmetric* theory. In other words, left- and right-movers in our K3 theory arise on equal footing in every respect. In [Wen02], this property was used in order to argue that the SCFT $(\widetilde{2})^4$ is *mirror self-dual*, in fact it has this property with respect to several versions of mirror symmetry. That different incarnations of that quantum symmetry may be applied to this theory is already a harbinger of its special properties. Prompted by the results of [DMC16], in the present work, we argue that we are confronted with yet another surprising, special property of this SCFT: without destroying mathematical consistency, one may *reflect* all right-movers and view them as holomorphic states, instead, while leaving left-movers untouched. This reflection property, which we do not expect to be shared by many SCFTs, is responsible for the beautiful result [DMC16, Prop. 11.1], which we lift to an isomorphism between modules of $\mathfrak{u}(1)$ extensions of the Virasoro algebras in Section 4. This extension follows also from the results of [CDR18], which have been obtained independently from ours.

The current section is devoted to a discussion of the process of *reflecting* all states in $\mathcal{H}^{\mathrm{GTVW}}$ so that they become holomorphic, and of its limitations, in a more general context. Therefore, in the following, let $\mathbb{H} = \mathbb{H}^{\mathrm{NS}} \oplus \mathbb{H}^{\mathrm{R}}$ denote the space of states of a SCFT at central charges $c, \overline{c}$, according to the description in Appendix A. We wish to collect some necessary and sufficient conditions for our SCFT, so that $\mathbb{H}^{\mathrm{NS}}$ can become a self-dual, C_2-cofinite super vertex operator algebra of CFT type, and $\mathbb{H}^{\mathrm{R}}$ can become an admissible twisted $\mathbb{H}^{\mathrm{NS}}$-module[13], by means of an appropriate process of reflecting all states in $\mathbb{H}$.

Below, we argue that among the necessary conditions, we find the following restrictions on the central charges $c, \overline{c}$ of our SCFT:

$$(3.1) \qquad c, \overline{c} \in 6\mathbb{N} \quad \text{and} \quad c - \overline{c} \equiv 0 \mod 24.$$

Examples of SCFTs with $c = \overline{c} \in 3\mathbb{N}$ are expected to arise from supersymmetric non-linear sigma model constructions in the context of superstring

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[13]As in Section 1, we refer the reader to the literature for the definition of all notions concerning super vertex operator algebras and their modules. We particularly recommend the introductory sections of [DMC15, DMC16] for a very accessible presentation.

theory. In such constructions, the quantum field theory emerges from the study of differentiable maps from some Riemann surface Σ, known as the *worldsheet*, into a compact Calabi-Yau manifold, known as the *target space*. Classically, one would restrict attention to worldsheets that are embedded into the target space with (locally) minimal area. The equations of motion governing the coordinates on the target space are wave equations, whose solutions, for boundary conditions corresponding to closed strings, decompose into contributions solely depending *holomorphically* or *anti-holomorphically* on the complex coordinates of Σ, i.e. comprising *left-* and *right-moving* waves, respectively. In the resulting quantum field theory, the latter descend to the *left-* and *right-moving* degrees of freedom mentioned in Appendix A. Therefore, a reflection which renders all states of a SCFT holomorphic should be reminiscent of a complex conjugation for the right-moving degrees of freedom. However, our description of SCFTs in Appendix A should leave the reader in no doubt that the passage from string theory, with its interpretation in terms of left- and right-moving waves, into mathematically well-defined superconformal field theories tunnels through a number of black boxes. In particular, as detailed in Appendix A, for fermionic fields, complex conjugation entails the introduction of additional cocycle factors, and the many consistency conditions alluded to in Appendix A need to be taken into account when attempting a manipulation akin to a *reflection* on the right-moving degrees of freedom. The circumstances under which one can consistently perform such a procedure on $\mathbb{H}$ are by no means trivial.

3.1. Necessary spectral conditions

On the level of representations of the Virasoro algebra, reflecting right-movers to become holomorphic is very simple. It amounts to viewing $\mathbb{H} = \mathbb{H}^{\mathrm{NS}} \oplus \mathbb{H}^{\mathrm{R}}$, as assumed above, as a representation of a Virasoro algebra at central charge $c + \bar{c}$ which is generated, as a Lie algebra, by the $L_{(n)} := L_n + \overline{L}_n$, $n \in \mathbb{Z}$, and $c + \bar{c}$. Unitarity of the SCFT at the outset implies that this representation is unitary, and compatible with the real structure on $\mathbb{H}$. All additional structures that we like to impose on $\mathbb{H}$ in the context of superconformal field theory or super vertex operator algebras depend crucially on the fine structure of these representations.

Recall that the partition function $Z(\tau, z)$ of our SCFT, defined as in (A.1), is invariant under the special Möbius transform $(\tau, z) \mapsto (\tau + 1, z)$ of (A.2). Since uniqueness of the vacuum in our theory implies that the leading

order term of $Z(\tau, z)$ is $q^{-c/24}\overline{q}^{-\overline{c}/24}$, we conclude

$$(3.2) \qquad\qquad\qquad c - \overline{c} \equiv 0 \mod 24,$$

as announced in (3.1). The same reasoning for each summand

$$M \cdot y^Q \overline{y}^{\overline{Q}} q^{h-c/24} \overline{q}^{\overline{h}-\overline{c}/24}$$

of $Z(\tau, z)$ with $M \in \mathbb{N} \setminus \{0\}$ moreover implies that all conformal spins of bosonic states $v \in \mathbb{H}_{\text{bos}}$ are integral. In other words, for $h, \overline{h} \in \mathbb{R}$,

if $v \in \mathbb{H}_{\text{bos}}$ exists with $v \neq 0$, $L_0 v = hv$ and $\overline{L}_0 v = \overline{h}v$,   then $h - \overline{h} \in \mathbb{Z}$.

Furthermore, semilocality, together with conformal covariance, forces all conformal spins to be integral or half integral, where states in $\mathbb{H}_{\text{ferm}}^{\text{NS}}$ have half integral conformal spin, i.e. for $h, \overline{h} \in \mathbb{R}$,

if $v \in \mathbb{H}_{\text{ferm}}$ exists with
$v \neq 0$, $L_0 v = hv$ and $\overline{L}_0 v = \overline{h}v$,   then $h - \overline{h} \in \frac{1}{2}\mathbb{Z}$;
if in addition, $v \in \mathbb{H}_{\text{ferm}}^{\text{NS}}$,   then $h - \overline{h} \in \frac{1}{2} + \mathbb{Z}$.

For $\mathbb{H}^{\text{NS}}$ to become a self-dual super vertex operator algebra of CFT type with respect to the action of the $L_{(n)}$, and for $\mathbb{H}^{\text{R}}$ to become an admissible twisted $\mathbb{H}^{\text{NS}}$-module, by the very definition of these notions[14], all eigenvalues of $L_{(0)}$ on $\mathbb{H}$ must be integral or half integral, and those for bosonic states in the space $\mathbb{H}^{\text{NS}}$ must be integral, while those for fermionic states in $\mathbb{H}^{\text{NS}}$ must be half integral [DMC15, §2.1, Axiom 8]. Hence, as a necessary condition on our SCFT we find, for $h, \overline{h} \in \mathbb{R}$:
(3.3)
If $v \in \mathbb{H}$ exists with
$v \neq 0$, $L_0 v = hv$ and $\overline{L}_0 v = \overline{h}v$,   then $h, \overline{h} \in \frac{1}{4}\mathbb{N}$ and $h + \overline{h} \in \frac{1}{2}\mathbb{N}$;

if in addition, $\begin{cases} v \in \mathbb{H}_{\text{bos}}^{\text{NS}}, & \text{then } h, \overline{h} \in \frac{1}{2}\mathbb{N} \quad \text{and } h + \overline{h} \in \mathbb{N}; \\ v \in \mathbb{H}_{\text{ferm}}^{\text{NS}}, & \text{then } h, \overline{h} \in \frac{1}{2}\mathbb{N} \quad \text{and } h + \overline{h} \in \frac{1}{2} + \mathbb{N}; \\ v \in \mathbb{H}_{\text{bos}}^{\text{R}}, & \text{then } h - \overline{h} \in \mathbb{Z}. \end{cases}$

Recall that our original SCFT was assumed to enjoy space-time supersymmetry. Hence in particular, the vacuum $\Omega \in \mathbb{H}^{\text{NS}}$, under *spectral flow* (A.5),

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[14]A C_2-cofinite super vertex operator algebra of this type is called *nice*, according to Höhn [Höh96].

is mapped to a non-zero state

$$\widetilde{\Omega}_{\mathrm{tw}} \in \mathbb{H}^{\mathrm{R}} \quad \text{with } L_0\widetilde{\Omega}_{\mathrm{tw}} = \tfrac{c}{24}\widetilde{\Omega}_{\mathrm{tw}}, \quad \overline{L}_0\widetilde{\Omega}_{\mathrm{tw}} = \tfrac{\bar{c}}{24}\widetilde{\Omega}_{\mathrm{tw}}.$$

Thus, if $\mathbb{H}^{\mathrm{R}}$ meets the spectral requirements (3.3), then $c, \overline{c} \in 6\mathbb{N}$ follows, which together with the above condition (3.2) confirms (3.1).

Our assumption of space-time supersymmetry allows us to further restrict the spectrum in (3.3): with notations as in Appendix A, consider a non-zero state $v \in \mathbb{H}^{\mathcal{S}}_{h,Q;\overline{h},\overline{Q}}$, $\mathcal{S} \in \{\mathrm{NS}, \mathrm{R}\}$. Space-time supersymmetry implies that $v \in \mathbb{H}_{\mathrm{bos}}$ iff $Q - \overline{Q} \in 2\mathbb{Z}$ and $v \in \mathbb{H}_{\mathrm{ferm}}$ iff $Q - \overline{Q} \in 2\mathbb{Z} + 1$, in other words, we have

$$(3.4) \qquad\qquad (-1)^F = (-1)^{J_0 - \overline{J}_0}.$$

The operator of spectral flow, in general, has $U(1)$ charges $(Q_{\mathrm{sf}}; \overline{Q}_{\mathrm{sf}}) = (\frac{c}{6}; \frac{\bar{c}}{6})$, so by the above and (3.1), it is bosonic. We thus may conclude that spectral flow maps each of $\mathbb{H}_{\mathrm{bos}}$ and $\mathbb{H}_{\mathrm{ferm}}$ isomorphically onto itself. According to Appendix A, it also induces an isomorphism $\mathbb{H}^{\mathrm{NS}}_{h,Q;\overline{h},\overline{Q}} \cong \mathbb{H}^{\mathrm{R}}_{h',Q';\overline{h}',\overline{Q}'}$ with $(h', Q'; \overline{h}', \overline{Q}')$ as in (A.5). In particular, $\mathbb{H}^{\mathrm{NS}}_{h,Q;\overline{h},\overline{Q}} \subset \mathbb{H}_{\mathrm{ferm}}$ is mapped isomorphically to $\mathbb{H}^{\mathrm{R}}_{h',Q';\overline{h}',\overline{Q}'} \subset \mathbb{H}_{\mathrm{ferm}}$ with $(h' - \overline{h}') \in (h - \overline{h}) + \frac{1}{2} + \mathbb{Z}$. So (3.3) implies for $h, \overline{h} \in \mathbb{R}$,

$$(3.5) \qquad \text{if } v \in \mathbb{H}^{\mathrm{R}} \text{ exists with } v \neq 0, \ L_0 v = hv \text{ and } \overline{L}_0 v = \overline{h}v,$$
$$\text{then } h - \overline{h} \in \mathbb{Z}.$$

The necessary spectral conditions on L_0, $\overline{L}_0$ obtained so far immediately show that the SCFTs for which a reflection procedure may work are very sparse within any of the known moduli spaces of SCFTs, but our claim is that the K3 theory with space of states $\mathcal{H}^{\mathrm{GTVW}}$ is one of them. Indeed, the free fermion description of this theory allows to break up $\mathcal{H}^{\mathrm{GTVW}}$ into contributions that are constructed from three octuplets of free Majorana fermions, each with coupled spin structures, according to (1.26) and (1.27). Since ground states in the sectors $\mathcal{H}_{\mathcal{S}}$, $\mathcal{S} \in \{0, v, s, c\}$ of Section 1.3 have conformal weights $(0; 0)$, $(\frac{1}{2}; 0)$ or $(0; \frac{1}{2})$, $(\frac{1}{4}; \frac{1}{4})$, $(\frac{1}{4}; \frac{1}{4})$, respectively, inspection of (1.26) and (1.27) immediately shows that the spectral conditions (3.3), (3.5) are indeed fulfilled. We stress that our identification of $\mathcal{H}^{\mathrm{GTVW}}$ with the space of states of the Conway Moonshine Module exploits the fact that the two underlying theories enjoy a free fermion description.

3.2. Vertex algebra and module structure

By assumption, $\mathbb{H}$ comes equipped with the n-point functions of a SCFT, where the resulting maps $z \mapsto \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle$, for $\phi_1, \ldots, \phi_n \in \mathbb{H}$, in general, are only real analytic on their domains of definition. By the delicate consistency conditions of SCFTs, the n-point functions encode all operator product expansions of the theory. Vice versa, reflection positivity (A.8) determines the two-point functions $\langle \phi_1(z_1)\phi_2(z_2) \rangle$ entirely by means of the scalar product $\langle \cdot, \cdot \rangle$ together with the real structure and Virasoro representations on $\mathbb{H}$, and all other n-point functions are determined by the two-point functions together with the operator product expansion.

The requirements for $\mathbb{H}$ to comprise a super vertex operator algebra $\mathbb{H}^{\mathrm{NS}}$ together with a twisted $\mathbb{H}^{\mathrm{NS}}$-module $\mathbb{H}^{\mathrm{R}}$ after reflection are somewhat weaker: we only need to fix an operator product expansion between fields $\phi(z)$, $\upsilon(w)$ in terms of formal power series, where $\phi \in \mathbb{H}^{\mathrm{NS}}$ and $\upsilon \in \mathbb{H}$. We require expansions in $(z-w)^{\pm 1}$ if $\phi \in \mathbb{H}^{\mathrm{NS}}_{\mathrm{bos}}$ or $\upsilon \in \mathbb{H}^{\mathrm{NS}}$ and in $(z-w)^{\pm \frac{1}{2}}$ if $\phi \in \mathbb{H}^{\mathrm{NS}}_{\mathrm{ferm}}$ and $\upsilon \in \mathbb{H}^{\mathrm{R}}$. All this is encoded in the rules for assigning modes

$$(3.6) \quad \forall \phi \in \mathbb{H}^{\mathrm{NS}}_{\mathrm{bos}}: \quad n \mapsto \phi_{(n)} \ \forall n \in \mathbb{Z}, \quad \forall \phi \in \mathbb{H}^{\mathrm{NS}}_{\mathrm{ferm}}: \quad n \mapsto \phi_{(n)} \ \forall n \in \tfrac{1}{2}\mathbb{Z},$$

as was detailed for the particular example of the Conway Moonshine Module in Section 2.

Nevertheless, in general one cannot expect to obtain the required operator product expansions on the space of states that arises from $\mathbb{H}$ by reflection, not even with these weaker requirements, and this is due to the real analytic behaviour of the n-point functions of our SCFT. However, if the spectral conditions (3.3), (3.5) on the eigenvalues of L_0, $\overline{L}_0$ hold, then conformal covariance of the n-point functions severely restricts the form of the power series describing the operator product expansions in the theory: with notations as in Appendix A, assume that $\phi_i \in \mathbb{H}^{\mathcal{S}_i}_{h_i; \overline{h}_i}$, $\phi_j \in \mathbb{H}^{\mathcal{S}_j}_{h_j; \overline{h}_j}$, where $\mathcal{S}_i, \mathcal{S}_j \in \{\mathrm{NS}, \mathrm{R}\}$. Then all summands in the operator product expansion between $\phi_i(z_i)$ and $\phi_j(z_j)$, by conformal covariance, have the form

$$(3.7) \qquad \frac{\phi_k(z_i)}{(z_i - z_j)^{h_i + h_j - h_k}(\overline{z}_i - \overline{z}_j)^{\overline{h}_i + \overline{h}_j - \overline{h}_k}}$$

$$\text{with} \quad \phi_k \in \mathbb{H}^{\mathcal{S}_k}_{h_k; \overline{h}_k}, \ \mathcal{S}_k = \mathrm{NS} \ \text{if} \ \mathcal{S}_i = \mathcal{S}_j, \ \mathcal{S}_k = \mathrm{R} \ \text{otherwise.}$$

Thus (3.3) implies that all such OPEs $\phi_i(z_i)\phi_j(z_j)$ are encoded in terms of formal power series in $|z_i - z_j|^{\pm \frac{1}{2}}$ and $(z_i - z_j)^{\pm \frac{1}{2}}$ with $i \neq j$. This means that replacing each $|z_i - z_j|^{\pm \frac{1}{2}}$ by $(z_i - z_j)^{\pm \frac{1}{2}}$, one obtains an ansatz for a

"reflected" operator product expansion between the fields $\phi_i(z_i)$ and $\phi_j(z_j)$, which after reflection should be viewed as (holomorphic) fields in a super vertex operator algebra. But the spectral conditions (3.3), (3.5) do not ensure that the OPE between $\phi_i \in \mathbb{H}^{\text{NS}}_{\text{bos}}$ and $\phi_j \in \mathbb{H}^{\text{R}}$, after replacing all $|z_i - z_j|^{\pm\frac{1}{2}}$ by $(z_i - z_j)^{\pm\frac{1}{2}}$, yields a formal power series in $(z_i - z_j)^{\pm 1}$, as it should. We therefore impose one additional, very natural assumption on our original SCFT: we require that all eigenvalues of J_0 and of $\overline{J}_0$ are integral, i.e. for $Q, \overline{Q} \in \mathbb{R}$,

(3.8)   if $v \in \mathbb{H}$ exists with $v \neq 0$, $J_0 v = Q v$ and $\overline{J}_0 v = \overline{Q} v$,
    then $Q, \overline{Q} \in \mathbb{Z}$.

This assumption is equivalent to the requirement that the theory is invariant under the *purely holomorphic* and *anti-holomorphic two-fold spectral flows*[15]. This condition holds for every K3 theory by definition, according to [Wen15, Def. 8]; there, the operators of two-fold spectral flow, together with the $U(1)$ currents, comprise the $\widehat{\mathfrak{su}}(2)_{L,1} \oplus \widehat{\mathfrak{su}}(2)_{R,1}$-subalgebra for the left- and the right-moving $N = 4$ superconformal algebras. In particular, this condition holds for the K3 theory of [GTVW14] described in Section 1. By [EOTY89], it should hold for all SCFTs that obey (3.1) and arise from a non-linear sigma model construction with a Calabi-Yau target space.

If the additional spectral condition (3.8) holds, then the properties (3.1), (3.3) and (3.5) further restrict the spectrum of L_0 and $\overline{L}_0$, since spectral flow yields a multigraded isomorphism $\mathbb{H}^{\text{NS}} \xrightarrow{\cong} \mathbb{H}^{\text{R}}$, which obeys (A.5): let $h, \overline{h} \in \mathbb{R}$.

(3.9) If $v \in \mathbb{H}^{\text{R}}$ exists with $v \neq 0$, $L_0 v = h v$ and $\overline{L}_0 v = \overline{h} v$,

$$\text{then } (h; \overline{h}) = \left(\frac{c}{24}; \frac{c}{24}\right) + \left(\frac{m}{2}; \frac{\overline{m}}{2}\right) \text{ with } m, \overline{m} \in \mathbb{Z}, \ m \equiv \overline{m} \mod 2.$$

With the help of (3.7) as well as the spectral conditions (3.3), (3.5), (3.9), a case by case analysis for $\phi_i \in \mathbb{H}^{\text{NS}}_{p_i}$, $\phi_j \in \mathbb{H}^{\mathcal{S}_j}_{p_j}$ with $\mathcal{S}_j \in \{\text{NS, R}\}$, $p_i, p_j \in \{\text{bos, ferm}\}$ reveals two facts: first that the OPE between $\phi_i(z_i)$ and $\phi_j(z_j)$ is encoded in terms of a formal power series in the $(z_i - z_j)^{\pm\frac{1}{2}}, (\overline{z}_i - \overline{z}_j)^{\pm\frac{1}{2}}$, and second that by replacing all $(\overline{z}_i - \overline{z}_j)^{\pm\frac{1}{2}}$ by $(z_i - z_j)^{\pm\frac{1}{2}}$, these become formal power series in $(z_i - z_j)^{\pm 1}$ if $\phi_i \in \mathbb{H}^{\text{NS}}_{\text{bos}}$ or $\phi_j \in \mathbb{H}^{\text{NS}}$.

The above yields an ansatz for an OPE after reflection between the fields associated with any $\phi_i \in \mathbb{H}^{\text{NS}}$, $\phi_j \in \mathbb{H}$, which are viewed as states in

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[15]See for example [Gre97, §3.4] or [Wen00, §3.1.1].

a super vertex operator algebra and its admissible modules. However, due to the occurrence of half integral exponents in our formal power series, the construction leaves room for ambiguities of signs, which may destroy the consistency of our operator product expansions. It is tempting to try an ansatz by which one chooses a basis of the vector space $\mathbb{H}$ such that every state comes with a decomposition into a left and a right-moving contribution. In practice, in a given SCFT, this is regularly the case. Then, operator product expansions can be defined by specifying left-moving and right-moving contributions separately. If reflection acts by complex conjugation on the contributions $(\overline{z}_i - \overline{z}_j)^{\overline{\nu}}$ with $\overline{\nu} \in \frac{1}{2}\mathbb{Z}$ in the operator product expansions arising from right-movers, then the reflection should be given by an anti-$\mathbb{C}$-linear map on purely anti-holomorphic fields, and thus also on the right-moving contributions to every field. However, this introduces further ambiguities of phases for our operator product expansions, since there is no canonical way to assign complex scalar factors to a left- or a right-moving contribution, respectively. Moreover, *associativity* of a would-be vertex algebra structure obtained by this procedure is by no means clear.

Hence instead of attempting to separate left-movers from right-movers in every state in $\mathbb{H}$, with notations as in Appendix A, we choose a real basis, say, of every $\mathbb{H}^{\mathcal{S}}_{h;\overline{h}} \subset \mathbb{H}^{\mathcal{S}}$. The compatibility (A.6) of our n-point functions with the real structure on $\mathbb{H}$ together with the unitarity of the representation of the operator product expansion ensures that all coefficients in the formal power series of the OPE between two fields corresponding to real basis elements are real. Then, the first step of reflection on $\mathbb{H}$ can indeed be implemented by replacing all contributions $(\overline{z}_i - \overline{z}_j)^{\overline{\nu}}$ by $(z_i - z_j)^{\overline{\nu}}$ when $\overline{\nu} \in \frac{1}{2}\mathbb{Z}$ in the operator product expansion between $\phi_i(z_i)$ and $\phi_j(z_j)$ for real $\phi_i, \phi_j \in \mathbb{H}$. Since the resulting formal power series agrees with the operator product expansion of our original SCFT if $z_i = \overline{z}_i$ and $z_j = \overline{z}_j$, i.e. for entries in $\mathbb{R}^n \setminus \cup_{i \neq j}\{z \in \mathbb{R}^n | z_i = z_j\}$ in our n-point functions, associativity for this ansatz for an OPE is guaranteed. Interestingly, additional choices of signs are required. Indeed, if $\nu, \overline{\nu} \in \frac{1}{2} + \mathbb{Z}$ for contributions $(z_i - z_j)^{\nu}(\overline{z}_i - \overline{z}_j)^{\overline{\nu}}$ of an OPE in our original SCFT, then our prescription for reflection changes the parity of this function. Thus, semilocality of some n-point functions may be destroyed. Whether or not these signs can be implemented consistently, in general, is a highly nontrivial question which so far, has to be resolved on a case by case basis. These signs have the same origin as the cocycle factor κ_ϕ introduced in the formula for $\phi^\dagger$ in (A.9). Note however that these sign issues do not occur as long as the OPE involves a chiral or an antichiral field. Indeed, contributions $(z_i - z_j)^{\nu}(\overline{z}_i - \overline{z}_j)^{\overline{\nu}}$ to the OPE then only yield $\nu, \overline{\nu} \in \mathbb{Z}$. The above-mentioned sign ambiguity therefore does not arise when

one restricts attention to the structure of a potential bulk SCFT on $\mathbb{H}$, as is suggested in [CDR18]. Note that by construction, if (3.3) and (3.8) hold, then the operator product expansion between any two fields corresponding to real states in $\mathbb{H}$, on restriction to $z_i = \overline{z}_i$ and $z_j = \overline{z}_j$, can be described in terms of the super vertex algebra formalism.

3.3. Reflecting: some necessary and sufficient conditions, and consequences

As a result of the discussions in Sections 3.1 and 3.2, we arrive at a set of some necessary and sufficient conditions for our reflection procedure to yield the desired structures: assume that $\mathbb{H}$ is the space of states of a SCFT as before, such that the necessary spectral conditions (3.1), (3.3) on the central charges c, $\overline{c}$ and eigenvalues of L_0, $\overline{L}_0$ hold. To obtain a well-defined structure of a super vertex operator algebra and admissible module on $\mathbb{H}$ by the reflection procedure described above, one needs to require that after replacing all contributions $(\overline{z}_i - \overline{z}_j)^{\pm \frac{1}{2}}$, $|z_i - z_j|^{\pm \frac{1}{2}}$ by $(z_i - z_j)^{\pm \frac{1}{2}}$ in the OPE between $\phi_i(z_i)$ and $\phi_j(z_j)$ for any $\phi_i \in \mathbb{H}^{\mathrm{NS}}_{\mathrm{bos}}$, $\phi_j \in \mathbb{H}^{\mathrm{R}}$, the result is a formal power series in $(z_i - z_j)^{\pm 1}$. By the discussion of Section 3.2, a sufficient condition to ensure this behaviour is invariance under the two-fold holomorphic and anti-holomorphic spectral flows, or equivalently, (3.8). Furthermore, we must require that the necessary implementation of additional signs mentioned at the end of Section 3.2 can be performed consistently. Then the reflection procedure as described above is well-defined. All states become holomorphic, and reflection yields a consistent super vertex operator algebra $\mathbb{H}^{\mathrm{NS}}$ of CFT type, along with an admissible twisted $\mathbb{H}^{\mathrm{NS}}$-module $\mathbb{H}^{\mathrm{R}}$. Indeed, once a consistent OPE has been implemented on the reflected $\mathbb{H}$ along the lines described above, the remaining axioms, like the *state-field correspondence*, the *vacuum* and the *translation axiom* are immediate as a heritage from the corresponding properties of the original SCFT. If both the chiral and the antichiral algebra are self-dual and C_2-cofinite then this guarantees that reflection yields a self-dual, C_2-cofinite super vertex operator algebra.

For the resulting super vertex operator algebra and its twisted module, one can still define the partition function as a formal power series in terms of its four parts by means of (A.3). However, in this definition, $\overline{q}$ must then be replaced by q, to take into account the fact that $\mathbb{H}$ is now viewed as a Virasoro module under the action of the $L_{(n)} = L_n + \overline{L}_n$, $n \in \mathbb{Z}$. So indeed,

we obtain a formal power series

$$Z^{\mathrm{refl}}(\tau, z, \zeta) := \mathrm{tr}_{\mathbb{H}_{\mathrm{bos}}} \left(y^{J_0} \tilde{y}^{\overline{J}_0} q^{L_{(0)} - (c + \overline{c})/24} \right)$$

in $y := e^{2\pi i z}$, $\tilde{y} := e^{2\pi i \zeta}$ and $q := e^{2\pi i \tau}$. Since for $q = \overline{q}$, we have $Z(\tau, z) = Z^{\mathrm{refl}}(\tau, z, -\overline{z})$, the latter is a convergent power series in $q \in \mathbb{R}_{>0}$ and $y \in \mathbb{C}$, where each summand

$$M y^Q \overline{y}^{\overline{Q}} q^{h + \overline{h} - (c + \overline{c})/24}$$

obeys $M \in \mathbb{N}$. Thus $Z^{\mathrm{refl}}(\tau, z, \zeta)$ converges absolutely and can be viewed as a function in complex variables τ, z, $\zeta \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ and $\zeta = -\overline{z}$. By the identity theorem for holomorphic functions, it is uniquely determined by its values for purely imaginary τ and $\zeta = -\overline{z}$. However, there is no reason to expect the partition function $Z^{\mathrm{refl}}(\tau, z, \zeta)$ to behave like the partition functions of SCFTs under $\mathrm{SL}(2, \mathbb{Z})$, since

$$(3.10) \qquad \left\{ (\tau, z, \zeta) \in \mathbb{C}^3 \mid \mathrm{Re}(\tau) = 0, \ \mathrm{Im}(\tau) > 0, \ \zeta = -\overline{z} \right\}$$

is not mapped to itself under $\tau \mapsto \tau + 1$. But as the set (3.10) is invariant under the map $(\tau, z, \zeta) \mapsto (-1/\tau, z/\tau, -\zeta/\tau)$, $Z^{\mathrm{refl}}(\tau, z, \zeta)$ will exhibit modular behaviour under this transformation. In Section 4.1.5 we will confirm that for the model obtained from the K3 theory of Section 1 by reflecting right-movers to holomorphic states on $\mathcal{H}^{\mathrm{GTVW}}$, the resulting partition function shares its modular behaviour under $S \colon (\tau, z, \zeta) \mapsto (-1/\tau, z/\tau, -\zeta/\tau)$ with that of SCFTs, while under the transformation $T \colon \tau \mapsto \tau + 1$, the usual invariance properties are broken. Note that invariance under $T^2 \colon \tau \mapsto \tau + 2$ is immediate due to our spectral assumptions (3.3) on the eigenvalues of L_0 and $\overline{L}_0$ on $\mathbb{H}$ and $c + \overline{c} \in 12\mathbb{N}$ according to (3.1). Hence $Z^{\mathrm{refl}}(\tau, z, \zeta)$ exhibits modular behaviour under the Hecke group $\mathfrak{G}(2)$ - also known as the Theta Group - that is, the level 2, index 3 congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ generated by S and T^2. Since the level two principal congruence subgroup $\Gamma(2) \subset \mathfrak{G}(2)$ has genus 0, so does the Hecke group $\mathfrak{G}(2)$. By construction, the reflected theory obeys the spectral condition that all eigenvalues of $L_{(0)}$ lie in $\frac{1}{2}\mathbb{N}$. Thus we have recovered the modular behaviour found by Höhn [Höh96] for "nice" super vertex operator algebras. In our setting, the modular behaviour under $\mathfrak{G}(2)$ is naturally inherited, via reflection, from the original SCFT, and C_2-cofiniteness does not enter as a necessary condition.

In addition to modular transformations, the reflected partition function exhibits elliptic behaviour,

$$Z^{\text{refl}}\left(\tau, z + \frac{\tau}{2}, \zeta + \frac{\tau}{2}\right) = q^{-(c+\bar{c})/24} y^{-c/6} \widetilde{y}^{-\bar{c}/6} Z^{\text{refl}}(\tau, z, \zeta),$$

$$Z^{\text{refl}}\left(\tau, z + \frac{1}{2}, \zeta + \frac{1}{2}\right) = Z^{\text{refl}}(\tau, z, \zeta)$$

as a consequence of (A.4).

Recalling that for our original SCFT, $Z_{\widetilde{R}}(\tau, z)$ on its own transforms like $Z(\tau, z)$ under $\text{SL}(2, \mathbb{Z})$, its reflected version

$$Z^{\text{refl}}_{\widetilde{R}}(\tau, z, \zeta) := \text{tr}_{\mathbb{H}^{\text{R}}}\left((-1)^F y^{J_0} \widetilde{y}^{\bar{J}_0} q^{L_{(0)} - (c+\bar{c})/24}\right)$$

should also have interesting modular properties. Indeed, by the same reasoning as for $Z^{\text{refl}}(\tau, z, \zeta)$, it exhibits modular behaviour under the Hecke group $\mathfrak{G}(2)$. In addition, (3.1) and (3.9) imply for $h, \bar{h} \in \mathbb{R}$, $v \in \mathbb{H}^{\text{R}}$ with $v \neq 0$, $L_0 v = hv$, $\overline{L}_0 v = \bar{h}v$:

$$\left(L_{(0)} - \frac{c+\bar{c}}{24}\right) v = av, \quad a \in \mathbb{Z},$$

where actually $a \geq 0$ due to the unitarity requirements of the original theory [LVW89]. In other words, $Z^{\text{refl}}_{\widetilde{R}}(\tau, z, \zeta)$ is a power series in q, $y^{\pm 1}$, $\widetilde{y}^{\pm 1}$ and thus invariant under T. Altogether, $Z^{\text{refl}}_{\widetilde{R}}(\tau, z, \zeta)$ exhibits modular behaviour under the full modular group $\text{SL}(2, \mathbb{Z})$.

In closing this section, we emphasize once again that the structure of a self-dual, C_2-cofinite super vertex operator algebra of CFT type on $\mathbb{H}^{\text{NS}}$ together with an admissible twisted $\mathbb{H}^{\text{NS}}$-module structure on $\mathbb{H}^{\text{R}}$, obtained by reflecting all states in a SCFT with space of states $\mathbb{H} = \mathbb{H}^{\text{NS}} \oplus \mathbb{H}^{\text{R}}$, is much weaker than that of the original SCFT. By our prescription of the reflection, a priori, we solely obtain the formal power series expansions required for the definition of the super vertex algebra and twisted module structures on $\mathbb{H}^{\text{NS}}$, $\mathbb{H}^{\text{R}}$. That it should yield well-defined n-point functions as in a full-fledged SCFT, is by no means guaranteed and is also not required in the definition of a super vertex operator algebra and its admissible modules.

For the Conway Moonshine Module $V^{s\natural} \oplus V^{s\natural}_{\text{tw}}$, this lack of SCFT-structure may well mean that its role for Conway or Mathieu Moonshine should not be expected to match that of the Moonshine Module $V^\natural$ of Frenkel, Lepowsky and Meurman [FLM84, FLM85, FLM88] for Monstrous Moonshine. However, the beautiful results of [Dun07, DMC15] show that the analogues of the McKay-Thompson series for this module are indeed

normalized principal moduli for genus zero subgroups of $\mathrm{SL}(2,\mathbb{R})$. On the other hand, the K3 theory of Section 1 built on $\mathcal{H}^{\mathrm{GTVW}}$ offers the more powerful structures of SCFT. In particular, we have revealed the genus zero property of $Z^{\mathrm{refl}}(\tau, z = 0, \zeta = 0)$ as a heritage from this theory under reflection. It would be interesting to know whether the genus zero properties of the Conway Moonshine Module, in general, are inherited from its underlying K3 theory.

4. Reflecting the K3 theory with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry

In this section, we show that the reflection procedure described in Section 3 transforms the K3 theory with space of states $\mathcal{H}^{\mathrm{GTVW}}$ of Section 1 into Duncan's Conway Moonshine Module $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$, whose construction we have recalled in Section 2. On the level of Virasoro modules for the respective natural Virasoro algebras at central charge $c = 12$, agreement was already shown by Duncan and Mack-Crane in [DMC16, Prop. 1.11]. In Section 4.1, we lift this result to the level of modules of the extensions of these Virasoro algebras by the zero modes J_0 and $\overline{J}_0$ of two commuting $U(1)$ currents, where it turns out that we have to reverse the role of bosons and fermions in the Ramond sector, in comparison to [DMC16]. In Section 4.2, we show that after reflection, $\mathcal{H}^{\mathrm{GTVW}}$ agrees with $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$ as a super vertex operator algebra with an admissible module. This allows us to uncover a considerably more elaborate structure on the space $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$, which it inherits from the K3 theory on $\mathcal{H}^{\mathrm{GTVW}}$. In Section 4.3 we discuss the conclusions on Moonshines that we draw from our results.

4.1. Comparison of multigraded modules

With notations as in (1.26), (1.27), let $\mathcal{H}^{\mathrm{NS}} := \mathcal{H}_{\mathrm{bos}}^{\mathrm{NS}} \oplus \mathcal{H}_{\mathrm{ferm}}^{\mathrm{NS}}$ denote the Neveu-Schwarz sector of the K3 theory with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry of [GTVW14], while $\mathcal{H}^{\mathrm{R}} := \mathcal{H}_{\mathrm{bos}}^{\mathrm{R}} \oplus \mathcal{H}_{\mathrm{ferm}}^{\mathrm{R}}$ denotes the Ramond sector. As Virasoro modules with respect to the Virasoro algebra at central charge $c = 12$, which on $\mathcal{H}^{\mathrm{GTVW}}$ is generated by the diagonal $L_{(n)}$, $n \in \mathbb{Z}$, of Section 3,

$$\mathcal{H}^{\mathrm{NS}} \cong V^{s\natural}, \quad \mathcal{H}^{\mathrm{R}} \cong V_{\mathrm{tw}}^{s\natural},$$

according to [DMC16, Prop. 11.1]. Duncan and Mack-Crane obtain this beautiful result by means of *triality*, which we will come back to in Section 4.2. It is a priori not clear whether under the triality map, the $U(1)$ charges introduced in Section 2.2 agree with the ones obtained from our

choices of left- and right-moving $U(1)$-currents of the K3 theory. This issue is not addressed in [DMC16] and is studied in detail here. It amounts to a comparison between $\mathcal{H}^{\mathrm{GTVW}}$ and $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$ as modules of the extensions of the respective Virasoro algebras at central charge $c = 12$ by two commuting Lie algebras of type $\mathfrak{u}(1)$. We continue to denote the generators of the latter by J_0, $\overline{J}_0$, where on $\mathcal{H}^{\mathrm{GTVW}}$, the $U(1)$ currents are chosen according to (1.19), while on $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$, we use (2.5), (2.6). Both J_0 and $\overline{J}_0$ are central in the extended Lie algebras. To prove agreement as modules of the extended Lie algebras, it therefore suffices to show that the multigraded traces of $y^{J_0} \widetilde{y}^{\overline{J}_0} q^{L_{(0)}-1/2}$ over the respective sectors of $\mathcal{H}^{\mathrm{GTVW}}$ and $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$ agree. We prove this in each sector separately:

4.1.1. Neveu-Schwarz bosons. In our K3 theory, the bosonic contributions to $\mathrm{tr}_{\mathcal{H}^{\mathrm{NS}}} \left(y^{J_0} \widetilde{y}^{\overline{J}_0} q^{L_{(0)}-1/2} \right)$ from the sector $(\mathrm{NS}, \mathrm{NS}, \mathrm{NS})$, according to (1.29), are

$$\frac{1}{4} \left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)} \right)^4 + \left(\frac{\vartheta_4(\tau)}{\eta(\tau)} \right)^4 \right)^2 \cdot \frac{1}{2} \left(\frac{\vartheta_3(\tau,z)^2 \vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4} + \frac{\vartheta_4(\tau,z)^2 \vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4} \right)$$

$$\overset{(2.7)}{=} B(\tau, z, \zeta) B(\tau)^2.$$

This agrees with the contributions from U_{000} (see (2.2)) to (2.8).

The bosonic contributions from the sector $(\mathrm{NS}, \mathrm{R}, \mathrm{R})$, by (1.30), are

$$\frac{1}{4} \left(\frac{\vartheta_2(\tau)}{\eta(\tau)} \right)^8 \cdot \frac{1}{2} \left(\frac{\vartheta_3(\tau,z)^2 \vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4} + \frac{\vartheta_4(\tau,z)^2 \vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4} \right)$$

$$\overset{(2.7),(\mathrm{C}.3)}{=} B(\tau, z, \zeta) F(\tau)^2.$$

In (2.8), these agree with the contributions from the sector U_{0vv}.

The bosons in the final sector $(\mathrm{R}, \mathrm{NS}, \mathrm{R}) + (\mathrm{R}, \mathrm{R}, \mathrm{NS})$ by (1.30) yield

$$\left(\frac{\vartheta_2(\tau)}{\eta(\tau)} \right)^4 \cdot \frac{1}{2} \left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)} \right)^4 + \left(\frac{\vartheta_4(\tau)}{\eta(\tau)} \right)^4 \right)$$

$$\times \frac{1}{2} \left(\frac{\vartheta_2(\tau,z)^2 \vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4} + \frac{\vartheta_1(\tau,z)^2 \vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4} \right)$$

$$\overset{(2.7),(\mathrm{C}.3),(\mathrm{C}.14)}{=} \widehat{F}(\tau, z, \zeta) \cdot 2 B(\tau) F(\tau).$$

In (2.8), these are the contributions from the sectors $U_{v0v} \oplus U_{vv0}$.

Altogether, we find agreement with the result of (2.8).

4.1.2. Neveu-Schwarz fermions. In our K3 theory, the fermionic contributions to $\mathrm{tr}_{\mathcal{H}^{\mathrm{NS}}}\left(y^{J_0}\widetilde{y}^{\widetilde{J}_0}q^{L_{(0)}-1/2}\right)$ from the sector $(\mathrm{NS},\mathrm{NS},\mathrm{NS})$, according to (1.29), are

$$\frac{1}{4}\left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4-\left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4\right)^2\cdot\frac{1}{2}\left(\frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4}-\frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(2.7)}{=}F(\tau,z,\zeta)F(\tau)^2.$$

In (2.9), these are the contributions from the sector U_{ccc}.

The fermionic contributions from the sector $(\mathrm{NS},\mathrm{R},\mathrm{R})$, according to (1.30), are

$$\frac{1}{4}\left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^8\cdot\frac{1}{2}\left(\frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4}-\frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(2.7),(C.3)}{=}F(\tau,z,\zeta)F(\tau)^2.$$

In (2.9), these are the contributions from the sector U_{css}.

From (1.31), fermions in the final sector $(\mathrm{R},\mathrm{NS},\mathrm{R})+(\mathrm{R},\mathrm{R},\mathrm{NS})$ yield

$$\left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^4\cdot\frac{1}{2}\left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4-\left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4\right)$$

$$\times\frac{1}{2}\left(\frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4}-\frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(2.7),(C.3),(C.10)}{=}F(\tau,z,\zeta)\cdot 2F(\tau)^2.$$

In (2.9), these are the contributions from $U_{scs}\oplus U_{ssc}$.

Altogether, we find agreement with the result of (2.9).

4.1.3. Ramond bosons. In our K3 theory, the bosonic contributions to the multigraded trace $\mathrm{tr}_{\mathcal{H}^{\mathrm{R}}}\left(y^{J_0}\widetilde{y}^{\widetilde{J}_0}q^{L_{(0)}-1/2}\right)$ in the sector $(\mathrm{R},\mathrm{NS},\mathrm{NS})$, according to (1.32), amount to

$$\frac{1}{4}\left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4+\left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4\right)^2\cdot\frac{1}{2}\left(\frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4}+\frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(2.7),(C.14)}{=}\widehat{F}(\tau,z,\zeta)B(\tau)^2.$$

In (2.11), these are the contributions from the sector U_{v00}.

The bosonic contributions from the sector $(\mathrm{R},\mathrm{R},\mathrm{R})$, by (1.33), are

$$\frac{1}{4}\left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^8 \cdot \frac{1}{2}\left(\frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4} + \frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(2.7),(\mathrm{C.3}),(\mathrm{C.14})}{=} \widehat{F}(\tau,z,\zeta)F(\tau)^2.$$

In (2.11), these are the contributions from the sector U_{vvv}.

The bosons in the final sector $(\mathrm{NS},\mathrm{NS},\mathrm{R}) + (\mathrm{NS},\mathrm{R},\mathrm{NS})$, by (1.34), yield

$$\left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^4 \cdot \frac{1}{2}\left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4 + \left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4\right)$$

$$\times \frac{1}{2}\left(\frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4} + \frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(\mathrm{C.3})}{=} B(\tau,z,\zeta) \cdot 2B(\tau)F(\tau).$$

In (2.11), these are the contributions from the sectors $U_{00v} \oplus U_{0v0}$.

Altogether, we find agreement with (2.11), which however counts the *fermions* in the twisted module $V_{\mathrm{tw}}^{s\natural}$ of [DMC16]. Indeed, we have obtained the multigraded trace over $V_{\mathrm{tw,ferm}}^{s\natural}$, which according to (2.2) is given by $U_{v00} \oplus U_{vvv} \oplus U_{00v} \oplus U_{0v0}$, where every state receives one or three tensor factors from the vector representation v of $\widehat{\mathfrak{so}}(8)_1$. This explains why from the viewpoint of the Conway Moonshine Module, it seems natural to dub these states fermionic. On the other hand, in the K3 theory, they arise from $\mathcal{H}_{\mathrm{bos}}^{\mathrm{R}} = \mathcal{H}_{s00} \oplus \mathcal{H}_{sss} \oplus \mathcal{H}_{00s} \oplus \mathcal{H}_{0s0}$, thus solely receiving contributions from the vacuum representation 0 and the spinor representation s of $\widehat{\mathfrak{so}}(8)_1$, all of which are naturally interpreted as being bosonic. Moreover, the total $U(1)$ charge with respect to $J_0 + \overline{J}_0$ of each of the states in $\mathcal{H}_{\mathrm{bos}}^{\mathrm{R}}$ is even, as is the eigenvalue of $J_0 - \overline{J}_0$ by the spectral condition (3.8). Therefore, according to (3.4), space-time supersymmetry implies that $(-1)^F$ acts by multiplication with $+1$ on $\mathcal{H}_{\mathrm{bos}}^{\mathrm{R}}$. We therefore continue to interpret these states as bosons, that is, we choose to interchange the roles of bosons and fermions in $V_{\mathrm{tw}}^{s\natural}$. This solely introduces a difference by a global factor of (-1) for the action of $(-1)^F$ on $V_{\mathrm{tw}}^{s\natural}$, which we will come back to in Subsection 4.1.5.

4.1.4. Ramond fermions. In our K3 theory, the fermionic contributions from the sector $(\mathrm{R},\mathrm{NS},\mathrm{NS})$, by (1.32), are

$$\frac{1}{4}\left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4\right)^2 \cdot \frac{1}{2}\left(\frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4} - \frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(2.7),(C.10)}{=} F(\tau,z,\zeta)F(\tau)^2.$$

In (2.10), these are the contributions from the sector U_{scc}.

The fermions in the sector $(\mathrm{R},\mathrm{R},\mathrm{R})$, according to (1.33), are counted by

$$\frac{1}{4}\left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^8 \cdot \frac{1}{2}\left(\frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4} - \frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(2.7),(C.3),(C.10)}{=} F(\tau,z,\zeta)F(\tau)^2.$$

In (2.10), these are the contributions from the sector U_{sss}.

The fermionic contributions in the final sector $(\mathrm{NS},\mathrm{NS},\mathrm{R}) + (\mathrm{NS},\mathrm{R},\mathrm{NS})$, by (1.34), yield

$$\left(\frac{\vartheta_2(\tau)}{\eta(\tau)}\right)^4 \cdot \frac{1}{2}\left(\left(\frac{\vartheta_3(\tau)}{\eta(\tau)}\right)^4 - \left(\frac{\vartheta_4(\tau)}{\eta(\tau)}\right)^4\right)$$

$$\times \frac{1}{2}\left(\frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4} - \frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}\right)$$

$$\overset{(C.3)}{=} F(\tau,z,\zeta) \cdot 2F(\tau)^2.$$

In (2.10), these are the contributions from the sectors $U_{ccs} \oplus U_{csc}$.

Altogether, we find agreement with the result of (2.10).

4.1.5. Reflected partition function. From the above, we collect the contributions to the reflected partition function in each sector:

$$(4.1)\qquad Z_{\widetilde{\mathrm{NS}}}^{\mathrm{refl}}(\tau,z,\zeta) := \mathrm{tr}_{\mathcal{H}^{\mathrm{NS}}}\left((-1)^F y^{J_0}\widetilde{y}^{\overline{J}_0}q^{L_{(0)}-1/2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^{4}\left(\frac{\vartheta_k(\tau)}{\eta(\tau)}\right)^8 \cdot \frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}\right.$$

$$+ \left(\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$+ \left(\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$\left.+ \left(\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4}\right),$$

$$(4.2) \qquad Z_{\mathrm{NS}}^{\mathrm{refl}}(\tau, z, \zeta) := \mathrm{tr}_{\mathcal{H}^{\mathrm{NS}}}\left(y^{J_0}\widetilde{y}^{\overline{J}_0}q^{L_{(0)}-1/2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^{4}\left(\frac{\vartheta_k(\tau)}{\eta(\tau)}\right)^8 \cdot \frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4}\right.$$

$$+ \left(\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$+ \left(\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$\left.+ \left(\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}\right),$$

$$(4.3) \qquad Z_{\widetilde{\mathrm{R}}}^{\mathrm{refl}}(\tau, z, \zeta) := \mathrm{tr}_{\mathcal{H}^{\mathrm{R}}}\left((-1)^F y^{J_0}\widetilde{y}^{\overline{J}_0}q^{L_{(0)}-1/2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^{4}\left(\frac{\vartheta_k(\tau)}{\eta(\tau)}\right)^8 \cdot \frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}\right.$$

$$+ \left(\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$+ \left(\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$\left.+ \left(\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4}\right),$$

$$(4.4) \qquad Z_{\mathrm{R}}^{\mathrm{refl}}(\tau, z, \zeta) := \mathrm{tr}_{\mathcal{H}^{\mathrm{R}}}\left(y^{J_0}\widetilde{y}^{\overline{J}_0}q^{L_{(0)}-1/2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^{4}\left(\frac{\vartheta_k(\tau)}{\eta(\tau)}\right)^8 \cdot \frac{\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2}{\eta(\tau)^4}\right.$$

$$+ \left(\frac{\vartheta_3(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$+ \left(\frac{\vartheta_2(\tau)\vartheta_3(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2}{\eta(\tau)^4}$$

$$\left.+ \left(\frac{\vartheta_2(\tau)\vartheta_4(\tau)}{\eta^2(\tau)}\right)^4 \cdot \frac{\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2}{\eta(\tau)^4}\right).$$

Of these contributions to the partition function, $Z_{\widetilde{\mathrm{R}}}^{\mathrm{refl}}(\tau, z, \zeta = 0)$ agrees with the result of [DMC16, (9.14)], if there, one inserts the identity element for g. This reproduces the elliptic genus of K3, as it should, and as was already confirmed in [EOTY89]. It also reinforces our suggestion to interchange the

roles of bosons and fermions in $V_{\text{tw}}^{s\natural}$, since otherwise, the graded trace of $(-1)^F y^{J_0} \overline{y}^{\overline{J}_0} q^{L_{(0)}-1/2}$ over $V_{\text{tw}}^{s\natural}$ yields the negative of the elliptic genus of K3. Indeed, between [DMC16, (8.7)] and [DMC16, (9.10),(9.14)], an additional factor of (-1) was introduced by hand. The parameter ζ has not been introduced in [DMC16], since there, the $U(1)$ current $\overline{J}$ of (2.6) was not considered. Concerning the partition functions for the other three sectors, solely $Z_{\widetilde{\text{NS}}}^{\text{refl}}(\tau, z = 0, \zeta = 0)$ can be read from [DMC16, (8.6)].

Inspecting the relations between the four parts of the reflected partition function, note first of all that as usual, $Z_{\widetilde{\text{NS}}}^{\text{refl}}(\tau, z, \zeta)$ and $Z_{\text{NS}}^{\text{refl}}(\tau, z, \zeta)$, on the one hand, and $Z_{\widetilde{\text{R}}}^{\text{refl}}(\tau, z, \zeta)$ and $Z_{\text{R}}^{\text{refl}}(\tau, z, \zeta)$, on the other, are related by $(z, \zeta) \mapsto (z + \frac{1}{2}, \zeta + \frac{1}{2})$. This is a heritage from the space-time supersymmetry of the K3 theory on $\mathcal{H}^{\text{GTVW}}$, since there, $(-1)^F = (-1)^{J_0 - \overline{J}_0}$, as was noted in (3.4). Space-time supersymmetry of the underlying K3 theory also ensures that $Z_{\widetilde{\text{R}}}^{\text{refl}}(\tau, z, \zeta)$ and $Z_{\widetilde{\text{NS}}}^{\text{refl}}(\tau, z, \zeta)$, on the one hand, and $Z_{\text{R}}^{\text{refl}}(\tau, z, \zeta)$ and $Z_{\text{NS}}^{\text{refl}}(\tau, z, \zeta)$, on the other, are related by spectral flow

$$f(\tau, z, \zeta) \mapsto q^{\frac{1}{2}}(y\tilde{y}) f(\tau, z + \tfrac{\tau}{2}, \zeta + \tfrac{\tau}{2}),$$

as they should, according to (A.4).

Under modular transformations, we find the following behaviour. Up to the expected elliptic prefactor, $(\tau, z, \zeta) \mapsto (-1/\tau, z/\tau, -\zeta/\tau)$ leaves $Z_{\text{NS}}^{\text{refl}}(\tau, z, \zeta)$ and $Z_{\widetilde{\text{R}}}^{\text{refl}}(\tau, z, \zeta)$ invariant, while it interchanges $Z_{\widetilde{\text{NS}}}^{\text{refl}}(\tau, z, \zeta)$ with $Z_{\text{R}}^{\text{refl}}(\tau, z, \zeta)$. On the other hand, the transformation $\tau \mapsto \tau + 1$ interchanges $Z_{\widetilde{\text{NS}}}^{\text{refl}}(\tau, z, \zeta)$ with $-Z_{\text{NS}}^{\text{refl}}(\tau, z, \zeta)$, while $Z_{\text{R}}^{\text{refl}}(\tau, z, \zeta)$ and $Z_{\widetilde{\text{R}}}^{\text{refl}}(\tau, z, \zeta)$ are invariant. This means that $Z_{\widetilde{\text{R}}}^{\text{refl}}(\tau, z, \zeta)$ shares its modular transformation properties under $\text{SL}(2, \mathbb{Z})$ with that of the partition functions of SCFTs, while the sum $Z^{\text{refl}}(\tau, z, \zeta)$ of the four parts of the partition function does *not*; the latter transforms like the partition functions of a SCFT only under the Hecke group $\mathfrak{G}(2)$.

Altogether, the total partition function $Z^{\text{refl}}(\tau, z, \zeta)$ and $Z_{\widetilde{\text{R}}}^{\text{refl}}(\tau, z, \zeta)$ exhibit the expected transformation properties, as we explained in Section 3.3.

4.2. The Conway Moonshine Module as reflection of a particular K3 theory

By the results of Section 3, our reflection procedure is well-defined on the space of states $\mathcal{H}^{\text{GTVW}}$ of the K3 theory with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry of [GTVW14], if the additional sign choices mentioned at the end of Section 3.2 can be implemented consistently. However, our description of the K3 theory

as a lattice CFT, built on a half integral lattice (Section 1.3), guarantees that such sign choices solely amount to the choice of cocycles. Indeed, as we shall argue below, the reflected theory continues to allow a lattice theory description, built on a half integral charge lattice that meets the requirements used in Appendix B. The existence of cocycles and thereby of consistent sign choices after reflection thus follows from the constructions given in Appendix B. Thus indeed, reflection is well-defined for our K3 theory. On the Neveu-Schwarz sector $\mathcal{H}^{\mathrm{NS}}$, this introduces the structure of a super vertex operator algebra at central charge $c = 12$, while the Ramond sector $\mathcal{H}^{\mathrm{R}}$ carries the structure of an admissible $\mathcal{H}^{\mathrm{NS}}$-module. By inspection of the partition functions $Z^{\mathrm{refl}}_{\mathcal{S}}$, $Z^{\mathrm{refl}}_{\widetilde{\mathcal{S}}}$, $\mathcal{S} \in \{\mathrm{NS}, \mathrm{R}\}$, of (4.1)–(4.4), one confirms that the $L_{(0)}$-eigenspace in the reflected $\mathcal{H}^{\mathrm{GTVW}}$ at eigenvalue $\frac{1}{2}$ is trivial. Using the uniqueness result of [Dun07, Thm. 5.15], we may conclude that the reflected $\mathcal{H}^{\mathrm{GTVW}}$ agrees with $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ as a super vertex operator algebra plus admissible module, if we show self-duality and C_2-cofiniteness. Since our theory is described in terms of a lattice vertex operator algebra, self-duality is immediately checked. C_2-cofiniteness follows by the techniques developed in [DLM00, §12]. Instead of going through the details of the proof of self-duality and C_2-cofiniteness, we will show by direct comparison that after reflection, $\mathcal{H}^{\mathrm{NS}}$ is isomorphic to $V^{s\natural}$ as a super vertex operator algebra, while $\mathcal{H}^{\mathrm{R}}$ is isomorphic to $V^{s\natural}_{\mathrm{tw}}$ as an admissible module of $\mathcal{H}^{\mathrm{NS}} \cong V^{s\natural}$.

As already indicated in Section 3.3, we may thereby induce a considerably richer structure on $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ than what has been investigated, so far. First, this space is in fact a module of an $N = (4, 4)$ super Virasoro algebra at central charges $(c, \overline{c}) = (6, 6)$, since this is the case for $\mathcal{H}^{\mathrm{GTVW}}$. As described in Section 2.1, one obtains an elegant description of the super vertex algebra and module structure in terms of a charge lattice Γ^{refl}, which after reflection of the superconformal field theory built on $\mathcal{H}^{\mathrm{GTVW}}$ governs all OPEs. Indeed, it is straightforward to apply the first step of the reflection procedure of Section 3 to the OPEs between real or imaginary parts of all momentum-winding fields V_γ, $\gamma \in \Gamma$, in (1.15). It amounts to replacing all contributions $(\overline{z} - \overline{w})^{\overline{\boldsymbol{Q} \cdot \boldsymbol{Q}}'}$ by $(z - w)^{\overline{\boldsymbol{Q} \cdot \boldsymbol{Q}}'}$. For the charge lattice Γ of our D_4-torus theory in Section 1.3, which is equipped with the scalar product $\bullet$ of (1.2), this amounts to using the standard Euclidean scalar product on $\mathbb{R}^{12}$, instead. Hence the reflection procedure changes the signature of this lattice from $(6, 6)$ to $(12, 0)$.

Using the description given in Section 1.3 and the notations of Section 2.1, after reflection, the various sectors of the space of states are given

by

$$(4.5) \quad \left(\Gamma_{\text{bos}}^{\text{NS}}\right)^{\text{refl}} \oplus \left(\Gamma_{\text{ferm}}^{\text{NS}}\right)^{\text{refl}} \oplus \left(\Gamma_{\text{bos}}^{\text{R}}\right)^{\text{refl}} \oplus \left(\Gamma_{\text{ferm}}^{\text{R}}\right)^{\text{refl}} \subset \widetilde{\Gamma}_{2,2}^{\text{refl}} \oplus \widetilde{\Gamma}_{2,2}^{\text{refl}} \oplus \widetilde{\Gamma}_{2,2}^{\text{refl}}$$

with

$$(4.6) \qquad \left(\mathcal{H}_p^{\mathcal{S}}\right)^{\text{refl}} = \bigoplus_{\gamma \in \left(\Gamma_p^{\mathcal{S}}\right)^{\text{refl}}} \mathcal{H}_\gamma, \quad \mathcal{S} \in \{\text{NS}, \text{R}\}, \; p \in \{\text{bos}, \text{ferm}\},$$

where

$$\begin{aligned}
\left(\Gamma_{\text{bos}}^{\text{NS}}\right)^{\text{refl}} &= (\widetilde{\Gamma}_0^{\text{refl}})^3 \cup \left(\widetilde{\Gamma}_0^{\text{refl}} \oplus \widetilde{\Gamma}_2^{\text{refl}} \oplus \widetilde{\Gamma}_2^{\text{refl}}\right) \\
&\quad \cup \left(\widetilde{\Gamma}_2^{\text{refl}} \oplus \widetilde{\Gamma}_0^{\text{refl}} \oplus \widetilde{\Gamma}_2^{\text{refl}}\right) \cup \left(\widetilde{\Gamma}_2^{\text{refl}} \oplus \widetilde{\Gamma}_2^{\text{refl}} \oplus \widetilde{\Gamma}_0^{\text{refl}}\right), \\
\left(\Gamma_{\text{ferm}}^{\text{NS}}\right)^{\text{refl}} &= (\widetilde{\Gamma}_1^{\text{refl}})^3 \cup \left(\widetilde{\Gamma}_1^{\text{refl}} \oplus \widetilde{\Gamma}_3^{\text{refl}} \oplus \widetilde{\Gamma}_3^{\text{refl}}\right) \\
&\quad \cup \left(\widetilde{\Gamma}_3^{\text{refl}} \oplus \widetilde{\Gamma}_1^{\text{refl}} \oplus \widetilde{\Gamma}_3^{\text{refl}}\right) \cup \left(\widetilde{\Gamma}_3^{\text{refl}} \oplus \widetilde{\Gamma}_3^{\text{refl}} \oplus \widetilde{\Gamma}_1^{\text{refl}}\right), \\
\left(\Gamma_{\text{bos}}^{\text{R}}\right)^{\text{refl}} &= \left(\widetilde{\Gamma}_2^{\text{refl}} \oplus \widetilde{\Gamma}_0^{\text{refl}} \oplus \widetilde{\Gamma}_0^{\text{refl}}\right) \cup (\widetilde{\Gamma}_2^{\text{refl}})^3 \\
&\quad \cup \left(\widetilde{\Gamma}_0^{\text{refl}} \oplus \widetilde{\Gamma}_0^{\text{refl}} \oplus \widetilde{\Gamma}_2^{\text{refl}}\right) \cup \left(\widetilde{\Gamma}_0^{\text{refl}} \oplus \widetilde{\Gamma}_2^{\text{refl}} \oplus \widetilde{\Gamma}_0^{\text{refl}}\right), \\
\left(\Gamma_{\text{ferm}}^{\text{R}}\right)^{\text{refl}} &= \left(\widetilde{\Gamma}_3^{\text{refl}} \oplus \widetilde{\Gamma}_1^{\text{refl}} \oplus \widetilde{\Gamma}_1^{\text{refl}}\right) \cup (\widetilde{\Gamma}_3^{\text{refl}})^3 \\
&\quad \cup \left(\widetilde{\Gamma}_1^{\text{refl}} \oplus \widetilde{\Gamma}_1^{\text{refl}} \oplus \widetilde{\Gamma}_3^{\text{refl}}\right) \cup \left(\widetilde{\Gamma}_1^{\text{refl}} \oplus \widetilde{\Gamma}_3^{\text{refl}} \oplus \widetilde{\Gamma}_1^{\text{refl}}\right).
\end{aligned}$$

The final step in the reflection procedure of Section 3, namely the introduction of appropriate signs in the resulting OPEs, now amounts to the implementation of cocycle factors. Their existence is guaranteed by the results of Appendix B. There, we also provide an explicit formula (B.13) for representatives of the cocycles obeying the compatibility conditions (B.5), (B.12) which are required by the role that the cocycles play in the OPEs, as detailed at the end of Section 1.1.2. Using formula (B.13) to construct the cocycles both before and after reflection also shows that the required additional signs that occur in the final step of the reflection procedure are governed by cocycles, as is expected from our prescription of the Hermitian conjugate in (A.9).

In Appendix B, we also show that there are precisely two inequivalent choices of cocycles in the lattice theory obtained through reflection. However, the resulting super vertex operator algebra structure after reflection is independent of that choice, since the two cocycles differ solely by a relative sign which plays no role in the super vertex algebra. Indeed, this relative

sign only affects the comparison of the OPEs for $\psi(z)\phi(w)$ and $\phi(w)\psi(z)$ involving fields $\psi(z)$, $\phi(w)$, of which at least one creates a state in the Ramond sector from the vacuum. Therefore, at least one of the OPEs $\psi(z)\phi(w)$ and $\phi(w)\psi(z)$ is not considered within the structure of the super vertex operator algebra plus admissible module. This uniqueness result (up to equivalence) for the induced super vertex operator algebra plus module structure on $\left(\mathcal{H}^{\mathrm{GTVW}}\right)^{\mathrm{refl}}$ is in accord with John Duncan's theorem on the uniqueness of the Conway Moonshine Module [Dun07, Thm. 5.15].

> *In summary, the above proves our claim that the Conway Moonshine Module is the reflection of the K3 theory with $\mathbb{Z}_2^8 \colon \mathbb{M}_{20}$ symmetry.*

This also shows that the spaces of states $\mathcal{H}^{\mathrm{GTVW}}$ and $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ are isomorphic as $N = (4,4)$ super Virasoro modules, since for OPEs that involve a chiral or an antichiral field, by construction, our reflection procedure is only a formal manipulation. In fact, by the same argument, we have given an independent proof of [CDR18, Prop. 5.7].

It is instructive to study the transition from $\mathcal{H}^{\mathrm{GTVW}}$ to $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ under reflection more closely. Indeed, comparison to (2.2) reveals an apparent difference which on the level of Virasoro modules was already explained by Duncan and Mack-Crane in [DMC16, §11], invoking *triality*. Here, triality amounts to a lattice automorphism of the lattice $\widetilde{\Gamma}^{\mathrm{refl}}_{2,2}$, which maps $\widetilde{\Gamma}^{\mathrm{refl}}_0$ to itself, and

$$(4.7) \qquad \widetilde{\Gamma}^{\mathrm{refl}}_1 \longrightarrow \widetilde{\Gamma}^{\mathrm{refl}}_3 \longrightarrow \widetilde{\Gamma}^{\mathrm{refl}}_2 \longrightarrow \widetilde{\Gamma}^{\mathrm{refl}}_1.$$

On the level of representations of $\widehat{\mathfrak{so}}(8)_1$, triality thus induces isomorphisms $v \to c \to s \to v$. This indeed transforms (1.26), (1.27) into to (2.2), up to interchanging the roles of bosons and fermions in the Ramond sector, which we have already discussed in Section 4.1, above. On the level of lattices, (4.7) transforms $\left(\Gamma^{\mathrm{NS}}_{\mathrm{bos}}\right)^{\mathrm{refl}}$ and $\left(\Gamma^{\mathrm{R}}_{\mathrm{bos}}\right)^{\mathrm{refl}}$ of (4.6) into the subsets of $\Gamma^{\mathrm{refl}}_{\mathrm{bos}}$ of (2.4) labelling states from the Neveu-Schwarz and Ramond sector, respectively. Analogously, the states labelled by $\left(\Gamma^{\mathrm{NS}}_{\mathrm{ferm}}\right)^{\mathrm{refl}}$ and $\left(\Gamma^{\mathrm{R}}_{\mathrm{ferm}}\right)^{\mathrm{refl}}$ in (4.6) are transformed to those labelled by $\Gamma^{\mathrm{refl}}_{\mathrm{ferm}}$ in (2.4).

That triality plays an important role in the context of bosonization and fermionization, which we have made repeated use of in our constructions, was probably first noticed by Shankar [Sha80]. A detailed discussion can be found in [GO84, GOS85]. Moreover, in [FFR91, Thm. 5.7], it is shown that triality yields an automorphism between the super vertex operator algebras corresponding to charge vectors in each of the lattices $\widetilde{\Gamma}^{\mathrm{refl}}_0 \cup \widetilde{\Gamma}^{\mathrm{refl}}_a$

with $a \in \{1, 2, 3\}$. In fact, the description of these super vertex operator algebras as lattice theories yields an alternative proof of this result, since for integral charge lattices, up to equivalence, cocycles are unique [FK81], [Kac98, Thm. 5.5] (see Appendix B).

Let us discuss triality in some greater detail. To do so, we may restrict our attention to a single summand $\widetilde{\Gamma}_{2,2}^{\text{refl}} \subset \mathbb{R}^4$ in (4.5). With respect to the standard basis of $\mathbb{R}^4$, the triality automorphism (4.7) can be expressed by the matrix

$$\Theta = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \in \text{SO}(4),$$

mapping the cosets $\widetilde{\Gamma}_a^{\text{refl}}$ according to (4.7) and inducing a lattice automorphism on $\widetilde{\Gamma}_0^{\text{refl}}$. In accordance with the results of Appendix B, we may introduce cocycles ε on $\widetilde{\Gamma}_{2,2}^{\text{refl}}$ by means of formula (B.13), where here we state the matrix M with respect to the standard basis as

$$(4.8) \qquad M := \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

Using

$$A := \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

as a generating matrix of the lattice $\widetilde{\Gamma}_0^{\text{refl}}$, one easily checks

$$A^T M A \equiv (\Theta A)^T M (\Theta A) \mod 2.$$

Since $\Theta \in \text{SO}(4)$, this proves that on restriction to the sublattice $\widetilde{\Gamma}_0^{\text{refl}}$, triality yields an automorphism of the OPE (1.15), thus confirming the result of [FFR91, Thm. 5.7].

The explicit description of triality, above, also justifies our choices of $U(1)$ currents in (2.5), (2.6). They are the images of the $U(1)$ currents in (1.19), after reflection and application of triality. To show this, it suffices to restrict attention to the lattice $\widetilde{\Gamma}_{2,2}$ which governs the pair of two left- and two right-moving Dirac fermions $\chi_k, \chi_k^*, \overline{\chi}_k, \overline{\chi}_k^*$ with $k \in \{1, 2\}$ as in Section 1.2. Reflection leaves the purely holomorphic fields untouched, such

that a comparison with the structures introduced in Section 2.2 allows us to set

$$a_X^+ = \chi_1, \quad a_X^- = \chi_1^*, \quad a_Z^+ = \chi_2, \quad a_Z^- = \chi_2^*.$$

After reflection to $\widetilde{\Gamma}_{2,2}^{\text{refl}} \subset \mathbb{R}^4$ and with the notations of (1.20), using bosonization like there, the images of the $U(1)$ currents $J, \overline{J}$ in (1.19) read

$$J^{\text{refl}} = \mathfrak{j}_1 + \mathfrak{j}_2, \quad \overline{J}^{\text{refl}} = \mathfrak{j}_3 + \mathfrak{j}_4.$$

To calculate the images under triality, we simply determine the images under Θ of the corresponding vectors in $\mathbb{R}^4$,

$$\Theta \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Theta \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus J^{refl} is invariant under triality, yielding the image

$$\mathfrak{j}_1 + \mathfrak{j}_2 = {:}\mathbf{a}_X^+ \mathbf{a}_X^-{:} + {:}\mathbf{a}_Z^+ \mathbf{a}_Z^-{:}$$

as claimed in (2.5). On the other hand, $\overline{J}^{\text{refl}}$ is mapped to

$$\mathfrak{j}_1 - \mathfrak{j}_2 = {:}\mathbf{a}_X^+ \mathbf{a}_X^-{:} - {:}\mathbf{a}_Z^+ \mathbf{a}_Z^-{:}$$

as claimed in (2.6).

4.3. Moon Shines on K3

As already mentioned in [DMC16], one may hope that a detailed understanding of the Conway Moonshine Module and its relation to K3 theories might help unveil the mysteries of Mathieu Moonshine. Our findings make this relation precise.

Indeed, the Conway Moonshine Module arises by reflection from a particular K3 theory with space of states $\mathcal{H}^{\text{GTVW}}$. As already emphasized in Section 3.3, the very fact that the reflection procedure yields a well-defined super vertex operator algebra plus admissible module structure on $\mathcal{H}^{\text{GTVW}}$ requires very special properties of this SCFT. It would be interesting to determine all K3 theories that allow such a procedure – we do not expect that there are many, although further examples may arise from the potential bulk SCFTs of [CDR18]. However, the example presented in [CDR18,

§5.4] requires the notion of a *quasi-potential bulk superconformal field theory*, introduced in [CDR18]. This in particular weakens the requirements on pairs of holomorphic and anti-holomorphic conformal weights, allowing them to differ by arbitrary rational numbers, in contrast to any well-defined SCFT. Indeed, the quasi-potential $N = (2, 2)$ bulk superconformal field theory in question cannot arise as the image under reflection of any well-defined SCFT, and it is not a *quasi-potential $N = (4, 4)$ bulk superconformal field theory*.

Reflection always yields a super vertex operator algebra plus admissible module that obeys some of the additional properties required by Höhn for "nice" theories, namely the spectral ones. It would be interesting to know whether all reflected SCFTs are nice, i.e. whether C_2-cofiniteness is immediate, and whether vice versa, all self-dual nice super vertex operator algebras at central charge $6N$, $N \in \mathbb{N}$, plus admissible modules arise by reflection from some SCFTs. According to the classification result of [CDR18, Thm. 3.1], there are only three such super vertex operator algebras at central charge 12 and only one at central charge 6.

For the special K3 theory with space of states $\mathcal{H}^{\mathrm{GTVW}}$, reflection becomes straightforward due to our description in terms of lattice vertex operator algebras. Here, the underlying SCFT on $\mathcal{H}^{\mathrm{GTVW}}$ induces a considerably richer structure by including OPEs between pairs of fields from the admissible module. In other words, the super vertex operator language yields a *forgetful description*. Further restricting attention to OPEs in our SCFT that involve holomorphic or antiholomorphic fields, which up to a formal manipulation remain unchanged under reflection, one obtains precisely the structure that defines the potential bulk SCFTs of [CDR18].

The interpretation of the Conway Moonshine Module as image of a K3 theory under reflection elucidates the modular properties of its partition function. Indeed, that $Z_{\widetilde{R}}^{\mathrm{refl}}(\tau, z, \zeta)$ is invariant under the full modular group $\mathrm{SL}(2, \mathbb{Z})$, to our knowledge, had not been noticed, so far. It would also be interesting to know whether the genus zero property for all analogues of the McKay-Thompson series for Conway Moonshine can be traced back to K3, as is the case for $Z_{\widetilde{R}}^{\mathrm{refl}}(\tau, z, \zeta = 0)$.

In [DMC16], the identification of the Virasoro modules $\mathcal{H}^{\mathrm{GTVW}}$ and $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$ is used to arrive at a procedure that realizes all possible symmetries of K3 theories, and more, within the Conway Moonshine Module. Our findings allow a precise interpretation of this process.

Indeed, the symmetries of our K3 theory are naturally described in terms of lattice automorphisms of certain indefinite lattices. For our special K3 theory with $\mathbb{Z}_2^8 \colon \mathbb{M}_{20}$ symmetry and those symmetries that respect

the $\widehat{\mathfrak{su}}(2)_{1,L}^6 \oplus \widehat{\mathfrak{su}}(2)_{1,R}^6$ structure, this can be done by means of the indefinite charge lattice $\Gamma_{\mathrm{bos}} \cong \Gamma_{6,6} \hookrightarrow \mathbb{R}^{6,6}$ introduced in Section 1.3. Using the ideas of [NW01] and [TW15a, App. A], one may translate this into the traditional description by automorphisms of the lattice of integral cohomology of K3, which has signature $(4, 20)$. To do so, one relates the lattice Γ_{bos} back to the charge lattice of the underlying toroidal theory, which captures the dependence on moduli. This amounts to dropping contributions from the first summand $\widetilde{\Gamma}_{2,2}$ in (1.28) which governs the "external free fermions" $\chi_j, \chi_j^*, \overline{\chi}_j, \overline{\chi}_j^*$ with $j \in \{1,\, 2\}$ introduced in Section 1.2, and then reducing to $\Gamma_{4,4} \cong \left(\widetilde{\Gamma}_0\right)^2 \cup \left(\widetilde{\gamma}^{(2)} + \widetilde{\Gamma}_0\right)^2$. The result is the image of the traditional charge lattice of [Nar86] under the triality map described in [NW01]. $\mathbb{Z}_2$-orbifolding induces a map into the even self-dual lattice of signature $(4, 20)$ on which the moduli space of K3 theories is modelled. This map is determined explicitly in [NW01]. By the transition from $\mathcal{H}^{\mathrm{GTVW}}$ to $V^{s\natural} \oplus V_{\mathrm{tw}}^{s\natural}$, the latter, a priori, also serves as a medium to capture the symmetries of the K3 theory. At its heart, reflection then amounts to replacing the charge lattices with signature (d, d), $d \in \mathbb{N}$, which govern the behaviour of the particular K3 theory in question, by the positive definite lattices that are used in the realm of super vertex operator algebras. Though the precise mechanism certainly deserves further investigation, we remark that within this process, the even self-dual K3 lattice with signature $(4, 20)$ is replaced by a positive definite even self-dual lattice Λ, to be specified below. Reflection thus is an implementation of the beautiful strategy of Kondo's proof [Kon98] of Mukai's classification result for symplectic automorphisms of K3 surfaces [Muk88].

The lattice Λ most naturally features in the construction by generating the space $\mathfrak{a}$ over $\mathbb{C}$, which underlies the Conway Moonshine Module. The authors of [DMC16] choose the Leech lattice for Λ and then extend the discussion of symmetries to *all* automorphisms of the Leech lattice. This in particular includes all possible symmetry groups of K3 theories, by the results of [GHV12]. Only very few K3 theories lend themselves to transition from the K3 lattice to Λ through reflection, i.e. by means of a map from the SCFT to some super vertex operator algebra and admissible module. But since we have one K3 theory where this is possible, the reflection procedure does employ the Leech lattice as a medium that collects symmetries of K3 theories from distinct points of the moduli space of such theories. The idea thus reveals itself as an incarnation of *symmetry surfing* as advocated in [TW13, TW15b]. We therefore do not regard the Conway Moonshine Module as a *universal* object, but rather as the reflected version of one special K3

theory which is a particularly convenient point of reference in symmetry
surfing.

Note that the Conway Moonshine Module possesses an *infinite* symme-
try group. Indeed, in the notations of Section 2, the symmetry group is a
$\mathbb{Z}_2$-quotient of $\mathrm{Spin}(\mathfrak{a})$, by [Dun07, Prop. 4.6]. This is a consequence of the
forgetful description, alluded to, above: the weaker the structure that the
symmetries are required to preserve, the more symmetries one expects to
find. Nevertheless, $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$ exhibits Conway Moonshine with respect to
a natural action of Co_0, since by [DMC15, Prop. 3.1], realizing the Leech
lattice with respect to some choice of real structure as a lattice in the real
part of $\mathfrak{a}$, the action of its automorphism group lifts to a finite subgroup
of the symmetry group. More precisely, by [Dun07, Thm. 4.11], Co_0 is the
automorphism group that leaves invariant a choice of $N = 1$ structure on
$V^{f\natural} := A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^0_{\mathrm{tw}}$ (with the notations of Sect. 2.1), factorizing through
Co_1. In [CDD$^+$15], this result is generalized to larger extended chiral alge-
bras, yielding Mock Modular Moonshine for various subgroups of Co_0.

From the viewpoint of Mathieu Moonshine, however, we find it more
natural to realize $\mathfrak{a}$ as a complex vector space generated by the Niemeier
lattice Λ of type A_1^{24}, whose symmetry group is an extension of the largest
Mathieu group M_{24}. Since M_{24} has trivial Schur multiplier, and it is a simple
group, the proof of [DMC15, Prop. 3.1] can be applied to this group just as
well, showing that the lattice automorphisms in M_{24} lift to form a symmetry
group of the super vertex operator algebra and admissible module structure
on $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$. Symmetry surfing the moduli space of K3 theories allows to
generate the action of the entire group M_{24} on $V^{s\natural} \oplus V^{s\natural}_{\mathrm{tw}}$. In the process,
one should keep in mind that it has long become clear that the symmetry
groups of K3 theories cannot explain Mathieu Moonshine, since these groups,
in general, need not even form subgroups of M_{24}. As in our earlier work
[TW13, TW15b, TW15a], we emphasize that this problem can possibly be
cured by restricting attention to *geometric* symmetry groups of K3 theories[16]
rather than including all quantum symmetries. The resulting M_{24}-twining
elliptic genera agree with the ones obtained by Duncan and Mack-Crane in
[DMC16]. Note that seven of these twining elliptic genera differ from the ones
of Mathieu Moonshine according to [DMC16, §1.4]. As was emphasized in
[TW13, TW15a], symmetry surfing by merely employing lattice techniques

---

[16]Here and in the references [TW13, TW15b, TW15a] by a *geometric symmetry*
we mean a symmetry which induces a lattice automorphism on the K3 lattice
$H^*(\mathrm{K3}, \mathbb{Z})$ which in some geometric interpretation leaves $H^0(\mathrm{K3}, \mathbb{Z})$ and $H^4(\mathrm{K3}, \mathbb{Z})$
pointwise invariant.

cannot be expected to yield the M_{24}-modules of Mathieu Moonshine. Indeed, the results of [TW15b, GKP17] show that the construction of the relevant representations of M_{24} by symmetry surfing must involve a twist, which is not implemented in the Conway Moonshine Module.

Appendix A. Necessary ingredients from conformal field theory

To pave the way for a meaningful comparison between the K3 theory studied in [GTVW14] and the Conway Moonshine Module, in this Appendix, we collect some of the main ingredients to (super-)conformal field theory. Throughout this work, by a *conformal field theory* (CFT) we mean a (compact) Euclidean two-dimensional unitary conformal field theory. In fact, we restrict our attention to superconformal field theories (SCFTs) with at least $N = 2$ *worldsheet supersymmetry* both for the left- and for the right-movers,

and in addition, we assume *spacetime supersymmetry* to hold. Our presentation is by no means complete, and we refer to the literature for further details, see e.g. [BPZ84, Nah87, Gin88, Gra92, DMS96, Kac98, Gaw99, Gan00, Gab00, BB02, FBZ04, Sch08, Wen15, Wen], and references therein.

The first main ingredient of a *conformal field theory* is a complex vector space *of states* $\mathbb{H}$, equipped with a positive definite scalar product $\langle \cdot, \cdot \rangle$ and a compatible real structure $\mathbb{H} \longrightarrow \mathbb{H}$, $v \mapsto v^*$. For a *superconformal field theory* as specified above, the space of states $\mathbb{H}$ is assumed to decompose into a direct sum of simultaneous unitary representations of two super-commuting copies of an $N = 2$ super-Virasoro algebra at central charges c and $\overline{c}$, respectively, compatible with the real structure of $\mathbb{H}$. The other even standard generators of these super-Lie algebras are traditionally denoted L_n, J_n; $\overline{L}_n$, $\overline{J}_n$ with $n \in \mathbb{Z}$. These two super-Lie algebras and all structures arising from them, in the physics literature, are known as *left-moving* or *holomorphic*, and *right-moving* or *anti-holomorphic*, respectively, where the latter are denoted by overlined letters, in general. By our assumptions on the worldsheet supersymmetry of our SCFTs, $\mathbb{H}$ enjoys a $\mathbb{Z}_2$-grading by $(-1)^F$, where F denotes the worldsheet fermion number operator. The eigenspaces of $(-1)^F$ with eigenvalues ± 1 contain the *worldsheet bosons* and *fermions*, respectively, and they are denoted $\mathbb{H}_{\text{bos}}$, $\mathbb{H}_{\text{ferm}}$, hence $\mathbb{H} = \mathbb{H}_{\text{bos}} \oplus \mathbb{H}_{\text{ferm}}$. In addition, spacetime supersymmetry imposes a second, compatible $\mathbb{Z}_2$-grading by fermion boundary conditions, which decomposes the space of states[17] into a *Neveu-Schwarz sector* $\mathbb{H}^{\text{NS}}$ and a *Ramond sector* $\mathbb{H}^{\text{R}}$, hence $\mathbb{H} = \mathbb{H}^{\text{NS}} \oplus \mathbb{H}^{\text{R}}$. Both $\mathbb{Z}_2$-gradings are compatible with the real structure on $\mathbb{H}$.

The linear operators L_0, J_0; $\overline{L}_0$, $\overline{J}_0$ are assumed to restrict to pairwise commuting self-adjoint linear operators on each of the four sectors

$$\mathbb{H}_p^{\mathcal{S}} := \mathbb{H}_p \cap \mathbb{H}^{\mathcal{S}}, \quad p \in \{\text{bos}, \text{ferm}\}, \ \mathcal{S} \in \{\text{NS}, \text{R}\}.$$

They are simultaneously diagonalizable with finite dimensional simultaneous eigenspaces of L_0 and $\overline{L}_0$, and with one-dimensional

$$\ker{(L_0)} \cap \ker{\left(\overline{L}_0\right)} = \ker{(L_0)} \cap \ker{\left(\overline{L}_0\right)} \cap \ker{(J_0)} \cap \ker{\left(\overline{J}_0\right)} \subset \mathbb{H}_{\text{bos}}^{\text{NS}}.$$

The latter condition is known as the *uniqueness of the vacuum*; one chooses a real $\Omega \in \ker{(L_0)} \cap \ker{\left(\overline{L}_0\right)}$ with $\langle \Omega, \Omega \rangle = 1$ and calls it the *vacuum*.

---

[17]In general, one should include an R-NS and an NS-R sector, but these are trivial in our examples. We therefore use the shortcut notation R for the R-R sector, and NS for the NS-NS sector in this work.

The second main ingredient of a SCFT is a *system of n-point functions*, that is, for every $n \in \mathbb{N}$, there are maps

$$\mathbb{H}_{\text{bos}}^{\otimes n} \longrightarrow \text{Maps}(\mathbb{C}^n \setminus \cup_{i \neq j}\{z \in \mathbb{C}^n | z_i = z_j\}, \mathbb{C}),$$

$$(\mathbb{H}^{\text{NS}})^{\otimes n} \longrightarrow \text{Maps}(\mathbb{C}^n \setminus \cup_{i \neq j}\{z \in \mathbb{C}^n | z_i = z_j\}, \mathbb{C}),$$

both denoted by $\quad \phi_1 \otimes \cdots \otimes \phi_n \mapsto (\ z \mapsto \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle\ ),$

since they agree on $(\mathbb{H}_{\text{bos}}^{\text{NS}})^{\otimes n}$. These maps form a *semilocal, Poincaré covariant, conformally covariant* system, which is a *unitary representation* of an *operator product expansion* (OPE). If $n \geq 2$, then the n-point functions have extensions to

$$(\mathbb{H}_{\text{bos}}^{\text{NS}})^{\otimes (n-2)} \otimes \mathbb{H}^{\otimes 2} \longrightarrow \text{Maps}(\mathbb{C}^n \setminus \cup_{i \neq j}\{z \in \mathbb{C}^n | z_i = z_j\}, \mathbb{C})$$

obeying all the above-mentioned properties. Moreover, for any choice of a contractible open subset $U \subset \mathbb{C}^n \setminus \cup_{i \neq j}\{z \in \mathbb{C}^n | z_i = z_j\}$, they can be extended to maps from $\mathbb{H}^{\otimes n}$ into $\text{Maps}(U, \mathbb{C})$. For details concerning this terminology, along with a list of the many consistency conditions and properties that the above-mentioned structures are assumed to obey in a full-fledged SCFT, we need to refer the reader to the literature, since a full account would lead way beyond the scope of this work. In the following, we will, however, collect some of the consequences of these consistency conditions which turn out to be most crucial to us.

For example, one assumes that every SCFT has a well-defined *partition function* $Z(\tau, z)$. That is, with $q := e^{2\pi i \tau}$, $y := e^{2\pi i z}$ for all $\tau, z \in \mathbb{C}$ with $\text{Im}(\tau) > 0$, and with $\overline{q}, \overline{y} \in \mathbb{C}$ denoting the complex conjugates of $q, y \in \mathbb{C}$,

$$\text{(A.1)} \qquad Z(\tau, z) := \text{tr}_{\mathbb{H}} \left(\frac{1}{2}(1 + (-1)^F)\, y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right)$$

$$= \text{tr}_{\mathbb{H}_{\text{bos}}} \left(y^{J_0} \overline{y}^{\overline{J}_0} q^{L_0 - c/24} \overline{q}^{\overline{L}_0 - \overline{c}/24} \right)$$

is convergent. Moreover, under "integral" Möbius transformations

$$\text{(A.2)} \qquad (\tau, z) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}),$$

$Z(\tau, z)$ transforms like the product of a weak Jacobi form of weight 0 and index $\frac{c}{6}$ with the complex conjugate of such a weak Jacobi form at index $\frac{\overline{c}}{6}$. By our assumptions on supersymmetries, we may decompose $Z(\tau, z)$ according

to

$$
\text{(A.3)}\quad
\begin{cases}
Z(\tau, z) = \frac{1}{2}\left(Z_{\mathrm{NS}}(\tau, z) + Z_{\widetilde{\mathrm{NS}}}(\tau, z) + Z_{\mathrm{R}}(\tau, z) + Z_{\widetilde{\mathrm{R}}}(\tau, z) \right), \\
\qquad \text{where for } \mathcal{S} \in \{\mathrm{NS}, \mathrm{R}\}, \\
Z_{\mathcal{S}}(\tau, z) := \operatorname{tr}_{\mathbb{H}^{\mathcal{S}}}\left(y^{J_0}\overline{y}^{\overline{J}_0} q^{L_0 - c/24}\overline{q}^{\overline{L}_0 - \overline{c}/24} \right), \\
Z_{\widetilde{\mathcal{S}}}(\tau, z) := \operatorname{tr}_{\mathbb{H}^{\mathcal{S}}}\left((-1)^F y^{J_0}\overline{y}^{\overline{J}_0} q^{L_0 - c/24}\overline{q}^{\overline{L}_0 - \overline{c}/24} \right),
\end{cases}
$$

and the functions $Z_{\mathcal{S}}(\tau, z)$, $Z_{\widetilde{\mathcal{S}}}(\tau, z)$ are well-defined for all $\tau, z \in \mathbb{C}$ with $\operatorname{Im}(\tau) > 0$, as well. As detailed, for example, in [Wen00, Thm. 3.1.4], since we have assumed worldsheet *and* spacetime supersymmetry, the above four summands of the partition function are related as follows:

$$
\text{(A.4)}\quad
\begin{aligned}
Z_{\mathrm{R}}(\tau, z) &= q^{c/24}\overline{q}^{\overline{c}/24} y^{c/6}\overline{y}^{\overline{c}/6} Z_{\mathrm{NS}}\left(\tau, z + \frac{\tau}{2}\right), \\
Z_{\mathrm{NS}}(\tau, z) &= q^{c/24}\overline{q}^{\overline{c}/24} y^{c/6}\overline{y}^{\overline{c}/6} Z_{\mathrm{R}}\left(\tau, z + \frac{\tau}{2}\right), \\
Z_{\widetilde{\mathrm{NS}}}(\tau, z) &= Z_{\mathrm{NS}}\left(\tau, z + \frac{1}{2}\right), \quad Z_{\widetilde{\mathrm{R}}}(\tau, z) = Z_{\mathrm{R}}\left(\tau, z + \frac{1}{2}\right).
\end{aligned}
$$

Moreover, $Z_{\widetilde{\mathrm{R}}}(\tau, z)$, on its own, transforms like $Z(\tau, z)$ under the "integral" Möbius transformations stated above.

The first two lines of (A.4) are an immediate consequence of our assumption of space-time supersymmetry. The latter implies that the theory is invariant under *spectral flow* (see, for example, [Sen86, Sen87], or [Gre97, §3.4]), which induces a multigraded isomorphism $\mathbb{H}^{\mathrm{NS}} \xrightarrow{\cong} \mathbb{H}^{\mathrm{R}}$. On the eigenspaces $\mathbb{H}^{\mathcal{S}}_{h,Q;\overline{h},\overline{Q}} \subset \mathbb{H}^{\mathcal{S}}$, $\mathcal{S} \in \{\mathrm{NS}, \mathrm{R}\}$, with eigenvalues $(h, Q; \overline{h}, \overline{Q})$ with respect to $(L_0, J_0; \overline{L}_0, \overline{J}_0)$, spectral flow induces $\mathbb{H}^{\mathrm{NS}}_{h,Q;\overline{h},\overline{Q}} \xrightarrow{\cong} \mathbb{H}^{\mathrm{R}}_{h',Q';\overline{h}',\overline{Q}'}$ with

$$
\text{(A.5)}\quad (h', Q'; \overline{h}', \overline{Q}') = \left(h + \tfrac{Q}{2} + \tfrac{c}{24}, Q + \tfrac{c}{6}\; ; \; \overline{h} + \tfrac{\overline{Q}}{2} + \tfrac{\overline{c}}{24}, \overline{Q} + \tfrac{\overline{c}}{6}\right).
$$

By what was said above, each subspace

$$
\mathbb{H}^{\mathcal{S}}_{h;\overline{h}} := \left\{ v \in \mathbb{H}^{\mathcal{S}} \mid L_0 v = h v, \ \overline{L}_0 v = \overline{h} v \right\} = \bigoplus_{Q, \overline{Q}} \mathbb{H}^{\mathcal{S}}_{h,Q;\overline{h},\overline{Q}} \subset \mathbb{H}^{\mathcal{S}}
$$

is finite dimensional and obeys $(\mathbb{H}^{\mathcal{S}}_{h;\overline{h}})^* \cong \mathbb{H}^{\mathcal{S}}_{h;\overline{h}}$.

The assumption that the system of n-point functions in a CFT represents an OPE *unitarily* implies *reflection positivity*. This property of a quantum field theory amounts to a compatibility condition between the

scalar product, the real structure, and the n-point functions on $\mathbb{H}$, as we shall recall now (see also [Gab00, §3.5], for example, which however focusses on bosonic CFTs). First, for every n-point function $z \mapsto \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle$, $\phi_1, \ldots, \phi_n \in \mathbb{H}$, that is well-defined on a domain $U \subset \mathbb{C}^n \setminus \cup_{i \neq j} \{z \in \mathbb{C}^n | z_i = z_j\}$ with $\overline{z} \in U$ for all $z \in U$, the following compatibility condition with the real structure holds:

$$(A.6) \qquad \forall z \in U: \quad \overline{\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle} = \langle \phi_1^*(\overline{z}_1) \cdots \phi_n^*(\overline{z}_n) \rangle.$$

To give the full statement of reflection positivity, we introduce the following notation: for a complex vector space V, let $V\{z\}$ denote the vector space of formal power series of the form

$$\sum_{(r,\overline{r}) \in R} C_{r,\overline{r}} z^r \overline{z}^{\overline{r}}, \quad C_{r,\overline{r}} \in V, \quad \text{with finite } R \subset \left\{ (h, \overline{h}) \in \mathbb{R}^2_{\geq 0} \mid h - \overline{h} \in \tfrac{1}{2}\mathbb{Z} \right\}.$$

Then, reflection positivity amounts to the existence of an anti-$\mathbb{C}$-linear map $\mathbb{H} \longrightarrow \mathbb{H}\{x\}$, $\phi \longmapsto \phi^\dagger$, which induces a map

$$\phi(z) \longmapsto (\phi(z))^\dagger = \phi^\dagger(\overline{z}^{-1}) \quad \text{with } \left((\phi(z))^\dagger \right)^\dagger = \phi(z)$$

on the level of the associated fields (usually called *Hermitian conjugation* in the physics literature), where

$$(A.7) \qquad \overline{\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle} = \langle (\phi_n(z_n))^\dagger \cdots (\phi_1(z_1))^\dagger \rangle$$

for $\phi_1, \ldots, \phi_n$ and $z_1, \ldots, z_n$ as in (A.6). For $\psi, \phi \in \mathbb{H}$, with $(\psi(z))^\dagger = \psi^\dagger(\overline{z}^{-1})$ as above, and by linear extension of $\langle \cdots \rangle$ to $\mathbb{H}\{x\} \otimes \mathbb{H}$,

$$(A.8) \qquad \langle \psi, \phi \rangle = \lim_{x,w \to 0} \langle \psi^\dagger(\overline{x}^{-1}) \phi(w) \rangle,$$

where (A.7) ensures the compatibility of (A.8) with the Hermitian product structure $\langle \psi, \phi \rangle = \overline{\langle \phi, \psi \rangle}$. For the field $\phi(z)$, in general, $(\phi(z))^\dagger$ is the image of $(\phi(z))^*$ under the conformal transformation $z \mapsto z^{-1}$. If $\phi \in \mathbb{H}$ is *quasi-primary*, i.e. $L_{-1}\phi = 0$ and $\overline{L}_{-1}\phi = 0$, and if $L_0 \phi = h\phi$ and $\overline{L}_0 \phi = \overline{h}\phi$, then

$$(A.9) \qquad \phi^\dagger = x^{2h} \overline{x}^{2\overline{h}} \kappa_\phi \phi^*,$$

where $\kappa_\phi = (-1)^{h - \overline{h}}$ if $\phi \in \mathbb{H}_{\text{bos}}$. In general, κ_ϕ is an operator that plays the role of an additional cocycle factor[18], thus the notation, reminiscent of

---

[18]We remark at this point that already on the level of bosonic fields, the factor $(-1)^{h - \overline{h}}$ is forgotten in various standard conformal field theory texts.

the *Kleinian transformations*[19] [Kle38]. It takes into account the fact that the very definition of $(\phi(z))^*$ requires the specification of complex conjugation, which on the Riemann surface Σ that parametrizes a bosonic field $\phi(z)$, $z \in \Sigma$, reverses the co-orientation. For fermionic fields, we effectively work on a 2:1 cover of Σ, which entails the choice of a lift of the complex conjugation on Σ. This choice introduces additional phase factors[20] which may be consistently implemented by means of the cocycle factor κ_ϕ.

Note that the two-point functions $\langle \psi^*(\overline{x}^{-1})\phi(w)\rangle$ with $\psi \in \mathbb{H}_p^{\mathcal{S}}$, $\phi \in \mathbb{H}_{\widetilde{p}}^{\widetilde{\mathcal{S}}}$, $\mathcal{S}, \widetilde{\mathcal{S}} \in \{\mathrm{NS, R}\}$, $kp, \widetilde{p} \in \{\mathrm{bos, ferm}\}$, can only be non-vanishing if $\mathcal{S} = \widetilde{\mathcal{S}}$ and $p = \widetilde{p}$. Hence, despite the restrictions on the validity of (A.6), equation (A.8) can be used to extract the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}$ from these two-point functions, with $\langle \psi, \phi \rangle = 0$ for ψ, ϕ as above if $\mathcal{S} \neq \widetilde{\mathcal{S}}$ or $p \neq \widetilde{p}$. Unitarity then implies that $\langle \cdot, \cdot \rangle$ is positive definite.

Reflection positivity together with conformal covariance of the n-point functions ensures that the *state-field correspondence* holds in our theory, i.e. that there is a linear map associating to every state $\upsilon \in \mathbb{H}$ a *field* $\upsilon(z)$ which *creates υ from the vacuum*.

As a main result of this work, we show that for particular SCFTs, one can consistently *reflect* all fields, transforming them into holomorphic fields to obtain a superconformal vertex operator algebra along with a twisted module from the original superconformal field theory. To do so, we need to pay special attention to the above-mentioned consequences of unitarity. Let us illustrate this for the left- and right-moving components $(\psi, \overline{\psi})$ of a free Majorana fermion, like in the Ising model, where we follow the normalisations used in [BPZ84, DMS96]:

$$\psi(z)\psi(w) \sim \frac{1}{z-w}, \quad \overline{\psi}(\overline{z})\overline{\psi}(\overline{w}) \sim \frac{1}{\overline{z}-\overline{w}}.$$

By definition, the bosonic field $\varepsilon(z,\overline{z}) := i:\psi(z)\overline{\psi}(\overline{z}):$ obeys

$$\langle \varepsilon(z,\overline{z})\varepsilon(w,\overline{w})\rangle = \frac{1}{|z-w|^2},$$

---

[19] James Tener has explained this to us; the cocycle factor κ_ϕ is indispensable for fermionic vertex operators $\phi(z)$ in order to consistently define adjoint intertwining operators [Ten17]. In the literature, incarnations of κ_ϕ can already be found, for example, in [Yam13, AL17, Ten19].

[20] André Henriques has calculated the lift of the complex conjugation to a 2:1 cover of $\Sigma = \mathbb{C}^*$, confirming the occurrence of κ_ϕ [Hen17].

and $\varepsilon^\dagger = x\overline{x}\varepsilon$. As is customary, we choose our real structure on the space of states such that $\psi^* = \psi$, but then $\overline{\psi}^* = -\overline{\psi}$ follows. This can also be seen as a consequence of the 'reality' condition [Gin88, Gra92] for a Majorana spinor (ξ_1, ξ_2), $\xi_1^+ = i\xi_2$, $\xi_2^+ = i\xi_1$, and thus $\xi_2^* = \xi_2$ implies $\xi_1^* = -\xi_1$. To have a purely real left-moving field $\psi(z)$, we have chosen the convention $(\psi, \overline{\psi}) = i(\xi_1, \xi_2)$.

In contrast, for two left-moving free Majorana fermions ψ_k, $k \in \{1, 2\}$, with coupled spin structures, the assumption that $\psi_k^* = \psi_k$ for both $k \in \{1, 2\}$ is consistent with the required $\varepsilon_{12}^\dagger = x^2 \varepsilon_{12}$ for $\varepsilon_{12}(z) := i : \psi_1(z)\psi_2(z):$. In other words, the *real structure* $v \mapsto v^*$ of the underlying spaces of states is not compatible with a map $\psi \mapsto \psi_1$ and $\overline{\psi} \mapsto \psi_2$. One readily checks that for the left-moving component ψ of the free Majorana fermion, with our conventions, $\psi^\dagger = ix\psi$. More generally, as is explained at the end of Section 3.2, reflection indeed entails the occurrence of additional cocycle factors, as may well be expected by the above discussion.

Appendix B. Cocycle construction for certain half integral lattices

In this Appendix, we provide a construction of consistent 2-cocycles obeying a number of additional conditions, for a certain type of lattice which is central to our work. Our presentation extends to certain half integral lattices the classical results of [FK81, Seg81, GO84], which apply to lattice vertex operator algebras built on integral lattices. Our analysis is inspired by [GNOS86, GNO+87], though we found it useful to include a proof of consistency for 2-cocycles on the lattices that are relevant to our work. Indeed, to extend the solutions offered, for example, in [KLL+88] for the (integral!) lattices governing certain current algebras to the half integral lattices that govern the fermionic fields, additional consistency requirements are necessary. In [BPZ16], a solution similar to ours is presented for the lattice we denote $\widetilde{\Gamma}_{2,2}$ in (1.21).

Let $\Gamma \subset \mathbb{R}^D$ denote a lattice of rank D. We begin by recalling the general definition of 2-cocycles on Γ, following the exposition in [Kac98, §5.5]. Let $Z \subset \mathbb{C}^*$ denote a multiplicative finite subgroup with $-1 \in Z$. We call a map

$$\varepsilon \colon \Gamma \times \Gamma \longrightarrow Z$$

a 2-*cocycle on* Γ *with values in* Z, if the following coboundary condition holds:

(B.1) $\qquad \forall \alpha, \beta, \gamma \in \Gamma: \quad \varepsilon(\alpha, \beta)\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \beta + \gamma)\varepsilon(\beta, \gamma).$

Two such 2-cocycles ε, $\widetilde{\varepsilon}$ are said to be *equivalent* if there is a map $\eta \colon \Gamma \to Z$, $\alpha \mapsto \eta_\alpha$, such that

(B.2) $\qquad \forall \alpha, \beta \in \Gamma: \quad \widetilde{\varepsilon}(\alpha, \beta) = \eta_\alpha \eta_\beta \eta_{\alpha+\beta}^{-1} \varepsilon(\alpha, \beta).$

In other words, the equivalence classes of 2-cocycles with values in Z are the elements of $H^2(\Gamma, Z)$, the second group cohomology of Γ with values in the trivial Γ-module Z. Note also that every 2-cocycle ε with values in Z defines a central extension $\widetilde{\Gamma}$ of Γ by Z, where as a set, $\widetilde{\Gamma} = \Gamma \times Z$, and one has the group law $(\alpha, \lambda) \cdot (\beta, \mu) := (\alpha + \beta, \varepsilon(\alpha, \beta)\lambda\mu)$ for (α, λ), $(\beta, \mu) \in \Gamma \times Z$. This induces an isomorphism between $H^2(\Gamma, Z)$ and the equivalence classes of central extensions of Γ by Z.

Given a 2-cocycle ε with values in Z, it is convenient to introduce the *symmetry factor*

$$S \colon \Gamma \times \Gamma \longrightarrow Z, \quad (\alpha, \beta) \mapsto S(\alpha, \beta) := \varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha)^{-1}.$$

One immediately checks that S satisfies the following conditions:

(B.3) $\qquad \begin{cases} \forall \alpha, \beta, \gamma \in \Gamma: & S(\alpha, \alpha) = 1, \\ & S(\alpha, \beta)S(\beta, \alpha) = 1, \\ & S(\alpha + \beta, \gamma) = S(\alpha, \gamma)S(\beta, \gamma). \end{cases}$

According to [Kac98, Lemma 5.5], the above yields a 1: 1 correspondence between symmetry factors S obeying (B.3) and equivalence classes of 2-cocycles with values in Z. Note that (B.3) allows us to express S in terms of its values on a choice $\alpha_1, \ldots, \alpha_D$ of generators of Γ, since

(B.4) for $\alpha = \displaystyle\sum_{j=1}^{D} a_j \alpha_j$, $\beta = \displaystyle\sum_{k=1}^{D} b_k \alpha_k \in \Gamma: \quad S(\alpha, \beta) = \displaystyle\prod_{j,k=1}^{D} S(\alpha_j, \alpha_k)^{a_j b_k}.$

The discussion, so far, is independent of any quadratic form that we may choose on Γ. In our applications, however, we are interested in special choices of 2-cocycles, where in addition to the above, we assume that the lattice Γ is

equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ with $\langle \alpha, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Gamma$. Furthermore, for our 2-cocycles, we require the following:

(B.5) $\forall \alpha, \beta \in \Gamma$: if $\langle \alpha, \beta \rangle \in \mathbb{Z}$, then $\varepsilon(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \cdot \langle \beta, \beta \rangle} \varepsilon(\beta, \alpha)$.

In other words, the associated symmetry factor S must obey

(B.6) $\qquad \forall \alpha, \beta \in \Gamma$: $\quad S(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \cdot \langle \beta, \beta \rangle} \quad$ if $\langle \alpha, \beta \rangle \in \mathbb{Z}$.

Since for all $\alpha \in \Gamma$, we have assumed that $\langle \alpha, \alpha \rangle \in \mathbb{Z}$, equation (B.6) ensures $S(\alpha, \alpha) = 1$, as required by (B.3).

If the lattice Γ is *integral*, i.e. if the bilinear form $\langle \cdot, \cdot \rangle$ takes values in $\mathbb{Z}$ only, then by the above, there is a unique equivalence class of 2-cocycles with values in Z that obeys the additional condition (B.5). We now show how this statement must be modified for certain half integral lattices.

From now on, we restrict our attention to lattices where the following holds: the lattice

$$\Gamma_0 := \Gamma^* = \left\{ \alpha \in \mathbb{R}^D \mid \langle \alpha, \beta \rangle \in \mathbb{Z} \, \forall \beta \in \Gamma \right\}$$

is an even sublattice $\Gamma_0 \subset \Gamma$ of index 4, such that for each of the cosets Γ_a in Γ / Γ_0, $a \in \{0, \ldots, 3\}$, the lattice $\Gamma_0 \cup \Gamma_a$ is integral (and thus self-dual if $a \neq 0$). We write $\Gamma_a = \gamma^{(a)} + \Gamma_0$, $a \in \{0, \ldots, 3\}$, with $\gamma^{(0)} := 0$. In other words, (Γ_0, Γ) is a $\mathbb{Z}_2$ lattice pair with $\Gamma / \Gamma_0 \cong \mathbb{Z}_2^2$ in the terminology of [GNOS86, GNO+87].

We claim that Γ possesses precisely two distinct equivalence classes of 2-cocycles with values in Z that obey (B.5). Indeed, by our assumptions on Γ, we have $\langle \alpha, \beta \rangle \in \frac{1}{2}\mathbb{Z}$ for all $\alpha, \beta \in \Gamma$, hence given such a 2-cocycle ε with symmetry factor S, by (B.3) and (B.6),

(B.7) $\qquad S(\alpha, \beta) = \overline{S(\beta, \alpha)} \in \{\pm i\} \quad$ if $\langle \alpha, \beta \rangle \in \frac{1}{2} + \mathbb{Z}$.

Moreover, by the assumptions on our lattice Γ, we can choose generators $\alpha_1, \ldots, \alpha_D$ such that

$$\alpha_1 = \gamma^{(1)}, \; \alpha_2 = \gamma^{(2)}, \quad \alpha_j \in \Gamma_0 \quad \forall j \in \{3, \ldots, D\}.$$

Then $\langle \alpha_a, \alpha_j \rangle \in \mathbb{Z}$ for $a \in \{1, 2\}$ and all $j \geq 3$, since $\Gamma_0 \cup \Gamma_a$ is an integral lattice by assumption. Moreover, $\langle \alpha_1, \alpha_2 \rangle \in \frac{1}{2} + \mathbb{Z}$, since our assumptions imply that Γ is *not* an integral lattice. In fact, by replacing α_2 by $-\alpha_2$ if

need be, we may assume that

$$(B.8) \qquad\qquad \langle \alpha_1, \alpha_2 \rangle \in \frac{1}{2} + 2\mathbb{Z}.$$

Now $S(\alpha_j, \alpha_k)$ is uniquely determined by (B.6) for all $(j,k) \notin \{(1,2), (2,1)\}$, and $S(\alpha_1, \alpha_2) = \overline{S(\alpha_2, \alpha_1)} = \pm i$ by (B.7). It hence follows from (B.4) that there are at most two distinct solutions for the symmetry factor S, and thus, there are at most two inequivalent 2-cocycles on Γ that obey the additional condition (B.5).

Note however that the existence of any such 2-cocycles is not immediate. Indeed, if $\alpha, \alpha', \beta \in \Gamma$ are such that $S(\alpha, \beta)$ and $S(\alpha', \beta)$ obey (B.6) and (B.7), then (B.3) implies $S(\alpha + \alpha', \beta) = S(\alpha, \beta) S(\alpha', \beta)$, and one needs to check that $S(\alpha + \alpha', \beta)$ obeys (B.6) and (B.7) as well. This is immediate if $\langle \alpha + \alpha', \beta \rangle \in \frac{1}{2} + \mathbb{Z}$. If on the other hand, $\langle \alpha, \beta \rangle, \langle \alpha', \beta \rangle \in \mathbb{Z}$, then one proves that $\langle \alpha, \alpha' \rangle \langle \beta, \beta \rangle \in \mathbb{Z}$ by showing that either $\langle \beta, \beta \rangle \in 2\mathbb{Z}$ or $\alpha, \alpha', \beta \in \Gamma_0 \cup \Gamma_a$ for some $a \in \{1, 2, 3\}$. From this, the claim

$$S(\alpha + \alpha', \beta) = (-1)^{\langle \alpha + \alpha', \beta \rangle + \langle \alpha + \alpha', \alpha + \alpha' \rangle \langle \beta, \beta \rangle},$$

as required by (B.6), follows. Finally, if $\langle \alpha, \beta \rangle, \langle \alpha', \beta \rangle \in \frac{1}{2} + \mathbb{Z}$, by what was already shown, one may assume without loss of generality that $\alpha, \alpha', \beta \in \operatorname{span}_{\mathbb{Z}} \{\alpha_1, \alpha_2\}$. One then checks that (B.6) holds for $S(\alpha + \alpha', \beta)$ by a direct calculation for both choices $S(\alpha_1, \alpha_2) = i$ and $S(\alpha_1, \alpha_2) = -i$. Since as explained above, symmetry factors S are in $1 : 1$ correspondence to equivalence classes of 2-cocycles on Γ, the claim follows.

Given a choice of symmetry factor S that obeys (B.6), for any lattice Γ of rank D with symmetric bilinear form $\langle \cdot, \cdot \rangle$ and generators $\alpha_1, \ldots, \alpha_D$, following [GNOS86, GNO$^+$87, Kac98], we obtain 2-cocycles
(B.9)

$$\text{for } \alpha = \sum_{j=1}^{D} a_j \alpha_j, \ \beta = \sum_{k=1}^{D} b_k \alpha_k \in \Gamma: \quad \varepsilon(\alpha, \beta) := \prod_{j,k=1, j>k}^{D} S(\alpha_j, \alpha_k)^{a_j b_k}$$

that obey the additional condition (B.5). We introduce a $D \times D$ matrix $M = (M_{jk})$ such that

$$\forall j, k \in \{1, \ldots, D\}: \quad \exp(\pi i M_{jk}) = \begin{cases} S(\alpha_j, \alpha_k) & \text{if } j > k, \\ 1 & \text{if } j \leq k, \end{cases}$$

then with notations as in (B.9), we obtain

(B.10) $\qquad \boldsymbol{a} := (a_1, \ldots, a_D)^T, \ \boldsymbol{b} := (b_1, \ldots, b_D)^T,$

$$\varepsilon(\alpha, \beta) = \exp\left(\pi i \boldsymbol{a}^T M \boldsymbol{b}\right).$$

The 2-cocycles thus inherit the *bimultiplicative* behaviour from S,

(B.11) $\qquad \forall \alpha, \alpha', \beta, \beta' \in \Gamma: \quad \varepsilon(\alpha + \alpha', \beta) = \varepsilon(\alpha, \beta)\varepsilon(\alpha', \beta),$

$$\varepsilon(\alpha, \beta + \beta') = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \beta').$$

In our applications, however, we require a special gauge for our choice of representative $\widetilde{\varepsilon}$ in an equivalence class of 2-cocycles as above,

(B.12) $\qquad \forall \alpha, \beta \in \Gamma: \quad \widetilde{\varepsilon}(\alpha, 0) = \widetilde{\varepsilon}(0, \alpha) = \widetilde{\varepsilon}(\alpha, -\alpha) = 1,$

$$\widetilde{\varepsilon}(\alpha, \beta) = \overline{\widetilde{\varepsilon}(-\beta, -\alpha)}.$$

As is explained in [GNOS86, GNO$^+$87, Kac98], this condition can always be met. Indeed, given the representative ε of our 2-cocycle constructed in (B.9), any choice of map

$$\eta \colon \Gamma \longrightarrow Z, \quad \alpha \mapsto \eta_\alpha \quad \text{with } \eta_0 = 1, \quad \forall \alpha \in \Gamma \colon \ \eta_\alpha \eta_{-\alpha} = \varepsilon(\alpha, \alpha)$$

and some $Z \subset \mathbb{C}^*$ as above yields a representative $\widetilde{\varepsilon}$ under the gauge transformation (B.2) which obeys (B.12). Using (B.10), and for all $\alpha \in \Gamma$ and $\boldsymbol{a} \in \mathbb{R}^D$ as in (B.9) and (B.10), the special choice

$$\eta_\alpha := \exp\left(\tfrac{\pi i}{2} \boldsymbol{a}^T M \boldsymbol{a}\right)$$

yields representatives $\widetilde{\varepsilon}$ that obey the special gauge (B.12) and inherit the bimultiplicativity property (B.11). Indeed, with the notations of (B.9) and (B.10),

(B.13) $\qquad \widetilde{\varepsilon}(\alpha, \beta) = \exp\left(\tfrac{\pi i}{2} \boldsymbol{a}^T (M - M^T) \boldsymbol{b}\right).$

At this point we would like to emphasize that our entire analysis is independent of the signature of the lattice Γ. The explicit formulas, of course, are dependent on the details of each lattice Γ.

It is also worth mentioning that although we proved the existence and gave a construction of two inequivalent sets of cocyles for our special K3 theory earlier in this appendix, the choice one makes in the context of the

present work is not crucial. Actually, there are two inequivalent choices of co-cycles after reflecting the K3 theory too, leading to two super vertex operator algebras with admissible module and the cocycles can be fixed consistently before and after reflection. On the level of the structure of a super vertex algebra with admissible module alone, one cannot distinguish between the two choices, as we mention in Section 4.2.

Appendix C. Theta function identities

In this Appendix, we fix our conventions for the various modular functions that we shall use, and we introduce some helpful identities.

We shall always use the parametrisation $q := e^{2\pi i \tau}$ and $y := e^{2\pi i z}$, some-times along with $\widetilde{y} := e^{2\pi i \zeta}$. The Dedekind eta function is defined as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \ ,$$

while the Jacobi theta functions have product formula presentations of the form

$$\vartheta_1(\tau, z) := i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}}$$

$$= iq^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-1}y)(1 - q^n y^{-1}),$$

$$\vartheta_2(\tau, z) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n-\frac{1}{2})^2} y^{n-\frac{1}{2}}$$

(C.1)
$$= q^{\frac{1}{8}} y^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-1}y)(1 + q^n y^{-1}),$$

$$\vartheta_3(\tau, z) := \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} y^n = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}}y)(1 + q^{n-\frac{1}{2}}y^{-1}),$$

$$\vartheta_4(\tau, z) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n$$

$$= \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}}y)(1 - q^{n-\frac{1}{2}}y^{-1}).$$

We always use the shorthand $\vartheta_k(\tau) := \vartheta_k(\tau, 0)$, $k = 1, \ldots, 4$.

The following identities can be proved using the Jacobi triple identity, and they are standard:

(C.2)
$$\vartheta_2(\tau)\vartheta_3(\tau)\vartheta_4(\tau) = 2\eta(\tau)^3,$$

(C.3)
$$\vartheta_2(\tau)^4 - \vartheta_3(\tau)^4 + \vartheta_4(\tau)^4 = 0,$$

(C.4)
$$\vartheta_2(\tau)^2 = 2\vartheta_2(2\tau)\vartheta_3(2\tau),$$

(C.5)
$$\vartheta_3(\tau)^2 = \vartheta_3(2\tau)^2 + \vartheta_2(2\tau)^2,$$

(C.6)
$$\vartheta_4(\tau)^2 = \vartheta_3(2\tau)^2 - \vartheta_2(2\tau)^2.$$

Using standard theta function techniques, one finds the following generalizations of (C.3), see e.g. [Töl66] or [Wen00, (A3.1)]:

(C.7)
$$\vartheta_2(\tau)^2\vartheta_2(\tau,z)^2 - \vartheta_3(\tau)^2\vartheta_3(\tau,z)^2 + \vartheta_4(\tau)^2\vartheta_4(\tau,z)^2 = 0,$$
$$\vartheta_2(\tau)^2\vartheta_1(\tau,z)^2 + \vartheta_4(\tau)^2\vartheta_3(\tau,z)^2 - \vartheta_3(\tau)^2\vartheta_4(\tau,z)^2 = 0.$$

Moreover, by [Töl66] or [Wen00, (A3.3),(A3.5),(A3.6)], we have

(C.8)
$$2\vartheta_2(2\tau,z)\vartheta_3(2\tau,z) = \vartheta_2(\tau)\vartheta_2(\tau,z),$$
$$\vartheta_3(\tau)^2\vartheta_2(2\tau,2z) = \vartheta_2(2\tau)\vartheta_3(\tau,z)^2 - \vartheta_3(2\tau)\vartheta_1(\tau,z)^2,$$
$$\vartheta_3(\tau)^2\vartheta_3(2\tau,2z) = \vartheta_3(2\tau)\vartheta_3(\tau,z)^2 + \vartheta_2(2\tau)\vartheta_1(\tau,z)^2,$$

and

(C.9)
$$\vartheta_3(2\tau,2z)\vartheta_3(2\tau,2\zeta) + \vartheta_2(2\tau,2z)\vartheta_2(2\tau,2\zeta) = \vartheta_3(\tau,z+\zeta)\vartheta_3(\tau,z-\zeta),$$
$$\vartheta_3(2\tau,2z)\vartheta_3(2\tau,2\zeta) - \vartheta_2(2\tau,2z)\vartheta_2(2\tau,2\zeta) = \vartheta_4(\tau,z+\zeta)\vartheta_4(\tau,z-\zeta).$$

We deduce

$$\left(\vartheta_2(\tau,z)^2\vartheta_2(\tau,\zeta)^2 - \vartheta_1(\tau,z)^2\vartheta_1(\tau,\zeta)^2\right)\vartheta_2(\tau)^4$$

$$\overset{(C.7)}{=} \left(\vartheta_3(\tau)^2\vartheta_3(\tau,z)^2 - \vartheta_4(\tau)^2\vartheta_4(\tau,z)^2\right)\left(\vartheta_3(\tau)^2\vartheta_3(\tau,\zeta)^2 - \vartheta_4(\tau)^2\vartheta_4(\tau,\zeta)^2\right)$$
$$\quad - \left(\vartheta_3(\tau)^2\vartheta_4(\tau,z)^2 - \vartheta_4(\tau)^2\vartheta_3(\tau,z)^2\right)\left(\vartheta_3(\tau)^2\vartheta_4(\tau,\zeta)^2 - \vartheta_4(\tau)^2\vartheta_3(\tau,\zeta)^2\right)$$
$$= \vartheta_3(\tau)^4\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2 + \vartheta_4(\tau)^4\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2$$
$$\quad - \vartheta_3(\tau)^4\vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2 - \vartheta_4(\tau)^4\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2$$
$$= \left(\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2 - \vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2\right)\left(\vartheta_3(\tau)^4 - \vartheta_4(\tau)^4\right)$$
$$\overset{(C.3)}{=} \left(\vartheta_3(\tau,z)^2\vartheta_3(\tau,\zeta)^2 - \vartheta_4(\tau,z)^2\vartheta_4(\tau,\zeta)^2\right)\vartheta_2(\tau)^4,$$

implying the following very useful generalization of (C.3),

$$(C.10) \qquad \vartheta_2(\tau, z)^2 \vartheta_2(\tau, \zeta)^2 - \vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2$$
$$= \vartheta_3(\tau, z)^2 \vartheta_3(\tau, \zeta)^2 - \vartheta_4(\tau, z)^2 \vartheta_4(\tau, \zeta)^2.$$

Moreover,

$$\vartheta_3(\tau, z + \zeta)\vartheta_3(\tau, z - \zeta)\vartheta_3(\tau)^4$$
$$\overset{(C.8),(C.9)}{=} \left(\vartheta_2(2\tau)^2 + \vartheta_3(2\tau)^2\right)\left(\vartheta_3(\tau, z)^2 \vartheta_3(\tau, \zeta)^2 + \vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2\right),$$

from which by (C.5) we find

$$(C.11) \qquad \vartheta_3(\tau, z + \zeta)\vartheta_3(\tau, z - \zeta)\vartheta_3(\tau)^2$$
$$= \vartheta_3(\tau, z)^2 \vartheta_3(\tau, \zeta)^2 + \vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2.$$

Similarly,

$$\vartheta_4(\tau, z + \zeta)\vartheta_4(\tau, z - \zeta)\vartheta_3(\tau)^4 \vartheta_4(\tau)^2$$
$$\overset{(C.8),(C.9)}{=} \left(\vartheta_3(2\tau)^2 - \vartheta_2(2\tau)^2\right)\left(\vartheta_3(\tau, z)^2 \vartheta_3(\tau, \zeta)^2 - \vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2\right)\vartheta_4(\tau)^2$$
$$+ 2\vartheta_2(2\tau)\vartheta_3(2\tau)\left(\vartheta_3(\tau, z)^2 \vartheta_1(\tau, \zeta)^2 + \vartheta_1(\tau, z)^2 \vartheta_3(\tau, \zeta)^2\right)\vartheta_4(\tau)^2$$
$$\overset{\substack{(C.4),(C.6),\\(C.7)}}{=} \left(\vartheta_3(\tau, z)^2 \vartheta_3(\tau, \zeta)^2 - \vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2\right)\vartheta_4(\tau)^4$$
$$+ \vartheta_3(\tau, z)^2 \left(\vartheta_3(\tau)^2 \vartheta_4(\tau, \zeta)^2 - \vartheta_4(\tau)^2 \vartheta_3(\tau, \zeta)^2\right)\vartheta_4(\tau)^2$$
$$+ \vartheta_1(\tau, z)^2 \left(\vartheta_3(\tau)^2 \vartheta_4(\tau, \zeta)^2 - \vartheta_2(\tau)^2 \vartheta_1(\tau, \zeta)^2\right)\vartheta_2(\tau)^2$$
$$= -\vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2 \left(\vartheta_4(\tau)^4 + \vartheta_2(\tau)^4\right)$$
$$+ \vartheta_4(\tau, \zeta)^2 \left(\vartheta_4(\tau)^2 \vartheta_3(\tau, z)^2 + \vartheta_2(\tau)^2 \vartheta_1(\tau, z)^2\right)\vartheta_3(\tau)^2$$
$$\overset{(C.3),(C.7)}{=} \left(\vartheta_4(\tau, z)^2 \vartheta_4(\tau, \zeta)^2 - \vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2\right)\vartheta_3(\tau)^4,$$

from which by (C.10) we find

$$(C.12) \qquad \vartheta_4(\tau, z + \zeta)\vartheta_4(\tau, z - \zeta)\vartheta_4(\tau)^2$$
$$= \vartheta_4(\tau, z)^2 \vartheta_4(\tau, \zeta)^2 - \vartheta_1(\tau, z)^2 \vartheta_1(\tau, \zeta)^2$$
$$= \vartheta_3(\tau, z)^2 \vartheta_3(\tau, \zeta)^2 - \vartheta_2(\tau, z)^2 \vartheta_2(\tau, \zeta)^2.$$

From (C.11) and (C.12) we obtain

$$(C.13) \qquad \vartheta_3(\tau, z + \zeta)\vartheta_3(\tau, z - \zeta)\vartheta_3(\tau)^2 + \vartheta_4(\tau, z + \zeta)\vartheta_4(\tau, z - \zeta)\vartheta_4(\tau)^2$$
$$= \vartheta_3(\tau, z)^2 \vartheta_3(\tau, \zeta)^2 + \vartheta_4(\tau, z)^2 \vartheta_4(\tau, \zeta)^2,$$

$$(C.14) \qquad \vartheta_3(\tau, z+\zeta)\vartheta_3(\tau, z-\zeta)\vartheta_3(\tau)^2 - \vartheta_4(\tau, z+\zeta)\vartheta_4(\tau, z-\zeta)\vartheta_4(\tau)^2$$
$$= \vartheta_2(\tau, z)^2\vartheta_2(\tau, \zeta)^2 + \vartheta_1(\tau, z)^2\vartheta_1(\tau, \zeta)^2.$$

Again similary,

(C.15)

$$\vartheta_2(\tau, z+\zeta)\vartheta_2(\tau, z-\zeta)\vartheta_2(\tau)^2$$
$$\overset{(C.8)}{=} 4 \cdot \vartheta_2(2\tau, z+\zeta)\vartheta_3(2\tau, z+\zeta)\vartheta_2(2\tau, z-\zeta)\vartheta_3(2\tau, z-\zeta)$$
$$= \left(\vartheta_3(2\tau, z+\zeta)\vartheta_3(2\tau, z-\zeta) + \vartheta_2(2\tau, z+\zeta)\vartheta_2(2\tau, z-\zeta) \right)^2$$
$$\quad - \left(\vartheta_3(2\tau, z+\zeta)\vartheta_3(2\tau, z-\zeta) - \vartheta_2(2\tau, z+\zeta)\vartheta_2(2\tau, z-\zeta) \right)^2$$
$$\overset{(C.9)}{=} \vartheta_3(\tau, z)^2\vartheta_3(\tau, \zeta)^2 - \vartheta_4(\tau, z)^2\vartheta_4(\tau, \zeta)^2.$$

References

[ABD+76] M. Ademollo, L. Brink, A. D'Adda, R. D'Auria, E. Napolitano, S. Sciuto, E. Del Giudice, P. Di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, and R. Pettorino, *Supersymmetric strings and color confinement*, Phys. Lett. **B62** (1976), 105–110.

[AL17] Chunrui Ai and Xingjun Lin, *On the unitary structures of vertex operator superalgebras*, J. Algebra **487** (2017), 217–243.

[AM94] P. S. Aspinwall and D. R. Morrison, *String theory on K3 surfaces*, in: Mirror symmetry II, B. Greene and S.-T. Yau, eds., AMS, (1994), pp. 703–716.

[BB02] R. E. Borcherds and A. Barnard, *Quantum field theory*, `arXiv: math-ph/0204014`.

[BPZ84] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two–dimensional quantum field theory*, Nucl. Phys. **B241** (1984), 333–380.

[BPZ16] B. A. Burrington, A. Peet, and I. G. Zadeh, *Bosonization, cocycles, and the D1-D5 CFT on the covering surface*, Phys. Rev. **D93** (2016), no. 2, 026004, 31.

[CDD+15] M. C. N. Cheng, X. Dong, J. F. R. Duncan, S. Harrison, S. Kachru, and T. Wrase, *Mock modular Mathieu Moonshine*

modules, Research in the Mathematical Sciences **2** (2015), 13.

[CDH14a] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, *Umbral Moonshine*, Commun. Number Theory Phys. **8** (2014), 101–242.

[CDH14b] M. C. N. Cheng, J. F. R. Duncan, and J. A. Harvey, *Umbral Moonshine and the Niemeier lattices*, Research in the Mathematical Sciences **1** (2014), 3.

[CDR18] Th. Creutzig, J. Duncan, and W. Riedler, *Self-dual vertex operator superalgebras and superconformal field theory*, J. Phys. A **51** (2018), no. 3.

[Che10] M. C. N. Cheng, *K3 surfaces, N = 4 dyons, and the Mathieu group M_{24}*, Commun. Number Theory Phys. **4** (2010), 623–657.

[DGH88] L. J. Dixon, P. Ginsparg, and J. Harvey, *Beauty and the beast: superconformal symmetry in a Monster module*, Commun. Math. Phys. **119** (1988), no. 2, 221–241.

[DGO15] J. F. R. Duncan, M. J. Griffin, and K. Ono, *Proof of the umbral moonshine conjecture*, Res. Math. Sci. **2** (2015), Art. 26, 47.

[DLM00] C. Dong, H. Li, and G. Mason, *Modular-invariance of trace functions in orbifold theory and generalized Moonshine*, Commun. Math. Phys. **214** (2000), no. 1, 1–56.

[DMC15] J. F. R. Duncan and S. Mack-Crane, *The moonshine module for Conway's group*, Forum Math. Sigma **3** (2015), e10, 52.

[DMC16] J. F. R. Duncan and S. Mack-Crane, *Derived equivalences of K3 surfaces and twined elliptic genera*, Res. Math. Sci. **3** (2016), Art. 1, 47.

[DMS96] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York Berlin Heidelberg Barcelona Hong Kong London Milan Paris Singapore Tokyo, (1996).

[Dun07] J. F. R. Duncan, *Super-moonshine for Conway's largest sporadic group*, Duke Math. J. **139(2)** (2007), 255–315.

[DW09] R. Donagi and K. Wendland, *On orbifolds and free fermion constructions*, J. Geom. Phys. **59** (2009), 942–968.

[EH11] T. Eguchi and K. Hikami, *Note on twisted elliptic genus of K3 surface*, Phys. Lett. **B694** (2011), 446–455.

[EOT11]   T. Eguchi, H. Ooguri, and Y. Tachikawa, *Notes on the K3 surface and the Mathieu group M_{24}*, Exp. Math. **20** (2011), no. 1, 91–96.

[EOTY89] T. Eguchi, H. Ooguri, A. Taormina, and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with $SU(n)$ holonomy*, Nucl. Phys. **B315** (1989), 193–221.

[ET87]   T. Eguchi and A. Taormina, *Unitary representations of the $N = 4$ superconformal algebra*, Phys. Lett. **B196** (1987), 75–81.

[ET88a]   T. Eguchi and A. Taormina, *Character formulas for the $N = 4$ superconformal algebra*, Phys. Lett. **B200** (1988), 315–322.

[ET88b]   T. Eguchi and A. Taormina, *On the unitary representations of $N = 2$ and $N = 4$ superconformal algebras*, Phys. Lett. **210** (1988), 125–132.

[FBZ04]   E. Frenkel and D. Ben-Zvi, Vertex Algebras and Algebraic Curves, second ed., Mathematical Surveys and Monographs, Vol. 88, American Mathematical Society, Providence, RI, (2004).

[FFR91]   A. J. Feingold, I. B. Frenkel, and J. F. X. Ries, Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$, Contemporary Mathematics, Vol. 121, American Mathematical Society, Providence, RI, (1991).

[FK81]   I. B. Frenkel and V. G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980/81), no. 1, 23–66.

[FLM84]   I. Frenkel, J. Lepowsky, and A. Meurman, *A natural representation of the Fischer-Griess Monster with the modular function J as character*, Proc. Nat. Acad. Sci. U.S.A. **81** (1984), no. 10, Phys. Sci., 3256–3260.

[FLM85]   I. Frenkel, J. Lepowsky, and A. Meurman, *A moonshine module for the monster*, in: Vertex Operators in Mathematics and Physics, J. Lepowsky, S. Mandelstam, and I. M. Singer, eds., Springer Verlag, (1985), pp. 231–273.

[FLM88]   I. Frenkel, J. Lepowsky, and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Appl. Math., Vol. 134, Academic Press, Boston, (1988).

[FS04]   E. Frenkel and M. Szczesny, *Twisted modules over vertex alge-
         bras on algebraic curves*, Adv. Math. **187** (2004), no. 1, 195–227.

[Gab00] M. R. Gaberdiel, *An introduction to conformal field theory*,
         Rept. Prog. Phys. **63** (2000), 607–667.

[Gan00] T. Gannon, *Monstrous Moonshine and the classification of CFT*,
         in: Conformal Field Theory (Istanbul, 1998), Vol. 102 of Front.
         Phys., Adv. Book Program, Perseus Publ., Cambridge, MA,
         2000, p. 66.

[Gan16] T. Gannon, *Much ado about Mathieu*, Adv. Math. **301** (2016),
         322–358.

[Gaw99] K. Gawędzki, *Lectures on conformal field theory*, in: Quantum
         Fields and Strings: A Course for Mathematicians, Vol. 1, 2
         (Princeton, NJ, 1996/1997), Amer. Math. Soc., Providence, RI,
         (1999), pp. 727–805.

[GHV10a] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, *Mathieu moon-
         shine in the elliptic genus of K3*, JHEP **1010** (2010), 062.

[GHV10b] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, *Mathieu twin-
         ing characters for K3*, JHEP **1009** (2010), 058.

[GHV12] M. R. Gaberdiel, S. Hohenegger, and R. Volpato, *Symmetries
         of K3 sigma models*, Commun. Number Theory Phys. **6** (2012),
         1–50.

[Gin88] P. Ginsparg, *Applied conformal field theory*, Lectures Given at
         the Les Houches Summer School in Theoretical Physics 1988
         (Les Houches, France), pp. 1–168.

[GK10]   S. Govindarajan and K. G. Krishna, *BKM Lie superalgebras
         from dyon spectra in Z_N-CHL orbifolds for composite N*, JHEP
         05 (2010), 014.

[GKP17] M. R. Gaberdiel, Ch. Keller, and H. Paul, *Mathieu Moonshine
         and symmetry surfing*, J. Phys. **A50** (2017), no. 47, 474002.

[GNO+87] P. Goddard, W. Nahm, D. Olive, H. Ruegg, and A. Schwimmer,
         Fermions and octonions, Commun. Math. Phys. **112** (1987),
         385–408.

[GNOS86] P. Goddard, W. Nahm, D. Olive, and A. Schwimmer, *Vertex
         operators for non-simply-laced algebras*, Commun. Math. Phys.
         107 (1986), 179–212.

[GO84]   P. Goddard and D. Olive, *Algebras, lattices and strings*, in: Vertex Operators in Mathematics and Physics, Vol. 3 of Math. Sci. Res. Inst. Publ., Springer, Heidelberg, (1984), pp. 51–96.

[GO86]   P. Goddard and D. Olive, *Kac–Moody and Virasoro algebras in relation to quantum physics*, Int. J. Mod. Phys. **A2** (1986), 303–414.

[GOS85] P. Goddard, D. Olive, and A. Schwimmer, *The Heterotic string and a fermionic construction of the E(8) Kac-Moody algebra*, Phys. Lett. **B157** (1985), 393–399.

[Gov11] S. Govindarajan, *BKM Lie superalgebras from counting twisted CHL dyons*, JHEP **05** (2011), 089.

[Gra92] Y. Grandati, *Eléments d'introduction à l'invariance conforme*, Ann. Phys. Fr. **17** (1992), 159–300.

[Gre97] B.R. Greene, *String theory on Calabi-Yau manifolds*, in: Fields, Strings and Duality (Boulder, CO, 1996), World Sci. Publishing, River Edge, NJ, (1997), pp. 543–726.

[GTVW14] M. R. Gaberdiel, A. Taormina, R. Volpato, and K. Wendland, *A K3 sigma model with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry*, JHEP **1402:022** (2014).

[Hen17] A. Henriques, *private communication*.

[HM96]   J. A. Harvey and G. Moore, *Algebras, BPS states, and strings*, Nucl. Phys. **B463** (1996), 315–368.

[Höh96] G. Höhn, *Selbstduale Vertexoperatorsuperalgebren und das Babymonster*, Bonner Mathematische Schriften [Bonn Mathematical Publications], **286**, Universität Bonn, Mathematisches Institut, Bonn, (1996), Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, (1995).

[Huy16] D. Huybrechts, *On derived categories of K3 surfaces, symplectic automorphisms and the Conway group*, Development of moduli theory – Kyoto 2013, Adv. Stud. Pure Math. **69**, 387–405 (2016).

[Huy06] D. Huybrechts, Fourier-Mukai Transforms in Algebraic Geometry, Oxford Mathematical Monographs, Oxford University Press, Oxford, (2006).

[Huy14] D. Huybrechts, *Introduction to stability conditions*, in: Moduli Spaces. Based on Lectures of A Programme on Moduli Spaces at the Isaac Newton Institute for Mathematical Sciences, Cambridge, UK, January 4 – July 1, 2011, Cambridge: Cambridge University Press, (2014), pp. 179–229.

[Kac98] V. G. Kac, Vertex Algebras for Beginners, second ed., University Lecture Series, Vol. 10, American Mathematical Society, Providence, RI, (1998).

[Kle38] O. Klein, *Quelques remarques sur le traitement approximatif du problème des electrons dans un réseau cristallin par la mécanique quantique*, J. Phys. Radium, VII. Ser. **9** (1938), 1–12.

[KLL⁺88] V. A. Kostelecky, O. Lechtenfeld, W. Lerche, S. Samuel, and S. Watamura, *Conformal techniques, bosonization and tree-level string amplitudes*, Nucl. Phys. **B288** (1988), 173–232.

[Kon98] S. Kondo, *Niemeier lattices, Mathieu groups and finite groups of symplectic automorphisms of K3 surfaces*, Duke Math. J. **92** (1998), 593–603, Appendix by S. Mukai.

[LL04] J. Lepowsky and H. Li, Introduction to Vertex Operator Algebras and Their Representations, Progress in Mathematics, Vol. 227, Birkhäuser Boston Inc., Boston, MA, (2004).

[LVW89] W. Lerche, C. Vafa, and N. P. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nucl. Phys. **B324** (1989), 427–474.

[Muk88] S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.

[Nah87] W. Nahm, *Quantum field theories in one and two dimensions*, Duke Math. J. **54** (1987), no. 2, 579–613.

[Nar86] K. S. Narain, *New heterotic string theories in uncompactified dimensions < 10*, Phys. Lett. **169B** (1986), 41–46.

[NW01] W. Nahm and K. Wendland, *A hiker's guide to K3 – Aspects of $N = (4, 4)$ superconformal field theory with central charge $c = 6$*, Commun. Math. Phys. **216** (2001), 85–138.

[Sch08] M. Schottenloher, A Mathematical Introduction to Conformal Field Theory, 2nd ed., Lecture Notes in Physics, Vol. 759, Springer-Verlag, Berlin, (2008).

[Seg81] G. Segal, *Unitary representations of some infinite dimensional groups*, Commun. Math. Phys. **80** (1981), 301–342.

[Sen86] A. Sen, $(2,0)$ *supersymmetry and space-time supersymmetry in the heterotic string theory*, Nucl. Phys. **B278** (1986), 289–308.

[Sen87] A. Sen, *Heterotic string theory on Calabi-Yau manifolds in the Green-Schwarz formalism*, Nucl. Phys. **B284** (1987), 423–448.

[Sha80] R. Shankar, *Some novel features of the Gross–Neveu model*, Phys. Lett. **92B** (1980), 333–336.

[Som68] C. M. Sommerfield, *Currents as dynamical variables*, Phys. Rev. **176** (1968), 2019–2025.

[Sug68] H. Sugawara, *A field theory of currents*, Phys. Rev. **170** (1968), 1659–1662.

[Ten19] J. Tener, *Geometric realization of algebraic conformal field theories*, Adv. Math. **349** (2019), 488–563.

[Ten17] J. Tener, *private communication*.

[Tho79] J. G. Thompson, *Some numerology between the Fischer-Griess Monster and the elliptic modular function*, Bull. London Math. Soc. **11** (1979), no. 3, 352–353.

[Töl66] F. Tölke, Praktische Funktionenlehre, Zweiter Band: Theta–Funktionen und spezielle Weierstraßsche Funktionen, Springer–Verlag, Berlin Heidelberg New York, (1966).

[TW13] A. Taormina and K. Wendland, *The overarching finite symmetry group of Kummer surfaces in the Mathieu group M_{24}*, JHEP **08** (2013), 125.

[TW15a] A. Taormina and K. Wendland, *Symmetry-surfing the moduli space of Kummer K3s*, Proc. of the Conference String-Math 2012, Proceedings of Symposia in Pure Mathematics, no. 90, pp. 129–153.

[TW15b] A. Taormina and K. Wendland, *A twist in the M_{24} moonshine story*, Confluentes Mathematici **7** (2015), 83–113.

[Wen] K. Wendland, *Conformal field theory for mathematicians*, book manuscript based on lecture notes for an MSc course taught by the author, currently 255 pages, preliminarily accepted for publication in the AMS-Series "Graduate Studies in Mathematics".

[Wen00] K. Wendland, *Moduli spaces of unitary conformal field theories*, Ph.D. thesis, University of Bonn, (2000), supervisor: W. Nahm, available on request.

[Wen02] K. Wendland, *Orbifold constructions of K3: A link between conformal field theory and geometry*, in: Orbifolds in Mathematics and Physics, Vol. 310 of Contemporary Mathematics, AMS, Providence R.I., 2002, pp. 333–358.

[Wen06] K. Wendland, *A family of SCFTs hosting all 'very attractive' relatives of the* $(2)^4$ *Gepner model*, JHEP **03** (2006), 102.

[Wen15] K. Wendland, *Snapshots of conformal field theory*, in: Mathematical Aspects of Quantum Field Theories, D. Calaque and Th. Strobl, eds., Springer-Verlag, (2015), pp. 89–129.

[Yam13] Hiroshi Yamauchi, *Extended Griess algebras, conformal designs and Matsuo-Norton trace formulae*, in: Symmetries and Groups in Contemporary Physics, Vol. 11 of Nankai Ser. Pure Appl. Math. Theoret. Phys., World Sci. Publ., Hackensack, NJ, (2013), pp. 423–428.

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