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# A Laplace transform approach to linear equations with infinitely many derivatives and zeta-nonlocal field equations

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We study existence, uniqueness and regularity of solutions for linear equations in infinitely many derivatives. We develop a natural framework based on Laplace transform as a correspondence between appropriate $L^p$ and Hardy spaces: this point of view allows us to interpret rigorously operators of the form $f(\partial_t)$ where $f$ is an analytic function such as (the analytic continuation of) the Riemann zeta function. We find the most general solution to the equation

$$f(\partial_t)\phi = J(t)\ , \quad t \geq 0\ ,$$

in a convenient class of functions, we define and solve its corresponding initial value problem, and we state conditions under which the solution is of class $C^k$, $k \geq 0$. More specifically, we prove that if some a priori information is specified, then the initial value problem is well-posed and it can be solved using only a *finite number* of local initial data. Also, motivated by some intriguing work by Dragovich and Aref'eva-Volovich on cosmology, we solve explicitly field equations of the form

$$\zeta(\partial_t + h)\phi = J(t)\ , \quad t \geq 0\ ,$$

in which $\zeta$ is the Riemann zeta function and $h > 1$. Finally, we remark that the $L^2$ case of our general theory allows us to give a precise meaning to the often-used interpretation of $f(\partial_t)$ as an operator defined by a power series in the differential operator $\partial_t$.

## 1. Introduction

Equations with an infinite number of derivatives have appeared recently as field equations of motion in particle physics [35], string theory [9, 23–25, 33, 40, 42–44], and (quantum) gravity and cosmology [1–6, 10, 12, 13, 32, 34]. For instance, an important equation in this class is

$$p^{a\,\partial_t^2}\phi = \phi^p\ , \qquad a > 0\ , \tag{1.1}$$

where $p$ is a prime number. Equation (1.1) describes the dynamics of the open $p$-adic string for the scalar tachyon field (see [1, 6, 22, 34, 42, 43] and references therein) and it can be understood, at least formally, as an equation in an infinite number of derivatives if we expand the entire function (called below the "symbol" of the equation)

$$f(s) := p^{as^2} = e^{as^2\,log(p)}$$

as a power series around zero and we replace $s$ for $\partial_t$. This equation has been studied by Vladimirov via integral equations of convolution type in [42, 43] (see also [1, 34]), and it has been also noted that in the limit $p \to 1$, Equation (1.1) becomes the *local* logarithmic Klein-Gordon equation [7, 26]. Moreover, Dragovich, see [22], has used (the Lagrangian formulation of) Equation (1.1) and the Euler product for the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}\ , \qquad Re(s) > 1\ ,$$

see [31], to deduce a string theory field equation of the form

$$\zeta\left(-\frac{1}{m^2}\,\partial_t^2 + h\right)\phi = U(\phi)\ , \tag{1.2}$$

in which $m, h$ are constant real numbers and $U$ is a nonlinear function of $\phi$. Equation (1.2) is called a zeta-nonlocal field equation in [22].

In this paper we focus on the solvability and functional analytic properties of *linear* equations of the form

$$f(\partial_t)\phi = J(t)\ , \quad t \geq 0\ , \tag{1.3}$$

in which $f$ is an analytic function and $J$ belongs to an adequate class of functions to be specified in Section 2. We also investigate carefully the formulation, existence and solution of initial value problems, and we explain precisely in what sense Equation (1.3) really *is* an ordinary differential equation in infinitely many derivatives. Motivated mainly by Dragovich's work (see also [2]), we endeavor to develop a theory for $f(\partial_t)$ able to deal with the case in which $f$ is, for instance, the Riemann zeta function. In the previous papers [27, 28] we used as our basic arena the Banach space of exponentially bounded locally integrable functions on $\mathbb{R}_+$ and Widder space (see *loc. cit.* for references). It does not appear straightforward to apply this functional analytic framework to equations of the form $\zeta(\partial_t + h)\phi = J$, as we explain in Section 2. Hence the need to develop an alternative approach: in this paper we take as our basic setup Lebesgue ($L^p$) and Hardy ($H^q$) spaces, and we show that indeed these spaces are flexible enough to allow for explicit non-trivial computations.

We remark that linear equations in infinitely many derivatives appeared in mathematics already in the final years of the XIX Century, see for instance [11, 15] and further references in [5]. It appears to us, however, that a truly fundamental stimulus for their study has been the realization that linear and nonlinear equations in this class play an important role in contemporary physical theories, as witnessed by [38] and the more recent papers cited above.

Now, a serious problem for the development of a rigorous theory for equations in infinitely many derivatives has been the difficulties inherent in the understanding of the initial value problem for equations such as (1.1) and (1.3). An interpretative complication considered already in the classical paper [25] by Eliezer and Woodard is the frequently stated argument (see for instance [3, 34] and the rigorous approach of [16]) that if an $n$th order ordinary differential equation requires $n$ initial conditions, then the "infinite order" equation $F(t, q', q'', \ldots, q^{(n)}, \ldots) = 0$ requires infinitely many initial conditions, and therefore the solution $q$ would be determined a priori (via power series) without reference to the actual equation.

In order to deal with this difficulty, we investigate initial value problems from scratch. Our approach is to emphasize the role played by the Laplace transform as an operator between appropriate $L^p$ and $H^q$ spaces, see [21], in the spirit of [1, 2, 5, 6, 27, 28]. Our solution (Section 3.2 below) is that if an a priori data directly connected with our interpretation of $f(\partial_t)$ is specified (see Definition 3.5) then the initial value problem is well-posed and it requires only a *finite number* of initial conditions.

Due to the preeminence of Laplace transform in our theory, it is natural for us to speak indistinctly of "nonlocal equations or equations in an infinite number of derivatives", and we do so hereafter. We also note that we have explained in [28] why it is not necessarily appropriate to use pseudo-differential operators for studying operators $f(\partial_t)$; we will not dwell on this issue here.

This paper is organized as follows. In Section 2 we present a rigorous interpretation of nonlocal ordinary differential equations via Laplace transform, considered as an operator between appropriate Lebesgue ($L^p(\mathbb{R}_+)$) and Hardy ($H^q(\mathbb{C}_+)$) spaces. In Section 3 we investigate (and propose a method for the solution of) initial value problems for linear equations of the form (1.3). In Section 4 we solve a class of zeta nonlocal equations, and in Section 5 we provide an $L^2$-theory for operators $f(\partial_t)$. taking advantage of the Paley-Wiener theorem: the Laplace transform provides an isomorphism between $L^2(\mathbb{R}_+)$ and $H^2(\mathbb{C}_+)$, and this fact allows us to better understand nonlocal operators, see for instance Proposition 5.5 and ensuing discussion. Finally in Section 6 we discuss some problems which are beyond the reach of the theory presented here; they will be considered in the companion article [18].

## 2. Linear nonlocal equations

In this section we introduce a rigorous and computationally useful framework for the analytic study of nonlocal equations, including the possibility of setting up meaningful initial value problems. We consider linear nonlocal equations of the form

$$f(\partial_t)\phi(t) - J(t) = 0 \,, \quad t \geq 0 \,, \tag{2.1}$$

in which $f$ is an analytic function.

### 2.1. Preliminaries

We begin by fixing our notation and stating some preliminary facts; our main source is the classical treatise [20] by G. Doetsch.

We recall that a function $g$ belongs to $L^1_{loc}(\mathbb{R}_+)$, in which $\mathbb{R}_+$ is the interval $[0,\infty)$, if and only if

$$\int_K |g(t)|dt$$

exists for any compact set $K \subseteq \mathbb{R}_+$.

**Definition 2.1.** A function $g : \mathbb{R} \to \mathbb{C}$ in $L^1_{loc}(\mathbb{R}_+)$ belongs to the class $\mathcal{T}_a$ if and only if $g(t)$ is identically zero for $t < 0$ and the integral

$$\int_0^\infty e^{-s\,t}\, g(t)\, dt$$

converges absolutely for $Re(s) > a$.

$\mathcal{T}_a$ is simply a vector space; we do not need to endow it with a topology. The *Laplace transform* of a function $g : \mathbb{R} \to \mathbb{C}$ in $\mathcal{T}_a$ is the integral

$$\mathcal{L}(g)(s) = \int_0^\infty e^{-s\,t}\, g(t)\, dt\ , \qquad Re(s) > a\ .$$

As proven in [20, Theorem 3.1], if $g \in \mathcal{T}_a$, then $\mathcal{L}(g)(s)$ converges absolutely and uniformly for $Re(s) \geq x_0 > a$ (a finer statement is in [20, Theorem 23.1]). It is also known (see [20, Theorem 6.1]) that the Laplace transform $\mathcal{L}(g)$ is an analytic function for $Re(s) > a$. We also record the following:

**Proposition 2.2.**

1) *Let us fix a function $g \in \mathcal{T}_a$. If the derivative $D^k g(t)$ belongs to $\mathcal{T}_a$ for $0 \leq k \leq n-1$, then $D^n g(t)$ belongs to $\mathcal{T}_a$ and*

$$\mathcal{L}(D^n g)(s) = s^n\, \mathcal{L}(g)(s) - \sum_{j=1}^{n} (D^{j-1}g)(0) s^{n-j}\ .$$

2) *If $g \in \mathcal{T}_a$ is a continuous function, then for any $\sigma > a$ we have the identity*

$$g(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\,t}\, \mathcal{L}(g)(s)\, ds \tag{2.2}$$

*for all $t \geq 0$.*

The second part of this proposition is in [20, Theorem 24.4]. The class $\mathcal{T}_a$ appears to be the simplest class of functions on which the Laplace transform is defined and for which there exists an inversion formula (2.2). The inverse Laplace transform is denoted by $\mathcal{L}^{-1}$.

We recall that we are interested in a theory able to deal with (2.1) in which, for instance, $f(s) = \zeta(s+h)$. This is not a trivial requirement: let us suppose that we are interested in applying Theorem 3.1 of [27] (or Theorem 2.1 of [28] on classical initial value problems to the equation $\zeta(\partial_t + h)\phi(t) = J(t)$. Then, we would need to check that $\mathcal{L}(J)(s)/\zeta(s+h)$ belongs to the Widder space

$$C_W^\infty(\omega,\infty) = \left\{ r : (\omega,\infty) \to \mathbb{C} / \|r\|_W = \sup_{n\in\mathbb{N}_0} \sup_{s>\omega} \left| \frac{(s-\omega)^{n+1}}{n!}\, r^{(n)}(s) \right| < \infty \right\},$$

in which $\omega > 0$ and $r^{(n)}(s)$ denotes the $n$th derivative of $r(s)$. Certainly, such a check does not look straightforward. It is important therefore to develop an alternative approach to nonlocal equations.

The class $\mathcal{T}_a$ appears to be a good starting point. Regretfully, because of reasons to be explained in Section 4, it is not adequate for defining and solving initial value problems. We have found that, instead of the spaces used in [27, 28] a better alternative is to consider Laplace transform as a correspondence from Lebesgue spaces $L^p(\mathbb{R}_+)$ into Hardy spaces $H^q(\mathbb{C}_+)$, which we now define.

We write $\mathbb{C}_+$ for the right half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$. The space $L^p(\mathbb{R}_+)$, $1 \le p < \infty$, is the Lebesgue space of measurable functions $\phi$ on $[0,\infty)$ such that

$$\|\phi\|_{L^p(\mathbb{R}_+)} := \left( \int_0^\infty |\phi(x)|^p dx \right)^{\frac{1}{p}} < \infty ,$$

and the $q$th Hardy space $H^q(\mathbb{C}_+)$ is the space of all functions $\Phi$ which are analytic on $\mathbb{C}_+$ and such that the integral $\mu_q(\Phi, x)$ given by

$$\mu_q(\Phi, x) := \left( \frac{1}{2\pi} \int_{-\infty}^\infty |\Phi(x+iy)|^q dy \right)^{\frac{1}{q}} ,$$

is uniformly bounded for $x > 0$. We note that $H^q(\mathbb{C}_+)$ becomes a Banach space with the norm $\|\Phi\|_{H^q(\mathbb{C}_+)} := \sup_{x>0} \mu_q(\Phi, x)$.

The following classic Representation theorem was first presented by Doetsch in [21, pp. 276 and 279]:

**Theorem 2.3.** *(Doetsch's Representation theorem)*

(i) *If* $\phi \in L^p(\mathbb{R}_+)$, *where* $1 < p \leq 2$, *and* $\Phi = \mathcal{L}(\phi)$, *then* $\Phi \in H^{p'}(\mathbb{C}_+)$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$*. Moreover if* $x > 0$ *there exists a positive constant* $C(p)$ *such that:*

$$\mu_{p'}(\Phi, x) \leq C(p) \left( \int_0^\infty e^{-pxt} |\phi(t)|^p dt \right)^{\frac{1}{p}} .$$

(ii) *If* $\Phi \in H^p(\mathbb{C}_+)$, *where* $1 < p \leq 2$, *then there exists* $\phi \in L^{p'}(0, \infty)$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$ *suth that* $\Phi = \mathcal{L}(\phi)$*. The function* $\phi$ *is given by the inversion formula*

$$\phi(t) := \lim_{v \to \infty} \frac{1}{2\pi} \int_{-v}^{v} e^{(\sigma + i\eta)t} \Phi(\sigma + i\eta) d\eta , \qquad \sigma \geq 0 ,$$

*in which the limit is understood in* $L^{p'}(\mathbb{R}_+)$, *and for* $x > 0$ *there exists a positive constant* $K(p)$ *such that*

$$\left( \int_0^\infty e^{-p'xt} |\phi(t)|^{p'} dt \right)^{\frac{1}{p'}} \leq K(p) \mu_p(\Phi, x).$$

We remark that the restrictions on $p$ appearing in both parts of the theorem imply that neither *(i)* is the converse of *(ii)* nor *(ii)* is the converse of *(i)*, except in the case $p = p' = 2$. This fact reflects itself in the enunciates of our main theorems (Theorems 2.8 and 3.4). If $p = p' = 2$, Doetsch's Representation theorem is the important Paley-Wiener theorem, which states that the Laplace transform is a unitary isomorphism between $L^2(\mathbb{R}_+)$ and $H^2(\mathbb{C}_+)$; a precise statement is in Section 5, after [29, 45]. We also note that there exist generalizations of Theorem 2.3: it has been observed that the Laplace transform determine correspondences between appropriate weighted Lebesgue and Hardy spaces, see [8, 39] and references therein, and there also exists a representation theorem for functions $\Phi$ which decay to zero in any closed half-plane $Re(s) \geq \delta > a$, $a \in \mathbb{R}$ and satisfy the additional requirement that $\Phi(\sigma + i(\cdot)) \in L^1(\mathbb{R})$ (roughly speaking, an "$L^1$-case" of Theorem 2.3, see [19, 20]). Interestingly, as we explain in Section 3, in this "$L^1$-situation" we can formulate theorems on existence and uniqueness of solutions for nonlocal linear equations but regretfully, we cannot develop a meaningful theory of initial value problems.

The Doetsch Representation theorem is our main tool for the understanding of nonlocal equations.

### 2.2. The operator $f(\partial_t) : L^p(\mathbb{R}_+) \longrightarrow H^q(\mathbb{C}_+)$

As a motivation, we calculate $f(\partial_t)\phi$ formally: we take a function $f$ analytic around zero, and suppose that $\phi \in L^{p'}(0,\infty)$ is smooth. Let us write

$$f(\partial_t)\phi = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, \partial_t^n \phi \,.$$

We note that this expansion is actually a rigorous result if $f$ extends to an entire function and $\phi$ is a $f$-analytic vector for $\partial_t$ on the space of square integrable functions $L^2(\mathbb{R}_+)$ (see Section 5 below); also, this expansion is a formal theorem if $f$ is an entire function and $\phi$ is an entire function of exponential type (see [16]; a generalization of this result appears in the companion paper [18]). Standard properties of the Laplace transform (see Proposition 2.2 or [20]) yield, formally,

$$\mathcal{L}(f(\partial_t)\phi)(s) = f(s)\mathcal{L}(\phi)(s) - \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{f^{(n)}(0)}{n!}\, s^{n-j}\phi^{(j-1)}(0)\,. \tag{2.3}$$

Now we recall from [27, 28] that we can define the formal series

$$r(s) = \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{f^{(n)}(0)}{n!}\, d_{j-1}\, s^{n-j}\ , \tag{2.4}$$

in which $d = \{d_j : j \geq 0\}$ is a sequence of complex numbers; thus, we can write (2.3) as

$$\mathcal{L}(f(\partial_t)\phi)(s) = f(s)\mathcal{L}(\phi)(s) - r(s)\ , \tag{2.5}$$

where $r(s)$ is given by (2.4) with $d_j = \phi^{(j)}(0)$.

The following lemma asserts that for some choices of the sequences $\{d_{j-1}\}_{j\geq 1}$, the series defined in (2.4) is in fact an analytic function. We include it here mainly because of its relation with the theory of Borel transforms, see the Remark appearing after the proof.

**Lemma 2.4.** *Let $R_1 > 1$ be the maximum radius of convergence of the Taylor series $f_T(s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, s^n$ . Set $0 < R < 1$ and suppose that the* *series*

$$\sum_{j=1}^{\infty} d_{j-1} \frac{1}{s^j} \tag{2.6}$$

*is uniformly convergent on compact sets for* $|s| > R$*. Then the series* (2.4) *is analytic on the disk* $|s| < R_1$.

*Proof.* Let us write the series (2.4) in the following form

$$r(s) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} s^n \sum_{j=1}^{n} d_{j-1}\, s^{-j} \ ,$$

and define $r_N(s)$ as the partial sum

$$r_N(s) := \sum_{n=1}^{N} \frac{f^{(n)}(0)}{n!} s^n \sum_{j=1}^{n} d_{j-1}\, s^{-j} \ .$$

Let $K \subset \{s : R < |s| < R_1\}$ be a compact set; there exists a positive constant $L_K$ such that for any $s \in K$ we have:

$$\sum_{j=1}^{\infty} \left| \frac{d_{j-1}}{s^j} \right| \leq L_K \ ;$$

also, given $\epsilon > 0$, there exist $N_0 = N_0(\epsilon)$ such that

$$\sum_{n=N_0+1}^{\infty} \left| \frac{f^{(n)}(0)}{n!} s^n \right| < \epsilon \ ;$$

from these inequalities we deduce that

$$|r(s) - r_N(s)| = \left| \sum_{n=N+1}^{\infty} \frac{f^{(n)}(0)}{n!} s^n \sum_{j=1}^{n} \frac{d_{j-1}}{s^j} \right| < \epsilon L_K \ ,$$

for any $N \geq N_0$ and uniformly for $s \in K$. Therefore, the partial sums $r_N(s)$ converge uniformly to $r(s)$ on $K$; since the compact subset $K$ is arbitrary, we have that $r(s)$ is analytic in $\{s : R < |s| < R_1\}$. Now we prove that $r(s)$

is analytic in $|s| \leq R$. We write $r(s)$ as

$$r(s) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} P_n(s) \;,$$

in which,

$$P_n(s) := \sum_{j=1}^{n} d_{j-1}\, s^{n-j} \; = d_0 S_{n-1} + d_1 S_{n-2} + \cdots + d_{n-1} S_0 \;,$$

where $S_n = \sum_{k=1}^{n} s^k$. Then, the convergence of the series $\sum_{j=1}^{\infty} d_{j-1}$ and the inequalities $R < 1 < R_1$ imply that the following three assertions hold:

- $\displaystyle\sum_{n=1}^{\infty} \left| \frac{f^{(n)}(0)}{n!} - \frac{f^{(n+1)}(0)}{(n+1)!} \right| < \infty \,,$
- $\lim_{n\to\infty} \frac{f^{(n)}(0)}{n!} = 0 \,,$
- $P_n(s)$ is uniformly bounded on $|s| \leq R$.

The result now follows from [30, Theorem 5.1.8]. □

As a corollary we recover a previously known lemma, see [27, Lemma 2.1]:

**Corollary 2.5.** *Let $f$ be an entire function. Set $R < 1$ and suppose that the series*

$$\sum_{j=1}^{\infty} d_{j-1} \frac{1}{s^j} \;,$$

*is convergent for $|s| > R$. Then the series* (2.4) *is an entire function.*

**Remark.** There exists a large class of series satisfying conditions of Lemma 2.4. Indeed, let $r > 0$ and denote by $Exp_r(\mathbb{C})$ the space of entire functions of exponential type $\tau < r$ (see [16, 41] and references therein). It is well know that if $\phi \in Exp_r(\mathbb{C})$ and it is of exponential type $\tau < r$, then its Borel transform

$$\mathcal{B}(\phi)(s) = \sum_{j=0}^{\infty} \frac{\phi^j(0)}{s^{j+1}}$$

converges uniformly on $|s| > \tau$ (see again [16, 41]). In our case, if $\phi \in Exp_1(\mathbb{C})$, its Borel transform $\mathcal{B}(\phi)(s)$ is precisely (2.6) for $d_j = \phi^j(0)$. We will come back to the class of functions of exponential type in our forthcoming paper [18].

Motivated by the previous computations and Doetsch's representation theorem (Theorem 2.3), we make the following definition:

**Definition 2.6.** Let $f$ be an analytic function on a region which contains the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$, and let $\mathcal{H}$ be the space of all $\mathbb{C}$-valued functions on $\mathbb{C}$ which are analytic on (regions of) $\mathbb{C}$. We fix $p$ and $p'$ such that $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and we consider the subspace $D_f$ of $L^{p'}(0, \infty) \times \mathcal{H}$ consisting of all the pairs $(\phi, r)$ such that

$$\widehat{(\phi, r)} = f\,\mathcal{L}(\phi) - r \tag{2.7}$$

belongs to the class $H^p(\mathbb{C}_+)$ . The domain of $f(\partial_t)$ as a linear operator from the product $L^{p'}(0, \infty) \times \mathcal{H}$ into $L^{p'}(0, \infty)$ is $D_f$. If $(\phi, r) \in D_f$ then we define

$$f(\partial_t)\,(\phi, r) = \mathcal{L}^{-1}(\ \widehat{(\phi, r)}\ ) = \mathcal{L}^{-1}(f\,\mathcal{L}(\phi) - r)\ . \tag{2.8}$$

Definition 2.6 is inspired by work in [27, 28]. However, this definition is not equivalent to the ones appearing therein, because in the present paper we are developing an $L^p$-theory instead of the very abstract "Widder space theory" of [27, 28]. This definition will be further generalized in [18] by means of the Borel transform, in order to capture an even larger class of symbols $f$. However, as we will see below, the foregoing interpretation for $f(\partial_t)$ already gives us a satisfactory way to deal with the initial value problem for linear nonlocal equations in many interesting cases.

### 2.3. Linear nonlocal equations

In this subsection we solve the nonlocal equation

$$f(\partial_t)(\phi, r) = J\ , \qquad J \in L^{p'}(0, \infty)\ , \tag{2.9}$$

in which we are using the above interpretation of $f(\partial_t)$. We assume hereafter that a suitable function $r \in \mathcal{H}$ has been fixed; consequently, we understand Equation (2.9) as an equation for $\phi \in L^{p'}(0, \infty)$ such that $(\phi, r) \in D_f$. We simply write $f(\partial_t)\phi = J$ instead of (2.9). First of all, we formalize what we mean by a solution in the present $L^p$-context.

**Definition 2.7.** Let us fix a function $r \in \mathcal{H}$. We say that $\phi \in L^{p'}(0, \infty)$ is a solution to the equation $f(\partial_t)\phi = J$ if and only if

1) $\widehat{\phi} = f\,\mathcal{L}(\phi) - r \in H^p(\mathbb{C}_+)$ ; ( i.e., $(\phi, r) \in D_f$ );

2) $f(\partial_t)(\phi) = \mathcal{L}^{-1}(\widehat{(\phi, r)}) = \mathcal{L}^{-1}(f\,\mathcal{L}(\phi) - r) = J\,.$

Our main theorem on existence and uniqueness of the solution to the linear problem (2.9) is the following abstract result:

**Theorem 2.8.** *Let us fix a function $f$ which is analytic in a region $D$ which contains the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$. We also fix $p$ and $p'$ such that $1 < p \le 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and we consider a function $J \in L^{p'}(\mathbb{R}_+)$ such that $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$. We assume that the function $(\mathcal{L}(J) + r)/f$ is in the space $H^p(\mathbb{C}_+)$. Then, the linear equation*

$$f(\partial_t)\phi = J \tag{2.10}$$

*can be uniquely solved on $L^{p'}(0, \infty)$. Moreover, the solution is given by the explicit formula*

$$\phi = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(J) + r}{f}\right)\,. \tag{2.11}$$

*Proof.* We set $\phi = \mathcal{L}^{-1}\left((\mathcal{L}(J) + r)/f\,\right)$. Since $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$, it follows that the pair $(\phi, r)$ is in the domain $D_f$ of the operator $f(\partial_t)$: indeed, an easy calculation using Theorem 2.3 shows that $\widehat{\phi} = \mathcal{L}(J)$, which is an element of $H^p(\mathbb{C}_+)$ by hypothesis. We can then check (using Theorem 2.3 again) that $\phi$ defined by (2.11) is a solution of (2.10).

We prove uniqueness using Definition 2.7: let us assume that $\phi$ and $\psi$ are solutions to Equation (2.10). Then, item 2 of Definition 2.7 implies $f\,\mathcal{L}(\phi - \psi) = 0$ on $\{s \in \mathbb{C} : Re(s) > a\}$. Set $h = \mathcal{L}(\phi - \psi)$ and suppose that $h(s_0) \neq 0$ for $s_0$ in the half-plane just defined. By analyticity, $h(s) \neq 0$ in a suitable neighborhood $U$ of $s_0$. But then $f = 0$ in $U$, so that (again by analyticity) $f$ is identically zero. □

**Remark.** Interestingly, the above theorem on solutions to Equation (2.10) is a fully rigorous version of theorems stated long ago, see for instance the classical papers [11, 15] by Bourlet and Carmichael, and also the recent work [5] in which further references appear. We recover a theorem proven by Carmichael in [15] in Corollary 2.10 below. We also note that the proof of Theorem 2.8 is different from the proofs of similarly worded results appearing in [27, 28] in the context of Widder space, because in our present $L^p$-setup the Laplace transform is not an isomorphism between fixed Banach spaces unless $p = 2$ as we explain in Section 5.

In Section 3 we impose further conditions on $J$ and $f$ which assure us that (2.11) is smooth at $t = 0$, and we use these conditions to study the initial value problem for (2.10).

We recall that we have fixed an analytic function $r$. Hereafter we assume the natural decay condition

$$\left|\frac{r(s)}{f(s)}\right| \leq \frac{C}{|s|^q} \tag{2.12}$$

for $|s|$ sufficiently large and some real number $q > 0$ (see for instance [20]): this condition allows us to give very explicit descriptions of the solution 2.11. These descriptions depend on the poles of $r/f$, and therefore it is natural to state corollaries of Theorem 2.8 for the following three cases: (a) the function $r/f$ has no poles; (b) the function $r/f$ has a finite number of poles; (c) the function $r/f$ has an infinite number of poles. The proofs of the corollaries below use arguments from [20] and proceed as in [28]; we omit them.

**Corollary 2.9.** *Assume that the hypotheses of Theorem* 2.8 *hold, that* $\mathcal{L}(J)/f$ *is in* $H^p(\mathbb{C}_+)$*, and that* $r/f$ *is an entire function such that* (2.12) *holds. Then, solution* (2.11) *to Equation* (2.10) *is simply* $\phi = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(J)}{f}\right)$.

**Corollary 2.10.** *Assume that the hypotheses of Theorem* 2.8 *hold, and that* $r/f$ *has a finite number of poles* $\omega_i$ $(i = 1, \ldots, N)$ *of order* $r_i$ *to the left of* $Re(s) = 0$*. Suppose also that* $\mathcal{L}(J)/f$ *is in* $H^p(\mathbb{C}_+)$*, and that the growth condition* (2.12) *holds. Then, the solution* $\phi \in L^{p'}(\mathbb{R}_+)$ *given by* (2.11) *can be represented in the form*

$$\phi(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{s\,t}\left(\frac{\mathcal{L}(J)}{f}\right)(s)\,ds + \sum_{i=1}^{N} P_i(t)\,e^{\omega_i t}\,, \qquad \sigma > 0\,, \tag{2.13}$$

*in which* $P_i(t)$ *are polynomials of degree* $r_i - 1$.

The explicit formula (2.13) for the solution $\phi(t)$ is precisely Carmichael's formula appearing in [15], as quoted in [5]. It will appear prominently in the study of the initial value problem we carry out in Section 3.

**Corollary 2.11.** *Assume that the hypotheses of Theorem* 2.8 *hold, that* $\mathcal{L}(J)/f$ *is in* $H^p(\mathbb{C}_+)$*, that the quotient* $r/f$ *has an infinite number of poles* $\omega_n$ *of order* $r_n$ *to the left of* $Re(s) = 0$*, and that* $|\omega_n| \leq |\omega_{n+1}|$ *for* $n \geq 1$*. Suppose that there exist curves* $\sigma_n$ *in the half-plane* $Re(s) \leq 0$ *satisfying the following:*

*The curves $\sigma_n$ connect points $+ib_n$ and $-ib_n$ , where the numbers $b_n$ are such that the closed curve formed by $\sigma_n$ together with the segment of the line $Re(s) = 0$ between the points $+ib_n$ and $-ib_n$ , encloses exactly the first $n$ poles of $r(s)/f(s)$ and, moreover, $\lim_{n\to\infty} b_n = \infty$.*

*If the condition*

$$\lim_{n\to\infty} \int_{\sigma_n} e^{st} \left(\frac{r}{f}\right)(s)ds = 0$$

*(almost everywhere in $t$) holds, then the solution $\phi \in L^{p'}(\mathbb{R}_+)$ given by (2.11), can be represented in the form*

$$\phi(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\,t} \left(\frac{\mathcal{L}(J)}{f}\right)(s)\, ds + \sum_{n=1}^{\infty} P_n(t)\, e^{\omega_n t} \ , \quad \sigma > 0 \ , \tag{2.14}$$

*in which $P_n(t)$ are polynomials of degree $r_n - 1$.*

**Remark.** The solution (2.11) is not necessarily differentiable, see for instance the example appearing in [28, p. 9]. This means, in particular, that in complete generality we cannot even formulate initial value problem for equations of the form $f(\partial_t)\phi = J$.

## 3. The initial value problem

We discuss the existence of solutions to the initial value problem for equations of the form

$$f(\partial_t)\phi = J \ , \qquad t \geq 0 \ , \tag{3.1}$$

in which $f$ is an (in principle arbitrary) analytic function, in the context of the theory developed in Section 2.

### 3.1. Generalized initial conditions

Our first observation is that the foregoing analysis was carried out by fixing the function $r$ appearing in (2.11), see Subsection 2.3 As remarked in [28], we think of $r$ as a “generalized initial condition”:

**Definition 3.1.** A generalized initial condition for the equation

$$f(\partial_t)\phi = J \tag{3.2}$$

is an analytic function $r_0$ such that $(\phi, r_0) \in D_f$ for some $\phi \in L^{p'}(0,\infty)$. A generalized initial value problem is an equation such as (3.2) together with a generalized initial condition $r_0$. A solution to a given generalized initial value problem $\{(3.2), r_0\}$ is a function $\phi$ satisfying the conditions of Definition 2.7 with $r = r_0$.

An application of this definition can be found in [28, Section 3] and [17]. As remarked after Corollary 2.11, the unique solution (2.11) to (3.2) is not necessarily analytic: all we know is that the solution belongs to the class $L^{p'}(0,\infty)$ for some $p' > 1$. This is why Definition 3.1 is important. Nonetheless, we now show that we *can* define initial value problems subject to a finite number of a priori local data.

## 3.2. Classical initial value problems

In this subsection we point out that we can define and solve initial value problems depending on a finite number of initial local data, and not on "initial functions". Our main tool is Corollary 2.10. We have already motivated our approach in [28] and therefore we go directly to our new technical results. We note that, in appearence, the analysis carried out in this subsection is similar to the one in [28]. However, this is not so, because our present functional analytic framework of Lebesgue ($L^p$) and Hardy ($H^q$) spaces is completely different to the one appearing therein, even in the Hilbert space case $p = q = 2$.

**Lemma 3.2.** *Let $J$ be a function such that $\mathcal{L}(J)$ exists and let $f$ be an analytic function. Suppose that there exist an integer $M \geq 0$ and a real number $\sigma > 0$ such that*

$$y \mapsto y^n \left( \frac{\mathcal{L}(J)(\sigma + iy)}{f(\sigma + iy)} \right) \text{ belongs to } L^1(\mathbb{R}) \tag{3.3}$$

*for each $n = 0, \ldots, M$; then the function*

$$t \mapsto \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{s\,t} \left( \frac{\mathcal{L}(J)}{f} \right)(s)\, ds \tag{3.4}$$

*is of class $C^M$.*

This technical lemma is proven in [17], see also [28]. Lemma 3.2 implies the following:

**Lemma 3.3.** *Let the functions $f$ and $J$ satisfy the conditions of Corollary* 2.10*, and also that they satisfy* (3.3) *for some $\sigma > 0$. Then, the solution* (2.13) *to the nonlocal equation*

$$f(\partial_t)\phi(t) = J(t) \tag{3.5}$$

*is of class $C^M$ for $t \geq 0$, and it satisfies the identities*

$$\phi^{(n)}(0) = L_n + \sum_{i=1}^{N} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \omega_i^k \, \frac{d^{n-k}}{dt^{n-k}}\bigg|_{t=0} P_i(t)\,, \quad n = 0, \ldots, M, \tag{3.6}$$

*for some numbers $L_n$ .*

This result is proven in [27, 28]. The numbers $L_n$, for $n = 0, \ldots, M$, are computed to be

$$L_n = \frac{d^n}{dt^n}\bigg|_{t=0} \left( \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \left( \frac{\mathcal{L}(J)}{f} \right)(s)\, ds \right). \tag{3.7}$$

Our main result on initial value problems is the following. (The word "generic" appearing in the enunciate will be explained in the proof, see paragraph below Equation (33) ).

**Theorem 3.4.** *We fix real numbers $1 < p \leq 2$ and $p' > 0$ such that $1/p + 1/p' = 1$, and we also fix an integer $N \geq 0$. Let $f$ be a function which is analytic in a region $D$ which contains $\{s \in \mathbb{C} : Re(s) > 0\}$, and let $J$ be a function in $L^{p'}(\mathbb{R}_+)$ satisfying the conditions $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$ and $\mathcal{L}(J)/f \in H^p(\mathbb{C}_+)$. We choose points $\omega_i$, $i = 1, \ldots, N$, to the left of $Re(s) = 0$, and positive integers $r_i$ , $i = 1, ..., N$. Set $K = \sum_{i=1}^N r_i$ and assume that for some $\sigma > 0$ condition* (3.3) *holds for each $n = 0, \ldots, M$, $M \geq K$. Then, generically, given $K$ values $\phi_0, \ldots, \phi_{K-1}$, there exists a unique analytic function $r_0$ such that*

($\alpha$) *$\dfrac{r_0}{f} \in H^p(\mathbb{C}_+)$ and it has a finite number of poles $\omega_i$ of order $r_i$, $i = 1, \ldots, N$;*

($\beta$) *$\dfrac{\mathcal{L}(J) + r_0}{f} \in H^p(\mathbb{C}_+)$;*

($\gamma$) $\left|\dfrac{r_0}{f}(s)\right| \leq \dfrac{M}{|s|^q}$ *for some* $q \geq 1$ *and* $|s|$ *sufficiently large.*

*Moreover, the unique solution* $\phi = \mathcal{L}^{-1}\left(\frac{\mathcal{L}(J)+r_0}{f}\right)$ *to Equation* (3.5) *belongs to* $L^{p'}(\mathbb{R}_+)$*, is of class* $C^K$*, and it satisfies* $\phi(0) = \phi_0, \ldots, \phi^{(K-1)}(0) = \phi_{K-1}$ *.*

*Proof.* We consider the $K$ arbitrary numbers $\phi_n$, $n = 0, 1, \ldots, K-1$. Recalling (3.6), Lemma 3.2, and Lemma 3.3, we set up the linear system

$$\phi_n = L_n + \sum_{i=1}^{N}\sum_{k=0}^{n}\binom{n}{k}\omega_i^k \left.\frac{d^{n-k}}{dt^{n-k}}\right|_{t=0} P_i(t)\,, \quad n = 0, \ldots, K-1, \tag{3.8}$$

in which the unknowns are the coefficients of polynomials

$$P_i(t) = a_{1,i} + a_{2,i}\,\frac{t}{1!} + \cdots + a_{r_i,i}\,\frac{t^{r_i-1}}{(r_i-1)!}\,,$$

and the numbers $L_n$ are given by (3.7).

System (3.8) can be solved (generically, this is, away from the variety in $\mathbb{R}^N$ defined by all the points $\omega_i$ for which the main determinant of the linear system (33) vanishes) uniquely in terms of the data $\phi_n$ . We define $r_0$ on the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$ as follows:

$$r_0(s) = f(s)\,\mathcal{L}\left(\sum_{i=1}^{N} P_i(t)e^{\omega_i t}\right)(s)\,. \tag{3.9}$$

Then, on this half-plane we have the identity

$$\mathcal{L}^{-1}(r_0/f)(t) = \sum_{i=1}^{N} P_i(t)e^{\omega_i t} \tag{3.10}$$

for all $t \in \mathbb{R}_+$. Let us prove that $\frac{r_0}{f}$ belongs to $H^p(\mathbb{C}_+)$. First of all, we have the standard formula

$$\mathcal{L}(bt^n e^{\omega_j t})(s) = \frac{bn!}{(s-\omega_j)^{n+1}}$$

for any $n \in \mathbb{N}$ and $b \in \mathbb{C}$, and so the function $\frac{r_0}{f}$ is analytic on the half-plane $Re(s) > 0$. We show that $\mu_p(\frac{r_0}{f}, x)$ is uniformly bounded on $x > 0$. For this, it is enough to prove that the function $\mu_p(\mathcal{L}(bt^n e^{\omega_j t}), x)$ is uniformly bounded for $x > 0$. Indeed, since the poles $\omega_j := a_j + ib_j$ of $\frac{r_0}{f}$ satisfy $a_j = Re(\omega_j) < 0$, we have $|x - a_j| > |a_j| > 0$ for $x > 0$. Therefore:

$$
\begin{aligned}
\int_{-\infty}^{-\infty} \left|\mathcal{L}(bt^n e^{\omega_j t})(x+iy))\right|^p dy &= |b| n! \int_{-\infty}^{-\infty} \frac{dy}{|x+iy-\omega_j|^{(n+1)p}} \\
&= |b| n! \int_{-\infty}^{\infty} \frac{dy}{((x-a_j)^2 + (y-b_j)^2)^{\frac{n+1}{2}p}} \\
&= |b| n! \int_{-\infty}^{\infty} \frac{d\xi}{((x-a_j)^2 + \xi^2)^{\frac{n+1}{2}p}} \\
&< 2|b| n! \int_{0}^{\infty} \frac{d\xi}{(a_j^2 + \xi^2)^{\frac{n+1}{2}p}} \qquad (3.11)\\
&< \infty.
\end{aligned}
$$

Thus, $\mu_p(\mathcal{L}(bt^n e^{\omega_j t}), x)$ is uniformly bounded for $x > 0$. This shows that $\frac{r_0}{f}$ belongs to $H^p(\mathbb{C}_+)$, and $(\alpha)$ follows.

We easily check that $r_0/f$ also satisfies conditions $(\beta)$, and $(\gamma)$ appearing in the enunciate of the theorem; we omit the details.

Now we define the function

$$
\phi(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \left(\frac{\mathcal{L}(J)}{f}\right)(s)\, ds + \sum_{i=1}^{N} P_i(t)\, e^{\omega_i t} \qquad (3.12)
$$

in $L^{p'}(\mathbb{R}_+)$. We claim that this function solves Equation (3.5) and that it satisfies the conditions appearing in the enunciate of the theorem. In fact, the foregoing analysis implies that

$$
\phi(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \left(\frac{\mathcal{L}(J)}{f}\right)(s)\, ds + \mathcal{L}^{-1}(r_0/f)(t),
$$

and this is precisely the unique solution to (3.5) appearing in Corollary 2.10 for $r = r_0$.

It remains to show that this solution satisfies $\phi^{(n)}(0) = \phi_n$ for $n = 0, \ldots, K-1$. Indeed, condition (3.3) tells us that $\phi(t)$ is at least of class $C^K$ and clearly

$$
\phi^{(n)}(0) = L_n + \sum_{i=1}^{N} \sum_{k=0}^{n} \binom{n}{k} \omega_i^k \left. \frac{d^{n-k}}{dt^{n-k}} \right|_{t=0} P_i(t) \qquad (3.13)
$$

in which

$$L_n = \frac{d^n}{dt^n}\bigg|_{t=0} \left( \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \left( \frac{\mathcal{L}(J)}{f} \right)(s)\, ds \right) . \tag{3.14}$$

Comparing (3.8) and (3.13) we obtain $\phi^{(n)}(0) = \phi_n$, $n = 0, \dots, K-1$. □

**Remark.**

- The proof of Theorem 3.4 breaks down if $p = 1$, as we lose the uniform bound (3.11). This is the reason why, as advanced in Subsection 2.1, "$L^1$-correspondence theorems" (see [19, Theorem 2] and [20, Theorem 28.2]) do not allow us to obtain a meaningful theory of initial value problems.
- Theorem 3.4 tells us that we can freely chose the first $K$ derivatives $\phi^{(n)}(0)$, $n = 0, \dots, K-1$ of the solution $\phi$ to Equation (3.5) but, from $n = K$ onward, if the derivative $\phi^{(n)}(0)$ exists, it is completely determined by (3.13) and (3.14). Thus, it does not make sense to formulate an initial value problem $f(\partial_t)\phi(t) = J(t)$, $\phi^{(n)}(0) = \phi_n$, with more than $K$ arbitrary initial conditions.

The above proof shows that the *a priori* given points $\omega_i$ become the poles of the quotient $r_0/f$, that the *a priori* given numbers $r_i$ are their respective orders, and that this information is essential in order to have meaningful initial value problems. If no points $\omega_i$ are present, the solution to the nonlocal equation (3.5) is simply

$$\phi = \mathcal{L}^{-1}(\mathcal{L}(J)/f) \ ,$$

a formula which fixes completely ( for $f$ and $J$ satisfying (3.3) ) the values of the derivatives of $\phi$ at zero. This discussion motivates the following definition:

**Definition 3.5.** A classical initial value problem for nonlocal equations is a triplet formed by a nonlocal equation

$$f(\partial_t)\phi = J \ , \tag{3.15}$$

a finite set of data:

$$\left\{N \geq 0\,;\ \{\omega_i \in \mathbb{C}\}_{1\leq i\leq N} \qquad \{r_i \in \mathbb{Z}, r_i > 0\}_{1\leq i\leq N}\,;\ \{\{\phi_n\}_{0\leq n\leq K-1}, K = \textstyle\sum_{i=1}^{N} r_i\}\right\}, \tag{3.16}$$

and the conditions

$$\phi(0) = \phi_0\,, \quad \phi'(0) = \phi_1\,, \quad \cdots .\phi^{(K-1)}(0) = \phi_{K-1}\,. \tag{3.17}$$

A solution to a classical initial value problem given by (3.15), (3.16) and (3.17) is a pair $(\phi, r_0) \in D_f$ satisfying the conditions of Definition 2.7 with $r = r_0$ such that $\phi$ is differentiable at zero and (3.17) holds.

Theorem 3.4 implies that this definition makes sense. From its proof we also deduce the following important corollaries:

**Corollary 3.6.** *We fix $1 < p \leq 2$ and $p' > 0$ such that $1/p + 1/p' = 1$. Let $f$ be a function which is analytic on the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$, and fix a function $J$ in $L^{p'}(0,\infty)$ such that $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$ and $\mathcal{L}(J)/f \in H^p(\mathbb{C}_+)$. Then, generically (in the sense of Theorem* 3.4*), the classical initial value problem* (3.15)–(3.17) *has a unique solution which depends smoothly on the initial conditions* (3.17).

*Proof.* The function (3.12) is the unique solution to (3.15) and it satisfies (3.17). Moreover, the coefficients of the polynomials $P_i$ appearing in (3.12) depend smoothly on the data $\phi_n$ and $\omega_i$, since they are solutions to the linear problem (3.8). □

**Corollary 3.7.** *We fix $1 < p \leq 2$ and $p' > 0$ such that $1/p + 1/p' = 1$. Let $f$ be a function which is analytic on the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$, and fix $J$ in $L^{p'}(0,\infty)$ such that $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$ and $\mathcal{L}(J)/f \in H^p(\mathbb{C}_+)$. Suppose also that there exist a positive integer $K$ such that for some $\sigma > 0$ condition* (3.3) *holds, and consider a set of complex numbers $\phi_0, \phi_1, \ldots, \phi_{K-1}$. Then, there exists a classical initial value problem* (3.15)–(3.17).

*Proof.* Since $K > 0$ we can find positive integers $N, r_1, \ldots, r_N$ such that $K = \sum_{i=1}^{N} r_i$. Also, we can choose $N$ complex numbers $\omega_i, 1 \leq i \leq N$ to the left of $Re(s) = 0$. Thus, we have constructed the data (3.16). The nonlocal equation (3.15) is $f(\partial_t)\phi(t) = J(t)$, and conditions (3.17) are determined by the given complex numbers $\phi_0, \ldots, \phi_{K-1}$. □

**Remark.** We note that there is at least one natural way to choose the poles $\omega_i$ appearing in the proof of Corollary 3.7: if the symbol $f$ has $K$ zeroes (counting with multiplicities), say $\{z_1, z_2, \dots, z_K\}$, to the left of $Re(s) < 0$, and condition (3.3) holds for $n = 0, \dots, K$, we can obtain a unique solution to the initial value problem (3.15)–(3.17) if we choose $\omega_i = z_i$. In this case, the determinant of the linear system (3.8) is precisely the non-zero determinant of the $K \times K$-Vandermonde matrix $A_K = (a_{ji})$ with $a_{ji} = z_i^{j-1}$, $1 \le j, i \le K$.

## 4. Zeta nonlocal equations

In this section we apply our previous approach to zeta nonlocal equations of the form

$$\zeta_h(\partial_t)\phi(t) = J(t) \;, \qquad t \ge 0 \;, \tag{4.1}$$

in which $h$ is a real parameter and the symbol $\zeta_h$ is the shifted Riemann zeta function

$$\zeta_h(s) := \zeta(h+s) = \sum_{n=1}^{\infty} \frac{1}{n^{s+h}} \;.$$

These equations are motivated by the cosmological models appearing in [2, 22–24]. In turn, it has been called to the authors' attention by A. Koshelev that the approach of [2] is at least partially based on [32], in which the author introduces a formal schemme for analyzing some nonlocal equations of interest for cosmology.

We recall some properties of the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$, $Re(s) > 1$, following [31]. It is analytic on its domain of definition and it has an analytic continuation to the whole complex plane with the exception of the point $s = 1$, at which it has a simple pole with residue 1. The analytic continuation of the Riemann zeta function will be also denoted by $\zeta$, and we will refer to it also as the Riemann zeta function.

From standard properties of the Riemann zeta function (see [31]) we have that the shifted Riemann zeta function $\zeta_h$ is analytic for $Re(s) > 1 - h$, and uniformly and absolutely convergent for $Re(s) \ge \sigma_0 > 1 - h$. We also find (see [31, Chp. I.6]) that $\zeta_h$ has infinite zeroes at the points $\{-2n - h : n \in \mathbb{N}\}$ (we call them "trivial zeroes") and that it also has "nontrivial" zeroes in the region $-h < Re(s) < 1 - h$ (we call this region the "critical region" of $\zeta_h$).

The Euler product expansion for the shifted Riemann zeta function is

$$\zeta_h(s) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^{s+h}}\right)^{-1},$$

where $\mathcal{P}$ is the set of the prime numbers. Therefore, for $Re(s) = \sigma > 1 - h$, we have

$$\begin{aligned} \left|\frac{1}{\zeta_h(s)}\right| &= \left|\prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^{s+h}}\right)\right| = \left|\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+h}}\right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+h}} \leq 1 + \int_1^{\infty} \frac{dx}{x^{\sigma+h}} = \frac{\sigma+h}{\sigma+h-1}, \end{aligned} \tag{4.2}$$

where $\mu(\cdot)$ is the Möebius function defined as follows: $\mu(1) = 1$, $\mu(n) = 0$ if $n$ is divisible by the square of a prime, and $\mu(n) = (-1)^k$ if n is the product of $k$ distinct prime numbers, see [31, Chp. II.2].

We study Equation (4.1) for values of $h$ in the region $(1, \infty)$, since in this case $\zeta_h$ is analytic for $Re(s) > 0$ and the theory developed in the previous sections apply. We start with the following lemma:

**Lemma 4.1.** *We fix $1 < p \leq 2$ and $p' > 0$ such that $1/p + 1/p' = 1$. Let us assume that $J \in L^{p'}(\mathbb{R}_+)$ and that $\mathcal{L}(J)$ is in the space $H^p(\mathbb{C}_+)$. Then, the function $F = \frac{\mathcal{L}(J)}{\zeta_h}$ belongs to $H^p(\mathbb{C}_+)$.*

*Proof.* We have:

1. The function $F$ is clearly analytic for every $s$ such that $Re(s) > 0$.
2. Since $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$ we have that $\mu_p(\mathcal{L}(J), x)$ is uniformly bounded for $x > 0$, Now for $x > 0$ and using inequality (4.2), we obtain

$$\begin{aligned} \mu_p(F, x) &= \left(\int_{\mathbb{R}} \left|\frac{1}{\zeta_h(x+iy)}\mathcal{L}(J)(\sigma+iy)\right|^p dy\right)^{\frac{1}{p}} \\ &\leq \frac{x+h}{x+h-1}\left(\int_{\mathbb{R}} |\mathcal{L}(J)(x+iy)|^p dy\right)^{\frac{1}{p}} < \infty. \end{aligned}$$

Since the function $x \to \frac{x+h}{x+h-1}$ is uniformly bounded for $x \geq 0$, the result follows. $\square$

Now we note that we can replace the general condition (3.3) for the following assumption on the function $J$:

**(H)** For some $M \geq 0$, for each $n = 1, 2, 3, \ldots, M$ and for some $\sigma > 0$ we have,

$$y \to y^n \mathcal{L}(J)(\sigma + iy) \in L^1(\mathbb{R}).$$

Condition (**H**) is enough to ensure differentiability of the function

$$t \to \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} \left( \frac{\mathcal{L}(J)}{\zeta_h} \right)(s) ds \ ,$$

as asked in the hypotheses of Lemma 3.2. Our main theorem on classical initial value problems for the zeta non-local equation (4.1) is the following

**Theorem 4.2.** *We fix $1 < p \leq 2$ and $p' > 0$ such that $1/p + 1/p' = 1$. Let $\zeta_h$ be the shifted Riemann zeta function, and assume that $J \in L^{p'}(\mathbb{R}_+)$ with $\mathcal{L}(J) \in H^p(\mathbb{C}_+)$. We also fix a number $N \geq 0$, a finite number of points $\omega_i$, $i = 1, ..., N$, to the left of $Re(s) = 0$, and a finite number of positive integers $r_i$. We set $K = \sum_{i=1}^{N} r_i$ and we assume that condition* (**H**) *holds for all $n = 0, \ldots, M$, $M \geq K$. Then, generically, given $K$ initial conditions, $\phi_0, \ldots, \phi_{K-1}$, there exists a unique analytic function $r_0$ such that*

($\alpha$) $\dfrac{r_0}{\zeta_h} \in H^p(\mathbb{C}_+)$ *and it has a finite number of poles $\omega_i$ of order $r_i$, $i = 1, \ldots, N$ to the left of $Re(s) = 0$;*

($\beta$) $\dfrac{\mathcal{L}(J) + r_0}{\zeta_h} \in H^p(\mathbb{C}_+)$;

($\gamma$) $\left| \dfrac{r_0}{\zeta_h}(s) \right| \leq \dfrac{M}{|s|^q}$ *for some $q \geq 1$ and $|s|$ sufficiently large.*

*Moreover, the unique solution $\phi$ to Equation* (4.1) *given by* (2.11) *with $r = r_0$ is of class $C^K$ and it satisfies $\phi(0) = \phi_0, \ldots, \phi^{(K-1)}(0) = \phi_{K-1}$.*

*Proof.* The proof consists in checking that the hypotheses of Theorem 3.4 hold. In fact, $\zeta_h$ is analytic on $\{s \in \mathbb{C} : Re(s) > 0\}$, and Lemma 4.1 tells us that $\mathcal{L}(J)/\zeta_h$ belongs to $H^p(\mathbb{C}_+)$. □

## 5. An $L^2(\mathbb{R}_+)$-theory for linear nonlocal equations

In this section we show that in the $p = p' = 2$ case of the foregoing theory, we can justify rigorously the interpretation of $f(\partial_t)$ as an operator in infinitely many derivatives on an appropriate domain. Our approach uses the technique of analytic vectors, motivated by Nelson's classical paper [36]. We have used analytic vectors in previous papers, see [28]; the novelty here is that now we can exploit the classical Paley-Wiener theorem stating that the Laplace transform is an isomorphism between the Hilbert spaces $L^2(\mathbb{R}_+)$ and $H^2(\mathbb{C}_+)$.

**Definition 5.1.** Let $A$ be a linear operator from a Banach space $\mathbb{V}$ to itself, and let $f : \mathbb{R} \to \mathbb{C}$ be a complex valued function, such that $f^{(n)}(0)$ exist for all $n \geq 0$. We say that $v \in \mathbb{V}$ is a $f$-analytic vector for $A$ if $v$ is in the domain of $A^n$ for all $n \geq 0$ and the series

$$\sum_{n=0} \frac{f^{(n)}(0)}{n!} A^n v \ ,$$

defines a vector in $\mathbb{V}$.

As stated in Section 2, the Paley-Wiener theorem (see [29, 45]) is the following special case of Doetsch's representation theorem:

**Theorem 5.2.** *The following assertions hold:*

1) *If $g \in L^2(\mathbb{R}_+)$, then $\mathcal{L}(g) \in H^2(\mathbb{C}_+)$.*
2) *Let $G \in H^2(\mathbb{C}_+)$. Then the function*

$$g(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} G(s) ds \ , \sigma > 0 \ ,$$

*is independent on $\sigma$, it belongs to $L^2(\mathbb{R}_+)$ and it satisfies $G = \mathcal{L}(g)$.*

*Moreover the Laplace transform $\mathcal{L} : L^2(\mathbb{R}_+) \to H^2(\mathbb{C}_+)$ is a unitary operator.*

We examine existence of analytic vectors for the operator $\partial_t$ on $L^2(\mathbb{R}_+)$.

**Lemma 5.3.** *Let $f$ be an analytic function on a region containing zero, and let $R_1$ be the maximun radius of convergence of the Taylor series $f_T(s) := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} s^n$ .*

a) *If $p$ is a polynomial on $\mathbb{R}_+$, and $I$ a finite interval on $\mathbb{R}_+$, then the function $\psi := p \cdot \chi_I$ is an $f$-analytic vector for $\partial_t$ on $L^2(\mathbb{R}_+)$.*
b) *Let $R_1 > 1$. If $\psi \in C^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ such that for all $n \geq 1$ and some $h \in L^2(\mathcal{I})$ for $\mathcal{I}$ equal to either $\mathbb{R}$ or $\mathbb{R}_+$, we have $\|\psi^{(n)}\|_{L^2(\mathbb{R}_+)} \leq c(n) \|h\|_{L^2(\mathcal{I})}$, with $\{c(n)\}_{n \in \mathbb{N}} =: c \in l^1(\mathbb{N})$. Then $\psi$ is an $f$-analytic vector for $\partial_t$ on $L^2(\mathbb{R}_+)$.*

*Proof.* Part $a)$ is immediate. For $b)$ we have: Since $R_1 > 1$, the sequence $\left\{\frac{f^{(n)}(0)}{n!}\right\}_{n\in\mathbb{N}}$ is bounded; therefore, there is a positive constant $C$ such that $|\frac{f^{(n)}(0)}{n!}| \leq C$ for every $n \in \mathbb{N}$. Now, by Minkowski inequality we have,

$$\begin{aligned}\left\|\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\psi^{(n)}\right\|_{L^2(\mathbb{R}_+)} &\leq \sum_{n=0}^{\infty}\left\|\frac{f^{(n)}(0)}{n!}\psi^{(n)}\right\|_{L^2(\mathbb{R}_+)} \\ &\leq C\sum_{n=0}^{\infty}\|\psi^{(n)}\|_{L^2(\mathbb{R}_+)} \\ &\leq C\|h\|_{L^2(\mathcal{A})}\sum_{n=0}^{\infty}c(n) \\ &\leq C\|c\|_{l^1(\mathbb{N})}\|h\|_{L^2(\mathcal{A})}\,.\end{aligned}$$

□

**Remark.** We present two large families of functions $\psi$ which meet conditions $b)$ of Lemma 5.3.

- Let $k > 1$ be a parameter and consider the functions $\psi(t) := e^{-\frac{t}{k}}, t \geq 0$; then we have $\psi^{(n)}(t) = \frac{(-1)^n}{k^n}e^{-\frac{t}{k}}$, therefore $|\psi^{(n)}(t)| = \frac{1}{k^n}e^{-\frac{t}{k}}$ and if we define $c(n) := \frac{1}{k^n}$, we have $\{c(n)\}_{n\in\mathbb{N}} \in l^1(\mathbb{N})$.
- We recall that an arbitrary entire function $\phi$ of exponential type $\tau$ which is also in $L^2(\mathbb{R})$, satisfies the generalized $L^2$-Berstein inequality

$$\|\phi^{(n)}\|_{L^2(\mathbb{R})} \leq \tau^n\|\phi\|_{L^2(\mathbb{R})}\ ,$$

see [37, Chp. 3]. We denote by $Exp_1^2(\mathbb{C})$ the space of entire functions of exponential type $\tau < 1$ which are $L^2$-functions on $\mathbb{R}$. Then, for any $\phi \in Exp_1^2(\mathbb{C})$ with exponential type $\tau_\phi > 0$, the function $\psi := \chi_{\mathbb{R}_+}\cdot\phi$ satisfies part $b)$ of the lemma with $h = \phi$. In particular, let $\phi$ be a smooth function on $\mathbb{R}$ with compact support in $[-\tau,\tau] \subset \mathbb{R}$ with $\tau < 1$. Then its Fourier transform

$$\mathcal{F}(\phi)(t) := (2\pi)^{-1/2}\int_{-\tau}^{\tau}e^{-ixt}\phi(x)dx$$

is an $L^2(\mathbb{R})$-function and it has an extension to an entire function $\Phi$ which is of exponential type $\tau$. The function $\psi := \chi_{\mathbb{R}_+}\cdot\Phi$ satisfies part $b)$ of the lemma.

Definition 2.6 of the operator $f(\partial_t)$ restricts to the present $L^2$-context. We state it explicitly for the reader's convenience.

**Definition 5.4.** Let $f$ be an analytic function on a region which contains zero and the half-plane $\{s \in \mathbb{C} : Re(s) > 0\}$, and let $\mathcal{H}$ be the space of all functions which are analytic on regions of $\mathbb{C}$. We consider the subspace $D_f$ of $L^2(\mathbb{R}_+) \times \mathcal{H}$ consisting of all the pairs $(\phi, r)$ such that

$$\widehat{(\phi, r)} = f\,\mathcal{L}(\phi) - r \tag{5.1}$$

belongs to the space $H^2(\mathbb{C}_+)$. The domain of $f(\partial_t)$ as a linear operator from $L^2(\mathbb{R}_+) \times \mathcal{H}$ to $L^2(\mathbb{R}_+)$ is the set $D_f$ . If $(\phi, r) \in D_f$ then

$$f(\partial_t)\,(\phi, r) = \mathcal{L}^{-1}(\ \widehat{(\phi, r)}\ ) = \mathcal{L}^{-1}(f\,\mathcal{L}(\phi) - r)\ . \tag{5.2}$$

We show that this definition is not empty, and that in fact the domain $D_f$ is quite large. The fact that the Laplace transform is an unitary operator plays an essential role at this point:

**Proposition 5.5.** *Let $f$ be a function which is analytic on a region containing $\mathbb{C}_+$, and let $R_1 > 1$ be the maximum radius of convergence of the Taylor series $f_T(s) := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, s^n$. Let $\phi$ be a smooth $f$-analytic vector for $\partial_t$ in $L^2(\mathbb{R}_+)$ and suppose that the sequence $\{d_j = \phi^{(j)}(0)\}$ satisfies the condition of Lemma 2.4. Then, there exists an analytic function $r_e$ on $\mathbb{C}_+$ such that $(\phi, r_e)$ is in the domain $D_f$ of $f(\partial_t)$.*

*Proof.* From Equation (2.4) and Lemma 2.4 we can define the following analytic function $\hat{\phi}$ on $\{z \in \mathbb{C} : |z| < R_1\}$ :

$$\begin{aligned}
\hat{\phi}(s) &:= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} s^n \mathcal{L}(\phi)(s) - \sum_{n=1}^{\infty}\sum_{j=1}^{n} \frac{f^{(n)}(0)}{n!}\, \phi^{(j-1)}(0)\, s^{n-j} \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \left( s^n \mathcal{L}(\phi)(s) - \sum_{j=1}^{n} \phi^{(j-1)}(0)\, s^{n-j} \right) \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mathcal{L}(\partial_t^n(\phi))(s) \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \int_0^{\infty} e^{-st} \partial_t^n(\phi)(t) dt\,.
\end{aligned}$$

Using this last equality and the fact that the Laplace transform is an isometric isomorphism from $L^2(\mathbb{R}_+)$ onto $H^2(\mathbb{C}_+)$, see Theorem 5.2, we have

$$\hat{\phi}(s) = \int_0^\infty e^{-st} \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \partial_t^n(\phi)(t) dt \ . \tag{5.3}$$

Therefore, on the disk $\{z \in \mathbb{C} : |z| < R_1\}$ we have the equation

$$\mathcal{L}\left( \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \partial_t^n(\phi)(t) \right)(s) = f(s)\mathcal{L}(\phi)(s) - r(s) \ , \tag{5.4}$$

where $r(s) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{f^{(n)}(0)}{n!} \, \phi^{(j-1)}(0) \, s^{n-j}$.

Now, we stress the fact that the right hand side of (5.3) belongs to the Hardy space $H^2(\mathbb{C}_+)$; it follows that we can extend $\hat{\phi}(s)$ via analytic continuation to a function $\hat{\phi}_e$ on the half-plane $Re(s) > 0$. This function belongs to the Hardy space $H^2(\mathbb{C}_+)$ by construction. Also, we note that Equation (5.4) implies

$$r(s) = f(s)\mathcal{L}(\phi)(s) - \hat{\phi}_e(s) \tag{5.5}$$

for $|s| < R_1$. However, the right hand side of (5.5) is defined on $Re(s) > 0$, and therefore it defines an analytic continuation $r_e$ of the series $r$ to the half-plane $Re(s) > 0$. Thus, on $Re(s) > 0$ we have the equation:

$$\hat{\phi}_e(s) = f(s)\mathcal{L}(\phi)(s) - r_e(s) \ .$$

Since $\hat{\phi}_e$ is in $H^2(\mathbb{C}_+)$, we have that $(\phi, r_e) \in D_f$. □

**Example 5.6.** It follows from our discussion on the function $\zeta_h$ introduced in Section 4, that $\zeta_h$ is analytic around zero for $h > 1$ large enough, and that therefore we have the power series expansion

$$\zeta_h(s) = \sum_{n=0}^{\infty} a_n(h) s^n \ .$$

Again for appropriate $h > 1$, we can assume that its maximum radius of convergence is $R_1 > 1$. Then, Lemma 5.3 implies that there exists a large class of $\zeta_h$-analytic vectors $\phi \in L^2(\mathbb{R}_+)$ and moreover, the above proposition applies. Thus, $\zeta_h(\partial_t)$ is a well-defined operator on (a subspace of) $L^2(\mathbb{R}_+) \times \mathcal{H}$.

An easy corollary of Proposition 5.5 is the following:

**Corollary 5.7.** *Let $f$ be an entire function and let $\phi$ be a smooth $f$-analytic vector for $\partial_t$ in $L^2(\mathbb{R}_+)$. Suppose that the sequence $\{d_j = \phi^{(j)}(0)\}$ satisfies*

$$d_j \leq C R^j$$

*for $0 < R < 1$. Then $(\phi, r) \in D_f$, in which $r$ is the series defined in the above proof.*

The proof of Corollary 5.7 consists in noting that the stated hypotheses allow us to apply Lemma 2.3.

Proposition 5.5 and Corollary 5.7 imply that (if $f$ is entire) the operator $f(\partial_t)$ is an operator in infinitely many derivatives on the space of smooth $f$-analytic vectors. In fact, let $\phi$ be a $f$-analytic vector for $\partial_t$ in $L^2(\mathbb{R}_+)$ and let

$$r(s) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{f^{(n)}(0)}{n!}\, \phi^{(j-1)}(0)\, s^{n-j} \; .$$

If conditions of Corollary 5.7 hold, then

$$\mathcal{L}\left( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \partial_t^n(\phi) \right) = f\mathcal{L}(\phi) - r = \mathcal{L}(f(\partial_t)\phi) \; ,$$

and therefore

$$f(\partial_t)\phi = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \partial_t^n(\phi) \; .$$

## 6. Discussion

As mentioned in Section 1, the following nonlocal equation appears naturally in the study of a zeta nonlocal scalar field model in string theory (See [22–24]; recall that we are using signature so that in the $1+0$ dimensional case the d'Alambert operator is $\partial_t^2$):

$$\zeta(\partial_t^2 + h)\phi = \mathcal{AC} \sum_{n=0}^{\infty} n^{-h} \phi^n \; , \quad t \geq 0 \; , \tag{6.1}$$

where $\mathcal{AC}$ means analytic continuation. We stress, after [1], that it is natural to consider the restriction $t \geq 0$ since classical versions of cosmological models contain singularities at the beginning of time.

Equation (6.1) motivates the study of the following nonlocal linear equations

$$\zeta(\partial_t^2 + h)\phi = J\ , \quad t \geq 0\ , \tag{6.2}$$

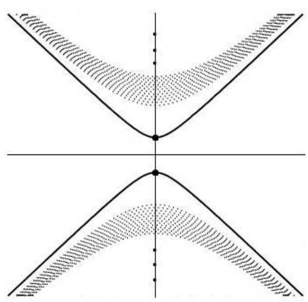
for appropriate functions $J$. Interestingly, the behavior of the symbol $\zeta(s^2 + h)$ is quite different to the behavior of the symbol $\zeta(s + h)$ appearing in Section 4. We show here that a study of Equation (6.2) requires a generalization of the theory developed in the above sections.

First of all, from the properties of the Riemann zeta function, we observe that the symbol

$$\zeta(s^2 + h) = \sum_{n=0}^{\infty} \frac{1}{n^{s^2+h}} \tag{6.3}$$

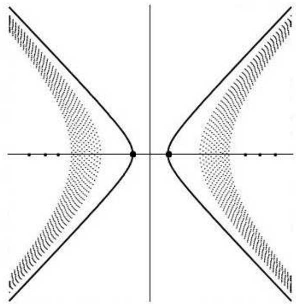
is analytic in the region $\Gamma := \{s \in \mathbb{C} : Re(s)^2 - Im(s)^2 > 1 - h\}$, which is not a half-plane; also we can note that its poles are the vertices of the hyperbolas $Re(s)^2 - Im(s)^2 = 1 - h$ and its critical region is the set $\{s \in \mathbb{C} : -h < Re(s)^2 - Im(s)^2 < 1 - h\}$. In fact, according to the value of $h$ we have:

i) For $h > 1$, $\Gamma$ is the region limited by the interior of the dark hyperbola $Re(s)^2 - Im(s)^2 = 1 - h$ containing the real axis:



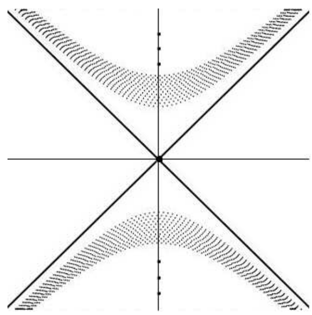
The poles of $\zeta(s^2 + h)$ are the vertices of dark hyperbola, indicated by two thick dots. The trivial zeroes of $\zeta(s^2 + h)$ are indicated by thin dots on the imaginary axis; and the non-trivial zeroes are located on the darker painted region (critical region).

ii) For $h < 1$, $\Gamma$ is the interior of the dark hyperbola $Re(s)^2 - Im(s)^2 = 1 - h$ containing the imaginary axis:



The poles of $\zeta(s^2+h)$ are the vertices of dark hyperbola, indicated by two thick dots. The trivial zeroes of $\zeta(s^2+h)$ are indicated by thin dots on the real axis; the non-trivial zeroes are located on the darker painted region (critical region).

iii) For $h = 1$, $\Gamma$ is the interior of the cones limited by the curves $y = |x|, y = -|x|$.



The pole of $\zeta(s^2+1)$ is the origin (vertex of dark curves $y = |x|, y = -|x|$). The trivial zeroes of $\zeta(s^2+h)$ are indicated by thin dots on the imaginary axis; the non-trivial zeroes are located on the darker painted region (critical region).

On the other hand, since the Riemann zeta function has an infinite number of zeroes on the critical strip (as famously proven by Hadamard and Hardy, see [31] for original references), we have that the function $\zeta(s^2+h)$ also has an infinite number of zeroes; we denote the set of all such zeroes by $\mathcal{Z}$. Using i), ii) and iii) we have that

$$\sup_{z\in\mathcal{Z}} |Re(z)| = +\infty.$$

This analysis implies that the expression $\mathcal{L}^{-1}(\mathcal{L}(J)/\zeta_h)$ for the solution to Equation (6.2) does not always make sense, since the function $\mathcal{L}(J)/\zeta_h$ does not necessarily belongs to $H^p(\mathbb{C}_+)$.

These observations mean that a new approach for the study of Equation (6.1) is necessary. We will present a method based on the Borel transform, see [16, 41], in the fortcoming paper [18].

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