Magnetic curves in the real special linear group

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We investigate contact magnetic curves in the real special linear group of degree 2. They are geodesics of the Hopf tubes over the projection curve. We prove that periodic contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$ can be quantized in the set of rational numbers. Finally, we study contact homogeneous magnetic trajectories in $\mathrm{SL}_2\mathbb{R}$ and show that they project to horocycles in $\mathbb{H}^2(-4)$.

1. Introduction

A magnetic field on a Riemannian manifold is defined by a closed 2-form. This definition comes from the fact that a closed 2-form on a Riemannian manifold can be regarded as a generalization of static magnetic fields on a Euclidean 3-space. See e.g. [8, 30]. A magnetic curve is a trajectory of a magnetic field and it is a solution of a second order differential equation known as

the *Lorentz equation* associated to the magnetic field. Lorentz equation generalizes the equation of geodesics under arc length parametrization. Hence, we may say that magnetic trajectories are perturbations of geodesics. On the other hand, magnetic curves derive also from the variational problem of the Landau-Hall functional. See e.g. [5]. In the absence of a magnetic field, this functional is nothing but the kinetic energy functional. It is well known that geodesics are critical points of the energy functional. This is another argument for saying that magnetic trajectories are generalizations of geodesics. In this sense, the geometric properties of magnetic curves show features of the underlying manifold, exactly how geodesics do.

The relation between geometry and magnetic fields have a long history. As is well known, the notion of linking number can be traced back to Gauss' work on terrestrial magnetism (see [27]). The linking number connects topology and Ampère's law in magnetism. De Turck and Gluck studied magnetic curves and linking numbers in the 3-sphere $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$ [9, 10].

On the other hand, contact structures play a important role in 3-dimensional topology. In [11], we have studied magnetic trajectories in Sasakian manifolds with respect to the magnetic field derived from the contact structure (*contact magnetic field*). Even that we employ physical terms, when we study contact magnetic curves, only we need is to get perturbations of geodesics obtained from the (almost) contact structure on the manifold. For readers who are not familiar with magnetic fields, it is enough to consider that these trajectories are special curves obtained as solutions of the Lorentz equation, which generalizes the equation of geodesics.

In 2007, Taubes [31] proved the generalized Weinstein conjecture in dimension 3, namely, *on a compact, orientable, contact 3-manifold the Reeb vector field ξ has at least one closed integral curve.* Linked to this problem it is important to investigate the existence of periodic magnetic trajectories of the contact magnetic field defined by ξ in Sasakian manifolds, in particular in Sasakian space forms.

In 2009, Cabrerizo et al. [7] have been looked for periodic orbits of the contact magnetic field on the unit sphere $\mathbb{S}^3$. See also [2]. The present authors [18] studied periodicity of contact magnetic trajectories on the 3-dimensional Berger sphere equipped with the canonical homogeneous Sasakian structure of constant φ-sectional curvature $c > -3$. The Berger 3-sphere $\mathcal{M}^3(c)$ equipped with Sasakian structure of constant φ-sectional curvature c has the structure of a principal circle bundle. The base space of this fibering is the 2-sphere $\mathbb{S}^2(c+3)$ of curvature $c+3$. This fibering includes the classical Hopf fibering $\mathbb{S}^3(1) \to \mathbb{S}^2(4)$.

There are three classes of 3-dimensional simply connected Sasakian space forms of constant φ-sectional curvature c:

- the Berger 3-sphere if $c > -3$; in particular the unit 3-sphere $\mathbb{S}^3(1)$ if $c = 1$;

- the Heisenberg group Nil_3 if $c = -3$;

- the universal covering of $\mathrm{SL}_2\mathbb{R}$ if $c < -3$.

Note also that these spaces (with $c=1$, $c\leq-3$) are model spaces of Thurston geometries. As a *hyperbolic counterpart* of the Berger 3-sphere, the special linear group $\mathrm{SL}_2\mathbb{R}$ admits a structure of principal circle bundle over the hyperbolic 2-space $\mathbb{H}^2(c+3)$ of curvature $c + 3 < 0$. The special linear group equipped with canonical left invariant Sasakian structure has constant φ-sectional curvature c. This paper is a continuation of previous papers [11, 18]. Our aim is to study periodicity of contact magnetic trajectories of $\mathrm{SL}_2\mathbb{R}$.

This paper is structured as follows. Firstly, considering the Hopf fibering from $\mathrm{SL}_2\mathbb{R}$ to $\mathbb{H}^2(-4)$, we show that contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$ are geodesics of the Hopf tubes over the projection curve. Then, we write the differential equations satisfied by the magnetic trajectories in $\mathrm{SL}_2\mathbb{R}$. The key of this part is the use of Iwasawa decomposition. In the following, we find a periodicity condition for contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$. We show that periodic magnetic curves in $\mathrm{SL}_2\mathbb{R}$ can be quantized in the set of rational numbers. Here we emphasize that periodic contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$ are knots in a solid torus. Knots in solid tori have been paid attention of knots researchers, see e.g., [13]. For torus knots in ideal magnetohydrodynamics, see [25]. Periodic magnetic curves in $\mathrm{SL}_2\mathbb{R}$ provides nice examples of torus knots.

We pay a special attention on Legendre curves, that is those curves whose contact angle is $\pi/2$. It should be remarked that the notion of Legendre curve only depends on the contact structure of $\mathrm{SL}_2\mathbb{R}$.

Finally, we are also interested in the study of magnetic trajectories in $\mathrm{SL}_2\mathbb{R}$, which project to horocycles in $\mathbb{H}^2(-4)$. Thus, in Section 6, we study homogeneous magnetic trajectories in $\mathrm{SL}_2\mathbb{R}$, that is contact magnetic curves which are obtained from one-parameter subgroups $\exp(tX)$, for $X \in \mathfrak{sl}_2\mathbb{R}$.

2. Preliminaries

2.1. Magnetic curves

The motion of the charged particles in a Riemannian manifold under the action of the magnetic fields are known as magnetic curves. More precisely, a *magnetic field* F on a Riemannian manifold (M, g) is a closed 2-form F and the *Lorentz force* associated to F is a tensor field ϕ of type $(1, 1)$ such that

$$(2.1) \qquad F(X, Y) = g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

A curve γ on M that satisfies the *Lorentz equation*

$$(2.2) \qquad \nabla_{\dot\gamma} \dot\gamma = \phi(\dot\gamma),$$

is called *magnetic trajectory* of F or simply a *magnetic curve*. Here ∇ denotes the Levi-Civita connection associated to the metric g. A magnetic field F is said to be *uniform* if $\nabla F = 0$.

It is well-known that the magnetic trajectories have constant speed. When the magnetic curve $\gamma(s)$ is arc length parametrized, it is called a *normal magnetic curve.*

The dimension 3 is rather special, since it allows us to identify 2-forms with vector fields via the Hodge $\star$ operator and the volume form dv_g of the (oriented) manifold. In this way, magnetic fields may be identified with divergence free vector fields by

$$F_V = \iota_V dv_g.$$

Magnetic fields F corresponding to Killing vector fields are usually known as *Killing magnetic fields*. Their trajectories, called *Killing magnetic curves*, are of great importance since they are related to the Kirchhoff elastic rods. See *e.g.*, [3, 4].

2.2. Sasakian manifolds

A (φ, ξ, η) *structure* on a manifold M is defined by a field φ of endomorphisms of tangent spaces, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{I} + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

If (M, φ, ξ, η) admits a compatible Riemannian metric g, namely

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \text{ for all } X, Y \in \mathfrak{X}(M),$$

then M is said to have an *almost contact metric structure*, and $(M, \varphi, \xi, \eta, g)$ is called an *almost contact metric manifold*. Consequently, we have that ξ is unitary and $\eta(X) = g(\xi, X)$, for any $X \in \mathfrak{X}(M)$.

We define a 2-form Ω on $(M, \varphi, \xi, \eta, g)$ by

(2.3) $$\Omega(X, Y) = g(\varphi X, Y), \text{ for all } X, Y \in \mathfrak{X}(M),$$

called *the fundamental 2-form* of the almost contact metric structure (φ, ξ, η, g).

If $\Omega = d\eta$, then $(M, \varphi, \xi, \eta, g)$ is called a *contact metric manifold*. Here $d\eta$ is defined by $d\eta(X, Y) = \frac{1}{2}\big(X\eta(Y) - Y\eta(X) - \eta([X, Y])\big)$, for any $X, Y \in \mathfrak{X}(M)$. On a contact metric manifold M, the 1-form η is a contact form (see Blair's book [6]). The vector field ξ is called the *Reeb vector field* of M and it is characterized by $\iota_\xi \eta = 1$ and $\iota_\xi d\eta = 0$. Here ι denotes the interior product. In analytical mechanics, ξ is traditionally called the *characteristic vector field* of M.

An almost contact metric manifold M is said to be *normal* if the normality tensor $S(X, Y) = N_\varphi(X, Y) + 2d\eta(X, Y)\xi$ vanishes, where N_φ is the *Nijenhuis torsion* of φ defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],$$

for any $X, Y \in \mathfrak{X}(M)$.

A *Sasakian manifold* is defined as a normal contact metric manifold. Denoting by ∇ the Levi-Civita connection associated to g, the Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is characterized by

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \text{ for any } X, Y \in \mathfrak{X}(M).$$

As a consequence, we have

(2.4) $$\nabla_X \xi = \varphi X, \ \forall X \in \mathfrak{X}(M).$$

A contact metric structure (φ, ξ, η, g) is called *K-contact* if ξ is a Killing vector field. Due to (2.4) and the fact that φ is skew-symmetric, it follows that a Sasakian manifold is *K*-contact. The converse is not true in general. Yet, a 3-dimensional manifold is Sasakian if and only if it is *K*-contact.

A plane section Π at $p \in M^{2n+1}$ is called a φ-section if it is invariant under φ_p. The sectional curvature $K(\Pi)$ of a φ-section is called the φ-*sectional curvature* of M^{2n+1} at p. A Sasakian manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said to be a *Sasakian space form* if it has constant φ-sectional curvature.

Take a positive constant a and define a new Sasakian structure $(\varphi, \hat{\xi}, \hat{\eta}, \hat{g})$ on M by

$$\hat{\xi} := \frac{1}{a}\xi, \quad \hat{\eta} := a\eta, \quad \hat{g} := ag + a(a-1)\eta \otimes \eta.$$

This structure is called a *D-homothetic deformation* of (φ, ξ, η, g). In particular, if $M(c)$ is a Sasakian space form of constant φ-sectional curvature c, then deforming the structure we obtain also a Sasakian space form $M(\hat{c})$, where $\hat{c} = \frac{c+3}{a} - 3$. For every value of c there exists Sasakian space forms, as follows: the elliptic Sasakian space forms, also known as the *Berger spheres* if $c > -3$, the *Heisenberg space* $\mathbb{R}^{2n+1}(-3)$, if $c = -3$, and $B^{2n} \times \mathbb{R}$ when $c < -3$. See also [6, Theorem 7.15]. Note that the case $c > -3$ includes the standard unit sphere $\mathbb{S}^{2n+1}(1)$.

Example 2.1. Let us identify the complex ball B^2 of curvature $-c^2$ ($c > 0$) with the upper half plane

$$\mathbb{H}^2(-c^2) = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

equipped with the Poincaré metric $\bar{g} = (dx^2 + dy^2)/(c^2 y^2)$ of constant curvature $-c^2$. Then we have a global orthonormal frame field

$$cy\frac{\partial}{\partial x}, \quad cy\frac{\partial}{\partial y}.$$

The standard complex structure J of $\mathbb{H}^2(-4)$ is defined by

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

Then $\mathbb{H}^2(-c^2) = (\mathbb{H}^2(-c^2), J)$ is a Kähler manifold.

The Kähler form $\bar{\Omega}$ of $\mathbb{H}^2(-c^2)$ is defined by

$$\bar{\Omega}(X, Y) = \bar{g}(JX, Y).$$

Define the one-form ω on $\mathbb{H}^2(-c^2)$ by $\omega = 2dx/(c^2 y)$ then the Kähler form of $\mathbb{H}^2(-c^2)$ is $d\omega$.

On the product manifold $\mathbb{H}^2(-c^2) \times \mathbb{R}$, we equip the contact metric structure (φ, ξ, η, g) by

$$\eta = dt + \pi^*\omega = dt + \frac{2dx}{c^2 y}, \quad g = \pi^* g_B + \eta \otimes \eta = \frac{dx^2 + dy^2}{c^2 y^2} + \left(dt + \frac{2dx}{c^2 y}\right)^2,$$

$$\xi = \frac{\partial}{\partial t}, \quad \varphi \frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad \varphi \frac{\partial}{\partial y} = -\frac{\partial}{\partial x} + \frac{2}{c^2 y} \frac{\partial}{\partial t}, \quad \varphi \frac{\partial}{\partial t} = 0.$$

Then the resulting Sasakian manifold is a simply connected Sasakian space form of constant φ-sectional curvature $-c^2 - 3$.

Note that this Sasakian space form is the universal covering of $\mathrm{SL}_2\mathbb{R}$.

Remark 2.1. On the product manifold $\mathbb{H}^2(-c^2) \times \mathbb{R}$, we may consider the following one-parameter family of Riemannian metrics:

$$g_\nu := \frac{dx^2 + dy^2}{c^2 y^2} + \left(dt + \frac{\nu \, dx}{c^2 y}\right)^2, \quad \nu \in \mathbb{R}, \quad \nu \geq 0.$$

For $\nu = 2$, we recover the metric of the Sasakian space form of constant φ-sectional curvature $-c^2 - 3$. On the other hand, for $\nu = 0$, we obtain the *Riemannian product* $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$. Moreover, when $\nu = c^2$, we obtain the *Sasaki-lift metric* of the universal covering of the unit tangent sphere bundle $\mathrm{U}\mathbb{H}^2(-c^2)$ of $\mathbb{H}^2(-c^2)$.

2.3. Magnetic curves in Sasakian manifolds

Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold and let Ω be the fundamental 2-form defined by (2.3). Since $\Omega = d\eta$ on a contact metric manifold, Ω is a closed 2-form, thus we can define a magnetic field on M by

$$F_q(X, Y) = q\Omega(X, Y),$$

where $X, Y \in \mathfrak{X}(M)$ and q is a real constant. We call F_q the *contact magnetic field* with the *strength* q. Notice that if $q = 0$, then the contact magnetic field vanishes identically and the magnetic curves are the geodesics of M. In the sequel we assume $q \neq 0$.

The *Lorentz force* ϕ_q associated to the contact magnetic field F_q may be easily determined combining (2.3) and (2.1), namely

$$\phi_q = q\varphi,$$

where φ is the field of endomorphisms of the contact metric structure.

In this setting, the Lorentz equation (2.2) can be written as

$$(2.5) \qquad\qquad \nabla_{\gamma'}\gamma' = q\varphi\gamma',$$

where $\gamma : I \subseteq \mathbb{R} \to M^{2n+1}$ is a smooth curve parametrized by arc length. The solutions of (2.5) are called *normal magnetic curves* or *trajectories* for F_q.

2.4. Contact magnetic curves in 3-dimensional Sasakian manifolds

Now we assume that M is a 3-dimensional Sasakian manifold. A curve $\gamma(u)$, parametrized by arclength, is said to be *slant* if it makes constant angle with the Reeb vector flow, that is the *contact angle* $\sigma(u)$, defined by $\cos \sigma(u) = g(\gamma'(u), \xi_{\gamma(u)})$, is constant along γ.

Proposition 2.1 ([11], [17]). *Let M be a 3-dimensional Sasakian manifold. Then a contact magnetic curve $\gamma(u)$, parametrized by arclength, is a slant helix with first curvature $\kappa_1 = |q| \sin \sigma$ and second curvature $\kappa_2 = |q \cos \sigma - 1|$. The principal normal N and binormal B are given by*

$$N = \frac{\varepsilon}{|\sin\sigma|} \varphi\gamma', \quad B = \frac{\varepsilon}{|\sin\sigma|} (\xi - \cos\sigma\,\gamma').$$

Here ε is the signature of q.

3. The special linear group

3.1. Iwasawa decomposition

Let $\mathrm{SL}_2\mathbb{R}$ be the real special linear group of degree 2:

$$\mathrm{SL}_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; \middle| \; a,b,c,d \in \mathbb{R}, \ ad - bc = 1 \right\}.$$

By using the Iwasawa decomposition $\mathrm{SL}_2\mathbb{R} = NAK$ of $\mathrm{SL}_2\mathbb{R}$;

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \; \middle| \; x \in \mathbb{R} \right\}, \qquad \text{(Nilpotent part)}$$

$$A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \; \middle| \; y > 0 \right\}, \qquad \text{(Abelian part)}$$

$$K = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \; \middle| \; 0 \le \theta < 2\pi \right\} = \mathrm{SO}(2), \qquad \text{(Maximal torus)}$$

we can introduce the following global coordinate system (x, y, θ) of $\mathrm{SL}_2\mathbb{R}$:

$$
(3.1) \qquad (x, y, \theta) \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.
$$

The mapping

$$
\psi : \mathbb{H}^2(-4) \times \mathbb{S}^1 \to \mathrm{SL}_2\mathbb{R};
$$
$$
\psi(x, y, \theta) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
$$

is a diffeomorphism onto $\mathrm{SL}_2\mathbb{R}$. Hereafter, we shall refer (x, y, θ) as a global coordinate system of $\mathrm{SL}_2\mathbb{R}$. Hence $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^1$ and hence diffeomorphic to $\mathbb{R}^3 \setminus \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}^+$ is diffeomorphic to open unit disk $\mathbb{D}$, $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to open solid torus $\mathbb{D} \times \mathbb{S}^1$.

Proposition 3.1. *The Iwasawa decomposition of an element $p = (p_{ij}) \in \mathrm{SL}_2\mathbb{R}$ is given explicitly by $p = n(p)a(p)k(p)$, where*

$$
n(p) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a(p) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}, \quad k(p) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
$$

with

$$
x = \frac{p_{11}p_{21} + p_{12}p_{22}}{(p_{21})^2 + (p_{22})^2}, \quad y = \frac{1}{(p_{21})^2 + (p_{22})^2}, \quad e^{i\theta} = \frac{p_{22} - ip_{21}}{\sqrt{(p_{21})^2 + (p_{22})^2}}.
$$

3.2. Left invariant vector fields

As is well known, the Lie algebra $\mathfrak{sl}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$ is given explicitly by

$$
\mathfrak{sl}_2\mathbb{R} = \left\{ X \in \mathrm{M}_2\mathbb{R} \;\middle|\; \mathrm{tr}\, X = 0 \right\}.
$$

We take the following basis of $\mathfrak{sl}_2\mathbb{R}$:

$$
E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

This basis satisfies the commutation relations:

$$
[E, F] = H, \quad [F, H] = 2F, \quad [H, E] = 2E.
$$

The Lie algebra $\mathfrak{n}$, $\mathfrak{a}$ and $\mathfrak{k}$ of closed groups N, A and K are given by

$$\mathfrak{n} = \mathbb{R}E, \quad \mathfrak{a} = \mathbb{R}H, \quad \mathfrak{k} = \mathbb{R}(E - F).$$

The left invariant vector fields obtained by left translating E, F and H are denoted by the same letter E, F and G, respectively. These left invariant vector fields are given by

$$E = y\cos(2\theta)\frac{\partial}{\partial x} + y\sin(2\theta)\frac{\partial}{\partial y} + \sin^2\theta\frac{\partial}{\partial\theta},$$
$$F = y\cos(2\theta)\frac{\partial}{\partial x} + y\sin(2\theta)\frac{\partial}{\partial y} - \cos^2\theta\frac{\partial}{\partial\theta},$$
$$H = -2y\sin(2\theta)\frac{\partial}{\partial x} + 2y\cos(2\theta)\frac{\partial}{\partial y} + \sin(2\theta)\frac{\partial}{\partial\theta}.$$

One notices that

$$\frac{\partial}{\partial\theta} = E - F$$

is left invariant. On the other hand we have

$$E + F = \cos(2\theta)\left(2y\frac{\partial}{\partial x} - \frac{\partial}{\partial\theta}\right) + \sin(2\theta)\left(2y\frac{\partial}{\partial y}\right),$$
$$H = -\sin(2\theta)\left(2y\frac{\partial}{\partial x} - \frac{\partial}{\partial\theta}\right) + \cos(2\theta)\left(2y\frac{\partial}{\partial y}\right).$$

Here we introduce a frame field $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ by

(3.2) $\epsilon_1 = 2y\frac{\partial}{\partial x} - \frac{\partial}{\partial\theta}, \quad \epsilon_2 = 2y\frac{\partial}{\partial y}, \quad \epsilon_3 = \frac{\partial}{\partial\theta}.$

This frame field is related to $\{E + F, H, E - F\}$ by

$$(E + F, H, E - F) = (\epsilon_1, \epsilon_2, \epsilon_3)\begin{pmatrix} \cos(2\theta) & -\sin(2\theta) & 0 \\ \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 3.1. The Lie algebra $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}_2\mathbb{R}$. Moreover $\mathfrak{n}$ and $\mathbb{R}F$ are root spaces with respect to $\mathfrak{h}$. The decomposition $\mathfrak{sl}_2\mathbb{R} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathbb{R}F$ is the root space decomposition (known also as Gauss decomposition) of $\mathfrak{sl}_2\mathbb{R}$.

3.3. Linear fractional transformations

The special linear group $\mathrm{SL}_2\mathbb{R}$ acts transitively and isometrically on the upper half plane:

$$\mathbb{H}^2(-4) = \left(\{(x,y) \in \mathbb{R}^2 \mid y > 0\}, \frac{dx^2 + dy^2}{4y^2} \right)$$

of constant curvature -4 by the linear fractional transformation as

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot z = \frac{az + b}{cz + d}.$$

Here we regard a point $(x, y) \in \mathbb{H}^2(-4)$ as a complex number $z = x + yi$.

Remark 3.2. A linear fractional transformation determined by a matrix $\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \neq \pm \mathrm{Id}$ with $ad - bc = 1$ is said to be

- *elliptic* if $|a + d| < 2$;
- *parabolic* if $|a + d| = 2$;
- *hyperbolic* if $|a + d| > 2$.

The isotropy subgroup of $\mathrm{SL}_2\mathbb{R}$ at $i = (0,1)$ is the rotation group $\mathrm{SO}(2)$. The natural projection $\pi : (\mathrm{SL}_2\mathbb{R}, g) \to \mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2) = \mathbb{H}^2(-4)$ is given explicitly by

$$\pi(x, y, \theta) = (x, y) \in \mathbb{H}^2(-4)$$

in terms of the global coordinate system (3.1).

The tangent space $T_i\mathbb{H}^2(-4)$ at the origin $i = (0,1)$ is identified with the vector subspace $\mathfrak{m}$ defined by

$$\mathfrak{m} = \{X \in \mathfrak{sl}_2\mathbb{R} \mid {}^t X = X\}.$$

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$ has the orthogonal splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This splitting can be carried out explicitly as

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}}, \quad X_{\mathfrak{k}} = \frac{1}{2}(X - {}^t X), \quad X_{\mathfrak{m}} = \frac{1}{2}(X + {}^t X).$$

3.4. Left invariant metrics

Define a one-parameter family of inner products $\{\langle\cdot,\cdot\rangle_\lambda \mid \lambda \in \mathbb{R}^*\}$ on $\mathfrak{sl}_2\mathbb{R}$ so that $\{E,\ F,\ H/\lambda\}$ is orthonormal with respect to $\langle\cdot,\cdot\rangle_\lambda$. By left-translating these inner products, we equip a one parameter family $\{g_\lambda\}$ of left invariant Riemannian metrics on $\mathrm{SL}_2\mathbb{R}$. This family $\{g_\lambda\}_{\lambda\in\mathbb{R}^*}$ is different from the family $\{g_\nu\}_{\nu\geq 0}$ in Remark 2.1. With respect to the global coordinate system (x,y,θ), each g_λ is expressed as

$$\frac{1}{2y^2}\begin{pmatrix} 2(\cos^4\theta + \sin^4\theta + \lambda^2\sin^2\theta\cos^2\theta) & (1-\lambda^2/2)\sin 2\theta\cos 2\theta & 2y \\ (1-\lambda^2/2)\sin 2\theta\cos 2\theta & \sin^2 2\theta + (\lambda^2/2)\cos^2 2\theta & 0 \\ 2y & 0 & 4y^2 \end{pmatrix}.$$

In particular the x and the y-coordinate curves are orthogonal if and only if $\lambda = \pm\sqrt{2}$.

The left invariant metric $g_{\sqrt{2}}$ is given by

$$g_{\sqrt{2}} = 2\left\{\frac{dx^2 + dy^2}{4y^2} + \left(d\theta + \frac{dx}{2y}\right)^2\right\}.$$

Here we would like to remark that one-forms

$$\frac{dx}{2y},\ \frac{dy}{2y},\ d\theta + \frac{dx}{2y}$$

are globally defined on $\mathrm{SL}_2\mathbb{R}$.

For simplicity we shall restrict our attention to $g_{\sqrt{2}}$. In addition, to adapt our computations to Sasakian geometry, we use the following homothetical change of $g_{\sqrt{2}}$.

$$g := \frac{1}{2}g_{\sqrt{2}} = \frac{dx^2 + dy^2}{4y^2} + \left(d\theta + \frac{dx}{2y}\right)^2.$$

It is easy to see that the projection $\pi : (\mathrm{SL}_2\mathbb{R}, g) \to \mathrm{SL}_2\mathbb{R}/\mathrm{SO}(2) = \mathbb{H}^2(-4)$ is a Riemannian submersion with totally geodesic fibres. This submersion $\pi : (\mathrm{SL}_2\mathbb{R}, g) \to \mathbb{H}^2(-4)$ is called the *hyperbolic Hopf fibering* of $\mathbb{H}^2(-4)$.

On the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$, the inner product $\langle \cdot, \cdot \rangle$ at the identity induced from g is written as

$$\langle X, Y \rangle = \frac{1}{2} \operatorname{tr} (^t XY), \quad X, Y \in \mathfrak{sl}_2\mathbb{R}.$$

By using this formula, we can see that the metric g is not only invariant by $\mathrm{SL}_2\mathbb{R}$-left translation but also right translations by $\mathrm{SO}(2)$. Hence the Lie group $\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)$ with multiplication:

$$(a, b)(a', b') = (aa', bb')$$

acts isometrically on $\mathrm{SL}_2\mathbb{R}$ via the action:

$$(\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)) \times \mathrm{SL}_2\mathbb{R} \to \mathrm{SL}_2\mathbb{R}; \quad (a, b) \cdot X = aXb^{-1}.$$

Furthermore, this action of $\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)$ on $\mathrm{SL}_2\mathbb{R}$ is transitive, hence $\mathrm{SL}_2\mathbb{R}$ is a Riemannian homogeneous space of $\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)$. The isotropy subgroup of $\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2)$ at the identity matrix Id is the diagonal subgroup

$$\Delta K = \{(k, k) \mid k \in K\} \cong K$$

of $K \times K$. The coset space $(\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}(2))/\mathrm{SO}(2)$ is a naturally reductive homogeneous space.

The tangent space $T_{\mathrm{Id}}\mathrm{SL}_2\mathbb{R}$ is the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R}$. This tangent space is identified with the vector subspace $\mathfrak{p}$ defined by

$$\mathfrak{p} = \{(V + W, 2W) \mid V \in \mathfrak{m}, \ W \in \mathfrak{k}\}.$$

The Lie algebra of the product group $G \times K$ is $\mathfrak{g} \oplus \mathfrak{k}$. On the other hand the Lie algebra of ΔK is

$$\Delta \mathfrak{k} = \{(W, W) \mid W \in \mathfrak{k}\} \cong \mathfrak{k}.$$

The Lie algebra $\mathfrak{g} \oplus \mathfrak{k}$ is decomposed as $\mathfrak{g} \oplus \mathfrak{k} = \Delta(\mathfrak{k}) \oplus \mathfrak{p}$.

Every $(X, Y) \in \mathfrak{g} \oplus \mathfrak{k}$ is decomposed as

$$(X, Y) = (2X_\mathfrak{k} - Y, 2X_\mathfrak{k} - Y) + (X_\mathfrak{m} + (Y - X_\mathfrak{k}), 2(Y - X_\mathfrak{k})).$$

3.5. Levi-Civita connection

We choose an orthonormal basis of $(\mathfrak{sl}_2\mathbb{R}, \langle \cdot, \cdot \rangle)$ by

$$E_1 = \sqrt{2}E, \ E_2 = \sqrt{2}F, \ E_3 = H.$$

Then the commutation relations are

$$[E_1, E_2] = 2E_3, \ [E_2, E_3] = 2E_2, \ [E_3, E_1] = 2E_1.$$

Let us denote the Levi-Civita connection of $(\mathrm{SL}_2\mathbb{R}, g)$ by ∇. By using the Koszul formula:

$$2\langle \nabla_X Y, Z \rangle = -\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle, \quad X, Y, Z \in \mathfrak{g},$$

one can obtain the following formulas:

$$\nabla_{E_1} E_1 = 2E_3, \ \nabla_{E_1} E_2 = E_3, \ \nabla_{E_1} E_3 = -2E_1 - E_2,$$
$$\nabla_{E_2} E_1 = -E_3, \ \nabla_{E_2} E_2 = -2E_3, \ \nabla_{E_2} E_3 = E_1 + 2E_2,$$
$$\nabla_{E_3} E_1 = -E_2, \ \nabla_{E_3} E_2 = E_1, \ \nabla_{E_3} E_3 = 0.$$

The bi-invariance obstruction U defined by

$$2\langle \mathsf{U}(X, Y), Z \rangle = -\langle X, [Y, Z] \rangle + \langle Y, [Z, X] \rangle, \ X, Y \in \mathfrak{g}$$

is given by

$$(3.3) \qquad \mathsf{U}(E_1, E_1) = 2E_3, \ \mathsf{U}(E_1, E_2) = 0, \ \mathsf{U}(E_1, E_3) = -E_1 - E_2,$$
$$\mathsf{U}(E_2, E_2) = -2E_3, \ \mathsf{U}(E_2, E_3) = E_1 + E_2, \ \mathsf{U}(E_3, E_3) = 0.$$

The Levi-Civita connection is rewritten as

$$(3.4) \qquad \nabla_X Y = \frac{1}{2}[X, Y] + \mathsf{U}(X, Y), \quad X, Y \in \mathfrak{g}.$$

3.6. Curvature

We take the following orthonormal coframe field of $\mathrm{SL}_2\mathbb{R}$:

$$\omega^1 = \frac{dx}{2y}, \quad \omega^2 = \frac{dy}{2y}, \quad \omega^3 = d\theta + \frac{dx}{2y}.$$

The dual frame field of $\{\omega^1, \omega^2, \omega^3\}$ is the frame field $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ introduced by (3.2). Note that this orthonormal frame field is *not* left invariant with respect to the Lie group structure.

The Levi-Civita connection ∇ of g is given by the following formulas:

$$\nabla_{\epsilon_1}\epsilon_1 = 2\epsilon_2, \quad \nabla_{\epsilon_1}\epsilon_2 = -2\epsilon_1 - \epsilon_3, \quad \nabla_{\epsilon_1}\epsilon_3 = \epsilon_2,$$
$$\nabla_{\epsilon_2}\epsilon_1 = \epsilon_3, \quad \nabla_{\epsilon_2}\epsilon_2 = 0, \quad \nabla_{\epsilon_2}\epsilon_3 = -\epsilon_1,$$
$$\nabla_{\epsilon_3}\epsilon_1 = \epsilon_2, \quad \nabla_{\epsilon_3}\epsilon_2 = -\epsilon_1, \quad \nabla_{\epsilon_3}\epsilon_3 = 0.$$

The commutation relations of the basis are given by

$$[\epsilon_1, \epsilon_2] = -2\epsilon_1 - 2\epsilon_3, \quad [\epsilon_1, \epsilon_3] = 0, \quad [\epsilon_2, \epsilon_3] = 0.$$

The Riemannian curvature tensor R of the metric g defined by

$$R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(\mathrm{SL}_2\mathbb{R})$$

is described by the following formulas:

$$\begin{aligned}
R(\epsilon_1, \epsilon_2)\epsilon_1 &= 7\epsilon_2, \quad & R(\epsilon_1, \epsilon_2)\epsilon_2 &= -7\epsilon_1, \\
R(\epsilon_1, \epsilon_3)\epsilon_1 &= -\epsilon_3, \quad & R(\epsilon_1, \epsilon_3)\epsilon_3 &= \epsilon_1, \\
R(\epsilon_2, \epsilon_3)\epsilon_2 &= -\epsilon_3, \quad & R(\epsilon_2, \epsilon_3)\epsilon_3 &= \epsilon_2.
\end{aligned}$$

The other significant components are zero.

3.7. Canonical Sasakian structure of $\mathrm{SL}_2\mathbb{R}$

The one-form $\eta = d\theta + dx/(2y)$ is a *contact form* on $\mathrm{SL}_2\mathbb{R}$, *i.e.*, $d\eta \wedge \eta \neq 0$. The Reeb vector field of η is $\xi = \epsilon_3$. The contact distribution determined by η coincides the horizontal distribution of the Riemannian submersion $\pi : G \to \mathbb{H}^2(-4)$.

Remark 3.3. Under the identification $\mathfrak{k} \cong \mathbb{R}$, the contact form η is regarded as a connection form of the principal circle bundle $\mathrm{SL}_2\mathbb{R} \to \mathbb{H}^2(-4)$.

Let us define an endomorphism field φ by

$$\varphi\,\epsilon_1 = \epsilon_2, \quad \varphi\,\epsilon_2 = -\epsilon_1, \quad \varphi\,\epsilon_3 = 0.$$

Then (φ, ξ, η, g) satisfies the following relations:

$$\varphi^2 = -I + \eta \otimes \xi, \quad d\eta(X,Y) = g(\varphi X, Y),$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$\nabla_X \xi = \varphi X,$$

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X$$

for all X, $Y \in \mathfrak{X}(\mathrm{SL}_2\mathbb{R})$. Thus the structure (φ, ξ, η) is an almost contact structure compatible to the metric g. In other words, structure (φ, ξ, η, g) is an almost contact metric structure associated to the the contact manifold $(\mathrm{SL}_2\mathbb{R}, \eta)$ [16]. Since all the structure tensor fields are left invariant, the resulting almost contact metric manifold $(\mathrm{SL}_2\mathbb{R}, \varphi, \xi, \eta, g)$ is a homogeneous Sasakian manifold of constant φ-sectional curvature -7. The structure (φ, ξ, η, g) is called the *canonical Sasakian structure* of $\mathrm{SL}_2\mathbb{R}$.

Remark 3.4. The Riemannian curvature tensor R of $(\mathrm{SL}_2\mathbb{R}, g)$ is given explicitly by

$$
\begin{aligned}
R(X, Y)Z = {}& -g(Y, Z)X + g(Z, X)Y \\
& - 2\{\eta(Z)\eta(X)Y - \eta(Y)\eta(Z)X \\
& + g(Z, X)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\
& - g(Y, \varphi Z)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\}
\end{aligned}
$$

in terms of the canonical Sasakian structure.

For more informations on the canonical Sasakian structure of $\mathrm{SL}_2\mathbb{R}$, we refer to [16].

4. Hopf tubes

4.1. Fundamental equations of Hopf tubes

Let $\pi : (SL_2\mathbb{R}, g) \to (H^2(-4), \bar{g})$ be the hyperbolic Hopf fibering. Consider a regular curve $\beta : \mathbb{R} \longrightarrow \mathbb{H}^2(-4)$, $u \mapsto \beta(u)$. As usual, β will be parametrized by the arc length and let $\hat{\beta}$ be a horizontal lift of β. This means that $\pi(\hat{\beta}(u)) = \beta(u)$ for all $u \in \mathbb{R}$ and $\langle \hat{\beta}(u)^{-1}\hat{\beta}'(u), \epsilon_3 \rangle = 0$. If we represent $\beta(u)$ as $\beta(u) = (x(u)), y(u))$, then the horizontal lift $\hat{\beta}(u)$ is given by $\hat{\beta}(u) = (x(u), y(u), \theta(u))$ whose third coordinate $\theta(u)$ is determined by the ordinary differential equation

$$\frac{d\theta}{du} = -\frac{1}{y(u)}\frac{dx}{du}$$

with initial condition $\theta(0) = \theta_0$.

The complete lift of β, namely $\pi^{-1}(\beta)$ is a flat surface in $\mathrm{SL}_2\mathbb{R}$ and it is usually called the *Hopf tube* over β.

Denote $\pi^{-1}(\beta)$ by H_β. The Hopf tube H_β is represented as an immersion:

$$F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathrm{SL}_2\mathbb{R}, \quad (t, u) \longmapsto F(t, u) = \hat{\beta}(u)k(t),$$

where

$$k(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

In other words,

$$F(t, u) = (x(u), y(u), \theta(u) + t).$$

The derivatives are

$$F_u = \frac{x'}{2y}\epsilon_1 + \frac{y'}{2y}\epsilon_2, \quad F_t = \epsilon_3.$$

Hence the induced metric g_{H_β} is computed as

$$g_{H_\beta} = dt^2 + du^2.$$

Let us compute the second fundamental form of H_β. Let $\{\overline{T}(u), \overline{N}(u)\}$ the Frenet frame field of $\beta(u)$. As usual

$$\overline{T}(u) = \beta'(u), \quad \overline{N}(u) = J\overline{T}(u).$$

The signed curvature κ_β is defined by

$$\nabla_{\beta'}\overline{T} = \kappa_\beta \overline{N}.$$

Let us denote by $\hat{T}$ the horizontal lift of $\overline{T}$. Then T tangents to H_β. Moreover $\{\hat{T}, \xi\}$ is an orthonormal frame field of H_β. The horizontal lift $\hat{N}$ of N is a unit normal vector field of H_β. One can check that $\hat{N} = \varphi\hat{T}$. The Gauss formula of H_β is given by [15, §1.3]:

$$\nabla_{\hat{T}}\hat{T} = (\kappa_\beta \circ \pi)\,\hat{N}, \quad \nabla_{\hat{T}}\xi = \nabla_\xi\hat{T} = \hat{N}, \quad \nabla_\xi\xi = 0.$$

These formula imply that H_β is flat and the second fundamental form h derived from $\hat{N}$ is

$$h(\hat{T}, \hat{T}) = \kappa_\beta \circ \pi, \quad h(\hat{T}, \xi) = 1, \quad h(\xi, \xi) = 0.$$

Thus, the mean curvature of H_β is $(\kappa_\beta \circ \pi)/2$.

Hence we have proved the following fact.

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Proposition 4.1. *If β is a curve on $\mathbb{H}^2(-4)$ of length L, then the corresponding Hopf tube H_β is isometric to $\mathbb{S}^1(1) \times [0, L]$, where $\mathbb{S}^1(1)$ is the unit circle endowed with the metric dt^2. Moreover, its mean curvature in $\mathrm{SL}_2\mathbb{R}$ is $(\kappa_\beta \circ \pi)/2$, where κ_β is the signed curvature of β in $\mathbb{H}^2(-4)$.*

If β is a closed curve, i.e., $\beta(u + L) = \beta(u)$ for all $u \in \mathbb{R}$, then the relation $F(t, u) = \hat{\beta}(u)k(t)$ defines a covering of the (t, u) plane onto an immersed torus in $\mathrm{SL}_2\mathbb{R}$, called the *Hopf torus* corresponding to β.

Remark 4.1. The Hopf tube H_β can be parametrized by the following immersion.

$$F_1(u, v) = (x(u), y(u), v), \quad u \in \mathbb{R}, \ v \in \mathbb{S}^1.$$

Then by using the Iwasawa decomposition, the Hopf tube over β can be parametrized as an immersion $F_1 : \mathbb{R} \times \mathbb{S}^1 \to \mathrm{SL}_2\mathbb{R}$:

$$(4.1) \quad F_1(u, v) = \begin{pmatrix} 1 & x(u) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y(u)} & 0 \\ 0 & 1/\sqrt{y(u)} \end{pmatrix} \begin{pmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{pmatrix}.$$

Under this parametization the induced metric is computed as:

$$\left(dv + \frac{x'(u)}{2y(u)} du \right)^2 + du^2.$$

One can check that the induced metric on H_β is flat. This parametization is used in [14, 22].

For later use we recall here the classification of Hopf tubes with constant mean curvature [22] (see also Appendix A).

Proposition 4.2 (Classification of CMC Hopf tubes). *Let β be a unit speed curve in $\mathbb{H}^2(-4)$ with curvature κ and H_β the Hopf tube over β in $\mathrm{SL}_2\mathbb{R}$. Then H_β is of constant mean curvature if and only if β is a Riemannian circle in $\mathbb{H}^2(-4)$.*
    The Hopf cylinder H_β is classified in the following way:

(1) *If $\kappa_\beta = 0$, then H_β is a minimal Hopf tube over a geodesic.*

(2) *If $0 < \kappa_\beta^2 < 4$, then H_β is a Hopf tube over an open circle or a Hopf tube over a line segment $y = \pm(\sqrt{1 - 4\kappa^2}/(2\kappa))x$.*

(3) *If $\kappa_\beta^2 = 4$, then H_β is a Hopf tube over a horocycle or a Hopf tube over $y = $ constant.*

(4) If $\kappa_\beta^2 > 4$, then H_β is a Hopf tube over a closed circle. In this case, H_β is an embedded Hopf torus.

4.2. Contact magnetic curves and Hopf tubes

We investigate the projection image of contact magnetic curves. First we recall the following fundamental fact.

Proposition 4.3. *Let $\gamma(u)$ be an arclength parametrized contact magnetic curve in a 3-dimensional Sasakian manifold $(M, \varphi, \xi, \eta, g)$. Then, the contact angle $\sigma(u)$, defined by $\cos \sigma(u) = g(\xi, \gamma'(u))$, is constant along γ.*

Contact magnetic curves are included in some Hopf tubes. Moreover, we have the following.

Theorem 4.1. *A contact magnetic curve γ in $\mathrm{SL}_2\mathbb{R}$ is a geodesic of the Hopf tube H_β over $\beta = \pi \circ \gamma$.*

Proof. Let $\gamma(u)$ be a contact magnetic curve with strength q parametrized by arclength u. Set $\beta = \pi \circ \gamma$ then γ is contained in the Hopf tube H_β. Remark that u is not, in general, the arclength parameter of β. Now let $\hat{N}$ the unit normal vector field along H_β as before, then we have $\pi_* \hat{N} = \overline{N}$. The Gauss-formula of H_β implies

$$(4.2) \qquad \nabla_{\gamma'} \gamma' = \dot{\nabla}_{\gamma'} \gamma' + h(\gamma', \gamma') \hat{N}.$$

Here $\dot{\nabla}$ is the Levi-Civita connection of H_β and h is as before.

Let us express γ as $\gamma(u) = (x(u), y(u), \theta(u))$, then the velocity vector field γ' is given by

$$(4.3) \qquad \gamma' = \frac{x'}{2y}\epsilon_1 + \frac{y'}{2y}\epsilon_2 + \eta(\gamma')\epsilon_3, \quad \eta(\gamma') = \theta' + \frac{x'}{2y}.$$

Thus we get

$$\varphi\gamma' = -\frac{y'}{2y}\epsilon_1 + \frac{x'}{2y}\epsilon_2.$$

On the other hand $\hat{N}$ is expressed as

$$\hat{N} = \frac{1}{\sqrt{(x')^2 + (y')^2}} \left(-y'\epsilon_1 + x'\epsilon_2 \right).$$

It follows that $q\varphi\gamma'$ is collinear to ν. Comparing with the magnetic equation $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$ and (4.2) we find $\dot{\nabla}_{\gamma'}\gamma' = 0$. $\qquad \square$

Let $\gamma(u)$ be an arclength parametrized contact magnetic curve in $\mathrm{SL}_2\mathbb{R}$, then the projection curve $\beta(u) = \pi(\gamma(u))$ has the velocity $\beta'(u) = \pi_*\gamma'$. Since π is a Riemannian submersion, we have

$$|\beta'(u)|^2 = |\gamma'(u)|^2 - \eta(\gamma')^2 = \sin^2 \sigma.$$

4.3. Kähler magnetic curves

Let us consider the magnetic curve in $\mathbb{H}^2(-4)$ with respect to the magnetic field $\bar{F}_q := \bar{q}\bar{\Omega}$. Here $\bar{\Omega}$ is the Kähler form of $\mathbb{H}^2(-4)$ defined by $\bar{\Omega} = (dx \wedge dy)/(2y^2)$ as in Example 2.1. The corresponding Lorentz equation is $\overline{\nabla}_{\beta'}\beta' = \bar{q}J\beta'$. Here $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{H}^2(-4)$. Comparing the Lorentz equation with the Frenet equation, we obtain that β is a Riemannian circle in $\mathbb{H}^2(-4)$ satisfying $\bar{q} = \kappa_\beta$. Hence normal magnetic trajectories are closed if and only if $|q| > 2$.

By the fundamental equation of Riemannian submersion one can check the following result.

Proposition 4.4. *The projection image $\beta(u) = \pi(\gamma(u))$ of a contact magnetic curve is a Kähler magnetic curve in $\mathbb{H}^2(-4)$. More precisely, β satisfies the Lorentz equation $\overline{\nabla}_{\beta'}\beta' = (q - 2\cos\sigma)J\beta'$. Hence $\gamma(u)$ is a geodesic in a Hopf tube over a Riemannian circle.*

5. Magnetic trajectories in $\mathrm{SL}_2\mathbb{R}$

In our previous paper [11], we have proved that the classification of contact magnetic curves in Sasakian space forms of arbitrary dimension reduces to that in 3-dimensional Sasakian space forms. More precisely, we have the following results.

Theorem 5.1. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a Sasakian manifold and consider F_q, $q \neq 0$, the contact magnetic field on M^{2n+1}. Then γ is a normal magnetic curve associated to F_q in M^{2n+1} if and only if γ belongs to the following list:*

    a) *geodesics, obtained as integral curves of ξ;*

    b) *non-geodesic φ-circles of curvature $\kappa_1 = \sqrt{q^2 - 1}$, for $|q| > 1$, and of constant contact angle $\sigma = \arccos\frac{1}{q}$;*

    c) *Legendre φ-curves in M^{2n+1} with curvatures $\kappa_1 = |q|$ and $\kappa_2 = 1$, i.e. 1-dimensional integral submanifolds of the contact distribution;*

d) φ-helices of order 3 with axis ξ, having curvatures $\kappa_1 = |q| \sin \sigma$ and $\kappa_2 = |q \cos \sigma - 1|$, where $\sigma \neq \frac{\pi}{2}$ is the constant contact angle.

Theorem 5.2 ([11]). *Let $\gamma : I \subseteq \mathbb{R} \to B^{2n}(-4) \times \mathbb{R}$, be a smooth curve parametrized by arclength s and let $F_q = q\Omega$, $q \neq 0$ be the contact magnetic field. Then γ is a normal magnetic curve associated to F_q if and only if it belongs to the following list:*

a) *a geodesic obtained as integral curve of ξ;*

b) *the horizontal lift of a magnetic trajectory in $B^2(-4)$ corresponding to the Kähler magnetic field $\bar{F} = \bar{q}\Omega$;*

c) *a helix in the 3-dimensional Sasakian space form $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ identified with $B^2(-4) \times \mathbb{R}$ as totally geodesic submanifold in $B^{2n}(-4) \times \mathbb{R}$. Moreover, γ is a geodesic on a Hopf tube over a curve of constant curvature in $\mathbb{H}^2(-4)$.*

Let us now take a contact magnetic curve $\gamma(s) = (x(s), y(s), \theta(s))$ in $\mathrm{SL}_2\mathbb{R}$. Then the velocity vector field is given by (4.3).

The acceleration vector field is computed as

$$\nabla_{\gamma'}\gamma' = \left\{ \frac{x''y - x'y'}{2y^2} - \frac{x'y'}{2y^2} - \frac{y'}{y}\eta(\gamma') \right\} \epsilon_1$$
$$+ \left\{ \frac{y''y - (y')^2}{2y^2} + \frac{(x')^2}{2y^2} + \frac{x'}{y}\eta(\gamma') \right\} \epsilon_2 + \{\eta(\gamma')\}'\epsilon_3.$$

The the magnetic equation $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$ with strength q is the following system:

$$\frac{x''y - x'y'}{2y^2} - \frac{x'y'}{2y^2} - \frac{y'}{y}\eta(\gamma') = -\frac{qy'}{2y},$$
$$\frac{y''y - (y')^2}{2y^2} + \frac{(x')^2}{2y^2} + \frac{x'}{y}\eta(\gamma') = \frac{qx'}{2y},$$
$$\{\eta(\gamma')\}' = \left[\theta' + \frac{x'}{2y} \right]' = 0.$$

The third equation confirms that γ is a *slant curve*, that is γ' makes constant angle σ with the Reeb vector field ξ. By definition of σ, we notice that

$\eta(\gamma') = \cos\sigma \in [-1, 1]$. The equations of magnetic trajectory become

$$\frac{x''y - x'y'}{2y^2} - \frac{x'y'}{2y^2} - (\cos\sigma)\frac{y'}{y} = -\frac{qy'}{2y},$$
$$\frac{y''y - (y')^2}{2y^2} + \frac{(x')^2}{2y^2} + (\cos\sigma)\frac{x'}{y} = \frac{qx'}{2y}.$$

Put

$$X = \frac{x'}{2y}, \quad Y = \frac{y'}{2y}.$$

Then we have $X^2 + Y^2 + \cos^2\sigma = 1$, which implies that $X^2 + Y^2 = \sin^2\sigma$.
    Moreover, we find

$$X' = \frac{x''y - x'y'}{2y^2}, \quad Y' = \frac{y''y - (y')^2}{2y^2}.$$

Hence, the equations of magnetic trajectory yield the system

(5.1)
$$\begin{cases} X' - Y(2X + 2\cos\sigma - q) = 0, \\ Y' + X(2X + 2\cos\sigma - q) = 0, \end{cases}$$

together with

(5.2)
$$\theta' + \frac{x'}{2y} = \cos\sigma.$$

It should be remarked that the system (5.1) is nothing but the Kähler magnetic curves in $\mathbb{H}^2(-4)$ with strength $\bar{q} = q - 2\cos\sigma$.

Example 5.1 (Reeb flows). According to item a) of Theorem 5.1, Reeb flows are magnetic curves. Choose $\theta = 0$ or π in the magnetic equations, we have $x(s) = $ constant and $y(s) = $ constant. The coordinate θ is determined by $\theta' = \pm 1$. Hence θ is an affine function of s.

Example 5.2 (Legendre φ-curves). According to item c) of Theorem 5.1, Legendre φ-curves with $\kappa_1 = |q|$ and $\kappa_2 = 1$ are magnetic curves. The magnetic curve $\gamma(s)$ is a horizontal lift of a Riemannian circle $\beta(s) = (x(s), y(s))$ with $|\kappa_\beta| = |q|$. The third coordinate $\theta(s)$ is determined by the horizontal

lift condition (Legendre condition):

$$\theta'(s) = -\frac{x'(s)}{2y(s)}$$

under the prescribed initial condition.

To look for periodic trajectories, we restrict our attention to horizontal lifts of closed Riemannian circles. See also [21, Example 5.5].

For $|\kappa_\beta| > 2$, $\beta(s)$ is a closed circle parametrized as (see Appendix A):

$$(x(s), y(s)) = \left(r\sin\mu(s) + x_0, r\left(\frac{|q|}{2} - \cos\mu(s)\right) \right),$$

where r is a positive constant and $\mu(s)$ is a solution of the following ODE

$$\mu'(s) = |q| - 2\cos\mu(s).$$

Under the initial condition $\mu(0) = 0$, the solution $\mu(s)$ is given explicitly by

$$\tan\frac{\mu(s)}{2} = \sqrt{\frac{|q| - 2}{|q| + 2}}\tan\frac{\sqrt{q^2 - 4}\,s}{2},$$

which implies

$$\sin\mu(s) = \frac{\sqrt{q^2 - 4}\sin(\sqrt{q^2 - 4}\,s)}{|q| + 2\cos(\sqrt{q^2 - 4}\,s)}, \quad \cos\mu(s) = \frac{2 + |q|\cos(\sqrt{q^2 - 4}\,s)}{|q| + 2\cos(\sqrt{q^2 - 4}\,s)}.$$

Thus $\beta(s)$ has the fundamental period $\mathsf{T} = 2\pi/\sqrt{q^2 - 4}$. The θ-coordinate is given by

$$\theta(s) = \frac{1}{2}\mu(s) - \frac{|q|}{2}s,$$

under the initial condition $\theta(0) = 0$.

The horizontal lift is closed if and only if there exists a positive integer m such that

$$\theta\left(s + \frac{2m\pi}{\sqrt{q^2 - 4}}\right) \equiv \theta(s) \quad \mod 2\pi.$$

Hence, the periodicity condition is equivalent to

$$|q| = \frac{2}{\sqrt{1 - (m/k)^2}}$$

for some relatively prime positive integers m and k satisfying $m/k < 1$. This is precisely the criterion found by Kajigaya in [21]. Thus there exist countably many closed Legendre magnetic curves in $\mathrm{SL}_2\mathbb{R}$.

In the following we draw some pictures for a better understanding of periodic Legendre magnetic curves in $\mathrm{SL}_2\mathbb{R}$.

From the previous computations we have

$$\mu(s) = 2\arctan\left(\sqrt{\frac{|q|-2}{|q|+2}}\,\tan\frac{\sqrt{q^2-4}\,s}{2}\right) + 2h\pi,$$

if $s \in \left(-\frac{\mathsf{T}}{2}\,,\,\frac{\mathsf{T}}{2}\right) + h\mathsf{T}$, where $h \in \mathbb{Z}$.

Fix the integers m and k as in the periodicity condition. We are looking now for a positive integer h such that $\theta\left(\frac{\mathsf{T}}{2} + h\mathsf{T}\right) \equiv \theta\left(-\frac{\mathsf{T}}{2}\right)$ (mod 2π). This means that γ has $(h+1)$ "branches" to be periodic. The condition is equivalent to $(h+1)\left(1 - \frac{k}{m}\right)$ is an even number.

In the following we give some examples and draw the corresponding pictures on $\mathrm{SL}_2\mathbb{R}$ thought as a solid torus $\mathbb{S}^1 \times \mathbb{D}^2$. The pictures are drawn up to a homothetic deformation of the circle $\mathbb{S}^1$. Here $\mathbb{D}^2$ is obtained from the Poincaré half plane $\mathbb{H}^2$ via the Cayley transformation

$$f : \mathbb{H}^2 \to \mathbb{D}^2, \ f(z) = \frac{z-i}{z+i}\,, \ \text{where } z \in \mathbb{C}, \ \Im m(z) > 0.$$

Every figure in the next three examples is composed by four images:

- the first one represents the curve β represented in the upper half-plane;

- the second one represents the same curve β in the unit disc $\mathbb{D}^2$;

- the last two pictures represent the same curve γ on the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$ from different viewpoints.
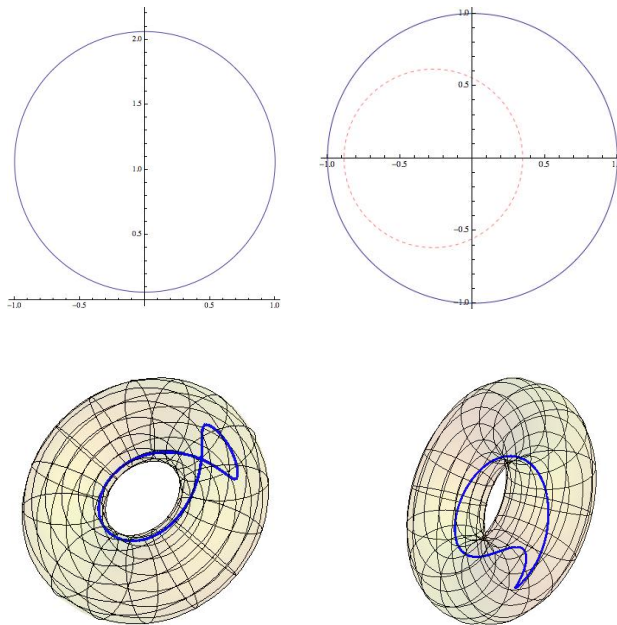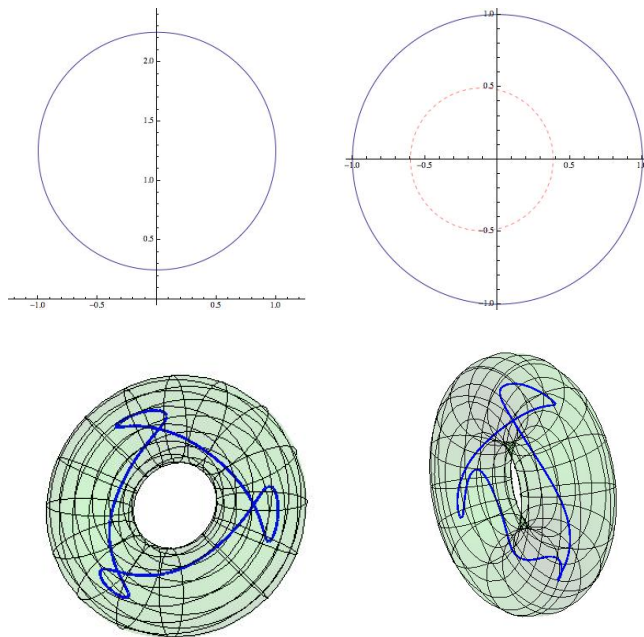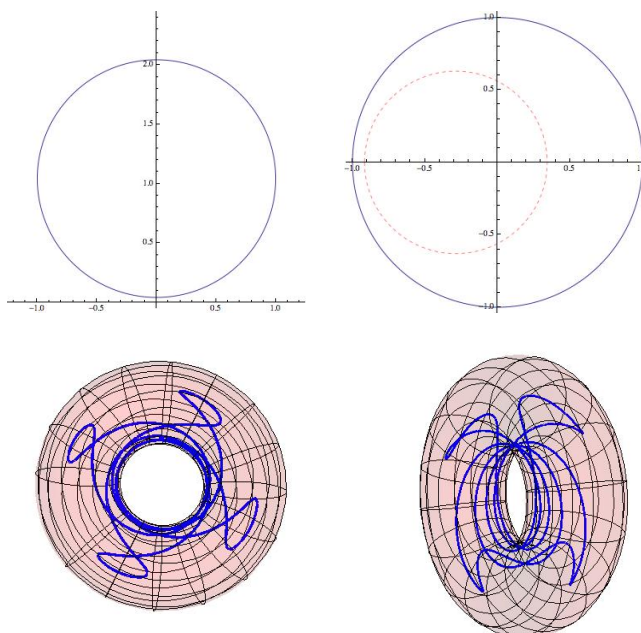
Figure 1: $m = 1$, $k = 3$, $h = 0$.

Figure 2: $m = 3$, $k = 5$, $h = 2$.

Figure 3: $m = 2$, $k = 7$, $h = 3$.

Our main interest is to classify periodic trajectories. Therefore, we consider magnetic curves whose projection images are closed circles. Nevertheless, we are also interested in other kinds of magnetic trajectories like contact magnetic trajectories over horocycles. This study will be done in the next section.

In the following we study periodicity of contact magnetic curves which are neither Reeb, nor Legendre. With this aim in view, we need to solve the system

$$X' - Y(2X - \bar{q}) = 0, \quad Y' + X(2X - \bar{q}) = 0, \quad \theta' + X = \cos\sigma,$$

where we put $\bar{q} := q - 2\cos\sigma$.

Since $X^2 + Y^2 = \sin^2\sigma$, we represent X and Y as

$$X = \sin\sigma \, \cos U, \quad Y = \sin\sigma \, \sin U,$$

for a certain function U. Then we have

$$X' = -(\sin\sigma \sin U)\, U', \quad Y' = (\sin\sigma \cos U)\, U',$$

The first equation of the system is

$$-(\sin \sigma \sin U)\, U' - \sin \sigma \sin U (2 \sin \sigma \cos U - \bar{q}) = 0$$

As γ is neither Reeb, nor Legendre, we assume that $\sin \sigma \neq 0$ and $\cos \sigma \neq 0$, so

$$\{U' + (2 \sin \sigma \cos U - \bar{q})\} \sin U = 0$$

The second equation becomes

$$\{U' + 2 \sin \sigma \cos U - \bar{q})\} \cos U = 0.$$

Hence we obtain

(5.3) $$U' + 2 \sin \sigma \cos U - \bar{q} = 0.$$

This ODE can be solved directly. With the new variable $t = \tan(U/2)$, the equation (5.3) can be rewritten as

$$2\frac{dt}{ds} = (\bar{q} + 2 \sin \sigma)t^2 + (\bar{q} - 2 \sin \sigma).$$

We have to distinguish several cases:
Case 1. $\bar{q} + 2 \sin \sigma = 0$
This is equivalent to $q = 2\sqrt{2} \sin(\sigma - \pi/4)$. Under the initial condition $U(0) = 0$, we get $t(s) = -2s \sin \sigma$. Thus we obtain

$$U(s) = -2 \arctan\left(2s \sin \sigma\right).$$

Case 2. $\bar{q} - 2 \sin \sigma = 0$
In this case, we have $q = 2\sqrt{2} \sin(\sigma + \pi/4)$ and $dt = (2 \sin \sigma)t^2\, ds$. The solution of this ODE with initial condition $U(0) = \pi/2$ is $t(s) = \frac{1}{1-2s \sin \sigma}$. Thus

$$U(s) = 2 \arctan \frac{1}{1 - 2s \sin \sigma}.$$

Case 3. $\bar{q}^2 - 4 \sin^2 \sigma > 0$
We need to solve the ODE: $\dfrac{dt}{ds} = \dfrac{\bar{q} + 2 \sin \sigma}{2}\left(t^2 + \dfrac{\bar{q} - 2 \sin \sigma}{\bar{q} + 2 \sin \sigma}\right)$, where $\frac{\bar{q}-2 \sin \sigma}{\bar{q}+2 \sin \sigma} > 0$. Solving this equation with the initial condition $U(0) = 0$, we

obtain

$$U(s) = 2 \arctan \left(\sqrt{\frac{\bar{q} - 2\sin\sigma}{\bar{q} + 2\sin\sigma}} \ \tan \frac{s \sqrt{\bar{q}^2 - 4\sin^2\sigma}}{2} \right).$$

Case 4. $\bar{q}^2 - 4\sin^2\sigma < 0$

We need to solve the ODE: $\dfrac{dt}{ds} = \dfrac{2\sin\sigma + \bar{q}}{2} \left(t^2 - \dfrac{2\sin\sigma - \bar{q}}{2\sin\sigma + \bar{q}} \right)$, where $\frac{2\sin\sigma - \bar{q}}{2\sin\sigma + \bar{q}} > 0$. Setting the initial condition $U(0) = 0$, we find

$$U(s) = -2 \arctan \left(\sqrt{\frac{2\sin\sigma + \bar{q}}{2\sin\sigma - \bar{q}}} \ \tanh \frac{s \sqrt{4\sin^2\sigma - \bar{q}^2}}{2} \right).$$

To look for closed trajectories, we need to demand that

$$|\bar{q}| = |q - 2\cos\sigma| > 2.$$

This condition immediately implies that cases 1, 2 and 4 cannot occur.

Let $\gamma(s) = (x(s), y(s), \theta(s))$ be a periodic contact magnetic curve which is neither Reeb nor Legendre. Let us denote by T the fundamental period of a periodic contact magnetic curve $\gamma(s) = (x(s), y(s), \theta(s))$. Namely

$$x(s + \mathsf{T}) = x(s), \quad y(s + \mathsf{T}) = y(s) \quad \text{and} \quad \theta(s + \mathsf{T}) \equiv \theta(s) \quad \text{mod } 2\pi.$$

The projected curve of $\beta(s) = (x(s), y(s))$ is a closed Riemannian circle determined by

$$\frac{x'(s)}{2y(s)} = \sin\sigma \ \cos U(s), \quad \frac{y'(s)}{2y(s)} = \sin\sigma \ \sin U(s).$$

The second equation implies that

$$\frac{\mathrm{d}}{\mathrm{d}s} \log y(s) = 2\sin\sigma \ \sin U(s).$$

Since

$$\frac{\mathrm{d}U}{\mathrm{d}s} = -2\sin\sigma \ \cos U + \bar{q},$$

we have

$$\log y(s) = \int \frac{2\sin\sigma \sin U}{\bar{q} - 2\sin\sigma \ \cos U} \ \mathrm{d}U = \log |\bar{q} - 2\sin\sigma \ \cos U(s)| + \text{constant}.$$

Thus we obtain

$$y(s) = \bar{r}\big(\bar{q} - 2\sin\sigma \ \cos U(s)\big)$$

for some $\bar{r} \neq 0$. From this we have

$$x'(s) = (2\sin\sigma)\cos U(s) \cdot (\bar{r}U'(s)),$$

which leads to

$$x(s) = \int (2\bar{r}\sin\sigma)\cos U \ \mathrm{d}U = (2\bar{r}\sin\sigma)\sin U + x_0.$$

We notice that $\bar{s} = (\sin\sigma)s$ is the arclength parameter of β. The curvature of β is $\kappa_\beta = \frac{\bar{q}}{\sin\sigma}$.

The θ-coordinate is determined by

$$\theta'(s) = \cos\sigma - \frac{x'(s)}{2y(s)} = \cos\sigma - \sin\sigma\cos U(s).$$

Since $U'(s) = -(2\sin\sigma)\cos U(s) + \bar{q}$, we get

$$\theta'(s) = \cos\sigma - \frac{\bar{q}}{2} + \frac{U'(s)}{2}.$$

Thus, the solution satisfying $\theta(0) = \theta_0$, is given by

$$\theta(s) = \left(\cos\sigma - \frac{\bar{q}}{2}\right)s + \frac{U(s)}{2} + \theta_0.$$

The periodicity of $x(s)$ and $y(s)$ implies that

$$x(s + \mathsf{T}) - x(s) = \ 2\bar{r}\sin\sigma \ \big[(\sin U(s + \mathsf{T}) - \sin U(s)\big],$$
$$y(s + \mathsf{T}) - y(s) = -2\bar{r}\sin\sigma \ \big[\cos U(s + \mathsf{T}) - \cos U(s)\big],$$

for all s. These formulas yield

$$\sin U(s + \mathsf{T}) = \sin U(s) \quad \text{and} \quad \cos U(s + \mathsf{T}) = \cos U(s), \quad \text{for all } s.$$

Thus we obtain

$$U(s + \mathsf{T}) \equiv U(s) \mod 2\pi.$$

Under the hypothesis $U(0) = 0$, we have

(5.4) $$U(\mathsf{T}) = 2k\pi$$

for some integer k. Since

$$(5.5) \qquad \tan \frac{U(s)}{2} = \sqrt{\frac{\bar{q} - 2\sin\sigma}{\bar{q} + 2\sin\sigma}} \tan\left(\frac{\sqrt{\bar{q}^2 - 4\sin^2\sigma}}{2} s\right),$$

we have

$$\tan\left(\frac{\mathsf{T}\sqrt{\bar{q}^2 - 4\sin^2\sigma}}{2}\right) = 0.$$

This is equivalent to

$$\frac{\mathsf{T}\sqrt{\bar{q}^2 - 4\sin^2\sigma}}{2} = m\pi$$

for some integer m.

The periodicity of $\theta(s)$ implies

$$(5.6) \quad \theta(s + \mathsf{T}) - \theta(s) = \left(\cos\sigma - \frac{\bar{q}}{2}\right)\mathsf{T} + \frac{U(s + \mathsf{T}) - U(s)}{2} \equiv 0 \quad \mathrm{mod}\ 2\pi.$$

From (5.4) and (5.6), we get

$$\left(\cos\sigma - \frac{\bar{q}}{2}\right)\mathsf{T} + k\pi \equiv 0 \quad \mathrm{mod}\ 2\pi.$$

which leads

$$(5.7) \qquad \mathsf{T} = \frac{2k\pi}{\bar{q} - 2\sin\sigma},$$

(possible for other integer k). From (5.5) and (5.7), we find

$$\sqrt{\bar{q}^2 - 4\sin^2\sigma} = \frac{m}{k}(\bar{q} - 2\cos\sigma).$$

Solving this equation we have

$$(5.8) \qquad q = \frac{2a\cos\sigma \pm \sqrt{2(1 - a\cos(2\sigma))}}{\frac{1+a}{2}},$$

where

$$(5.9) \qquad a = 1 - 2\left(\frac{m}{k}\right)^2.$$

We now state the following result.

Theorem 5.3. *The set of all periodic magnetic curves on the special linear group* $\mathrm{SL}_2\mathbb{R}$ *can be quantized in the set of rational numbers.*

Proof. The proof is a consequence of equations (5.8) and (5.9).   □

Remark 5.1. When $\cos\sigma = 0$, the strength q has the form

$$q = \frac{\pm 2}{\sqrt{1 - (m/k)^2}}.$$

This expression coincides with the Kajigaya's criterion.

In the following we draw some pictures of periodic non-Reeb and non-Legendre magnetic curves in $\mathrm{SL}_2\mathbb{R}$. Every figure in the next three examples is composed by four images, keeping the same convention as before.
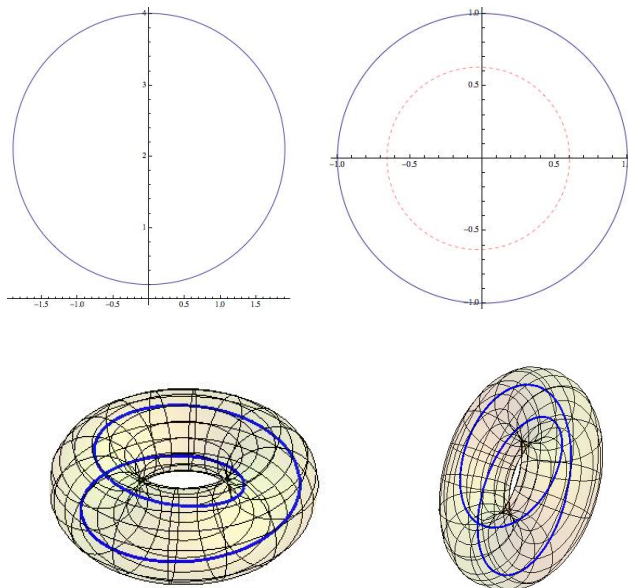

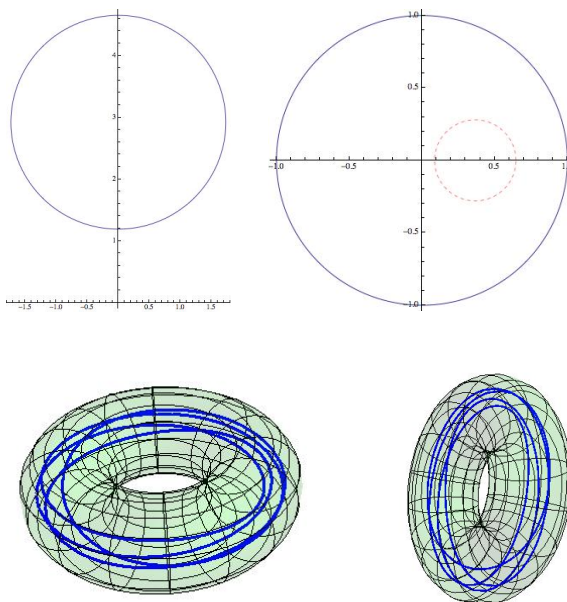Figure 4: $m = 1$, $k = 3$, $\sigma = \frac{2\pi}{5}$.

Figure 5: $m = 3$, $k = 5$, $\sigma = \frac{\pi}{3}$.

Remark 5.2. As we have mentioned in Remark 2.1, we may consider the one-parameter family of homogenous metrics g_ν. In [12], we have studied contact magnetic curves in cosymplectic manifolds. In particular, Nistor investigated contact magnetic curves in [24]. According to [24], contact magnetic curves in $\mathbb{H}^2(-4) \times \mathbb{R}$ are classified as follows:

- a geodesic line $(x_0, y_0, t_0 \pm s)$ through a point (x_0, y_0, t_0).

- a horocycle $\beta_0 \times \{t_0\}$ in every point $t_0 \in \mathbb{R}$, where β_0 denotes a (open) circle tangent to the ideal boundary or a horizontal line in $\mathbb{H}^2(-4)$, of constant curvature $\kappa^2 = 4$;

- a non-degenerate cylindrical helix on $\beta \times \mathbb{R}$, where β denotes both a Euclidean and a hyperbolic circle in $\mathbb{H}^2(-4)$.

One can unify the results in this paper and those in [24].

6. One-parameter subgroups

Let G be a Lie group equipped with a left invariant Riemannian metric and let $\mathfrak{g}$ be its Lie algebra. Then an arclength parametrized curve γ in G is said to be *homogenous* if there exists a one-parameter subgroup $\{\exp(tX)\}$ such

that $\gamma(t)$ is expressed as $\gamma(t) = a \, \exp(tX)$ for some $a \in G$ and unit vector $X \in \mathfrak{g}$. In this section we investigate homogeneous magnetic trajectories.

6.1. Homogeneous geodesics

In Lie groups with bi-invariant Riemannian metric, all the geodesics starting at the identity are one parameter subgroups. However, if the metric is only left invariant, one-parameter subgroups are not necessarily geodesics. Here we study geodesic in $SL_2\mathbb{R}$ which are one-parameter subgroups of $SL_2\mathbb{R}$.

Proposition 6.1. *The one-parameter subgroup* $\exp(tX)$ *of an element*

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11} \end{pmatrix}$$

of $\mathfrak{sl}_2\mathbb{R}$ *is given explicitly by as follows*:

- *If* $\det X = 0$, *then*

$$\exp(tX) = \begin{pmatrix} 1 + tX_{11} & tX_{12} \\ tX_{21} & 1 - tX_{11} \end{pmatrix}.$$

  Every $\exp(tX)$ *induces a parabolic transformation on* $\mathbb{H}^2(-4)$.

- $\det X = \delta^2 > 0$, *then*

$$\exp(tX) = \begin{pmatrix} \cos(\delta t) + \frac{X_{11}}{\delta}\sin(\delta t) & \frac{X_{12}}{\delta}\sin(\delta t) \\ \frac{X_{21}}{\delta}\sin(\delta t) & \cos(\delta t) - \frac{X_{11}}{\delta}\sin(\delta t) \end{pmatrix}.$$

  Every $\exp(tX)$ *induces an elliptic transformation on* $\mathbb{H}^2(-4)$.

- $\det X = -\delta^2 < 0$

$$\exp(tX) = \begin{pmatrix} \cosh(\delta t) + \frac{X_{11}}{\delta}\sinh(\delta t) & \frac{X_{12}}{\delta}\sinh(\delta t) \\ \frac{X_{21}}{\delta}\sinh(\delta t) & \cosh(\delta t) - \frac{X_{11}}{\delta}\sinh(\delta t) \end{pmatrix}.$$

  Every $\exp(tX)$ *induces a hyperbolic transformation on* $\mathbb{H}^2(-4)$.

Take an element $X \in \mathfrak{sl}_2\mathbb{R}$, then the acceleration vector field $\nabla_{\gamma'}\gamma'$ of $\gamma(t) = \exp(tX)$ at the origin is $U(X, X)$ because of (3.4). Thus we obtain the following well known criterion:

Proposition 6.2. *A one parameter subgroup* $\{\exp(tX)\}_{t\in\mathbb{R}}$, $X \in \mathfrak{sl}_2\mathbb{R}$ *is a geodesic if and only if* $U(X, X) = 0$.

Now we apply this criterion for $X = aE_1 + bE_2 + cE_3 \in \mathfrak{sl}_2\mathbb{R}$. By using (3.3), $\mathsf{U}(X,X)$ is computed as

$$\mathsf{U}(X,X) = 2c(b-a)E_1 + 2c(b-a)E_2 + 2(a^2 - b^2)E_3.$$

Thus we obtain

Corollary 6.1. *A one-parameter subgroup* $\{\exp(tX)\}$ *of* $X = aE_1 + bE_2 + cE_3$ *is a geodesic in* $\mathrm{SL}_2\mathbb{R}$ *if and only if either*

- $a = b$ *or*

- $c = 0$ *and* $a = -b$.

In particular $\exp(tE_3)$ is the only geodesic among $\exp(tE_i)$, $(i = 1, 2, 3)$. Here we describe the spurs of $\exp(tE_i)$. Direct computations show the following formula:

$$\exp(tE_1) = \begin{pmatrix} 1 & \sqrt{2}t \\ 0 & 1 \end{pmatrix}.$$

Thus the coordinate expression of $\exp(tE_1)$ is

$$x(t) = \sqrt{2}t, \quad y = 1, \quad \theta(t) = 0.$$

Hence the spur of $\exp(tE_1)$ in the universal covering $\widetilde{\mathrm{SL}_2\mathbb{R}} = \mathbb{H}^2(-4) \times \mathbb{R}$ is the line through $(0,1,0)$ parallel to the x-axis. The contact angle of $\exp(tE_1)$ is $\pi/4$.

Note that for all $t \in \mathbb{R}$,

(6.1)
$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \exp\left(\frac{x}{\sqrt{2}}E_1\right).$$

Hence the mapping

$$\exp\left(\frac{\bullet}{\sqrt{2}}E_1\right) : (\mathbb{R}(x), +) \longrightarrow N$$

is a Lie group isomorphism.

Next the trace of $\exp(tE_2)$ is given by

(6.2)
$$\exp(tE_2) = \begin{pmatrix} 1 & 0 \\ \sqrt{2}t & 1 \end{pmatrix}.$$

This curve has the parametrization

$$x(t) = \frac{\sqrt{2}t}{1 + 2t^2}, \quad y(t) = \frac{1}{1 + 2t^2}, \quad \theta(t) = \arctan(-\sqrt{2}t)$$

and contact angle $3\pi/4$. The projected curve $(x(t), y(t))$ of $\exp(tE_2)$ in $\mathbb{H}^2(-4)$ is the horocycle

$$x^2 + \left(y - \frac{1}{2} \right)^2 = \frac{1}{4}.$$

These two one-parameter subgroups $\{\exp(tE_1)\}$ and $\{\exp(tE_2)\}$ are *not* geodesics. However as we will see later, they are contact magnetic curves.

The one-parameter subgroup

$$\exp(tE_3) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

is a Legendre geodesic with parametrization

$$x(t) = 0, \quad y(t) = e^t, \quad \theta(t) = 0.$$

The spur of $\exp(tE_3)$ is the y-axis in the universal covering $\mathbb{H}^2(-4) \times \mathbb{R}$. The projected curve $(x(t), y(t))$ is a vertical line, *i.e.*, a horocycle with base point ∞.

Remark 6.1. A homogeneous Riemannian manifold $M = G/K$ is called a *space with homogeneous geodesics* or a *Riemannian g.o. space* if every geodesic $\gamma(t)$ of M is an orbit of a one-parameter subgroup of G [23]. Naturally reductive homogenous spaces are typical examples of Riemannian g.o. spaces. (For more informations, we refer to [1]).

Explicit parametrization of geodesics in $\mathbb{H}^2(-1) \times \mathbb{R}$ have been obtained in [29] (see also [26]). Note that in [26, 29], $\widetilde{\mathrm{SL}_2\mathbb{R}}$ is realized as the product manifold of unit disk and the real line.

6.2. Homogeneous magnetic trajectories

Let us consider magnetic equation for one-parameter subgroups. First we prepare the following proposition.

Proposition 6.3. *The endomorphism field φ satisfies*

$$\varphi(aE_1 + bE_2 + cE_3) = -\frac{c}{\sqrt{2}}(E_1 + E_2) + \frac{1}{\sqrt{2}}(a + b)E_3$$

for any $X = aE_1 + bE_2 + cE_3 \in \mathfrak{sl}_2\mathbb{R}$.

From this proposition, the magnetic equation $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$ for $\gamma(t) = \exp(tX)$ may be rewritten as

$$2c(b - a) = -\frac{cq}{\sqrt{2}}, \quad 2(a^2 - b^2) = \frac{q}{\sqrt{2}}(a + b).$$

Proposition 6.4. *The one parameter subgroup $\exp(tX)$ of $X = aE_1 + bE_2 + cE_3 \in \mathfrak{sl}_2\mathbb{R}$ is a magnetic curve with strength $q \neq 0$ if and only if $a - b = q/(2\sqrt{2})$.*

Remark 6.2. If $b = -a$, in order to have a non-geodesic magnetic curve with strength q, we need to ask that $c \neq 0$. In this situation, $a = q/(4\sqrt{2})$.

Since the Reeb vector field is

$$\xi = \frac{1}{\sqrt{2}}(E_1 - E_2),$$

the contact angle σ of the one parameter subgroup $\exp(tX)$ of $X = aE_1 + bE_2 + cE_3 \in \mathfrak{sl}_2\mathbb{R}$ defined by

$$\cos\sigma = \frac{a - b}{\sqrt{2}\sqrt{a^2 + b^2 + c^2}}.$$

Therefore, if $\exp(tX)$ is a Legendre curve, it necessarily should be a geodesic.

Example 6.1. Let $X = aE_1 + bE_2 + cE_3 \in \mathfrak{sl}_2\mathbb{R}$ such that $\det X = 0$. We shall emphasize two situations: (i) $b = -a$ and $c \neq 0$ and (ii) $c = 0$. In the

first case (i), $c = \pm\sqrt{2}a$ and

$$\exp(tX) = \left(\begin{array}{cc} 1 \pm \sqrt{2}at & \sqrt{2}at \\ -\sqrt{2}at & 1 \mp \sqrt{2}at \end{array} \right).$$

The Iwasawa decomposition of $\exp(tX)$ is given by

$$x(t) = \frac{\mp 4a^2t^2}{2a^2t^2 + (1 \mp \sqrt{2}at)^2}, \quad y(t) = \frac{1}{2a^2t^2 + (1 \mp \sqrt{2}at)^2},$$

$$e^{i\theta(t)} = \frac{1 \mp \sqrt{2}at + i\sqrt{2}at}{\sqrt{2a^2t^2 + (1 \mp \sqrt{2}at)^2}}.$$

The contact angle σ satisfies $\cos\sigma = \operatorname{sgn}(a)/\sqrt{2}$, thus $\sigma = \pi/4$ or $3\pi/4$. The projected curve is the horocycle $(x \pm 1)^2 + (y - 1)^2 = 1$. The curve $\exp(tX)$ is a contact magnetic curve with strength q if and only if $a = q/(4\sqrt{2})$.

In the second case (ii), we consider $Y = aE_1 + bE_2 \neq 0$. Then $\det Y = 0$ if and only if $a = 0$ or $b = 0$, namely $Y = aE_1$ or $Y = bE_2$.

(ii.1) In the case $Y = aE_1$, from (6.1), we obtain that $\exp(taE_1)$ is parametrized as $(x(t), y(t), \theta(t)) = (a\sqrt{2}t, 1, 0)$. It is a contact magnetic curve with strength q if and only if $a = q/(2\sqrt{2})$.

The trajectory lies in the nilpotent subgroup N. The contact angle is $\cos\sigma = \pm 1/\sqrt{2}$. Thus $\sigma = \pi/4$ or $3\pi/4$. The projected curve is the horizontal line $y = 1$ (horocycle with base point ∞).

(ii.2) In the case $Y = bE_2$, using (6.2), it follows that $\exp(tbE_2)$ is parametrized as

$$x = \frac{\sqrt{2}bt}{2b^2t^2 + 1}, \quad y = \frac{1}{2b^2t^2 + 1}, \quad e^{i\theta} = \frac{1 - i\sqrt{2}bt}{\sqrt{2b^2t^2 + 1}}.$$

From these relations we obtain $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. Thus $(x(t), y(t))$ is a horocycle. Note that $\exp(tY)$ has contact angle $\pi/4$ or $3\pi/4$. It is a contact magnetic curve with strength q if and only if $b = -q/(2\sqrt{2})$.

Remark 6.3. We can write the projection curve β for the general situation $\exp(tX)$ with $X = aE_1 + bE_2 + cE_3$ with $\det X = -(c^2 + 2ab) = 0$:

$$x(t) = \frac{\sqrt{2}\{(a + b)t + c(b - a)t^2\}}{2b^2t^2 + (1 - tc)^2}, \quad y(t) = \frac{1}{2b^2t^2 + (1 - tc)^2}.$$

If the parameter t us eliminated, we obtain the following equation for β:

(1) If $b = 0$ (and hence also $c = 0$), then $y - 1 = 0$.

(2) If $b \neq 0$, then

$$\left(x - \frac{c}{b\sqrt{2}}\right)^2 + \left(y - \frac{b - a}{2b}\right)^2 = \frac{(b - a)^2}{4b^2}.$$

These formulas actually show that β is a horocycle.

Note that t is not, in general, the arc length parameter for $\exp(tX)$. In case t is the arc length parameter for $\exp(tX)$, then $a^2 + b^2 + c^2 = 1$. The condition $\det X = 0$ implies $|a - b| = 1$ and thus, the contact angle σ is obtained as $\cos \sigma = \pm 1/\sqrt{2}$ meaning that $\sigma = \pi/4$ or $3\pi/4$. The magnetic condition is $q = 2\text{sgn}(a - b)\sqrt{2}$. Computing $\bar{q} = q - 2\cos\sigma = \sqrt{2}\text{sgn}(a - b)$. Hence the necessarily condition for periodicity, namely $|\bar{q}| > 2$ fails. The curvature $\kappa_\beta = \bar{q}/\sin\sigma = 2\,\text{sgn}(a - b)$.

Example 6.2. Let us assume that $\det X > 0$. First we consider $X = a(E_1 - E_2) + cE_3$. The one-parameter subgroup is given by

$$\exp(tX) = \begin{pmatrix} \cos(\delta t) + \frac{c}{\delta}\sin(\delta t) & \frac{\sqrt{2}a}{\delta}\sin(\delta t) \\ -\frac{\sqrt{2}a}{\delta}\sin(\delta t) & \cos(\delta t) - \frac{c}{\delta}\sin(\delta t) \end{pmatrix}.$$

Note that $\delta^2 = 2a^2 - c^2$. We perform the Iwasawa decomposition.

$$x(t) = \frac{-\frac{2\sqrt{2}ac}{\delta^2}\sin^2(\delta t)}{\frac{2a^2}{\delta^2}\sin^2(\delta t) + (\cos(\delta t) - \frac{c}{\delta}\sin(\delta t))^2},$$

$$y(t) = \frac{1}{\frac{2a^2}{\delta^2}\sin^2(\delta t) + (\cos(\delta t) - \frac{c}{\delta}\sin(\delta t))^2},$$

$$e^{i\theta(t)} = \frac{\cos(\delta t) - \frac{c}{\delta}\sin(\delta t) + \frac{\sqrt{2}ai}{\delta}\sin(\delta t)}{\sqrt{\frac{2a^2}{\delta^2}\sin^2(\delta t) + (\cos(\delta t) - \frac{c}{\delta}\sin(\delta t))^2}}.$$

The projected curve β is expressed as

$$\left(x + \frac{c}{a\sqrt{2}}\right)^2 + (y - 1)^2 = \frac{c^2}{2a^2}.$$

Next we treat the case $Y = aE_1 + bE_2$, for which $\det Y = -2ab = \delta^2 > 0$. We have

$$\exp(tY) = \begin{pmatrix} \cos(\delta t) & \frac{\sqrt{2}a}{\delta}\sin(\delta t) \\ \frac{\sqrt{2}b}{\delta}\sin(\delta t) & \cos(\delta t) \end{pmatrix}.$$

Hence the projected curve β is

$$x(t) = \frac{\sqrt{2}(a+b)\sin(\delta t)\cos(\delta t)}{\frac{2b^2}{\delta}\sin^2(\delta t) + \delta\cos^2(\delta t)}, \quad y(t) = \frac{1}{\frac{2b^2}{\delta^2}\sin^2(\delta t) + \cos^2(\delta t)}.$$

The implicit form of β is

$$x^2 + \left(y - \frac{b-a}{2b}\right)^2 = \left(\frac{a+b}{2b}\right)^2.$$

Comparing this with (A.1), we obtain $\kappa_\beta = \dfrac{2(b-a)\operatorname{sgn}(b)}{|b+a|}$. The contact angle satisfies $\cos\sigma = \dfrac{a-b}{\sqrt{2}\sqrt{a^2+b^2}}$. In both cases, the magnetic curves are periodic.

Example 6.3. Assume now that $\det X = -\delta^2 < 0$, for $X = aE_1 + bE_2 + cE_3$. In this case we have

$$\exp(tX) = \begin{pmatrix} \cosh(\delta t) + \frac{c}{\delta}\sinh(\delta t) & \frac{\sqrt{2}a}{\delta}\sinh(\delta t) \\ \frac{\sqrt{2}b}{\delta}\sinh(\delta t) & \cosh(\delta t) - \frac{c}{\delta}\sinh(\delta t) \end{pmatrix}.$$

Let us compute the coordinates (x, y, θ) of $\exp(tX)$.

$$x(t) = \frac{\frac{\sqrt{2}(a+b)}{\delta}\sinh(\delta t)\cosh(\delta t) + \frac{\sqrt{2}(b-a)c}{\delta^2}\sinh(\delta t)\cosh(\delta t)}{\frac{2b^2}{\delta^2}\sinh^2(\delta t) + \left(\cosh(\delta t) - \frac{c}{\delta}\sinh(\delta t)\right)^2},$$

$$y(t) = \frac{1}{\frac{2b^2}{\delta^2}\sinh^2(\delta t) + \left(\cosh(\delta t) - \frac{c}{\delta}\sinh(\delta t)\right)^2}.$$

Then the projected curve β is described as follows:

(1) If $b \neq 0$, then β is a part of an open circle:

$$\left(x - \frac{c}{b\sqrt{2}}\right)^2 + \left(y + \frac{a-b}{2b}\right)^2 = \frac{(a+b)^2 + 2c^2}{4b^2}.$$

(2) If $b = 0$, then β is

$$\sqrt{2}\delta x + \operatorname{sgn}(c)a(1-y) = 0.$$

In both cases, the signed curvature of β is

$$\kappa_\beta = \frac{2\operatorname{sgn}(b)(b-a)}{\sqrt{(a+b)^2 + 2c^2}}.$$

Appendix A. Curve theory in $\mathbb{H}^2(-4)$

A.1. Frenet formula in $\mathbb{H}^2(-4)$

Let $\beta(s) = (x(s), y(s))$ be an arclength parametrized curve in $\mathbb{H}^2(-4)$. Take a global orthonormal frame field

$$\bar{\epsilon}_1 = 2y\,\frac{\partial}{\partial x}, \quad \bar{\epsilon}_2 = 2y\,\frac{\partial}{\partial y},$$

then the unit tangent vector field $\overline{T}(s) = \beta'(s)$ is represented by

$$\beta'(s) = \frac{1}{2y(s)}\left(x'(s)\bar{\epsilon}_1 + y'(s)\bar{\epsilon}_2\right).$$

The unit normal vector field $\overline{N}(s)$ is

$$\overline{N}(s) = J\overline{T}(s) = \frac{1}{2y(s)}\left(-y'(s)\bar{\epsilon}_1 + x'(s)\bar{\epsilon}_2\right).$$

Then we obtain

$$\overline{\nabla}_{\beta'}\overline{T} = \frac{1}{2y^2}\left\{(x''y - 2x'y')\bar{\epsilon}_1 + (y''y + (x')^2 - (y')^2)\bar{\epsilon}_2\right\} = \kappa_\beta J\overline{T}(s).$$

Thus the (signed) curvature κ_β is

$$\kappa_\beta = \frac{x'y'' - x''y'}{4y^2} + \frac{x'}{y}.$$

A.2. Riemannian circles in $\mathbb{H}^2(-4)$

In a Euclidean space, Riemannian circles are nothing but usual circles. Hence, every Riemannian circle in a Euclidean space is simple and closed. Besides, Riemannian circles in spheres are small circles and hence, every Riemannian circle in spheres is simple and closed, too. Nevertheless, in hyperbolic spaces, Riemannian circles are not necessarily closed.

Proposition A.1. *Every Riemannian circle β of curvature $\kappa_\beta = k$ in $\mathbb{H}^2(-4)$ is a horizontal line, a vertical line or a part of a circle, given by*

the following formula:

(A.1) $$(x - a)^2 + (y - rk)^2 = 4r^2.$$

In particular, a Riemannian circle is closed if and only if $|k| > 2$.

Sketch of proof. We solve the following equation:

(A.2) $$\frac{x'y'' - x''y'}{4y^2} + \frac{x'}{y} = k,$$

for nonzero constant k. Setting

$$X := \frac{x'}{2y}, \quad Y := \frac{y'}{2y},$$

the equation (A.2) becomes

$$k = XY' - X'Y + 2X.$$

Since β is parametrized by arclength parameter, then X and Y satisfies $X^2 + Y^2 = 1$. Consequently, we have to solve the following system

$$X^2 + Y^2 = 1, \quad XY' - X'Y + 2X = k.$$

To do this, we introduce the function $\mu = \mu(s)$, as follows

$$X = \cos\mu(s), \quad Y = \sin\mu(s), \quad \frac{d\mu}{ds} = k - 2\cos\mu.$$

In case μ is constant, we have $|k| \le 2$ and $\sin\mu = \pm\sqrt{4 - k^2}/2$.

If $k^2 = 4$, then $Y = 0$ and hence y is constant. Thus β is a horizontal line.

If $k^2 \neq 4$, then the system

(A.3) $$\frac{dx}{ds} = 2y\cos\mu,$$

(A.4) $$\frac{dy}{ds} = 2y\sin\mu$$

is solved as

$$x = (\cot\mu)y + x_0.$$

Thus β is an Euclidean line and it is usually known as a *hypercycle* or and *equidistant line*. Of course, the particular situation $\cos\mu = 0$ leads to a vertical (half) line, which is a geodesic.

Next we consider the case μ is non-constant. Then the derivative of x and y are given by

$$\frac{dx}{ds} = \frac{dx}{d\mu}(k - 2\cos\mu), \quad \frac{dy}{ds} = \frac{dy}{d\mu}(k - 2\cos\mu).$$

From these, we get

(A.5) $$\frac{dy}{y} = \frac{2\sin\mu}{k - 2\cos\mu}d\mu.$$

Solving (A.5), we get

(A.6) $$y = r(k - 2\cos\mu), \ r > 0.$$

Next, inserting (A.6) to (A.3), we have $dx = 2r\cos\mu \, d\mu$, which implies

$$x = 2r\sin\mu + a, \ a \in \mathbb{R}.$$

Hence the Riemannian circle of curvature $\kappa_\beta \neq 0$ is a horizontal line, an equidistant line or a part of the Euclidean circle:

$$(x - a)^2 + (y - rk)^2 = 4r^2.$$

It is straightforward that β is closed if and only if the circle lies entirely above the boundary line, equivalently to $|k| > 2$. Furthermore, β is a horocycle if and only if $|k| = 2$. When $|k| < 2$, the curve β is a portion of an Euclidean circle that makes non-right angles with the boundary line. $\quad\square$

Conclusion. We investigate periodic contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$ and give a criterion to have periodicity. In such a way, we obtain a quantization principle for periodic contact magnetic curves in $\mathrm{SL}_2\mathbb{R}$ over the set of rational numbers. This conclusion is similar to that for closed geodesics on a torus $\mathbb{T}^2$ and has a physical meaning: every closed geodesic corresponds to a discrete set of energy levels, "mirroring the analogous quantization of energy levels in the model of an atom" [20]. On the other hand, it is proved [19, 28] that every closed L-minimal Legendre curve in the 3-sphere $\mathbb{S}^3$ is a magnetic curve. These curves are known as torus knots and Kajigaya [21] proved that they are L-unstable.

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