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A construction of quarter BPS coherent states and Brauer algebras

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BPS coherent states have gravity dual descriptions in terms of semiclassical geometries. The half BPS coherent states have been well studied, however less is known about quarter BPS coherent states. Here we provide a construction of quarter BPS coherent states. They are coherent states built with two matrix fields, generalizing the half BPS case. These states are both the eigenstates of the annihilation operators and in the kernel of the anomalous dimension dilatation operator. Another useful labeling of quarter BPS states is by representations of Brauer algebras and their projection onto a subalgebra $\mathbb{C}[S_n \times S_m]$. Here, the Schur-Weyl duality for the Brauer algebra plays an important role in organizing the operators. One interesting subclass of these Brauer states are labeled by representations involving two Young tableaux. We obtain the overlap between quarter BPS Brauer states and quarter BPS coherent states, where the Schur polynomials are used. We also derive superposition formulas transforming quarter BPS coherent states to quarter BPS Brauer states. The entanglement entropy of Brauer states as well as the overlap between Brauer states and squeezed states are also computed.

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1. Introduction

The gauge/gravity correspondence [1–3] has provided a remarkable way to describe quantum gravity by quantum field theory on the boundary of the spacetime. It nontrivially relates a quantum system without gravity to a quantum theory with gravity. This correspondence reveals the emergence of spacetime geometry from the degrees of freedom on the boundary. The bulk spacetime dynamically emerges from the boundary quantum mechanical system [4–7]. This duality further provides us a way to explore interesting quantitative features of non-perturbative effects in string theory and quantum gravity, since it allows us to perform calculations pertaining to the gravity side by working in the quantum field theory side.

Based on very general arguments involving the supersymmetry algebra, one can calculate exactly the correlation functions of certain class of operators in $\mathcal{N} = 4$ SYM, namely the BPS operators. The properties of being protected from quantum correction make these quantities valuable in the study of the field theory and their relation with the dual gravity side. People now have a rather clear understanding of the dynamics of the half BPS operators and the dual gravity picture. In the gravity dual, there are back-reacted geometries that correspond to these BPS states in the field theory side [8–10].

Coherent state [11, 12] is a very important concept in quantum mechanics, often describing a state that most closely resembles the behavior of a semiclassical state. Coherent states arise in a wide range of physical systems and have applications in different fields such as quantum optics. It was realized previously [13] that coherent states also play an important role in the context of gauge/gravity duality. The half BPS coherent states have been constructed, and their various properties have been studied. And in [14], the half BPS coherent states in $\mathcal{N} = 4$ SYM are related with the phenomenon of topology change in quantum gravity.

However, the analog of the half BPS coherent states for the quarter BPS generalization has not been completely known in the literature. In this paper, we will extend the previous definition of half BPS coherent states to give a construction of quarter BPS coherent states, and study its relation with other operators in the quarter BPS sector, such as the Brauer operators [15, 16]. Then it is natural to consider more complicated quarter BPS states. However, a systematic understanding of quarter BPS operators is not an easy task. In this paper, we will consider the large N limit where the analysis gets simplified since the dilation operator [16, 17] can be simplified. And the construction of quarter BPS operators provides us necessary ingredients to construct quarter BPS coherent states, which is a main topic in this paper.

The quarter BPS states [18, 19] play important roles in gauge/gravity duality. Apart from the multi-trace basis, the quarter BPS states have representation bases, including the Brauer basis, the restricted Schur basis, and the flavor symmetry basis. These important aspects have been overviewed in for example [7, 20]. In addition to these labelings, the quarter BPS coherent states serve as another labeling of the operators or states in the Hilbert space. These states live in the same Hilbert space, hence one can superpose them and compute transition probabilities between states, and such operations have been performed in [14, 21–23]. Different states can be distinguished from each other, by carefully observing correlation functions [24–27].

Schur-Weyl duality and its generalizations provide us a powerful set of tools to organize the gauge theory operators and to relate them with interesting configurations in the dual gravity theory. In many previous examples, the representation of the symmetric group is used to construct gauge invariant operators that have interesting gravity interpretation [8–10, 14, 23]. In this paper, we make use of the generalization of Schur-Weyl duality involving Brauer algebras, see [15, 20]. The Brauer algebras and Walled Brauer algebras [28–30] can be regarded as a generalization of the group algebra of symmetric group and play a similar role in constructing gauge invariant operators. We will call the operators labeled by representations of Brauer algebra, Brauer operators, and the corresponding states in the Hilbert space, Brauer states. A subclass of Brauer operators gives us useful examples of quarter BPS operators. Brauer states share many features that are similar to Young tableau states in [8, 14, 23].

We will consider three sets of labelings of the states: the trace product basis, the coherent states, and the Brauer states. Their relations will be studied. We will calculate the overlap of these different states. It is interesting that we obtain many results involving the Brauer states which are very similar with respect to our previous results of Young tableau states in [23]. Besides, we can write superposition formulas transforming between quarter BPS Brauer states and quarter BPS coherent states that resemble our previous superposition formulas that describe topology change [14, 23].

The organization of this paper is as follows. In Section 2, we describe quarter BPS operators of the $\mathcal{N}=4$ SYM in the large N limit, and based on these results, we construct the quarter BPS coherent states. Then in Section 3, we analyze the Brauer states and their relation to other states including quarter BPS coherent states. Afterwards in Section 4, we analyze the squeezed states that generalize the above coherent states. In Section 5, we discuss our results and draw conclusions. Finally, we include Appendices A and B for more details on Brauer algebras.

2. Construction of quarter BPS coherent states

2.1. Trace product basis

To begin with, we describe the general form of a multi-trace operator built from two complex scalar fields. We can make use of symmetric group, since the trace structure can be captured by a permutation $\alpha \in S_{n+m}$. We consider operators of the form

$$\text{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}) = Z^{i_1}_{i_{\alpha(1)}} Z^{i_2}_{i_{\alpha(2)}} \cdots Z^{i_n}_{i_{\alpha(n)}} Y^{i_{n+1}}_{i_{\alpha(n+1)}} Y^{i_{n+m}}_{i_{\alpha(n+m)}}, \quad \alpha \in S_{n+m}. \tag{2.1}$$

For example, for the case $n=m=2$, $\alpha=1$ corresponds to $(\text{tr}Z)^2(\text{tr}Y)^2$, $\alpha=(1234)$ corresponds to $\text{tr}(Z^2Y^2)$, and $\alpha=(1324)$ corresponds to $\text{tr}(ZYZY)$. Note that if two permutations α, α' are conjugate to each other by an element $h \in S_n \times S_m$, they correspond to the same state $\text{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}) = \text{tr}(\alpha' Z^{\otimes n} \otimes Y^{\otimes m})$. Therefore we make use of $S_n \times S_m$ equivalence class of S_{n+m}: Two elements $\alpha, \alpha' \in S_{n+m}$ are in the same $S_n \times S_m$ equivalence class $[\alpha]=[\alpha']$ if and only if $\alpha = h\alpha h^{-1}$ for some $h \in S_n \times S_m$. As a special case, we note that for $m=0$, this basis gives us all multi-trace operators, and is a basis of the half BPS operators, see [14]. More precisely, the basis is labeled by conjugacy class of S_n, which is just given by a sequence $(w_1, w_2, \dots)$ where w_s means that there are w_s cycles of length s in the conjugacy class. The Brauer operators, which we will introduce later, can be expanded by the above basis. See Appendix A of [15] for some examples. Therefore in the following, we will consider all states labeled by $S_n \times S_m$ equivalence class of S_{n+m}.

The Hilbert space of all two-matrix multi-trace operators has a tensor product structure. For the case of half BPS operators, the Hilbert space has a tensor product structure given by the momentum number k and $\mathcal{H} =$

$\bigotimes_k \mathcal{H}_k$, where each $\mathcal{H}_k$ is created by $a_k^\dagger \leftrightarrow \mathrm{tr}(\frac{Z}{\sqrt{N}})^k$ corresponding to a single trace operator. Write the operator by a permutation, then a single trace corresponds to a permutation that has only one cycle. For the more general two-matrix case, we also expect that a factor of the tensor product is created by a single trace operator. A general single trace operator can be written as

$$\mathrm{tr}\left(\prod_j Z^{n_j} Y^{m_j}\right). \tag{2.2}$$

Note that since $\mathrm{tr}(Z^{k_1}Y^{k_2}\cdots Z^{k_n}) = \mathrm{tr}(Z^{k_1+k_n}Y^{k_2}\cdots)$, we can always take a state into the standard form $\mathrm{tr}(\prod_j Z^{n_j}Y^{m_j})$ where $n_j, m_j \geq 1$. To label these states, we define the following sets

$$K_0 = \{\vec{k} = (k_1, 0) \text{ or } (0, k_2) \mid k_1, k_2 \geq 1\}, \tag{2.3}$$

$$K_2 = \{\vec{k} = (k_1, k_2) \mid k_1, k_2 \geq 1\}, \tag{2.4}$$

$$K_4 = \{\vec{k} = (k_1, k_2, k_3, k_4) \mid k_1, k_2, k_3, k_4 \geq 1\}, \tag{2.5}$$

$$\cdots$$

$$K_{2p} = \{\vec{k} = (k_1, \ldots, k_{2p}) \mid k_i \geq 1,\ i = 1, \ldots, 2p\},\ \ldots \tag{2.6}$$

However, there is some redundancy in the above labeling since the traces have cyclic invariant property. For example, consider $\vec{k} = (1,1,3,1)$ and $\vec{k}' = (3,1,1,1)$, they are different as vectors in K_4. However

$$\mathrm{tr}(ZYZ^3Y) = \mathrm{tr}(Z^3YZY), \tag{2.7}$$

by cyclic property of trace. More generally, if two $\vec{k}$ are related by a cyclic permutation, they actually define the same multi-trace operator. Consider a group action of the abelian group $\mathbb{Z}_p$ on K_{2p} for $p \geq 1$. Write λ for the generator of $\mathbb{Z}_p$, or in other words $\mathbb{Z}_p = \langle\lambda\rangle$ with $\lambda^p = id$. Then define the action of λ on K_{2p}

$$\lambda \cdot (k_1, k_2, \ldots, k_{2p-1}, k_{2p}) = (k_{2p-1}, k_{2p}, k_1, k_2, \ldots, k_{2p-3}, k_{2p-2}). \tag{2.8}$$

And we define the quotient $\tilde{K}_{2p} = K_{2p}/\mathbb{Z}_p$ where two $\vec{k}, \vec{k}'$ are equivalent to each other if and only if $\vec{k} = \lambda^l \cdot \vec{k}'$ for some l. Then we define the equivalence class $[\vec{k}]$, where from the above $[\vec{k}'] = [\vec{k}]$. Note that $\mathbb{Z}_1 = \{id\}$ is trivial, therefore $K_2 = \tilde{K}_2$, and also $K_0 = \tilde{K}_0$. Then we define the set

$$\tilde{K} = K_0 \cup K_2 \cup \tilde{K}_4 \cup \tilde{K}_6 \cup \cdots \tag{2.9}$$

Then the Hilbert space can be written as

$$\mathcal{H} = \bigotimes_{[\vec{k}] \in \tilde{K}} \mathcal{H}_{[\vec{k}]}, \tag{2.10}$$

where $\mathcal{H}_{[\vec{k}]}$ is spanned by $\left[\mathrm{tr}\left(\frac{Z}{\sqrt{N}}\right)^{k_1}\left(\frac{Y}{\sqrt{N}}\right)^{k_2}\cdots\right]^{w_{[\vec{k}]}}$, $w_{[\vec{k}]} = 0, 1, 2, \ldots$. Now we identify the operator:

$$a^\dagger_{[\vec{k}]} \leftrightarrow O_{[\vec{k}]} = \mathrm{tr}\left[\left(\frac{Z}{\sqrt{N}}\right)^{k_1}\left(\frac{Y}{\sqrt{N}}\right)^{k_2}\cdots\right]. \tag{2.11}$$

A complete trace product basis is then provided by

$$\prod_{[\vec{k}] \in \tilde{K}} a^{\dagger w_{[\vec{k}]}}_{[\vec{k}]} |0\rangle. \tag{2.12}$$

Using this notation, the half BPS case corresponds to $\bigotimes_{[\vec{k}] \in K_0} \mathcal{H}_{[\vec{k}]}$. In the half BPS case, states $\prod_k a_k^{\dagger w_k}|0\rangle$ with $\sum_k k w_k = n$ correspond to multi-trace operators labeled by an equivalence class of S_n. In the two matrix case, states $\prod_{[\vec{k}] \in \tilde{K}} a^{\dagger w_{[\vec{k}]}}_{[\vec{k}]}|0\rangle$ with $\sum_{[\vec{k}]}((\sum_i k_{2i+1}) w_{[\vec{k}]}) = n$, $\sum_{[\vec{k}]}((\sum_i k_{2i}) w_{[\vec{k}]}) = m$ correspond to multi-trace operators labeled by an $S_n \times S_m$ equivalence class of S_{n+m}. That is, a sequence

$$\mathbf{w} = (w_{[\vec{k}]})_{[\vec{k}] \in \tilde{K}} \tag{2.13}$$

with $\sum_{[\vec{k}]}((\sum_i k_{2i+1}) w_{[\vec{k}]}) = n$, $\sum_{[\vec{k}]}((\sum_i k_{2i}) w_{[\vec{k}]}) = m$ uniquely corresponds to an $S_n \times S_m$ equivalence class of S_{n+m}. And each $w_{[\vec{k}]}$ is the number of cycles of type $[\vec{k}]$ in the permutation. We use notation $\mathbf{w}$ to distinguish our previous notation $\vec{w} = (w_1, w_2, \ldots)$ in the half BPS case, see for example [14, 23].

The inner product is a very important structure for the Hilbert space. Let $\alpha, \beta \in S_{n+m}$, with $n, m \geq 1$. we consider the inner product

$$\begin{aligned}(2.14)\quad &\langle \mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})^\dagger \mathrm{tr}(\beta Z^{\otimes n} \otimes Y^{\otimes m})\rangle \\ &= \sum_{\sigma \in S_n \times S_m} \prod_{k=1}^{n} \delta^{i_k}_{i_{\beta(\sigma(k))}} \delta^{i_{\sigma(k)}}_{i_{\alpha^{-1}(k)}} \prod_{h=n+1}^{n+m} \delta^{i_h}_{i_{\beta(\sigma(h))}} \delta^{i_{\sigma(h)}}_{i_{\alpha^{-1}(h)}} \\ &= \sum_{\sigma \in S_n \times S_m} \mathrm{tr}(\alpha^{-1}\sigma^{-1}\beta\sigma) \\ &= \sum_{\sigma \in S_n \times S_m} N^{C(\alpha^{-1}\sigma^{-1}\beta\sigma)},\end{aligned}$$

where in the second line, we sum over $\sigma \in S_n \times S_m$. And in the last line we use $C(\alpha)$ to represent the number of cycles in the permutation α. Note that this expression contains all orders of N, and highest order of N appears only when $\alpha^{-1}\sigma^{-1}\beta\sigma = 1$, $C(1) = n + m$. Therefore N^{m+n} appear only when α, β are in the same equivalence class, that is $[\alpha] = [\beta]$. Using appropriate normalization:

$$\begin{aligned}(2.15)\quad &\langle \mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})^\dagger \mathrm{tr}(\beta Z^{\otimes n} \otimes Y^{\otimes m})\rangle \\ &= \sum_{\sigma \in S_n \times S_m} N^{C(\alpha^{-1}\sigma^{-1}\beta\sigma)} \\ &= N^{m+n}(\delta_{[\alpha],[\beta]} \left(\sum_{\substack{\sigma \in S_n \times S_m \\ \alpha = \sigma^{-1}\alpha\sigma}} 1 \right) + O(1/N)).\end{aligned}$$

And in the last line, $\delta_{[\alpha],[\beta]}$ equals 1 only when α, β are in the same $S_n \times S_m$ equivalence class of S_{n+m}, which is also when $\mathrm{tr}(\alpha Z^{\otimes n} Y^{\otimes m}) = \mathrm{tr}(\beta Z^{\otimes n} Y^{\otimes m})$.

The coefficient $\sum_{\substack{\sigma \in S_n \times S_m \\ \alpha = \sigma^{-1}\alpha\sigma}} 1$ is determined by the $S_n \times S_m$ equivalence class of α. The equivalence class $[\alpha]$ can be represented by the sequence $\mathbf{w}$, then we have

$$(2.16)\quad \sum_{\substack{\sigma \in S_n \times S_m \\ \alpha = \sigma^{-1}\alpha\sigma}} 1 = \prod_{[\vec{k}] \in K} N([\vec{k}], w_{[\vec{k}]}),$$

where

$$(2.17)\quad N([\vec{k}], w_{[\vec{k}]}) = \begin{cases} k^{w_k} w_k! & [\vec{k}] \in K_0, \vec{k} = (k, 0) \text{ or } \vec{k} = (0, k) \\ w_{[\vec{k}]}! & [\vec{k}] \notin K_0. \end{cases}$$

Therefore we have

$$\left\langle \operatorname{tr}\left(\alpha\left(\frac{Z}{\sqrt{N}}\right)^{\otimes n}\otimes\left(\frac{Y}{\sqrt{N}}\right)^{\otimes m}\right)^{\dagger}\operatorname{tr}\left(\beta\left(\frac{Z}{N}\right)^{\otimes n}\otimes\left(\frac{Y}{\sqrt{N}}\right)^{\otimes m}\right)\right\rangle = \delta_{[\alpha],[\beta]}\prod_{[\vec{k}]\in\tilde{K}} N([\vec{k}],w_{[\vec{k}]}) + O(1/N), \tag{2.18}$$

where we have worked in the large N limit. The above formula completely determines the inner product in the Hilbert space of quarter BPS operators. Using our notation (2.12), the inner product can be written as

$$\langle 0|a_{[\vec{k}]}^{w_{[\vec{k}]}} a_{[\vec{k}]}^{\dagger w_{[\vec{k}]}}|0\rangle = N([\vec{k}],w_{[\vec{k}]}) + O(1/N), \tag{2.19}$$

and we get the commutation relation for the operators

$$[a_{[\vec{k}]}, a^{\dagger}_{[\vec{k}]'}] = N([\vec{k}],1)\delta_{[\vec{k}],[\vec{k}']} + O(1/N). \tag{2.20}$$

The trace product basis is very useful for the computation of inner products of other operators, since we can expand other operators in terms of trace product basis.

2.2. Coherent states in a general form

In the rest of this Section, we work in the infinite N limit. With the notation of Sec. 2.1, a general coherent state $|Coh\rangle$ is

$$\exp\left(\sum_{[\vec{k}]\in\tilde{K}} c_{[\vec{k}]}a^{\dagger}_{[\vec{k}]}\right)|0\rangle. \tag{2.21}$$

The condition for coherent state is

$$a_{[\vec{k}]}|Coh\rangle = c_{[\vec{k}]}|Coh\rangle \quad \text{for } [\vec{k}]\in\tilde{K}, \tag{2.22}$$

and we need to check this. Using the commutation relations

$$[a_{[\vec{k}]}, a^{\dagger}_{[\vec{k}']}] = \begin{cases} k\delta_{[\vec{k}],[\vec{k}']} & [\vec{k}]\in K_0, [\vec{k}]=(k,0) \text{ or } [\vec{k}]=(0,k) \\ \delta_{[\vec{k}],[\vec{k}']} & [\vec{k}]\notin K_0, \end{cases} \tag{2.23}$$

$$[a_{[\vec{k}]}, a_{[\vec{k}']}] = 0, \tag{2.24}$$

we have that

$$\begin{aligned}
a_{[\vec{k}]}|Coh\rangle &= a_{[\vec{k}]} \prod_{[\vec{l}]\in\tilde{K}} \exp(c_{[\vec{l}]}a^{\dagger}_{[\vec{l}]})|0\rangle \\
&= \prod_{\substack{[\vec{l}]\in\tilde{K}\\ [\vec{l}]\neq[\vec{k}]}} \exp(c_{[\vec{l}]}a^{\dagger}_{[\vec{l}]}) \left(a_{[\vec{k}]} \sum_j \frac{(c_{[\vec{k}]}a^{\dagger}_{[\vec{k}]})^j}{j!} \right)|0\rangle \\
&= \prod_{\substack{[\vec{l}]\in\tilde{K}\\ [\vec{l}]\neq[\vec{k}]}} \exp(c_{[\vec{l}]}a^{\dagger}_{[\vec{l}]}) \left(N([\vec{k}],1)c_{[\vec{k}]} \sum_j \frac{j(c_{[\vec{k}]}a^{\dagger}_{[\vec{k}]})^{j-1}}{j!} \right)|0\rangle \\
&= N([\vec{k}],1)c_{[\vec{k}]}|Coh\rangle, \qquad (2.25)
\end{aligned}$$

where in the third line we have used $[a_{[\vec{k}]}, a^{\dagger j}_{[\vec{k}]}] = N([\vec{k}],1)ja^{\dagger j-1}_{[\vec{k}]}$. And thus we have proved that the state $\exp(\sum_{[\vec{k}]\in\tilde{K}} c_{[\vec{k}]}a^{\dagger}_{[\vec{k}]})|0\rangle$ is a coherent state.

The coherent states have overlaps with the trace product states. Write a general trace product state as

$$\prod_{[\vec{k}]\in\tilde{K}} a^{\dagger w_{[\vec{k}]}}_{[\vec{k}]}|0\rangle. \qquad (2.26)$$

We consider overlap

$$\begin{aligned}
\langle 0| \prod_{[\vec{k}]\in\tilde{K}} a^{w_{[\vec{k}]}}_{[\vec{k}]}|Coh\rangle &= \langle 0| \prod_{[\vec{k}]\in\tilde{K}} a^{w_{[\vec{k}]}}_{[\vec{k}]} \prod_{[\vec{k}]\in K} \exp(c_{[\vec{k}]}a^{\dagger}_{[\vec{k}]})|0\rangle \\
&= \prod_{[\vec{k}]\in\tilde{K}} \sum_j \frac{(c_{[\vec{k}]})^j}{j!} \langle 0|a^{w_{[\vec{k}]}}_{[\vec{k}]} a^{\dagger j}_{[\vec{k}]}|0\rangle \\
&= \prod_{[\vec{k}]\in\tilde{K}} \frac{(c_{[\vec{k}]})^{w_{[\vec{k}]}}}{w_{[\vec{k}]}!} N([\vec{k}], w_{[\vec{k}]}), \qquad (2.27)
\end{aligned}$$

where

$$N([\vec{k}], w_{[\vec{k}]}) = \begin{cases} k^{w_k} w_k! & [\vec{k}] \in K_0, [\vec{k}] = (k,0) \text{ or } [\vec{k}] = (0,k) \\ w_{[\vec{k}]}! & [\vec{k}] \notin K_0. \end{cases} \qquad (2.28)$$

We will use notation $k \leftrightarrow (k,0), \bar{k} \leftrightarrow (0,k)$, and define

$$c_k = \frac{\Lambda_k}{k}, \quad c_{\bar{k}} = \frac{\Lambda_{\bar{k}}}{\bar{k}}. \tag{2.29}$$

We use this definition to make our results to be compatible with our previous half BPS case [23]. Then the above results can be written as

$$\left\langle 0 | \prod_{[\vec{k}]\in\tilde{K}} a_{[\vec{k}]}^{w_{[\vec{k}]}} | Coh \right\rangle = \prod_{k=1}^{\infty} \Lambda_k^{w_k} \prod_{\bar{k}=1}^{\infty} \Lambda_{\bar{k}}^{w_{\bar{k}}} \prod_{[\vec{k}]\in\tilde{K}-K_0} (c_{[\vec{k}]})^{w_{[\vec{k}]}}. \tag{2.30}$$

We then consider inner product between coherent states. To simplify notation, we define

$$\begin{aligned} |Coh_1\rangle &= \exp\left(\sum_{[\vec{k}]\in\tilde{K}} \alpha_{[\vec{k}]} a^\dagger_{[\vec{k}]}\right) |\, 0\rangle, \\ |Coh_2\rangle &= \exp\left(\sum_{\vec{k}\in\tilde{K}} \beta_{[\vec{k}]} a^\dagger_{[\vec{k}]}\right) |\, 0\rangle. \end{aligned} \tag{2.31}$$

And we also define

$$\alpha_k = \frac{\Lambda_k}{k}, \quad \alpha_{\bar{k}} = \frac{\Lambda_{\bar{k}}}{\bar{k}}, \quad \beta_k = \frac{B_k}{k}, \quad \beta_{\bar{k}} = \frac{B_{\bar{k}}}{\bar{k}}. \tag{2.32}$$

Then the overlap between two coherent states is

$$\begin{aligned} \langle Coh_2 | Coh_1 \rangle &= \sum_{\{w_{[\vec{k}]}\}} \langle 0| \prod_{[\vec{k}]\in\tilde{K}} \frac{(\beta^*_{[\vec{k}]})^{w_{[\vec{k}]}}}{w_{[\vec{k}]}!} a_{[\vec{k}]}^{w_{[\vec{k}]}} |Coh_1\rangle \\ &= \sum_{\{w_{[\vec{k}]}\}} \prod_{[\vec{k}]\in\tilde{K}} \frac{(\beta^*_{[\vec{k}]})^{w_{[\vec{k}]}}}{w_{[\vec{k}]}!} \frac{(\alpha_{[\vec{k}]})^{w_{[\vec{k}]}}}{w_{[\vec{k}]}!} N([\vec{k}], w_{[\vec{k}]}) \\ &= \sum_{\{w_{[\vec{k}]}\}} \prod_k \frac{(B_k^*\Lambda_k)^{w_k}}{k^{w_k} w_k!} \prod_{\bar{k}} \frac{(B_{\bar{k}}^*\Lambda_{\bar{k}})^{w_{\bar{k}}}}{\bar{k}^{w_{\bar{k}}} w_{\bar{k}}!} \prod_{[\vec{k}]\in\tilde{K}-K_0} \frac{(\beta^*_{[\vec{k}]}\alpha_{[\vec{k}]})^{w_{[\vec{k}]}}}{w_{[\vec{k}]}!} \\ &= \exp\left(\sum_k \frac{1}{k} B_k^*\Lambda_k + \sum_{\bar{k}} \frac{1}{\bar{k}} B_{\bar{k}}^*\Lambda_{\bar{k}} + \sum_{[\vec{k}]\in\tilde{K}-K_0} \beta^*_{[\vec{k}]}\alpha_{[\vec{k}]}\right). \end{aligned} \tag{2.33}$$

We can consider the case when

$$\Lambda_k = x_1^k + x_2^k + \cdots, \quad B_k = y_1^k + y_2^k + \cdots, \tag{2.34}$$
$$\Lambda_{\bar{k}} = x_1^{\bar{k}} + x_2^{\bar{k}} + \cdots, \quad B_{\bar{k}} = y_1^{\bar{k}} + y_2^{\bar{k}} + \cdots.$$

Then

$$\begin{aligned}\exp\left(\sum_k \frac{1}{k} B_k^* \Lambda_k\right) &= \exp\left(\sum_k \frac{1}{k}(y_1^{*k} + y_2^{*k} + \cdots)(x_1^k + x_2^k + \cdots)\right)\\ &= \prod_{i,j} \exp\left(\sum_k \frac{(x_i y_j^*)^k}{k}\right)\\ &= \prod_{i,j} \frac{1}{1 - x_i y_j^*}.\end{aligned} \tag{2.35}$$

This result agrees perfectly with our previous results about half BPS operators [23].

2.3. Quarter BPS coherent states and dilatation operator

In [18, 19], a systematic construction of quarter BPS operators is given, and the construction is based on the following symmetrized trace operator

$$A_{r_1 \cdots r_p} = \mathrm{tr}(W_{(r_1} \cdots W_{r_p)}) \tag{2.36}$$

where $r_i = 1, 2$ and we assume that $W_1 = Z, W_2 = Y$. Since all the indices are symmetrized, and the operator $A_{r_1 \cdots r_p}$ only depends on how many r_i are 1 and how many r_i are 2, we write

$$A_{(n,m)} = A_{\underbrace{1, \ldots, 1}_{n}, \underbrace{2, \ldots, 2}_{m}} \tag{2.37}$$

with $n + m = p$.

By definition

$$A_{r_1 \cdots r_p} = \sum_{\sigma \in S_p} \frac{1}{p!} \mathrm{tr}(W_{r_{\sigma(1)}} \cdots W_{r_{\sigma(p)}}). \tag{2.38}$$

Therefore $A_{(n,m)}$ is a linear combination of our previously defined operators in Section 2.1:

$$A_{(n,m)} = \sum_{[\vec{k}]\in\tilde{K}^{(n,m)}} C([\vec{k}])\mathrm{tr}(Z^{k_1}Y^{k_2}Z^{k_3}Y^{k_4}\dots) \tag{2.39}$$

where $C([\vec{k}])$ is some constant to be determined, and

$$\tilde{K}^{(n,m)} = \left\{[\vec{k}] \in \tilde{K} \mid \sum_i k_{2i-1} = n,\ \sum_i k_{2i} = m\right\}. \tag{2.40}$$

However, the labeling $\tilde{K}$ is less convenient for our later study, since we will consider the action of symmetric group on the operators, which takes a complicated form using the labeling $\tilde{K}$. Therefore, we will introduce a new but equivalent labeling of single trace operators. We see that the sequence of matrices $W_r = Z, Y$ resemble a $1d$ lattice, where each lattice site has spin up and down. So we define the set

$$V_{n,m} = \{f \in \mathrm{Map}(\{1,\dots,n+m\},\{\uparrow,\downarrow\}) \mid n = \#\{i \mid f(i) =\uparrow\},\ m = \#\{i \mid f(i) =\downarrow\}\}. \tag{2.41}$$

There is a left action of the group S_{n+m} on $V_{n,m}$:

$$(\sigma f)(i) = f(\sigma^{-1}(i)),\ \forall\sigma \in S_{n+m}. \tag{2.42}$$

Here we use σ^{-1} to define this action to make the action compatible with the group structure. We can check that

$$((\sigma_1\sigma_2)f)(i) = f(\sigma_2^{-1}\sigma_1^{-1}(i)) = (\sigma_1(\sigma_2 f))(i). \tag{2.43}$$

The cyclic group $\mathbb{Z}_{n+m}$ is a subgroup of S_{n+m}, therefore $\mathbb{Z}_{n+m} \subset S_{n+m}$ acts on $V_{n,m}$. Denote λ the generator of $\mathbb{Z}_{n+m}$, which satisfies $\lambda^{n+m} = 1$. We can write the group action as

$$(\lambda^k f)(i) = f(i-k). \tag{2.44}$$

The cyclic property of trace tells us that lattice configurations that differ only by a cyclic permutation should be considered as equivalent. Hence we define the quotient $\tilde{V}_{n,m} = V_{n,m}/\mathbb{Z}_{n+m}$, where we identify $f \sim \lambda^k f$, and write

$[f]$ for the corresponding equivalence class in $\tilde{V}_{n,m}$. For each $[f] \in \tilde{V}_{n,m}$, we have a corresponding operator

$$O_{[f]} = \operatorname{tr}\left(\prod_{i=1}^{n+m} W_{f_i}\right) \tag{2.45}$$

where we identify $W_\uparrow = Z, W_\downarrow = Y$. By the cyclic property of the trace, the definition of $O_{[f]}$ does not depend on the choice of representative f. The advantage of using $\tilde{V}$ to label the single trace operator is that we can easily write down the action of symmetric group S_{n+m} on the operators. First, the action of S_{n+m} on $V_{n,m}$ induces an action on the quotient $\tilde{V}_{n,m}$, and further induces an action on the operator labeled by $\tilde{V}_{n,m}$:

$$\sigma O_{[f]} = O_{[\sigma f]}, \quad \forall \sigma \in S_{n+m}. \tag{2.46}$$

Having shown the advantage of using $\tilde{V}$, we still need to show why this labeling is equivalent to our previous one $\tilde{K}$. To see this, we establish a bijection between the two sets: $\varphi : \tilde{K}^{(n,m)} \xrightarrow{\sim} \tilde{V}_{n,m}$. On the one hand, for a $[\vec{k}] \in \tilde{K}^{(n,m)}$, and write $\vec{k} = (k_1, k_2, \dots, k_{2j})$, we define the corresponding $\varphi([\vec{k}]) = [f] = [\uparrow^{k_1}\downarrow^{k_2}\uparrow^{k_3}\downarrow^{k_4}\cdots]$. The $f = \uparrow^{k_1}\downarrow^{k_2}\uparrow^{k_3}\downarrow^{k_4}\cdots$ is defined by the following:

$$\begin{aligned} &f(1) = \cdots f(k_1) = \uparrow, \quad f(k_1+1) = \cdots f(k_1+k_2) = \downarrow, \\ &f(k_1+k_2+1) = \cdots f(k_1+k_2+k_3) = \uparrow, \dots. \end{aligned} \tag{2.47}$$

On the other hand, for every $[f] \in \tilde{V}_{(n,m)}$, we can define the corresponding $[\vec{k}] = \varphi^{-1}([f]) \in \tilde{K}^{(n,m)}$ by the following:

(2.48)

$$\varphi^{-1}([f]) \in \tilde{K}^{(n,m)} = \begin{cases} [(n,0)], & \text{if } [f] = [\uparrow^n] \in \tilde{V}_{n,0} \\ [(0,m)], & \text{if } [f] = [\downarrow^m] \in \tilde{V}_{0,m} \\ [(k_1,k_2,k_3,k_4,\dots)], & \text{if } [f] = [\uparrow^{k_1}\downarrow^{k_2}\uparrow^{k_3}\downarrow^{k_4}\cdots] \\ & \qquad \in \bigcup_{n,m\geq 1} \tilde{V}_{n,m}. \end{cases}$$

The above formula is complete since $[\uparrow^n]$ is the only element in $\tilde{V}_{n,0}$, and $[\downarrow^m]$ is the only element in $\tilde{V}_{0,m}$, while every element in $\bigcup_{n,m\geq 1} \tilde{V}_{n,m}$ can be written in the form $[\uparrow^{k_1}\downarrow^{k_2}\cdots]$.

Then we can move on to consider the dilatation operator, it will turn out that our above notation will give a simple expression for the operator.

According to [16, 17], the dilatation operator can be expanded in power series of the coupling constant as

$$\hat{D} = \sum_{l=0} \left(\frac{g_{YM}^2}{16\pi^2} \right)^l \hat{D}_{2l}, \tag{2.49}$$

where $\hat{D}_{2l}$ is the l loop dilatation operator. We also consider the $N \to \infty$ limit. Since the coupling constants are related by $g_{YM}^2 N = \lambda$, where λ is the 't Hooft coupling, we can write

$$\hat{D} = \hat{D}_0 + \frac{\lambda}{16\pi^2 N} \hat{D}_2. \tag{2.50}$$

Therefore, for one loop dilatation operator acting on some operator, we will consider:

$$\frac{\lambda}{16\pi^2 N} \hat{D}_2 O(Z, Y). \tag{2.51}$$

The zero loop operator can be written as $\hat{D}_0 = \mathrm{tr}(Z \frac{d}{dZ} + Y \frac{d}{dY})$. And one can easily calculate that the action of the zero loop dilation operator on a general multi-trace operator $\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})$ for $\alpha \in S_{n+m}$ is:

$$\hat{D}_0 \mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}) = (n+m) \mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}). \tag{2.52}$$

We can write the above formula in our previous notation

$$\hat{D}_0 \prod_{[\vec{k}] \in \tilde{K}} a_{[\vec{k}]}^{\dagger w_{[\vec{k}]}} |0\rangle = \left(\sum_{[\vec{k}] \in \tilde{K}} w_{[\vec{k}]} \left(\sum_i k_i \right) \right) \prod_{[\vec{k}] \in \tilde{K}} c_{[\vec{k}]}^{\dagger w_{[\vec{k}]}} |0\rangle, \tag{2.53}$$

or we can write

$$[\hat{D}_0, a_{[\vec{k}]}^\dagger] = \sum_i k_i. \tag{2.54}$$

Now for the one loop operator:

$$\hat{D}_2 = -2 \mathrm{tr} \left([Z, Y] \left[\frac{d}{dZ}, \frac{d}{dY} \right] \right), \tag{2.55}$$

our aim is to calculate

$$\frac{\lambda}{16\pi^2 N} \hat{D}_2 O_{[f]} = \frac{\lambda}{16\pi^2 N} \hat{D}_2 \mathrm{tr} \left(\prod_{i=1}^{n+m} W_{f_i} \right). \tag{2.56}$$

In the $N \to \infty$ limit, we write $\hat{D}_2' = \frac{1}{N}\hat{D}_2$, which can be shown to satisfy the Leibniz rule. To write down the $\hat{D}_2'$ we first need to introduce some special permutation operators that will be useful, namely the swap operator $P_{i,i+1} \in S_{n+m}$, which is defined by:

$$P_{i,i+1}(i) = i+1,\ P_{i,i+1}(i+1) = i,\ P_{i,i+1}(j) = j,\ \text{for } j \neq i, i+1, \tag{2.57}$$

where we identify the indices $n+m+1 \sim 1$. Then using the notations (2.45)–(2.57), the one loop dilatation operator can be easily written as

$$\frac{1}{2}\hat{D}_2' = \sum_{i=1}^{n+m} (\mathbb{I} - P_{i,i+1}). \tag{2.58}$$

Another advantage of using $\tilde{V}$ for the labeling is that the symmetrized trace operator $A_{(n,m)}$ can be written in a simple form:

$$A_{(n,m)} = \frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} \sigma O_{[f_0]}, \tag{2.59}$$

where $[f_0] \in \tilde{V}_{n,m}$ can be given by

$$f_0(1) = \cdots = f_0(n) = \uparrow, \quad f_0(n+1) = \cdots = f_0(n+m) = \downarrow .$$

Here as before, we use $a^\dagger$ to represent the corresponding creation operator, for example

$$a^\dagger_{[f]} \leftrightarrow O_{[f]}, \quad a^\dagger_{(n,m)} \leftrightarrow A_{(n,m)}. \tag{2.60}$$

These are analogous to the relation (2.11).

Proposition 2.1. *The following operator defined by*

$$O_{(n,m)} = \sum_{\substack{l_i \geq 0 \\ l_1 + l_2 + \cdots + l_m = n}} \mathrm{tr}(Z^{l_1} Y Z^{l_2} Y \cdots Z^{l_m} Y) \tag{2.61}$$

for $m > 0$*, and* $O_{(n,0)} = \mathrm{tr}(Z^n)$ *for* $m = 0$*, is proportional to the symmetrized operator* $A_{(n,m)}$*.*

Proof. For $m > 0$, we can write $O_{(n,m)}$ as

$$O_{(n,m)} = \sum_{[f]\in \tilde{V}_{n,m}} c_{[f]} O_{[f]} \tag{2.62}$$

and we need to find the coefficient $c_{[f]}$.

First, we see that for any $[f] \in \tilde{V}_{(n,m)}$, $c_{[f]} \geq 1$. This is because we can describe a $[f] \in \tilde{V}_{(n.,m)}$ by saying that there are first l_1 Zs followed by a Y and then l_2 Zs followed by a Y and so on.

Second, for a generic sequence $l_1, l_2, \ldots, l_m$, the new sequence $l'_1 = l_{\lambda(1)}, l'_2 = l_{\lambda(2)}, \ldots, l'_m = l_{\lambda(m)}$ for a $\lambda \in \mathbb{Z}_m$ corresponds to the same operator. For each $\lambda \in \mathbb{Z}_m$ we have a new but equivalent sequence, thus if we forget the over counting, a generic operator $O_{[f]}$ is counted m times in the summation. However, there is possibility of over counting since it is possible that for some $l_1, l_2, \ldots, l_m$ and some λ we have $l_1 = l_{\lambda(1)}, l_2 = l_{\lambda(2)}, \ldots, l_m = l_{\lambda(m)}$. Therefore for general cases, the results should be m divided by the number of group elements in $\mathbb{Z}_m$ that keeps the sequence $l_1, \ldots, l_m$ unchanged. Remember that we use $[f] \in \tilde{V}_{(n,m)}$ to label the operator, and if the sequence $l_1, \ldots, l_m$ corresponds to $[f]$, then the number of group elements in $\mathbb{Z}_m$ that fixes the sequence $l_1, \ldots, l_m$ is the same as $|Stab_f(\mathbb{Z}_{n+m})|$, where

$$Stab_f(\mathbb{Z}_{n+m}) = \{\sigma \in \mathbb{Z}_{n+m} \mid \sigma f = f\}. \tag{2.63}$$

Therefore, for any $[f] \in \tilde{V}_{n,m}$, the coefficient $c_{[f]}$ can be written as

$$c_{[f]} = \frac{m}{|Stab_f(\mathbb{Z}_{n+m})|}. \tag{2.64}$$

And we can write $O_{(n,m)}$ as

$$O_{(n,m)} = \sum_{[f]\in \tilde{V}_{(n,m)}} \frac{m}{|Stab_f(\mathbb{Z}_{n+m})|} O_{[f]}. \tag{2.65}$$

For the case $m = 0$, $O_{(n,0)} = \mathrm{tr}(Z^n)$ by definition.

Now we consider the symmetrized operator $A_{(n,m)}$, which can be written in the form

$$A_{(n,m)} = \sum_{[f]\in \tilde{V}_{n,m}} c'_{[f]} O_{[f]}. \tag{2.66}$$

We can also write the definition of $A_{(n,m)}$

$$A_{(n,m)} = \frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} \sigma O_{[f_0]}. \tag{2.67}$$

For any f, we can always find a σ_f such that $\sigma_f f_0 = f$. Define $I \subset \{1, 2, \ldots, n+m\}$ by

$$I = \{i \mid f(i) = \uparrow\}. \tag{2.68}$$

And similarly define $J \subset \{1, 2, \ldots, n+m\}$ by

$$J = \{j \mid f(j) = \downarrow\}. \tag{2.69}$$

Then we define $S_I \subset S_{n+m}$ to be the group of permutation of indices in I and similarly for S_J. Then for any $\pi \in S_I \times S_J$, we have

$$\pi f = f, \quad \Rightarrow \quad (\pi\sigma_f) f_0 = f. \tag{2.70}$$

Therefore each f is counted $|S_I| \times |S_J|$ times. Then consider the equivalence class $[f]$. We need to count the number of elements in the equivalence class, which is given by $(n+m)/|Stab_f(\mathbb{Z}_{n+m})|$. The coefficient $c'_{[f]}$ is then the product of the overall $\frac{1}{(n+m)!}$ with the above two factors, which is

$$\begin{aligned} c'_{[f]} &= \frac{1}{(n+m)!} |S_I| \times |S_J| \frac{n+m}{|Stab_f(\mathbb{Z}_{n+m})|} \\ &= \frac{n!m!}{(n+m)!} \frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|}. \end{aligned} \tag{2.71}$$

And we have

$$A_{(n,m)} = \sum_{[f] \in \tilde{V}_{n,m}} \frac{n!m!}{(n+m)!} \frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|} O_{[f]}. \tag{2.72}$$

Therefore, for $m > 0$, we have

$$A_{(n,m)} = \frac{n!m!}{(n+m)!} \frac{(n+m)}{m} O_{(n,m)}. \tag{2.73}$$

Here, the formula is not symmetric in n and m since the definition of $O_{(n,m)}$ is not symmetric in n and m.

And for $m = 0$, $A_{(n,0)} = \mathrm{tr}(Z^n) = O_{(n,0)}$. □

From the calculation in the above proof, or specifically from the equation (2.72), we can easily express $a^\dagger_{(n,m)}$ by linear combination of $a^\dagger_{[f]}$ as follows:

$$a^\dagger_{(n,m)} = \sum_{[f]\in\tilde{V}_{n,m}} \frac{n!m!}{(n+m)!} \frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|} a^\dagger_{[f]}. \tag{2.74}$$

Then we can also write down the commutation relations

$$[a_{(n,m)}, a^\dagger_{(n',m')}] = \delta_{nn'}\delta_{mm'} \sum_{[f]\in\tilde{V}_{n,m}} \left(\frac{n!m!}{(n+m)!} \frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|} \right)^2 \tag{2.75}$$

for $n, m \geq 1$

and

$$[a_{(n,0)}, a^\dagger_{(n',0)}] = n\delta_{nn'}, \quad [a_{(0,m)}, a^\dagger_{(0,m')}] = m\delta_{mm'}. \tag{2.76}$$

It's also easy to see that the symmetrized trace operator $A_{(n,m)}$ is in the kernel of $\frac{1}{2}\hat{D}'_2$. Note that

$$\begin{aligned}(\mathbb{I} - P_{i,i+1}) \sum_{\sigma\in S_{n+m}} \sigma &= \sum_{\sigma\in S_{n+m}} \sigma - \sum_{\sigma\in S_{n+m}} P_{i,i+1}\sigma \\ &= \sum_{\sigma\in S_{n+m}} \sigma - \sum_{\sigma\in S_{n+m}} \sigma = 0.\end{aligned} \tag{2.77}$$

Therefore $(\mathbb{I} - P_{i,i+1})A_{(n,m)} = 0$ and we have

$$\frac{1}{2}\hat{D}'_2 A_{(n,m)} = 0. \tag{2.78}$$

Proposition 2.2. *The quarter BPS coherent states*

$$|Coh^{\frac{1}{4}}\rangle = \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^\dagger_{(n,m)} \right) |0\rangle \tag{2.79}$$

where $c_{(n,m)}$ are complex coefficients, are annihilated by $\hat{D}'_2$ and are eigenstates of the annihilation operators $a_{(n,m)}$.

Proof. The quarter BPS coherent states are

$$|Coh^{\frac{1}{4}}\rangle = \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle \tag{2.80}$$

where $c_{(n,m)}$ are complex coefficients. We can then easily prove that exponential of the symmetrized operator is still in the kernel of $\hat{D}'_2$

$$\begin{aligned}
&\frac{1}{2}\hat{D}'_2 \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle \\
&\quad = \frac{1}{2}\hat{D}'_2 \prod_{n,m\geq 0}\sum_{l\geq 0}\frac{1}{l!}(c_{(n,m)}a^{\dagger}_{(n,m)})^{l}|0\rangle \\
&\quad = \frac{1}{2}\hat{D}'_2 \sum_{l_{(n,m)}}\prod_{n,m\geq 0}\frac{1}{l_{(n,m)}!}(c_{(n,m)}a^{\dagger}_{(n,m)})^{l_{(n,m)}}|0\rangle \\
&\quad = \sum_{l_{(n,m)}}\frac{1}{2}\hat{D}'_2 \prod_{n,m\geq 0}\frac{1}{l_{(n,m)}!}(c_{(n,m)}a^{\dagger}_{(n,m)})^{l_{(n,m)}}|0\rangle = 0,
\end{aligned} \tag{2.81}$$

where in the last line we have used the fact that $\frac{1}{2}\hat{D}'_2 a^{\dagger}_{(n,m)} = 0$ and the action of $\frac{1}{2}\hat{D}'_2$ satisfies Leibniz rule.

Now we show that the state defined above is the eigenstate of the annihilation operators. First, consider the case when $n, m \geq 1$

$$\begin{aligned}
&a_{(n,m)} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle \\
&\quad = \left(\sum_{[f]\in\tilde{V}_{n,m}} \frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|} a_{[f]}\right) \\
&\qquad \times \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle.
\end{aligned} \tag{2.82}$$

And for $[f] \in \tilde{V}_{(n,m)}$

$$a_{[f]} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle \tag{2.83}$$

$$= a_{[f]} \exp\left(\sum_{n,m\geq 0}\sum_{[f]\in\tilde{V}_{n,m}} c_{(n,m)} \frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|} a^{\dagger}_{[f]}\right)|0\rangle \tag{2.84}$$

$$= \frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|} c_{(n,m)} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle.$$

Combining the above two formulas, we have

$$\begin{aligned} & a_{(n,m)} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle \\ &= c_{(n,m)} \sum_{[f]\in\tilde{V}_{n,m}} \left(\frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|}\right)^2 \\ &\quad \times \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle. \end{aligned} \tag{2.85}$$

Then we consider the case $m = 0$. In this situation, $V_{n,0}$ contains only one element $f = \uparrow^n$, and $|Stab_{\uparrow^n}(\mathbb{Z}_n)| = n$. Then we have $a^{\dagger}_{(n,0)} = a^{\dagger}_n$ which corresponds to our previous half BPS operator. The derivation is formally the same.

$$\begin{aligned} & a_{(n,0)} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle \\ &= \exp\left(\sum_{n\geq 1,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right) a_n \exp\left(\sum_k c_{(k,0)} a^{\dagger}_k\right)|0\rangle \\ &= \exp\left(\sum_{n\geq 1,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right) n c_{(n,0)} \exp\left(\sum_k c_{(k,0)} a^{\dagger}_k\right)|0\rangle \\ &= n c_{(n,0)} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^{\dagger}_{(n,m)}\right)|0\rangle. \end{aligned} \tag{2.86}$$

Similarly, for $n = 0$, we have

$$a_{(0,m)} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^\dagger_{(n,m)}\right) |\,0\rangle = m c_{(0,m)} \exp\left(\sum_{n,m\geq 0} c_{(n,m)} a^\dagger_{(n,m)}\right) |\,0\rangle. \tag{2.87}$$

□

The stabilizer subgroup $Stab_f(\mathbb{Z}_{n+m})$ plays an important role in our above results. Basically, it determines the normalization of the symmetrized operator $A_{(n,m)}$. It's easy to see that they satisfy the following formula

$$\sum_{[f]\in\tilde{V}_{n,m}} \frac{n!m!}{(n+m)!} \frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|} = 1. \tag{2.88}$$

2.4. Inner products with quarter BPS coherent states

The truncated coherent state is

$$|Coh^{\frac{1}{4}}(x,y,c)\rangle = \exp\left(\sum_n \frac{\Lambda_n}{n} a^\dagger_{(n,0)} + \sum_m \frac{\Lambda'_m}{m} a^\dagger_{(0,m)} + c a^\dagger_{(1,1)}\right)|0\rangle, \tag{2.89}$$

with

$$\Lambda_n = x_1^n + x_2^n + \cdots, \quad \Lambda'_m = y_1^m + y_2^m + \cdots. \tag{2.90}$$

Using equation (2.33) for the overlap of coherent states in a general form, we can directly write the overlap between two truncated coherent states as follows:

$$\langle Coh^{\frac{1}{4}}(x,y,c)|Coh^{\frac{1}{4}}(x',y',c')\rangle = \prod_{i,j} \frac{1}{1 - x'_i x^*_j} \prod_{i,j} \frac{1}{1 - y'_i y^*_j} \exp(c' c^*). \tag{2.91}$$

Then for the special case where $x' = x, y' = y, c' = c$, we get the norm-squared of a truncated coherent state:

$$\langle Coh^{\frac{1}{4}}(x,y,c)|Coh^{\frac{1}{4}}(x,y,c)\rangle = \prod_{i,j} \frac{1}{1 - x_i x^*_j} \prod_{i,j} \frac{1}{1 - y_i y^*_j} \exp(|c|^2). \tag{2.92}$$

We consider the overlap of the coherent states with trace product basis. Using notation $a^\dagger_{[f]}$ in this section, then a general basis can be written as

$$\prod_{[f]\in\tilde{V}} a^{\dagger w_{[f]}}_{[f]}|0\rangle, \tag{2.93}$$

where $\tilde{V} = \bigcup_{n,m} \tilde{V}_{n,m}$.

Then we consider

$$\langle 0|\prod_{[f]\in\tilde{V}} a^{w_{[f]}}_{[f]}|\,Coh^{\frac{1}{4}}\rangle, \tag{2.94}$$

where $|Coh^{\frac{1}{4}}\rangle = \exp(\sum_{n,m} c_{(n,m)}a^\dagger_{(n,m)})|0\rangle$. Since $|Coh^{\frac{1}{4}}\rangle$ is just a special case of our previous $|Coh\rangle = \exp(\sum_{[f]\in\tilde{V}} c_{[f]}a^\dagger_{[f]})|0\rangle$, we still use the equation (2.30). And we have that

$$\langle 0|\prod_{[f]\in\tilde{V}} a^{w_{[f]}}_{[f]}|\,Coh\rangle = \prod_n (nc_{[\uparrow^n]})^{w_{[\uparrow^n]}}\prod_m (mc_{[\downarrow^m]})^{w_{[\downarrow^m]}}\prod_{[f]\in\cup_{n,m\geq 1}\tilde{V}_{n,m}} (c_{[f]})^{w_{[f]}}, \tag{2.95}$$

where $\uparrow^n$ is defined to map every $i\in\{1,\ldots,n\}$ to $\uparrow$, and similarly for $\downarrow^m$. We can write $|Coh^{\frac{1}{4}}\rangle$ as

$$|\,Coh^{\frac{1}{4}}\rangle = \exp\left(\sum_{n,m\geq 0}\sum_{[f]\in\tilde{V}_{n,m}} c_{(n,m)}\frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|}a^\dagger_{[f]}\right)|\,0\rangle. \tag{2.96}$$

Therefore, replacing $c_{[f]}$ by $c_{(n,m)}\frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|}$ in (2.95), we have

$$\begin{aligned}\langle 0|\prod_{[f]\in\tilde{V}} a^{w_{[f]}}_{[f]}|\,Coh^{\frac{1}{4}}\rangle &= \prod_n (nc_{(n,0)})^{w_{[\uparrow^n]}}\prod_m (mc_{(0,m)})^{w_{[\downarrow^m]}}\\ &\quad\times\prod_{[f]\in\cup_{n,m\geq 1}\tilde{V}_{n,m}}\left(c_{(n,m)}\frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|}\right)^{w_{[f]}}.\end{aligned} \tag{2.97}$$

In some situations, the above formula will give zero, and whether the above overlap equals zero depends on which $c_{(n,m)}$ equals zero.

For the special case of half BPS states, we have

$$w_{[f]} = 0 \text{ for } [f] \notin \tilde{V}_{n,0}, \tag{2.98}$$

and the coherent state is

$$|Coh^{\frac{1}{2}}\rangle = \exp\left(\sum_n c_{(n,0)} a^\dagger_{[\uparrow^n]}\right)|\,0\rangle. \tag{2.99}$$

We have the overlap

$$\langle 0\,|\prod_n a_{[\uparrow^n]}^{w_{[\uparrow^n]}}|Coh^{\frac{1}{2}}\rangle = \prod_n (nc_{(n,0)})^{w_{[\uparrow^n]}}. \tag{2.100}$$

Using notation in [23]

$$a^\dagger_{[\uparrow^n]} = a^\dagger_n,\ w_{[\uparrow^n]} = w_n,\ c_{(n,0)} = \frac{\Lambda_n}{n}, \tag{2.101}$$

this reduces to our previous results in [23],

$$\langle 0\,|\prod_n a_n^{w_n}|Coh^{\frac{1}{2}}\rangle = \prod_n (\Lambda_n)^{w_n}. \tag{2.102}$$

Till now, we have established the basic ingredients related to the quarter BPS coherent states and studied their properties including inner product. In the following section we will consider other interesting operators and look at their relations.

3. Brauer states and their relation to quarter BPS coherent states

3.1. Brauer operators and trace product basis

The Brauer algebra [28] is a natural generalization of the symmetric group algebra. The Walled Brauer algebra [29, 31, 32] is a subalgebra of Brauer algebra. Kimura and Ramgoolam constructed basis of gauge invariant operators of two matrix fields, labelled by representations of Brauer algebras [15, 16]. The Walled Brauer algebra $B_N(n, m)$ is very natural to multi-matrix case. The Schur-Weyl duality for Walled Brauer algebra is

$$V^{\otimes n}\bigotimes \bar{V}^{\otimes m}=\bigoplus_{\gamma} V_{\gamma}^{U(N)}\bigotimes V_{\gamma}^{B_N(n,m)} \tag{3.1}$$

with V and $\bar{V}$ corresponding to the two matrices. In the above formula, γ runs over the set of all staircases. A staircase is defined to be a sequence of integers $(\gamma_1,\gamma_2,\ldots,\gamma_r)$ such that $\gamma_1\geq\gamma_2\geq\cdots\geq\gamma_r$. The sets of positive integers and negative integers determine two partitions respectively, hence we can equivalently write a staircase as $\gamma=(l,\gamma^+,\gamma^-)$ where l is lying in between 0 and $\min(n,m)$. And γ^+,γ^- are two Young tableaux. For more detail of the Walled Brauer algebra, see [15, 16].

We will consider b to be an element of a basis of the Brauer algebra $B_N(n,m)$. Although the basis can be more general in our following definitions, we will consider the basis to be the set of (n,m) diagrams $\mathcal{D}_{n,m}$ defined in Appendix B. The dimension of the $U(N)$ irreducible representation associated with the label γ, is denoted as $t_\gamma=\dim\gamma$. And the dual element b^* is defined and computed in [15], which is as follows

$$b^*=\frac{1}{N^{n+m}}\Sigma^{-1}(\Omega_{n+m}^{-1}(\Sigma(b))^{-1}) \tag{3.2}$$

where the map $\Sigma: B_N(n,m)\to\mathbb{C}[S_{n+m}]$ is defined in (3.19) in [15]. The operator Ω_{n+m} is defined by $\Omega_{n+m}=\sum_{\sigma\in S_{n+m}}N^{C(\sigma)-(n+m)}\sigma$. And its inverse is

$$\Omega_{n+m}^{-1}=\frac{N^{n+m}}{(n+m)!}\sum_T\frac{d_T^2}{\dim T}\sum_{\sigma\in S_{n+m}}\chi_T(\sigma)\sigma. \tag{3.3}$$

There are projection operators P^γ which can be expressed as

$$P^\gamma=t_\gamma\sum_b\chi^\gamma(b)b^*=t_\gamma\sum_b\chi_\gamma(b^*)b, \tag{3.4}$$

where χ^γ is the irreducible representation labeled by staircase γ.

The irreducible representation $V_\gamma^{B_N(n,m)}$ of $B_N(n,m)$ can be further decomposed into a number of representations of the subalgebra $\mathbb{C}[S_n\times S_m]$. We can introduce the operators $Q_{A,i,j}^\gamma$ as follows:

$$Q_{A,ij}^\gamma=t_\gamma\sum_b\chi_{A,ij}^\gamma(b^*)b. \tag{3.5}$$

Here A labels irreducible representations of $\mathbb{C}[S_n \times S_m]$ and i, j run over the multiplicity of the relevant decomposition. The $\chi^{\gamma}_{A,ij}$ is the restricted character. For more detail, see [16, 35]. We see that $P^{\gamma} = \sum_{A,i} Q^{\gamma}_{A,ii}$. The operators $P^{\gamma}, Q^{\gamma}_{A,ij}$ have the property

$$hP^{\gamma}h^{-1} = P^{\gamma},\ hQ^{\gamma}_{A,ij}h^{-1} = Q^{\gamma}_{A,ij}, \quad \text{for all } h \in S_n \times S_m \subset B_N(n,m). \tag{3.6}$$

By the equation (3.37) in [15], we have

$$\begin{aligned} &h\Sigma(P^{\gamma})h^{-1} = \Sigma(P^{\gamma}), \quad h\Sigma(Q^{\gamma}_{A,ij})h^{-1} = \Sigma(Q^{\gamma}_{A,ij}), \\ &\text{for all } h \in S_n \times S_m \subset B_N(n,m). \end{aligned} \tag{3.7}$$

Here, the inclusion $S_n \times S_m \subset B_N(n,m)$ means that we can take from the algebra $B_N(n,m)$ a subset $S_n \times S_m$ which is also a group. The Brauer operators take the form

$$O^{\gamma}_{A,ij}(Z,Y) = \mathrm{tr}(Q^{\gamma}_{A,ij} Z^{\otimes n} Y^{T\otimes m}). \tag{3.8}$$

There are also operators

$$O^{\gamma}(Z,Y) = \mathrm{tr}(P^{\gamma} Z^{\otimes n} Y^{T\otimes m}). \tag{3.9}$$

The inner products of the Brauer operators are described in Appendix A.

Properties of the map Σ is needed to compute various quantities, and an important formula is (3.36) in [15], which is

$$\mathrm{tr}(\Sigma(b) Z^{\otimes n} Y^{\otimes m}) = \mathrm{tr}(b Z^{\otimes n} Y^{T\otimes m}). \tag{3.10}$$

We will consider the inner product between $O^{\gamma}_{A,ij}(Z,Y)$ and our previous trace product basis. Using (3.4–3.10), the inner product can be computed

$$\begin{aligned} &\langle (\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}))^{\dagger} O^{\gamma}_{A,ij}(Z,Y) \rangle \\ &= \langle (\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}))^{\dagger} \mathrm{tr}(Q^{\gamma}_{A,ij} Z^{\otimes n} Y^{T\otimes m}) \rangle \\ &= t_{\gamma} \sum_{b} \chi_{\gamma}(Q^{\gamma}_{A,ij} b^{*}) \langle (\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}))^{\dagger} \mathrm{tr}(b Z^{\otimes n} Y^{T\otimes m}) \rangle \\ &= t_{\gamma} \sum_{b} \chi_{\gamma}(Q^{\gamma}_{A,ij} b^{*}) \langle (\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}))^{\dagger} \mathrm{tr}(\Sigma(b) Z^{\otimes n} Y^{\otimes m}) \rangle \\ &= t_{\gamma} \sum_{b} \chi_{\gamma}(Q^{\gamma}_{A,ij} b^{*}) \sum_{h \in S_n \times S_m} \mathrm{tr}(\alpha^{-1} h^{-1} \Sigma(b) h) \\ &= \sum_{h \in S_n \times S_m} \mathrm{tr}(\alpha^{-1} h^{-1} \Sigma(Q^{\gamma}_{A,ij}) h) \\ &= n! m! \mathrm{tr}(\alpha^{-1} \Sigma(Q^{\gamma}_{A,ij})). \end{aligned} \tag{3.11}$$

Similarly, we have formula for the operators $O^\gamma(Z,Y)$

$$\langle(\mathrm{tr}(\alpha Z^{\otimes n}\otimes Y^{\otimes m}))^\dagger O^\gamma(Z,Y)\rangle = n!m!\mathrm{tr}(\alpha^{-1}\Sigma(P^\gamma)). \tag{3.12}$$

Our next goal is to find more explicit expressions of our above results. However, t_γ, χ_γ, b^* all depend on N, therefore we need more formulas for the N dependence of these quantities.

The $t_\gamma = \dim\gamma$ is the dimension of the $U(N)$ irreducible representation associated with the label γ. The dimension can be computed as follows. First $\gamma = (l,\gamma^+,\gamma^-)$ can always be represented as a sequence $(\gamma_1,\gamma_2,\dots,\gamma_N)$ with $\gamma_1\geq\gamma_2\geq\cdots\geq\gamma_N$. We define $\lambda(\gamma) = (\lambda_1,\dots,\lambda_N)$ with $\lambda_i = \gamma_i-\gamma_N+1$. By definition, $\lambda(\gamma)$ is a partition, therefore it corresponds to an irreducible representation of $U(N)$ and we know the dimension formula for this representation, and we define

$$\dim\gamma = \dim\lambda(\gamma) = \prod_{(i,j)\in\lambda(\gamma)}\frac{N-i+j}{h_{ij}}. \tag{3.13}$$

We have that $\dim\gamma = O(N^{n+m-2l})$. It's interesting to know the leading order behavior of t_γ. Therefore we define $\tilde{t}_\gamma$ as

$$\tilde{t}_\gamma = \lim_{N\to\infty}\frac{1}{N^{n+m-2l}}t_\gamma, \tag{3.14}$$

where $\gamma=(l,\gamma^+,\gamma^-)$. We will give an explicit expression for the coefficient $\tilde{t}_\gamma$.

From the above definition, we see that $\dim\gamma$ does not explicitly depend on l. The $\dim\gamma$ only depends on l through its dependence on γ^+,γ^-, or in other words, $\dim\gamma$ is a function $f(N,\gamma^+,\gamma^-)$ of variables N,γ^+,γ^-. Then we rewrite the expression of $\tilde{t}_\gamma$:

$$\tilde{t}_\gamma = \lim_{N\to\infty}\frac{1}{N^{|\gamma^+|+|\gamma^-|}}f(N,\gamma^+,\gamma^-). \tag{3.15}$$

We find that $\tilde{t}_\gamma$ only depends on γ^+,γ^-,

$$\tilde{t}_\gamma = \tilde{t}(\gamma^+,\gamma^-). \tag{3.16}$$

We find a formula to calculate the coefficient $\tilde{t}_\gamma$ explicitly in the case $\gamma = (0,\gamma^+,\gamma^-)$. First, using equation (4.18) in [15]:

$$\frac{d_R^2 d_S^2}{\dim R\bar{S}} = \frac{m!^2 n!^2}{(m+n)!}\sum_{T\vdash(m+n)}\frac{d_T^2}{\dim T}g(R,S;T) \tag{3.17}$$

where in this notation, $\gamma = (0, R, S)$, and $\dim R\bar{S} = \dim \gamma$. The d_R for a Young tableau $R \vdash n$ is the dimension of the irreducible representation of S_n labeled by R. The $\dim T$ for $T \vdash (n+m)$ is the dimension of the irreducible representation of $U(N)$ labeled by T. In our notation, we have

$$\frac{d_{\gamma^+}^2 d_{\gamma^-}^2}{\dim \gamma} = \frac{m!^2 n!^2}{(m+n)!^2} \sum_{T \vdash (m+n)} \frac{d_T^2}{\dim T} g(\gamma^+, \gamma^-; T). \tag{3.18}$$

And we use formulas

$$\dim T = \prod_{(i,j) \in T} \frac{N - i + j}{h_{ij}}, \qquad d_T = (n+m)! \prod_{(i,j) \in T} \frac{1}{h_{ij}}. \tag{3.19}$$

This gives us

$$\begin{aligned} d_{\gamma^+}^2 d_{\gamma^-}^2 &= \frac{m!^2 n!^2}{(m+n)!} \sum_{T \vdash (m+n)} \dim \gamma \frac{d_T^2}{\dim T} g(\gamma^+, \gamma^-; T) \\ &= \frac{m!^2 n!^2}{(m+n)!^2} \sum_{T \vdash (m+n)} \dim \gamma d_T \frac{(n+m)!}{\prod_{(i,j) \in T} (N - i + j)} g(\gamma^+, \gamma^-; T) \\ &= \lim_{N \to \infty} \frac{m!^2 n!^2}{(m+n)!^2} \\ &\quad \times \sum_{T \vdash (m+n)} d_T \dim \gamma \frac{(n+m)!}{N^{n+m}} (1 + O(1/N)) g(\gamma^+, \gamma^-; T) \\ &= \frac{m!^2 n!^2}{(m+n)!} \tilde{t}_\gamma \sum_{T \vdash (m+n)} d_T g(\gamma^+, \gamma^-; T). \end{aligned} \tag{3.20}$$

Then we use equation (4.10) in [15]:

$$\frac{(m+n)!}{m! n!} d_R d_S = \sum_T g(R, S; T) d_T. \tag{3.21}$$

We find that

$$\tilde{t}_{(0, \gamma^+, \gamma^-)} = \frac{d_{\gamma^+} d_{\gamma^-}}{n! m!} = \frac{d_{\gamma^+} d_{\gamma^-}}{|\gamma^+|! |\gamma^-|!}. \tag{3.22}$$

From this we have

$$\tilde{t}_\gamma = \tilde{t}(\gamma^+, \gamma^-) = \frac{d_{\gamma^+} d_{\gamma^-}}{|\gamma^+|! |\gamma^-|!}. \tag{3.23}$$

For $\gamma = (l, \gamma^+, \gamma^-)$, $|\gamma^+| = n - l, |\gamma^-| = m - l$, therefore

$$\tilde{t}_\gamma = \tilde{t}_{(l,\gamma^+,\gamma^-)} = \frac{d_{\gamma^+} d_{\gamma^-}}{(n-l)!(m-l)!}. \tag{3.24}$$

3.2. Results for $l = 0$ Brauer states

As a warm up, we first consider the case $l = 0$. However, many results obtained in this subsection can be useful for the analysis of more general $l \neq 0$ case. The trace product operator is normalized as $N^{-(n+m)/2}\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})$), and the Brauer basis is normalized as $N^{-(n+m)/2}O^\gamma(Z, Y)$. Therefore we should consider

$$\frac{1}{N^{n+m}} \langle (\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m}))^\dagger O^\gamma(Z, Y)\rangle. \tag{3.25}$$

The projection operator has an expression

$$P^\gamma = t_\gamma \frac{1}{N^{n+m}} \sum_b \chi^\gamma(\Sigma^{-1}(\Omega_{n+m}^{-1}(\Sigma(b))^{-1}))b. \tag{3.26}$$

In the above expression, $t_\gamma, \chi^\gamma, \Omega_{n+m}$ all depend on N. And we will analyze the N dependence of each term. The Ω_{n+m}^{-1} has the expression

$$\Omega_{n+m}^{-1} = \frac{N^{n+m}}{(n+m)!^2} \sum_{T \vdash (n+m)} \frac{d_T^2}{\dim T} \sum_{\sigma \in S_{n+m}} \chi_T(\sigma)\sigma, \tag{3.27}$$

where the $\dim T$, d_T are

$$\dim T = \prod_{(i,j)\in T} \frac{N - i + j}{h_{ij}}, \qquad d_T = (n+m)! \prod_{(i,j)\in T} \frac{1}{h_{ij}}. \tag{3.28}$$

Therefore

$$\frac{d_T}{\dim T} = \frac{(n+m)!}{\prod_{(i,j)\in T}(N - i + j)} = \frac{(n+m)!}{N^{n+m}}(1 + O(1/N)). \tag{3.29}$$

Hence we have for Ω_{n+m}^{-1}

$$\Omega_{n+m}^{-1} = \frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} \sum_{T \vdash (n+m)} d_T \chi_T(\sigma)(1 + O(1/N))\sigma. \tag{3.30}$$

If we assume that $N \geq (n+m)$, then the above summation is over all Young tableaux with $T \vdash (n+m)$. From the representation theory of symmetric groups [33, 34], we know that $\sum_{T\vdash(n+m)} d_T \chi_T$ is just the character of the regular representation $\sum_{T\vdash(n+m)} d_T \chi_T(\sigma) = \chi_{\text{reg}}(\sigma) = (n+m)!\delta(\sigma)$, where $\delta(\sigma) = 1$ if and only if $\sigma = 1$ and $\delta(\sigma) = 0$ otherwise. Therefore we have

$$\Omega^{-1}_{n+m} = (1 + O(1/N)). \tag{3.31}$$

According to Theorem 7.20 in [30], the character χ^γ has expression

$$\chi^\gamma(\zeta) = N^h \sum_{\substack{\lambda\vdash m' \\ \pi\vdash n'}} \left(\sum_{\delta\vdash(l-h)} g(\delta,\gamma^+;\lambda) g(\delta,\gamma^-;\pi) \right) \chi^\lambda_{S_{n'}}(\zeta^+) \chi^\pi_{S_{m'}}(\zeta^-) \tag{3.32}$$

where $\gamma = (l, \gamma^+, \gamma^-)$, with $\gamma^+ \vdash n-l, \gamma^- \vdash m-l$. And $\zeta = (h, \zeta^+, \zeta^-)$, with $\zeta^+ \vdash n' = n-h, \zeta^- \vdash m' = m-h$. The ζ is a staircase that represents a character class of $B_N(n,m)$. The character classes of Brauer algebra are reviewed in Appendix B. The coefficient $g(\delta, \gamma^-; \pi)$ is the Littlewood-Richardson coefficient. The above formula tells us that $\chi^\gamma(\zeta) \neq 0$ only if $h \leq l$.

For $l = 0$, the character χ^γ can be further simplified. In this case, $\chi^\gamma(\zeta) \neq 0$ only if $h = 0$, which means that ζ represents an element $b \in S_n \times S_m$, where $\mathbb{C}[S_n \times S_m] \subset B_N(n,m)$, therefore we have

$$\chi^\gamma(b) = \begin{cases} 0, & b \notin S_n \times S_m \\ \chi^{\gamma^+}_{S_n}(b_1)\chi^{\gamma^-}_{S_m}(b_2), & b = b_1 \otimes b_2 \in S_n \times S_m. \end{cases} \tag{3.33}$$

Putting these together, we have

$$\begin{aligned} &\frac{1}{N^{n+m}} \langle \text{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})^\dagger O^\gamma(Z,Y) \rangle \\ &\quad = \tilde{t}_\gamma n! m! \sum_{b\in S_n\times S_m} \chi^\gamma(\Sigma^{-1}(\Sigma(b)^{-1})) \\ &\qquad \times \frac{1}{N^{n+m}} \text{tr}(\alpha^{-1}\Sigma(b))(1 + O(1/N)) \\ &\quad = \tilde{t}_\gamma n! m! \sum_{b_1\in S_n} \sum_{b_2\in S_m} \chi^\gamma(b_1^{-1} \otimes b_2^{-1}) \\ &\qquad \times \frac{1}{N^{n+m}} \text{tr}(\alpha^{-1} b_1 \otimes b_2^{-1})(1 + O(1/N)) \end{aligned} \tag{3.34}$$

$$= \tilde{t}_\gamma n!m! \sum_{b_1 \in S_n} \chi^{\gamma^+}(b_1^{-1}) \times \sum_{b_2 \in S_m} \chi^{\gamma^-}(b_2^{-1}) N^{C(\alpha^{-1} b_1 \otimes b_2^{-1}) - (n+m)} (1 + O(1/N)).$$

In the above formula, we have used the properties of Σ that

$$\Sigma(\sigma \otimes \tau) = \sigma \otimes \tau^{-1}, \quad \text{for } \sigma \otimes \tau \in S_n \times S_m. \tag{3.35}$$

This gives us:

$$\frac{1}{N^{n+m}} \langle \mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})^\dagger O^\gamma(Z, Y) \rangle = \begin{cases} O(1/N), & \alpha \notin S_n \times S_m \\ \tilde{t}_\gamma n!m! \chi^{\gamma^+}(\alpha_1) \chi^{\gamma^-}(\alpha_2)(1 + O(1/N)), & \alpha = \alpha_1 \otimes \alpha_2 \in S_n \times S_m \end{cases} \tag{3.36}$$

where we have used the fact that for the symmetric group S_n a permutation σ and its inverse σ^{-1} are in the same conjugacy class, or in other words, $\chi^{\gamma^+}(\alpha_1) = \chi^{\gamma^+}(\alpha_1^{-1})$.

We can write the Brauer state by

$$|\gamma\rangle \leftrightarrow O^\gamma \left(\frac{Z}{\sqrt{N}}, \frac{Y}{\sqrt{N}} \right) |0\rangle. \tag{3.37}$$

Using our previous notation, the trace product state is

$$|[\alpha]\rangle \leftrightarrow \mathrm{tr} \left(\alpha \left(\frac{Z}{\sqrt{N}} \right)^{\otimes n} \otimes \left(\frac{Y}{\sqrt{N}} \right)^{\otimes m} \right). \tag{3.38}$$

Hence we can equivalently write our above results as

$$\langle [\alpha] | (0, \gamma^+, \gamma^-) \rangle = \begin{cases} O(1/N), & \alpha \neq \alpha_1 \otimes \alpha_2 \\ d_{\gamma^+} d_{\gamma^-} \chi^{\gamma^+}(\alpha_1) \chi^{\gamma^-}(\alpha_2)(1 + O(1/N)), & \alpha = \alpha_1 \otimes \alpha_2 \end{cases} \tag{3.39}$$

$$|(0, \gamma^+, \gamma^-)\rangle = d_{\gamma^+} d_{\gamma^-} |\gamma^+\rangle \otimes |\gamma^-\rangle + O(1/N). \tag{3.40}$$

Our above derivation is performed for $l = 0$, but the analysis of t_γ and Ω_{n+m} is useful for general case. To consider the situation for $l \neq 0$, we only need to figure out the behavior of χ^γ, which is analyzed in the next subsection.

3.3. Results for $l \neq 0$ Brauer states

In the $l \neq 0$ case, we need to consider a normalization for the Brauer state. We take $|(l,\gamma^+,\gamma^-)\rangle \leftrightarrow N^l O^\gamma(\frac{Z}{\sqrt{N}},\frac{Y}{\sqrt{N}})$. The meaning of the factor N^l will be clear when we later discuss the character of Brauer algebra. Then we consider the overlap

$$\frac{1}{N^{n+m}}\langle \mathrm{tr}(\alpha Z^{\otimes n}\otimes Y^{\otimes m})^\dagger N^l O^\gamma(Z,Y)\rangle. \tag{3.41}$$

And we already have the expression (3.12)

$$\langle \mathrm{tr}(\alpha Z^{\otimes n}\otimes Y^{\otimes m})^\dagger O^\gamma(Z,Y)\rangle = n!m!\mathrm{tr}(\alpha^{-1}\Sigma(P^\gamma)), \tag{3.42}$$

where $P^\gamma = t_\gamma \frac{1}{N^{n+m}}\sum_b \chi^\gamma(\Sigma^{-1}(\Omega_{n+m}^{-1}(\Sigma(b))^{-1}))b$. We have leading order results for Ω_{n+m}^{-1} and t_γ,

$$\Omega_{n+m}^{-1} = 1 + O(1/N), \qquad t_\gamma = \tilde{t}_\gamma N^{n+m-2l}(1+O(1/N)). \tag{3.43}$$

Therefore we have

$$\begin{aligned}&\frac{1}{N^{n+m}}\langle \mathrm{tr}(\alpha Z^{\otimes n}\otimes Y^{\otimes m})^\dagger N^l O^\gamma(Z,Y)\rangle \\ &\quad = n!m!\tilde{t}_\gamma \sum_b \frac{1}{N^l}\chi^\gamma(\Sigma^{-1}(\Sigma(b)^{-1}))\frac{\mathrm{tr}(\alpha^{-1}\Sigma(b))}{N^{n+m}}(1+O(1/N)) \\ &\quad = n!m!\tilde{t}_\gamma \frac{1}{N^l}\chi^\gamma(\Sigma^{-1}(\alpha^{-1}))(1+O(1/N))\end{aligned} \tag{3.44}$$

where in the third line we have used that $\mathrm{tr}(\alpha^{-1}\Sigma(b)) = N^{n+m}$ only for $\Sigma(b) = \alpha$. Write $b_\alpha = \Sigma^{-1}(\alpha^{-1})$, we need to find the order N^l value of $\chi^\gamma(b_\alpha)$.

In Appendix B, we give a rather detailed introduction to the characters of representations of Brauer algebra. The main idea is that although we don't have the notion of conjugacy class, we can define a notion of character class which plays a similar role as conjugacy class. We summarize some main results here. A character class is represented by a staircase $\zeta = (h,\zeta^+,\zeta^-)$. And for an element d, its character class is determined by its cycle type. The characters of elements in the same character class have the same value up to a factor of exponent of N. In the equation (B.11) of the character, we have a sum over $\delta \vdash (l-h)$, which tells us that the character $\chi^\gamma(\zeta) = 0$ for

$h > l$. Besides, the character is of order $N^{z(d)}$, where $z(d)$ by definition is the number of zero cycles in d. And we must have $z(d) \leq h(d)$. Therefore we have an inequality $z(d) \leq h \leq l$. So we have $\frac{1}{N^l}\chi^\gamma(d) \geq O(N^0)$ if and only if $z(d) = h = l$. Furthermore, this condition $z(d) = h = l$ would mean that d has exactly l zero cycles, when each zero cycle only contains one vertex in each side of the wall, and all other cycles that is not a zero cycle must be completely contained in only one side of the wall. Take the diagram (B.1) as an example, it does not satisfy this condition since it has a cycle $6, 4', 3'$, which is not a zero cycle and contains vertices from both side of the wall.

From our previous discussion, we know that

$$\frac{1}{N^{n+m}}\langle \mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})^\dagger N^l O^\gamma\rangle \geq O(N^0)$$

if and only if $b_\alpha = \Sigma^{-1}(\alpha^{-1})$ has l zero cycles, with each zero cycle only containing one vertex in each side of the wall, and each non zero cycle is contained completely in one side of the wall. On the other hand, $\mathrm{tr}(\alpha Z^{\otimes n} \otimes Y^{\otimes m})$ is determined by the $S_n \times S_m$ equivalence class of α which we denoted by $[\alpha]$. Therefore we need to know how to start from the class $[\alpha]$ to obtain the cycle type of b_α.

We know that the class $[\alpha]$ is described by a sequence $\{w_{[\vec{k}]}\}_{[\vec{k}]\in\tilde{K}}$. And we will describe a general procedure to construct the cycle type of b_α from the sequence $\{w_{[\vec{k}]}\}_{[\vec{k}]\in\tilde{K}}$. We need to translate the above condition on b_α into a set of conditions on the sequence $\{w_{[\vec{k}]}\}$.

We obtain the following lemma and give its proof.

Lemma 3.1. *The condition that b_α has l zero cycles, and each zero cycle only contains one vertex in each side of the wall, while each non zero cycle is contained completely in one side of the wall, can be equivalently described by the condition on the sequence $\{w_{[\vec{k}]}\}_{[\vec{k}]\in\tilde{K}}$ (or on $\{w_{[f]}\}_{[f]\in\tilde{V}}$) associated to the class $[\alpha]$. The condition is*

(1) $w_{(1,1)} = l$

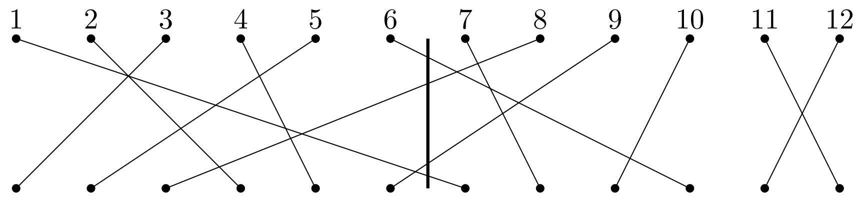
(2) $w_{[\vec{k}]} = 0$ *for* $[\vec{k}] \notin \tilde{K}_0 \bigcup \{(1,1)\}$

or equivalently

(1') $w_{[\uparrow\downarrow]} = l$

(2') $w_{[f]} = 0$ *for* $[f] \notin \bigcup_n \tilde{V}_{n,0} \bigcup_m \tilde{V}_{0,m} \bigcup \tilde{V}_{1,1}$ *where* $\uparrow\downarrow$ *is defined to be the map in* $V_{1,1}$ *that takes* 1 *to* $\uparrow$ *and* 2 *to* $\downarrow$.

Proof. Just as $d \in \mathcal{D}_{n,m}$ can be drawn as a diagram, elements of S_{n+m} can also be drawn as diagrams. For example
(3.45)

$\alpha =$ 

We follow the procedure

(1) Start with vertex $t_1^L(\alpha)$ if it exists.
(2) Follow the edge connected to this vertex. Upon reaching the other side of the edge, jump to the vertex directly above it if we are in $b(\alpha)$ or to the vertex below it if we are in $t(d)$, and continue following the edge connected to that vertex.
(3) Following the above procedure, we will end by returning to the starting vertex and complete a cycle in α. We denote such a cycle c_1.
(4) We start from another vertex that has not been visited and repeat the above process. Each time we finish the above process we will get a cycle c_i in α. And we end the process if we visited all vertices of α.

In this way, we decompose α into disjoint cycles. For example in the above diagram (3.45), we have 4 disjoint cycles. The first is on vertices $1, 7, 8, 3$, the second on vertices $2, 4, 5$, the third on $6, 10, 9$ and the fourth on $11, 12$. Note that this decomposition is just the cycle decomposition of α. In the example, the permutation is just $\alpha = (3, 1, 7, 8)(2, 4, 5)(6, 10, 9)(11, 12)$.

In each cycle, there are vertex on the left hand side of the wall and vertex on the right hand side of the wall, depend on $i \leq n$ or $i > n$. We label a cycle $(i_1, i_2, i_3, \dots)$ by a sequence $(k_1, k_2, k_3, k_4, \dots)$ in such a way that there are first k_1 vertex $i_1, i_2, \dots, i_{k_1}$ on the left hand side of the wall and followed by k_2 vertex on the right hand side of the wall $i_{k_1+1}, \dots, i_{k_1+k_2}$ and so on. For example, cycle $(3, 1, 7, 8)$ is labeled by $(k_1, k_2) = (2, 2)$ and $(2, 4, 5)$ is labeled by $k_1 = 3$. In this way, each cycle is labeled by a $[\vec{k}] \in \tilde{K}$. Also, we can label a cycle $(i_1, i_2, i_3, \dots)$ by a $[f] \in \tilde{V}$, where the corresponding $[f]$ is defined in the following way. We let $f(j) =\uparrow$ if i_j is on the left hand side of the wall and $f(j) =\downarrow$ if i_j is on the right hand side of the wall. If a permutation α has the properties that it has $w_{[\vec{k}]}$ cycles labeled by $[\vec{k}] \in \tilde{K}$ or has $w_{[f]}$ cycles labeled by $[f]$, then the operator satisfies

$$\operatorname{tr}(\alpha Z^{\otimes n} \bigotimes Y^{\otimes m}) = \prod_{[\vec{k}]\in\tilde{K}} \operatorname{tr}(Z^{k_1}Y^{k_2}Z^{k_3}Y^{k_4}\cdots)^{w_{[\vec{k}]}} = \prod_{[f]\in\tilde{V}} O_{[f]}^{w_{[f]}}. \tag{3.46}$$

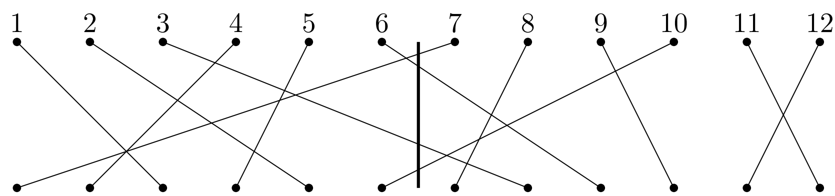
In the above example

$$\operatorname{tr}(\alpha Z^{\otimes n} \bigotimes Y^{\otimes m}) = \operatorname{tr}(Z^2Y^2)\operatorname{tr}(Z^3)\operatorname{tr}(ZY^2)\operatorname{tr}(Y^2). \tag{3.47}$$

Recall that we defined $b_\alpha = \Sigma^{-1}(\alpha^{-1})$. We first describe the α^{-1}. In diagram representation, the diagram of α^{-1} is obtained from the diagram by interchange $t(\alpha)$ and $b(\alpha)$ and keep the edge. In our example (3.45), the inverse is

(3.48)

$\alpha^{-1} =$



After the inverse, a cycle of type $[\vec{k}] = (k_1, k_2, \ldots, k_{2r-1}, k_{2r})$ becomes a cycle of type $[\vec{k'}] = (k_1, k_{2r}, k_{2r-1}, k_{2r-2}, \ldots, k_3, k_2)$. We thus define a map $\phi : K \to K$ by

$$\phi(k_1, k_2, \ldots, k_{2r-1}, k_{2r}) = (k_1, k_{2r}, k_{2r-1}, k_{2r-2}, \ldots, k_3, k_2). \tag{3.49}$$

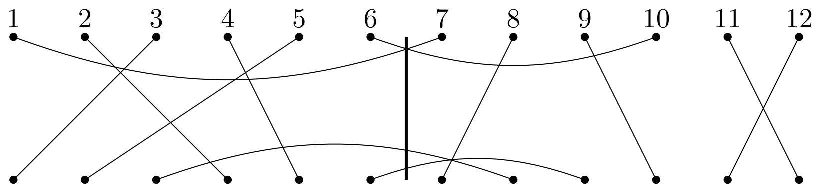
That is, a cycle of α labeled by $[\vec{k}]$ becomes a cycle of α^{-1} labeled by $[\phi(\vec{k})]$. Or using the notation of $[f]$, we define a map $\phi' : \bigcup_{n,m\geq 0} V_{n,m} \to \bigcup_{n,m\geq 0} V_{n,m}$ by

$$\phi'(f)(i) = f(m+n+1-i). \tag{3.50}$$

Then a cycle of α labeled by $[f]$ becomes a cycle of α^{-1} labeled by $[\phi'(f)]$. Note that $\phi(k_1, 0) = (k_1, 0)$, $\phi(0, k_2) = (0, k_2)$, $\phi(k_1, k_2) = (k_1, k_2)$. And $\phi'(\uparrow^n) = \uparrow^n$, $\phi'(\downarrow^m) = \downarrow^m$, $[\phi'(\uparrow\downarrow)] = [\uparrow\downarrow]$.

Now we describe Σ^{-1}. By definition, Σ^{-1} interchange $t^R(\alpha)$ and $b^R(\alpha)$ and keep the edge. In our example

(3.51)

$\Sigma^{-1}(\alpha) =$ 

Since Σ^{-1} keeps every edge, each cycle c in α is still a cycle c in $\Sigma^{-1}(\alpha)$. And for a cycle c labeled by $(k_1, k_2, \dots)$, the corresponding $type(c)$ can be calculated as

$$type(c) = \sum_{i\geq 1} k_{2i-1} - \sum_{i\geq 1} k_{2i}. \tag{3.52}$$

For the same reason, the type for a cycle c labeled by $[f] \in \tilde{V}_{n,m}$ is just $n - m$.

Therefore if a cycle in $\Sigma(\alpha)$ is a zero cycle, then $\sum_{i\geq 1} k_{2i-1} - \sum_{i\geq 1} k_{2i} = 0$ (or $n = m$). And if a cycle c in $\Sigma(\alpha)$ only contains one vertex in each side of the wall, then this cycle is labeled by $[\vec{k}] = [(1,1)]$ (or $[\uparrow\downarrow]$).

Therefore the condition that $b_\alpha = \Sigma^{-1}(\alpha^{-1})$ has l zero cycles, and each zero cycle only contains one vertex in each side of the wall, while each non zero cycle is contained completely in one side of the wall, is equivalent to

$$w_{\phi(1,1)} = w_{(1,1)} = l, \quad w_{[\vec{k}]} = 0 \text{ for } [\vec{k}] \notin \tilde{K}_0 \bigcup \{(1,1)\}, \tag{3.53}$$

or equivalently as

$$w_{[\phi'(\uparrow\downarrow)]} = w_{[\uparrow\downarrow]} = l, \quad w_{[f]} = 0 \text{ for } [f] \notin \bigcup_n \tilde{V}_{n,0} \bigcup_m \tilde{V}_{0,m} \bigcup \tilde{V}_{1,1}. \tag{3.54}$$

This is just the condition in the statement of the lemma. $\square$

Therefore we know that a trace product basis state $\prod_{[\vec{k}]\in K} a_{[\vec{k}]}^{\dagger w_{[\vec{k}]}} |0\rangle$ has nonzero overlap with Brauer state $|(l, \gamma^+, \gamma^-)\rangle$ if and only if the trace product basis state is of the form

$$\prod_{k\geq 1} a_k^{\dagger w_k} \prod_{\bar{k}\geq 1} a_{\bar{k}}^{\dagger w_{\bar{k}}} a_{(1,1)}^{\dagger l} |0\rangle, \tag{3.55}$$

with the constraints

$$\sum_k k w_k = n - l, \quad \sum_{\bar{k}} \bar{k} w_{\bar{k}} = m - l. \tag{3.56}$$

The above two constraints define two partitions of $n-l, m-l$ respectively, and we let $[\alpha_1], [\alpha_2]$ to represent the two partitions respectively, then we have

$$\begin{aligned} &\langle 0| \prod_{k\geq 1} a_k^{w_k} \prod_{\bar{k}\geq 1} a_{\bar{k}}^{w_{\bar{k}}} a_{(1,1)}^l |(l,\gamma^+,\gamma^-)\rangle \\ &\qquad = n!m!\tilde{t}_\gamma \chi^{\gamma^+}([\alpha_1])\chi^{\gamma^-}([\alpha_2])(1+O(1/N)). \end{aligned} \tag{3.57}$$

Using this results, we can write the Brauer state as

$$|(l,\gamma^+,\gamma^-)\rangle = \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{l!(n-l)!(m-l)!}|\gamma^+\rangle \otimes |\gamma^-\rangle \otimes a_{(1,1)}^{\dagger l}|0\rangle + O(1/N), \tag{3.58}$$

where $|\gamma^+\rangle$ is the Young tableau state associated with $a_k^\dagger$, and $|\gamma^-\rangle$ is the Young tableau state associated with $a_{\bar{k}}^\dagger$. Our analysis in the large N limit shows that to leading order, the Brauer operators can be expressed as the product of Young tableau operators with an extra operator mixing two matrices.

We can equivalently write

$$\begin{aligned} &\langle [\alpha] | (l,\gamma^+,\gamma^-)\rangle \\ &\quad = \begin{cases} O(1/N), & \alpha \neq \alpha_1 \otimes \alpha_2 \\ \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{l!(n-l)!(m-l)!}\chi^{\gamma^+}(\alpha_1)\chi^{\gamma^-}(\alpha_2)(1+O(1/N)), & \alpha = \alpha_1 \otimes \alpha_2. \end{cases} \end{aligned} \tag{3.59}$$

From equation (3.58), we see that the Brauer states $|(l,\gamma^+,\gamma^-)\rangle$ lie in the kernel of dilatation operator (2.58) in the infinite N limit. In this case $a_{(n,0)}^\dagger, a_{(0,m)}^\dagger$ and $a_{(1,1)}^\dagger$ all commute with the dilatation operator. For non-planar correction of the action of dilatation operator on more general Brauer operators $O^\gamma_{A;i,j}$, see [16, 36].

3.4. Coherent states and their overlaps with Brauer states

In the rest of this Section, we work in the infinite N limit. We now consider the overlap between coherent states and Brauer states. We first consider general coherent states and then quarter BPS coherent states.

First, the Brauer state $|\gamma\rangle$ takes the form

$$|(l,\gamma^+,\gamma^-)\rangle = \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{l!(n-l)!(m-l)!}|\gamma^+\rangle\otimes|\gamma^-\rangle \otimes a^{\dagger l}_{(1,1)}|0\rangle. \tag{3.60}$$

On the other hand, we consider a general coherent state which takes the form

$$|Coh\rangle = \exp\left(\sum_{[\vec{k}]\in\tilde{K}} c_{[\vec{k}]}a^\dagger_{[\vec{k}]}\right)|0\rangle. \tag{3.61}$$

As in Section 2, we denote

$$c_k = c_{(k,0)}, \quad c_{\bar{k}} = c_{(0,k)}. \tag{3.62}$$

Then the coherent state can be factorized into different modes and we mainly separate the following three parts

$$\exp\left(\sum_k c_k a^\dagger_k\right) = \left(\sum_{\vec{w}}\prod_k \frac{(c_k a^\dagger_k)^{w_k}}{w_k!}\right), \tag{3.63}$$

$$\exp\left(\sum_{\bar{k}} c_{\bar{k}} a^\dagger_{\bar{k}}\right) = \left(\sum_{\vec{w}}\prod_{\bar{k}} \frac{(c_{\bar{k}} a^\dagger_{\bar{k}})^{w_{\bar{k}}}}{w_{\bar{k}}!}\right), \tag{3.64}$$

and $\exp(c_{(1,1)}a^\dagger_{(1,1)}) = \sum_l \frac{(c_{(1,1)}a^\dagger_{(1,1)})^l}{l!}$.

In the following, we take a specific class of coherent states by letting

$$c_k = \frac{\Lambda_k}{k} = \frac{x_1^k + x_2^k + \cdots}{k}, \quad c_{\bar{k}} = \frac{\Lambda_{\bar{k}}}{\bar{k}} = \frac{y_1^{\bar{k}} + y_2^{\bar{k}} + \cdots}{\bar{k}}. \tag{3.65}$$

Then the overlap between the coherent state and Brauer state can be calculated as follows:

$$\begin{aligned}
\langle(l,\gamma^+,\gamma^-)|\,Coh\rangle &= \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{l!(n-l)!(m-l)!}(\langle\gamma^+|\otimes\langle\gamma^-|\otimes\langle 0|a^l_{(1,1)})|Coh\rangle \\
&= \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{l!(n-l)!(m-l)!}\langle\gamma^+|\sum_{\vec{w}}\prod_k \frac{1}{w_k!}\left(\frac{\Lambda_k a^\dagger_k}{k}\right)^{w_k}|0\rangle \\
&\quad\times\langle\gamma^-|\sum_{\vec{u}}\prod_{\bar{k}}\frac{1}{u_{\bar{k}}!}\left(\frac{\Lambda_{\bar{k}}a^\dagger_{\bar{k}}}{\bar{k}}\right)^{u_{\bar{k}}}|0\rangle \\
&\quad\times\langle 0|a^l_{(1,1)}\sum_j \frac{(c_{(1,1)}a^\dagger_{(1,1)})^j}{j!}|0\rangle.
\end{aligned} \tag{3.66}$$

In the above derivation, we only need to consider the three parts since we have

$$\langle (l, \gamma^+, \gamma^-)| a^{\dagger l_{[\vec{k}]}}_{[\vec{k}]} |0\rangle = 0, \text{ for } [\vec{k}] \notin K_0 \bigcup K_2. \tag{3.67}$$

This means that the value of the overlap between the coherent state and an $l = 0$ Brauer state only depends on the value of $c_{[\vec{k}]}$ for $[\vec{k}] \in K_0$.

To further simplify the above formula, we can use results from our previous paper [23] that

$$\langle \gamma^+ | \sum_{\vec{w}} \prod_k \frac{1}{w_k!} \left(\frac{\Lambda_k}{k} \right)^{w_k} | t_k^{w_k} \rangle = s_{\gamma^+}(x_1, x_2, \dots), \tag{3.68}$$

where $|t_k^{w_k}\rangle = a_k^{\dagger w_k}|0\rangle$ and s_{γ^+} is the Schur function corresponding to the Young tableau γ^+. For more about Schur functions, see [34]. Similarly, we denote $|s_{\bar{k}}^{u_{\bar{k}}}\rangle = a_{\bar{k}}^{\dagger u_{\bar{k}}}|0\rangle$ and we have

$$\langle \gamma^- | \sum_{\vec{u}} \prod_{\bar{k}} \frac{1}{u_{\bar{k}}!} \left(\frac{\Lambda_{\bar{k}}}{\bar{k}} \right)^{u_{\bar{k}}} | s_{\bar{k}}^{u_{\bar{k}}} \rangle = s_{\gamma^-}(y_1, y_2, \dots). \tag{3.69}$$

Therefore we have

$$\langle (l, \gamma^+, \gamma^-)| \, Coh \rangle = \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{l!(n-l)!(m-l)!} c^l_{(1,1)} s_{\gamma^+}(x_1, \dots) s_{\gamma^-}(y_1, \dots). \tag{3.70}$$

The above results may be regarded as a generalization of our previous results for the overlap between Young tableau states and half BPS coherent states.

As a special case, we consider $\gamma = (0, \gamma^+, \emptyset)$. In this case, we are considering the Brauer algebra $B_N(n, 0) = \mathbb{C}[S_n]$. The character of Brauer algebra just becomes the character of symmetric group. The Brauer state then becomes:

$$|(0, \gamma^+, \emptyset)\rangle = d_{\gamma^+}|\gamma^+\rangle, \tag{3.71}$$

which is just the Young tableau state. The overlap between the coherent state and the Brauer state becomes

$$\langle (0, \gamma^+, \emptyset)| Coh \rangle = d_{\gamma^+} s_{\gamma^+}(x_1, x_2, \dots). \tag{3.72}$$

This is just our previous result for the half BPS case [23].

We consider another special case with $l \neq 0$, $\gamma^+ = \emptyset, \gamma^- = \emptyset$. In this special case, we first give a different derivation of the form of the Brauer state $|(l, \emptyset, \emptyset)\rangle$. We see that in this case, $l = n = m$. We have expression for the projector P^γ(see (28) in [16]):

$$P^\gamma = \frac{1}{N^l}\Omega_l^{-1}C_{(l)}, \tag{3.73}$$

where $C_{(l)}$ is defined to be

$$C_{(l)} = \sum_{\sigma \in S_l} C_{\sigma(1)\bar{1}}C_{\sigma(2)\bar{2}}\cdots C_{\sigma(l)\bar{l}}. \tag{3.74}$$

And we have

$$\mathrm{tr}(C_{\sigma(1)\bar{1}}C_{\sigma(2)\bar{2}}\cdots C_{\sigma(l)\bar{l}}Z^{\otimes n}\otimes Y^{\otimes m}) = \mathrm{tr}(ZY)^l. \tag{3.75}$$

From our previous results, $\Omega_l^{-1} = 1 + O(1/N)$, we have that $N^l O^\gamma(Z, Y) = l!\mathrm{tr}(ZY)^l + O(1/N)$. Then in the infinite N limit,

$$|(l, \emptyset, \emptyset)\rangle = l! a^{\dagger l}_{(1,1)}|0\rangle. \tag{3.76}$$

This coincides with our equation (3.60) after taking $\gamma^+ = \gamma^- = \emptyset$. And the overlap with coherent state is

$$\langle 0|a^l_{(1,1)}|Coh\rangle = l! c^l_{(1,1)}. \tag{3.77}$$

Another special case is provided for $l \neq 0$, $\gamma^- = \emptyset$. In this case, $m - l = 0$, we have

$$|(l, \gamma^+, \emptyset)\rangle = \frac{n! d_{\gamma^+}}{(n-l)!}|\gamma^+\rangle \otimes a^{\dagger l}_{(1,1)}|0\rangle. \tag{3.78}$$

And the overlap with coherent state is

$$\langle (l, \gamma^+, \emptyset)|Coh\rangle = \frac{n! d_{\gamma^+}}{(n-l)!} c^l_{(1,1)} s_{\gamma^+}(x_1, \ldots). \tag{3.79}$$

Now we move on to consider the quarter BPS coherent states $|Coh^{\frac{1}{4}}\rangle$, which can be written as

$$|Coh^{\frac{1}{4}}\rangle = \exp\left(\sum_{n,m\geq 0} c_{(n,m)}a^\dagger_{(n,m)}\right)|\,0\rangle. \tag{3.80}$$

The symmetrized operator $a^\dagger_{(1,1)}$ is the same as the operator $a^\dagger_{[\vec{k}]}$ for $[\vec{k}] = (1, 1)$. Therefore the overlap between a Brauer state $|\gamma\rangle$ and a quarter BPS coherent state still takes the form:

$$\begin{aligned}(3.81)\qquad &\langle (l,\gamma^+,\gamma^-)|\,Coh^{\frac14}\rangle\\ &= \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{l!(n-l)!(m-l)!}c^l_{(1,1)}s_{\gamma^+}(x_1,x_2,\dots)s_{\gamma^-}(y_1,y_2,\dots),\end{aligned}$$

where

$$(3.82)\qquad c_{(n,0)} = \frac{\Lambda_n}{n} = \frac{x_1^n+x_2^n+\cdots}{n},\quad c_{(0,m)} = \frac{\Lambda'_m}{m} = \frac{y_1^m+y_2^m+\cdots}{m}.$$

We have considered the inner products $\langle\gamma|Coh^{\frac14}\rangle$ and $\langle Coh^{\frac14}|Coh^{\frac14}\rangle$. We considered similar quantity in our previous work since their quotient gives us normalized value of the overlap. And we find

$$\begin{aligned}(3.83)\qquad \frac{|\langle\gamma|Coh^{\frac14}\rangle|^2}{\langle\gamma|\gamma\rangle\langle Coh^{\frac14}|Coh^{\frac14}\rangle} &= \frac{1}{l!}\prod_{i,j}(1-x_ix_j^*)\prod_{i,j}(1-y_iy_j^*)\\ &\quad\times\exp(-|c|^2)|c|^{2l}\left|s_{\gamma^+}(x_1,\dots)s_{\gamma^-}(y_1,\dots)\right|^2.\end{aligned}$$

In the half BPS case, $l=0$ and $\gamma^-=\emptyset$, we have the normalized overlap

$$(3.84)\qquad \prod_{i,j}(1-x_ix_j^*)\left|s_{\gamma^+}(x_1,\dots)\right|^2,$$

which is the same as our previous results in [23], where we further analyzed the case with rectangular tableaux $\gamma^+=\square_{LM}$.

Although Brauer state of the form $|(l,\gamma^+,\gamma^-)\rangle$ does not span the whole Hilbert space, we can consider the truncated subspace spanned by $a^\dagger_{(n,0)}$, $a^\dagger_{(0,m)}$, $a^\dagger_{(1,1)}$. For this reason, we can also consider a subclass of coherent states defined by

$$(3.85)\qquad |Coh^{\frac14}\rangle = \exp\left(\sum_n\frac{\Lambda_n}{n}a^\dagger_{(n,0)}+\sum_m\frac{\Lambda'_m}{m}a^\dagger_{(0,m)}+ca^\dagger_{(1,1)}\right)|\,0\rangle.$$

And we use notation $|Coh^{\frac14}(x,y,c)\rangle$ by requiring that

$$(3.86)\qquad \Lambda_n = x_1^n+x_2^n+\cdots,\quad \Lambda'_m = y_1^m+y_2^m+\cdots.$$

The advantage of considering the truncated quarter BPS coherent state can be seen from the following proposition.

Proposition 3.1. *The Brauer state* $|\gamma\rangle = |(l,\gamma^+,\gamma^-)\rangle$ *and the coherent state* $|Coh^{\frac{1}{4}}(x,y,c)\rangle$ *can be transformed into each other by the following formulas*

$$|Coh^{\frac{1}{4}}(x,y,c)\rangle = \sum_{\gamma} \frac{(n-l)!(m-l)!}{n!m!d_{\gamma^+}d_{\gamma^-}} c^l s_{\gamma^+}(x)s_{\gamma^-}(y)|(l,\gamma^+,\gamma^-)\rangle \tag{3.87}$$

and

$$\begin{aligned}|(l,\gamma^+,\gamma^-)\rangle = {} & \frac{n!m!d_{\gamma^+}d_{\gamma^-}}{(n-l)!(m-l)!}\frac{1}{M_+!M_-!}\oint \frac{dc}{2\pi i c}\prod_{j=1}^{M_+}\frac{dx_j}{2\pi i x_j}\prod_{j=1}^{M_-}\frac{dy_j}{2\pi i y_j} \\ & \times c^{-l}\prod_{1\le i<j\le M_+}\left|1-\frac{x_i}{x_j}\right|^2\prod_{1\le i<j\le M_-}\left|1-\frac{y_i}{y_j}\right|^2 \\ & \times s_{\gamma^+}(x^{-1})s_{\gamma^-}(y^{-1})|Coh^{\frac{1}{4}}(x,y,c)\rangle \end{aligned} \tag{3.88}$$

where M_+ *is the number of rows of* γ^+ *and* M_- *is the number of rows of* γ^-*. The* x^{-1} *here is a short hand for* $(x_1^{-1},x_2^{-1},\dots)$*. And the integration* $\oint$ *is alone the circular paths which are equivalently defined by* $|c|=1$, $|x_j|=1$, $|y_j|=1$.

Proof. For the first formula, we need to note that Brauer states of the form $|\gamma\rangle$ form an orthogonal basis of the truncated subspace. This is because Young tableau states $|\gamma^+\rangle$ for all possible Young tableaux γ^+ form an orthonormal basis of the subspace $\bigcup_n \mathcal{H}_{(n,0)}$. And similarly for $|\gamma^-\rangle$. Also, $\{a^{\dagger l}_{(1,1)}|0\rangle\}_{l\ge 0}$ form an orthogonal basis of the subspace $\mathcal{H}_{(1,1)}$. So we have

$$|Coh^{\frac{1}{4}}(x,y,c)\rangle = \sum_{\gamma}\frac{\langle\gamma|Coh^{\frac{1}{4}}(x,y,c)\rangle}{\langle\gamma|\gamma\rangle}|\gamma\rangle. \tag{3.89}$$

Then, using the formula (3.81) for the overlap, we can derive equation (3.87).

The derivation of the second formula can be considered as performing inverse Fourier transform to the first one. We need to use the following results:

$$\oint \frac{dc}{2\pi i c}c^{-l}c^{l'} = \delta_{ll'}, \tag{3.90}$$

$$\frac{1}{M!}\oint\prod_{j=1}^{M}\frac{dx_j}{2\pi i x_j}\prod_{1\le i<j\le M}\left|1-\frac{x_i}{x_j}\right|^2 s_\lambda(x^{-1})s_\mu(x) = \delta_{\lambda\mu}. \tag{3.91}$$

The first formula is obvious by calculating the residue, and the second one is from [38], see also [23]. We then multiply the three factors

$$\frac{dc}{2\pi i c}c^{l'}, \quad \prod_{j=1}^{M}\frac{dx_j}{2\pi i x_j}\prod_{1\le i<j\le M}\left|1-\frac{x_i}{x_j}\right|^2 s_{\gamma^{+'}}(x^{-1})$$
$$\text{and} \quad \prod_{j=1}^{M}\frac{dy_j}{2\pi i y_j}\prod_{1\le i<j\le M}\left|1-\frac{y_i}{y_j}\right|^2 s_{\gamma^{-'}}(y^{-1})$$

to both sides of equation (3.87). After integration of c, x_i, y_i over the contour, we can find equation (3.88). □

3.5. Entanglement entropy of Brauer states

From our above results, we see that the Brauer states $|\gamma\rangle$ span a subspace of the Hilbert space:

(3.92)
$$\mathcal{H}_{\text{Brauer}\{\gamma\}} = \left(\bigotimes_{[k]\in K_0}\mathcal{H}_{[\vec{k}]}\right)\otimes\mathcal{H}_{(1,1)} = \left(\bigotimes_{k\ge 1}\mathcal{H}_{(k,0)}\otimes\mathcal{H}_{(0,k)}\right)\otimes\mathcal{H}_{(1,1)}.$$

In the following, we always assume that $\gamma = (l, \gamma^+, \gamma^-)$. And to simplify the notation, we write $\mathcal{H}_k = \mathcal{H}_{(k,0)}$, $\mathcal{H}_{\bar{k}} = \mathcal{H}_{(0,k)}$. The subspaces $\mathcal{H}_k, \mathcal{H}_{\bar{k}}$ are generated by t_k and s_k respectively, where we identify

$$t_k \leftrightarrow \operatorname{tr}\left(\frac{Z}{\sqrt{N}}\right)^k, \quad s_k \leftrightarrow \operatorname{tr}\left(\frac{Y}{\sqrt{N}}\right)^k. \tag{3.93}$$

Similar to the previous work of [14, 23], we define traces

$$\begin{aligned}\operatorname{tr}_j &= \operatorname{tr}_{\otimes_{k\ne j}\mathcal{H}_k \bigotimes \otimes_{\bar{k}}\mathcal{H}_{\bar{k}}\otimes\mathcal{H}_{(1,1)}},\\ \operatorname{tr}_{\bar{j}} &= \operatorname{tr}_{\otimes_k\mathcal{H}_k \bigotimes \otimes_{\bar{k}\ne\bar{j}}\mathcal{H}_{\bar{k}}\otimes\mathcal{H}_{(1,1)}},\\ \operatorname{tr}_{(1,1)} &= \operatorname{tr}_{\otimes_k\mathcal{H}_k \bigotimes \otimes_{\bar{k}}\mathcal{H}_{\bar{k}}}.\end{aligned} \tag{3.94}$$

The above notation should be distinguished from the traces $\operatorname{tr}_{\mathcal{H}_j}, \operatorname{tr}_{\mathcal{H}_{\bar{j}}}$. We then consider the entanglement spectrum and entanglement entropy of the Brauer state. After normalization, the Brauer state is

$$\frac{1}{\sqrt{l!}}|\gamma^+\rangle\otimes|\gamma^-\rangle\otimes a^{\dagger l}_{(1,1)}|0\rangle.$$

And we write $|l\rangle_{(1,1)} = \frac{1}{\sqrt{l!}} a^{\dagger l}_{(1,1)}|0\rangle$. Then we can calculate the density operator of a mode of a Brauer state as

$$\begin{aligned} \hat{\rho}_j(\gamma) &= \mathrm{tr}_j(|\gamma^+\rangle \otimes |\gamma^-\rangle \otimes |l\rangle_{(1,1)}\langle\gamma^+| \otimes \langle\gamma^-| \otimes \langle l|_{(1,1)}), \\ \hat{\rho}_{\bar{j}}(\gamma) &= \mathrm{tr}_{\bar{j}}(|\gamma^+\rangle \otimes |\gamma^-\rangle \otimes |l\rangle_{(1,1)}\langle\gamma^+| \otimes \langle\gamma^-| \otimes \langle l|_{(1,1)}), \\ \hat{\rho}_{(1,1)}(\gamma) &= \mathrm{tr}_{(1,1)}(|\gamma^+\rangle \otimes |\gamma^-\rangle \otimes |l\rangle_{(1,1)}\langle\gamma^+| \otimes \langle\gamma^-| \otimes \langle l|_{(1,1)}). \end{aligned} \tag{3.95}$$

Then we have

$$\begin{aligned} \hat{\rho}_j(\gamma) &= \mathrm{tr}_j(|\gamma^+\rangle \otimes |\gamma^-\rangle \otimes |l\rangle_{(1,1)}\langle\gamma^+| \otimes \langle\gamma^-| \otimes \langle l|_{(1,1)}) \\ &= \mathrm{tr}_j(|\gamma^+\rangle\langle\gamma^+|)\langle\gamma^-|\gamma^-\rangle\langle l|l\rangle_{(1,1)} \\ &= \mathrm{tr}_j(|\gamma^+\rangle\langle\gamma^+|) \\ &= \hat{\rho}_j(\gamma^+), \end{aligned} \tag{3.96}$$

where $\hat{\rho}_j(\gamma^+)$ is the density matrix for the Young tableau state $|\gamma^+\rangle$. Similarly

$$\hat{\rho}_{\bar{j}}(\gamma) = \hat{\rho}_{\bar{j}}(\gamma^-). \tag{3.97}$$

And for the $(1,1)$ mode, we have $\hat{\rho}_{(1,1)}(\gamma) = |l\rangle_{(1,1)}\langle l|_{(1,1)}$, which is a pure state density matrix.

Then the calculation of entanglement entropy [37] is straightforward:

$$\begin{aligned} s_j(\gamma) &= -\mathrm{tr}_{\mathcal{H}_j}(\hat{\rho}_j \log(\hat{\rho}_j)), \\ s_{\bar{j}}(\gamma) &= -\mathrm{tr}_{\mathcal{H}_{\bar{j}}}(\hat{\rho}_{\bar{j}} \log(\hat{\rho}_{\bar{j}})), \\ s_{(1,1)}(\gamma) &= -\mathrm{tr}_{\mathcal{H}_{((1,1))}}(\hat{\rho}_{(1,1)} \log(\hat{\rho}_{(1,1)})). \end{aligned} \tag{3.98}$$

We have

$$s_j(\gamma) = s_j(\gamma^+), \quad s_{\bar{j}}(\gamma) = s_{\bar{j}}(\gamma^-), \quad s_{(1,1)}(\gamma) = 0, \tag{3.99}$$

where $s_j(\gamma^+)$ is the entanglement entropy of the Young tableau state γ^+ for mode j and $s_{\bar{j}}(\gamma^-)$ is the entanglement entropy of the Young tableau state γ^- for mode $\bar{j}$. Detailed analysis of the entanglement entropy $s_j(\gamma^+)$ has been given in [14, 23]. And since $\hat{\rho}_{(1,1)}(\gamma)$ is a pure state density matrix, the entanglement entropy for the mode $(1,1)$ is zero.

4. Squeezed states and their relation to Brauer states

Motivated by our previous work [23], we define the squeezed state as follows

$$(4.1)\qquad |Squ_{n,m;n',m'}\rangle = \exp(\mu(a^{\dagger}_{(n,m)}a^{\dagger}_{(n'm')} - a_{(n,m)}a_{(n'm')}))|0\rangle.$$

In the special case where $m = 0$ and $m' = 0$, we can write $a_{(n,0)} = a_n, a^{\dagger}_{(n,0)} = a^{\dagger}_n$. And the above definition gives the half BPS squeezed state (6.1) defined in our previous work [23],

$$(4.2)\qquad |Squ_{nn'}\rangle = \exp(\mu(a^{\dagger}_n a^{\dagger}_{n'} - a_n a_{n'}))|0\rangle.$$

Since the case for $m = m' = 0$ has been discussed in our previous work, here we only consider the case $m, m', n, n' \geq 1$. In this case, we use the commutation relation

$$(4.3)\qquad [a_{(n,m)}, a^{\dagger}_{(n',m')}] = \delta_{nn'}\delta_{mm'} \sum_{[f]\in\tilde{V}_{n,m}} \left(\frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|}\right)^2.$$

To simplify the calculation, we write the above formula as

$$(4.4)\qquad \left[\frac{1}{\kappa_{(n,m)}}a_{(n,m)}, \frac{1}{\kappa_{(n',m')}}a^{\dagger}_{(n',m')}\right] = \delta_{nn'}\delta_{mm'},$$

where

$$(4.5)\qquad \kappa_{(n,m)} = \sqrt{\sum_{[f]\in\tilde{V}_{n,m}} \left(\frac{n!m!}{(n+m)!}\frac{(n+m)}{|Stab_f(\mathbb{Z}_{n+m})|}\right)^2}.$$

Then for $(n,m) \neq (n',m')$, the squeezed state can be expanded as follows:

$$(4.6)\qquad |Squ\rangle = \left(1 - \tanh^2(\mu\kappa_{(n,m)}\kappa_{(n',m')})\right)^{\frac{1}{2}} \times \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\tanh(\mu\kappa_{(n,m)}\kappa_{(n'm')})}{\kappa_{(n,m)}\kappa_{(n',m')}}\right)^j a^{\dagger j}_{(n,m)}a^{\dagger j}_{(n',m')}|0\rangle.$$

As a remark, we mention that the above formula also applies to the case when some of n, m, n', m' equal to zero. In some special cases, we would only need to have

$$\kappa_{(n,0)} = \sqrt{n}, \quad \kappa_{(0,m)} = \sqrt{m}. \tag{4.7}$$

And for the case $m = m' = 0$, the above results give the expansion of the half BPS squeezed state (4.2). It's also easy to see from the above expansion that squeezed states have inner products given by

$$\langle Squ_{n_1,m_1;n_2,m_2} | Squ_{n_3,m_3;n_4,m_4} \rangle = \delta_{n_1 n_3}\delta_{n_2 n_4}\delta_{m_1 m_3}\delta_{m_2 m_4} + \delta_{n_1 n_4}\delta_{n_2 n_3}\delta_{m_1 m_4}\delta_{m_2 m_3}. \tag{4.8}$$

One of the motivations for us to consider the squeezed state is that we can take a limit of the squeezed states to obtain EPR states. First define a new parameter

$$q = \tanh(\mu \kappa_{(n,m)} \kappa_{(n',m')}). \tag{4.9}$$

Then the squeezed state can be written as:

$$|Squ_{n,m;n',m'}\rangle = \left(1 - q^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{q}{\kappa_{(n,m)}\kappa_{(n',m')}} \right)^j a^{\dagger j}_{(n,m)} a^{\dagger j}_{(n',m')} |0\rangle. \tag{4.10}$$

And we can write the corresponding EPR limit as follows:

$$\begin{aligned} |EPR_{n,m;n',m'}\rangle &= \lim_{q \to 1} |Squ_{n,m;n',m'}\rangle \\ &= \mathcal{N}^{-\frac{1}{2}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{1}{\kappa_{(n,m)}\kappa_{(n',m')}} \right)^j a^{\dagger j}_{(n,m)} a^{\dagger j}_{(n',m')} |0\rangle, \end{aligned} \tag{4.11}$$

where $\mathcal{N}^{-\frac{1}{2}}$ is a normalization factor that tends to infinity as $q \to 1$. One can take an infinitesimal positive cutoff $\epsilon \to 0$, such that $1 - q = \epsilon$ and $\mathcal{N} = \frac{1}{2\epsilon}$.

It is interesting to consider the overlap between a squeezed state and a Brauer state. We can see from the expansion (4.6) that the overlap is zero when $n, m, n', m' > 1$. And it is not zero only for the following situations:

1. $(n, m) = (1, 1)$, $m' = 0$ and $\gamma = (l, \gamma^+, \emptyset)$ with $\gamma^+ \vdash ln'$.

In this situation,

$$\langle (l,\gamma^+,\emptyset)|Squ_{1,1;n',0}\rangle = \left(1-\tanh^2(\mu\sqrt{n'})\right)^{\frac{1}{2}}\left(\frac{\tanh(\mu\sqrt{n'})}{\sqrt{n'}}\right)^{l} \times \frac{(n'l+l)!}{(n'l)!} d_{\gamma^+}\chi_{\gamma^+}(\vec{w})|_{\substack{w_k=0 \text{ except}\\ w_{n'}=l}} \tag{4.12}$$

Similarly we can consider $(n,m)=(1,1)$, $n'=0$ and $\gamma=(l,\emptyset,\gamma^-)$ with $\gamma^-\vdash lm'$.

2. $m=0$, $m'=0$, and $\gamma=(0,\gamma^+,\emptyset)$ with $\gamma^+\vdash j(n+n')$.
   In this situation, we have that,

$$\langle (0,\gamma^+,\emptyset)|Squ_{n,0;n',0}\rangle = \left(1-\tanh^2(\mu\sqrt{nn'})\right)^{\frac{1}{2}}\frac{1}{j!} \times \left(\frac{\tanh(\mu\sqrt{nn'})}{\sqrt{nn'}}\right)^{j} d_{\gamma^+}\chi_{\gamma^+}(\vec{w})|_{\substack{w_k=0 \text{ except}\\ w_n=w_{n'}=j}} \tag{4.13}$$

This situation is the same as (6.7) in [23] except that the normalization of the Brauer state is different.
   Similarly we can consider $n=0$, $n'=0$, and $\gamma=(0,\emptyset,\gamma^-)$.

3. $m=0$, $n'=0$, and $\gamma=(0,\gamma^+,\gamma^-)$ with $\gamma^+\vdash jn,\gamma^-\vdash jm'$.
   We have that,

$$\langle (0,\gamma^+,\emptyset)|Squ_{n,0;0,m'}\rangle = \left(1-\tanh^2(\mu\sqrt{nm'})\right)^{\frac{1}{2}}\frac{1}{j!}\left(\frac{\tanh(\mu\sqrt{nm'})}{\sqrt{nm'}}\right)^{j} \times d_{\gamma^+}d_{\gamma^-}\chi_{\gamma^+}(\vec{w})|_{\substack{w_k=0 \text{ except}\\ w_n=j}}\chi_{\gamma^-}(\vec{w})|_{\substack{w_k=0 \text{ except}\\ w_{m'}=j}} \tag{4.14}$$

5. Discussion

In this paper, we constructed quarter BPS coherent states. The construction starts with a general construction of the Hilbert space of two-matrix gauge invariant operators. Then we consider the anomalous dimension dilation operator. In our case, we care about the kernel of the anomalous dimension dilatation operator, and this gives us the quarter BPS operators. Then the construction of quarter BPS coherent states generalize the construction of half BPS coherent states by taking exponential of the creation operators.

These quarter BPS coherent states are also the eigenstates of the annihilation operators. We also computed the inner products involving the quarter BPS coherent states.

The Brauer operators are also explored in this paper. The construction of Brauer operators involves characters of irreducible representations of Brauer algebra [15, 35]. And we calculated the inner product between Brauer operators and trace product operators. The construction of Brauer operators and the analysis of them have been carried out in many previous works, see [15, 35], and the explicit form of these operators are known in special cases, see for example [16, 39, 40]. We see that the Brauer operators are in some sense the generalization of Young tableau operators in quarter BPS case. This observation is also closely related to the dual gravity interpretation [40], where the droplet configuration of the dual gravity solution is described by the two Young tableaux in the Brauer basis. We also calculated the entanglement entropy of the Brauer states, and the results are very similar to the Young tableau states.

One of the motivations of constructing the coherent states is that they are important ingredients in the study of superposition-induced topology change in quantum gravity [14, 41]. With our previous work of a superposition formula that gives a Young tableau state by superposing half BPS coherent states, we considered here similar superposition formulas involving Brauer states and quarter BPS coherent states. We show that one can superpose quarter BPS coherent states to obtain Brauer states. Conversely, our superposition formulas show that one can also superpose Brauer states to obtain quarter BPS coherent states. The ideas of superposition of states on the gravity side have also been considered in [14, 23, 43–45]. It is useful to explore these ideas with the setup of this paper. Also, inspired by previous works [14, 23], it is very interesting to further study the relation between entanglement and the dual spacetime geometry.

We also generalized the squeezed states from our previous half BPS case [23] to quarter BPS case. The squeezed states itself can be regarded as a generalization of the coherent states, since they both satisfy the property that they can saturate the uncertainty principle. Moreover, taking certain limit of the squeezed state can give us a EPR pair state, which is important in the quantum information theory and quantum optics. And in our setup, it is interesting to study their entanglement properties and the dual geometric picture.

In the context of gauge/gravity correspondence, coherent states have gravity dual descriptions in terms of semiclassical geometries, and this has been studied in details in the half BPS case. These coherent states, in the dual gravity side, correspond to creating deformations [13, 24, 46–48] on the vacuum geometry. Some classes of these geometries can be reduced to lower dimensions and viewed as geometries in lower dimensional gravity [10, 49–51]. Geometries in lower dimensional gravity that are dual to coherent states have also been considered in [52]. As similar to the half BPS case, there are smooth spacetime geometries dual to quarter BPS states, see e.g. [49, 50, 53–55] and related discussions. These quarter BPS states include the quarter BPS coherent states that we describe in this paper. It would also be interesting to explore the gravity dual of the BPS coherent states further.

Our results may provide further insights into emergent spacetime geometry and other interesting related phenomena in gauge/gravity correspondence. Various other similar spacetime geometries in the context of string theory and quantum gravity have been analyzed, see for example [56]−[62] and their related discussions. Our methods and discussions may also be related to 2d Yang-Mills [63] and to fuzzball proposal [56]. It would also be good to understand more the relation to proposals of emergent spacetime geometries.

We know that the dynamics of half BPS sector of $\mathcal{N} = 4$ SYM is described by a matrix quantum mechanical model with harmonic oscillator potential, which itself is equivalent to the dynamics of N free fermions. And the dynamics of quarter and eighth BPS sector are investigated in, for example [13]. Therefore, our discussions are also related to the matrix model approach and other approaches for several matrix fields [13, 64–69].

We take the approach that first includes both BPS states and non-BPS states. Although we mainly studied the BPS states, it is also very interesting to consider other non-BPS states in this system, such as [70–72]. There are restricted Schur basis and flavor symmetry basis for example [73], which have their own distinct properties and can be transformed into each other. Therefore we can also study their relation to our setup.

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Appendix A. Orthogonality relation of Brauer states

For general $l \geq 0$, we take a normalization of Brauer state

$$|\gamma\rangle \leftrightarrow N^l O^\gamma \left(\frac{Z}{\sqrt{N}}, \frac{Y}{\sqrt{N}} \right),$$

where $\gamma = (l, \gamma^+, \gamma^-)$. We can write down the orthogonality relation. According to formula (7.15) in [15]

$$\langle O^{\gamma_1}(Y,Z)^\dagger O^{\gamma_2}(Z,Y)\rangle = m!n!\delta_{\gamma_1,\gamma_2} d_{\gamma_1} \dim \gamma_1. \tag{A.1}$$

We write the above formula in terms of notation $|\gamma_1\rangle = |(l_1, \gamma_1^+, \gamma_1^-)\rangle, |\gamma_2\rangle = |(l_2, \gamma_2^+, \gamma_2^-)\rangle$,

$$\begin{aligned} \langle\gamma_1|\gamma_2\rangle &= \frac{1}{N^{n+m}} \langle N^{l_1} O^{\gamma_1}(Y,Z)^\dagger N^{l_2} O^{\gamma_2}(Z,Y)\rangle \\ &= \frac{1}{N^{n+m-l_1-l_2}} m!n!\delta_{\gamma_1,\gamma_2} d_{\gamma_1} \dim \gamma_1 \\ &= m!n!\delta_{\gamma_1,\gamma_2} d_{\gamma_1} \frac{\dim \gamma_1}{N^{n+m-2l_1}}, \end{aligned} \tag{A.2}$$

where $\delta_{\gamma_1,\gamma_2} = \delta_{\gamma_1^+,\gamma_2^+}\delta_{\gamma_1^-,\gamma_2^-}\delta_{l_1,l_2}$. We can use the formula for calculating d_γ in equation (3.12) in [15]

$$d_\gamma = \frac{m!n!}{l!(m-l)!(n-l)!} d_{\gamma^+} d_{\gamma^-}, \qquad \text{for } \gamma = (l, \gamma^+, \gamma^-). \tag{A.3}$$

And we use the definition

$$\tilde{t}_\gamma = \lim_{N\to\infty} \frac{\dim \gamma}{N^{n+m-2l}}. \tag{A.4}$$

For general l, we also have an expression for $\tilde{t}_\gamma$ derived in Sec. 3.1, which is

$$\tilde{t}_\gamma = \frac{d_{\gamma^+} d_{\gamma^-}}{(n-l)!(m-l)!}. \tag{A.5}$$

Then we have

$$\langle\gamma_1|\gamma_2\rangle = \delta_{\gamma_1,\gamma_2} \frac{m!^2 n!^2}{l_1!(m-l_1)!(n-l_1)!} d_{\gamma_1^+} d_{\gamma_1^-} \tilde{t}_{\gamma_1} (1 + O(1/N)). \tag{A.6}$$

As a special case, for $l_1 = 0$, we have $\tilde{t}_{\gamma_1} = \frac{d_{\gamma_1^+} d_{\gamma_1^-}}{n!m!}$. Using this in the above formula we have

$$\langle \gamma_1 | \gamma_2 \rangle = \delta_{\gamma_1,\gamma_2} d^2_{\gamma_1^+} d^2_{\gamma_1^-} (1 + O(1/N)). \tag{A.7}$$

More generally, we have more Brauer states $O^{\gamma}_{A,ij}(Z, Y)$ which defined previously. We may identify

$$|\gamma; A, ij\rangle \leftrightarrow N^l O^{\gamma}_{A,ij} \left(\frac{Z}{\sqrt{N}}, \frac{Y}{\sqrt{N}} \right). \tag{A.8}$$

According to (7.12) in [15]

$$\langle O^{\gamma_1}_{A_1,i_1j_1}(Z,Y)^\dagger O^{\gamma_2}_{A_2,i_2j_2}(Z,Y)\rangle = \delta_{\gamma_1,\gamma_2} \delta_{A_1A_2} \delta_{i_1i_2} \delta_{j_1j_2} d_{A_1} \dim \gamma_1. \tag{A.9}$$

Therefore we have

$$\begin{aligned} \langle \gamma_1; A_1, i_1j_1 | \gamma_2; A_2, i_2j_2 \rangle &= \frac{1}{N^{n+m-l_1-l_2}} \delta_{\gamma_1,\gamma_2} \delta_{A_1A_2} \delta_{i_1i_2} \delta_{j_1j_2} d_{A_1} \dim \gamma_1 \\ &= \delta_{\gamma_1,\gamma_2} \delta_{A_1A_2} \delta_{i_1i_2} \delta_{j_1j_2} d_{A_1} \frac{\dim \gamma_1}{N^{n+m-2l_1}}. \end{aligned} \tag{A.10}$$

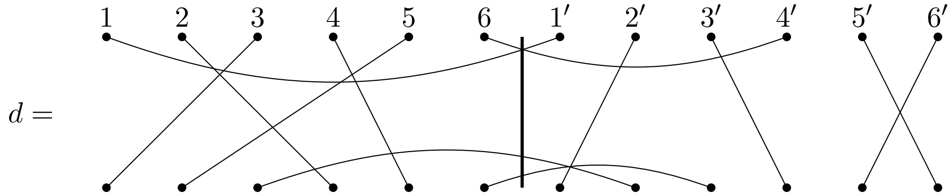
Appendix B. Characters of Brauer algebra

In this section, we present some results about the characters of Brauer algebra that will be useful in the study of Brauer states. We mainly refer to [30] in this section. Brauer algebras are extensively explored also in e.g. [15, 74–78].

Remember that the Brauer algebra $B_N(n, m)$ is not a group, therefore some familiar results in group representation theory may not hold in this case. For example, there is no notion of conjugacy class, since there is no inverse for every element in the algebra. For this reason, we cannot say that the character takes the same value on a conjugacy class for the Brauer algebra. However, we have an analogous notion of conjugacy class which [30] calls character class that shares similar feature of conjugacy class in group representation theory.

First we introduce some basic results and fix some notations related to the Brauer algebra. The Brauer algebra $B_N(n, m)$ has a basis given by (n, m) diagrams d. We write $\mathcal{D}_{n,m}$ for the set of all (n, m) diagrams. A $d \in \mathcal{D}_{n,m}$ is defined to be a diagram with a vertical wall between the nth and $(n + 1)$th vertices such that vertical edges never cross the wall and horizontal edges always begin and end on opposite side of the wall. For example, the diagram below is a $(6,6)$ diagram

(B.1)



We let $t_i^L(d), t_j^R(d)$ denote the ith and jth vertices in the top on the right and left side of the wall respectively, as denoted in the above diagram. And we let $b_i^L(d), b_j^R(d)$ denote the ith and jth vertices in the bottom on the right and left side of the wall respectively. We denote $t(d)$ the set of vertices in the top of the diagram, and $b(d)$ the set of vertices in the bottom of the diagram.

We then define a cycle type of a diagram d through traversing the diagram d as follows:

(1) Start with vertex $t_1^L(d)$ if it exists; otherwise start with $b_1^R(d)$.

(2) Follow the edge connected to this vertex. Upon reaching the other side of the edge, jump to the vertex directly above it if we are in $b(d)$ or to the vertex below it if we are in $t(d)$, and continue following the edge connected to that vertex.

(3) Following the above procedure, we will end by returning to the starting vertex and complete a cycle in d. We denote such a cycle c_1.

(4) We start from another vertex that has not been visited and repeat the above process. Each time we finish the above process we will get a cycle c_i in d. And we end the process if we visited all vertices of d.

In this way, we decompose d into disjoint cycles. For example in the above diagram (B.1), we have 4 disjoint cycles. The first is on vertices $1, 1', 2', 3$, the second on vertices $2, 4, 5$, the third on $6, 4', 3'$, and the fourth on $5', 6'$.

For each cycle c in d, we define $type(c)$ to be the the number of vertical edges in c on the left side of the wall minus the the number of vertical edges in c on the right side of the wall. The integer $type(c)$ is called the cycle type of c. We can always reorder all cycles in d in such a way that

$$type(c_1) \geq type(c_2) \geq \cdots \geq type(c_s). \tag{B.2}$$

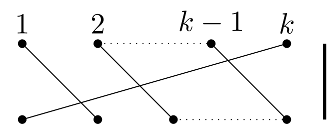
For example, in the above case (B.1), the cycle type of each cycle in d is

$$\begin{aligned} type(2,4,5) = 3 &\geq type(1,1',2',3) = 0 \\ &\geq type(6,4',3') = -1 \geq type(5',6') = -2. \end{aligned} \tag{B.3}$$

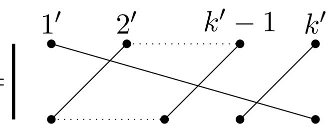
We then associate with $d \in \mathcal{D}_{n,m}$ a $(n+m)$-staircase $\zeta(d)$ obtained from (B.2) by inserting $(n+m-s)$ zeros between the positive values and negative values. That is to say that $\zeta(d) = (k, \zeta^+, \zeta^-)$ with ζ^+ the same as the positive part of $type(c_i)$ and ζ^- the same as the negative part of $type(c_i)$. And we call $\zeta(d)$ the cycle type of d. For example, in the above example, $\zeta(d) = (3, 0^9, -1, -2)$. Zero cycles contain the same number of vertices on each side of the wall. Thus there exists an integer $h(d)$ satisfying $\zeta(d)^+ \vdash (n - h(d))$ and $\zeta(d)^- \vdash (m - h(d))$. In our above example, $n = m = 6$, and $h(d) = 3$.

The above procedure gives us a way to assign each $d \in \mathcal{D}_{n,m}$ a (n,m)-staircase $\zeta(d)$. We then describe a way to assign each (n,m)-staircase ζ a element $d_\zeta \in \mathcal{D}_{n,m}$. First for each $k \in \mathbb{Z} - \{0\}$, we define an element

(B.4)

$d_k =$ 

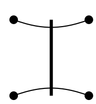
if $k > 0$,

$d_k =$ 

if $k' = -k > 0$.

And we also define element e:

(B.5)

$e =$ 

Now for a (n,m)-staircase $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_{n+m})$, which could also be written as $\zeta = (k, \zeta^+, \zeta^-)$. There exist a integer $h(\zeta)$ that $\zeta^+ \vdash (n - h(\zeta)), \zeta^- \vdash (m - h(\zeta))$. And assume that the length of positive and negative part of ζ are $l(\zeta^+) = i$ and $l(\zeta^-) = j$. Then we define

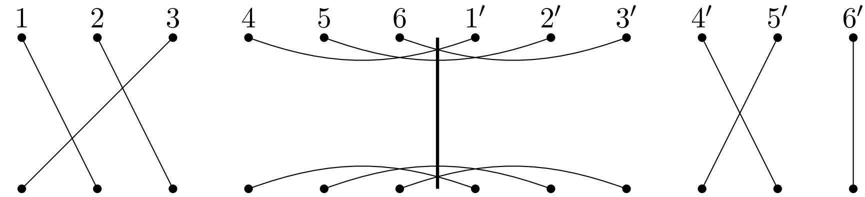
$$\begin{aligned} d_{\zeta^+} &= d_{\zeta_1} \otimes d_{\zeta_2} \otimes \cdots \otimes d_{\zeta_i}, \\ d_{\zeta^+} &= d_{\zeta_{m+n-j}} \otimes \cdots \otimes d_{\zeta_{m+n-1}} \otimes d_{\zeta_{m+n}}. \end{aligned} \tag{B.6}$$

And we define $d_\zeta \in B_N(n,m)$ to be

$$d_\zeta = d_{\zeta^+} \otimes e^{\otimes h(\zeta)} \otimes d_{\zeta^-}. \tag{B.7}$$

As an example, let $\zeta = (3, 0^9, -1, -2)$ as a $(6, 6)$ staircase. $h(\zeta) = 3$, $\zeta^+ = (3)$ and $\zeta^- = (2, 1)$. In this case, we have

(B.8)

$d_\zeta =$ 

Then the following results tells us that the character of a $d \in \mathcal{D}_{n,m}$ of a certain type ζ is related to the character of the standard diagram d_ζ.

Theorem B.1. *[30] Let $d \in \mathcal{D}_{n,m}$ with $\zeta = \zeta(d)$ and $h = h(d)$. Then for any character χ_B of the Brauer algebra $B_N(n, m)$, we have:*

$$\chi_B(d) = N^{z(d)-h(d)} \chi_B(d_\zeta), \tag{B.9}$$

where $z(d)$ is the number of zero-cycles in d.

The above formula tells us that if two $d, d' \in \mathcal{D}_{n,m}$ have the same cycle type $\zeta(d) = \zeta(d') = \zeta$. Then any character of Brauer algebra evaluated on the two elements are the same up to a constant that depends on N. For this reason we call the class labeled by (n, m)-staircase ζ character class.

Now we come to the irreducible representations of Brauer algebra. The irreducible representation of Brauer algebra is also labeled by (n, m)-staircase. We have the following formula:

(B.10)

$$\chi^\gamma(d_\zeta) = N^{h(\zeta)} \sum_{\substack{\lambda \vdash n' \\ \pi \vdash m'}} \left(\sum_{\delta \vdash (l-h)} g(\delta, \gamma^+; \lambda) g(\delta, \gamma^-; \pi) \right) \chi^\lambda_{S_{n'}}(\zeta^+) \chi^\pi_{S_{m'}}(\zeta^-),$$

where $\gamma = (l, \gamma^+, \gamma^-)$, $\zeta = (h, \zeta^+, \zeta^-)$. And g in the above formula is the Littlewood-Richardson coefficient. Therefore, for arbitrary $d \in \mathcal{D}_{n,m}$, we have

(B.11)

$$\chi^\gamma(d) = N^{z(d)} \sum_{\substack{\lambda \vdash n' \\ \pi \vdash m'}} \left(\sum_{\delta \vdash (l-h)} g(\delta, \gamma^+; \lambda) g(\delta, \gamma^-; \pi) \right) \chi^\lambda_{S_{n'}}(\zeta^+) \chi^\pi_{S_{m'}}(\zeta^-),$$

where $\zeta = \zeta(d)$.

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