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Coupled gravitational and electromagnetic perturbations of Reissner-Nordström spacetime in a polarized setting

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We derive a system of equations governing the coupled gravitational and electromagnetic perturbations of Reissner-Nordström spacetime. The equations are derived in the context of global non-linear stability of Reissner-Nordström under axially symmetric polarized perturbations, as a generalization of the recent work on non-linear stability of Schwarzschild spacetime of Klainerman-Szeftel ([9]). The main result consists in deriving, through a Chandrasekhar-type transformation, a gauge invariant quantity associated to the electromagnetic tensor that verifies a Regge-Wheeler equation. In this paper, we present the derivation of the main equations.

Introduction

One of the fundamental open problems in General Relativity is the one concerning the stability of the exterior of black holes under gravitational perturbations. The stability conjecture says that the class of metrics of the Kerr family is stable under small perturbations of initial data as solutions to the vacuum Einstein equation:

$$\mathrm{Ric}(g) = 0, \tag{0.1}$$

where $\mathrm{Ric}(g)$ is the Ricci curvature tensor of the metric g. The conjecture is also formulated for charged black holes, and states that the Kerr-Newman family of spacetimes is stable under small perturbation of initial data as solutions to the Einstein-Maxwell equation:

$$\mathrm{Ric}(g)_{\mu\nu} = T(F)_{\mu\nu} := 2F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{2}g_{\mu\nu}|F|^2 \tag{0.2}$$

where F is a 2-form satisfying Maxwell's equations

$$\nabla_{[\alpha}F_{\beta\gamma]} = 0, \qquad \nabla^{\alpha}F_{\alpha\beta} = 0. \tag{0.3}$$

In the case of Einstein-Maxwell equation, coupled gravitational and electromagnetic perturbations of initial data are considered, i.e. the Weyl curvature and the electromagnetic tensor F of the spacetime are both perturbed with respect to the initial spacetime.

The only known result concerning the full non-linear stability of a vacuum spacetime without symmetries is the celebrated stability of Minkowski space by Christodoulou-Klainerman ([5]). The result was generalized by Zipser to the non-linear stability of Minkowski space under gravitational and electromagnetic perturbations in [2]. The first linear stability of a black hole spacetime has been proved by Dafermos-Holzegel-Rodnianski in [7], in the case of Schwarzschild spacetime. The full non-linear stability of the Schwarzschild solution is still open, and would require a formulation of the stability not just for Schwarzschild, but for slowly rotating Kerr solutions. Indeed, even if one restricts to small perturbations of Schwarzschild, one should expect that generically the spacetime would evolve to a slowly rotating Kerr spacetime, with small but non-zero angular momentum.

A recent work by Klainerman-Szeftel ([9]) addresses the global non-linear stability of Schwarzschild black hole as solution to the Einstein vacuum equation (0.1). The authors in [9] consider a particular class of gravitational perturbations of Schwarzschild, namely axially symmetric polarized spacetimes.

This choice of perturbations forces the final state of the evolution to have zero angular momentum, therefore ruling out the general Kerr spacetime. They can therefore prove the global non-linear stability of Schwarzschild space, meaning that the axially symmetric polarized perturbed spacetime close to the initial Schwarzschild evolves to another Schwarzschild solution, with mass close to initial one.

In [9], the authors consider the celebrated Teukolsky equation (first derived in [13]) verified by the extreme null component of the Riemann curvature α, and apply a Chandrasekhar-type transformation to obtain a new quantity $\mathfrak{q}$, at the level of second derivative of α, that verifies a Regge-Wheeler equation. This transformation was first introduced in the physics literature in the cotext of mode stability by Chandrasekhar (see [4]), and it first appeared as a spacetime version in the context of linear stability of Schwarzschild in [7] to derive decay estimates for solution of Teukolsky equation. In [9], the decay estimates for $\mathfrak{q}$ are the starting point to derive decays for the curvature components and the connection coefficients, using the null structure equations and the Bianchi identities. The dynamical construction of the spacetime follows, along with many subtleties related to the non-linearity of the problem.

In the present paper, we address the problem of stability of charged black holes subject to coupled gravitational and electromagnetic perturbations. A particular class of these spacetimes are the spherically symmetric charged black holes, namely the Reissner-Nordström solution, corresponding to Kerr-Newman spacetime with zero angular momentum. In local coordinates (t, r, θ, φ) the metric is expressed as

$$\begin{aligned} \mathbf{g}_{RN} = & -\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 \\ & + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned}$$

where m is the mass and Q is the charge of the black hole.

Gravitational and electromagnetic perturbations of Reissner-Nordström black holes have been extensively considered in the setting of metric perturbations. Moncrief ([11]) reduced the governing equations to pair of decoupled one dimensional wave equations both for odd and for the even parity perturbations. This approach corresponds to the description of gravitational perturbations of Schwarzschild spacetime via Regge-Wheeler and Zerilli equations for metric perturbations (as in [8]), as opposed to the Teukolsky equations for curvature perturbations (as in [7]). Chandrasekhar ([3]), using the Newman-Penrose formalism, derived a pair of decoupled equations for perturbations of Reissner-Nordström which can be transformed to one dimensional wave equations for both parity perturbations. In order to decouple the equations, Chandrasekhar chooses a gauge (the phantom gauge, as discussed in Remark 4.7) and separates them in radial and angular parts. Our approach instead is based on the use of a gauge invariant quantity, and no separation of variables is needed to decouple the equations.

As for the vacuum Einstein equation, the general problem of stability of charged black holes would have to be solved in the context of global nonlinear stability of the Kerr-Newman spacetime as solution to the Einstein-Maxwell equation (0.2). However, as in the case of axially symmetric polarized perturbations of Schwarzschild in [9], restricting to axially symmetric polarized perturbations of Reissner-Nordström will force the final state of the evolution to a non-rotating black hole, excluding the general Kerr-Newman spacetime.

The present work is meant to be the initial step in order to extend the stability result of [9] to the case of electrovacuum perturbations of charged black holes. The main gauge-invariant quantities and the equations verified by them were unknown up to this point.

The equations governing the evolution of the curvature and the electromagnetic tensor in the case of electrovacuum spacetime are coupled. In particular, the Teukolsky equation verified by the extreme component of the Weyl curvature α is coupled with the electromagnetic components coming from the non-vanishing Ricci curvature. Applying a Chandrasekhar-type transformation we derive the corresponding new quantity $\mathfrak{q}$ veryfing a Regge-Wheeler type equation, coupled with electromagnetic terms. However, these additional terms are multiplied by the charge of the spacetime, so, provided that we have control on the electromagnetic part, they could in principle be absorbed as error terms for small enough charge.

The main new insight is the control of the electromagnetic part making use of a gauge invariant quantity depending on the electromagnetic components, whose null derivative appears in the Teukolsky equation for α. Applying a new Chandrasekhar-type transformation, at the level of one derivative only (as opposed to two derivatives as in the case of curvature), we are able to find a new quantity $\mathfrak{q}^{\mathbf{F}}$ verifying another Regge-Wheeler type equation, coupled with the curvature term $\mathfrak{q}$. Therefore, we have a coupled system of wave equations for the term encoding the curvature $\mathfrak{q}$ and for the term encoding the electromagnetic part $\mathfrak{q}^{\mathbf{F}}$ which at the linear level looks like[1]

$$(0.4)\qquad \begin{cases} \square_{\mathbf{g}}\mathfrak{q} = V_1\mathfrak{q} + e\cdot\mathcal{M}(\partial^{\leq 2}\mathfrak{q}^{\mathbf{F}}) + e(l.o.t.(\mathfrak{q}^{\mathbf{F}})) + e^2(l.o.t.(\mathfrak{q})) \\ \square_{\mathbf{g}}\mathfrak{q}^{\mathbf{F}} = V_2\mathfrak{q}^{\mathbf{F}} + e\cdot\mathcal{M}(\mathfrak{q}) + e^2(l.o.t.(\mathfrak{q}^{\mathbf{F}})) \end{cases}$$

where the operator $\square_{\mathbf{g}} = \mathbf{D}^\alpha\mathbf{D}_\alpha$ is the wave operator associated to the perturbed metric $\mathbf{g}$, e is the charge of the perturbed spacetime, and $\mathcal{M}$ is an expression of the arguments. We denote $l.o.t.(\mathfrak{q})$ and $l.o.t.(\mathfrak{q}^{\mathbf{F}})$ lower order terms with respect to $\mathfrak{q}$ and $\mathfrak{q}^{\mathbf{F}}$ respectively.

The two wave equations are coupled: on the right hand side of the equation for the curvature term $\mathfrak{q}$ we find an expression of the electromagnetic term $\mathfrak{q}^{\mathbf{F}}$, and similarly on the right hand side of the equation for $\mathfrak{q}^{\mathbf{F}}$ we find the curvature term $\mathfrak{q}$. Notice that those coupled terms are multiplied by the charge of the spacetime. The coupling is not symmetric in terms of dependence on derivatives: the presence of two derivatives of $\mathfrak{q}^{\mathbf{F}}$ on the right hand side of the first equation is a consequence of the Teukolsky equation for α, in which the derivative of the electromagnetic term appears. However, this asymmetry is good in terms of deriving estimates for such a system. Indeed, taking one derivative of the second equation and deriving Morawetz estimates for it, we would have a term for second derivative of $\mathfrak{q}$ and one term for first derivative of $\mathfrak{q}$, the latter multiplied by the charge. Those are exactly the kind of terms appearing in the Morawetz estimates obtained for the first equation (because of the presence of $\partial^{\leq 2}\mathfrak{q}^{\mathbf{F}}$), with the term for second derivative of $\mathfrak{q}^{\mathbf{F}}$ multiplied by the charge. Summing those estimates, the terms on the right hand side multiplied by the charge could be absorbed on the right hand side for small enough charge. In addition to the coupling, there is the presence of lower order terms, which will have to be treated either in the spirit of [10] in the case of slowly rotating Kerr (i.e. considering a system of equations for the lower order terms), or as in [6] (i.e. deriving decay for the lower order terms using transport equations).

In this paper, we derive the main equations leading to the system (0.4), as a first step towards the proof of non-linear stability of Reissner-Nordström spacetime under polarized perturbations. We remark that the structure of the system (0.4) does not depend on the polarization of the metric nor on the assumption of axial symmetry. We present the general result in the Appendix. The system can therefore be used to prove linear stability of Reissner-Nordström spacetime.

[1] The explicit form of the final system is (6.5).

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1. Electrovacuum axially symmetric polarized spacetimes

We consider electrovacuum spacetimes $(\mathbf{M}, \mathbf{g})$, namely solution to the Einstein-Maxwell equation (0.2). We denote $\mathbf{D}$ the Levi-Civita connection of the spacetime $(\mathbf{M}, \mathbf{g})$.

An axially symmetric spacetime $(\mathbf{M}, \mathbf{g}, \mathbf{Z})$ is a four dimensional simply connected manifold $\mathbf{M}$ with a Lorentzian metric $\mathbf{g}$ and an axial Killing vectorfield $\mathbf{Z}$ on $\mathbf{M}$ that preserves $\mathbf{F}$, i.e. $\mathcal{L}_{\mathbf{Z}}\mathbf{F} = 0$.

The Ernst potential of the spacetime is given by

$$\sigma_\mu := \mathbf{D}_\mu(-\mathbf{Z}^\alpha \mathbf{Z}_\alpha) - i \in_{\mu\beta\gamma\delta} \mathbf{Z}^\beta \mathbf{D}^\gamma \mathbf{Z}^\delta.$$

The 1-form $\sigma_\mu dx^\mu$ is closed and thus there exists a function $\sigma : \mathbf{M} \to \mathbb{C}$, called the $\mathbf{Z}$- Ernst potential, such that $\sigma_\mu = \mathbf{D}_\mu \sigma$. Note also that $\mathbf{D}_\mu \mathbf{g}(\mathbf{Z}, \mathbf{Z}) = 2\mathbf{G}_{\mu\lambda}\mathbf{Z}^\lambda = -Re(\sigma_\mu)$ where $\mathbf{G}_{\mu\nu} = \mathbf{D}_\mu \mathbf{Z}_\nu$. Hence we can choose the potential σ such that $Re(\sigma) = -X$, where $X = \mathbf{g}(\mathbf{Z}, \mathbf{Z})$.

Definition 1.1. An axially symmetric Lorentzian manifold $(\mathbf{M}, \mathbf{g}, \mathbf{Z})$ is said to be polarized if the Ernst potential σ is real, i.e. $\sigma = -X$. In this case, we can find coordinates (φ, x^a) such that $\mathbf{Z} = \partial_\varphi$, and the metric $\mathbf{g}$ can be written in the form

$$\mathbf{g} = X d\varphi^2 + g_{ab} dx^a dx^b = e^{2\Phi} d\varphi^2 + g_{ab} dx^a dx^b \tag{1.1}$$

where g_{ab} is a $1+2$ Lorentzian metric, and $\Phi = \frac{1}{2}\log(X)$. The axial symmetry implies that X and g are independent of φ.

Following [5], we define the null decomposition of the curvature and the electromagnetic tensor on a given $\mathbf{Z}$-invariant polarized S-foliation of an electrovacuum spacetime. We assume we have a fixed adapted null pair e_3, e_4, i.e. future directed $\mathbf{Z}$-invariant null vectors orthogonal to the leaves S of the foliation, such as $\mathbf{g}(e_3, e_4) = -2$, while on S we have an orthonormal frame e_1, e_2.

We define the spacetime Ricci coefficients, where the indices A, B take values $1, 2$

$$
\begin{aligned}
{}^{(1+3)}\chi_{AB} &:= \mathbf{g}(\mathbf{D}_A e_4, e_B), \qquad {}^{(1+3)}\xi_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_4, e_A),\\
{}^{(1+3)}\eta_A &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_4, e_A), \qquad {}^{(1+3)}\zeta_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_A e_4, e_3),\\
{}^{(1+3)}\omega &:= \frac{1}{4}\mathbf{g}(\mathbf{D}_4 e_4, e_3)
\end{aligned}
\tag{1.2}
$$

and interchanging e_3, e_4,

$$
\begin{aligned}
{}^{(1+3)}\underline{\chi}_{AB} &:= \mathbf{g}(\mathbf{D}_A e_3, e_B), \qquad {}^{(1+3)}\underline{\xi}_A := \frac{1}{2}\mathbf{g}(\mathbf{D}_3 e_3, e_A),\\
{}^{(1+3)}\underline{\eta}_A &:= \frac{1}{2}\mathbf{g}(\mathbf{D}_4 e_3, e_A), \qquad {}^{(1+3)}\zeta_A := -\frac{1}{2}\mathbf{g}(\mathbf{D}_A e_3, e_4),\\
{}^{(1+3)}\underline{\omega} &:= \frac{1}{4}\mathbf{g}(\mathbf{D}_3 e_3, e_4)
\end{aligned}
\tag{1.3}
$$

We define the spacetime null curvature components of the Weyl curvature $\mathbf{W}$,

$$
\begin{aligned}
{}^{(1+3)}\alpha_{AB} &:= \mathbf{W}_{A4B4}, \qquad {}^{(1+3)}\beta_A := \frac{1}{2}\mathbf{W}_{A434},\\
{}^{(1+3)}\rho &:= \frac{1}{4}\mathbf{W}_{3434}, \qquad {}^{(1+3)}\underline{\alpha}_{AB} := \mathbf{W}_{A3B3},\\
{}^{(1+3)}\underline{\beta}_A &:= \frac{1}{2}\mathbf{W}_{A334}, \qquad {}^{(1+3)\star}\rho := \frac{1}{4}{}^*\mathbf{W}_{3434}
\end{aligned}
\tag{1.4}
$$

We define the spacetime null electromagnetic components of the electromagnetic tensor $\mathbf{F}$ in the following way[2]:

$$
{}^{(F)}\beta_A := \mathbf{F}_{A4}, \qquad {}^{(F)}\underline{\beta}_A := \mathbf{F}_{A3}, \qquad {}^{(F)}\rho := \frac{1}{2}\mathbf{F}_{34}, \qquad {}^{(F)\star}\rho := \frac{1}{2}{}^*\mathbf{F}_{34}
\tag{1.5}
$$

The Ricci tensor can be expressed in terms of the electromagnetic null decomposition according to Einstein equation 0.2, and using the decomposition

---

[2]Note that we define the extreme components of the electromagnetic tensor using the ${}^{(F)}\beta$ notation, as opposed to the standard ${}^{(F)}\alpha$ for the spin ± 1 Teukolsky equation. This choice is meant to stress the fact that in electrovacuum background under gravitational and electromagnetic perturbation, the extreme components $\mathbf{F}_{A4}$ and $\mathbf{F}_{A3}$ are not gauge invariant (as the β component of the curvature is not invariant, as opposed to the extreme component α). See Remark 4.5.

of the Riemann curvature in Weyl curvature and Ricci tensor,

$$\mathbf{R}_{\alpha\beta\gamma\delta} = \mathbf{W}_{\alpha\beta\gamma\delta} + \frac{1}{2}(\mathbf{g}_{\beta\delta}\mathbf{R}_{\alpha\gamma} + \mathbf{g}_{\alpha\gamma}\mathbf{R}_{\beta\delta} - \mathbf{g}_{\beta\gamma}\mathbf{R}_{\alpha\delta} - \mathbf{g}_{\alpha\delta}\mathbf{R}_{\beta\gamma}),$$

we can express the full Riemann tensor of the perturbed spacetime in terms of the above decompositions.

Suppose now that the orthonormal frame on S is adapted to the axial symmetry as follows: $e_1 = e_\varphi = X^{-1/2}\mathbf{Z}$ with $X := \mathbf{g}(\mathbf{Z},\mathbf{Z})$, and $e_2 = e_\theta$. We define e_3, e_4, e_θ to be the reduced null frame, associated to the reduced metric g.

We can define the reduced Ricci coefficients as follows:

$$\begin{aligned} &\chi := {}^{(1+3)}\chi_{\theta\theta}, \quad \underline{\chi} := {}^{(1+3)}\underline{\chi}_{\theta\theta}, \quad \eta := {}^{(1+3)}\eta_\theta, \quad \underline{\eta} := {}^{(1+3)}\underline{\eta}_\theta, \\ &\xi := {}^{(1+3)}\xi_\theta, \quad \underline{\xi} := {}^{(1+3)}\underline{\xi}_\theta, \quad \zeta := {}^{(1+3)}\zeta_\theta, \\ &\omega := {}^{(1+3)}\omega, \quad \underline{\omega} := {}^{(1+3)}\underline{\omega} \end{aligned} \tag{1.6}$$

We define the reduced curvature and electromagnetic tensor as

$$\begin{aligned} &\alpha : = {}^{(1+3)}\alpha_{\theta\theta}, \quad \underline{\alpha} := {}^{(1+3)}\underline{\alpha}_{\theta\theta}, \quad \beta := {}^{(1+3)}\beta_\theta, \quad \underline{\beta} := -{}^{(1+3)}\underline{\beta}_\theta \\ &\rho : = {}^{(1+3)}\rho \quad {}^{(F)}\beta := {}^{(F)}\beta_\theta, \quad {}^{(F)}\underline{\beta} := {}^{(F)}\underline{\beta}_\theta \end{aligned} \tag{1.7}$$

Notice that the polarization implies that every $\mathbf{Z}$-invariant and $\mathbf{Z}$-polarized spacetime tensor U is such that its contraction with an odd number of $e_\varphi = X^{-\frac{1}{2}}\mathbf{Z}$ vanishes identically. Therefore, the polarization of the metric and the $\mathbf{Z}$-invariance of $\mathbf{F}$ imply that the remaining components of the Ricci coefficients, the curvature and the electromagnetic tensor are determined in the following way:

$$\begin{aligned} &{}^{(1+3)}\chi_{\theta\varphi} = {}^{(1+3)}\underline{\chi}_{\theta\varphi} = {}^{(1+3)}\eta_\varphi = {}^{(1+3)}\underline{\eta}_\varphi = {}^{(1+3)}\xi_\varphi = {}^{(1+3)}\underline{\xi}_\varphi = {}^{(1+3)}\zeta_\varphi = 0, \\ &{}^{(1+3)}\chi_{\varphi\varphi} = e_4(\Phi), \quad {}^{(1+3)}\underline{\chi}_{\varphi\varphi} = e_3(\Phi), \\ &{}^{(1+3)}\alpha_{\theta\varphi} = {}^{(1+3)}\underline{\alpha}_{\theta\varphi} = {}^{(1+3)}\beta_\varphi = {}^{(1+3)}\underline{\beta}_\varphi = 0, \\ &{}^{(1+3)}\alpha_{\varphi\varphi} = -\alpha, \quad {}^{(1+3)}\underline{\alpha}_{\varphi\varphi} = -\underline{\alpha} \end{aligned}$$

and

$${}^\star\rho = 0, \qquad {}^{(F)}\beta_\varphi = {}^{(F)}\underline{\beta}_\varphi = {}^{(F)\star}\rho = 0 \tag{1.8}$$

Remark 1.2. Since in Kerr-Newman spacetime, the components ρ and ${}^{(F)\star}\rho$ are different from zero for non-zero angular momentum, we see from

(1.8) that the hypothesis of polarization of the metric forces the final state of the evolution to be a non-rotating charged black hole.

Following the notation in [9], we define

$$
\begin{aligned}
\vartheta &:= \chi - e_4(\Phi), \qquad \kappa := {}^{(1+3)}\mathrm{tr}\chi = \chi + e_4(\Phi) \\
\underline{\vartheta} &:= \underline{\chi} - e_3(\Phi), \qquad \underline{\kappa} := {}^{(1+3)}\mathrm{tr}\underline{\chi} = \underline{\chi} + e_3(\Phi)
\end{aligned}
$$

Thus,

$$
{}^{(1+3)}\widehat{\chi}_{\theta\theta} = -\,{}^{(1+3)}\widehat{\chi}_{\varphi\varphi} = \frac{1}{2}\vartheta, \qquad {}^{(1+3)}\widehat{\underline{\chi}}_{\theta\theta} = -\,{}^{(1+3)}\widehat{\underline{\chi}}_{\varphi\varphi} = \frac{1}{2}\underline{\vartheta}
$$

where ${}^{(1+3)}\widehat{\chi}$ and ${}^{(1+3)}\widehat{\underline{\chi}}$ are the traceless part of ${}^{(1+3)}\chi$ and ${}^{(1+3)}\underline{\chi}$ respectively.

1.1. Reissner-Nordström spacetime

The Reissner-Nordström metric has the axial symmetric vector field $\mathbf{Z} = \partial_\varphi$ and the polarized form (1.1) of the metric in standard coordinates is given by

$$
\begin{aligned}
\mathbf{g} &= X^2 d\varphi^2 - \Upsilon dt^2 + \Upsilon^{-1} dr^2 + r^2 d\theta^2, \\
\Upsilon &:= 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \qquad X = r^2 \sin^2\theta
\end{aligned}
$$

The electromagnetic tensor is given by

$$
\mathbf{F} = -\frac{Q}{r^2} dr \wedge dt.
$$

Proposition 1.3. *All curvature and electromagnetic components of the Reissner-Nordström spacetime vanish identically except*

$$
{}^{(1+3)}\rho = -\frac{2m}{r^3} + \frac{2Q^2}{r^4}, \qquad {}^{(F)}\rho = \frac{Q}{r^2}
$$

2. Main equations in electrovacuum

Following [5] and [9], we derive the main equations in electrovacuum spacetimes and then obtain their reduction, i.e. their evaluation along e_θ, for axially symmetric polarized spacetimes.

2.1. Null structure equations

The spacetime[3] null structure equations are

$$
\begin{aligned}
\nabla\!\!\!\!/_3\underline{\chi}_{AB} &= 2\nabla\!\!\!\!/_B\underline{\xi}_A - 2\underline{\omega}\underline{\chi}_{AB} - \underline{\chi}_A^{\ C}\chi_{CB} + 2\eta_B\underline{\xi}_A + 2\underline{\eta}_A\underline{\xi}_B \\
&\quad - 4\zeta_B\underline{\xi}_A + \mathbf{R}_{A33B}, \\
\nabla\!\!\!\!/_4\underline{\chi}_{AB} &= 2\nabla\!\!\!\!/_B\underline{\eta}_A + 2\omega\underline{\chi}_{AB} - \chi_B^{\ C}\underline{\chi}_{AC} + 2(\xi_B\underline{\xi}_A + \underline{\eta}_B\underline{\eta}_A) \\
&\quad + \mathbf{R}_{A34B}, \\
\nabla\!\!\!\!/_3\zeta_A &= -2\nabla\!\!\!\!/_A\underline{\omega} - \underline{\chi}_A^{\ B}(\zeta_B + \eta_B) + 2\underline{\omega}(\zeta_A - \eta_A) + \chi_A^{\ B}\underline{\xi}_B \\
&\quad + 2\omega\underline{\xi}_A - \frac{1}{2}\mathbf{R}_{A334}, \\
\nabla\!\!\!\!/_4\underline{\xi} - \nabla\!\!\!\!/_3\underline{\eta} &= 4\omega\underline{\xi} + \widehat{\underline{\chi}}\cdot(\underline{\eta} - \eta) + \frac{1}{2}\mathrm{tr}\underline{\chi}(\underline{\eta} - \eta) - \frac{1}{2}\mathbf{R}_{A334}, \\
\nabla\!\!\!\!/_4\underline{\omega} + \nabla\!\!\!\!/_3\omega &= 4\omega\underline{\omega} + \xi\cdot\underline{\xi} + \zeta\cdot(\eta - \underline{\eta}) - \eta\cdot\underline{\eta} + \frac{1}{4}\mathbf{R}_{3434}, \\
\nabla\!\!\!\!/_C\underline{\chi}_{AB} + \zeta_B\underline{\chi}_{AC} &= \nabla\!\!\!\!/_B\underline{\chi}_{AC} + \zeta_C\underline{\chi}_{AB} + \mathbf{R}_{A3CB}, \\
\not{g}^{AC}\not{g}^{BD}\mathbf{R}_{ADCB} &= 2K + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - \widehat{\chi}\cdot\widehat{\underline{\chi}}
\end{aligned}
$$

The symmetric traceless part of the first equation in the reduced picture becomes

$$e_3(\underline{\vartheta}) + \underline{\kappa}\,\underline{\vartheta} = -2\,\not{d}_2^{\star}\underline{\xi} - 2\underline{\omega}\,\underline{\vartheta} + 2(\eta + \underline{\eta} - 2\zeta)\,\underline{\xi} - 2\underline{\alpha}$$

where $\not{d}_2^{\star}\underline{\xi} = -e_\theta\underline{\xi} + e_\theta\Phi\underline{\xi}$, while its trace gives

$$e_3(\underline{\kappa}) + \frac{1}{2}\underline{\kappa}^2 + 2\underline{\omega}\,\underline{\kappa} = 2\,\not{d}_1\underline{\xi} + 2(\eta + \underline{\eta} - 2\zeta)\underline{\xi} - \frac{1}{2}\underline{\vartheta}\,\underline{\vartheta} - 2\,{}^{(F)}\underline{\beta}^2$$

where $\not{d}_1\underline{\xi} = e_\theta\underline{\xi} + e_\theta(\Phi)\underline{\xi}$.

The symmetric traceless part of the second equation gives the reduced equation

$$e_4\underline{\vartheta} + \frac{1}{2}\kappa\,\underline{\vartheta} - 2\omega\underline{\vartheta} = -2\,\not{d}_2^{\star}\underline{\eta} - \frac{1}{2}\underline{\kappa}\,\vartheta + 2(\xi\,\underline{\xi} + \underline{\eta}^2) - {}^{(F)}\beta\,{}^{(F)}\underline{\beta}$$

and its trace gives

$$e_4(\underline{\kappa}) + \frac{1}{2}\kappa\,\underline{\kappa} - 2\omega\underline{\kappa} = 2\,\not{d}_1\underline{\eta} - \frac{1}{2}\vartheta\,\underline{\vartheta} + 2(\xi\,\underline{\xi} + \underline{\eta}\,\underline{\eta}) + 2\rho$$

[3] For convenience we drop the ${}^{(1+3)}$ labels in what follows.

The reduction of the third, fourth and fifth spacetime equation become

$$
\begin{aligned}
e_3\zeta + \frac{1}{2}\underline{\kappa}(\zeta+\eta) - 2\underline{\omega}(\zeta-\eta) &= \underline{\beta} - 2e_\theta(\underline{\omega}) + 2\omega\underline{\xi} + \frac{1}{2}\kappa\,\underline{\xi} - \frac{1}{2}\underline{\vartheta}(\zeta+\eta) \\
&\quad + \frac{1}{2}\vartheta\,\underline{\xi} - {}^{(F)}\rho\,{}^{(F)}\underline{\beta}, \\
e_4(\underline{\xi}) - e_3(\underline{\eta}) &= \underline{\beta} + 4\omega\underline{\xi} + \frac{1}{2}\underline{\kappa}(\underline{\eta}-\eta) \\
&\quad + \frac{1}{2}\underline{\vartheta}(\underline{\eta}-\eta) - {}^{(F)}\rho\,{}^{(F)}\underline{\beta}, \\
e_4\underline{\omega} + e_3\omega &= \rho + {}^{(F)}\rho^2 + 4\omega\underline{\omega} + \xi\,\underline{\xi} + \zeta(\eta-\underline{\eta}) - \eta\,\underline{\eta}
\end{aligned}
$$

The last two equations are Codazzi and Gauss equations, which in the reduction become

$$
\begin{aligned}
\not{d}_2\underline{\vartheta} &= e_\theta(\underline{\kappa}) - \underline{\kappa}\zeta + \underline{\vartheta}\zeta - 2\underline{\beta} - 2\,{}^{(F)}\rho\,{}^{(F)}\underline{\beta}, \\
K &= -\frac{1}{4}\kappa\underline{\kappa} + \frac{1}{4}\vartheta\underline{\vartheta} - \rho + {}^{(F)}\rho^2
\end{aligned}
$$

where $\not{d}_2\underline{\vartheta} = e_\theta(\underline{\vartheta}) + 2e_\theta(\Phi)\underline{\vartheta}$. By the symmetry $e_3 - e_4$ we derive the specular equations.

2.2. Bianchi identities

In electrovacuum spacetimes, the Bianchi identities for the Weyl curvature have non-homogeneous terms, in particular, using the notations in [5], we have

$$
\begin{aligned}
\mathbf{D}^\alpha \mathbf{W}_{\alpha\beta\gamma\delta} &= \frac{1}{2}(\mathbf{D}_\gamma \mathbf{R}_{\beta\delta} - \mathbf{D}_\delta \mathbf{R}_{\beta\gamma}) =: J_{\beta\gamma\delta} \\
\mathbf{D}_{[\sigma}\mathbf{W}_{\gamma\delta]\alpha\beta} &= \mathbf{g}_{\delta\beta}J_{\alpha\gamma\sigma} + \mathbf{g}_{\gamma\alpha}J_{\beta\delta\sigma} + \mathbf{g}_{\sigma\beta}J_{\alpha\delta\gamma} + \mathbf{g}_{\delta\alpha}J_{\beta\sigma\gamma} \\
&\quad + \mathbf{g}_{\gamma\beta}J_{\alpha\sigma\delta} + \mathbf{g}_{\sigma\alpha}J_{\beta\gamma\delta} \\
&:= \tilde{J}_{\sigma\gamma\delta\alpha\beta}
\end{aligned}
$$

The non-homogeneous terms $J_{\beta\gamma\delta}$ and $\tilde{J}_{\sigma\gamma\delta\alpha\beta}$ can be expressed in terms of the electromagnetic components ${}^{(F)}\beta$, ${}^{(F)}\underline{\beta}$, ${}^{(F)}\rho$ and their derivative.

The spacetime[4] Bianchi identities equations are

$$
\begin{aligned}
\nabla\!\!\!/_3\alpha_{AB}+\frac{1}{2}\mathrm{tr}\underline{\chi}\,\alpha_{AB} &= -2(\mathcal{D}\!\!\!/_2^*\,\beta)_{AB}+4\underline{\omega}\alpha_{AB}-3(\widehat{\chi}_{AB}\rho+{}^*\,\widehat{\chi}_{AB}\,{}^\star\rho)\\
&\quad+((\zeta+4\eta)\otimes\beta)_{AB}+\frac{1}{2}(\tilde{J}_{3A4B4}+\tilde{J}_{3B4A4}+J_{434}\delta_{AB}),\\
\nabla\!\!\!/_4\beta_A+2\mathrm{tr}\chi\beta_A &= \mathrm{d}\!\!\!/\mathrm{iv}\,\alpha_A-2\omega\beta_A+((2\zeta+\underline{\eta})\cdot\alpha)_A+3(\xi_A\rho\\
&\quad+{}^*\xi_A\,{}^\star\rho)-J_{4A4},\\
\nabla\!\!\!/_3\beta_A+\mathrm{tr}\underline{\chi}\beta_A &= \mathcal{D}\!\!\!/_1^*(-\rho,{}^\star\rho)_A+2(\widehat{\chi}\cdot\underline{\beta})_A+2\underline{\omega}\,\beta_A+(\underline{\xi}\cdot\alpha)_A\\
&\quad+3(\eta_A\rho+{}^*\eta_A\,{}^\star\rho)+J_{3A4},\\
\mathbf{D}_4\rho+\frac{3}{2}\mathrm{tr}\chi\rho &= \mathrm{d}\!\!\!/\mathrm{iv}\,\beta-\frac{1}{2}\widehat{\underline{\chi}}\cdot\alpha+\zeta\cdot\beta+2(\underline{\eta}\cdot\beta-\xi\cdot\underline{\beta})-\frac{1}{2}J_{434}
\end{aligned}
$$

These equations in the reduction are

(2.1)

$$
\begin{aligned}
e_3(\alpha)+\left(\frac{1}{2}\underline{\kappa}-4\underline{\omega}\right)\alpha &= -\,d\!\!\!/_2^*\beta-\frac{3}{2}\vartheta\rho+(\zeta+4\eta)\beta\\
&\quad+(2\zeta+3\underline{\eta}+2\eta)\,{}^{(F)}\!\rho\,{}^{(F)}\!\beta_\theta-\xi\,{}^{(F)}\!\rho\,{}^{(F)}\!\underline{\beta}_\theta\\
&\quad+e_\theta(2\,{}^{(F)}\!\rho\,{}^{(F)}\!\beta)+e_4({}^{(F)}\!\underline{\beta}\,{}^{(F)}\!\beta)+\chi\,{}^{(F)}\!\underline{\beta}\,{}^{(F)}\!\beta\\
&\quad-(\underline{\chi}+2\underline{\omega})\,{}^{(F)}\!\beta^2-\frac{1}{2}e_3({}^{(F)}\!\beta^2)\\
&\quad-\frac{1}{2}e_4({}^{(F)}\!\rho^2)-2\chi\,{}^{(F)}\!\rho^2,\\
e_4(\beta)+2\,(\kappa+\omega)\,\beta &= d\!\!\!/_2\alpha+(2\zeta+\underline{\eta})\alpha+3\xi\rho-e_\theta({}^{(F)}\!\beta^2)\\
&\quad+e_4({}^{(F)}\!\rho\,{}^{(F)}\!\beta)-(2\zeta+\underline{\eta})({}^{(F)}\!\beta^2)\\
&\quad+2(\omega+\chi)\,{}^{(F)}\!\rho\,{}^{(F)}\!\beta-2\xi\,{}^{(F)}\!\rho^2+\xi\,{}^{(F)}\!\beta\,{}^{(F)}\!\underline{\beta},\\
e_3(\beta)+(\underline{\kappa}-2\underline{\omega})\,\beta &= e_\theta(\rho)+3\eta\rho-\vartheta\underline{\beta}+\underline{\xi}\alpha+e_\theta({}^{(F)}\!\rho^2)\\
&\quad+e_4({}^{(F)}\!\rho\,{}^{(F)}\!\underline{\beta})+2\underline{\eta}\,{}^{(F)}\!\rho^2+(\chi-2\omega)\,{}^{(F)}\!\rho\,{}^{(F)}\!\underline{\beta}\\
&\quad-2\underline{\omega}\,{}^{(F)}\!\beta^2-\underline{\chi}\,{}^{(F)}\!\rho\,{}^{(F)}\!\beta-\underline{\eta}\,{}^{(F)}\!\beta\,{}^{(F)}\!\underline{\beta}+\xi\,{}^{(F)}\!\underline{\beta}^2,\\
e_4\rho+\frac{3}{2}\kappa\rho &= d\!\!\!/_1\beta-\frac{1}{2}\underline{\vartheta}\,\alpha+\zeta\,\beta+2(\underline{\eta}\,\beta+\xi\,\underline{\beta})-\frac{1}{2}e_3({}^{(F)}\!\beta^2)\\
&\quad+\frac{1}{2}e_4({}^{(F)}\!\rho^2)+2\underline{\omega}\,{}^{(F)}\!\beta^2+(2\eta-\underline{\eta})\,{}^{(F)}\!\rho\,{}^{(F)}\!\beta\\
&\quad+\xi\,{}^{(F)}\!\rho\,{}^{(F)}\!\underline{\beta}
\end{aligned}
$$

All other equations can be obtained by symmetry e_3-e_4.

[4]For convenience we drop the $^{(1+3)}$ labels in what follows.

2.3. Maxwell's equations

We write Maxwell's equations (0.3) in null decomposition. The spacetime Maxwell's equations $\mathbf{D}_\alpha \mathbf{F}_{\beta\gamma} + \mathbf{D}_\beta \mathbf{F}_{\gamma\alpha} + \mathbf{D}_\gamma \mathbf{F}_{\alpha\beta} = 0$ read

$$\begin{aligned}\nabla\!\!\!/_3\, {}^{(F)}\beta_A + \frac{1}{2}\mathrm{tr}\underline{\chi}\, {}^{(F)}\beta_A = {} & \nabla\!\!\!/_4\, {}^{(F)}\underline{\beta}_A + \frac{1}{2}\mathrm{tr}\chi\, {}^{(F)}\underline{\beta}_A + 2\underline{\omega}\, {}^{(F)}\beta_A - 2\omega\, {}^{(F)}\underline{\beta}_A \\ & + 2\, \mathcal{D}\!\!\!/_1^*\, {}^{(F)}\rho + 2(\eta_A + \underline{\eta}_A)\, {}^{(F)}\rho\end{aligned}$$

The spacetime Maxwell equations $\mathbf{D}^\alpha \mathbf{F}_{\alpha\beta} = 0$ in null decomposition are

$$\begin{aligned}\nabla\!\!\!/_3\, {}^{(F)}\beta_A + \nabla\!\!\!/_4\, {}^{(F)}\underline{\beta}_A = {} & -\left(\frac{1}{2}\mathrm{tr}\chi - 2\omega\right) {}^{(F)}\underline{\beta}_A - (\widehat{\chi}\cdot {}^{(F)}\underline{\beta})_A \\ & -\left(\frac{1}{2}\mathrm{tr}\underline{\chi} - 2\underline{\omega}\right) {}^{(F)}\beta_A \\ & - (\widehat{\underline{\chi}}\cdot {}^{(F)}\beta)_A + (\eta_A - \underline{\eta}_A)\, {}^{(F)}\rho, \\ \nabla\!\!\!/_3\, {}^{(F)}\rho + \mathrm{tr}\underline{\chi}\, {}^{(F)}\rho = {} & -\mathrm{d}\!\!\!/\mathrm{iv}\, {}^{(F)}\underline{\beta} + (\zeta - \eta)\cdot {}^{(F)}\underline{\beta} + \underline{\xi}\cdot {}^{(F)}\beta, \\ \nabla\!\!\!/_4\, {}^{(F)}\rho + \mathrm{tr}\chi\, {}^{(F)}\rho = {} & \mathrm{d}\!\!\!/\mathrm{iv}\, {}^{(F)}\beta + (\zeta + \underline{\eta})\cdot {}^{(F)}\underline{\beta} - \xi\cdot {}^{(F)}\underline{\beta}\end{aligned}$$

The reduced equations are therefore

$$\begin{aligned}e_4\, {}^{(F)}\underline{\beta} &= -e_\theta\, {}^{(F)}\rho - 2\underline{\eta}\, {}^{(F)}\rho + \left(-\frac{1}{2}\kappa + 2\omega\right) {}^{(F)}\underline{\beta} + \frac{1}{2}\underline{\vartheta}\, {}^{(F)}\beta, \\ e_3\, {}^{(F)}\beta &= e_\theta\, {}^{(F)}\rho + 2\eta\, {}^{(F)}\rho + \left(-\frac{1}{2}\underline{\kappa} + 2\underline{\omega}\right) {}^{(F)}\beta + \frac{1}{2}\vartheta\, {}^{(F)}\underline{\beta}, \\ e_3\, {}^{(F)}\rho + \mathrm{d}\!\!\!/_1\, {}^{(F)}\underline{\beta} &= -\underline{\kappa}\, {}^{(F)}\rho + (\zeta - \eta)\, {}^{(F)}\underline{\beta} + \underline{\xi}\, {}^{(F)}\beta, \\ -e_4\, {}^{(F)}\rho + \mathrm{d}\!\!\!/_1\, {}^{(F)}\beta &= \kappa\, {}^{(F)}\rho + (-\zeta - \underline{\eta})\, {}^{(F)}\beta + \xi\, {}^{(F)}\underline{\beta}\end{aligned} \tag{2.2}$$

3. Quasi-local mass and charge

Given a $\mathbf{Z}$-invariant polarized surface S we define its volume radius by the formula

$$|S| = 4\pi r^2$$

where $|S|$ is the volume of the surface using the volume form of the metric $g\!\!\!/$.

Definition 3.1. We define the quasi-local charge $e = e(S)$ of the foliated spacetime as

$$e = \frac{1}{4\pi}\int_S {}^{(F)}\rho$$

Recall that the standard Hawking mass $m_H = m_H(S)$ is defined by

$$\frac{2m_H}{r} = 1 + \frac{1}{16\pi}\int_S \kappa\underline{\kappa}$$

In Reissner-Nordström we have

$$e = \frac{1}{4\pi}\int_S {}^{(F)}\rho = \frac{1}{4\pi}\int_S \frac{Q}{r^2} = Q,$$
$$\frac{2m_H}{r} = 1 + \frac{1}{16\pi}\int_S \kappa\underline{\kappa} = 1 - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) = \frac{2m}{r} - \frac{Q^2}{r^2}$$

We see that the Hawking mass does not correspond to the usual mass. We need therefore a definition of a modified quasi-local mass.

Definition 3.2. We define[5] the modified Hawking mass $\varpi = \varpi(S)$ of the foliated spacetime as

$$\varpi = m_H + \frac{e^2}{2r},$$

Observe that in Reissner-Nordström we have

$$\frac{2\varpi}{r} = \frac{2m}{r} - \frac{Q^2}{r^2} + \frac{Q^2}{r^2} = \frac{2m}{r}$$

as desired.

Following [9], we define average quantities and the difference between a quantity and its average in the following way. Given a function f on S we denote

$$\bar{f} := \frac{1}{|S|}\int_S f, \qquad \check{f} := f - \bar{f}. \tag{3.1}$$

Observe that, by Definition 3.1 of the quasi-local charge e,

$$\overline{{}^{(F)}\rho} = \frac{e}{r^2}, \qquad {}^{(F)}\rho = \frac{e}{r^2} + {}^{(\check{F})}\rho, \tag{3.2}$$

[5] Observe that this corresponds to the standard definition of modified Hawking mass ϖ in spherical symmetry.

4. Perturbations of Reissner-Nordström spacetime and invariant quantities

We recall that by Proposition 1.3, in Reissner-Nordström spacetime the Ricci coefficients $\xi, \underline{\xi}, \vartheta, \underline{\vartheta}, \eta, \underline{\eta}, \zeta$, the curvature components $\alpha, \underline{\alpha}, \beta, \underline{\beta}$ and the electromagnetic components ${}^{(F)}\beta$, ${}^{(F)}\underline{\beta}$ vanish identically. Thus, roughly, we expect that in perturbations of Reissner-Nordström these quantities stay small, i.e. of order $O(\epsilon)$ for a sufficiently small ϵ. Moreover, recall that under axially symmetric polarized perturbations, we know that ${}^{\star}\rho$, ${}^{(F)}\beta_{\varphi}$, ${}^{(F)}\underline{\beta}_{\varphi}$, ${}^{(F)\star}\rho = 0$, as derived in (1.8).

Definition 4.1. We say that a smooth, electrovacuum, **Z**-invariant, polarized spacetime is an $O(\epsilon)$-perturbation of Reissner-Nordström if the following are true:

$$\xi, \underline{\xi}, \vartheta, \underline{\vartheta}, \eta, \underline{\eta}, \zeta, \qquad \alpha, \underline{\alpha}, \beta, \underline{\beta}, \qquad {}^{(F)}\beta,\ {}^{(F)}\underline{\beta} = O(\epsilon), \tag{4.1}$$

Also,

$$\kappa - \overline{\kappa}, \quad \underline{\kappa} - \overline{\underline{\kappa}}, \quad \omega - \overline{\omega}, \quad \underline{\omega} - \overline{\underline{\omega}}, \quad \rho - \overline{\rho}, \quad {}^{(F)}\rho - \overline{{}^{(F)}\rho} = O(\epsilon) \tag{4.2}$$

where $\overline{\kappa}, \overline{\underline{\kappa}}, \overline{\omega}, \overline{\underline{\omega}}, \overline{\rho}$ denote spacetime averages as defined in equation (3.1). Moreover the modified Hawking mass and the quasi-local charge are nearly constant, i.e.

$$d\varpi = O(\epsilon^2), \qquad de = O(\epsilon^2) \tag{4.3}$$

Finally,

$$e_3(r) = \frac{r}{2}\overline{\underline{\kappa}} + O(\epsilon), \qquad e_4(r) = \frac{r}{2}\overline{\kappa} + O(\epsilon), \qquad e_\theta(r) = 0.$$

We summarize the linear terms of the null structure equations, the Bianchi identities and the Maxwell equations for perturbations of Reissner-Nordström in the following proposition. Remark that we used Maxwell equations to simplify the Bianchi identities.

Proposition 4.2. *Modulo $O(\epsilon^2)$, the null structure equations, Bianchi identities* (2.1) *and Maxwell's equations* (2.2) *are*

$$
\begin{aligned}
e_3(\underline{\vartheta}) + \underline{\kappa}\, \underline{\vartheta} &= 2(e_\theta(\underline{\xi}) - e_\theta(\Phi)\underline{\xi}) - 2\underline{\omega}\, \underline{\vartheta} - 2\underline{\alpha} \\
e_3(\underline{\kappa}) + \frac{1}{2}\underline{\kappa}^2 + 2\underline{\omega}\, \underline{\kappa} &= 2(e_\theta \underline{\xi} + e_\theta(\Phi)\underline{\xi}) \\
e_4 \underline{\vartheta} + \frac{1}{2}\kappa\, \underline{\vartheta} - 2\omega \underline{\vartheta} &= 2(e_\theta \underline{\eta} - e_\theta(\Phi)\underline{\eta}) - \frac{1}{2}\underline{\kappa}\, \vartheta \\
e_4(\underline{\kappa}) + \frac{1}{2}\kappa\, \underline{\kappa} - 2\omega \underline{\kappa} &= 2(e_\theta \underline{\eta} + e_\theta(\Phi)\underline{\eta}) + 2\rho \\
e_3 \zeta + \frac{1}{2}\underline{\kappa}(\zeta + \eta) - 2\underline{\omega}(\zeta - \eta) &= \underline{\beta} - 2e_\theta(\underline{\omega}) + 2\omega \underline{\xi} + \frac{1}{2}\kappa\, \underline{\xi} - {}^{(F)}\rho\, {}^{(F)}\underline{\beta} \\
e_4(\underline{\xi}) - e_3(\underline{\eta}) &= \underline{\beta} + 4\omega \underline{\xi} + \frac{1}{2}\underline{\kappa}(\underline{\eta} - \eta) - {}^{(F)}\rho\, {}^{(F)}\underline{\beta} \\
e_4 \underline{\omega} + e_3 \omega &= \rho + {}^{(F)}\rho^2 + 4\omega \underline{\omega} \\
e_\theta(\underline{\vartheta}) + 2e_\theta(\Phi)\underline{\vartheta} &= -2\underline{\beta} + (e_\theta(\underline{\kappa}) - \zeta \underline{\kappa}) - 2\, {}^{(F)}\rho\, {}^{(F)}\underline{\beta} \\
K &= -\rho - \frac{1}{4}\kappa\, \underline{\kappa} + {}^{(F)}\rho^2 \\
e_3(\alpha) + \frac{1}{2}\underline{\kappa}(\alpha) &= (e_\theta(\beta) - (e_\theta \Phi)\beta) + 4\underline{\omega}(\alpha) - \frac{3}{2}\vartheta \rho \\
&\quad + {}^{(F)}\rho(e_\theta\, {}^{(F)}\beta - e_\theta \Phi\, {}^{(F)}\beta - \vartheta\, {}^{(F)}\rho) \\
e_4(\beta) + 2\kappa \beta &= (e_\theta \alpha + 2e_\theta \Phi \alpha) - 2\omega \beta + 3\xi \rho \\
&\quad + {}^{(F)}\rho(e_4\, {}^{(F)}\beta + 2\omega\, {}^{(F)}\beta - 2\xi\, {}^{(F)}\rho) \\
e_3(\beta) + \underline{\kappa}\beta &= e_\theta(\rho) + 2\underline{\omega}\beta + 3\eta \rho \\
&\quad + {}^{(F)}\rho(e_\theta\, {}^{(F)}\rho - \kappa\, {}^{(F)}\underline{\beta} - \frac{\underline{\kappa}}{2}\, {}^{(F)}\beta) \\
e_4 \rho + \frac{3}{2}\kappa \rho &= (e_\theta(\beta) + (e_\theta \Phi)\beta) \\
&\quad + {}^{(F)}\rho(-\kappa\, {}^{(F)}\rho + e_\theta\, {}^{(F)}\beta + e_\theta(\Phi)\, {}^{(F)}\beta) \\
e_4\, {}^{(F)}\underline{\beta} &= -e_\theta\, {}^{(F)}\rho - 2\underline{\eta}\, {}^{(F)}\rho + \left(-\frac{1}{2}\kappa + 2\omega\right) {}^{(F)}\underline{\beta} \\
e_3\, {}^{(F)}\beta &= e_\theta\, {}^{(F)}\rho + 2\eta\, {}^{(F)}\rho + \left(-\frac{1}{2}\underline{\kappa} + 2\underline{\omega}\right) {}^{(F)}\beta, \\
e_3\, {}^{(F)}\rho &= -\underline{\kappa}\, {}^{(F)}\rho - \not{d}_1\, {}^{(F)}\underline{\beta}, \\
e_4\, {}^{(F)}\rho &= -\kappa\, {}^{(F)}\rho + \not{d}_1\, {}^{(F)}\beta
\end{aligned}
$$

and the other equations are obtained through the symmetry $e_3 - e_4$.

The definition of $O(\epsilon)$-Reissner-Nordström perturbations does not specify a particular frame. In what follows we investigate how the main Ricci and curvature quantities change relative to frame transformations, i.e linear transformations of the form $e'_\alpha = \Omega_\alpha{}^\beta e_\beta$ which take null frames into null frames. We will use the fact that a general frame transformation can be decomposed into the following three elementary types:

- Transformations which fix e_3,

$$e'_3 = e_3, \qquad e'_\theta = e_\theta + \frac{1}{2} f e_3, \qquad e'_4 = e_4 + f e_\theta + \frac{1}{4} f^2 e_3 \tag{4.4}$$

- Transformations which fix e_4,

$$e'_3 = \left(e_3 + \underline{f} e_\theta + \frac{1}{4} \underline{f}^2 e_4 \right), \qquad e'_\theta = e_\theta + \frac{1}{2} \underline{f} e_4, \qquad e'_4 = e_4 \tag{4.5}$$

- Transformation which preserve the directions of e_3, e_4, i.e confomal transformations of the form $e'_3 = \lambda e_3$, $e'_4 = \lambda^{-1} e_4$.

where $f, \underline{f}$ are reduced 1-forms and λ is a reduced scalar. A transformation consistent with $O(\epsilon)$-perturbations of Reissner-Nordstrom spacetimes must have $f, \underline{f} = O(\epsilon)$ and $a := \log \lambda = O(\epsilon)$.

Lemma 4.3 (Lemma 2.3.1. of [9]). *A general composite transformation type(3) $\circ$ type(1) $\circ$ type(2) has the form,*

$$\begin{aligned}
e'_3 &= \lambda \left(e_3 + \underline{f} e_\theta + \frac{1}{4} \underline{f}^2 e_4 \right) \\
e'_\theta &= \left(1 + \frac{1}{2} f \underline{f} \right) e_\theta + \frac{1}{2} f e_3 + \frac{1}{2} \left(\underline{f} + \frac{1}{4} f \underline{f}^2 \right) e_4 \\
e'_4 &= \lambda^{-1} \left(\left(1 + \frac{1}{2} f \underline{f} + \frac{1}{16} f^2 \underline{f}^2 \right) e_4 + \left(f + \frac{1}{4} f^2 \underline{f} \right) e_\theta + \frac{1}{4} f^2 e_3 \right)
\end{aligned} \tag{4.6}$$

Proposition 4.4. *Under a general transformation of type* (4.6) *the curvature and electromagnetic components transform as follows:*

$$\begin{aligned}
&\alpha' = \alpha + O(\epsilon^2), \qquad \underline{\alpha}' = \underline{\alpha} + O(\epsilon^2) \\
&\beta' = \lambda^{-1} \left(\beta + \frac{3}{2} \rho f \right) + O(\epsilon^2), \qquad \underline{\beta}' = \lambda \left(\underline{\beta} + \frac{3}{2} \rho \underline{f} \right) + O(\epsilon^2), \\
&{}^{(F)}\beta' = \lambda^{-1} \left({}^{(F)}\beta + f\, {}^{(F)}\rho \right) + O(\epsilon^2), \qquad {}^{(F)}\underline{\beta}' = \lambda \left({}^{(F)}\underline{\beta} - \underline{f}\, {}^{(F)}\rho \right) + O(\epsilon^2), \\
&\rho' = \rho + O(\epsilon^2), \qquad {}^{(F)}\rho' = {}^{(F)}\rho + O(\epsilon^2)
\end{aligned}$$

Proof. Straighforward calculations using the definitions (1.7) and Lemma 4.3. See also Proposition 2.3.4 of [9]. □

Notice that the only quantities which vanish in the background and which are $O(\epsilon^2)$ invariant are the extreme curvature components $\alpha, \underline{\alpha}$. These components verify the Teukolsky equation, which is the first step in the deriving the Regge-Wheeler type equation for the curvature term $\mathfrak{q}$.

Remark 4.5. As a consequence of Proposition 4.4, the extreme components of the electromagnetic tensor ${}^{(F)}\beta$, ${}^{(F)}\underline{\beta}$ are not $O(\epsilon^2)$ invariant. Moreover, they transform under change of frame similarly to the $\beta, \underline{\beta}$ component of the curvature. This motivates the notation.

4.1. The new invariant quantity $\mathfrak{f}$

To consider the non-linear electromagnetic perturbation of Reissner-Nordström we need an equation for a $O(\epsilon^2)$-invariant quantity. By Remark 4.5, we can't make direct use of the spin ± 1 Teukolsky equation verified by ${}^{(F)}\beta$ and ${}^{(F)}\underline{\beta}$ (see for example [12]), as compared to the electromagnetic perturbation of Schwarzschild, treated in Section 6.5. We will make use instead of a new quantity for the electromagnetic part of the curvature. From the Bianchi identity for $e_3\alpha$, we identify the following quantity $\mathfrak{f}$ defined by

$$\mathfrak{f} := \mathcal{D}_2^{\star}\, {}^{(F)}\beta + \vartheta\, {}^{(F)}\rho = -e_\theta\, {}^{(F)}\beta + e_\theta\Phi\, {}^{(F)}\beta + \vartheta\, {}^{(F)}\rho$$

This new quantity turns out to play a fundamental role in the equations governing the coupled gravitational and electromagnetic perturbations. Indeed, it appears in the Teukolsky equation for the extreme curvature component α in electrovacuum. Moreover, there exists a Chandrasekhar-type transformation which transforms $\mathfrak{f}$ into $\mathfrak{q}^{\mathbf{F}}$, and $\mathfrak{q}^{\mathbf{F}}$ verifies a Regge-Wheeler type equation coupled with the curvature as in (0.4). And most importantly, $\mathfrak{f}$ is a $O(\epsilon^2)$ invariant quantity.

Lemma 4.6. *The quantity* $\mathfrak{f}$ *is* $O(\epsilon^2)$ *invariant.*

Proof. Using Lemma 4.3 together with the definition of ϑ, we have that $\vartheta' = \vartheta + e_\theta(f) - fe_\theta(\Phi) + O(\epsilon^2)$. Using Proposition 4.4 and that $e_\theta\, {}^{(F)}\rho = O(\epsilon)$

as a consequence of Maxwell equations, we have

$$\begin{aligned}
\mathfrak{f}' &= -e'_\theta\, {}^{(F)}\beta' + e'_\theta\Phi\, {}^{(F)}\beta' + \vartheta'\, {}^{(F)}\rho' \\
&= -e_\theta({}^{(F)}\beta + f\, {}^{(F)}\rho) + e_\theta\Phi({}^{(F)}\beta + f\, {}^{(F)}\rho) + (\vartheta + e_\theta(f) \\
&\quad - f e_\theta(\Phi))\, {}^{(F)}\rho + O(\epsilon^2) \\
&= \mathfrak{f} - e_\theta(f)\, {}^{(F)}\rho + e_\theta\Phi(f\, {}^{(F)}\rho) + (e_\theta(f) - f e_\theta(\Phi))\, {}^{(F)}\rho + O(\epsilon^2) = \mathfrak{f} + O(\epsilon^2)
\end{aligned}$$

therefore $\mathfrak{f}$ is $O(\epsilon^2)$ invariant. □

Remark 4.7. To the knowledge of the author, the quantity $\mathfrak{f}$ seems to not have been noticed or used so far in the literature. One main reason could be found in the choice of gauge of Chandrasekhar in [4]. In treating the gravitational and electromagnetic perturbation of Reissner-Nordström, the author picks the phantom gauge $\phi_0 = 0$, corresponding to ${}^{(F)}\beta = {}^{(F)}\underline{\beta} = 0$. From Proposition 4.4, we can see that the choice of gauge with $f = -\, {}^{(F)}\rho^{-1}\, {}^{(F)}\beta$ and $\underline{f} = -\, {}^{(F)}\rho^{-1}\, {}^{(F)}\underline{\beta}$ gives ${}^{(F)}\beta$, ${}^{(F)}\underline{\beta} = O(\epsilon^2)$ in the non-linear setting. This choice of gauge would reduce $\mathfrak{f}$ to $\vartheta\, {}^{(F)}\rho$.

Applying Maxwell's equations and null structure equations, and using that $[e_\theta, e_3] = \frac{1}{2}\underline{\kappa} e_\theta + O(\epsilon)$ and $[e_\theta, e_4] = \frac{1}{2}\kappa e_\theta + O(\epsilon)$ we write

$$\begin{aligned}
e_3(\mathfrak{f}) &= -(e_\theta e_3\, {}^{(F)}\beta - e_\theta\Phi e_3\, {}^{(F)}\beta) + \frac{1}{2}\underline{\kappa}(e_\theta\, {}^{(F)}\beta - e_\theta\Phi\, {}^{(F)}\beta) \\
&\quad - {}^{(F)}\rho\left(\left(\frac{3}{2}\underline{\kappa} - 2\underline{\omega}\right)\vartheta - 2(e_\theta\eta - e_\theta(\Phi)\eta) + \frac{1}{2}\kappa\,\underline{\vartheta}\right), \\
e_4(\mathfrak{f}) &= -(e_\theta e_4\, {}^{(F)}\beta - e_\theta\Phi e_4\, {}^{(F)}\beta) + \frac{1}{2}\kappa(e_\theta\, {}^{(F)}\beta - e_\theta\Phi\, {}^{(F)}\beta) \\
&\quad - {}^{(F)}\rho\Big(2\kappa\,\vartheta - 2(e_\theta(\xi) - e_\theta(\Phi)\xi) + 2\omega\,\vartheta + 2\alpha\Big)
\end{aligned} \tag{4.7}$$

to be used later.

5. Teukolsky equations for the $O(\epsilon^2)$-invariant quantities α and $\mathfrak{f}$

Let f be a **Z**-invariant scalar function. Then, by definition $\Box_{\mathbf{g}}$ for a polarized metric $\mathbf{g}$, we have

$$\begin{aligned}
\Box_{\mathbf{g}} f &= -e_4(e_3(f)) + e_\theta(e_\theta(f)) - \frac{1}{2}\underline{\kappa} e_4(f) + \left(-\frac{1}{2}\kappa + 2\omega\right) e_3(f) \\
&\quad + e_\theta(\Phi) e_\theta(f) + 2\underline{\eta} e_\theta(f).
\end{aligned} \tag{5.1}$$

as in Lemma 2.4.1. of [9]. Using this formula, we derive the wave equations for the invariant quantities α and $\mathfrak{f}$.

Proposition 5.1. *[Teukolsky equation for α] The $O(\epsilon^2)$ invariant quantity α verifies the following wave equation:*

$$\begin{aligned}\square_{\mathbf{g}}\alpha &= -4\underline{\omega}e_4(\alpha) + (2\kappa + 4\omega)e_3(\alpha)\\ &\quad + \left(\frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} - 4\rho + 4\,{}^{(F)}\rho^2 - 4e_4\underline{\omega} - 10\kappa\underline{\omega} - 8\omega\underline{\omega} + 4e_\theta(\Phi)^2\right)\alpha\\ &\quad + {}^{(F)}\rho\,(2e_4(\mathfrak{f}) + (2\kappa + 4\omega)\mathfrak{f}) + O(\epsilon^2)\end{aligned}$$

Proof. Using the Bianchi identity

$$e_3(\alpha) + \frac{1}{2}\underline{\kappa}(\alpha) = (e_\theta(\beta) - (e_\theta\Phi)\beta) + 4\underline{\omega}(\alpha) - \frac{3}{2}\vartheta\rho - {}^{(F)}\rho\mathfrak{f}$$

and applying Bianchi identities, null structure equations and Maxwell's equations as in Proposition 4.2 and formulas (4.7), we have

$$\begin{aligned}e_4(e_3(\alpha)) &= e_4(e_\theta(\beta)) - e_\theta(\Phi)e_4(\beta) - e_4(e_\theta(\Phi))\beta - \left(\frac{\kappa}{2}\right)e_4(\alpha) - \left(\frac{e_4(\underline{\kappa})}{2}\right)\alpha\\ &\quad - \frac{3}{2}\vartheta e_4(\rho) - \frac{3}{2}e_4(\vartheta)\rho + 4e_4\underline{\omega}\alpha + 4\underline{\omega}e_4\alpha - e_4\,{}^{(F)}\rho\mathfrak{f} - {}^{(F)}\rho e_4\mathfrak{f}\\ &= e_4(e_\theta(\beta)) - e_\theta(\Phi)\big(e_\theta(\alpha) + 2e_\theta(\Phi)\alpha - 2(\kappa+\omega)\beta + 3\xi\rho\\ &\quad + {}^{(F)}\rho(e_4\,{}^{(F)}\beta + 2\omega\,{}^{(F)}\beta - 2\xi\,{}^{(F)}\rho)\big) - \left(-\frac{\kappa}{2}e_\theta\Phi\right)\beta - \left(\frac{\kappa}{2}\right)e_4(\alpha)\\ &\quad - \left(\frac{e_4(\underline{\kappa})}{2}\right)\alpha - \frac{3}{2}\vartheta e_4(\rho) - \frac{3}{2}e_4(\vartheta)\rho + 4e_4\underline{\omega}\alpha + 4\underline{\omega}e_4\alpha +\\ &\quad + {}^{(F)}\rho\Bigg(\kappa\mathfrak{f} + (e_\theta e_4\,{}^{(F)}\beta - e_\theta\Phi e_4\,{}^{(F)}\beta) - \frac{1}{2}\kappa(e_\theta\,{}^{(F)}\beta - e_\theta\Phi\,{}^{(F)}\beta)\\ &\quad + {}^{(F)}\rho(2\kappa\,\vartheta - 2(e_\theta(\xi) - e_\theta(\Phi)\xi) + 2\omega\,\vartheta + 2\alpha)\Bigg)\\ &= e_4(e_\theta(\beta)) - e_\theta(\Phi)(e_\theta(\alpha) + 2e_\theta(\Phi)\alpha) + 2e_\theta(\Phi)(\kappa+\omega)\beta\\ &\quad - 3e_\theta(\Phi)\xi\rho + \frac{\kappa}{2}e_\theta(\Phi)\beta - \left(\frac{\kappa}{2}\right)e_4(\alpha) - \left(\frac{e_4(\underline{\kappa})}{2}\right)\alpha - \frac{3}{2}\vartheta e_4(\rho)\\ &\quad - \frac{3}{2}e_4(\vartheta)\rho + 4e_4\underline{\omega}\alpha + 4\underline{\omega}e_4\alpha + {}^{(F)}\rho\Bigg((e_\theta e_4\,{}^{(F)}\beta - 2e_\theta\Phi e_4\,{}^{(F)}\beta)\\ &\quad - \frac{3}{2}\kappa e_\theta\,{}^{(F)}\beta + \left(\frac{3}{2}\kappa - 2\omega\right)e_\theta\Phi\,{}^{(F)}\beta\\ &\quad + {}^{(F)}\rho(3\kappa\,\vartheta - 2(e_\theta(\xi) - 2e_\theta(\Phi)\xi) + 2\omega\,\vartheta + 2\alpha)\Bigg)\end{aligned}$$

Using that $e_\theta(\kappa), e_\theta(\omega), e_\theta(\rho), e_\theta({}^{(F)}\rho) = O(\epsilon)$, we have

$$
\begin{aligned}
e_\theta(e_\theta(\alpha)) &= e_\theta\big(e_4(\beta) + 2(\kappa+\omega)\beta - 2e_\theta(\Phi)\alpha - 3\xi\rho - {}^{(F)}\rho e_4\, {}^{(F)}\beta \\
&\quad - 2\omega\, {}^{(F)}\rho\, {}^{(F)}\beta + 2\xi\, {}^{(F)}\rho^2\big) \\
&= e_\theta(e_4(\beta)) + 2(\kappa+\omega)e_\theta(\beta) - 2e_\theta(\Phi)e_\theta(\alpha) - 2e_\theta(e_\theta(\Phi))\alpha \\
&\quad - 3e_\theta(\xi)\rho + {}^{(F)}\rho(-e_\theta e_4\, {}^{(F)}\beta - 2\omega e_\theta\, {}^{(F)}\beta + 2e_\theta\xi\, {}^{(F)}\rho) \\
&= e_\theta(e_4(\beta)) + 2(\kappa+\omega)\left(e_\theta(\Phi)\beta + e_3(\alpha) + \left(\frac{\underline{\kappa}}{2}\right)\alpha + \frac{3}{2}\vartheta\,\rho - 4\underline{\omega}\alpha\right) \\
&\quad - 2e_\theta(\Phi)e_\theta(\alpha) - 2\left(\rho - {}^{(F)}\rho^2 - (e_\theta\Phi)^2 + \frac{1}{4}\kappa\underline{\kappa}\right)\alpha - 3e_\theta(\xi)\rho \\
&\quad + {}^{(F)}\rho\bigg(- e_\theta e_4\, {}^{(F)}\beta + (-2\kappa - 4\omega)e_\theta\, {}^{(F)}\beta + (2\kappa+2\omega)e_\theta\Phi\, {}^{(F)}\beta \\
&\quad + {}^{(F)}\rho(2\kappa\vartheta + 2\omega\vartheta + 2e_\theta\xi)\bigg)
\end{aligned}
$$

Using (5.1), we have

$$
\begin{aligned}
\Box_{\mathbf{g}}\alpha &= -e_4(e_3(\alpha)) + e_\theta(e_\theta(\alpha)) - \frac{1}{2}\underline{\kappa}e_4(\alpha) + \left(-\frac{1}{2}\kappa + 2\omega\right)e_3(\alpha) \\
&\quad + e_\theta(\Phi)e_\theta(\alpha) \\
&= [e_\theta, e_4](\beta) + 2e_\theta(\Phi)^2\alpha + 3e_\theta(\Phi)\xi\rho - \frac{\kappa}{2}e_\theta(\Phi)\beta + \left(\frac{\underline{\kappa}}{2}\right)e_4(\alpha) \\
&\quad + \left(\frac{e_4(\underline{\kappa})}{2}\right)\alpha + \frac{3}{2}\vartheta e_4(\rho) + \frac{3}{2}e_4(\vartheta)\rho - 4e_4\underline{\omega}\alpha - 4\underline{\omega}e_4\alpha \\
&\quad + 2(\kappa+\omega)e_3(\alpha) + 2(\kappa+\omega)\left(\frac{\underline{\kappa}}{2}\right)\alpha \\
&\quad + 3(\kappa+\omega)\vartheta\,\rho - 8(\kappa\underline{\omega} + \omega\underline{\omega})\alpha - 2\left(\rho - {}^{(F)}\rho^2 - (e_\theta\Phi)^2 + \frac{1}{4}\kappa\underline{\kappa}\right)\alpha \\
&\quad - 3e_\theta(\xi)\rho - \frac{1}{2}\underline{\kappa}e_4(\alpha) + \left(-\frac{1}{2}\kappa + 2\omega\right)e_3(\alpha) \\
&\quad + {}^{(F)}\rho\bigg(- 2(e_\theta e_4\, {}^{(F)}\beta - e_\theta\Phi e_4\, {}^{(F)}\beta) \\
&\quad - \left(\frac{1}{2}\kappa + 4\omega\right)(e_\theta\, {}^{(F)}\beta - e_\theta\Phi\, {}^{(F)}\beta) \\
&\quad - {}^{(F)}\rho(\kappa\,\vartheta - 2(2e_\theta(\xi) - 2e_\theta(\Phi)\xi) + 2\alpha)\bigg)
\end{aligned}
$$

Using that modulo $O(\epsilon^2)$,

$$
\begin{aligned}
[e_\theta, e_4](\beta) &= \frac{\kappa}{2} e_\theta(\beta)\\
&= \frac{\kappa}{2}\Bigg(e_\theta(\Phi)\beta + e_3(\alpha) + \left(\frac{\underline{\kappa}}{2}\right)\alpha - 4\underline{\omega}\alpha + \frac{3}{2}\vartheta\,\rho\\
&\quad + {}^{(F)}\rho(\vartheta\,{}^{(F)}\rho - e_\theta\,{}^{(F)}\beta + e_\theta\Phi\,{}^{(F)}\beta)\Bigg)
\end{aligned}
$$

we have

$$
\begin{aligned}
\square_{\mathbf{g}}\alpha &= 3e_\theta(\Phi)\xi\rho - 3e_\theta(\xi)\rho + \frac{3}{2}\vartheta e_4(\rho) + \frac{3}{2}e_4(\vartheta)\rho + 3(\kappa+\omega)\vartheta\,\rho + \frac{3}{4}\kappa\vartheta\rho\\
&\quad + (-4\underline{\omega})\, e_4(\alpha) + (2\kappa + 4\omega)e_3(\alpha) + \Bigg(\frac{e_4(\underline{\kappa})}{2} + 4e_\theta(\Phi)^2 - 4e_4\underline{\omega}\\
&\quad + \frac{3}{4}\kappa\underline{\kappa} + \omega\underline{\kappa} - 10\kappa\underline{\omega} - 8\omega\underline{\omega} - 2\rho + 2\,{}^{(F)}\rho^2\Bigg)\alpha\\
&\quad + {}^{(F)}\rho\Bigg(- 2(e_\theta e_4\,{}^{(F)}\beta - e_\theta\Phi e_4\,{}^{(F)}\beta) - (\kappa+4\omega)(e_\theta\,{}^{(F)}\beta - e_\theta\Phi\,{}^{(F)}\beta)\\
&\quad - {}^{(F)}\rho(\frac{1}{2}\kappa\,\vartheta - 2(2e_\theta(\xi) - 2e_\theta(\Phi)\xi) + 2\alpha)\Bigg)
\end{aligned}
$$

Using equations for $e_4(\vartheta), e_4\rho, e_4(\underline{\kappa})$ we infer

$$
\begin{aligned}
\square_{\mathbf{g}}\alpha &= -4\underline{\omega}e_4(\alpha) + (2\kappa+4\omega)e_3(\alpha)\\
&\quad + \left(4e_\theta(\Phi)^2 - 4e_4\underline{\omega} + \frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} - 10\kappa\underline{\omega} - 8\omega\underline{\omega} - 4\rho\right)\alpha\\
&\quad + {}^{(F)}\rho\Bigg(- 2(e_\theta e_4\,{}^{(F)}\beta - e_\theta\Phi e_4\,{}^{(F)}\beta) - (\kappa+4\omega)(e_\theta\,{}^{(F)}\beta - e_\theta\Phi\,{}^{(F)}\beta)\\
&\quad - {}^{(F)}\rho(2\kappa\,\vartheta - 2(2e_\theta(\xi) - 2e_\theta(\Phi)\xi)\Bigg)
\end{aligned}
$$

Using (4.7), we can write the term multiplying ${}^{(F)}\rho$ on the right hand side as $(2\kappa+4\omega)\mathfrak{f} + 2e_4\mathfrak{f} + 4\,{}^{(F)}\rho\alpha$ giving therefore the desired expression. $\square$

It is remarkable that the new quantity $\mathfrak{f}$ also verifies a Teukolsky equation.

Proposition 5.2 (Teukolsky equation for $\mathfrak{f}$). *The $O(\epsilon^2)$-invariant quantity $\mathfrak{f}$ verifies the following wave equation:*

$$
\begin{aligned}
\square_{\mathbf{g}}\mathfrak{f} &= \left(\frac{3}{2}\kappa + 2\omega\right) e_3\mathfrak{f} \\
&\quad + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right) e_4\mathfrak{f} + \left(\frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} - 2\rho - 4\kappa\underline{\omega} - 2e_4\underline{\omega} + 4e_\theta\Phi^2\right)\mathfrak{f} \\
&\quad + {}^{(F)}\rho\Big(- 2e_3(\alpha) - (2\underline{\kappa} - 8\underline{\omega})\alpha\Big) + O(\epsilon^2)
\end{aligned}
$$

Proof. We derive the Teukolsky equation verified by ${}^{(F)}\beta$. Consider the Maxwell's equations:

$$
\begin{aligned}
e_3\,{}^{(F)}\beta - e_\theta\,{}^{(F)}\rho - 2\eta\,{}^{(F)}\rho + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right){}^{(F)}\beta = 0,\\
e_4\,{}^{(F)}\rho + \kappa\,{}^{(F)}\rho - e_\theta\,{}^{(F)}\beta - e_\theta(\Phi)\,{}^{(F)}\beta = 0
\end{aligned}
$$

and apply the operator $(e_4 + \frac{3}{2}\kappa)$ to the first equation and the operator $(e_\theta + 2\eta)$ to the second and add them, keeping only the linear terms. We are left with:

$$
\begin{aligned}
0 &= -\Bigg(- e_4e_3\,{}^{(F)}\beta + e_\theta e_\theta\,{}^{(F)}\beta + e_\theta(\Phi)e_\theta\,{}^{(F)}\beta - \frac{1}{2}\underline{\kappa}e_4\,{}^{(F)}\beta \\
&\quad + \left(- \frac{1}{2}\kappa + 2\omega\right)e_3\,{}^{(F)}\beta\Bigg) \\
&\quad + \left(\frac{1}{2}\kappa\underline{\kappa} + \omega\underline{\kappa} + \rho - 3\kappa\underline{\omega} - 2e_4\underline{\omega} - e_\theta e_\theta(\Phi)\right){}^{(F)}\alpha_\theta - 2\underline{\omega}e_4\,{}^{(F)}\beta \\
&\quad + (\kappa + 2\omega)e_3\,{}^{(F)}\beta + [e_\theta, e_4]\,{}^{(F)}\rho - \frac{1}{2}\kappa e_\theta\,{}^{(F)}\rho + (-\kappa\eta + e_\theta\kappa - 2e_4\eta)\,{}^{(F)}\rho \\
&= -\square_{\mathbf{g}}\,{}^{(F)}\beta + \left(\frac{1}{4}\kappa\underline{\kappa} + \omega\underline{\kappa} - 3\kappa\underline{\omega} - 2e_4\underline{\omega} + {}^{(F)}\rho^2 + (e_\theta\Phi)^2\right){}^{(F)}\beta \\
&\quad - 2\underline{\omega}e_4\,{}^{(F)}\beta + (\kappa + 2\omega)e_3\,{}^{(F)}\beta + (e_\theta(\vartheta) + 2e_\theta(\Phi)\vartheta \\
&\quad - 2e_3(\xi) + \xi\underline{\kappa} + 8\underline{\omega}\xi + 4\beta)\,{}^{(F)}\rho
\end{aligned}
$$

which gives an expression for $\square_{\mathbf{g}}\,{}^{(F)}\beta$. Recall that $\mathfrak{f} = -e_\theta\,{}^{(F)}\beta + e_\theta\Phi\,{}^{(F)}\beta + \vartheta\,{}^{(F)}\rho$. Using that

$$
\begin{aligned}
[\Box_{\mathbf{g}}, e_\theta] f &= \frac{1}{2}\kappa e_3 e_\theta(f) + \frac{1}{2}\underline{\kappa} e_4 e_\theta f + \left({}^{(F)}\rho^2 + (e_\theta\Phi)^2 + \frac{1}{4}\kappa\underline{\kappa}\right) e_\theta(f),\\
\Box(e_\theta\Phi) &= e_\theta\Phi \left({}^{(F)}\rho^2 + (e_\theta\Phi)^2 - \frac{1}{4}\kappa\underline{\kappa}\right),\\
\Box(\vartheta) &= e_\theta e_\theta \vartheta + e_\theta\Phi e_\theta\vartheta + \left(-\frac{1}{4}\kappa\underline{\kappa} + \rho + \underline{\omega}\kappa + 8\omega\underline{\omega} - 2e_4\underline{\omega}\right)\vartheta\\
&\quad + \left(-\frac{1}{4}\kappa^2 - \omega\kappa\right)\underline{\vartheta} - 2(e_\theta e_3(\xi) - e_\theta(\Phi)e_3(\xi)) + 4\underline{\omega}(e_\theta\xi - e_\theta\Phi\xi)\\
&\quad + (\kappa + 4\omega)(e_\theta\eta - e_\theta(\Phi)\eta) + 4\underline{\omega}\alpha + 2(e_\theta\beta - e_\theta(\Phi)\beta)\\
&\quad + 2\,{}^{(F)}\rho(e_\theta\,{}^{(F)}\alpha_\theta - e_\theta(\Phi)\,{}^{(F)}\alpha_\theta),\\
\Box_{\mathbf{g}}\,{}^{(F)}\rho &= {}^{(F)}\rho\left(-\frac{1}{2}\kappa\underline{\kappa} + 2\rho\right) + O(\epsilon)
\end{aligned}
$$

we have

$$
\begin{aligned}
\Box(e_\theta\,{}^{(F)}\beta) &= e_\theta(\Box\,{}^{(F)}\beta) + [\Box, e_\theta]\,{}^{(F)}\beta\\
&= \left(\frac{1}{4}\kappa\underline{\kappa} - 3\kappa\underline{\omega} + \omega\underline{\kappa} + {}^{(F)}\rho^2 + e_\theta\Phi^2 - 2e_4\underline{\omega}\right) e_\theta\,{}^{(F)}\beta\\
&+ 2e_\theta\Phi e_\theta e_\theta\Phi\,{}^{(F)}\beta + (\kappa + 2\omega)\, e_\theta e_3({}^{(F)}\beta) - 2\underline{\omega} e_\theta e_4\,{}^{(F)}\beta\\
&+ \Big(e_\theta e_\theta(\vartheta) + 2e_\theta(\Phi)e_\theta\vartheta + 2e_\theta e_\theta(\Phi)\vartheta + e_\theta\xi\underline{\kappa} - 2e_\theta e_3(\xi)\\
&\quad + 8\underline{\omega} e_\theta\xi + 4e_\theta\beta\Big)\,{}^{(F)}\rho + \frac{1}{2}\kappa e_3 e_\theta({}^{(F)}\beta) + \frac{1}{2}\underline{\kappa} e_4 e_\theta\,{}^{(F)}\beta\\
&\quad + \left({}^{(F)}\rho^2 + (e_\theta\Phi)^2 + \frac{1}{4}\kappa\underline{\kappa}\right) e_\theta({}^{(F)}\beta)\\
&= \left(\frac{3}{2}\kappa + 2\omega\right) e_\theta e_3\,{}^{(F)}\alpha + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right) e_\theta e_4\,{}^{(F)}\alpha\\
&\quad + \Big(-3\kappa\underline{\omega} + \omega\underline{\kappa} + 2\,{}^{(F)}\rho^2 + 2e_\theta\Phi^2 - 2e_4\underline{\omega}\Big) e_\theta\,{}^{(F)}\alpha_\theta\\
&\quad + \left(2\rho - 2\,{}^{(F)}\rho^2 - 2(e_\theta\Phi)^2 + \frac{1}{2}\kappa\underline{\kappa}\right) e_\theta\Phi\,{}^{(F)}\alpha_\theta\\
&\quad + {}^{(F)}\rho\Bigg(e_\theta e_\theta(\vartheta) + 2e_\theta(\Phi)e_\theta\vartheta + \left(2\rho - 2\,{}^{(F)}\rho^2 - 2(e_\theta\Phi)^2 + \frac{1}{2}\kappa\underline{\kappa}\right)\vartheta\\
&\quad + e_\theta\xi\underline{\kappa} - 2e_\theta e_3(\xi) + 8\underline{\omega} e_\theta\xi + 4e_\theta\beta\Bigg)
\end{aligned}
$$

and

$$\begin{aligned}
\square(e_\theta\Phi\,{}^{(F)}\beta) &= \square_{\mathbf{g}}(e_\theta\Phi)\,{}^{(F)}\beta + e_\theta\Phi\square_{\mathbf{g}}({}^{(F)}\beta)\\
&\quad - e_3 e_\theta\Phi e_4\,{}^{(F)}\beta - e_4 e_\theta\Phi e_3\,{}^{(F)}\beta + 2e_\theta e_\theta\Phi e_\theta\,{}^{(F)}\beta\\
&= \left(2\rho - 2\,{}^{(F)}\rho^2 - 2e_\theta\Phi^2 + \frac{1}{2}\kappa\underline{\kappa}\right) e_\theta\,{}^{(F)}\beta\\
&\quad + e_\theta\Phi\Bigg(\left(\frac{3}{2}\kappa + 2\omega\right) e_3({}^{(F)}\beta) + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right) e_4\,{}^{(F)}\beta\\
&\quad + \left(-3\kappa\underline{\omega} - 2e_4\underline{\omega} + 2\,{}^{(F)}\rho^2 + 2(e_\theta\Phi)^2 + \omega\underline{\kappa}\right)\,{}^{(F)}\beta\\
&\quad + (e_\theta(\vartheta) + 2e_\theta(\Phi)\vartheta + \xi\underline{\kappa} - 2e_3(\xi) + 8\underline{\omega}\xi + 4\beta)\,{}^{(F)}\rho\Bigg)
\end{aligned}$$

Putting all this together we get

$$\begin{aligned}
&\square_{\mathbf{g}}(\mathfrak{f}) = \square_{\mathbf{g}}(-e_\theta\,{}^{(F)}\beta + e_\theta\Phi\,{}^{(F)}\beta + \vartheta\,{}^{(F)}\rho)\\
&= \square_{\mathbf{g}}(-e_\theta\,{}^{(F)}\beta + e_\theta\Phi\,{}^{(F)}\beta) + \square(\vartheta)\,{}^{(F)}\rho + \vartheta\square({}^{(F)}\rho)\\
&\quad - e_3(\vartheta)e_4({}^{(F)}\rho) - e_4(\vartheta)e_3({}^{(F)}\rho)\\
&= -\left(\frac{3}{2}\kappa + 2\omega\right)(e_\theta e_3\,{}^{(F)}\beta - e_\theta\Phi e_3\,{}^{(F)}\beta)\\
&\quad - \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right)(e_\theta e_4\,{}^{(F)}\beta - e_\theta\Phi e_4\,{}^{(F)}\beta)\\
&\quad - \left(-\frac{1}{2}\kappa\underline{\kappa} - 3\kappa\underline{\omega} + \omega\underline{\kappa} - 2\rho + 2\,{}^{(F)}\rho^2 + 4e_\theta\Phi^2 - 2e_4\underline{\omega}\right)(e_\theta\,{}^{(F)}\beta - e_\theta\Phi\,{}^{(F)}\beta)\\
&\quad - {}^{(F)}\rho\Bigg(\left(\frac{11}{4}\kappa\underline{\kappa} + 2\omega\underline{\kappa} - \rho - 2\,{}^{(F)}\rho^2 - 4(e_\theta\Phi)^2 - 3\underline{\omega}\kappa - 8\omega\underline{\omega} + 2e_4\underline{\omega}\right)\vartheta\\
&\quad + \left(\frac{3}{4}\kappa^2 + \omega\kappa\right)\underline{\vartheta} + (-\underline{\kappa} + 4\underline{\omega})(e_\theta\xi - e_\theta\Phi\xi) + (-3\kappa - 4\omega)(e_\theta\eta - e_\theta(\Phi)\eta)\\
&\quad + 2(e_\theta\beta - e_\theta\Phi\beta) + (2\underline{\kappa} - 4\underline{\omega})\alpha\Bigg)
\end{aligned}$$

Using once again (4.7) and

$$\begin{aligned}
(e_\theta(\beta) - (e_\theta\Phi)\beta) &= e_3(\alpha) + \frac{1}{2}\underline{\kappa}\alpha - 4\underline{\omega}\alpha + \frac{3}{2}\vartheta\rho + \vartheta\,{}^{(F)}\rho^2\\
&\quad - {}^{(F)}\rho(e_\theta\,{}^{(F)}\beta - e_\theta\Phi\,{}^{(F)}\beta)
\end{aligned}$$

we obtain

$$
\begin{aligned}
\square_{\mathbf{g}}(\mathfrak{f}) &= \left(\frac{3}{2}\kappa + 2\omega\right) e_3\mathfrak{f} + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right) e_4\mathfrak{f} \\
&\quad - \left(\frac{1}{2}\kappa\underline{\kappa} - 4\kappa\underline{\omega} + 2\omega\underline{\kappa} - 2\rho + 4e_\theta\Phi^2 - 2e_4\underline{\omega}\right)(e_\theta\, {}^{(F)}\beta - e_\theta\Phi\, {}^{(F)}\beta) \\
&\quad - {}^{(F)}\rho\Bigg(\left(-\frac{1}{2}\kappa\underline{\kappa} - 2\omega\underline{\kappa} + 2\rho - 4(e_\theta\Phi)^2 + 4\underline{\omega}\kappa + 2e_4\underline{\omega}\right)\vartheta \\
&\quad + 2e_3(\alpha) + (2\underline{\kappa} - 8\underline{\omega})\alpha\Bigg) \\
&= \left(\frac{3}{2}\kappa + 2\omega\right) e_3\mathfrak{f} + \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right) e_4\mathfrak{f} \\
&\quad + \left(\frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} - 2\rho - 4\kappa\underline{\omega} - 2e_4\underline{\omega} + 4e_\theta\Phi^2\right)\mathfrak{f} \\
&\quad + 2\, {}^{(F)}\rho\Big(- e_3(\alpha) - (\underline{\kappa} - 4\underline{\omega})\alpha\Big)
\end{aligned}
$$

as desired. $\square$

Remark 5.3. Observe that equations (5.1) and (5.2), respectively for the Weyl curvature α and for the electromagnetic tensor $\mathfrak{f}$,

$$
\begin{aligned}
\square_{\mathbf{g}}\alpha &= -4\underline{\omega}e_4(\alpha) + (2\kappa + 4\omega)e_3(\alpha) \\
&\quad + \left(\frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} - 4\rho + 4\, {}^{(F)}\rho^2 - 4e_4\underline{\omega} - 10\kappa\underline{\omega} - 8\omega\underline{\omega} + 4e_\theta(\Phi)^2\right)\alpha \\
&\quad + {}^{(F)}\rho\Big(2e_4(\mathfrak{f}) + (2\kappa + 4\omega)\mathfrak{f}\Big) + O(\epsilon^2), \\
\square_{\mathbf{g}}\mathfrak{f} &= \left(\frac{1}{2}\underline{\kappa} - 2\underline{\omega}\right) e_4\mathfrak{f} \\
&\quad + \left(\frac{3}{2}\kappa + 2\omega\right) e_3\mathfrak{f} + \left(\frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} - 2\rho - 4\kappa\underline{\omega} - 2e_4\underline{\omega} + 4e_\theta\Phi^2\right)\mathfrak{f} \\
&\quad + {}^{(F)}\rho\Big(- 2e_3(\alpha) - (2\underline{\kappa} - 8\underline{\omega})\alpha\Big) + O(\epsilon^2)
\end{aligned}
$$

are coupled. As in [5], signature arguments apply. The component α has signature 2 and the quantity $\mathfrak{f}$ has signature 1, therefore in the wave equation for α, $\mathfrak{f}$ has to appear with an e_4 derivative. On the other hand, in the wave equation for $\mathfrak{f}$ the component α has to appear with an e_3 derivative. Moreover, this coupling comes with a multiplication for ${}^{(F)}\rho$, and recall that from (3.2), ${}^{(F)}\rho$ can be interpreted as a weighted quasi-local charge of the spacetime.

6. System of equations for the coupled gravitational and electromagnetic perturbations

In Schwarzschild spacetime, the Chandrasekhar's transformation applied to the extreme curvature component α gives a quantity at the level of the second derivative along the ingoing null direction of α that verifies a Regge-Wheeler equation (see for example the quantity P in [7], or the quantity $\mathfrak{q}$ in [9]). In Reissner-Nordström spacetime, we will get a Regge-Wheeler type equation, i.e. a wave equation with a good potential and no lower order terms, but with additional terms giving the coupling with the electromagnetic tensor. The new result is that there exists a transformation similar to Chandrasekhar's one, at the level of one derivative along the ingoing null direction, that can be applied to the new quantity $\mathfrak{f}$ to obtain a Regge-Wheeler type equation for the electromagnetic term $\mathfrak{q}^{\mathbf{F}}$, with additional terms giving the coupling with the curvature.

Inspired by the system of three equations for the extreme curvature component and its two derivatives in slowly rotating Kerr as obtained in [10], we write a system of five equations for suitably chosen rescaled quantities depending on the curvature and on the electromagnetic components, from the two quantities α and $\mathfrak{f}$.

- The first three equations are equations for the rescaled α, its first and its second derivative in the ingoing null direction, respectively. The third quantity corresponds to the $\mathfrak{q}$ obtained by Chandrasekhar transformation in [9] which verifies the Regge-Wheeler type equation with a new right hand side depending on the electromagnetic components.
- The last two equations are equations for the rescaled $\mathfrak{f}$ and its first derivative in the ingoing null directions $\mathfrak{q}^{\mathbf{F}}$. This last quantity turns out to verify a Regge-Wheeler type equation too, with a right hand side depending on the curvature.

The first three equations correspond to the equations obtained by Ma in [10] in the case of Kerr spacetime. On the right hand side, his equations have lower order terms in the curvature multiplied by the angular momentum. In the case of small angular moment, the author is able to absorb the error terms coming from the lower error terms. In the case of coupled gravitational and electromagnetic perturbations, the right hand side of the first three equations is not given by lower order terms, but from a non trivial dependence on the electromagnetic parts, which are independent to the curvature part.

6.1. Definition of rescaled quantities and operators

As suggested in [1], we introduce the following operators as a rescaled version of the derivative in the ingoing and in the outgoing null directions[6]:

$$\underline{P}(f) = r\underline{\kappa}^{-1}e_3 f + \frac{1}{2}rf, \tag{6.1}$$

$$Q(f) = r\underline{\kappa}e_4 f + \frac{1}{2}r\kappa\underline{\kappa}f \tag{6.2}$$

The operator $\underline{P}$ is fundamentally used to define the various quantities in (6.3), while Q is introduced to simplify the right hand side of the Teukolsky equation for α. Observe that the operators $\underline{P}$ and Q, even if consist in derivatives along the e_3 and e_4 directions, do not change the signature of the quantity they are applied to.

We compute $\Box_{\mathbf{g}}(\underline{P}(f))$ and $e_3(Qf)$, which will be useful in the derivation of the main equations.

Lemma 6.1. *We have, modulo $O(\epsilon^2)$,*

$$\begin{aligned}\Box_{\mathbf{g}}(\underline{P}f) &= \frac{1}{r}(-\kappa\underline{\kappa}+2\rho)\underline{P}(\underline{P}(f)) + \left(\frac{1}{2}\kappa\underline{\kappa} - 4\rho - 2\,{}^{(F)}\!\rho^2\right)\underline{P}(f)\\ &+ \left(\frac{1}{2}\rho + {}^{(F)}\!\rho^2\right)rf + \frac{3}{2}r\Box_{\mathbf{g}}(f) + \underline{\kappa}^{-1}re_3(\Box_{\mathbf{g}}(f))\end{aligned}$$

Proof. Writing $\underline{P}f = r\underline{\kappa}^{-1}e_3 f + \frac{1}{2}rf$, we have

$$\begin{aligned}\Box_{\mathbf{g}}(\underline{P}f) = \frac{1}{2}\Big(&\Box_{\mathbf{g}}(r)f + r\Box_{\mathbf{g}}(f) - e_3(r)e_4(f) - e_4(r)e_3(f)\Big)\\ &+ \Box_{\mathbf{g}}(\underline{\kappa}^{-1}r)e_3(f) + \underline{\kappa}^{-1}r\Box_{\mathbf{g}}(e_3(f)) - e_3(\underline{\kappa}^{-1}r)e_4e_3(f)\\ &- e_4(\underline{\kappa}^{-1}r)e_3e_3(f)\end{aligned}$$

---

[6] To be consistent with the previous definitions, we define the operator as $\underline{P}$, since the bar quantities refer to e_3. The operator Q differ from the operator $P = r\kappa^{-1}e_4 f + \frac{1}{2}rf$ which would be used in the treatment of the corresponding system for the spin -2 quantity $\underline{\alpha}$. See Remark 6.11.

Using that

$$\begin{aligned}
\square_{\mathbf{g}}(r) &= r\left(-\frac{1}{2}\kappa\underline{\kappa}-\rho\right),\\
e_3(\underline{\kappa}^{-1}) &= -\frac{1}{\underline{\kappa}^2}e_3(\underline{\kappa}) = -\frac{1}{\underline{\kappa}^2}\left(-\frac{1}{2}\underline{\kappa}^2-2\underline{\omega}\underline{\kappa}\right) = \frac{1}{2}+2\underline{\omega}\underline{\kappa}^{-1},\\
e_4(\underline{\kappa}^{-1}) &= -\frac{1}{\underline{\kappa}^2}e_4(\underline{\kappa}) = -\frac{1}{\underline{\kappa}^2}\left(-\frac{1}{2}\kappa\underline{\kappa}+2\omega\underline{\kappa}+2\rho\right)\\
&= \frac{1}{2}\kappa\underline{\kappa}^{-1}-2\omega\underline{\kappa}^{-1}-2\rho\underline{\kappa}^{-2},\\
\square_{\mathbf{g}}(\underline{\kappa}) &= \underline{\kappa}\rho+2\underline{\kappa}e_4\underline{\omega}+4\underline{\omega}\rho,\\
\square_{\mathbf{g}}(e_3(f)) &= e_3(\square_{\mathbf{g}}(f))+[\square_{\mathbf{g}},e_3]f\\
&= e_3(\square_{\mathbf{g}}(f))+\underline{\kappa}\square_{\mathbf{g}}f-2\omega e_3(e_3(f))+(\underline{\kappa}+2\underline{\omega})e_4(e_3(f))\\
&\quad+\left(\frac{1}{4}\kappa\underline{\kappa}-3\omega\underline{\kappa}+\underline{\omega}\kappa-8\omega\underline{\omega}-\rho-2\,{}^{(F)}\rho^2+2e_4(\underline{\omega})\right)e_3(f)\\
&\quad+\frac{1}{4}\underline{\kappa}^2e_4(f)
\end{aligned}$$

we have

$$\begin{aligned}
\square_{\mathbf{g}}(\underline{P}f) = r\Bigg[&(-\kappa\underline{\kappa}^{-1}+2\rho\underline{\kappa}^{-2})e_3e_3(f)+\left(-\frac{1}{4}\kappa\underline{\kappa}-\frac{1}{2}\rho\right)f+\frac{3}{2}\square_{\mathbf{g}}(f)\\
&+\left(-\frac{3}{2}\kappa-2\underline{\omega}\kappa\underline{\kappa}^{-1}+4\underline{\omega}\rho\underline{\kappa}^{-2}-2\,{}^{(F)}\rho^2\underline{\kappa}^{-1}\right)e_3(f)\\
&+\underline{\kappa}^{-1}e_3(\square_{\mathbf{g}}(f))\Bigg]
\end{aligned}$$

Writing

$$\begin{aligned}
e_3(f) &= \frac{1}{r}\underline{\kappa}(\underline{P}f)-\frac{1}{2}\underline{\kappa}f,\\
e_3e_3f &= \frac{1}{r^2}\underline{\kappa}^2\underline{P}(\underline{P}(f))-\frac{2}{r}(\underline{\kappa}^2+\underline{\omega}\underline{\kappa})(\underline{P}f)+\left(\frac{1}{2}\underline{\kappa}^2+\underline{\omega}\underline{\kappa}\right)f
\end{aligned}$$

then

$$\begin{aligned}
\square_{\mathbf{g}}(\underline{P}f) = r\Bigg(&\frac{1}{r^2}(-\kappa\underline{\kappa}+2\rho)\underline{P}(\underline{P}(f))+\frac{1}{r}\left(\frac{1}{2}\kappa\underline{\kappa}-4\rho-2\,{}^{(F)}\rho^2\right)\underline{P}(f)\\
&+\left(\frac{1}{2}\rho+{}^{(F)}\rho^2\right)f+\frac{3}{2}\square_{\mathbf{g}}(f)+\underline{\kappa}^{-1}e_3(\square_{\mathbf{g}}(f))\Bigg)
\end{aligned}$$

as desired. □

Lemma 6.2. *We have, modulo $O(\epsilon^2)$,*

$$e_3(Q(f)) = \frac{1}{r}\underline{\kappa}(Q(\underline{P}(f))) - \frac{1}{2}\underline{\kappa}Q(f) + (-\kappa\underline{\kappa}^2 + 2\rho\underline{\kappa})\underline{P}(f)$$

Proof. We have

$$\begin{aligned}
e_3(Qf) &= e_3\left(\underline{\kappa}re_4(f) + \frac{1}{2}\kappa\underline{\kappa}rf\right)\\
&= e_3(\underline{\kappa})re_4(f) + \underline{\kappa}e_3(r)e_4(f) + \underline{\kappa}re_3e_4(f) + \frac{1}{2}e_3(\kappa)\underline{\kappa}rf + \frac{1}{2}\kappa e_3(\underline{\kappa})rf\\
&\quad + \frac{1}{2}\kappa\underline{\kappa}e_3(r)f + \frac{1}{2}\kappa\underline{\kappa}re_3(f)\\
&= \underline{\kappa}re_4e_3(f) + \left(\frac{1}{2}\kappa\underline{\kappa} - 2\omega\underline{\kappa}\right)re_3(f) + \left(-\frac{1}{4}\kappa\underline{\kappa}^2 + \rho\underline{\kappa}\right)rf
\end{aligned}$$

Writing $e_3(f) = \frac{1}{r}\underline{\kappa}\ \underline{P}(f) - \frac{1}{2}\underline{\kappa}f$, we have

$$\begin{aligned}
e_3(Qf) &= \underline{\kappa}re_4\left(\frac{1}{r}\underline{\kappa}\ \underline{P}f - \frac{1}{2}\underline{\kappa}f\right) + \left(\frac{1}{2}\kappa\underline{\kappa}^2 - 2\omega\underline{\kappa}^2\right)\underline{P}(f)\\
&\quad + \left(-\frac{1}{2}\kappa\underline{\kappa}^2 + \omega\underline{\kappa}^2 + \rho\underline{\kappa}\right)rf\\
&= \underline{\kappa}r\Bigg(-\frac{1}{2r}\kappa\underline{\kappa}\underline{P}f + \frac{1}{r}\left(-\frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} + 2\rho\right)\underline{P}f + \frac{1}{r}\underline{\kappa}e_4(\underline{P}f)\\
&\quad - \frac{1}{2}\left(-\frac{1}{2}\kappa\underline{\kappa} + 2\omega\underline{\kappa} + 2\rho\right)f - \frac{1}{2}\underline{\kappa}e_4(f)\Bigg)\\
&\quad + \left(\frac{1}{2}\kappa\underline{\kappa}^2 - 2\omega\underline{\kappa}^2\right)\underline{P}f + \left(-\frac{1}{2}\kappa\underline{\kappa}^2 + \omega\underline{\kappa}^2 + \rho\underline{\kappa}\right)rf
\end{aligned}$$

which gives

$$e_3(Qf) = \underline{\kappa}^2e_4(\underline{P}f) - \frac{1}{2}\underline{\kappa}^2re_4(f) + \left(-\frac{1}{2}\kappa\underline{\kappa}^2 + 2\rho\underline{\kappa}\right)\underline{P}(f) + \left(-\frac{1}{4}\kappa\underline{\kappa}^2\right)rf$$

and writing $e_4(f) = \frac{1}{r}\underline{\kappa}^{-1}Qf - \frac{1}{2}\kappa f$, and $e_4(\underline{P}f) = \frac{1}{r}\underline{\kappa}^{-1}Q\underline{P}f - \frac{1}{2}\kappa\underline{P}f$ we have the desired expression □

As suggested in [1], we define the following new scaling of the extreme components of the curvature α and of the electromagnetic quantity $\mathfrak{f}$ as the following:

$$
\begin{aligned}
\Phi_0 &= r^2\underline{\kappa}^2\alpha,\\
\Phi_1 &= \underline{P}(\Phi_0),\\
\Phi_2 &= \underline{P}(\Phi_1) = \underline{P}(\underline{P}(\Phi_0)),\\
\Phi_3 &= r^2\underline{\kappa}\mathfrak{f},\\
\Phi_4 &= \underline{P}(\Phi_3)
\end{aligned}
\tag{6.3}
$$

The quantities Φ_0, Φ_1, Φ_2 contain information about the gravitational perturbation of the metric, i.e. about the Weyl curvature of the perturbed spacetime. The first quantity Φ_0 is a rescaled version of α, and being multiplied by $\underline{\kappa}^2$ it is of signature 0. Then, Φ_1 and Φ_2 are respectively the first and the second derivative, through the operator $\underline{P}$, of Φ_0, giving other two signature 0 quantities. Observe that the last quantity Φ_2 defined as

$$
\begin{aligned}
\Phi_2 = P(P(\Phi_0)) &= r^2\underline{\kappa}^{-2}e_3e_3\Phi_0 + 2r^2\underline{\kappa}^{-1}(1+\underline{\omega}\underline{\kappa}^{-1})e_3\Phi_0 + \frac{1}{2}r^2\Phi_0 =\\
&= r^4\left(e_3e_3\alpha + (2\underline{\kappa} - 6\underline{\omega})e_3\alpha + \left(\frac{1}{2}\underline{\kappa}^2 - 8\underline{\kappa}\underline{\omega} - 4e_3\underline{\omega} + 8\underline{\omega}^2\right)\alpha\right) = \mathfrak{q}
\end{aligned}
$$

coincides with the $\mathfrak{q}$ obtained by Chandrasekhar transformation in [9].

The quantities Φ_3, Φ_4 contain information about the electromagnetic perturbation of the metric, i.e. about the Ricci curvature (or electromagnetic tensor) of the perturbed spacetime. The quantity Φ_3 is a rescaled version of $\mathfrak{f}$, and being multiplied by $\underline{\kappa}$ it is of signature 0. Then Φ_4 is the first derivative, through the same operator $\underline{P}$, of Φ_3, giving another signature 0 quantity. The last quantity Φ_4, in analogy to Φ_2, is called $\mathfrak{q}^{\mathbf{F}}$.

6.2. Wave equations for the curvature quantities Φ_0, Φ_1, Φ_2

We will derive the wave equations for the quantities Φ_0, Φ_1, Φ_2.

Proposition 6.3. *Modulo $O(\epsilon^2)$,*

$$
\begin{aligned}
\square_{\mathbf{g}}\Phi_0 = \frac{1}{r}\left(2\kappa\underline{\kappa} - 4\rho\right)\Phi_1 + \left(-\frac{1}{2}\kappa\underline{\kappa} - 4\rho + 4\,{}^{(F)}\rho^2 + 4e_\theta(\Phi)^2\right)\Phi_0\\
+\,{}^{(F)}\rho\left(\frac{2}{r}Q(\Phi_3) - 4\rho\Phi_3\right)
\end{aligned}
$$

Proof. We have

$$\begin{aligned}\square_{\mathbf{g}}\Phi_0=\square_{\mathbf{g}}(r^2\underline{\kappa}^2\alpha)=\square_{\mathbf{g}}(r^2\underline{\kappa}^2)\alpha+r^2\underline{\kappa}^2\square_{\mathbf{g}}\alpha\\
-e_3(r^2\underline{\kappa}^2)e_4(\alpha)-e_4(r^2\underline{\kappa}^2)e_3(\alpha)\end{aligned}$$

and using that

$$\begin{aligned}e_3(r^2\underline{\kappa}^2)&=e_3(r^2)\underline{\kappa}^2+r^2e_3(\underline{\kappa}^2)=r^2\underline{\kappa}^3+r^2(-\underline{\kappa}^3-4\underline{\omega}\underline{\kappa}^2)=-4\underline{\omega}r^2\underline{\kappa}^2,\\
e_4(r^2\underline{\kappa}^2)&=e_4(r^2)\underline{\kappa}^2+r^2e_4(\underline{\kappa}^2)=(r^2\kappa)\underline{\kappa}^2+r^2(-\kappa\underline{\kappa}^2+4\omega\underline{\kappa}^2+4\rho\underline{\kappa})\\
&=4\omega r^2\underline{\kappa}^2+4\rho r^2\underline{\kappa},\\
e_4(e_3(r^2\underline{\kappa}^2))&=e_4(-4\underline{\omega}r^2\underline{\kappa}^2)=-4e_4(\underline{\omega})r^2\underline{\kappa}^2-4\underline{\omega}e_4(r^2\underline{\kappa}^2)\\
&=-4e_4(\underline{\omega})r^2\underline{\kappa}^2-16\underline{\omega}\omega r^2\underline{\kappa}^2-16\underline{\omega}\rho r^2\underline{\kappa}\end{aligned}$$

we have

$$\begin{aligned}\square_{\mathbf{g}}(r^2\underline{\kappa}^2)&=-e_4(e_3(r^2\underline{\kappa}^2))+e_\theta(e_\theta(r^2\underline{\kappa}^2))-\frac{1}{2}\underline{\kappa}e_4(r^2\underline{\kappa}^2)\\
&\quad+\left(-\frac{1}{2}\kappa+2\omega\right)e_3(r^2\underline{\kappa}^2)+e_\theta(\Phi)e_\theta(r^2\underline{\kappa}^2)\\
&=-(-4e_4(\underline{\omega})r^2\underline{\kappa}^2-16\underline{\omega}\omega r^2\underline{\kappa}^2-16\underline{\omega}\rho r^2\underline{\kappa})\\
&\quad-\frac{1}{2}\underline{\kappa}(4\omega r^2\underline{\kappa}^2+4\rho r^2\underline{\kappa})+\left(-\frac{1}{2}\kappa+2\omega\right)(-4\underline{\omega}r^2\underline{\kappa}^2)\\
&=(2\underline{\omega}\kappa-2\omega\underline{\kappa}+4e_4(\underline{\omega})+8\underline{\omega}\omega)\,r^2\underline{\kappa}^2+(16\underline{\omega}-2\underline{\kappa})\rho r^2\underline{\kappa}\end{aligned}$$

Therefore,

$$\begin{aligned}\square_{\mathbf{g}}\Phi_0&=\big((2\underline{\omega}\kappa-2\omega\underline{\kappa}+4e_4(\underline{\omega})+8\underline{\omega}\omega)\,r^2\underline{\kappa}^2+(16\underline{\omega}-2\underline{\kappa})\rho r^2\underline{\kappa}\big)\,\alpha\\
&\quad+r^2\underline{\kappa}^2\bigg(-4\underline{\omega}e_4(\alpha)+(2\kappa+4\omega)\,e_3(\alpha)\\
&\quad+(-4e_4(\underline{\omega})+\frac{1}{2}\kappa\underline{\kappa}-10\kappa\underline{\omega}+2\underline{\kappa}\omega-8\omega\underline{\omega}-4\rho+4e_\theta(\Phi)^2)\alpha\bigg)\\
&\quad-\big(-4\underline{\omega}r^2\underline{\kappa}^2\big)\,e_4(\alpha)-\big(4\omega r^2\underline{\kappa}^2+4\rho r^2\underline{\kappa}\big)\,e_3(\alpha)\\
&\quad+2r^2\underline{\kappa}^2\,{}^{(F)}\!\rho\,(e_4(\mathfrak{f})+(\kappa+2\omega)\mathfrak{f})\end{aligned}$$

giving

$$\square_{\mathbf{g}}\Phi_0 = r^2\Bigg(\left(2\kappa\underline{\kappa}^2 - 4\rho\underline{\kappa}\right) e_3(\alpha) + \left(\frac{1}{2}\kappa\underline{\kappa}^3 - 6\rho\underline{\kappa}^2 - 8\underline{\omega}\kappa\underline{\kappa}^2 + 16\rho\underline{\kappa}\underline{\omega} + 4\underline{\kappa}^2\, {}^{(F)}\!\rho^2 + 4\underline{\kappa}^2 e_\theta(\Phi)^2\right)\alpha + 2\underline{\kappa}^2\, {}^{(F)}\!\rho\left(e_4(\mathfrak{f}) + (\kappa + 2\omega)\mathfrak{f}\right)\Bigg)$$

Using the following relations

$$\begin{aligned} r^2\underline{\kappa}^2\alpha &= \Phi_0, \\ e_3\alpha &= \frac{1}{r^2}\underline{\kappa}^{-2}e_3(\Phi_0) + 4\underline{\omega}\alpha, \\ e_3\Phi_0 &= \frac{1}{r}\underline{\kappa}\Phi_1 - \frac{1}{2}\underline{\kappa}\Phi_0, \\ r^2\underline{\kappa}^2 e_4(\mathfrak{f}) + r^2(\kappa\underline{\kappa}^2 + 2\omega\underline{\kappa}^2)\mathfrak{f} &= \frac{1}{r}\underline{\kappa}e_4(r^3\underline{\kappa}\mathfrak{f}) - 2\rho\underline{\kappa}r^2\mathfrak{f} = \frac{1}{r}Q(\Phi_3) - 2\rho\Phi_3 \end{aligned}$$

we obtain the desired identity. $\square$

Remark 6.4. Comparing Proposition 5.1 to Proposition 6.3, we can notice that the rescaled quantity Φ_0 verifies a wave equation independent of the quantities $\omega, \underline{\omega}$, one of which can be made small in the ingoing or outgoing geodesic null frame. Therefore, the wave equation for Φ_0 is more natural and frame-independent compared to the one for α.

Proposition 6.5. *We have modulo $O(\epsilon^2)$,*

$$\square_{\mathbf{g}}(\Phi_1) = \frac{1}{r}(\kappa\underline{\kappa} - 2\rho)\Phi_2 + (-\kappa\underline{\kappa} + 6\, {}^{(F)}\!\rho^2 + 4e_\theta\Phi^2)\Phi_1 + \left(\frac{3}{2}\rho + {}^{(F)}\!\rho^2\right) r\Phi_0 + {}^{(F)}\!\rho\left(\frac{2}{r}Q(\Phi_4) - Q(\Phi_3) - 2\kappa\underline{\kappa}\Phi_4 + r\left(6\rho + 4\, {}^{(F)}\!\rho^2\right)\Phi_3\right)$$

Proof. We first compute $e_3(\square_{\mathbf{g}}\Phi_0)$, using Proposition 6.3.

$$
\begin{aligned}
e_3(\square_{\mathbf{g}}\Phi_0) &= e_3\Bigg(\frac{1}{r}\left(2\kappa\underline{\kappa}-4\rho\right)\Phi_1+\left(-\frac{1}{2}\kappa\underline{\kappa}-4\rho+4\,{}^{(F)}\rho^2+4e_\theta(\Phi)^2\right)\Phi_0\\
&\quad+{}^{(F)}\rho\left(\frac{2}{r}Q(\Phi_3)-4\rho\Phi_3\right)\Bigg)\\
&=-\frac{1}{2}r^{-1}\underline{\kappa}\left(2\kappa\underline{\kappa}-4\rho\right)\Phi_1+\frac{1}{r}\Bigg(2\left(-\frac{1}{2}\kappa\underline{\kappa}+2\underline{\omega}\kappa+2\rho\right)\underline{\kappa}\\
&\quad+2\kappa\left(-\frac{1}{2}\underline{\kappa}^2-2\underline{\omega}\underline{\kappa}\right)-4\left(-\frac{3}{2}\underline{\kappa}\rho-\underline{\kappa}\,{}^{(F)}\rho^2\right)\Bigg)\Phi_1\\
&\quad+\frac{1}{r}\left(2\kappa\underline{\kappa}-4\rho\right)e_3\Phi_1+e_3\left(-\frac{1}{2}\kappa\underline{\kappa}-4\rho+4\,{}^{(F)}\rho^2+4e_\theta(\Phi)^2\right)\Phi_0\\
&\quad+\left(-\frac{1}{2}\kappa\underline{\kappa}-4\rho+4\,{}^{(F)}\rho^2+4e_\theta(\Phi)^2\right)e_3\Phi_0\\
&\quad+{}^{(F)}\rho\Bigg(-\underline{\kappa}\left(\frac{2}{r}Q(\Phi_3)-4\rho\Phi_3\right)-\frac{1}{r}\underline{\kappa}Q(\Phi_3)+\frac{2}{r}e_3(Q(\Phi_3))\\
&\quad-4\left(-\frac{3}{2}\underline{\kappa}\rho-\underline{\kappa}\,{}^{(F)}\rho^2\right)\Phi_3-4\rho e_3\Phi_3\Bigg)
\end{aligned}
$$

and since

$$
e_3\left(-\frac{1}{2}\kappa\underline{\kappa}-4\rho+4\,{}^{(F)}\rho^2+4e_\theta(\Phi)^2\right)=\frac{1}{2}\kappa\underline{\kappa}^2+5\rho\underline{\kappa}-4\underline{\kappa}e_\theta(\Phi)^2-4\underline{\kappa}\,{}^{(F)}\rho^2
$$

applying Lemma 6.2 and writing $e_3\Phi_0=\frac{1}{r}\underline{\kappa}\Phi_1-\frac{1}{2}\underline{\kappa}\Phi_0$, $e_3\Phi_1=\frac{1}{r}\underline{\kappa}\Phi_2-\frac{1}{2}\underline{\kappa}\Phi_1$ and $e_3\Phi_3=\frac{1}{r}\underline{\kappa}\Phi_4-\frac{1}{2}\underline{\kappa}\Phi_3$ we have

$$
\begin{aligned}
e_3(\square_{\mathbf{g}}\Phi_0) &= \frac{1}{r}\left(-\frac{9}{2}\kappa\underline{\kappa}^2+10\rho\underline{\kappa}+8\underline{\kappa}\,{}^{(F)}\rho^2+4\underline{\kappa}e_\theta\Phi^2\right)\Phi_1\\
&\quad+\frac{1}{r^2}\left(2\kappa\underline{\kappa}^2-4\rho\underline{\kappa}\right)\Phi_2+\left(\frac{3}{4}\kappa\underline{\kappa}^2+7\rho\underline{\kappa}-6\underline{\kappa}e_\theta(\Phi)^2-6\underline{\kappa}\,{}^{(F)}\rho^2\right)\Phi_0\\
&\quad+{}^{(F)}\rho\Bigg(\frac{2}{r^2}\underline{\kappa}Q(\Phi_4)-\frac{4}{r}\underline{\kappa}Q(\Phi_3)-\frac{2}{r}\kappa\underline{\kappa}^2\Phi_4+(12\rho\underline{\kappa}+4\underline{\kappa}\,{}^{(F)}\rho^2)\Phi_3\Bigg)
\end{aligned}
$$

We have for $\Phi_1 = \underline{P}(\Phi_0)$, applying Lemma 6.1 to $f = \Phi_0$, and using Lemma 6.3 ,

$$\begin{aligned}
\square_{\mathbf{g}}(\Phi_1) &= \frac{1}{r}(-\kappa\underline{\kappa} + 2\rho)\underline{P}(\underline{P}(\Phi_0)) + \left(\frac{1}{2}\kappa\underline{\kappa} - 4\rho - 2\,{}^{(F)}\rho^2\right)\underline{P}(\Phi_0) \\
&\quad + \left(\frac{1}{2}\rho + {}^{(F)}\rho^2\right) r\Phi_0 + \frac{3}{2}r\square_{\mathbf{g}}(\Phi_0) + r\underline{\kappa}^{-1}e_3(\square_{\mathbf{g}}(\Phi_0)) \\
&= \frac{1}{r}(\kappa\underline{\kappa} - 2\rho)\Phi_2 + (-\kappa\underline{\kappa} + 6\,{}^{(F)}\rho^2 + 4e_\theta\Phi^2)\Phi_1 + \left(\frac{3}{2}\rho + {}^{(F)}\rho^2\right) r\Phi_0 \\
&\quad + {}^{(F)}\rho\left(\frac{2}{r}Q(\Phi_4) - Q(\Phi_3) - 2\kappa\underline{\kappa}\Phi_4 + r\left(6\rho + 4\,{}^{(F)}\rho^2\right)\Phi_3\right)
\end{aligned}$$

as desired. $\square$

We derive the Regge-Wheeler type equation for the curvature term $\Phi_2 = \mathfrak{q}$ with right hand side coupled to the electromagnetic components Φ_3 and Φ_4, multiplied by ${}^{(F)}\rho$.

Proposition 6.6. *We have modulo $O(\epsilon^2)$,*

$$\begin{aligned}
\square_{\mathbf{g}}(\Phi_2) &= \Big(-\kappa\underline{\kappa} + 6\,{}^{(F)}\rho^2 + 4e_\theta\Phi^2\Big)\Phi_2 \\
&\quad + {}^{(F)}\rho\bigg(\frac{2}{r}(Q(\underline{P}(\Phi_4))) - 2Q(\Phi_4) + (-4\kappa\underline{\kappa} + 4\rho)\underline{P}(\Phi_4) \\
&\quad + r(3\kappa\underline{\kappa} + 4\,{}^{(F)}\rho^2)\Phi_4 + r^2(-6\rho - 12\,{}^{(F)}\rho^2)\Phi_3\bigg) \\
&\quad + {}^{(F)}\rho^2\left(-4r\Phi_1 - 2r^2\Phi_0\right)
\end{aligned}$$

Proof. We first compute $e_3(\square_{\mathbf{g}}\Phi_1)$, using Proposition 6.5. We get

$$\begin{aligned}
e_3(\square_{\mathbf{g}}\Phi_1) &= e_3\Bigg(\frac{1}{r}(\kappa\underline{\kappa} - 2\rho)\Phi_2 \\
&\quad + (-\kappa\underline{\kappa} + 6\,{}^{(F)}\rho^2 + 4e_\theta\Phi^2)\Phi_1 + \left(\frac{3}{2}\rho + {}^{(F)}\rho^2\right) r\Phi_0 \\
&\quad + {}^{(F)}\rho\left(\frac{2}{r}Q(\Phi_4) - Q(\Phi_3) - 2\kappa\underline{\kappa}\Phi_4 + r\left(6\rho + 4\,{}^{(F)}\rho^2\right)\Phi_3\right)\Bigg)
\end{aligned}$$

which gives

$$
\begin{aligned}
e_3(\square_{\mathbf{g}}\Phi_1) &= \frac{1}{r}\left(-\frac{3}{2}\kappa\underline{\kappa}^2+6\rho\underline{\kappa}+2\underline{\kappa}\,{}^{(F)}\!\rho^2\right)\Phi_2+\frac{1}{r}\left(\kappa\underline{\kappa}-2\rho\right)e_3\Phi_2\\
&+\left(\kappa\underline{\kappa}^2-2\rho\underline{\kappa}-12\underline{\kappa}\,{}^{(F)}\!\rho^2-4\underline{\kappa}e_\theta\Phi^2\right)\Phi_1\\
&+\Big(-\kappa\underline{\kappa}+6\,{}^{(F)}\!\rho^2+4e_\theta\Phi^2\Big)e_3\Phi_1+r\left(-\frac{3}{2}\underline{\kappa}\rho-3\underline{\kappa}\,{}^{(F)}\!\rho^2\right)\Phi_0\\
&+r\left(\frac{3}{2}\rho+{}^{(F)}\!\rho^2\right)e_3(\Phi_0)+{}^{(F)}\!\rho\Bigg[-\frac{3}{r}\underline{\kappa}Q(\Phi_4)+\underline{\kappa}Q(\Phi_3)\\
&+(4\kappa\underline{\kappa}^2-4\rho\underline{\kappa})\Phi_4+r(-12\underline{\kappa}\rho-16\underline{\kappa}\,{}^{(F)}\!\rho^2)\Phi_3\\
&+\frac{2}{r}e_3Q(\Phi_4)-e_3Q(\Phi_3)-2\kappa\underline{\kappa}\;e_3\Phi_4+r\left(6\rho+4\,{}^{(F)}\!\rho^2\right)e_3\Phi_3\Bigg]
\end{aligned}
$$

Using Lemma 6.2 and writing $e_3\Phi_0=\frac{1}{r}\underline{\kappa}\Phi_1-\frac{1}{2}\underline{\kappa}\Phi_0$, $e_3\Phi_1=\frac{1}{r}\underline{\kappa}\Phi_2-\frac{1}{2}\underline{\kappa}\Phi_1$, $e_3\Phi_2=\frac{1}{r}\underline{\kappa}\underline{P}(\Phi_2)-\frac{1}{2}\underline{\kappa}\Phi_2$, and $e_3\Phi_3=\frac{1}{r}\underline{\kappa}\Phi_4-\frac{1}{2}\underline{\kappa}\Phi_3$, $e_3\Phi_4=\frac{1}{r}\underline{\kappa}\underline{P}(\Phi_4)-\frac{1}{2}\underline{\kappa}\Phi_4$ we have

$$
\begin{aligned}
e_3(\square_{\mathbf{g}}\Phi_1) &= \frac{1}{r^2}\Big(\kappa\underline{\kappa}^2-2\rho\underline{\kappa}\Big)\underline{P}(\Phi_2)\\
&+\frac{1}{r}\Big(-3\kappa\underline{\kappa}^2+7\rho\underline{\kappa}+8\underline{\kappa}\,{}^{(F)}\!\rho^2+4\underline{\kappa}e_\theta\Phi^2\Big)\Phi_2\\
&+\left(\frac{3}{2}\kappa\underline{\kappa}^2-\frac{1}{2}\rho\underline{\kappa}-14\underline{\kappa}\,{}^{(F)}\!\rho^2-6\underline{\kappa}e_\theta\Phi^2\right)\Phi_1\\
&+r\left(-\frac{9}{4}\underline{\kappa}\rho-\frac{7}{2}\underline{\kappa}\,{}^{(F)}\!\rho^2\right)\Phi_0+{}^{(F)}\!\rho\Bigg[\frac{2}{r^2}\underline{\kappa}Q\underline{P}\Phi_4-\frac{5}{r}\underline{\kappa}Q\Phi_4\\
&+\frac{1}{r}(-4\kappa\underline{\kappa}^2+4\rho\underline{\kappa})\underline{P}\Phi_4+\frac{3}{2}\underline{\kappa}Q(\Phi_3)\\
&+(6\kappa\underline{\kappa}^2+4\underline{\kappa}\,{}^{(F)}\!\rho^2)\Phi_4+r(-15\underline{\kappa}\rho-18\underline{\kappa}\,{}^{(F)}\!\rho^2)\Phi_3\Bigg]
\end{aligned}
$$

Applying Lemma 6.1 to $\Phi_2=P(\Phi_1)$,

$$
\begin{aligned}
\square_{\mathbf{g}}(\Phi_2) &= \frac{1}{r}(-\kappa\underline{\kappa}+2\rho)\underline{P}(\underline{P}(\Phi_1))+\left(\frac{1}{2}\kappa\underline{\kappa}-4\rho-2\,{}^{(F)}\!\rho^2\right)\underline{P}(\Phi_1)\\
&+\left(\frac{1}{2}\rho+{}^{(F)}\!\rho^2\right)r\Phi_1+\frac{3}{2}r\square_{\mathbf{g}}(\Phi_1)+r\underline{\kappa}^{-1}e_3(\square_{\mathbf{g}}(\Phi_1))
\end{aligned}
$$

which gives

$$\begin{aligned}\square_{\mathbf{g}}(\Phi_2) &= \Big(-\kappa\underline{\kappa} + 6\,{}^{(F)}\rho^2 + 4e_\theta\Phi^2\Big)\Phi_2 \\ &+ {}^{(F)}\rho\bigg(\frac{2}{r}(Q(\underline{P}(\Phi_4))) - 2Q(\Phi_4) + (-4\kappa\underline{\kappa} + 4\rho)\underline{P}(\Phi_4) \\ &+ r(3\kappa\underline{\kappa} + 4\,{}^{(F)}\rho^2)\Phi_4 + r^2(-6\rho - 12\,{}^{(F)}\rho^2)\Phi_3\bigg) \\ &+ {}^{(F)}\rho^2\left(-4r\Phi_1 - 2r^2\Phi_0\right)\end{aligned}$$

as desired. $\square$

Remark 6.7. Notice that the wave equation given by Proposition 6.6, for $\Phi_2 = \mathfrak{q}$ and $\Phi_4 = \mathfrak{q}^{\mathbf{F}}$ has the form

$$\square_{\mathbf{g}}\mathfrak{q} = V_1\mathfrak{q} + e\mathcal{M}(\mathfrak{q}^{\mathbf{F}}, \partial\mathfrak{q}^{\mathbf{F}}, \partial\partial\mathfrak{q}^{\mathbf{F}}) + e(l.o.t.(\mathfrak{q}^{\mathbf{F}})) + e^2(l.o.t.(\mathfrak{q}))$$

of the first equation of (0.4). Indeed, Φ_3 is a lower order term with respect to $\mathfrak{q}^{\mathbf{F}}$.

6.3. Wave equations for the electromagnetic quantities Φ_3, Φ_4

We compute the wave equations for the electromagnetic terms Φ_3, Φ_4.

Proposition 6.8. *We have modulo $O(\epsilon^2)$,*

$$\square_{\mathbf{g}}\Phi_3 = \frac{1}{r}\Big(\kappa\underline{\kappa} - 2\rho\Big)\Phi_4 + \Big(-\kappa\underline{\kappa} - 3\rho + 4e_\theta\Phi^2\Big)\Phi_3 + {}^{(F)}\rho\left(-\frac{2}{r}\Phi_1 - \Phi_0\right)$$

Proof. We have for $\Phi_3 = r^2\underline{\kappa}\mathfrak{f}$,

$$\begin{aligned}\square_{\mathbf{g}}\Phi_3 &= \square_{\mathbf{g}}(r^2)\underline{\kappa}\mathfrak{f} + r^2(\square(\underline{\kappa})\mathfrak{f} + \underline{\kappa}\square(\mathfrak{f}) - e_3(\underline{\kappa})e_4(\mathfrak{f}) - e_4(\underline{\kappa})e_3(\mathfrak{f})) \\ &- e_3(r^2)(e_4(\underline{\kappa})\mathfrak{f} + \underline{\kappa}e_4(\mathfrak{f})) - e_4(r^2)(e_3(\underline{\kappa})\mathfrak{f} + \underline{\kappa}e_3(\mathfrak{f}))\end{aligned}$$

As in Proposition 6.3, using Proposition 5.2, we have

$$\begin{aligned}\square_{\mathbf{g}}\Phi_3 &= \Big(-5\rho - 2\underline{\omega}\kappa + 4\rho\underline{\omega}\underline{\kappa}^{-1} + 4e_\theta\Phi^2\Big)\Phi_3 + r^2\Big(\kappa\underline{\kappa} - 2\rho\Big)e_3(\mathfrak{f}) \\ &+ 2r^2\,{}^{(F)}\rho\Big(-\underline{\kappa}e_3(\alpha) + (-\underline{\kappa}^2 + 4\underline{\omega}\underline{\kappa})\alpha\Big)\end{aligned}$$

Writing $e_3(\mathfrak{f}) = \frac{1}{r^3}\Phi_4 - \frac{1}{r^2}(1-2\underline{\omega}\underline{\kappa}^{-1})\Phi_3$ and $e_3\alpha = \frac{1}{r^2}\underline{\kappa}^{-2}e_3(\Phi_0) + 4\underline{\omega}\alpha$, and $e_3\Phi_0 = \frac{1}{r}\underline{\kappa}\Phi_1 - \frac{1}{2}\underline{\kappa}\Phi_0$ we have

$$\begin{aligned}\Box_{\mathbf{g}}\Phi_3 &= \frac{1}{r}\Big(\kappa\underline{\kappa} - 2\rho\Big)P\Phi_3 + \Big(-\kappa\underline{\kappa} - 3\rho + 4e_\theta\Phi^2\Big)\Phi_3\\ &\quad + {}^{(F)}\rho\left(-\frac{2}{r}\Phi_1 - \Phi_0\right)\end{aligned}$$

as desired. $\square$

We derive the Regge-Wheleer type equation for the quantity $\Phi_4 = \mathfrak{q}^{\mathbf{F}}$, with on the right hand side the curvature multiplied by ${}^{(F)}\rho$.

Proposition 6.9. *We have modulo $O(\epsilon^2)$,*

$$\Box_{\mathbf{g}}(\Phi_4) = \left(-\kappa\underline{\kappa} - 3\rho + 4e_\theta\Phi^2\right)\Phi_4 + {}^{(F)}\rho\left(-\frac{2}{r}\Phi_2 + {}^{(F)}\rho\,(4r\Phi_3)\right)$$

Proof. We first compute $e_3(\Box_{\mathbf{g}}\Phi_3)$, using Proposition 6.8,

$$\begin{aligned}e_3(\Box_{\mathbf{g}}\Phi_3) &= e_3\left(\frac{1}{r}\Big(\kappa\underline{\kappa} - 2\rho\Big)\Phi_4 + \Big(-\kappa\underline{\kappa} - 3\rho + 4e_\theta\Phi^2\Big)\Phi_3\right.\\ &\qquad\left. + {}^{(F)}\rho\left(-\frac{2}{r}\Phi_1 - \Phi_0\right)\right)\\ &= -\frac{1}{2}r^{-1}\underline{\kappa}\left(\kappa\underline{\kappa} - 2\rho\right)\Phi_4 + \frac{1}{r}\left(\left(-\frac{1}{2}\kappa\underline{\kappa} + 2\underline{\omega}\kappa + 2\rho\right)\underline{\kappa}\right.\\ &\qquad\left. + \kappa\left(-\frac{1}{2}\underline{\kappa}^2 - 2\underline{\omega}\underline{\kappa}\right) - 2\left(-\frac{3}{2}\underline{\kappa}\rho - \underline{\kappa}\,{}^{(F)}\rho^2\right)\right)\Phi_4\\ &\qquad + \frac{1}{r}\left(\kappa\underline{\kappa} - 2\rho\right)e_3\Phi_4 + e_3\Big(-\kappa\underline{\kappa} - 3\rho + 4e_\theta(\Phi)^2\Big)\Phi_3\\ &\qquad + \Big(-\kappa\underline{\kappa} - 3\rho + 4e_\theta(\Phi)^2\Big)e_3\Phi_3\\ &\qquad + {}^{(F)}\rho\left(\frac{2}{r}\underline{\kappa}\Phi_1 + \underline{\kappa}\Phi_0 - e_3\left(\frac{2}{r}\right)\Phi_1 - \frac{2}{r}e_3\Phi_1 - e_3\Phi_0\right)\end{aligned}$$

Since

$$
\begin{aligned}
&e_3\Big(- \kappa\underline{\kappa} - 3\rho + 4e_\theta(\Phi)^2\Big) = -e_3\kappa\underline{\kappa} - \kappa e_3\underline{\kappa} - 3e_3\rho + 8e_\theta(\Phi)e_3e_\theta\Phi \\
&= -\left(-\frac{1}{2}\kappa\underline{\kappa} + 2\underline{\omega}\kappa + 2\rho\right)\underline{\kappa} - \kappa\left(-\frac{1}{2}\underline{\kappa}^2 - 2\underline{\omega}\underline{\kappa}\right) \\
&\quad - 3\left(-\frac{3}{2}\underline{\kappa}\rho - \underline{\kappa}\,{}^{(F)}\rho^2\right) - 4\underline{\kappa}e_\theta(\Phi)^2 \\
&= \kappa\underline{\kappa}^2 + \frac{5}{2}\rho\underline{\kappa} + 3\underline{\kappa}\,{}^{(F)}\rho^2 - 4\underline{\kappa}e_\theta(\Phi)^2
\end{aligned}
$$

and writing $e_3\Phi_0 = \frac{1}{r}\underline{\kappa}\Phi_1 - \frac{1}{2}\underline{\kappa}\Phi_0$, $e_3\Phi_1 = \frac{1}{r}\underline{\kappa}\Phi_2 - \frac{1}{2}\underline{\kappa}\Phi_1$, $e_3\Phi_3 = \frac{1}{r}\underline{\kappa}\Phi_4 - \frac{1}{2}\underline{\kappa}\Phi_3$, $e_3\Phi_4 = \frac{1}{r}\underline{\kappa P}\Phi_4 - \frac{1}{2}\underline{\kappa}\Phi_4$ we have

$$
\begin{aligned}
e_3(\Box_{\mathbf{g}}\Phi_3) &= \frac{1}{r}\left(-3\kappa\underline{\kappa}^2 + 4\rho\underline{\kappa} + 2\underline{\kappa}\,{}^{(F)}\rho^2 + 4\underline{\kappa}e_\theta\Phi^2\right)\Phi_4 \\
&\quad + \frac{1}{r^2}\left(\kappa\underline{\kappa}^2 - 2\rho\underline{\kappa}\right)\underline{P}\Phi_4 \\
&\quad + \left(\frac{3}{2}\kappa\underline{\kappa}^2 + 4\rho\underline{\kappa} + 3\underline{\kappa}\,{}^{(F)}\rho^2 - 6\underline{\kappa}e_\theta(\Phi)^2\right)\Phi_3 \\
&\quad + {}^{(F)}\rho\left(-\frac{2}{r^2}\underline{\kappa}\Phi_2 + \frac{3}{r}\underline{\kappa}\Phi_1 + \frac{3}{2}\underline{\kappa}\Phi_0\right)
\end{aligned}
$$

Applying Lemma 6.1, we have

$$
\begin{aligned}
\Box_{\mathbf{g}}(\Phi_4) &= \frac{1}{r}(-\kappa\underline{\kappa} + 2\rho)\underline{P}(\underline{P}(\Phi_3)) + \left(\frac{1}{2}\kappa\underline{\kappa} - 4\rho - 2\,{}^{(F)}\rho^2\right)\underline{P}(\Phi_3) \\
&\quad + \left(\frac{1}{2}\rho + {}^{(F)}\rho^2\right)r\Phi_3 + \frac{3}{2}r\Box_{\mathbf{g}}(\Phi_3) + \underline{\kappa}^{-1}re_3(\Box_{\mathbf{g}}(\Phi_3)) \\
&= \frac{1}{r}(-\kappa\underline{\kappa} + 2\rho)\underline{P}(\Phi_4) + \left(\frac{1}{2}\kappa\underline{\kappa} - 4\rho - 2\,{}^{(F)}\rho^2\right)\Phi_4 \\
&\quad + \left(\frac{1}{2}\rho + {}^{(F)}\rho^2\right)r\Phi_3 + \frac{3}{2}r\Bigg(\frac{1}{r}\Big(\kappa\underline{\kappa} - 2\rho\Big)\Phi_4 \\
&\quad + \Big(- \kappa\underline{\kappa} - 3\rho + 4e_\theta\Phi^2\Big)\Phi_3 + {}^{(F)}\rho\left(-\frac{2}{r}\Phi_1 - \Phi_0\right)\Bigg) +
\end{aligned}
$$

$$
\begin{aligned}
&+\underline{\kappa}^{-1}r\left(\frac{1}{r}\left(-3\kappa\underline{\kappa}^2+4\rho\underline{\kappa}+2\underline{\kappa}\,{}^{(F)}\rho^2+4\underline{\kappa}e_\theta\Phi^2\right)\Phi_4\right.\\
&+\frac{1}{r^2}\left(\kappa\underline{\kappa}^2-2\rho\underline{\kappa}\right)\underline{P}\Phi_4\\
&+\left(\frac{3}{2}\kappa\underline{\kappa}^2+4\rho\underline{\kappa}+3\underline{\kappa}\,{}^{(F)}\rho^2-6\underline{\kappa}e_\theta(\Phi)^2\right)\Phi_3\\
&\left.+\,{}^{(F)}\rho\left(-\frac{2}{r^2}\underline{\kappa}\Phi_2+\frac{3}{r}\underline{\kappa}\Phi_1+\frac{3}{2}\underline{\kappa}\Phi_0\right)\right)
\end{aligned}
$$

which gives

$$
\square_{\mathbf{g}}(\Phi_4)=(-\kappa\underline{\kappa}-3\rho+4e_\theta\Phi^2)\Phi_4+\,{}^{(F)}\rho\left(-\frac{2}{r}\Phi_2+4r\,{}^{(F)}\rho\Phi_3\right)
$$

as desired. $\square$

Remark 6.10. Notice that the wave equation given by Proposition 6.9, for $\Phi_2=\mathfrak{q}$ and $\Phi_4=\mathfrak{q}^{\mathbf{F}}$ has the form

$$
\square_{\mathbf{g}}\mathfrak{q}^{\mathbf{F}}=V_2\mathfrak{q}^{\mathbf{F}}+e\mathcal{M}(\mathfrak{q})+e^2(l.o.t.(\mathfrak{q}^{\mathbf{F}}))
$$

of the second equation of (0.4).

Using the wave equation for Φ_4, we can simplify the wave equation for Φ_2 in Proposition 6.6, since the derivative $\underline{P}P$ is related to $\square_2:=\square_{\mathbf{g}}-(2)^2e_\theta\Phi^2$ in the following way:

$$
\frac{1}{r^2}Q(\underline{P}(\Phi_4))=-\square_2\Phi_4+\frac{1}{r}(\kappa\underline{\kappa}-2\rho)\underline{P}(\Phi_4)+\not{\Delta}_2\Phi_4+\rho\Phi_4,
$$

where $\not{\Delta}_2$ is the Laplacian on the spheres S of the foliation of the spacetime. Using Proposition 6.9, we can write

$$
\begin{aligned}
Q(\underline{P}(\Phi_4))&=r(\kappa\underline{\kappa}-2\rho)\underline{P}(\Phi_4)+r^2\not{\Delta}_2\Phi_4\\
&\quad+r^2(\kappa\underline{\kappa}+4\rho)\Phi_4+2r\,{}^{(F)}\rho\Phi_2-4r^3\,{}^{(F)}\rho^2\Phi_3
\end{aligned}
$$

giving

$$
\begin{aligned}
\square_{\mathbf{g}}\Phi_2 = & \Big(-\kappa\underline{\kappa} + 10\,{}^{(F)}\rho^2 + 4e_\theta\Phi^2\Big)\Phi_2 \\
& + {}^{(F)}\rho\Big(2r\not{\Delta}_2\Phi_4 - 2Q(\Phi_4) - 2\kappa\underline{\kappa}\underline{P}(\Phi_4) \\
& + r(5\kappa\underline{\kappa} + 8\rho + 4\,{}^{(F)}\rho^2)\Phi_4 + r^2(-6\rho - 20\,{}^{(F)}\rho^2)\Phi_3\Big) \\
& + {}^{(F)}\rho^2\left(-4r\Phi_1 - 2r^2\Phi_0\right)
\end{aligned}
$$

6.4. The system of coupled wave equations

Writing the five equations together, using (3.2) to write ${}^{(F)}\rho = \frac{e}{r^2} + O(\epsilon)$ in the coupling terms, we found the following system of equations modulo $O(\epsilon^2)$:

$$
\begin{aligned}
\left(\square_2 + \frac{1}{2}\kappa\underline{\kappa} + 4\rho - 4\,{}^{(F)}\rho^2\right)\Phi_0 = & \frac{1}{r}\left(2\kappa\underline{\kappa} - 4\rho\right)\Phi_1 \\
& + e\left(\frac{2}{r^3}Q(\Phi_3) - \frac{4}{r^2}\rho\Phi_3\right) \\
\left(\square_2 + \kappa\underline{\kappa} - 6\,{}^{(F)}\rho^2\right)\Phi_1 = & \frac{1}{r}(\kappa\underline{\kappa} - 2\rho)\Phi_2 + r\left(\frac{3}{2}\rho + {}^{(F)}\rho^2\right)\Phi_0 \\
& + e\Big(\frac{2}{r^3}Q(\Phi_4) - \frac{1}{r^2}Q(\Phi_3) - \frac{2}{r^2}\kappa\underline{\kappa}\Phi_4 \\
& + \frac{1}{r}(6\rho + 4\,{}^{(F)}\rho^2)\Phi_3\Big), \\
\left(\square_2 + \kappa\underline{\kappa} - 10\,{}^{(F)}\rho^2\right)\Phi_2 = & e\Big(\frac{2}{r}\not{\Delta}_2\Phi_4 - \frac{2}{r^2}Q(\Phi_4) - \frac{2}{r^2}\kappa\underline{\kappa}\underline{P}(\Phi_4) \\
& + \frac{1}{r}\left(5\kappa\underline{\kappa} + 8\rho + 4\,{}^{(F)}\rho^2\right)\Phi_4 - 6\rho\Phi_3 \\
& + e\left(-\frac{4}{r^3}\Phi_1 - \frac{2}{r^2}\Phi_0\right) + e^2\left(-\frac{20}{r^4}\Phi_3\right)\Big), \\
\left(\square_2 + \kappa\underline{\kappa} + 3\rho\right)\Phi_3 = & \frac{1}{r}\left(\kappa\underline{\kappa} - 2\rho\right)\Phi_4 + e\left(-\frac{2}{r^3}\Phi_1 - \frac{1}{r^2}\Phi_0\right), \\
\left(\square_2 + \kappa\underline{\kappa} + 3\rho\right)\Phi_4 = & e\left(-\frac{2}{r^3}\Phi_2 + \frac{4e}{r^3}\Phi_3\right)
\end{aligned}
\tag{6.4}
$$

where $\square_2 = \square_{\mathbf{g}} - (2)^2 e_\theta\Phi^2$ is the wave operator applied to 2-reduced scalars.

Remark 6.11. A complete analogous system holds for the spin -2 quantities $\underline{\alpha}$. Defining the operator $Pf = r\kappa^{-1}e_4 f + \frac{1}{2}r\underline{f}$ and the $O(\epsilon^2)$ invariant quantity $\underline{\mathfrak{f}} = \not{d}_2^\star {}^{(F)}\underline{\beta} + \underline{\vartheta}\,{}^{(F)}\rho$, then the quantities $\widetilde{\Phi_0} = r^2\kappa^2\underline{\alpha}$, $\widetilde{\Phi_1} = P(\widetilde{\Phi_0})$, $\widetilde{\Phi_2} = P(\widetilde{\Phi_1})$, $\widetilde{\Phi_3} = r^2\kappa\underline{\mathfrak{f}}$, $\widetilde{\Phi_4} = P(\widetilde{\Phi_3})$ verify the same system above, with $\underline{Q}f = r\kappa e_3 f + \frac{1}{2}r\kappa\underline{\kappa}f$.

Selecting the third and fifth equation we have the system of Regge-Wheeler type equations for $\mathfrak{q} = \Phi_2$ and $\mathfrak{q}^{\mathbf{F}} = \Phi_4$ modulo $O(\epsilon^2)$, as announced in (0.4). We summarize it in the following theorem.

Theorem 6.12. *Let* $(\mathbf{M}, \mathbf{g}, \mathbf{Z})$ *be an axially symmetric polarized spacetime solution of the Einstein-Maxwell equation* (0.2)*, which is a* $O(\epsilon)$*-perturbation of Reissner-Nordström spacetime. Then there exist* $O(\epsilon^2)$*-invariant quantities* $\mathfrak{q}$ *and* $\mathfrak{q}^{\mathbf{F}}$ *related to the Weyl curvature and to the Ricci curvature respectively that verify the following coupled system of wave equations, modulo* $O(\epsilon^2)$ *terms,*

$$
\begin{cases}
\Big(\square_2 + \kappa\underline{\kappa} - 10\,{}^{(F)}\rho^2\Big)\mathfrak{q} = e\Big(\frac{2}{r}\not{\Delta}_2\mathfrak{q}^{\mathbf{F}} - \frac{2}{r^2}Q\mathfrak{q}^{\mathbf{F}} - \frac{2}{r^2}\kappa\underline{\kappa}\underline{P}\mathfrak{q}^{\mathbf{F}} \\
\qquad\qquad\qquad\qquad\qquad + \frac{1}{r}\left(5\kappa\underline{\kappa} + 8\rho + 4\,{}^{(F)}\rho^2\right)\mathfrak{q}^{\mathbf{F}}\Big) \\
\Big(\square_2 + \kappa\underline{\kappa} + 3\rho\Big)\mathfrak{q}^{\mathbf{F}} = e\left(-\frac{2}{r^3}\mathfrak{q}\right)
\end{cases}
\tag{6.5}
$$

where $\underline{P}$ *and* Q *are rescaled null derivatives, as defined in* (6.1)*, and additional lower order terms with respect to* $\mathfrak{q}$ *and* $\mathfrak{q}^{\mathbf{F}}$ *appear to both, explicitely,*

$$
\begin{aligned}
(l.o.t.)_1 &= -6\rho\Phi_3 + e\left(-\frac{4}{r^3}\Phi_1 - \frac{2}{r^2}\Phi_0\right) + e^2\left(-\frac{20}{r^4}\Phi_3\right), \\
(l.o.t.)_2 &= \frac{4}{r^3}\Phi_3
\end{aligned}
$$

Remark 6.13. The structure of the coupling in (6.5) does not depend on the polarization of the metric, as observed in [1]. See Appendix.

6.5. Case of perturbation of Schwarzschild spacetime

Coupled gravitational and electromagnetic perturbations of Reissner-Nordström spacetime are clearly a generalization of gravitational perturbations of Schwarzschild spacetime as solution to the vacuum Einstein equation (0.1), as treated in [9]. In this case, the electromagnetic quantities and the quasi-local charge in (6.4) vanish identically, and the system reduces to the first three equations:

$$\left(\square_2 + \frac{1}{2}\kappa\underline{\kappa} + 4\rho\right)\Phi_0 = \frac{1}{r}\left(2\kappa\underline{\kappa} - 4\rho\right)\Phi_1 + O(\epsilon^2),$$
$$\left(\square_2 + \kappa\underline{\kappa}\right)\Phi_1 = \frac{1}{r}(\kappa\underline{\kappa} - 2\rho)\Phi_2 + \frac{3}{2}\rho r\Phi_0 + O(\epsilon^2),$$
$$\left(\square_2 + \kappa\underline{\kappa}\right)\Phi_2 = O(\epsilon^2)$$

which are the linear parts of the equations obtained in Appendix A.3.2 of [9]. In particular, the last equation, for $\mathfrak{q} = \Phi_2$ is

$$\square_2\mathfrak{q} + \kappa\underline{\kappa}\mathfrak{q} = \mathrm{Err}[\square_{\mathbf{g}}\mathfrak{q}]$$

which is the main equation used in [9], to derive decay estimates for $\mathfrak{q}$, and subsequently for α and all other curvature and connection coefficients quantities.

In the case of coupled gravitational and electromagnetic perturbation of Schwarzschild spacetime, namely perturbation of Schwarzschild as solution to the Einstein-Maxwell equation (0.2), the system (6.5) simplifies. Perturbing the background Schwarzschild, being a vacuum spacetime, we have ${}^{(F)}\rho = O(\epsilon)$. Therefore, the main equation for the curvature $\mathfrak{q}$ is unchanged, but the right hand side is given by quadratic terms only, i.e.

$$\square_2\mathfrak{q} + \kappa\underline{\kappa}\mathfrak{q} = O(\epsilon^2)$$

Again since ${}^{(F)}\rho = O(\epsilon)$, using Proposition 4.4, in the case of perturbation of Schwarzschild the extreme components of the electromagnetic tensor ${}^{(F)}\beta$, ${}^{(F)}\underline{\beta}$ are $O(\epsilon^2)$-invariant quantities, and verify the Teukolsky equation

$$\begin{aligned}\square_{\mathbf{g}}\,{}^{(F)}\beta &= -2\underline{\omega}e_4\,{}^{(F)}\beta + (\kappa + 2\omega)\,e_3({}^{(F)}\beta)\\ &\quad + \left(\frac{1}{4}\kappa\underline{\kappa} - 3\kappa\underline{\omega} + \omega\underline{\kappa} + e_\theta\Phi^2 - 2e_4\underline{\omega}\right){}^{(F)}\beta + O(\epsilon^2)\end{aligned}$$

as derived in Proposition 5.2. As in [12], we can define a Chandrasekhar-type transformation at the level of one derivative along the ingoing null direction to obtain a Regge-Wheeler equation. Defining $\mathfrak{l} = r^2(e_3\,{}^{(F)}\alpha_\theta + \frac{1}{2}\underline{\kappa}\,{}^{(F)}\alpha_\theta)$, in the case of ${}^{(F)}\rho = O(\epsilon)$ in a frame for which $\omega = O(\epsilon)$, then $\mathfrak{l}$ verifies the equation

$$\square_1\mathfrak{l} = -\frac{1}{4}\kappa\underline{\kappa}\mathfrak{l} + O(\epsilon^2)$$

The system is therefore given by equations which are decoupled at the linear level

$$\begin{cases} \square_2 \mathfrak{q} + \kappa \underline{\kappa} \mathfrak{q} = O(\epsilon^2), \\ \square_1 \mathfrak{l} + \frac{1}{4} \kappa \underline{\kappa} \mathfrak{l} = O(\epsilon^2) \end{cases}$$

Remark 6.14. It is only in the case of gravitational and electromagnetic perturbations of Reissner-Nordström spacetime that we find a non-trivial coupling for the linear terms of the equations for $\mathfrak{q}$ and $\mathfrak{q}^{\mathbf{F}}$ as described in system (6.5). If the coupled gravitational and electromagnetic perturbations of Kerr-Newman spacetime would have a structure similar to the one here presented is an open question to be addressed.

Appendix A. System of equations without polarization

In this appendix, we will not assume polarization of the metric or axial symmetry. This appendix is based on computations done through computer algebra by Steffen Aksteiner.

Consider a null pair e_3, e_4 on $(\mathbf{M}, \mathbf{g})$ and, at every point $p \in \mathbf{M}$ the horizontal space $S = \{e_3, e_4\}^\perp$. Let γ the metric induced on S. By definition, for all $X, Y \in T_S\mathbf{M}$, i.e. vectors in $\mathbf{M}$ tangent to S, $\gamma(X, Y) = \mathbf{g}(X, Y)$. For any $Y \in T(\mathbf{M})$ we define its horizontal projection by

$$Y^\perp = Y + \frac{1}{2}\mathbf{g}(Y, e_3)e_4 + \frac{1}{2}\mathbf{g}(Y, e_3)e_4 \tag{A.1}$$

Definition A.1. A k-covariant tensor-field U is said to be S-horizontal, $U \in \mathbf{T}^k_S(\mathbf{M})$, if for any $X_1, \dots X_k$ we have,

$$U(Y_1, \dots Y_k) = U(Y_1^\perp, \dots Y_k^\perp)$$

Definition A.2. Given $X \in \mathbf{T}(\mathbf{M})$ and $Y \in \mathbf{T}_S(\mathbf{M})$ we define,

$$\dot{\mathbf{D}}_X Y := (\mathbf{D}_X Y)^\perp$$

Remark A.3. In the particular case when S is integrable and both $X, Y \in \mathbf{T}_S\mathbf{M}$ then $\dot{\mathbf{D}}_X Y$ is the standard induced covariant differentiation on S.

Definition A.4. Given a general, covariant, S- horizontal tensor-field U we define its horizontal covariant derivative according to the formula,

$$\begin{aligned}\dot{\mathbf{D}}_X U(Y_1, \dots Y_k) = X(U(Y_1, \dots Y_k)) - U(\dot{\mathbf{D}}_X Y_1, \dots Y_k) \\ - \dots - U(Y_1, \dots \dot{\mathbf{D}}_X Y_k).\end{aligned}$$

where $X \in \mathbf{TM}$ and $Y_1, \dots Y_k \in \mathbf{T}_S\mathbf{M}$.

Definition A.5. Given Ψ a 2 S-horizontal tensor, we define the wave operator $\dot{\square}_{\mathbf{g}}$ applied to Ψ by

$$\dot{\square}_{\mathbf{g}} \Psi_{AB} := \mathbf{g}^{\mu\nu} \dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu \Psi_{AB}$$

Recall the definition of spacetime Ricci coefficients and spacetime null curvature components in (1.2), (1.3), (1.4). In particular recall

$$^{(1+3)}\alpha_{AB} = W_{A4B4},$$

The tensorial version[7] of the invariant quantity $\mathfrak{f}$ introduced in Section 4.1 is given by

$$^{(1+3)}\mathfrak{f}_{AB} = 2\, \not{\mathcal{D}}_2^\star\, {}^{(F)}\beta_{AB} + 2\, {}^{(F)}\rho \widehat{\underline{\chi}}_{AB}$$

where $\not{\mathcal{D}}_2^\star\, {}^{(F)}\beta_{AB} = -\not{\nabla}_{(A}\, {}^{(F)}\beta_{B)} + \mathbf{g}_{AB} \not{\mathrm{d}}\mathrm{iv}\, {}^{(F)}\beta$, as introduced in [9].

We define the tensorial versions of the operators $\underline{P}$ and Q introduced in Section 6.1 as

$$\underline{P}(\Psi_{AB}) = r\underline{\kappa}^{-1} e_3 \Psi_{AB} + \frac{1}{2} r \Psi_{AB}, \tag{A.2}$$

$$Q(\Psi_{AB}) = r\underline{\kappa} e_4 \Psi_{AB} + \frac{1}{2} r \kappa \underline{\kappa} \Psi_{AB} \tag{A.3}$$

where $\kappa = \mathrm{tr}\chi$ and $\underline{\kappa} = {}^{(1+3)}\mathrm{tr}\underline{\chi}$.

We finally define the main quantities that verify the system of wave equations.

Definition A.6. The tensorial quantities $\mathfrak{q}_{AB}$ and $\mathfrak{q}^{\mathbf{F}}_{AB}$ are defined by

$$^{(1+3)}\mathfrak{q}_{AB} = \underline{P}\left(\underline{P}\left(r^2 \underline{\kappa}^2\, {}^{(1+3)}\alpha_{AB}\right)\right), \qquad ^{(1+3)}\mathfrak{q}^{\mathbf{F}}_{AB} = \underline{P}\left(r^2 \underline{\kappa}\, {}^{(1+3)}\mathfrak{f}_{AB}\right)$$

where $\underline{\kappa} = {}^{(1+3)}\mathrm{tr}\underline{\chi}$.

---

[7]Indeed, $^{(1+3)}\mathfrak{f}_{\theta\theta} = -\, {}^{(1+3)}\mathfrak{f}_{\varphi\varphi} = \mathfrak{f}$, $^{(1+3)}\mathfrak{f}_{\theta\varphi} = 0$ in the case of polarized metric.

We summarize in the following theorem the system of tensorial wave equation that is verified by the two quantities $\mathfrak{q}_{AB}$ and $\mathfrak{q}^{\mathbf{F}}_{AB}$.

Theorem A.7. *Let* $(\mathbf{M}, \mathbf{g})$ *be a spacetime solution of the Einstein-Maxwell equation* (0.2)*, which is a* $O(\epsilon)$*-perturbation of Reissner-Nordström spacetime. Then the tensorial quantities* $\mathfrak{q}_{AB}$ *and* $\mathfrak{q}^{\mathbf{F}}_{AB}$ *verify the following coupled system of wave equations, for* $A, B = 1, 2$*, modulo* $O(\epsilon^2)$*,*

$$
\begin{aligned}
\left(\dot{\Box}_{\mathbf{g}} + \kappa\underline{\kappa} - 10\,{}^{(F)}\rho^2\right)\mathfrak{q}_{AB} &= e\Bigg(\frac{2}{r}\triangle\!\!\!\!/_2\mathfrak{q}^{\mathbf{F}}_{AB} - \frac{2}{r^2}Q(\mathfrak{q}^{\mathbf{F}}_{AB}) - \frac{2}{r^2}\kappa\underline{\kappa}\underline{P}(\mathfrak{q}^{\mathbf{F}}_{AB}) \\
&\quad + \frac{1}{r}\left(5\kappa\underline{\kappa} + 8\rho + 4\,{}^{(F)}\rho^2\right)\mathfrak{q}^{\mathbf{F}}_{AB}\Bigg) + e(l.o.t.)_1, \qquad \text{(A.4)}\\
\left(\dot{\Box}_{\mathbf{g}} + \kappa\underline{\kappa} + 3\,{}^{(1+3)}\rho\right)\mathfrak{q}^{\mathbf{F}}_{AB} &= e\left(-\frac{2}{r^3}\mathfrak{q}_{AB}\right) + e^2(l.o.t.)_2
\end{aligned}
$$

where P *and* $\underline{P}$ *are tensorial null derivatives, as defined in* (A.2)*,* $(\triangle\!\!\!\!/_2\Psi)_{AB} = \gamma^{CD}\nabla\!\!\!\!/_C\nabla\!\!\!\!/_D\Psi_{AB}$ *and* $(l.o.t.)_1$ *and* $(l.o.t.)_2$ *are lower order terms with respect to* $\mathfrak{q}_{AB}$ *and* $\mathfrak{q}^{\mathbf{F}}_{AB}$*.*

The system has the same structure as in the case of polarized metrics. Since no symmetries are assumed in this case, the analysis of the system (A.4) can be used to derive linear stability of Reissner-Nordström spacetime, as a generalization to [7].

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