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Perturbation theory for critical points of causal variational principles

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The perturbation theory for critical points of causal variational principles is developed. We first analyze the class of perturbations obtained by multiplying the universal measure by a weight function and taking the push-forward under a diffeomorphism. Then the constructions are extended to convex combinations of such measures, leading to perturbation expansions for the mean and the fluctuation of the measure, both being coupled in higher order perturbation theory. It is explained how our methods and results apply to the causal action principle for causal fermion systems. It is shown how the perturbation expansion in the continuum limit and the effect of microscopic mixing are recovered in specific limiting cases.

1. Introduction

The theory of causal fermion systems is a recent approach to fundamental physics. Giving quantum mechanics, general relativity and quantum field theory as limiting cases, it is a candidate for a unified physical theory (see [8] or the survey article [14]). So far, the connection to perturbative quantum field theory has been established by first taking the continuum limit [8] and then including the mechanism of microscopic mixing (see [7]). Although this procedure gives the correct limiting case with an interaction described by a unitary time evolution on Fock spaces (see [7, Section 8]), the derivation is not quite convincing conceptually because it is based on the perturbation expansion for solutions of the Dirac equation coupled to classical bosonic fields as obtained in the continuum limit (see [8, §3.8.4] and [7, Section 2]). In order to clarify the mathematical structure of the theory, it is desirable to perform the perturbation expansion directly for the universal measure of the causal fermion system, without referring to specific limiting cases (for more details see Section 2 below). Analyzing this problem also opens up the research program to explore how the perturbation theory for causal fermion systems differs from perturbative quantum field theory, with the goal of making experimental predictions.

In this paper the general perturbation theory for causal fermion systems is developed. We thus succeed in extending the methods of perturbative quantum field theory to non-smooth situations where space-time has a non-trivial, possibly discrete microscopic structure and the physical equations are no longer obtained by quantizing differential equations. We work in the jet formalism introduced in [15] in the more general and at the same time more convenient framework of causal variational principles in the non-compact setting. Our perturbation expansion has the nice feature that the bosonic and fermionic perturbations are described on the same footing in terms of jet spaces containing bosonic and fermionic subspaces.

In the setting of causal variational principles, the basic object is a measure ρ on a manifold $\mathcal{F}$ (for the necessary preliminaries see Section 3). Our methods for perturbing this measure are developed in two steps. In the first step, our method is to multiply ρ by a non-negative function f and to take the push-forward under a mapping F,

$$\tilde{\rho} = F_*\big(f\,\rho\big)\,. \tag{1.1}$$

We then compute f and F order by order in a formal power expansion in a "coupling parameter" λ. In the second step, we consider more generally a

convex combination of a finite number of measures of the form (1.1),

$$\tilde{\rho} = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} (F_\mathfrak{a})_* \big(f_\mathfrak{a}\, \rho\big)\,. \tag{1.2}$$

This ansatz allows for the possibility that the measure is "decomposed" into several components and the support of the measure is "enlarged" (see Figure 3 on page 586). We refer to this effect as a *fragmentation* of the measure. In analogy to the perturbation theory for degenerate eigenvalues of a linear operator, where the perturbation must be "diagonalized on the degenerate eigenspace" before performing the perturbation expansion, the perturbation theory with fragmentation makes it necessary to choose jets which describe how the fragmentation forms (see (5.23) and (5.24) on page 595).

The paper is organized as follows. In Section 2 we give a brief physical motivation and put our perturbation expansion into the context of the ongoing research program on causal fermion systems. Section 3 provides the necessary background on causal variational principles and the jet formalism. In Section 4 the perturbation theory without fragmentation is developed. After bringing the combinatorics into a convenient form (Section 4.1), we invert the linearized equations with Green's operators (see Definition 4.2 in Section 4.2). The resulting perturbation expansion is summarized in Section 4.3. In Section 4.4 it is explained how, starting from a linearized solution, one can construct a one-parameter family of nonlinear solutions of the field equations. In Sections 4.5 it is shown how, perturbing the vacuum by an inhomogeneity, one can construct a corresponding nonlinear solution of the field equations.

In Section 5 the perturbation theory with fragmentation is developed. The method is to decompose suitable jets describing the perturbation into their mean and the fluctuations (see (5.4) in Section 5.1). A technical complication is that, if fragmentation occurs, the unperturbed Laplacian can no longer be inverted. This is illustrated in Section 5.2 in a concrete example. The method to overcome this problem is to invert instead the perturbed Laplacian (see Section 5.3).

In Section 6 we explain how our methods and results apply to the setting of causal fermion systems. After the necessary preliminaries (Section 6.1), the perturbation expansion for the wave evaluation operator is derived (Section 6.2). After identifying jets with perturbations of the wave evaluation operator, the general perturbation expansion applies in a straightforward way (Section 6.3).

In Section 7, it is shown that by a suitable choice of the jet spaces one recovers the analysis in the continuum limit as carried out in [8]. Finally, in Section 8 we describe how to incorporate the effect of microscopic mixing as analyzed in [7].

2. Physical motivation and significance of the perturbation expansion

Before delving into the constructions, we give a physical motivation of the perturbation expansion and explain its significance within the research program on causal fermion systems and causal variational principles.

We begin with a brief introduction and an outline of the present status of the theory. Causal fermion systems are based on a novel *mathematical model of space-time*, where the basic object is a measure on linear operators of a Hilbert space:

Definition 2.1. *(causal fermion system)* Given a separable complex Hilbert space $\mathcal{H}$ with scalar product $\langle .|. \rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (the *"spin dimension"*), we let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all selfadjoint operators on $\mathcal{H}$ of finite rank, which (counting multiplicities) have at most n positive and at most n negative eigenvalues. On $\mathcal{F}$ we are given a positive measure ρ (defined on a σ-algebra of subsets of $\mathcal{F}$), the so-called *universal measure*. We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system*.

This definition gives rise to a space-time together with structures therein, most notably a causal structure, spinorial wave functions and geometric objects like connection and curvature. The resulting abstract setting has been worked out in a satisfying way (see for example [8, Section 1.1]). Moreover, it is clear how the abstract structures are related to the usual objects in Minkowski space or on a Lorentzian manifold (see [8, Section 1.2] or the introduction and survey in [10]). In order to see the correspondence, one must keep in mind that the objects of the causal fermion system involve an ultraviolet regularization on a length scale $\varepsilon > 0$. Thus we always consider the regularized quantities as those having mathematical and physical significance. The corresponding objects in Minkowski space or on a Lorentzian manifold are obtained in a certain limiting case $\varepsilon \searrow 0$ in which the ultraviolet regularization is removed.

In the theory of causal fermion systems, the physical equations are formulated via a variational principle, the so-called *causal action principle*. It is defined as follows (for more details see [F1, §1.1.1] or [15]). Given $x, y \in \mathcal{F}$,

we denote the non-trivial eigenvalues of the operator product xy counting algebraic multiplicities by $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy} \in \mathbb{C}$. We introduce the *spectral weight* $|\,.\,|$ of an operator as the sum of the absolute values of its eigenvalues. In particular, the spectral weights of the operator products xy and $(xy)^2$ are defined by

$$|xy| = \sum_{i=1}^{2n} \left|\lambda_i^{xy}\right| \qquad \text{and} \qquad \left|(xy)^2\right| = \sum_{i=1}^{2n} \left|\lambda_i^{xy}\right|^2 . \tag{2.1}$$

We introduce the Lagrangian and the causal action by

$$\textit{Lagrangian:} \quad \mathcal{L}(x,y) = \left|(xy)^2\right| - \frac{1}{2n}\,|xy|^2 \tag{2.2}$$

$$\textit{causal action:} \quad \mathcal{S}(\rho) = \iint_{\mathcal{F}\times\mathcal{F}} \mathcal{L}(x,y)\, d\rho(x)\, d\rho(y)\,. \tag{2.3}$$

The *causal action principle* is to minimize $\mathcal{S}$ by varying the universal measure under the following constraints,

$$\begin{aligned} \textit{volume constraint:} \quad & \rho(\mathcal{F}) = \text{const} \\ \textit{trace constraint:} \quad & \int_{\mathcal{F}} \mathrm{tr}(x)\, d\rho(x) = \text{const} \\ \textit{boundedness constraint:} \quad & \mathcal{T}(\rho) := \iint_{\mathcal{F}\times\mathcal{F}} |xy|^2\, d\rho(x)\, d\rho(y) \leq C\,, \end{aligned}$$

where C is a given parameter (and tr denotes the trace of a linear operator on $\mathcal{H}$). The form of the above Lagrangian is the result of long considerations and many computations (for a systematic account see [4, Chapter 5]). The constraints are needed in order to obtain a mathematically well-defined variational principle with non-trivial minimizers. A simple way of understanding the structure of the Lagrangian is the following connection to *causality*: Writing the Lagrangian as (see [8, eq. (1.1.9)])

$$\mathcal{L} = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(\left|\lambda_i^{xy}\right| - \left|\lambda_j^{xy}\right| \right)^2 ,$$

one sees that $\mathcal{L}$ vanishes if the eigenvalues λ_i^{xy} all have the same absolute value. Defining spacelike separation by this property, pairs of points with spacelike separation do not enter the action. This can be seen in analogy to the usual notion of causality where points with spacelike separation cannot influence each other. It turns out that in suitable limiting cases, the

above definition of causality indeed agrees with the usual causal structure of Minkowski space or of a Lorentzian manifold (for details see [8, §1.2.5] or [12, Sections 4 and 5]).

Causal variational principles are a mathematical generalization of the causal action principle, with the aim of restricting attention to the essential analytic structures. The existence theory and the general structure of the corresponding Euler-Lagrange equations have been worked out in [1, 5]. The connection to physics is made in [8, Chapter 3-5], where it is shown that in a well-defined limiting case, the so-called *continuum limit*, the interaction given by the causal action principle can be described effectively by the Dirac equation coupled to classical field equations for gauge fields and the gravitational field. In this limiting case, one obtains all the interactions of the standard model plus classical gravity.

The next challenge is to understand how *quantum field theory* arises from the causal action principle. Indeed, in the paper [7] the connection between the causal action principle and a second-quantized dynamics on Fock spaces has been made in a certain limiting case. But some of the assumptions and constructions remain to be justified and understood better. Moreover, a number of important open questions still need to be addressed:

(a) Since an ultraviolet regularization on the scale ε is built in, the theory of causal fermion systems is ultraviolet finite. Nevertheless, it is an important task to understand the asymptotics of interacting systems for small ε. In particular, is it possible to take limit $\varepsilon \searrow 0$ with renormalization techniques? Is the effective theory obtained in this limit *renormalizable*?

(b) The fact that the continuum limit also gives the Einstein equations raises the question to which extent and how precisely the constructions in [7] relate to *quantum gravity*. Do causal fermion systems really give a mathematically well-defined setting for describing quantized gravitational fields? Can the resulting "geometry of quantum gravity" be described by the geometric structures of the causal fermion system?

Answering these questions in the affirmative would show that causal fermion systems are a mathematically consistent, non-perturbative quantum field theory. The ultimate goal is to understand the quantum field theory limit of causal fermion systems in a way where it becomes possible to go beyond quantum field theory in the following sense:

(c) How does the dynamics described by the causal action principle differ from quantum field theory? How can the deviations be quantified? Can they be detected in experiments?

The present paper is an important step towards answering these questions, as we now explain. The procedure in [7] is closely tied to the continuum limit analysis and to the Dirac equation coupled to classical field equations obtained in this limit. The method to go beyond the continuum limit is referred to as *microscopic mixing*. In the present formulation with measures, microscopic mixing can be understood by taking universal measures $\rho_1, \ldots, \rho_L$, each describing a system of Dirac particles in Minkowski space with an interaction via classical bosonic potentials $A_1, \ldots, A_L$. Then the convex combination of the measures

$$\tilde{\rho} = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \rho_{\mathfrak{a}} \tag{2.4}$$

is again a measure. It contains information on the bosonic potentials $A_1, \ldots, A_L$ of the "subsystems" described by $\rho_1, \ldots, \rho_L$. As observed in [7] (based on preliminary considerations in [6]), the resulting collection of bosonic potentials can be described effectively by a *second-quantized* bosonic field. Moreover, taking into account an interaction of the subsystems described by $\rho_1, \ldots, \rho_L$ by combining the perturbation expansion for classical fields with some features of microscopic mixing, one obtains an effective interaction described by a second-quantized Hamiltonian acting on fermionic and bosonic Fock spaces (see [7, Section 8]).

Although these constructions are an important first step, there is the major shortcoming that the connection to Fock spaces is based on the perturbation expansion for classical fields in each subsystem. The fact that the description with classical fields is valid only approximately makes it difficult to justify the validity and to quantify the error of the Fock space dynamics. Moreover, the description makes it necessary to assume that there are subsystems, but it remains unclear how the subsystems form dynamically. In [7], this open problem was bypassed by considering the so-called limiting case of an *instantaneous recombination* of subsystems. But in order to tackle the above open questions, the validity of this limiting case must be justified, and the errors of the approximation must be quantified.

The main point of the constructions of the present paper is to overcome the shortcomings of the treatment in [7]. The perturbation expansion developed here only uses the Euler-Lagrange equations of the causal action, but

it does not rely on the classical field equations obtained in the continuum limit. The possibility for the formation of subsystems (2.4) is now taken into account by the fragmentation of the measure (1.2). In contrast to the ad-hoc ansatz (2.4), the perturbation theory with fragmentation makes it possible to analyze in detail whether and how fragmentation forms. Moreover, the mutual interaction of the resulting subsystems can be quantified. Intuitively speaking, the fragmentation of the measure means that space-time does not stay classical, but becomes a "quantum space-time" which can be thought of as a "superposition" or "mixture" of the space-times described by the individual subsystems. In view of the general scope and applicability of our constructions, the methods and results of this paper are a promising starting point for addressing the above questions (a)–(c) in a precise mathematical setting.

3. Preliminaries

3.1. Causal variational principles in the non-compact setting

We consider causal variational principles in the non-compact setting as introduced in [15, Section 2] (the connection to causal fermion systems will be made in Section 6 below). Thus let $\mathcal{F}$ be a (possibly non-compact) smooth manifold of dimension $m \geq 1$. Moreover, we are given a non-negative function $\mathcal{L} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}_0^+$ (the *Lagrangian*) with the following properties:

(i) $\mathcal{L}$ is lower semi-continuous, i.e. for all sequences $x_n \to x$ and $y_{n'} \to y$,

$$\mathcal{L}(x,y) \leq \liminf_{n,n' \to \infty} \mathcal{L}(x_n, y_{n'})\,.$$

(ii) $\mathcal{L}$ is symmetric: $\mathcal{L}(x,y) = \mathcal{L}(y,x)$ for all $x, y \in \mathcal{F}$.

Next, we let ρ be a (positive) Borel measure on $\mathcal{F}$ (the *universal measure*). The *causal variational principle* is to minimize the action

$$\mathcal{S} = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y)\, \mathcal{L}(x,y)$$

under variations of the measure ρ, keeping the total volume $\rho(\mathcal{F})$ fixed. If the total volume is infinite, one can make mathematical sense of variations of $\mathcal{S}$ by considering variations of ρ of finite total variation and zero volume (for details see [15, Section 2]). Here we do not enter the details of the minimization procedure and of the properties of the minimizing measure.

Instead, we restrict attention to the resulting Euler-Lagrange (EL) equations as derived in [15, Lemma 2.3]:

Definition 3.1. *A Borel measure ρ on $\mathcal{F}$ is a* **minimizer** *of the causal variational principle if it has the following properties:*

- *The measure ρ is locally finite.*
- *The function $\mathcal{L}(x,.)$ is ρ-integrable for all $x \in \mathcal{F}$.*
- *For a suitable value of the parameter $\mathfrak{s} > 0$, the function ℓ defined by*

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{s} \tag{3.1}$$

is minimal and vanishes on the support of ρ,

$$\ell|_{\text{supp}\,\rho} \equiv \inf_{\mathcal{F}} \ell = 0\,. \tag{3.2}$$

We remark that the value of the parameter $\mathfrak{s}$ can be changed arbitrarily by rescaling the measure according to

$$\rho \to \nu\rho \qquad \text{with} \quad \nu > 0\,. \tag{3.3}$$

With this in mind, we shall always keep $\mathfrak{s}$ fixed when varying or perturbing the measure.

3.2. The weak Euler-Lagrange equations

Let ρ be a critical point of the causal variational principle. We introduce *space-time* M as the support of this measure,

$$M := \text{supp}\,\rho \subset \mathcal{F}\,.$$

The idea behind the formulation of the weak EL equations is to use only part of the information contained in the EL equations (3.2). Namely, we evaluate them only on M, taking into account first derivatives. Moreover, we restrict attention to directions where ℓ is differentiable. And finally, we want to have the freedom to restrict attention to the part of information needed for the application. This leads to the following construction: By $C^\infty(M,\mathbb{R})$ we denote all real-valued functions on M which have a smooth extension to $\mathcal{F}$.

Likewise, by

$$\Gamma = C^\infty(M, T\mathcal{F})$$

we denote the smooth vector fields on $\mathcal{F}$ along M (thus every $u \in \Gamma$ is a mapping from M to $T\mathcal{F}$ with $u(x) \in T_x\mathcal{F}$ for all $x \in M$, which can be extended to a smooth vector field on $\mathcal{F}$). We define the jet space on M as the vector space

$$\mathfrak{J} := \big\{ \mathfrak{u} = (a, u) \text{ with } a \in C^\infty(M, \mathbb{R}) \text{ and } u \in \Gamma \big\} .$$

Moreover, we let Γ^{diff} be those vector fields for which the directional derivative of the function ℓ exists,

$$\Gamma^{\text{diff}} = \big\{ u \in C^\infty(M, T\mathcal{F}) \;\big|\; D_u \ell(x) \text{ exists for all } x \in M \big\} .$$

Next, we introduce the subspace of jets

$$\mathfrak{J}^{\text{diff}} := C^\infty(M, \mathbb{R}) \oplus \Gamma^{\text{diff}} \;\subset\; \mathfrak{J} .$$

For a jet $\mathfrak{u} = (a, u) \in \mathfrak{J}^{\text{diff}}$ we define $\nabla_\mathfrak{u}$ as the linear combination of scalar multiplication and directional derivative, i.e.

$$\nabla_\mathfrak{u} \ell(x) := a(x)\, \ell(x) + \big(D_u \ell\big)(x) .$$

Finally, we choose a linear subspace $\mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}}$ with the property that its scalar and vector components are both vector spaces,

$$\mathfrak{J}^{\text{test}} = C^{\text{test}}(M, \mathbb{R}) \oplus \Gamma^{\text{test}} \;\subset\; \mathfrak{J}^{\text{diff}} ,$$

and the scalar component is nowhere trivial in the sense that

$$\text{for all } x \in M \text{ there is } a \in C^{\text{test}}(M, \mathbb{R}) \text{ with } a(x) \neq 0 .$$

Then the *weak EL equations* read (for details cf. [15, (eq. (4.10))])

$$\nabla_\mathfrak{u} \ell|_M = 0 \qquad \text{for all } \mathfrak{u} \in \mathfrak{J}^{\text{test}} . \tag{3.4}$$

The purpose of introducing $\mathfrak{J}^{\text{test}}$ is that it gives the freedom to restrict attention to the portion of information in the EL equations which is relevant for the application in mind. For example, if one is interested only in the macroscopic dynamics, one can choose $\mathfrak{J}^{\text{test}}$ to be composed of jets which disregard the microscopic fluctuations of ℓ.

We finally point out that the weak EL equations (3.4) do not hold only for minimizers, but also for critical points of the causal action. With this in mind, all methods and results of this paper do not apply only to minimizers, but more generally to critical points of the causal variational principle. For brevity, we also refer to a measure with satisfies the weak EL equations (3.4) as a *critical measure*.

3.3. Jet spaces and the linearized field equations

For the detailed study of the weak EL equations it is most convenient work with Taylor expansions of the component functions in a given chart. Therefore, for any $x \in M$ we choose a chart of $\mathcal{F}$ around x and work in components x^α. For ease in notation, we usually omit the index α as well as all vector and tensor indices. But one should keep in mind that from now on, we always work in suitably chosen charts.

We now introduce useful jet spaces. We begin with the space of *dual jets* $(\mathfrak{J}^{\text{test}})^*$. To this end, we denote the continuous global one-jets taking values in the cotangent bundle restricted to M by

$$\mathfrak{J}^* := C^0(M, \mathbb{R}) \oplus C^0(M, T^*\mathcal{F})\,.$$

We let $(\mathfrak{J}^{\text{test}})^*$ be the quotient space

$$(\mathfrak{J}^{\text{test}})^* := \mathfrak{J}^* \Big/ \Big\{ (g,\varphi) \in \mathfrak{J}^* \,\Big|\, g(x)\, a(x) + \langle \varphi(x), u(x) \rangle = 0 \\ \text{for all } \mathfrak{u} = (a,u) \in \mathfrak{J}^{\text{test}} \text{ and all } x \in M \Big\}\,,$$

where $\langle .,. \rangle$ denotes the dual pairing of $T_x^*\mathcal{F}$ and $T_x\mathcal{F}$. Here we take equivalence classes simply because it is convenient to disregard dual jets which are trivial on $\mathfrak{J}^{\text{test}}$.

We next introduce the spaces $\mathfrak{J}^\ell$, where the superscript $\ell \in \mathbb{N}_0 \cup \{\infty\}$ can be thought of as the order of differentiability if the derivatives act simultaneously on both arguments of the Lagrangian:

Definition 3.2. *For any $\ell \in \mathbb{N}_0 \cup \{\infty\}$, the jet space $\mathfrak{J}^\ell \subset \mathfrak{J}$ is defined as the vector space of test jets with the following properties:*

(i) *For all $y \in M$ and all x in an open neighborhood of M, the directional derivatives*

$$\big(\nabla_{1,\mathfrak{v}_1} + \nabla_{2,\mathfrak{v}_1}\big) \cdots \big(\nabla_{1,\mathfrak{v}_p} + \nabla_{2,\mathfrak{v}_p}\big) \mathcal{L}(x,y) \tag{3.5}$$

(computed componentwise in charts around x and y) exist for all $p \in \{1, \ldots, \ell\}$ and all $\mathfrak{v}_1, \ldots, \mathfrak{v}_p \in \mathfrak{J}^\ell$.

(ii) *The functions in* (3.5) *are ρ-integrable in the variable y, giving rise to locally bounded functions in x. More precisely, these functions are in the space*

$$L^\infty_{\text{loc}}\Big(L^1\big(M, d\rho(y)\big), d\rho(x)\Big)\,.$$

(iii) *Integrating the expression* (3.5) *in y over M with respect to the measure ρ, the resulting function (defined for all x in an open neighborhood of M) is continuously differentiable in the direction of every jet $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$.*

Here and throughout this paper, we use the following conventions for partial derivatives and jet derivatives:

- Partial and jet derivatives with an index $i \in \{1, 2\}$, as for example in (3.5), only act on the respective variable of the function $\mathcal{L}$. This implies, for example, that the derivatives commute,

$$\nabla_{1,\mathfrak{v}} \nabla_{1,\mathfrak{u}} \mathcal{L}(x, y) = \nabla_{1,\mathfrak{u}} \nabla_{1,\mathfrak{v}} \mathcal{L}(x, y)\,.$$

- The partial or jet derivatives which do not carry an index act as partial derivatives on the corresponding argument of the Lagrangian. This implies, for example, that

$$\nabla_{\mathfrak{u}} \int_{\mathcal{F}} \nabla_{1,\mathfrak{v}}\, \mathcal{L}(x, y)\, d\rho(y) = \int_{\mathcal{F}} \nabla_{1,\mathfrak{u}} \nabla_{1,\mathfrak{v}}\, \mathcal{L}(x, y)\, d\rho(y)\,.$$

We point out that, in contrast to the method and conventions used in [15], *jets are never differentiated*.

The combination of derivatives in (3.5) requires a brief explanation. In the case $p = 1$, the combination of directional derivatives in (3.5) is defined by

$$\big(D_{1,v} + D_{2,v}\big)\mathcal{L}(x, y) := \frac{d}{d\tau} \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\big|_{\tau=0}\,,$$

where F_τ is the flow of the vector field v. The higher derivatives are defined inductively. However, we use the convention that the partial derivatives act only on the arguments of $\mathcal{L}$, but not on any other jets. This means that one

must subtract the terms involving derivatives of the jets. For example,

$$\big(D_{1,v} + D_{2,v}\big)^2 \mathcal{L}(x,y) := \frac{d^2}{d\tau^2} \mathcal{L}\big(F_\tau(x), F_\tau(y)\big)\big|_{\tau=0} \\ - \big(D_{1,D_v v} + D_{2,D_v v}\big) \mathcal{L}(x,y)\,,$$

and similarly for higher derivatives. The condition in Definition 3.2 (i) implies that all the resulting terms must exist.

Linearized solutions are linear perturbations of ρ which preserve the weak EL equations (3.4). We now give the precise definition (for more details see [15, Section 4.2]).

Definition 3.3. *A jet $\mathfrak{v} \in \mathfrak{J}^1$ is referred to as a* **solution of the linearized field equations** *if*

$$\nabla_{\mathfrak{u}} \int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathcal{L}(x,y)\, d\rho(y) = \nabla_{\mathfrak{u}} \nabla_{\mathfrak{v}}\, \mathfrak{s} \quad \text{for all } \mathfrak{u} \in \mathfrak{J}^{\text{test}} \text{ and } x \in M\,.$$

The vector space of all linearized solutions is denoted by $\mathfrak{J}^{\text{lin}} \subset \mathfrak{J}^1$.

4. The abstract perturbation expansion

4.1. Perturbation expansion for the universal measure

Let ρ be a measure (not necessarily a critical point of the causal variational principle). We want to construct a measure $\tilde{\rho}$ which satisfies the weak EL equations. To this end, we make the ansatz

$$\tilde{\rho} = F_*\big(f\, \rho\big)\,, \tag{4.1}$$

where f and F are smooth,

$$f \in C^\infty\big(M, \mathbb{R}^+\big) \qquad \text{and} \qquad F \in C^\infty\big(M, \mathcal{F}\big) \tag{4.2}$$

(where smooth on M again means that there exists a smooth extension to $\mathcal{F}$). This ansatz is motivated mainly by its simplicity. More general perturbations of the universal measure will be studied in Section 5.

We denote the test space for the measure $\tilde{\rho}$ by $\tilde{\mathfrak{J}}^{\text{test}}$, i.e.

$$\tilde{\mathfrak{J}}^{\text{test}} \subset \big\{ \mathfrak{u} = (a, u) \text{ with } a \in C^\infty(\tilde{M}, \mathbb{R}) \text{ and } u \in C^\infty(\tilde{M}, T\mathcal{F}) \big\},$$

where $\tilde{M} := \operatorname{supp} \tilde{\rho}$ is the support of the varied measure. We write the weak EL equations (3.4) for the measure $\tilde{\rho}$ as

$$\nabla_{1,\tilde{\mathfrak{u}}(F(x))} \Big(\int_M \mathcal{L}\big(F(x), F(y)\big)\, f(y)\, d\rho(y) - \mathfrak{s} \Big) = 0 \quad \text{for all } \tilde{\mathfrak{u}} \in \tilde{\mathfrak{J}}^{\text{test}}, \tag{4.3}$$

to be evaluated pointwise for all $x \in M$. Here the notation $\nabla_{1,\tilde{\mathfrak{u}}}$ clarifies that the derivative acts on the first argument of the Lagrangian. On the constant $\mathfrak{s}$ it acts by multiplication with the scalar component,

$$\nabla_{1,\tilde{\mathfrak{u}}(F(x))}\, \mathfrak{s} = \nabla_{\tilde{\mathfrak{u}}(F(x))}\, \mathfrak{s} = a\big(F(x)\big)\, \mathfrak{s},$$

where we again denote the components by $\mathfrak{u} = (a, u)$. Note that, being defined on $\tilde{M}$, the jet $\tilde{\mathfrak{u}}$ can be evaluated at $x \in M$ only after composing it with F. In order to rewrite this equation in a way where x and y are treated in a more symmetric way, we multiply (4.3) by the function $f(x)$ and write this function inside the brackets,

$$\nabla_{1,\tilde{\mathfrak{u}}(F(x))} \Big(\int_M f(x)\, \mathcal{L}\big(F(x), F(y)\big)\, f(y)\, d\rho(y) - \mathfrak{s}\, f(x) \Big) = 0$$
$$\text{for all } \tilde{\mathfrak{u}} \in \tilde{\mathfrak{J}}^{\text{test}}.$$

Working in charts makes it possible to identify the tangent spaces at different points simply by identifying the components. In particular, we use this method in order to identify $\tilde{\mathfrak{u}}(F(x))$ with a jet $\mathfrak{u}(x)$. We choose the jet space $\tilde{\mathfrak{J}}^{\text{test}}$ such that, under this identification, it coincides with $\mathfrak{J}^{\text{test}}$. Then the weak EL equations can be written as

$$\nabla_{1,\mathfrak{u}} \Big(\int_M f(x)\, \mathcal{L}\big(F(x), F(y)\big)\, f(y)\, d\rho(y) - \mathfrak{s}\, f(x) \Big) = 0, \tag{4.4}$$

to be satisfied for all $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$ and all $x \in M$. We again point out that the derivative $\nabla_{1,\mathfrak{u}}$ acts on the first argument of the Lagrangian and on the constant $\mathfrak{s}$, but the factor $f(x)$ is not differentiated.

In physical applications, it is relatively easy to construct an approximate solution of the EL equations (typically by regularizing Dirac sea structures in the presence of a classical bosonic potential; for details see [8]). With this

in mind, we now assume that the measure ρ is close to a critical point in the sense that

$$\nabla_{\mathrm{u}} \Big(\int_M \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{s} \Big) = \lambda\, \nabla_{\mathrm{u}} E^{(1)} \tag{4.5}$$

with an error term $E^{(1)}$, where $\lambda \in \mathbb{R}$ is a small parameter. We expand both f and F in a power series in λ. For the function f, we make the perturbation ansatz

$$f(x) = \sum_{p=0}^{\infty} \lambda^p\, f^{(p)}(x) \qquad \text{with} \quad f^{(0)}(x) = 1\,, \tag{4.6}$$

where the choice of $f^{(0)}$ will ensure that the measure $\tilde{\rho}$ goes over to the unperturbed measure ρ in the limit $\lambda \to 0$. For the expansion of F, we choose a chart around x and write $F(x)$ in components as $(F(x)^\alpha)_{\alpha=1,\ldots,m}$. Then we can expand F componentwise,

$$F(x)^\alpha = \sum_{p=0}^{\infty} \lambda^p\, F^{(p)}(x)^\alpha \qquad \text{with} \quad F^{(0)}(x)^\alpha = x^\alpha\,. \tag{4.7}$$

For ease in notation, we shall omit the index α from now on. But one should keep in mind that the expansion of $F(x)$ always involves the choice of a chart around x.

In the next lemma we evaluate (4.4) to any order $p = 1, 2, \ldots$ in λ. In order to simplify the combinatorics, it turns out to be convenient to work instead of the function f with its logarithm

$$c(x) := \log f(x)\,, \tag{4.8}$$

which, similar to (4.6), we again expand in powers of λ,

$$c(x) = \sum_{p=0}^{\infty} \lambda^p\, c^{(p)}(x) \qquad \text{with} \quad c^{(0)}(x) = 0\,. \tag{4.9}$$

Moreover, we combine the $c^{(p)}$ and $F^{(p)}$ to jets $\mathfrak{w}^{(p)}$, i.e.

$$\mathfrak{w}^{(p)} := \big(c^{(p)}, F^{(p)}\big) \qquad \text{for } p = 1, 2, \ldots. \tag{4.10}$$

Lemma 4.1. *To every order $p = 1, 2, \ldots$, the weak EL equations* (4.4) *can be written as*

$$0 = \nabla_{\mathrm{u}} \sum_{\ell=1}^{p} \frac{1}{\ell!} \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} \times \left\{ \int_M \big(\nabla_{1,\mathfrak{w}^{(q_1)}} + \nabla_{2,\mathfrak{w}^{(q_1)}}\big) \cdots \big(\nabla_{1,\mathfrak{w}^{(q_\ell)}} + \nabla_{2,\mathfrak{w}^{(q_\ell)}}\big) \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{s}\, c^{(q_1)}(x) \cdots c^{(q_\ell)}(x) \right\}. \tag{4.11}$$

Proof. The combinatorics can be handled elegantly by working with exponentials. We explain the method in the example of a function $h(F(x))$. We first expand in a Taylor series,

$$h\big(F(x)\big) = h\big(x + (F(x)-x)\big) = \sum_{p=0}^{\infty} \frac{1}{p!}\, D^p_{F(x)-x} h(x) = \exp\Big(D^p_{F(x)-x}\Big) h(x)\,.$$

The exponential on the right simply is a short notation for the formal power series. Multiplying by $f(x)$ and using (4.8), we can combine the exponentials to obtain a jet derivative,

$$f(x)\, h\big(F(x)\big) = e^{c(x)}\, \exp\Big(D^p_{F(x)-x}\Big) h(x) = \exp\Big(\nabla^p_{\tilde{\mathfrak{w}}}\Big) h(x)\,,$$

where the jet $\tilde{\mathfrak{w}}$ has the components

$$\tilde{\mathfrak{w}}(x) = \big(c(x), F(x) - x\big)\,.$$

Applying the same method to the integrand in (4.4) gives

$$f(x)\, \mathcal{L}\big(F(x), F(y)\big)\, f(y) = \exp\Big(\nabla_{1,\tilde{\mathfrak{w}}} + \nabla_{2,\tilde{\mathfrak{w}}}\Big) \mathcal{L}(x,y)\,, \tag{4.12}$$

where we used that the derivatives all act on the arguments of $\mathcal{L}(x,y)$, making it possible to simplify the prefactors with the usual computation rules of the exponential. Using the abbreviation

$$\overline{\nabla}_{\mathfrak{w}} := \big(\nabla_{1,\mathfrak{w}} + \nabla_{2,\mathfrak{w}}\big)\,,$$

the identity (4.12) can be written in the compact form

$$f(x)\, \mathcal{L}\big(F(x), F(y)\big)\, f(y) = e^{\overline{\nabla}_{\tilde{\mathfrak{w}}}}\, \mathcal{L}(x,y)\,. \tag{4.13}$$

It remains to expand the exponential in (4.13) in powers of λ. Inserting the perturbation expansion of $\tilde{\mathfrak{w}}$, we obtain

$$e^{\overline{\nabla}_{\tilde{\mathfrak{w}}}} = \exp\left(\lambda\, \overline{\nabla}_{\mathfrak{w}^{(1)}} + \lambda^2\, \overline{\nabla}_{\mathfrak{w}^{(2)}} + \cdots\right).$$

Let us compute the p^{th} λ-derivative of this exponential at $\lambda = 0$. We consider the contribution involving the factors $\overline{\nabla}_{\mathfrak{w}^{q_1}}, \ldots, \overline{\nabla}_{\mathfrak{w}^{q_\ell}}$. Since each factor $\overline{\nabla}_{\mathfrak{w}^q}$ comes with a factor λ^q, we clearly get a contribution only if $q_1 + \cdots + q_\ell = p$. We thus obtain

$$\frac{d^p}{d\lambda^p} e^{\overline{\nabla}_{\tilde{\mathfrak{w}}}}\Big|_{\lambda=0} = \sum_{\ell=1}^{p} \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} c_{q_1,\ldots,c_\ell}\; \overline{\nabla}_{\mathfrak{w}^{(q_1)}} \cdots \overline{\nabla}_{\mathfrak{w}^{(q_\ell)}},$$

where $c_{q_1,\ldots,c_\ell}$ are combinatorial factors which can be determined as follows. Clearly, each λ-derivative annihilates one of the factors λ of the monomial $\lambda^{q_1} \cdots \lambda^{q_\ell}$. We must count the number of possibilities with which this can occur. We first distinguish those λ-derivatives which act on the exponential according to

$$\frac{d}{d\lambda}\, e^{\lambda^q\, \overline{\nabla}_{\mathfrak{w}^{(q)}}} = q\, \lambda^{q-1}\, \overline{\nabla}_{\mathfrak{w}^{(q)}}\, e^{\lambda^q\, \overline{\nabla}_{\mathfrak{w}^{(q)}}}.$$

Note that each such derivative generates a factor $\overline{\nabla}_{\mathfrak{w}^{(q)}}$. We use the convention that, carrying out the λ-derivatives consecutively, the first λ-derivative acting on the exponential generates the factor $\overline{\nabla}_{\mathfrak{w}^{(q_1)}}$, the second such derivative generates the factor $\overline{\nabla}_{\mathfrak{w}^{(q_2)}}$, and so on. Dropping this convention gives a factor $1/\ell!$, i.e.

$$\frac{d^p}{d\lambda^p} e^{\overline{\nabla}_{\tilde{\mathfrak{w}}}}\Big|_{\lambda=0} = \sum_{\ell=1}^{p} \frac{1}{\ell!} \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} \tilde{c}_{q_1,\ldots,c_\ell}\; \overline{\nabla}_{\mathfrak{w}^{(q_1)}} \cdots \overline{\nabla}_{\mathfrak{w}^{(q_\ell)}}$$

with new combinatorial factors $\tilde{c}_{q_1,\ldots,c_\ell}$ which are obtained simply by counting the number of possibilities of forming groups of λ-derivatives acting

on λ^{q_1}, λ^{q_2}, and so on. These combinatorial factors are given by the multinomial theorem. We thus obtain

$$\begin{aligned}\frac{d^p}{d\lambda^p} e^{\overline{\nabla}_{\tilde{\mathfrak{w}}}}\Big|_{\lambda=0} &= \sum_{\ell=1}^p \frac{1}{\ell!} \sum_{\substack{q_1,\ldots,q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} \binom{p}{q_1 \cdots q_\ell} \\ &\qquad \times \left(\frac{d^{q_1}}{d\lambda^{q_1}} e^{\lambda^{q_1} \overline{\nabla}_{\mathfrak{w}^{(q_1)}}}\right) \cdots \left(\frac{d^{q_\ell}}{d\lambda^{q_\ell}} e^{\lambda^{q_\ell} \overline{\nabla}_{\mathfrak{w}^{(q_\ell)}}}\right)\Big|_{\lambda=0} \\ &= p! \sum_{\ell=1}^p \frac{1}{\ell!} \sum_{\substack{q_1,\ldots,q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} \overline{\nabla}_{\mathfrak{w}^{(q_1)}} \cdots \overline{\nabla}_{\mathfrak{w}^{(q_\ell)}}\,.\end{aligned}$$

Using this formula in (4.13) gives

$$\begin{aligned}&\frac{1}{p!} \frac{d^p}{d\lambda^p} \Big(f(x)\, \mathcal{L}\big(F(x), F(y)\big)\, f(y) \Big) \\ &= \sum_{\ell=0}^p \frac{1}{\ell!} \sum_{\substack{q_1,\ldots,q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} \overline{\nabla}_{\mathfrak{w}^{(q_1)}} \cdots \overline{\nabla}_{\mathfrak{w}^{(q_\ell)}}\, \mathcal{L}(x,y)\,.\end{aligned}$$

Similarly, one derives the identity

$$\frac{1}{p!} \frac{d^p}{d\lambda^p}\, e^{c(x)} = \sum_{\ell=0}^p \frac{1}{\ell!} \sum_{\substack{q_1,\ldots,q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} c^{(q_1)}(x) \cdots c^{(q_\ell)}(x)\,.$$

Employing these formulas in (4.4) gives the result. □

4.2. Green's operators

In Lemma 4.1 we rewrote the weak EL equations as the system of equations (4.11), to be satisfied for every $p = 1, 2, \ldots$. In order to solve this system of equations, we bring the contribution involving $\mathfrak{w}^{(p)}$ to the left. We thus obtain the equation

$$\nabla_{\mathfrak{u}} \bigg(\int_M \big(\nabla_{1,\mathfrak{w}^{(p)}} + \nabla_{2,\mathfrak{w}^{(p)}}\big) \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{s}\, c^{(p)}(x) \bigg) = -\nabla_{\mathfrak{u}} E^{(p)}(x), \tag{4.14}$$

where $E^{(1)}$ is given by (4.5), whereas for $p > 1$ we have

(4.15)
$$E^{(p)} = \sum_{\ell=2}^{p} \frac{1}{\ell!} \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1+\cdots+q_\ell=p}} \Big\{ -\mathfrak{s}\, c^{(q_1)}(x) \cdots c^{(q_\ell)}(x) \\ + \int_M \big(\nabla_{1,\mathfrak{w}^{(q_1)}} + \nabla_{2,\mathfrak{w}^{(q_1)}}\big) \cdots \big(\nabla_{1,\mathfrak{w}^{(q_\ell)}} + \nabla_{2,\mathfrak{w}^{(q_\ell)}}\big) \mathcal{L}(x,y)\, d\rho(y) \Big\} .$$

Before solving for $\mathfrak{w}^{(p)}$, we need to specify the jet space used for varying the measure: We denote the continuous global one-jets of the cotangent bundle restricted to M by

$$\mathfrak{J}^* := C^0(M, \mathbb{R}) \oplus C^0(M, T^*\mathcal{F}) .$$

We let $(\mathfrak{J}^{\text{test}})^*$ be the quotient space

$$(\mathfrak{J}^{\text{test}})^* := \mathfrak{J}^* \Big/ \big\{ (g,\varphi) \in \mathfrak{J}^* \,\big|\, g(x)\, a(x) + \langle \varphi(x), u(x) \rangle = 0 \\ \text{for all } \mathfrak{u} = (a,u) \in \mathfrak{J}^{\text{test}} \text{ and } x \in M \big\} ,$$

where $\langle ., . \rangle$ denotes the dual pairing of $T_x^*\mathcal{F}$ and $T_x\mathcal{F}$ (the reason for taking equivalence classes simply is that it is convenient to disregard dual jets which are trivial on $\mathfrak{J}^{\text{test}}$). We thus obtain a mapping

(4.16)
$$\Delta_\ell : \underbrace{\mathfrak{J}^\infty \times \cdots \times \mathfrak{J}^\infty}_{\ell \text{ factors}} \to (\mathfrak{J}^{\text{test}})^* , \\ \big\langle \mathfrak{u}, \Delta_\ell \big[\mathfrak{v}_1, \ldots, \mathfrak{v}_\ell\big] \big\rangle(x) \\ = \frac{1}{\ell!}\, \nabla_{\mathfrak{u}} \Big(\int_M \big(\nabla_{1,\mathfrak{v}_1} + \nabla_{2,\mathfrak{v}_1}\big) \cdots \big(\nabla_{1,\mathfrak{v}_\ell} + \nabla_{2,\mathfrak{v}_\ell}\big) \mathcal{L}(x,y)\, d\rho(y) \\ - \mathfrak{s}\, b_1(x) \cdots b_\ell(x) \Big) ,$$

valid for any $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$. We remark for clarity that the mapping Δ_ℓ is symmetric in its ℓ arguments. Choosing $\ell = 1$, we obtain the mapping $\Delta \equiv \Delta_1 : \mathfrak{J}^\infty \to (\mathfrak{J}^{\text{test}})^*$ given by

$$\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) = \nabla_{\mathfrak{u}} \Big(\int_M \big(\nabla_{1,\mathfrak{v}} + \nabla_{2,\mathfrak{v}}\big) \mathcal{L}\big(x,y\big)\, d\rho(y) - \nabla_{\mathfrak{v}}\, \mathfrak{s} \Big) .$$

Definition 4.2. *A linear mapping* $S : (\mathfrak{J}^{\text{test}})^* \to \mathfrak{J}^\infty$ *is referred to as a* **Green's operator** *if*

$$\Delta S \mathfrak{v} = -\mathfrak{v} \qquad \text{for all } \mathfrak{v} \in (\mathfrak{J}^{\text{test}})^* . \tag{4.17}$$

Clearly, a Green's operator exists if and only if the mapping Δ is surjective. In analogy to the situation for hyperbolic PDEs, the Green's operators need not be unique. Indeed, just as in classical field theory, the difference of two Green's operators is a solution of the linearized field equations (see Definition 3.3). We remark that, similar as in classical field theory and quantum field theory, one could work with specific Green's operators determined by support properties (like retarded or advanced Green's operators) or by microlocal properties (like the Feynman propagator). However, at this stage, where we merely seek for solutions of the weak EL equations without specifying initial conditions, we cannot and need not specify the Green's operators.

With the above notions, we can write (4.14) as

$$\Delta \mathfrak{w}^{(p)} = -E^{(p)} \in (\mathfrak{J}^{\text{test}})^* . \tag{4.18}$$

Having a Green's operator to our disposal, we can solve this equation for $\mathfrak{w}^{(p)}$,

$$\mathfrak{w}^{(p)} = S\, E^{(p)} . \tag{4.19}$$

Combining this equation with (4.5) and (4.15), we have obtained an iterative procedure for constructing measures which satisfy the weak EL equations (4.4). We again point out that the Green's operator S is not unique. Indeed, there is the freedom to choose a different Green's operator to every order in perturbation theory. Exactly as in the analogous situation for hyperbolic PDEs, taking this freedom into account gives rise to the *general* solution to the weak EL equations. In order to make this non-uniqueness manifest, we prefer to write (4.19) as

$$\mathfrak{w}^{(p)} = S^{(p)}\, E^{(p)} , \tag{4.20}$$

where $S^{(1)}, S^{(2)}, \ldots$ are arbitrary Green's operators.

4.3. Diagrams and Feynman rules

We now summarize the above construction and formulate it in a diagrammatic language. For simplicity, we leave out the parameter λ, which was

used merely as a book-keeping device in order to keep track of the different orders in perturbation theory. We introduce the operators Δ_ℓ by (see (4.16))

$$\Delta_0(x) = \int_M \mathcal{L}(x,y)\, d\rho(y) - \mathfrak{s} \tag{4.21}$$

$$\Delta_\ell\big[\mathfrak{w}_1, \ldots, \mathfrak{w}_\ell\big](x) = \frac{1}{\ell!}\bigg(\int_M \big(\nabla_{1,\mathfrak{w}_1} + \nabla_{2,\mathfrak{w}_1}\big) \cdots \big(\nabla_{1,\mathfrak{w}_\ell} + \nabla_{2,\mathfrak{w}_\ell}\big) \mathcal{L}(x,y)\, d\rho(y)$$

$$- \mathfrak{s}\, c_1(x) \cdots c_\ell(x) \bigg) \qquad (\text{for } \ell \geq 1) \tag{4.22}$$

and choose Green's operators $S^{(p)}$ with $p = 1, 2, \ldots$ as minus the inverse of $\Delta \equiv \Delta_1$ (see Definition 4.2),

$$\Delta\, S^{(p)}\, \mathfrak{v} = -\mathfrak{v} \qquad \text{for all } v \in (\mathfrak{J}^{\text{test}})^* . \tag{4.23}$$

Then the jets $\mathfrak{w}^{(p)}$ are defined iteratively by (see (4.20))

$$\mathfrak{w}^{(p)} = S^{(p)}\, E^{(p)} , \tag{4.24}$$

where $E^{(p)}$ depends on the previous jets $\mathfrak{w}^{(1)}, \ldots, \mathfrak{w}^{(p-1)}$ by (see (4.5) and (4.15))

$$E^{(1)}(x) = \Delta_0(x) \tag{4.25}$$

$$E^{(p)}(x) = \sum_{\ell=2}^{p} E_\ell^{(p)}(x) \qquad (\text{for } p \geq 2) \tag{4.26}$$

$$E_\ell^{(p)}(x) = \sum_{\substack{q_1, \ldots, q_\ell \geq 1 \\ \text{with } q_1 + \cdots + q_\ell = p}} \Delta_\ell\big[\mathfrak{w}^{(q_1)}, \ldots, \mathfrak{w}^{(q_\ell)}\big](x) . \tag{4.27}$$

The universal measure $\tilde{\rho}$ is obtained by (see (4.1), (4.6), (4.7), (4.8) and (4.10))

$$\tilde{\rho} = F_*\big(e^c\, \rho\big) \qquad \text{where} \quad (c, F)(x) = (0, x) + \sum_{p=1}^{\infty} \mathfrak{w}^{(p)}(x) . \tag{4.28}$$

For the graphical representation, we denote the Green's operator by a wiggled line and the operators Δ_ℓ by semicircles (see Figure 1). Then the contributions to the perturbation expansion can be depicted by Feynman diagrams as illustrated in Figure 2. The combinatorics is given in (4.26), (4.27) and (4.28). We point out that our perturbation expansion only involves tree diagrams.

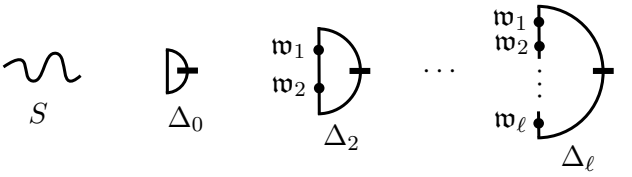

Figure 1: Building blocks of Feynman diagrams.

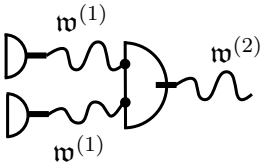

Figure 2: A simple Feynman diagram.

Remark 4.3. (alternative form of the perturbation expansion) For completeness, we now give an alternative form of the perturbation expansion will be used in Section 5 and might be useful for future applications. Namely, dividing by $f(x)$, the weak EL equations (4.4) can be written alternatively as

$$\nabla_{1,\mathfrak{u}} \left(\int_M \mathcal{L}\big(F(x), F(y)\big)\, f(y)\, d\rho(y) - \mathfrak{s} \right) = 0\,, \tag{4.29}$$

to be satisfied for all $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$. Expanding the equations in this form, one obtains the same perturbation expansion as above, except that the operator Δ_ℓ in (4.22) is to be modified to

$$\begin{aligned} &\breve{\Delta}_\ell\big[\mathfrak{w}_1, \ldots, \mathfrak{w}_\ell\big](x) \\ &= \frac{1}{\ell!} \int_M \big(D_{1,w_1} + \nabla_{2,\mathfrak{w}_1}\big) \cdots \big(D_{1,w_\ell} + \nabla_{2,\mathfrak{w}_\ell}\big) \mathcal{L}(x,y)\, d\rho(y)\,. \end{aligned} \tag{4.30}$$

This formulation has the advantage that the Lagrange multiplier $\mathfrak{s}$ drops out. Moreover, it becomes clearer that the scalar component of the jets only enters at the point y (as is obvious in (4.29) where only $f(y)$ appears). The disadvantage is that (4.30) is less symmetric in the variables x and y (in particular, the form (4.22) is of advantage for the derivation of conservation laws for surface layer integrals in [15, 16]).

4.4. Constructing nonlinear solutions of the field equations

We now explain how the general construction of Section 4.1 can be adapted in order to construct nonlinear solutions of the field equations. We consider the setting that ρ is a minimizing measure, and we again assume that we are given a Green's operator S (see Definition 4.2). Moreover, we are given a jet $\mathfrak{w}^{(1)} \in \mathfrak{J}^\infty$ being a solution of the linearized field equations (see Definition 3.3). Our goal is to construct a family of solutions $(\tilde{\rho}_\tau)_{\tau \in \mathbb{R}}$ of the weak EL equations of the form (4.1) whose first variation coincides with $\mathfrak{w}^{(1)}$, i.e.

$$\tilde{\rho}_\tau|_{\tau=0} = \rho \qquad \text{and} \qquad \big(\partial_\tau f, \partial_\tau F\big)\big|_{\tau=0} = \mathfrak{w}^{(1)} .$$

To this end, we construct the jets $\mathfrak{w}^{(2)}, \mathfrak{w}^{(3)}, \ldots$ iteratively again by (4.24), (4.26) and (4.27) with Δ_ℓ according to (4.22) (note that now $\Delta_0 = \Delta_1[\mathfrak{w}^{(1)}] = 0$). The desired family of measures $(\tilde{\rho}_\tau)$ is then defined similar to (4.28) by inserting powers of τ, i.e.

$$\tilde{\rho}_\tau = (F_\tau)_*\big(e^{c_\tau}\, \rho\big) \qquad \text{with} \quad (c_\tau, F_\tau)(x) = (0, x) + \sum_{p=1}^{\infty} \tau^p\, \mathfrak{w}^{(p)}(x) .$$

4.5. Perturbing a vacuum measure

In the applications, one often knows a critical measure which typically describes the vacuum of the system. Then the system is modified, for example by introducing particles and/or fields. The task is to construct a solution of the weak EL equations, starting from the modified system. We now adapt the construction of Section 4.1 to this setting. To this end, we assume that ρ is a measure which satisfies the EL equations (3.2). Moreover, we assume that we are given a Green's operator S (see Definition 4.2). The modified system is described by a measure $\hat{\rho}$ which, in analogy to (4.1), we assume to be of the form

$$\hat{\rho} = H_*\big(h\, \rho\big) , \tag{4.31}$$

where h and H are smooth,

$$h \in C^\infty\big(M, \mathbb{R}^+\big) \qquad \text{and} \qquad H \in C^\infty\big(M, \mathcal{F}\big) .$$

Clearly, the measure $\hat{\rho}$ is no longer a solution of the weak EL equations. Similar to (4.8), (4.10) and (4.28), we expand h and H and rewrite the

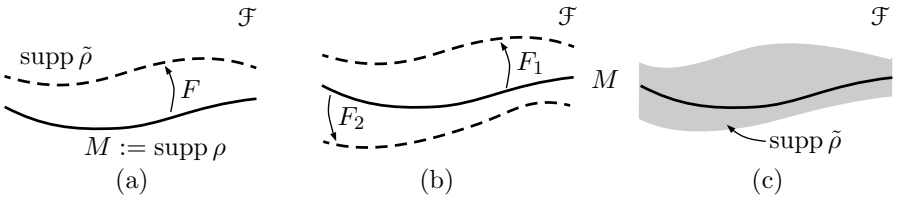

Figure 3: Fragmentation of the measure ρ.

coefficients with jets,

$$\big(\log h, H\big)(x) = (0, x) + \sum_{p=1}^{\infty} \mathfrak{v}^{(p)}(x) \qquad \text{with} \quad \mathfrak{v}^{(p)} \in \mathfrak{J}^{\infty}\,. \tag{4.32}$$

Here we need to assume that the resulting jets $\mathfrak{v}^{(p)}$ are in $\mathfrak{J}^{\infty}$.

In order to construct a corresponding solution of the EL equations, we again make the ansatz (4.1) and describe f and F by jets $\mathfrak{w}^{(p)}$ (see (4.7), (4.8), (4.9) and (4.10)). Now we perform the perturbation expansion similar to (4.26)–(4.28), taking into account the inhomogeneity $\mathfrak{v}^{(p)}$ to every order in perturbation theory. More precisely, (4.24) is to be replaced by

$$\mathfrak{w}^{(p)} = \mathfrak{v}^{(p)} + S^{(p)} \Big(E^{(p)} + \Delta \mathfrak{v}^{(p)} \Big)\,.$$

Indeed, applying the operator Δ and using the defining equation of the Green's operator (4.17), one sees that the relations (4.18) again hold.

5. Perturbation theory with fragmentation

The perturbation expansion of the previous section was based on the ansatz that the perturbed measure $\tilde{\rho}$ should be of the form (4.1) with f and F according to (4.2). Intuitively speaking, this ansatz means that the support of the measure is changed smoothly as a whole (see Figure 3 (a)), but it is impossible to model a situation where the measure ρ "disintegrates" into several "components" which are perturbed differently (see Figure 3 (b)). We now extend the constructions Section 4 such as to allow for such a so-called *fragmentation* of the universal measure.

We consider the following setting. Similar as in Section 4.4 we want to construct nonlinear solutions of the field equations. Therefore, we assume that ρ is a measure which satisfies the weak EL equations (3.4). We choose

a parameter $L \in \mathbb{N}$ and consider mappings

$$f_{\mathfrak{a}} \in C^\infty\big(M, \mathbb{R}^+\big)\,, \quad F_{\mathfrak{a}} \in C^\infty\big(M, \mathcal{F}\big) \qquad \text{with } \mathfrak{a} = 1, \ldots, L\,.$$

For the *universal measure with fragmentation* we make the ansatz

$$\tilde{\rho} = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} (F_{\mathfrak{a}})_* \big(f_{\mathfrak{a}}\, \rho\big)\,. \tag{5.1}$$

We refer to L as the *number of subsystems* and to $\mathfrak{a}$ as the *subsystem index*. Clearly, for one subsystem, (5.1) reduces to our earlier ansatz (4.1). The larger L is chosen, the more freedom we have in perturbing the measure ρ. We point out that we may choose L arbitrarily large. In the limit $L \to \infty$, one can even describe situations where the support of the measure ρ is "enlarged" by the perturbation as shown in Figure 3 (c). We also note that a universal measure of the form (5.1) is closely related to the mechanism of microscopic mixing as introduced in [7]; this will be explained further in Section 8.

5.1. Linearized field equations for fluctuations

It is a bit easier to perform the perturbation expansion with fragmentation in the alternative formulation introduced in Remark 4.3, because then the scalar component of the jets appears only as a function of the variable y (but of course, all our results can be rewritten in a straightforward way in the formulation (4.4)). Adapted to the measure (5.1), the weak EL equations (4.29) read

$$\nabla_{1,\mathfrak{u}_{\mathfrak{a}}} \left(\frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \int_M \mathcal{L}\big(F_{\mathfrak{a}}(x), F_{\mathfrak{b}}(y)\big)\, f_{\mathfrak{b}}(y)\, d\rho(y) - \mathfrak{s} \right) = 0\,,$$

to be satisfied for all jets $\mathfrak{u} \in (\mathfrak{J}^{\text{test}})^L$ as well as for all $x \in M$ and $\mathfrak{a} \in \{1, \ldots, L\}$. Since in finite dimension, pointwise evaluation is the same as weak evaluation, we can write this equation equivalently as

$$\frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \nabla_{1,\mathfrak{u}_{\mathfrak{a}}} \left(\frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \int_M \mathcal{L}\big(F_{\mathfrak{a}}(x), F_{\mathfrak{b}}(y)\big)\, f_{\mathfrak{b}}(y)\, d\rho(y) - \mathfrak{s} \right) = 0\,, \tag{5.2}$$

which must hold for all $\mathfrak{u} \in (\mathfrak{J}^{\text{test}})^L$ and all $x \in M$.

In preparation of the perturbation expansion, we derive the corresponding linearized field equations. To this end, we again expand f and F according to (4.6) and (4.7). To first order, the EL equations (5.2) become

$$\begin{aligned}(5.3)\quad \big\langle \mathfrak{u}, \Delta\big[\mathfrak{w}^{(1)}\big]\big\rangle(x) &:= \frac{1}{L^2} \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \\ &\times \nabla_{\mathfrak{u}_\mathfrak{a}(x)} \int_M \Big(\big(D_{1,w_\mathfrak{a}^{(1)}} + D_{2,w_\mathfrak{b}^{(1)}}\big)\mathcal{L}(x,y) + \mathcal{L}(x,y)\, f_\mathfrak{b}^{(1)}(y) \Big)\, d\rho(y)\end{aligned}$$

with $\mathfrak{w}_\mathfrak{a}^{(1)} := (f_\mathfrak{a}^{(1)}, F_\mathfrak{a}^{(1)})$. Note that the vector component of the jet $\mathfrak{w}_\mathfrak{a}^{(1)}$ shifts the support of the universal measure in each subsystem independently (as shown in Figure 3 (b)).

At this point, it is helpful to decompose the jets into components independent of the subsystem index and components whose mean vanishes, i.e.

$$(5.4)\qquad \mathfrak{u} = \bar{\mathfrak{u}} + \mathfrak{u}_\mathrm{F} \qquad \text{with} \quad \bar{\mathfrak{u}}_\mathfrak{a}(x) := \frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \mathfrak{u}_\mathfrak{b}(x)\,.$$

Here the subscript "F" can be thought of as referring to the "fragmentation" of the universal measure or as describing the "fluctuations" of the jets in the subsystems. For a convenient notation, we usually omit the subsystem index of $\bar{\mathfrak{u}}$. The above splitting gives rise to a direct sum decomposition of the jet spaces, which we write as

$$\mathfrak{J}^L = \bar{\mathfrak{J}} \oplus \mathfrak{J}_\mathrm{F}$$

and similarly for the jet spaces $\mathfrak{J}^\mathrm{test}$ and $\mathfrak{J}^\infty$.

Using these notions, we can carry out the $\mathfrak{b}$-sum in (5.3) to obtain

$$\begin{aligned}\big\langle \mathfrak{u}, \Delta\big[\mathfrak{w}^{(1)}\big]\big\rangle(x) = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \nabla_{\mathfrak{u}_\mathfrak{a}(x)} \int_M \Big(&\big(D_{1,w_\mathfrak{a}^{(1)}} + D_{2,\bar{w}^{(1)}}\big)\mathcal{L}(x,y) \\ &+ \mathcal{L}(x,y)\, \bar{f}^{(1)}(y) \Big)\, d\rho(y)\,.\end{aligned}$$

The fluctuations drop out completely when testing in $\bar{\mathfrak{J}}^\mathrm{test}$,

$$\begin{aligned}\big\langle \bar{\mathfrak{u}}, \Delta\big[\mathfrak{w}^{(1)}\big]\big\rangle(x) = \nabla_{\bar{\mathfrak{u}}(x)} \int_M \Big(&\big(D_{1,\bar{w}^{(1)}} + D_{2,\bar{w}^{(1)}}\big)\mathcal{L}(x,y) \\ &+ \mathcal{L}(x,y)\, \bar{f}^{(1)}(y) \Big)\, d\rho(y)\,,\end{aligned}$$

giving back the linearized field equations without fragmentation. But clearly, the fluctuations are visible when testing in $\mathfrak{J}_{\mathrm{F}}^{\mathrm{test}}$ because

$$\big\langle \mathfrak{u}_{\mathrm{F}}, \Delta\big[\mathfrak{w}^{(1)}\big]\big\rangle(x) = \frac{1}{L}\sum_{\mathfrak{a}=1}^{L} \nabla_{\mathfrak{u}_{\mathrm{F},\mathfrak{a}}(x)} \int_M D_{1,w^{(1)}_{\mathrm{F},\mathfrak{a}}} \mathcal{L}(x,y)\, d\rho(y)\,.$$

Using that the first derivative of ℓ vanishes in view of the EL equations, we can write this equation in the more compact form

$$\big\langle \mathfrak{u}_{\mathrm{F}}, \Delta\big[\mathfrak{w}^{(1)}\big]\big\rangle(x) = \frac{1}{L}\sum_{\mathfrak{a}=1}^{L} D_{u_{\mathrm{F},\mathfrak{a}}(x)} D_{w^{(1)}_{\mathrm{F},\mathfrak{a}}} \ell(x) \qquad \text{for all } x \in M\,.$$

These findings lead to the following definition:

Definition 5.1. *A jet $\mathfrak{v} \in (\mathfrak{J}^1)^L$ is referred to as a* **solution of the linearized field equations with fragmentation** *if its mean $\bar{\mathfrak{v}}$ and fluctuation $\mathfrak{v}_{\mathrm{F}}$ satisfy for all $\mathfrak{u} \in (\mathfrak{J}^{\mathrm{test}})^L$ and all $x \in M$ the equations*

$$\nabla_{\bar{\mathfrak{u}}(x)} \int_M \Big(\big(D_{1,\bar{v}} + \nabla_{2,\bar{\mathfrak{v}}}\big)\mathcal{L}(x,y)\Big)\, d\rho(y) = 0 \tag{5.5}$$

$$\frac{1}{L}\sum_{\mathfrak{a}=1}^{L} D_{u_{\mathrm{F},\mathfrak{a}}(x)} D_{v_{\mathrm{F},\mathfrak{a}}} \ell(x) = 0\,. \tag{5.6}$$

The vector space of all linearized solutions is denoted by

$$\mathfrak{J}^{\mathrm{lin}} = \bar{\mathfrak{J}}^{\mathrm{lin}} \oplus \mathfrak{J}_{\mathrm{F}}^{\mathrm{lin}} \;\subset\; (\mathfrak{J}^1)^L\,.$$

We point out that the linearized field equations with fragmentation do not involve all the components of the jets, neither of the test jet $\mathfrak{u}$ nor of the linearized field $\mathfrak{v}$. Indeed, only the vector component of the fluctuations comes into play, but their scalar component does not enter. Moreover, if u_{F} is chosen as a linearized solution, then (5.6) is satisfied, no matter how v_{F} is chosen. In other words, testing in the direction of fluctuating linearized solutions, the equation (5.6) does not give any information. Hence in the linearized field equations with fragmentation (5.5) and (5.6), the jets $\mathfrak{u}$ can be changed freely in $\mathfrak{J}_{\mathrm{F}}^{\mathrm{lin}}$.

In order to implement these findings in a compact notation, it is useful to decompose the fluctuating jets as

$$\mathfrak{J}^1_{\mathrm{F}} = \mathfrak{J}^{\mathrm{c}}_{\mathrm{F}} \oplus \mathfrak{J}^{\mathrm{lin}}_{\mathrm{F}}\,,$$

where $\mathfrak{J}^{\mathrm{c}}_{\mathrm{F}}$ is a (non-orthogonal) complement of $\mathfrak{J}^{\mathrm{lin}}_{\mathrm{F}}$ in $\mathfrak{J}^1_{\mathrm{F}}$. We thus obtain the decomposition of the jet spaces

$$(\mathfrak{J}^1)^L = \bar{\mathfrak{J}} \oplus \mathfrak{J}^{\mathrm{c}}_{\mathrm{F}} \oplus \mathfrak{J}^{\mathrm{lin}}_{\mathrm{F}}\,.$$

Using a block matrix notation, the unperturbed operator Δ takes the form

$$\langle \mathfrak{u}, \Delta \mathfrak{v} \rangle(x) = \left\langle \begin{pmatrix} \bar{\mathfrak{u}} \\ \mathfrak{u}^{\mathrm{c}}_{\mathrm{F}} \\ \mathfrak{u}^{\mathrm{lin}}_{\mathrm{F}} \end{pmatrix}, \begin{pmatrix} \bar{\Delta} & 0 & 0 \\ 0 & \Delta_{\mathrm{F}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathfrak{v}} \\ \mathfrak{v}^{\mathrm{c}}_{\mathrm{F}} \\ \mathfrak{v}^{\mathrm{lin}}_{\mathrm{F}} \end{pmatrix} \right\rangle, \tag{5.7}$$

where the operators $\bar{\Delta}$ and Δ_{F} are defined by

$$\bar{\Delta} : \bar{\mathfrak{J}}^1 \to (\bar{\mathfrak{J}}^{\mathrm{test}})^*$$

$$\langle \bar{\mathfrak{u}}, \bar{\Delta} \bar{\mathfrak{v}} \rangle(x) = \nabla_{\bar{\mathfrak{u}}} \int_M \Big(\big(D_{1,\bar{v}} + D_{2,\bar{v}}\big) \mathcal{L}(x,y) + \mathcal{L}(x,y)\, \bar{b}(y) \Big)\, d\rho(y) \tag{5.8}$$

$$\Delta_{\mathrm{F}} : \mathfrak{J}^{\mathrm{c}}_{\mathrm{F}} \to (\mathfrak{J}^{\mathrm{c}}_{\mathrm{F}} \cap \mathfrak{J}^{\mathrm{test}}_{\mathrm{F}})^*$$

$$\langle \mathfrak{u}_{\mathrm{F}}, \Delta_{\mathrm{F}} v_{\mathrm{F}} \rangle(x) = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} D_{u_{\mathrm{F},\mathfrak{a}}(x)} D_{v_{\mathrm{F},\mathfrak{a}}} \ell(x)\,. \tag{5.9}$$

We finally remark that, disregarding differentiability issues, the jet space $\mathfrak{J}^{\mathrm{lin}}_{\mathrm{F}}$ can also be understood from the perspective of stability. If ρ is a minimizer, then then the Hessian of ℓ is non-negative and thus gives rise to the positive semi-definite bilinear form (for details see [11, Section 4])

$$\frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \int_M \nabla^2 \ell|_x(.,.)\, d\rho \;:\; (\mathfrak{J}^{\mathrm{test}})^L \times (\mathfrak{J}^1)^L \to \mathbb{R}\,.$$

The space $\mathfrak{J}^{\mathrm{lin}}_{\mathrm{F}}$ is obtained by all fluctuating jets which are in the neutral subspace of this positive semi-definite bilinear form. This means that fragmentation can occur only in directions in which the Hessian of the causal action vanishes.

5.2. An explicit example

Similar to the procedure in Section 4.2, we may assume that the operators $\bar{\Delta}$ and Δ_{F} in (5.7) can be inverted by corresponding Green's operators:

Definition 5.2. *A linear mapping* $\bar{S} : (\bar{\mathfrak{J}}^{\text{test}})^* \to \bar{\mathfrak{J}}^\infty$ *is referred to as an* **Green's operator for the mean** *if*

$$\bar{\Delta}\, \bar{S}\, \bar{\mathfrak{v}} = -\bar{\mathfrak{v}} \quad \text{for all } \bar{\mathfrak{v}} \in (\bar{\mathfrak{J}}^{\text{test}})^* .$$

A linear mapping $S_{\text{F}} : (\mathfrak{J}_{\text{F}}^{\text{c}} \cap \mathfrak{J}_{\text{F}}^{\text{test}})^* \to \mathfrak{J}_{\text{F}}^{\text{c}} \cap \mathfrak{J}_{\text{F}}^\infty$ *is referred to as a* **Green's operator for fluctuations** *if*

$$\Delta_{\text{F}}\, S_{\text{F}}\, v_{\text{F}} = -v_{\text{F}} \quad \text{for all } v_{\text{F}} \in (\mathfrak{J}_{\text{F}}^{\text{c}} \cap \mathfrak{J}_{\text{F}}^{\text{test}})^* .$$

Before we can perform the perturbation expansion, we must analyze how to invert the field equations on the subspace $\mathfrak{J}_{\text{F}}^{\text{lin}}$. As one sees in (5.7), the linearized operator Δ vanishes on this subspace. This means that the operator on this subspace is determined by the perturbation itself. This situation resembles the perturbation theory with degeneracies for the eigenvalues of a linear operator. In this case, the procedure is to diagonalize the perturbation on the degenerate subspaces (without using perturbation theory) before performing the perturbation expansion. In order to explain how to proceed in our setting, we begin with a simple concrete example.

Example 5.3. Let $\mathcal{F} = \mathbb{R}^2$ and

$$\mathcal{L}\big((x_1, x_2), (y_1, y_2)\big) = (x_1 - y_1)^4 + (x_2 - y_2)^2 - (x_2 + y_2)^2\, (x_1 - y_1)^2$$

(for the moment, we disregard that this Lagrangian is unbounded from below; this shortcoming will be removed after (5.22) below). Moreover, we let ρ be the Dirac measure supported at the origin. The jet spaces are

$$\mathfrak{J} = \mathfrak{J}^{\text{test}} = \mathbb{R} \times \mathbb{R}^2 \ \ni\ \mathfrak{u} = (a, u^1, u^2) .$$

Obviously, all first and second partial derivatives of the Lagrangian vanish at the origin. Therefore, ρ is a critical measure, and the EL equations (5.2) are satisfied for the unperturbed system with the Lagrange multiplier $\mathfrak{s}$ chosen to be zero.

We now consider a fragmentation with two subsystems $L = 2$, i.e.

$$\mathfrak{J}_\mathrm{F} = \big\{(\mathfrak{u}_\mathfrak{a})_{\mathfrak{a}=1,2}\,,\quad \mathfrak{u}_1 = -\mathfrak{u}_2 = \big(a, u^1, u^2\big) \in \mathbb{R} \times \mathbb{R}^2\big\}\,. \tag{5.10}$$

In order to determine $\mathfrak{J}_\mathrm{F}^\mathrm{lin}$, we first compute the Hessian of ℓ,

$$\ell(x_1, x_2) = \mathcal{L}\big((x_1, x_2), (0,0)\big) = x_1^4 + x_2^2 - x_2^2\, x_1^2\,, \qquad D^2\ell|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Therefore

$$\mathfrak{J}_\mathrm{F}^\mathrm{lin} = \Big\{(\mathfrak{u}_\mathfrak{a})_{\mathfrak{a}=1,2}\,,\quad \mathfrak{u}_1 = -\mathfrak{u}_2 = \big(a, u^1, 0\big) \in \mathbb{R} \times \mathbb{R}\Big\}\,, \tag{5.11}$$

showing that fragmentation can occur only in the x_1-direction.

We now prescribe the leading orders of the transformation of the universal measure (5.1) and verify if this gives a suitable starting point for a perturbative treatment. In order to preserve the total volume, we choose $f_\mathfrak{a} = f_\mathfrak{a}^{(0)}$ with

$$0 < f_1^{(0)} < 2 \qquad \text{and} \qquad f_2^{(0)} = 2 - f_1^{(0)}\,. \tag{5.12}$$

The transformation $F_\mathfrak{a}$, on the other hand, is chosen as

$$F_\mathfrak{a}(0) = \lambda\, w_\mathfrak{a}^{(1)} \tag{5.13}$$

with the vector component

$$w_1^{(1)} = (w, 1) \qquad \text{and} \qquad w_1^{(2)} = (-w, 1)$$

and $w \in \mathbb{R}$.

Let us verify if this family of measures satisfies the weak EL equations, and if not, what the resulting error is. The support of the perturbed measures consists of the two points

$$p_1 := \lambda\,(w, 1) \qquad \text{and} \qquad p_2 := \lambda\,(-w, 1)\,. \tag{5.14}$$

Moreover, a direct computation gives

$$\begin{aligned}
\ell(p_1) &= -8\,\lambda^4 \left(2 - f_1^{(0)}\right) w^2 \left(w^2-1\right) \\
\ell(p_2) &= -8\,\lambda^4\, f_1^{(0)}\, w^2 \left(w^2-1\right) \\
D\ell|_{p_1} &= -8\,\lambda^3 \left(2 - f_1^{(0)}\right) \Big(-w \left(2w^2-1\right),\, w^2 \Big) \\
D\ell|_{p_2} &= -8\,\lambda^3\, f_1^{(0)} \Big(w \left(2w^2-1\right), w^2 \Big) .
\end{aligned}$$

Hence, testing with the average and the fluctuation gives

$$\frac{1}{2} \sum_{\mathfrak{a}=1}^{2} \nabla_{\bar{u}_\mathfrak{a}} \ell(p_\mathfrak{a}) = -8\,\lambda^3 \begin{pmatrix} \bar{a} \\ \bar{u}^1 \\ \bar{u}^2 \end{pmatrix} \cdot \begin{pmatrix} \lambda\, w^2 \left(w^2-1\right) \\ \left(f_1^{(0)} - 1\right) w \left(2w^2-1\right) \\ w^2 \end{pmatrix} \tag{5.15}$$

$$\frac{1}{2} \sum_{\mathfrak{a}=1}^{2} \nabla_{(u_\mathrm{F})_\mathfrak{a}} \ell(p_\mathfrak{a}) = -8\,\lambda^3 \begin{pmatrix} a \\ u^1 \\ u^2 \end{pmatrix} \cdot \begin{pmatrix} \lambda \left(f_1^{(0)} - 1\right) w^2 \left(w^2-1\right) \\ -w \left(2w^2-1\right) \\ -\left(f_1^{(0)} - 1\right) w^2 \end{pmatrix} \tag{5.16}$$

(where in the last line we parametrized the fluctuating jets as in (5.10)).

We now restrict attention to the subspace $\mathfrak{J}_\mathrm{F}^\mathrm{lin}$ on which the unperturbed operator Δ in (5.7) vanishes. Again parametrizing according to (5.11), we obtain

$$\frac{1}{2} \sum_{\mathfrak{a}=1}^{2} \nabla_{(u_\mathrm{F}^\mathrm{lin})_\mathfrak{a}} \ell(p_\mathfrak{a}) = -8\,\lambda^3 \begin{pmatrix} a \\ u^1 \end{pmatrix} \cdot \begin{pmatrix} \lambda \left(f_1^{(0)} - 1\right) w^2 \left(w^2-1\right) \\ -w \left(2w^2-1\right) \end{pmatrix} \tag{5.17}$$

(here we simply dropped the last component in (5.16)). Moreover, the Laplacian on $\mathfrak{J}_\mathrm{F}^\mathrm{lin}$ is computed by

$$\begin{aligned}
&\langle \mathfrak{u}_\mathrm{F}^\mathrm{lin}, \tilde{\Delta} \mathfrak{v}_\mathrm{F}^\mathrm{lin} \rangle \\
&= 4 \left\langle \begin{pmatrix} a \\ u^1 \end{pmatrix}, \begin{pmatrix} 2w^2 \left(w^2-1\right) \lambda^4 & 4 \left(f_1^{(0)} - 1\right) w \left(2w^2-1\right) \lambda^3 \\ -w \left(2w^2-1\right) \lambda^3 & -\left(f_1^{(0)} - 4\right) \left(6w^2-1\right) \lambda^2 \end{pmatrix} \begin{pmatrix} b \\ v^1 \end{pmatrix} \right\rangle_{\mathbb{C}^2} .
\end{aligned} \tag{5.18}$$

The basic question is whether the error in the linearized field equations (5.17) can be compensated by perturbations of $f_\mathfrak{a}$ and $F_\mathfrak{a}$. Having prescribed the leading orders by (5.12) and (5.13), the next orders are perturbations of the form

$$\lambda\, f_\mathfrak{a}^{(1)} \qquad \text{and} \qquad \lambda^2\, F_\mathfrak{a}^{(2)} .$$

Substituting into (5.18) gives a contribution scaling like

$$\langle \mathfrak{u}_{\mathrm{F}}^{\mathrm{lin}}, \tilde{\Delta} \mathfrak{v}_{\mathrm{F}}^{\mathrm{lin}} \rangle = a\, c_5\, \lambda^5 + u^1\, c_4\, \lambda^4 + \mathcal{O}(\lambda^6)\,.$$

This contribution is by a factor of λ *smaller* than the error in the linearized field equations (5.17). This shows that at this stage, a perturbation expansion is not sensible. This can be understood similar to the problem in the perturbation theory for linear operators when applying the naive perturbation expansion to a degenerate subspace.

The method to cure this problem is to choose $f_1^{(0)}$ and w appropriately. Indeed, setting

$$f_1^{(0)} = 1 \qquad \text{and} \qquad w = \frac{1}{\sqrt{2}}\,, \tag{5.19}$$

we obtain

$$\frac{1}{2} \sum_{\mathfrak{a}=1}^{2} \nabla_{(u_{\mathrm{F}}^{\mathrm{lin}})_{\mathfrak{a}}}\, \ell(p_{\mathfrak{a}}) = \begin{pmatrix} a \\ u^1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5.20}$$

$$\langle \mathfrak{u}_{\mathrm{F}}^{\mathrm{lin}}, \tilde{\Delta} \mathfrak{v}_{\mathrm{F}}^{\mathrm{lin}} \rangle = \left\langle \begin{pmatrix} a \\ u^1 \end{pmatrix}, \begin{pmatrix} 2\lambda^4 & 0 \\ 0 & 24\,\lambda^2 \end{pmatrix} \begin{pmatrix} b \\ v^1 \end{pmatrix} \right\rangle_{\mathbb{C}^2}. \tag{5.21}$$

Now the linearized field equations are satisfied. This can be understood immediately by the plot of $\tilde{\ell}(x^1, \lambda)$ in Figure 4, which shows that the minima

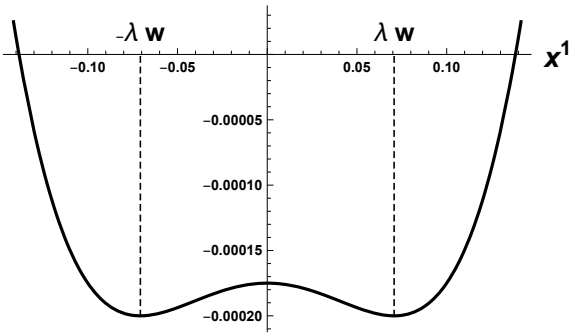

Figure 4: The function $\ell(x_1, \lambda)$ for the fragmented measure and $\lambda = 0.1$.

of ℓ are precisely at the support points (5.14).

Moreover, one sees that for the resulting system a perturbation expansion is sensible, provided that the error in the linearized field equations scales

like

$$\frac{1}{2} \sum_{\mathfrak{a}=1}^{2} \nabla_{(u_{\mathrm{F}}^{\mathrm{lin}})_{\mathfrak{a}}} \ell(p_{\mathfrak{a}}) \lesssim \begin{pmatrix} a \\ u^1 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{O}(\lambda^5) \\ \mathcal{O}(\lambda^4) \end{pmatrix} .$$

If this condition holds, the perturbation expansion consists in determining the jets

$$\big(f_{\mathfrak{a}}^{(1)}, F_{\mathfrak{a}}^{(2)}\big) , \quad \big(f_{\mathfrak{a}}^{(2)}, F_{\mathfrak{a}}^{(3)}\big) , \quad \big(f_{\mathfrak{a}}^{(3)}, F_{\mathfrak{a}}^{(4)}\big) , \ldots$$

iteratively. For example, we may modify the Lagrangian by adding a potential of sixth order

$$\mathcal{L}\big((x_1, x_2), (y_1, y_2)\big) \;\to\; \mathcal{L}\big((x_1, x_2), (y_1, y_2)\big) + \big(x_1^6 + x_2^6 + y_1^6 + y_2^6\big) . \tag{5.22}$$

After this modification, the Lagrangian is bounded below. By adding an irrelevant constant, it can even be arranged to be non-negative.

5.3. The perturbation expansion

After these preparations, we now give the general construction. We choose the unperturbed scalar components such as to preserve the total volume, i.e.

$$f_{\mathfrak{a}}^{(0)} \geq 0 \qquad \text{and} \qquad \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} f_{\mathfrak{a}}^{(0)} = 1 . \tag{5.23}$$

Next, we choose the linearized solution which triggers the fragmentation. In order to allow for a more general scaling, we make the ansatz

$$\mathfrak{w}^{(1)} = \lambda^p\, \bar{\mathfrak{v}}^{\mathrm{lin}} + \lambda^q\, v_{\mathrm{F}}^{\mathrm{lin}} \tag{5.24}$$

with parameters $p, q > 0$ and

$$\min(p, q) = 1$$

(here $v_{\mathrm{F}}^{\mathrm{lin}}$ denotes a jet with vanishing scalar component).

For the perturbation expansion, we again work with the function c defined by (4.8) and expand according to (4.7) and (4.9). We also again use the notation (4.10) (but of course, now all objects carry additional subsystem indices $\mathfrak{a}$ or $\mathfrak{b}$). Our ansatz (5.23) and (5.24) means that the following jets

are already determined:

$$c_{\mathfrak{a}}^{(0)}(x) = \log f_{\mathfrak{a}}^{(0)}(x)\,, \qquad F_{\mathfrak{a}}^{(0)}(x) = x$$

and

$$\begin{cases} \mathfrak{w}^{(p)} = \bar{\mathfrak{v}}^{\text{lin}} + v_{\text{F}}^{\text{lin}} & \text{if } p = q \\ \mathfrak{w}^{(p)} = \bar{\mathfrak{v}}^{\text{lin}} \quad \text{and} \quad \mathfrak{w}^{(q)} = v_{\text{F}}^{\text{lin}} & \text{if } p \neq q\,. \end{cases}$$

We set all other jets $\mathfrak{w}^{(\ell)}$ to zero. Now we proceed in two steps. We first perturb only in the jet spaces $\bar{\mathfrak{J}} \oplus \mathfrak{J}_{\text{F}}^{\text{c}}$, beginning to the order $p+1$. Thus we modify the jets according to

$$\begin{cases} \mathfrak{w}^{(n)} = \bar{\mathfrak{w}}^{(n)} + (\mathfrak{w}_{\text{F}}^{\text{c}})^{(n)} & \text{if } n > p,\, n \neq q \\ \mathfrak{w}^{(n)} = v_{\text{F}}^{\text{lin}} + \bar{\mathfrak{w}}^{(n)} + (\mathfrak{w}_{\text{F}}^{\text{c}})^{(n)} & \text{if } n > p,\, n = q\,. \end{cases}$$

The jets $\bar{\mathfrak{w}}^{(n)}$ and $(\mathfrak{w}_{\text{F}}^{\text{c}})^{(n)}$ can be computed iteratively for $n = p+1, p+2, \ldots$ by multiplying the error in the weak EL equations by the Green's operators in Definition 5.2. We let $\tilde{\rho}$ be the measure obtained from this perturbation expansion according to (5.1). By construction, this measure satisfies the weak EL equations (5.2) if tested in the direction of $\bar{\mathfrak{J}} \oplus \mathfrak{J}_{\text{F}}^{\text{c}}$, i.e.

$$\frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \nabla_{\mathfrak{u}_{\mathfrak{a}}} \tilde{\ell}\big(F_{\mathfrak{a}}(x)\big) = 0 \qquad \text{for all } \mathfrak{u} \in (\bar{J} \oplus \mathfrak{J}_{\text{F}}^{\text{c}}) \cap \mathfrak{J}^{\text{test}}\,,$$

where the tilde refers to the perturbed measure,

$$\tilde{\ell}(F_{\mathfrak{a}}(x)) := \frac{1}{L} \sum_{\mathfrak{b}=1}^{L} \int_M \mathcal{L}\big(F_{\mathfrak{a}}(x), F_{\mathfrak{b}}(y)\big)\, f_{\mathfrak{b}}(y)\, d\rho(y) - \mathfrak{s}\,.$$

However, the weak EL equations will not hold in general if we test in the direction of $\mathfrak{J}_{\text{F}}^{\text{lin}}$. In order to obtain a well-defined perturbation expansion, we need to assume that the error in the EL equation is small compared to the size of the Laplacian on $\mathfrak{J}_{\text{F}}^{\text{lin}}$, as is made precise in the following definition.

Definition 5.4. *The Laplacian on $\mathfrak{J}_{\text{F}}^{\text{lin}}$ is* **definite of order** *r if there is an operator $T_{\text{F}} : (\mathfrak{J}_{\text{F}}^{\text{lin}} \cap \mathfrak{J}_{\text{F}}^{\text{test}})^* \to \mathfrak{J}_{\text{F}}^{\text{lin}}$ with the property that for all $\mathfrak{u} \in \mathfrak{J}_{\text{F}}^{\text{lin}} \cap \mathfrak{J}^{\text{test}}$ and $\mathfrak{v} \in (\mathfrak{J}_{\text{F}}^{\text{lin}} \cap \mathfrak{J}_{\text{F}}^{\text{test}})^*$,*

$$\Big\langle \big(a_{\text{F}}^{\text{lin}}, \lambda^q\, u_{\text{F}}^{\text{lin}}\big), \tilde{\Delta}\, T_{\text{F}}\, \big(b_{\text{F}}^{\text{lin}}, \lambda^q\, v_{\text{F}}^{\text{lin}}\big) \Big\rangle = \lambda^r\, \big\langle \mathfrak{u}_{\text{F}}^{\text{lin}}, \mathfrak{v}_{\text{F}}^{\text{lin}} \big\rangle \Big(1 + \mathcal{O}(\lambda)\Big)\,. \tag{5.25}$$

The ansatz (5.23) *and* (5.24) *gives rise to a* **well-posed fragmentation** *if there is $r > q$ such that the Laplacian on $\mathfrak{J}_{\text{F}}^{\text{lin}}$ is definite of order r and if for*

all $\mathfrak{u} \in \mathfrak{J}^{\text{lin}}_{\text{F}} \cap \mathfrak{J}^{\text{test}}$,

$$\frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \Big(\big(a^{\text{lin}}_{\text{F}}\big)_{\mathfrak{a}}(x) + \lambda^q D_{\big(u^{\text{lin}}_{\text{F}}\big)_{\mathfrak{a}}} \Big) \tilde{\ell}\big(F_{\mathfrak{a}}(x)\big) = \mathcal{O}\big(\lambda^{r+1}\big)\,. \tag{5.26}$$

If the condition in this definition holds, the weak EL equations can also be satisfied in the direction of $\mathfrak{J}^{\text{lin}}_{\text{F}}$ by changing the perturbation ansatz according to

$$\begin{aligned} \tilde{\mathfrak{w}} \to \tilde{\mathfrak{w}} &+ \lambda \Big(\big(c^{\text{lin}}_{\text{F}}\big)^{(1)}, \lambda^q \big(w^{\text{lin}}_{\text{F}}\big)^{(q+1)} \Big) \\ &+ \lambda^2 \Big(\big(c^{\text{lin}}_{\text{F}}\big)^{(2)}, \lambda^q \big(w^{\text{lin}}_{\text{F}}\big)^{(q+2)} \Big) + \cdots\,. \end{aligned} \tag{5.27}$$

Now the error in the EL equations in the direction of $\mathfrak{J}^{\text{lin}}_{\text{F}}$ can be compensated order by order by multiplying with the corresponding Green's operator T_{F}. Clearly, the higher order jets $\bar{\mathfrak{w}}^{(n)}$ and $(\mathfrak{w}^{\text{c}}_{\text{F}})^{(n)}$ are also affected by the jets added in (5.27), but the resulting error can be compensated again using the Green's operators in Definition 5.2. In this way, we obtain a perturbation expansion for the universal measure with fragmentation. The expansion is well-defined as a formal power series in λ.

We note that the different fragments of the measure are separated by $\tilde{w}^{\text{lin}}_{\text{F}} \sim \lambda^q$. Therefore, the "size" of the microstructure obtained by fragmentation is of order $\sim \lambda^q$. Consequently, differentiating this microstructure gives a scaling factor λ^{-q}. This is the reason why on the left side of (5.25) and (5.26), the vector components of the jets are multiplied by scaling factors λ^q. In Example (5.3), this scaling behavior can be seen explicitly from the different powers of λ in (5.21).

For clarity, we also point out that in the applications, the delicate step is to choose the weights $f^{(0)}_{\mathfrak{a}}$ as well as the ansatz (5.24) correctly such as to obtain a well-posed fragmentation. This difficulty already became clear in Example 5.3, where we had to come up with the ansatz (5.19) and choose $p = q$, giving a well-posed fragmentation with $r = 4$. Once the correct ansatz for the fragmentation has been found, the perturbation expansion can be performed in a straightforward way as outlined above. We postpone the combinatorial details to the physical applications in [2].

6. Perturbation expansion for causal fermion systems

6.1. Preliminaries

We briefly recall how the causal action principle for causal fermion systems fits into the framework of causal variational principles in the non-compact setting (see also [15, Section 2.3]). Compared to the setting in Section 2 and [8, Section 1.1], we incorporate the trace constraint by restricting attention to operators of fixed trace. Moreover, we treat the boundedness constraint with a Lagrange multiplier κ. Finally, by assuming that the unperturbed measure has the property that all space-time points are regular (see [8, Definition 1.1.5]), we may assume that all operators have exactly n positive and n negative eigenvalues. This leads to the following setting:

Let $(\mathcal{H}, \langle .|. \rangle_{\mathcal{H}})$ be a complex Hilbert space. Moreover, we are given parameters $n \in \mathbb{N}$ (the spin dimension), $c > 0$ (the constraint for the local trace) and $\kappa > 0$ (the Lagrange multiplier of the boundedness constraint). We let $\mathcal{F} \subset \mathrm{L}(\mathcal{H})$ be the set of all operators on $\mathcal{H}$ with the following properties:

- F is selfadjoint, has finite rank and (counting multiplicities) has n positive and n negative eigenvalues.
- The trace is constant, i.e.

$$\mathrm{tr}(F) = c\,. \tag{6.1}$$

On $\mathcal{F}$ we consider the topology induced by the sup-norm on $\mathrm{L}(\mathcal{H})$. If $\mathcal{H}$ is finite-dimensional, then $\mathcal{F}$ has a smooth manifold structure (see the concept of a flag manifold in [18] or the detailed construction in [17, Section 2.4]).

We introduce the Lagrangian $\mathcal{L}_\kappa$ by adding a Lagrange multiplier term to (2.2),

$$\mathcal{L}_\kappa : \mathcal{F} \times \mathcal{F} \to \mathbb{R}\,, \qquad \mathcal{L}_\kappa(x,y) = \big|(xy)^2\big| - \frac{1}{2n}\,|xy|^2 + \kappa\,|xy|^2\,. \tag{6.2}$$

Clearly, this Lagrangian is non-negative and continuous on $\mathcal{F} \times \mathcal{F}$. Thus we are back in the setting of Section 3.1. The EL equations in Definition 3.1 agree with the EL equations as derived for the causal action principle with constraints in [1] (see [1, Theorem 1.1]).

Before going on, we make a few remarks. Since in the present setting, the Lagrange multiplier term $\kappa\,|xy|^2$ in (6.2) is always present, we can simplify the notation by always omitting the subscript κ. We also point out that we

shall always keep the constants c and κ in (6.1) and (6.2) fixed when varying or perturbing the measure ρ. This is justified as follows. The constant c can be changed arbitrarily by rescaling the measure according to

$$\rho(\Omega) \to \rho\Big(\big\{\alpha x \,\big|\, x \in \Omega\big\}\Big) \qquad \text{with} \quad \alpha \in \mathbb{R}\,.$$

Combining this transformation with our previous transformation (3.3), the freedom in rescaling the universal measure is exhausted. Therefore, the parameter κ must be regarded as a physical parameter of the system. The reason for keeping it fixed is that we want to describe *localized physical systems*, meaning that the perturbations of ρ are spatially compact or that the resulting space-time is asymptotically flat. In such situations, the parameter κ is determined by the asymptotic form of the universal measure at infinity, which is kept fixed in our variations and perturbations. More generally, κ can be kept fixed if we assume that there is a macroscopic region in space-time where no interaction takes place.

We now recall the definition of a few other basic objects needed for the analysis of causal fermion systems (for more details see [8, Section 1.1]). For every $x \in \mathcal{F}$ we define the *spin space* S_x by $S_x = x(\mathcal{H})$; it is a subspace of $\mathcal{H}$ of dimension $2n$. On the spin space S_x, the *spin scalar product* $\prec .|.\succ_x$ is defined by

$$\prec u|v\succ_x = -\langle u|xu\rangle_{\mathcal{H}} \qquad (\text{for all } u, v \in S_x)\,.$$

We let π_x be the orthogonal projection on $S_x \subset \mathcal{H}$. Then, for any $x, y \in M$ we define the *kernel of the fermionic projector* $P(x,y)$ by

$$P(x,y) = \pi_x\, y|_{S_y} \,:\, S_y \to S_x\,.$$

The kernel of the fermionic projector is very useful because, forming the closed chain A_{xy} by

$$A_{xy} := P(x,y)\, P(y,x) = \pi_x\, y\, x|_{S_x} \,:\, S_x \to S_x\,,$$

the eigenvalues of A_{xy} coincide with the eigenvalues $\lambda_1^{xy}, \ldots, \lambda_{2n}^{xy}$ in (2.1). In this way, the Lagrangian can be expressed in terms of the kernel of the fermionic projector.

A *wave function* ψ is defined as a mapping which to every $x \in M$ associates a vector of the corresponding spin space,

$$\psi \,:\, M \to \mathcal{H} \qquad \text{with} \quad \psi(x) \in S_x M \quad \text{for all } x \in M\,.$$

A wave function is said to be *continuous* at x if for every $x \in M$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$\big\| \sqrt{|y|}\, \psi(y) - \sqrt{|x|}\, \psi(x) \big\|_{\mathcal{H}} < \varepsilon \qquad \text{for all } y \in M \text{ with } \|y - x\| \leq \delta\,.$$

The vector space of continuous wave functions is denoted by $C^0(M, SM)$. For every $u \in \mathcal{H}$, the corresponding *physical wave function* ψ^u is the wave function obtained by projecting to the spin spaces, i.e.

$$\psi^u(x) := \pi_x u \in S_x M\,.$$

The physical wave functions can be understood as describing the "occupied states" of the system (for details see [8, §1.1.4 and §1.2.4]). The physical wave functions are all continuous. The *wave evaluation operator* Ψ is the linear operator which to every Hilbert space vector associates the corresponding physical wave function,

$$\Psi \,:\, \mathcal{H} \to C^0(M, SM)\,, \qquad u \mapsto \psi^u\,. \tag{6.3}$$

Evaluating at a fixed space-time point gives the mapping

$$\Psi(x) \,:\, \mathcal{H} \to S_x M\,, \qquad u \mapsto \psi^u(x)\,.$$

The operator x as well as the kernel of the fermionic projector can be expressed in terms of the wave evaluation operator by (see [8, Lemma 1.1.3])

$$x = -\Psi(x)^*\, \Psi(x) \qquad \text{and} \qquad P(x,y) = -\Psi(x)\, \Psi(y)^*\,. \tag{6.4}$$

6.2. Perturbation expansion for the wave evaluation operator

The perturbation expansion in Section 4 was performed in a chart on $\mathcal{F}$. We now explain how to construct such a chart. Working in this chart will also immediately give a perturbation expansion for the wave evaluation operator. Given $x \in M$, we consider the mapping

$$R \,:\, \big\{ \psi \in \mathrm{L}(\mathcal{H}, S_x) \,\big|\, \mathrm{tr}(\psi^* \psi) \neq 0 \big\} \to \mathrm{L}(\mathcal{H})\,, \qquad \psi \mapsto \frac{c}{\mathrm{tr}(\psi^* \psi)}\, \psi^* \psi\,. \tag{6.5}$$

The operators in the image of R are selfadjoint, have finite rank and at most n positive and at most n negative eigenvalues. Moreover, due to the rescaling by the prefactor $c/\mathrm{tr}(\psi^*\psi)$, they satisfy the trace condition (6.1). Let us verify that the image of R contains all operators in $\mathcal{F}$: By definition

of $\mathcal{F}$, a given operator $F \in \mathcal{F}$ is selfadjoint and has n positive eigenvalues (which we denote by $\nu_1, \ldots, \nu_n > 0$) and n negative eigenvalues (denoted by $(\nu_{n+1}, \ldots, \nu_{2n})$). Diagonalizing F gives a representation

$$F = U \,\mathrm{diag}(\nu_1, \ldots, \nu_{2n})\, U^*$$

where $U : \mathbb{C}^{2n} \to \mathcal{H}$ is an isometric embedding. It is useful to rewrite this equation as

$$F = V^* \,\mathrm{diag}(\underbrace{1, \ldots, 1}_{n \text{ entries}}, \underbrace{-1, \ldots, -1}_{n \text{ entries}})\, V$$

with $V := \mathrm{diag}(\sqrt{|\nu_1|}, \ldots, \sqrt{|\nu_{2n}|})\, U^*$. Then, choosing a pseudo-orthogonal basis $(e_\alpha)_{\alpha=1,\ldots,2n}$ of S_x, the mapping

$$\psi : \mathcal{H} \to S_x\,, \qquad \psi(u) := \sum_{\alpha=1}^{2n} (Vu)^\alpha\, e_\alpha$$

has the desired property $F = \psi^* \psi$.

But the mapping R is not injective for two reasons: First, due to the rescaling, multiplying ψ by a complex number leaves $R(\psi)$ unchanged. Second, a local unitary transformation

$$\psi \to U\, \psi \qquad \text{with} \quad U \in \mathrm{U}(S_x) \tag{6.6}$$

preserves the combination $\psi^* \psi$ and thus leaves $R(\psi)$ unchanged.

The mapping R can be used to construct a chart of $\mathcal{F}$ around x: Since the image of R contains $\mathcal{F}$, the operator x can be written as $x = R(\psi)$ with $\psi \in \mathrm{L}(\mathcal{H}, S_x)$ (more explicitly, we can choose $\psi = \Psi(x)$). By continuity, the numbers of positive and negative eigenvalues of the operator $R(\phi)$ are again equal to n for all ϕ in a small neighborhood $V \subset \mathrm{L}(\mathcal{H}, S_x)$ of ψ. Thus the restriction of R to this neighborhood maps to $\mathcal{F}$,

$$R|_V \;:\; \mathrm{L}(\mathcal{H}, S_x) \cap V \to \mathcal{F}\,.$$

Differentiating at ψ gives a linear operator $DR|_\psi : \mathrm{L}(\mathcal{H}, S_x) \to T_x \mathcal{F}$. This operator is not injective (because infinitesimal scalings and unitary transformations (6.6) lie in its kernel). Therefore, we choose a proper subspace $E \subset$

$\mathrm{L}(\mathcal{H}, S_x)$ such that the restriction to E is invertible,

$$DR|_\psi\big|_E \,:\, E \subset \mathrm{L}(\mathcal{H}, S_x) \to T_x\mathcal{F} \qquad \text{is continuously invertible}$$

(if $\mathcal{H}$ is finite-dimensional, such a subspace E always exist; in the infinite-dimensional setting the condition that the inverse be continuous poses constraints which we shall not analyze here). As a consequence, the restriction of R is a local diffeomorphism, meaning that there is an open neighborhood $V' \subset V \subset \mathrm{L}(\mathcal{H}, S_x)$ of ψ and an open neighborhood $U \subset \mathcal{F}$ of x such that the restriction

$$R|_{\psi+(E\cap V')} \,:\, \psi + (E \cap V') \to U \subset \mathcal{F}$$

is a diffeomorphism (here $\psi + E$ denotes the affine subspace through ψ). Its inverse

$$X := (R|_{\psi+(E\cap V')})^{-1} \,:\, U \subset \mathcal{F} \to \psi + E$$

defines a chart (X, U) around x. Choosing a basis $(e_1, \ldots, e_m)$ of E, we write the mapping $F : M \to \mathcal{F}$ in components $F(x)^\alpha$, i.e.

$$X\big(F(x)\big) = \psi + \sum_{\alpha=1}^m F(x)^\alpha\, e_\alpha\,.$$

Choosing for every $x \in M$ a chart of this form and choosing a suitable jet space $\mathfrak{J}^{\text{test}}$, we are back in the setting of Section 4.1. After determining the $F^{(p)}$, the corresponding perturbation of the wave evaluation operator is given simply by the component in our chart, i.e.

$$\Psi^{(p)}(x) = F^{(p)}(x)^\alpha\, e_\alpha \;\in\; E \subset \mathrm{L}(\mathcal{H}, S_x) \qquad (p \geq 1)\,.$$

6.3. Perturbing the vacuum

We now explain how the construction in Section 4.5 applies to causal fermion systems. Let ρ be a universal measure describing the vacuum (for example, a regularized Dirac sea configuration as constructed in [8, Section 1.2]). Introducing particles and/or anti-particles (as described in [8, Section §2.1.7]) amounts to modifying the wave evaluation operator Ψ to

$$\hat{\Psi} := \Psi + \Delta\Psi \,:\, \mathcal{H} \to C^0(M, SM)\,. \tag{6.7}$$

At this point, the complication arises that the local correlation operators defined in analogy to (6.4) by $\hat{F}(x) = -\hat{\Psi}(x)^* \hat{\Psi}(x)$ (see [8, eq. (1.4.12)]) will

in general violate our trace condition (6.1). In order to resolve this problem, we rescale the local correlation operators similar as in (6.5) by setting

$$\hat{H}(x) := \frac{c}{\mathrm{tr}\left(\hat{\Psi}(x)^* \hat{\Psi}(x)\right)}\, \hat{\Psi}(x)^* \hat{\Psi}(x)\,. \tag{6.8}$$

We now introduce the corresponding universal measure $\hat{\rho}$ as the push-forward of $\hat{H}$,

$$\hat{\rho} := \hat{H}_* \rho\,. \tag{6.9}$$

Now we are back in the setting of Section 4.5. We remark that the rescaling (6.8) seems unproblematic because in physical applications it affects only the higher orders in ε relative to the length scale of macroscopic physics (for details on this point see [8, Section 2.5]).

7. Example: Perturbation expansion in the continuum limit

7.1. Preliminaries

We now recall a few constructions of the continuum limit analysis in [8] which will be of relevance here. In [8, §1.4.1] the EL equations are written in a form which is particularly convenient for a detailed analysis. These EL equations are obtained by considering a special class of variations of the wave evaluation operator Ψ:

Definition 7.1. *A* **variation of the physical wave functions** $(\Psi_\tau)_{\tau \in (-\tau_{\max}, \tau_{\max})}$ *with* $\tau_{\max} > 0$ *and* $\Psi_0 = \Psi$ *is* **smooth** *and* **compact** *if the family of operators has the following properties:*

(a) *The variation is trivial on the orthogonal complement of a finite-dimensional subspace* $I \subset \mathcal{H}$*, i.e.*

$$\Psi_\tau|_{I^\perp} = \Psi \qquad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max})\,.$$

(b) *There is a compact subset* $K \subset M$ *outside which the variation is trivial, i.e.*

$$\big(\Psi_\tau(u)\big)\big|_{M\setminus K} = \big(\Psi(u)\big)\big|_{M\setminus K} \qquad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) \text{ and } u \in \mathcal{H}\,.$$

(c) *The Lagrangian is continuously differentiable in the sense that the derivative*

$$\frac{d}{d\tau} \mathcal{L}\big(x, F_\tau(y)\big)\big|_{\tau=0} \qquad \text{with} \quad F_\tau(x) := \frac{c}{\mathrm{tr}(\Psi_\tau(x)^* \Psi_\tau(x))}\, \Psi_\tau(x)^* \Psi_\tau(x)$$

exists and is continuous on $M \times M$.

For clarity, we point out that, similar to (6.5), the factor

$$c/\,\mathrm{tr}(\Psi_\tau(x)^* \Psi_\tau(x))$$

is again needed in order to the trace condition (6.1). For the derivation of the EL equations, it is more convenient to disregard this condition in the variation, and to realize it instead by a Lagrange multiplier term. Then, according to (6.4), the first variation $\delta\Psi = \partial_\tau \Psi|_{\tau=0}$ defines a corresponding variation of the kernel of the fermionic projector given by

$$\delta P(x,y) = -\delta\Psi(x)\, \Psi(y)^* - \Psi(x)\, \delta\Psi(y)^* \,. \tag{7.1}$$

The resulting first variation of the Lagrangian can be written as (see [4, Section 5.2] and [8, eq. (1.4.16)])

$$\delta\mathcal{L}(x,y) = \mathrm{Tr}_{S_y}\big(Q(y,x)\, \delta P(x,y)\big) + \mathrm{Tr}_{S_x}\big(Q(x,y)\, \delta P(x,y)^*\big) \tag{7.2}$$

with a kernel $Q(x,y) : S_y \to S_x$ which is symmetric in the sense that

$$Q(x,y)^* = Q(y,x)$$

(a more explicit formula for $Q(x,y)$ is given in [8, Lemma 3.6.2]). Then the EL equations corresponding to the above variations can be written as (see [8, Proposition 1.4.3])

$$\int_M Q(x,y)\, \psi^u(y)\, d\rho(y) = \frac{\lambda}{2}\, \psi^u(x) \qquad \text{for all } u \in \mathcal{H} \text{ and } x \in M\,, \tag{7.3}$$

where λ is the Lagrange multiplier needed in order to arrange the trace condition (6.1). The connection to the weak EL equations (3.4) is not obvious and will be explained in Section 7.2 below.

In the *continuum limit* (for details see [8, §3.5.2]), the EL equations (7.3) are evaluated for a physical wave function ψ^u having the form of an *ultra-relativistic wave packet* of negative energy, meaning that the wave packet has frequency of the order $|\Omega|$ and is spatially localized on the scale δ (as

measured in a chosen reference frame). Moreover, we assume that the spatial distance of the ultrarelativistic wave packet from the space-time point x is on the scale ℓ with (see [8, eq. (3.5.28) and Figure 3.1])

$$\varepsilon \ll |\Omega|^{-1} \ll \delta \ll \ell, \ell_{\text{macro}}, m^{-1} \tag{7.4}$$

(where m^{-1} is the Compton scale and ℓ_{macro} denotes the length scales of atomic or high energy physics). Moreover, the equations (7.3) are evaluated weakly with a test function ϕ which is supported in a δ-neighborhood of the point x (with Euclidean distances measured again in a chosen reference frame). Then the supports of ϕ and ψ^u are disjoint, so that the right side of (7.3) vanishes (see [8, eqs (3.5.24) and (3.5.29)])

$$\int_M d\rho(x) \int_M d\rho(y) \prec \phi(x) \,|\, Q(x,y)\, \psi^u(y) \succ_x = 0\,. \tag{7.5}$$

Written in this form, the main contribution to the EL equations comes from the behavior of $Q(x,y)$ on the light cone, making it possible to analyze the equations in detail in the formalism of the continuum limit (for details see [8, Section 2.4 and Chapters 3-5]).

In the resulting continuum description, the kernel of the fermionic projector is a solution of the Dirac equation in the presence of a classical gauge field. In order to keep the setting as simple as possible, we here restrict attention to one type of elementary particles and a U(1) gauge field (the generalizations to several generations and more general gauge fields are carried out in detail in [8, Chapters 3–5]). Then the Dirac equation reads

$$\big(i\partial\!\!\!/ + A\!\!\!/ - m\big)\, P(x,y) = 0\,, \tag{7.6}$$

where A can be thought of as an electromagnetic potential, but it does not need to satisfy Maxwell's equations. In order to construct the kernel of the fermionic projector in the presence of the electromagnetic potential, one expands the Dirac equation (7.6) in powers of the potential and solves the equations iteratively with the help of Dirac Green's operators s defined by

$$(i\partial\!\!\!/ - m)\, s_m(x,y) = \delta^4(x-y)\,. \tag{7.7}$$

The resulting *causal perturbation expansion* becomes unique by making use of the underlying causal structure (for details see [8, Section 2.1]).

7.2. Choosing the jet spaces and the Green's operator

In this section we explain how the weak EL equations (3.4) and the perturbation expansion of Section 4 are related to the analysis in the continuum limit. Our first task is to introduce the jet spaces. It is useful that, similar as explained in (6.7) and (6.8) for finite variations, tangent vectors to $\mathcal{F}$ on M can be described by infinitesimal variations of the wave evaluation operator. Thus we describe a tangent vector $u \in T_x\mathcal{F}$ at a space-time point $x \in M$ as

$$u = \delta\hat{H}(x) = -\delta\Psi(x)^*\,\Psi(x) - \Psi(x)^*\,\delta\Psi(x) + \frac{x}{c}\,\mathrm{tr}\big(\delta\Psi(x)^*\,\Psi(x) + \Psi(x)^*\,\delta\Psi(x)\big) \tag{7.8}$$

(where we used that $\mathrm{tr}\,x = -\,\mathrm{tr}\,\Psi(x)^*\Psi(x) = c$) with

$$\delta\Psi \;:\; \mathcal{H} \to C^\infty(M, SM)\,. \tag{7.9}$$

Our next goal is to introduce the space of test jets $\mathfrak{J}^{\text{test}}$ in such a way that the weak EL equations (3.4) agree with the EL equations in the continuum limit (7.5) for ψ^u an ultrarelativistic wave packet (7.4). We say that a physical wave function ψ^u is *macroscopic* if its energy and momentum is much smaller than the Planck energy. We choose u such that ψ^u is macroscopic and is an *ultrarelativistic wave packet* as defined before (7.4). Next, we choose $\delta\psi^u$ as a wave function with compact support such that its spatial distance to the ultrarelativistic wave packet scales like

$$\varepsilon \ll \mathrm{dist}\big(\,\mathrm{supp}\,\delta\psi^u, \mathrm{supp}\,\psi^u\big) \ll \ell_{\text{macro}}\,. \tag{7.10}$$

We define the corresponding variation of the wave evaluation operator $\delta\Psi$ as the unique linear mapping with the properties that

$$\delta\Psi \;:\; v \mapsto \begin{cases} \delta\psi^u & \text{if } v = u \\ 0 & \text{if } v \perp u\,. \end{cases}$$

Since by construction, ψ^u and $\delta\psi^u$ have disjoint supports, the trace in (7.8) vanishes. Therefore, the vector field described by $\delta\Psi$ is given by

$$u = \delta\hat{H}(x) = -\delta\Psi(x)^*\,\Psi(x) - \Psi(x)^*\,\delta\Psi(x)\,.$$

We choose Γ^{test} as the span of all the vector fields u for $\delta\Psi$ as specified above. Since in the weak evaluation on the light cone, only variations of the wave

functions are considered, we choose the scalar component of $\mathfrak{J}^{\text{test}}$ trivially,

$$\mathfrak{J}^{\text{test}} = \{0\} \oplus \Gamma^{\text{test}} \subset C^\infty(M, \mathbb{R}) \oplus C^\infty(M, T\mathcal{F})\,. \tag{7.11}$$

We remark that there is no point in making (7.10) mathematically more precise, because in the formalism of the continuum limit one also works merely with the scaling behavior.

The next lemma gives the connection between the weak EL equations (3.4) and their continuum limit (7.5).

Lemma 7.2. *For any $\mathfrak{u} \in \mathfrak{J}^{\text{test}}$ and all $x \in M$,*

$$\nabla_{\mathfrak{u}} \ell(x) = -2\,\text{Re} \int_M \text{tr}\left(\delta\Psi(x)^*\, Q(x,y)\, \Psi(y)\right) d\rho(y)\,.$$

Proof. Since $\mathfrak{u}$ has no scalar component, the term involving $\mathfrak{s}$ in (3.1) drops out. Using (7.1) together with the fact that the jet $\mathfrak{u}$ acts only on x,

$$\nabla_{\mathfrak{u}(x)} P(x,y) = -\delta\Psi(x)\, \Psi(y)^*\,.$$

Using this formula in (7.2), we obtain

$$\begin{aligned} \nabla_{\mathfrak{u}(x)} \mathcal{L}(x,y) &= -\,\text{Tr}_{S_y}\left(Q(y,x)\, \delta\Psi(x)\, \Psi(y)^*\right) - \text{Tr}_{S_x}\left(Q(x,y)\, \Psi(y)\, \delta\Psi(x)^*\right) \\ &= -2\;\text{Re}\,\text{tr}\left(\delta\Psi(x)^*\, Q(x,y)\, \Psi(y)\right), \end{aligned}$$

where in the last step we cyclically commuted the factors inside the trace. Integrating over y gives the result. □

We next turn attention to the jets used for perturbing the measure. The abstract Definition 3.2 is intended to make $\mathfrak{J}^\infty$ as large as possible, giving the largest possible freedom for the perturbations. But not all of the degrees of freedom of $\mathfrak{J}^\infty$ are needed in the applications. Therefore, we must specify those subspaces of $\mathfrak{J}^\infty$ which are of relevance here. We first consider jets which are needed to describe particle and anti-particle states.

Definition 7.3. *A vector field u of the form (7.8) where the variation $\delta\Psi$ is a mapping of finite rank with the property that for every $u \in \mathcal{H}$, either Ψu or $\delta\Psi u$ is macroscopic, is called* **fermionic vector field**. *The vector space of fermionic vector fields is referred to as Γ^{f}. The* **fermionic jets** *are defined by*

$$\mathfrak{J}^{\text{f}} = \{0\} \oplus \Gamma^{\text{f}}\,.$$

In the next definition we introduce the jets describing the bosons, for simplicity for an electromagnetic potential.

Definition 7.4. *Let $A \in C^\infty(M, T^*M)$ be a smooth one-form. A vector field u of the form* (7.8) *with*

$$\big(\delta\Psi\big)(x) = -\int_M s_m(x,y)\, \A(y)\, \Psi(y)\, d\rho(y) \tag{7.12}$$

is called **bosonic vector field** *(here $s_m(x,y)$ is a Dirac Green's function* (7.7)*). The vector space of bosonic vector fields is referred to as Γ^{b}. The* **bosonic jets** *are defined by*

$$\mathfrak{J}^{\mathrm{b}} = \{0\} \oplus \Gamma^{\mathrm{b}}\,.$$

Clearly, the fermionic and bosonic jets are subspaces of $\mathfrak{J}^\infty$,

$$\mathfrak{J}^{\mathrm{f}}, \mathfrak{J}^{\mathrm{b}} \subset \mathfrak{J}^\infty\,.$$

We now explain how the perturbative description in the continuum limit is described in our setting. In the formalism of Section 4.5, the particles and anti-particles as introduced in [8, §3.4.3] correspond to a perturbation H of the vacuum measure in (4.31). The corresponding jets in (4.32) are fermionic,

$$\mathfrak{v}^{(p)} \in \mathfrak{J}^{\mathrm{f}}\,.$$

The resulting contributions to the weak EL equations are compensated by bosonic fields. Consequently, we here introduce the Green's operator S (see Definition 4.2) as a mapping to the bosonic jets,

$$S\,:\,(\mathfrak{J}^{\mathrm{test}})^* \to \mathfrak{J}^{\mathrm{b}} \subset \mathfrak{J}^\infty\,. \tag{7.13}$$

The condition (4.17) means that the potential $\mathcal{B}$ in (7.12) satisfies the inhomogeneous classical field equations. In the example of an electromagnetic potential (7.6) a Maxwell field, these equations become

$$\partial_{jk}(S\mathfrak{v})^k - \Box(S\mathfrak{v})_j = -c\, v_j$$

(or equivalently with differential forms $\delta d\, S\, \mathfrak{v} = -c\, \mathfrak{v}$, where the constant c depends on the detailed form of the regularization parameters in [8, Chapter 3]). This is the usual equation for the Maxwell propagator. It involves the freedom in choosing a gauge. For example, in the Lorenz gauge, one may

choose S as the multiplication operator in momentum space $S(k) = c/k^2$. But S can also be given in any other gauge. More generally, the choice of the Green's operator (7.13) always involves a choice of gauge.

7.3. Discussion and remarks

We now clarify the previous constructions by a few remarks. We first note that, in order to simplify the computations, it is often convenient to assume that the rescaling term in (7.8) vanishes, i.e.

$$\text{tr}\left(\Psi(x)^*\, \delta\Psi(x)\right) = 0 \qquad \text{for all } x \in M\,. \tag{7.14}$$

This can be arranged for example by the transformation

$$\delta\Psi \to \delta\Psi + \text{tr}\left(\Psi(x)^*\, \delta\Psi(x)\right) \frac{\Psi}{c}\,.$$

Thinking in terms of the charts constructed in Section 6.2, with the condition (7.14) one restricts attention to a special class of charts around x.

We next point out that, as explained in [8, Section 2.5], the rescaling terms in (7.8) give rise to terms of higher order in $\varepsilon/l_{\text{macro}}$. With this in mind, in many applications it is admissible to simply leave out the rescaling and to ignore the condition (7.14).

We also remark that all the above jet spaces have a natural *complex structure*. In order to understand how this comes about, we recall that according to (7.8) the vector fields on M were described by variations of the wave evaluation operator (7.9). Since the spin spaces are complex vector spaces, pointwise multiplication by complex scalars gives a natural complex structure on $\delta\Psi$. Using the notation (7.8), we thus obtain a corresponding almost complex structure J on $T_x\mathcal{F}$ given by

$$\begin{aligned} J\,\delta\hat{F}[\delta\Psi](x) &= \delta\hat{F}[i\delta\Psi](x) \\ &= i\delta\Psi(x)^*\,\Psi(x) - i\Psi(x)^*\,\delta\Psi(x) \\ &\quad + \frac{x}{c}\,\text{tr}\left(-i\delta\Psi(x)^*\,\Psi(x) + i\Psi(x)^*\,\delta\Psi(x)\right). \end{aligned} \tag{7.15}$$

This also gives rise to a complex structure on the vector spaces of vectorial jets on M like $\mathfrak{J}^{\text{f}}$ and $\mathfrak{J}^{\text{b}}$. This complex structure is of no relevance for the constructions in [8] but might be of importance for future developments. Indeed, in [13] an almost-complex structure was constructed on the jet spaces in the more general setting of causal variational principles. It was

used to deduce a unitary time evolution on bosonic Fock spaces. The detailed connection to the almost-complex structure in (7.15) and generalization to fermionic Fock spaces still need to be worked out.

We finally point out that the choice of test jets in (7.11) is very restrictive. In other words, the analysis of the continuum limit only uses very little of the information contained in the EL equations. On the other hand, this information seems to capture precisely what is needed in order to describe the effective macroscopic interaction. One shortcoming of the analysis in the continuum limit is that the test jets do not contain the bosonic jets,

$$\mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{b}} = \varnothing\,.$$

This implies that the symplectic form as introduced in [15] is undefined for the bosonic jets. Moreover, since the intersection of the test jets with the fermionic jets only contains the very restrictive class of jets formed of ultrarelativistic wave packets, also the conserved surface layer integrals in [16] cannot be evaluated for interesting fermionic jets. This last shortcoming is closely related to the fact that the Green's operator (7.13) is purely bosonic, whereas the fermionic dynamics is encoded in the Dirac equation (7.6). Taking into account that in [8, Section 3.10] the validity of the Dirac equation is justified from the causal action principle by ruling out nonlocal potentials, this procedure is conceptually convincing as a first step. But eventually, one would like to have more general test jets, giving rise to a unified description of the interaction in terms of Green's operators composed of a fermionic and a bosonic component. A first step in this direction is the computation of surface layer integrals for bosonic and fermionic jets in [9].

8. Example: Perturbation expansion with microscopic mixing

8.1. Preliminaries

The method of microscopic mixing of wave functions was introduced in [7] (based on preliminary considerations in [6]). Using our present notation, the basic construction is summarized as follows. One first decomposes space-time into disjoint subsystems $M_1, \ldots, M_L$,

$$M = M_1 \cup \cdots \cup M_L \qquad \text{and} \qquad M_{\mathfrak{a}} \cap M_{\mathfrak{b}} = \varnothing \quad \text{if } a \neq b\,.$$

For each subsystem, one introduces a unitary operator $V_{\mathfrak{a}}$ with the property that $\mathbb{1} - V_{\mathfrak{a}}$ is an operator of finite rank which maps particle and anti-particle

states to sea states and vice versa (for details see [7, Section 2.2]). Then the kernel of the fermionic projector with microscopic mixing is introduced by

$$P^\varepsilon(x,y) = \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \chi_{M_\mathfrak{a}}(x)\, P^{\mathfrak{a},\mathfrak{b}}(x,y)\, \chi_{M_\mathfrak{b}}(y) \tag{8.1}$$

$$P^{\mathfrak{a},\mathfrak{b}}(x,y) = -\Psi(x)\, V_\mathfrak{a}\, V_\mathfrak{b}^*\, \Psi(y)^* \tag{8.2}$$

(where $\chi_{M_\mathfrak{a}}$ is the characteristic function). In [7], this kernel of the fermionic projector is used as the starting point for a perturbative treatment based on the methods of the analysis in the continuum limit. It is shown that in a suitable limiting case, one obtains an effective interaction in terms of bosonic and fermionic field operators acting on Fock spaces.

8.2. A synchronization mechanism

In preparation for getting a connection to the setting of Section 5, we recast microscopic mixing in terms of the universal measure (for a similar construction see [8, §1.5.3]). To this end, for a unitary operator $V \in \mathrm{U}(\mathcal{H})$ we define the measure $V(\rho)$ by

$$(V\rho)(\Omega) = \rho\big(V \Omega V^{-1}\big)\,. \tag{8.3}$$

We introduce the measure $\hat{\rho}$ as the convex combination

$$\hat{\rho} = \frac{1}{L} \sum_{\mathfrak{a}=1}^{L} \rho_\mathfrak{a} \qquad \text{with} \quad \rho_\mathfrak{a} = V_\mathfrak{a} \rho\,.$$

Then the resulting space-time $\hat{M} := \operatorname{supp} \hat{\rho}$ is given by

$$\hat{M} = \bigcup_{\mathfrak{a}=1}^{L} M_\mathfrak{a} \qquad \text{with} \quad M_\mathfrak{a} := V_\mathfrak{a}\, M\, V_\mathfrak{a}^{-1}\,.$$

Comparing the unitary transformation $x \to V x V^{-1}$ in (8.3) with the first equation in (6.4), one sees that the wave evaluation operator (6.3) is transformed to

$$\hat{\Psi} \,:\, \mathcal{H} \to C^0\big(\hat{M}, S\hat{M}\big)\,, \qquad \hat{\Psi}(x_\mathfrak{a}) = \Psi(x)\, V_\mathfrak{a}\,.$$

Applying this relation in the second equation in (6.4), one recovers (8.2).

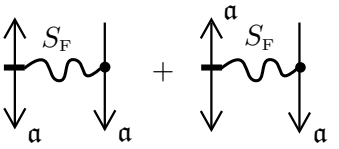

Figure 5: Synchronization of fluctuations.

Next, we rewrite the wave evaluation operator of the $\mathfrak{a}^{\text{th}}$ subsystem as

$$\hat{\Psi}_{\mathfrak{a}} = \Psi + \Delta\Psi_{\mathfrak{a}} \qquad \text{with} \quad \Delta\Psi_{\mathfrak{a}} = \Psi(x)\left(V_{\mathfrak{a}} - \mathbb{1}\right).$$

Exactly as explained in Section 6.3, the resulting transformation of the universal measure can be written as (cf. (6.7) and (6.9))

$$\rho_{\mathfrak{a}} = (H_{\mathfrak{a}})_* \rho\,.$$

Expanding $H_{\mathfrak{a}}$ in a given chart on $\mathcal{F}$ similar to (4.32), one obtains inhomogeneities $\mathfrak{v}_{\mathfrak{a}}^{(p)}$ in the EL equations which depend on the subsystem. Following the constructions in Section 5.1 for the linearized inhomogeneity $\mathfrak{v} = \mathfrak{v}^{(1)}$, one gets a corresponding linearized solution of the field equations $\mathfrak{w}^{(1)}$ which involves fluctuations. The higher orders in perturbation theory are obtained just as in Section 5.3. The crucial condition for the construction to work is that the resulting fragmentation must be well-posed (see Definition 5.4).

Choosing the jet spaces as in the continuum limit in Section 7.2, the above construction simplifies because the jet spaces do not have a scalar component. In this limiting case, one recovers the perturbation expansion in [7] with one important exception: the perturbation expansion with fragmentation gives rise to an additional *synchronization mechanism*. Indeed, according to Definition 5.2, the Green's operators S_{F} acts on each subsystem separately,

$$\left(S_{\mathrm{F}}\right)_{\mathfrak{b}}^{\mathfrak{a}} = \delta_{\mathfrak{b}}^{\mathfrak{a}}\, S_{\mathrm{F}}\,. \tag{8.4}$$

From (5.9) one sees that it couples only to the current generated by Dirac wave functions in the subsystem $\mathfrak{a}$ (see the Feynman diagrams in Figure 5). This seems to make it unnecessary to consider the stochastic background field in [7, Section 4] for synchronization. Also, the recombination of subsystems in [7, Section 7] needs to be reconsidered. The consequences of this

synchronization mechanism will be analyzed in detail in a separate publication [2].

8.3. Gauge potentials are subsystem-diagonal

The previous constructions yield an interaction described by a Dirac equation which, according to (8.4), is coupled to an electromagnetic potential for each subsystem, i.e.

$$(i\partial\!\!\!/ + A\!\!\!/_{\mathfrak{a}} - mY)\,\psi^{\mathfrak{a}}(x) = 0 \qquad \text{for all } \mathfrak{a} = 1, \ldots, L\,.$$

More generally, one could consider a matrix potential which mixes the subsystems, i.e.

$$\sum_{\mathfrak{b}=1}^{L} (i\partial\!\!\!/ + A\!\!\!/_{\mathfrak{b}}^{\mathfrak{a}} - mY)\,\psi^{\mathfrak{b}} = 0 \qquad \text{for all } \mathfrak{a} = 1, \ldots, L\,. \tag{8.5}$$

We now give an independent general argument which conveys a good intuitive understanding for why such subsystem-mixing potentials must not occur.

The matrix potential in (8.5) can be regarded as a $\mathrm{U}(L)$ gauge potential. To leading degree on the light cone, this gauge potentials affects the kernel of the fermionic projector via generalized phase transformations (for details see [3] or [8, §3.6.2 and §4.3.2]). Considering for simplicity the special case of a gauge transformation, the Dirac wave functions transforms according to

$$\psi_{\mathfrak{a}}(x) \to \sum_{\mathfrak{b}=1}^{L} U_{\mathfrak{b}}^{\mathfrak{a}}(x)\, \psi_{\mathfrak{b}}(x)\,.$$

Using this transformation law in (8.2) in the special case with trivial mixing matrices $V_1 = \cdots = V_L = \mathbb{1}$, one finds that the kernel of the fermionic projector transforms according to

$$P^{\mathfrak{a},\mathfrak{b}}(x,y) \to (U(x)\,v)^{\mathfrak{a}}\, P(x,y)\, \overline{(U(y)\,v)^{\mathfrak{b}}}\,,$$

where

$$v = (1, \ldots, 1) \in \mathbb{C}^L\,.$$

Since the Lagrangian is homogeneous of degree four in $P(x,y)$, it transforms like

$$\mathcal{L}(x,y) \to \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \left|(U(x)\,v)^{\mathfrak{a}}\right|^4 \left|(U(y)\,v)^{\mathfrak{b}}\right|^4 \mathcal{L}(x,y)\,.$$

Thus, seeking for minimizers of the causal action, one must

$$\text{minimize} \qquad \sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4. \tag{8.6}$$

We would like to show that the minimizers of this functional are precisely the subsystem-diagonal potentials. However, the situation is not quite so simple, as the following counter example shows:

Example 8.1. Choose $L = 2$ and consider the one-parameter group of unitary matrices $(U_t)_{t\in\mathbb{R}}$

$$U_t = \exp\left(\frac{it}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right).$$

Using that the matrix in the exponent is twice a projection operator, a short computation yields

$$U_t = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{e^{it}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Thus

$$U_t\, v = e^{it}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence the relations

$$\left|(U_t\, v)^{\mathfrak{a}}\right| = 1 \qquad \text{for all } \mathfrak{a} = 1, 2$$

hold, although the unitary operators U_t are not diagonal. This shows that the diagonal unitary matrices cannot be singled out by minimizing (8.6).

We now enter the general analysis. Given a compact connected Lie subgroup $\mathcal{G} \subset \mathrm{U}(L)$, we set

$$\mathcal{G}v := \{Uv \,|\, U \in \mathcal{G}\} \subset \mathbb{C}^L .$$

Moreover, we introduce the *diagonal* and *orthogonal subgroups* by

$$\mathcal{G}^{\mathrm{d}} = \left\{ U \in \mathcal{G} \,\middle|\, U = \left(e^{i\varphi_1}, \ldots, e^{i\varphi_L}\right) \text{ with } \varphi_{\mathfrak{a}} \in \mathbb{R} \right\} \tag{8.7}$$

$$\mathcal{G}^{\perp} = \left\{ U \in \mathcal{G} \,\middle|\, U|_{\mathcal{G}v} = \mathbb{1}_{\mathcal{G}v} \right\}. \tag{8.8}$$

The vector v is called *cyclic* if $\mathcal{G}v = \mathbb{C}^L$. Clearly, if v is cyclic, then $\mathcal{G}^{\perp}$ is trivial.

Proposition 8.2. *The infimum of the functional in* (8.6) *is given by*

$$\inf_{U \in \mathcal{G}} \sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4 = \inf_{U \in \mathrm{U}(L)} \sum_{\mathfrak{a}=1}^{L} \left|(Uv)^{\mathfrak{a}}\right|^4 = L . \tag{8.9}$$

Moreover, if this functional is minimal on all of $\mathcal{G}$, i.e.

$$\sum_{\mathfrak{a}=1^L} \left|(Uv)^{\mathfrak{a}}\right|^4 = L \qquad \text{for all } U \in \mathcal{G} ,$$

then every $U \in \mathcal{G}$ has a unique decomposition into a diagonal and an orthogonal element,

$$U = U^{\mathrm{d}}\, U^{\perp} \qquad \text{with} \quad U^{\mathrm{d}} \in \mathcal{G}^{\mathrm{d}} \text{ and } U^{\perp} \in \mathcal{G}^{\perp} . \tag{8.10}$$

Before giving the proof, we explain what this result means. Generally speaking, this proposition gives strong constraints for the form of the subsystem-mixing gauge potentials. Indeed, such potentials may be nontrivial only if the vector v is not cyclic. But the vector v will be cyclic whenever each subsystem has its own dynamics. Namely, in this case, the subsystem-diagonal gauge potentials will be different in each subsystems, giving rise to different U(1)-phases in each subsystem. As a consequence, the group $\mathcal{G}$ will contain the abelian subgroup of all diagonal unitary matrices, implying that v is cyclic.

Proof of Proposition 8.2. We first prove (8.9). Since the rows of a unitary matrix are unit vectors, we know that

$$\sum_{\mathfrak{b}=1}^{L} |U^{\mathfrak{a}}_{\mathfrak{b}}|^2 = 1\,.$$

As a consequence, using the Schwarz inequality,

$$L = \sum_{\mathfrak{a},\mathfrak{b}=1}^{L} \big|U^{\mathfrak{a}}_{\mathfrak{b}}\big|^2 = \sum_{\mathfrak{a}=1}^{L} \big|(Uv)^{\mathfrak{a}}\big|^2 \leq \sqrt{L} \left(\sum_{\mathfrak{a}=1}^{L} \big|(Uv)^{\mathfrak{a}}\big|^4 \right)^{\frac{1}{2}}, \tag{8.11}$$

implying that

$$\sum_{\mathfrak{a}=1}^{L} \big|(Uv)^{\mathfrak{a}}\big|^4 \geq L\,.$$

Equality is attained in the case $U = \mathbb{1}$, proving (8.9). More generally, equality holds if and only if all the summands in (8.11) coincide, i.e.

$$\big|(Uv)^{\mathfrak{a}}\big| = 1 \qquad \text{for all } \mathfrak{a} = 1, \ldots, L\,. \tag{8.12}$$

Next, we prove uniqueness of the decomposition (8.10). Suppose that a unitary operator U has the representation (8.10). Then, using (8.8), we know that $Uv = U^{\mathrm{d}} v$. This relation uniquely determines all the phases $\varphi_1, \ldots, \varphi_L$ in (8.7). Hence U^{d} is unique, which also determines $U^{\perp}$ uniquely by $U^{\perp} = (U^{\mathrm{d}})^{-1} U$.

It remains to construct the decomposition (8.10). Let $A \in \mathfrak{g} \subset \mathfrak{u}(L)$ be a vector of the Lie algebra of $\mathcal{G}$. Then (8.12) implies that for any vector $w \in \mathcal{G}v$, the equation

$$\big|(e^{itA} w)^{\mathfrak{a}}\big| = 1 \qquad \text{holds for all } t \in \mathbb{R} \text{ and all } \mathfrak{a} = 1, \ldots, L\,.$$

Employing a spectral decomposition of the Hermitian matrix A,

$$A = \sum_{k=1}^{K} \lambda_k\, E_k\,, \qquad e^{itA} = \sum_{k=1}^{K} e^{i\lambda_k t}\, E_k\,,$$

we obtain

$$1 = \big|(e^{itA} w)^{\mathfrak{a}}\big|^2 = \sum_{k,k'=1}^{K} e^{i(\lambda_k - \lambda_{k'})t}\, \overline{\big(E_{k'} w\big)^{\mathfrak{a}}} \big(E_k w\big)^{\mathfrak{a}}\,. \tag{8.13}$$

We want to conclude that at most one summand is non-zero, i.e.

$$\big(E_k w\big)^{\mathfrak{a}} = 0 \qquad \text{for all } k \neq \ell$$

and a suitable $\ell = \ell(\mathfrak{a}, w)$. To this end, assume conversely that $(E_k w)^{\mathfrak{a}}$ and $(E_{k'} w)^{\mathfrak{a}}$ are both non-zero for $k \neq k'$. We choose k and k' such that $\lambda_k - \lambda_{k'}$ is maximal. Then the right side of (8.13) involves non-zero Fourier terms $\sim e^{\pm i(\lambda_k - \lambda'_k)t}$, a contradiction.

Let us show that ℓ can be chosen independent of w. We proceed indirectly and assume that $k := \ell(\mathfrak{a}, w_1) \neq \ell(\mathfrak{a}, w_2) =: k'$. Then, evaluating (8.13) for $w = w_1 + w_2$, one gets a non-zero contribution

$$\text{Re}\left(e^{i(\lambda_k - \lambda_{k'})t}\, \overline{\big(E_{k'} w_2\big)^{\mathfrak{a}}} \big(E_k w_1\big)^{\mathfrak{a}} \right).$$

Varying the phase of w_2, one again gets a contradiction. We conclude that

$$\big(E_k w\big)^{\mathfrak{a}} = 0 \qquad \text{for all } k \neq \ell(\mathfrak{a}) \text{ and all } w \in \mathcal{G}v\,. \tag{8.14}$$

Using the completeness of the spectral projectors, we obtain

$$w^{\mathfrak{a}} = \sum_{j=1}^{L} \big(E_j w\big)^{\mathfrak{a}} = \sum_{j=\ell(\mathfrak{a})} \big(E_j w\big)^{\mathfrak{a}} = \big(E_{\ell(\mathfrak{a})} w\big)^{\mathfrak{a}}\,.$$

Combining this relation with (8.14), it follows that

$$(E_k w)^{\mathfrak{a}} = \delta_{k, \ell(\mathfrak{a})}\, w^{\mathfrak{a}}\,.$$

Since $w \in \mathcal{G}v$ is arbitrary, we can also write this relation as

$$E_k\big|_{\mathcal{G}v} = E_k^{\mathrm{d}}\big|_{\mathcal{G}v} \qquad \text{with} \quad \big(E_k^{\mathrm{d}}\big)_{\mathfrak{b}}^{\mathfrak{a}} = \delta_{\mathfrak{b}}^{\mathfrak{a}}\, \delta_{k, \ell(\mathfrak{a})}\,.$$

As a consequence, the matrix

$$U^{\mathrm{d}} := \sum_{k=1}^{K} e^{i\lambda_k t}\, E_k^{\mathrm{d}}$$

is diagonal. Moreover, the matrix $U^{\perp} := (U^{\mathrm{d}})^{-1} U$ is trivial on $\mathcal{G}v$, giving the desired decomposition (8.10). □

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