ADV. THEOR. MATH. PHYS.
Volume 23, Number 3, 767–802, 2019

Some applications of the mirror theorem for toric stacks

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We use the mirror theorem for toric Deligne–Mumford stacks, proved recently by the authors and by Cheong–Ciocan-Fontanine–Kim, to compute genus-zero Gromov–Witten invariants of a number of toric orbifolds and gerbes. We prove a mirror theorem for a class of complete intersections in toric Deligne–Mumford stacks, and use this to compute genus-zero Gromov–Witten invariants of an orbifold hypersurface.

1. Introduction

Given a symplectic orbifold or Deligne–Mumford stack $\mathcal{X}$, one might want to calculate the Gromov–Witten invariants of $\mathcal{X}$:

$$\left\langle a_1\bar{\psi}^{k_1}, \ldots, a_n\bar{\psi}^{k_n} \right\rangle^{\mathcal{X}}_{g,n,d}$$

where $a_1, \ldots, a_n$ are classes in the Chen–Ruan orbifold cohomology of $\mathcal{X}$ and $k_1, \ldots, k_n$ are non-negative integers. Gromov–Witten invariants carry information about the enumerative geometry of $\mathcal{X}$: roughly speaking they count the number of orbifold curves in $\mathcal{X}$, of genus g and degree d, that pass through certain cycles (recorded by the classes a_i) and satisfy certain constraints on their complex structure. Computing Gromov–Witten invariants is in general hard, but one can often compute genus-zero Gromov–Witten invariants using mirror symmetry. A mirror theorem for toric Deligne–Mumford stacks was proved recently by the authors [13] and, independently, by Cheong–Ciocan-Fontanine–Kim [8]. In what follows we give various applications of this mirror theorem. We compute genus-zero Gromov–Witten invariants of a number of toric Deligne–Mumford stacks; prove a mirror theorem (Theorem 25) for certain complete intersections in toric Deligne–Mumford stacks; and use this to compute genus-zero Gromov–Witten invariants of an orbifold hypersurface. Along the way we make a technical point that may be useful elsewhere: showing that one can apply Coates–Givental/Tseng-style hypergeometric modifications to I-functions, rather than just to J-functions (Theorem 22). The mirror theorem proved here (Theorem 25) is new, and contains all known mirror theorems for toric complete intersections as special cases (see e.g. [12, 21]). It plays a key role in the recent proof of the Crepant Transformation Conjecture for complete intersections in toric Deligne–Mumford stacks [16].

This paper is written with two purposes in mind. It provides a reasonably self-contained guide that should help the reader to apply our mirror theorems to new examples. It also increases the number of explicit, nontrivial calculations of orbifold Gromov–Witten invariants in the literature. Orbifold Gromov–Witten theory is fraught with technical subtleties, and we hope that our calculations will be useful for others, as test examples for more sophisticated theories. The examples also demonstrate a practical advantage of our mirror theorems over existing methods [17, 24, 38–40], in that they often allow the direct determination of genus-zero Gromov–Witten invariants with insertions from twisted sectors, without needing to resort to the WDVV equation or reconstruction theorems [26, 36].

Let $\mathcal{X}$ be an algebraic Deligne–Mumford stack equipped with the action of a (possibly-trivial) torus $\mathbb{T}$. Suppose that $\mathcal{X}$ is sufficiently nice that one can define $\mathbb{T}$-equivariant Gromov–Witten invariants; this is the case, for example, if $\mathcal{X}$ is smooth as a stack and the coarse moduli space X of $\mathcal{X}$ is semi-projective (projective over affine). Let $H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$ denote the $\mathbb{T}$-equivariant Chen–Ruan cohomology of $\mathcal{X}$ (see §2.2). Let $\mathbf{\Lambda}(R)$ denote the Novikov ring of $\mathcal{X}$; this is a completion of the group ring $R[H_2(\mathcal{X};\mathbb{Z}) \cap \mathrm{NE}(\mathcal{X})]$ of the semigroup $H_2(\mathcal{X};\mathbb{Z}) \cap \mathrm{NE}(\mathcal{X})$ generated by classes of effective curves. Following Givental [22], Tseng has defined a symplectic structure on:

$$\mathcal{H} := H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \otimes_{H^{\bullet}_{\mathbb{T}}(\mathrm{pt})} \mathbf{\Lambda}\left(H^{\bullet}_{\mathbb{T}}(\mathrm{pt})(\!(z^{-1})\!)\right)$$

and a Lagrangian submanifold $\mathcal{L}$ of $\mathcal{H}$ that encodes all genus-zero Gromov–Witten invariants of $\mathcal{X}$ [37]. We will not give a precise definition of $\mathcal{L}$ in this paper, referring the reader to [13, §2] for a detailed discussion. For us, what will be important is that $\mathcal{L}$ determines and is determined by Givental's J-function:

(1)
$$J_{\mathcal{X}}(t,z) = z + t + \sum_{d \in H_2(X;\mathbb{Z})} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\alpha} \frac{Q^d}{n!} \left\langle t, t, \ldots, t, \phi_\alpha \psi^k \right\rangle^{\mathcal{X}}_{0,n+1,d} \phi^\alpha z^{-k-1}$$

where $t \in H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$; z is a formal variable; Q^d is the representative of d in the Novikov ring $\mathbf{\Lambda}$; the correlator denotes a Gromov–Witten invariant, exactly as in [13, §2]; and $\{\phi_\alpha\}$, $\{\phi^\alpha\}$ denote bases for $H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$ which are dual with respect to the pairing on Chen–Ruan cohomology. The submanifold $\mathcal{L}$ determines the J-function because $J_{\mathcal{X}}(t,-z)$ is the unique point on $\mathcal{L}$ of the form $-z + t + O(z^{-1})$, where $O(z^{-1})$ is a power series in z^{-1}. The J-function determines $\mathcal{L}$ because it determines all genus-zero Gromov–Witten invariants of $\mathcal{X}$ with descendant insertions at one or fewer marked points; it thus determines all genus-zero invariants with descendant insertions at two marked points via [19, Proposition 2.1], and determines all other genus-zero invariants via the Topological Recursion Relations [37, §2.5.7]. To determine the genus-zero Gromov–Witten invariants of $\mathcal{X}$, therefore, it suffices to determine the J-function $J_{\mathcal{X}}(t,z)$. In §3 and §5 below we use mirror theorems to determine the J-function of a number of Deligne–Mumford stacks $\mathcal{X}$.

The reader may be interested in the quantum orbifold cohomology ring of $\mathcal{X}$. Recovering quantum cohomology from the J-function is straightforward: general theory implies that $J_{\mathcal{X}}(t,z)$ satisfies a system of differential equations:

$$z \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^\beta} J_{\mathcal{X}}(t,z) = \sum_{\gamma} c_{\alpha\beta}{}^{\gamma}(t) \frac{\partial}{\partial t^\gamma} J_{\mathcal{X}}(t,z)$$

where $t = \sum_\alpha t^\alpha \phi_\alpha$ and the coefficients $c_{\alpha\beta}{}^{\gamma}(t)$ are the structure constants of the orbifold quantum product [3, 6, 23]. Thus:

$$z \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^\beta} J_{\mathcal{X}}(t,z) = \phi_\alpha \star_t \phi_\beta + O(z^{-1})$$

where $\star_t$ denotes the big orbifold quantum product with parameter $t \in H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$.

Acknowledgments

We thank Ionut Ciocan-Fontanine for useful discussions. T.C. was supported in part by a Royal Society University Research Fellowship, ERC Starting Investigator Grant number 240123, and the Leverhulme Trust. A.C. was supported in part by EPSRC grants EP/E022162/1 and EP/I008128/1. H.I. was supported in part by EPSRC grant EP/E022162/1 and JSPS Grant-in-Aid for Scientific Research 25400069, 16K05127, 16H06335, 16H06337, 17H06127. H.-H.T. was supported in part by a Simons Foundation Collaboration Grant.

2. The mirror theorem for toric Deligne–Mumford stacks

We assume that the reader is familiar with toric Deligne–Mumford stacks. A quick summary of the relevant material can be found in [13, §3]; the theory is developed in detail in [4, 18, 28–30].

2.1. Stacky fans

A toric Deligne–Mumford stack is defined by a *stacky fan* $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where N is a finitely generated abelian group, $\Sigma \subset N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$ is a rational simplicial fan, and $\rho\colon \mathbb{Z}^n \to N$ is a homomorphism with finite cokernel such that the images of the standard basis vectors in $\mathbb{Z}^n$ under the composition $\mathbb{Z}^n \xrightarrow{\ \rho\ } N \longrightarrow N_\mathbb{Q}$ generate the 1-dimensional cones of Σ. Let $\mathbb{L} \subset \mathbb{Z}^n$ be the kernel of ρ. The exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^n \xrightarrow{\ \rho\ } N$$

is called the *fan sequence*. Let $\rho_i \in N$ denote the image under ρ of the ith standard basis vector in $\mathbb{Z}^n$. Let $\rho^\vee\colon (\mathbb{Z}^*)^n \to \mathbb{L}^\vee := H^1(\mathrm{Cone}(\rho)^*)$ be the Gale dual [4] of ρ. There is an exact sequence

$$0 \longrightarrow N^* \longrightarrow (\mathbb{Z}^*)^n \xrightarrow{\ \rho^\vee\ } \mathbb{L}^\vee$$

called the *divisor sequence*. The toric Deligne–Mumford stack associated to the stacky fan $\mathbf{\Sigma}$ admits a canonical action of the torus $\mathbb{T} := N \otimes \mathbb{C}^\times$.

2.2. Chen–Ruan cohomology

Let $\mathcal{X}$ denote the toric Deligne–Mumford stack defined by the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$. Let N_{tor} denote the torsion subgroup of N, let $\overline{N} := N/N_{\mathrm{tor}}$, and let $\overline{c} \in \overline{N}$ denote the image of $c \in N$ under the canonical projection $N \to \overline{N}$. The *box* of $\mathbf{\Sigma}$ is:

$$\mathrm{Box}(\mathbf{\Sigma}) := \left\{ b \in N : \exists \sigma \in \Sigma \text{ such that} \right. \left. \bar{b} = \sum_{i:\bar{\rho}_i \in \sigma} a_i \bar{\rho}_i \text{ for some } a_i \text{ with } 0 \leq a_i < 1 \right\}$$

Components of the inertia stack $I\mathcal{X}$ are indexed by elements of $\mathrm{Box}(\mathbf{\Sigma})$, and we write $I\mathcal{X}_b$ for the component of inertia corresponding to $b \in \mathrm{Box}$.

The $\mathbb{T}$-equivariant Chen–Ruan orbifold cohomology [7, 34] of $\mathcal{X}$ is:

$$H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) := H^{\bullet}_{\mathbb{T}}(I\mathcal{X})$$

with a grading and product defined as follows. Let $R_{\mathbb{T}} := \mathrm{Sym}^{\bullet}_{\mathbb{C}}(N^* \otimes \mathbb{C}) = H^{2\bullet}_{\mathbb{T}}(\mathrm{pt})$, noting that elements of $H^{2k}(\mathrm{pt})$ are taken to have degree k. As an $R_{\mathbb{T}}$-module, we have:

$$H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X}) \cong \frac{R_{\mathbb{T}}[N]}{\left\{\chi - \sum_{i=1}^{n} \chi(\rho_i) y^{\rho_i} : \chi \in N^* \otimes \mathbb{C} \cong H^2_{\mathbb{T}}(\mathrm{pt})\right\}} \tag{2}$$

This is a graded ring with respect to the Chen–Ruan orbifold cup product [4, 31, 34]; here if $b \in N$ is such that $\bar{b} = \sum_{\bar{\rho}_i \in \sigma} m_i \bar{\rho}_i$ where σ is the minimal cone in containing $\bar{b}$, then y^b has degree $\sum_{\bar{\rho}_i \in \sigma} m_i$. The degree of y^b is known as the *age* of b. For $b \in \mathrm{Box}(\mathbf{\Sigma})$, the unit class supported on the component $I\mathcal{X}(\mathbf{\Sigma})_b$ of the inertia stack corresponds under (2) to y^b. The fact that $I\mathcal{X}_0 = \mathcal{X}$ gives a canonical inclusion $H^{\bullet}_{\mathbb{T}}(\mathcal{X}; \mathbb{C}) \subset H^{\bullet}_{\mathrm{CR},\mathbb{T}}(\mathcal{X})$, and the class $u_i \in H^2_{\mathbb{T}}(\mathcal{X})$ given by the $\mathbb{T}$-equivariant Poincaré-dual to the ith toric divisor in $\mathcal{X}$ corresponds under (2) to y^{ρ_i}.

2.3. Extended stacky fans

Let $\mathbf{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan, write $N_{\Sigma} := \{c \in N : \bar{c} \in |\Sigma|\}$, and let S be a finite set equipped with a map $S \to N_{\Sigma}$. We label the finite set S by $\{1, \ldots, m\}$, where $m = |S|$, and write $s_j \in N$ for the image of the jth element of S. The *S-extended stacky fan* is (N, Σ, ρ^S) where $\rho^S \colon \mathbb{Z}^{n+m} \to N$

is defined by:

$$\rho^S(e_i) = \begin{cases} \rho_i & 1 \leq i \leq n \\ s_{i-n} & n < i \leq n+m \end{cases}$$

and e_i denotes the ith standard basis vector for $\mathbb{Z}^n$. This gives an S-extended fan sequence

$$0 \longrightarrow \mathbb{L}^S \longrightarrow \mathbb{Z}^{n+m} \xrightarrow{\rho^S} N$$

and, by Gale duality, an S-extended divisor sequence:

$$0 \longrightarrow N^* \longrightarrow (\mathbb{Z}^*)^{n+m} \xrightarrow{\rho^{S\vee}} \mathbb{L}^{S\vee}$$

The toric Deligne–Mumford stacks associated to the stacky fan (N, Σ, ρ) and the S-extended stacky fan (N, Σ, ρ^S) are canonically isomorphic [30].

2.4. Extended degrees for toric stacks

Consider an S-extended stacky fan $\mathbf{\Sigma}$ as in §2.3, and let $\mathcal{X}$ be the corresponding toric Deligne–Mumford stack. The inclusion $\mathbb{Z}^n \to \mathbb{Z}^{n+m}$ of the first n factors induces an exact sequence:

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{L}^S \longrightarrow \mathbb{Z}^m$$

This splits over $\mathbb{Q}$, via the map $\mu \colon \mathbb{Q}^m \to \mathbb{L}^S \otimes \mathbb{Q}$ that sends the jth standard basis vector to

$$e_{j+n} - \sum_{i:\bar{\rho}_i \in \sigma(j)} s_{ji} e_i \in \mathbb{L}^S \otimes \mathbb{Q} \subset \mathbb{Q}^{n+m} \tag{3}$$

where $\sigma(j)$ is the minimal cone containing $\bar{s}_j$ and the positive numbers s_{ji} are determined by $\sum_{i:\bar{\rho}_i \in \sigma(j)} s_{ji} \bar{\rho}_i = \bar{s}_j$. Thus we obtain an isomorphism:

$$\mathbb{L}^S \otimes \mathbb{Q} \cong (\mathbb{L} \otimes \mathbb{Q}) \oplus \mathbb{Q}^m \tag{4}$$

Recall that $\mathrm{Pic}(\mathcal{X}) \cong \mathbb{L}^\vee$, and hence that the Mori cone $\mathrm{NE}(\mathcal{X})$ is a subset of $\mathbb{L} \otimes \mathbb{R}$. The *$S$-extended Mori cone* is the subset of $\mathbb{L}^S \otimes \mathbb{R}$ given by:

$$\mathrm{NE}^S(\mathcal{X}) = \mathrm{NE}(\mathcal{X}) \times (\mathbb{R}_{\geq 0})^m \qquad \text{via (4).}$$

The S-extended Mori cone can be thought of as the cone spanned by the "extended degrees" of certain orbifold stable maps $f \colon \mathcal{C} \to \mathcal{X}$: see [13, §4].

Notation 1. We denote the fractional part of x by $\langle x \rangle$.

Definition 2. Recall that $\mathbb{L}^S \subset \mathbb{Z}^{n+m}$, where $m = |S|$. For a cone $\sigma \in \Sigma$, denote by $\Lambda_\sigma^S \subset \mathbb{L}^S \otimes \mathbb{Q}$ the subset consisting of elements

$$\lambda = \sum_{i=1}^{n+m} \lambda_i e_i$$

such that $\lambda_{n+j} \in \mathbb{Z}, 1 \leq j \leq m$, and $\lambda_i \in \mathbb{Z}$ if $\bar{\rho}_i \notin \sigma$ and $i \leq n$. Set $\Lambda^S := \bigcup_{\sigma \in \Sigma} \Lambda_\sigma^S$.

Definition 3. The *reduction function* is

$$\begin{aligned} v^S \colon \Lambda^S &\longrightarrow \operatorname{Box}(\boldsymbol{\Sigma}) \\ \lambda &\longmapsto \sum_{i=1}^{n} \lceil \lambda_i \rceil \rho_i + \sum_{j=1}^{m} \lceil \lambda_{n+j} \rceil s_j \end{aligned}$$

The reduction function takes values in $\operatorname{Box}(\boldsymbol{\Sigma})$: for $\lambda \in \Lambda_\sigma^S$ we have $\overline{v^S(\lambda)} = \sum_{i=1}^{n} \langle -\lambda_i \rangle \bar{\rho}_i \in \sigma$.

Definition 4. For a box element $b \in \operatorname{Box}(\boldsymbol{\Sigma})$, we set:

$$\Lambda_b^S := \{\lambda \in \Lambda^S : v^S(\lambda) = b\}$$

and define:

$$\Lambda E^S := \Lambda^S \cap \mathrm{NE}^S(\mathcal{X}) \qquad \Lambda E_b^S := \Lambda_b^S \cap \mathrm{NE}^S(\mathcal{X})$$

Notation 5. Recall that Q^d denotes the representative of $d \in H_2(X;\mathbb{Z})$ in the Novikov ring $\boldsymbol{\Lambda}$. Given $\lambda \in \Lambda E^S$ write $\lambda = (d, k)$ via (4), so that $d \in \mathrm{NE}(\mathcal{X}) \cap H_2(X, \mathbb{Z})$ and $k \in (\mathbb{Z}_{\geq 0})^m$. We set:

$$\widetilde{Q}^\lambda = Q^d x^k = Q^d x_1^{k_1} \cdots x_m^{k_m} \in \boldsymbol{\Lambda}[\![x_1, \ldots, x_m]\!]$$

2.5. Mirror theorem

Once again, consider an S-extended stacky fan $\boldsymbol{\Sigma}$ as in §2.3. Let $\mathcal{X}$ be the corresponding toric Deligne–Mumford stack.

Definition 6. The *S-extended $\mathbb{T}$-equivariant I-function* of $\mathcal{X}$ is:

$$I^S(t,x,z) := ze^{\sum_{i=1}^n u_i t_i/z} \sum_{b\in \mathrm{Box}(\mathbf{\Sigma})} \sum_{\lambda \in \Lambda E_b^S} \widetilde{Q}^\lambda e^{\lambda t} \times \left(\prod_{i=1}^{n+m} \frac{\prod_{\langle a\rangle = \langle \lambda_i\rangle, a\leq 0}(u_i + az)}{\prod_{\langle a\rangle = \langle \lambda_i\rangle, a\leq \lambda_i}(u_i + az)} \right) y^b$$

Here:

- $t = (t_1, \ldots, t_n)$ are variables, and $e^{\lambda t} := \prod_{i=1}^n e^{(u_i\cdot d)t_i}$.
- $x = (x_1, \ldots, x_m)$ are variables: see Notation 5.
- for each $\lambda \in \Lambda E_b^S$, we write λ_i for the ith component of λ as an element of $\mathbb{Q}^{n+m}$; in particular $\langle \lambda_i\rangle = 0$ for $n < i \leq n+m$.
- For $1 \leq i \leq n$, u_i is the $\mathbb{T}$-equivariant Poincaré dual to the ith toric divisor: see Section 2.2. For $n < i \leq n+m$, u_i is defined to be zero.
- y^b is the unit class supported on the component of inertia $I\mathcal{X}(\mathbf{\Sigma})_b$ associated to $b \in \mathrm{Box}(\mathbf{\Sigma})$: see Section 2.2.

The I-function $I^S(t,x,z)$ is a formal power series in Q, x, t with coefficients in $H^\bullet_{\mathrm{CR},\mathbb{T}}(\mathcal{X})(\!(z^{-1})\!)$.

Theorem 7 (The mirror theorem for toric Deligne–Mumford stacks [8, 13]). *Let $\mathbf{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan, and let $\mathcal{X}$ be the corresponding toric Deligne–Mumford stack. Let S be a finite set equipped with a map to N_Σ. Suppose that the coarse moduli space of $\mathcal{X}$ is semi-projective (projective over affine). Then $I^S(t,x,-z) \in \mathcal{L}$.*

Remark 8. The statement that $I^S(t,x,-z) \in \mathcal{L}$ has a precise meaning in formal geometry: see §2.3 and Theorem 31 in [13]. The reader may want to work with a slightly vague but more intuitive interpretation of this statement: that $I^S(t,x,-z) \in \mathcal{L}$ for all values of the parameters t and x. No confusion should result, and the statements that we make are valid in the above, precise sense.

Remark 9. In the work of Cheong–Ciocan-Fontanine–Kim [8], a toric orbifold $\mathcal{X}$ is represented by a triple (V, T, θ) where T is a torus, V is a representation of T, and θ is a a character of T such that V has no strictly θ-semistable points. The toric orbifold $\mathcal{X}$ is the stack quotient $[V/\!\!/_\theta T]$. In this language, S-extending the stacky fan $\mathbf{\Sigma}$ corresponds to changing the GIT presentation (V, T, θ) of $\mathcal{X}$. Thus Theorem 7 with non-trivial S-extension (not just Theorem 7 with $S = \varnothing$) can be obtained from [8] by considering an appropriate GIT presentation of $\mathcal{X}$.

2.6. Condition $\sharp$ and condition S-$\sharp$

Recall that the J-function (1) is characterized by the fact that $J_{\mathcal{X}}(t, -z)$ is the unique point on $\mathcal{L}$ of the form $-z + t + O(z^{-1})$. Let $\mathbf{\Sigma}$ be a stacky fan and let S be a finite set equipped with a map $\kappa \colon S \to \mathrm{Box}(\mathbf{\Sigma})$. Label the elements of S by $\{1, \ldots, m\}$, where $m = |S|$, and let $s_j = \kappa(j)$. We say that the S-extended stacky fan $\mathbf{\Sigma}$ satisfies *condition* S-$\sharp$ if and only if

$$I^S(t, x, -z) = -z + t + \sum_{j=1}^{m} x_j y^{s_j} + O(z^{-1})$$

If condition S-$\sharp$ holds then $J_{\mathcal{X}}(\tau, z) = I^S(t, x, z)$ where $\tau = t + \sum_{j=1}^{m} x_j y^{s_j}$. We say that a stacky fan $\mathbf{\Sigma}$ satisfies *condition* $\sharp$ if and only if it satisfies condition S-$\sharp$ with $S = \varnothing$.

Condition $\sharp$ is equivalent to the statement: for all $b \in \mathrm{Box}(\mathbf{\Sigma})$ and all non-zero $\lambda \in \Lambda E_b^{\varnothing}$ we have:

$$-K_{\mathcal{X}} \cdot \lambda + \mathrm{age}(b) + \#\big\{i \mid \lambda_i < 0 \text{ and } \lambda_i \in \mathbb{Z}\big\} \geq 2$$

This is not automatically satisfied for Fano stacks or even for Fano orbifolds: see §3.7. The surface $\mathbb{F}_2$ is nonsingular and weak Fano and it does not satisfy condition $\sharp$.

Lemma 10 (A simple criterion for condition $\sharp$ to hold). *Let $\mathbf{\Sigma} = (N, \Sigma, \rho)$ be a stacky fan, and let $\mathcal{X}$ be the corresponding toric Deligne–Mumford stack. Suppose that $\mathcal{X}$ is smooth and has semi-projective coarse moduli space. If $\mathcal{X}$ is Fano and has* canonical singularities, *that is, if* $\mathrm{age}(b) \geq 1$ *for all non-zero elements $b \in \mathrm{Box}(\mathbf{\Sigma})$, then $\mathcal{X}$ satisfies condition $\sharp$.*

Proof. The key observation is that if $\mathcal{X}$ is a toric stack and $\lambda \in \Lambda E_b^{\varnothing} \setminus \{0\}$ is the class of a *compact* curve, then there are at least two toric divisors $D_i \subset \mathcal{X}$ such that $\lambda \cdot D_i > 0$. Set:

$$N_\lambda = \#\big\{i \mid \lambda_i < 0 \text{ and } \lambda_i \in \mathbb{Z}\big\}$$

The quantity $\mathrm{ord}\, \lambda := -K_{\mathcal{X}} \cdot \lambda + \mathrm{age}(b) + N_\lambda$ is an integer. If $N_l \geq 1$ then, since $\mathcal{X}$ is Fano, $-K_{\mathcal{X}} \cdot \lambda > 0$ and $\mathrm{ord}\, \lambda \geq 2$. If $N_\lambda = 0$ and $\mathrm{age}(b) \geq 1$, the same argument applies. Otherwise $b = 0$, and then $\lambda_i \in \mathbb{Z}$ for all i, and all $\lambda_i \geq 0$. By the key observation, at least two of the λ_i are strictly positive and we are done. □

Remark 11. Note that the criterion in Lemma 10 is far from best possible: the more positive the anticanonical class is, the worse the singularities are allowed to be.

3. Applying the mirror theorem

3.1. Example 1: $B\mu_3$

This is the toric Deligne–Mumford stack $\mathcal{X}$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where $N = \frac{1}{3}\mathbb{Z}/\mathbb{Z}$, $\Sigma = \{0\}$, and $\rho\colon (0) \to N$ is the zero map. We have $\mathrm{Box}(\mathbf{\Sigma}) = \left\{0, \frac{1}{3}, \frac{2}{3}\right\}$. We consider the S-extended I-function where $S = \mathrm{Box}(\mathbf{\Sigma})$ and $S \to N_\Sigma$ is the canonical inclusion. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} : \mathbb{Z}^3 \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} = \mathbb{Q}^3$ and $\mathbb{L}^S$ is the lattice of vectors:

$$\begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} \in \mathbb{Z}^3 \quad \text{such that} \quad k_1 + 2k_2 \equiv 0 \bmod 3$$

The S-extended Mori cone is the positive octant. We have $\Lambda^S = \mathbb{Z}^3$, and the reduction function is

$$v^S : \begin{pmatrix} k_0 \\ k_1 \\ k_2 \end{pmatrix} \mapsto \left\langle \frac{k_1}{3} + \frac{2k_2}{3} \right\rangle.$$

The S-extended I-function is:

$$I^S(x, z) = z \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{x_0^{k_0} x_1^{k_1} x_2^{k_2}}{z^{k_0+k_1+k_2} k_0! k_1! k_2!} \mathbf{1}_{\langle \frac{k_1}{3} + \frac{2k_2}{3} \rangle}$$

This is homogeneous of degree 1 if we set $\deg x_0 = \deg x_1 = \deg x_2 = \deg z = 1$. Since:

$$I^S(x, z) = z + x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{\frac{1}{3}} + x_2 \mathbf{1}_{\frac{2}{3}} + O(z^{-1})$$

condition S-$\sharp$ holds, and Theorem 7 implies that:

$$J_{\mathcal{X}}\big(x_0\mathbf{1}_0 + x_1\mathbf{1}_{\frac{1}{3}} + x_2\mathbf{1}_{\frac{2}{3}}, z\big) = I^S(x,z)$$

3.2. Example 2: $\frac{1}{3}(1,1)$

This is the toric Deligne–Mumford stack $\mathcal{X}$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where:

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z}^2 \to N = \mathbb{Z}^2 + \tfrac{1}{3}(1,1)\mathbb{Z}.$$

and Σ is the positive quadrant in $N_{\mathbb{Q}}$. We have $\operatorname{Box}(\mathbf{\Sigma}) = \big\{0, \frac{1}{3}(1,1), \frac{2}{3}(1,1)\big\}$; to streamline the notation we will identify $\operatorname{Box}(\mathbf{\Sigma})$ with the set $\big\{0, \frac{1}{3}, \frac{2}{3}\big\}$ via the map κ that sends x to $x(1,1)$. We consider the S-extended I-function where $S = \big\{0, \frac{1}{3}\big\}$ and S maps to N_{Σ} via κ. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \end{pmatrix} : \mathbb{Z}^{2+2} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^2$ is identified as a subset of $\mathbb{Q}^{2+2}$ via the inclusion:

$$\begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\frac{1}{3} \\ 0 & -\frac{1}{3} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix}$$

The S-extended Mori cone is the positive quadrant. We see that $\Lambda^S \subset \mathbb{L}^S_{\mathbb{Q}}$ is the lattice of vectors:

$$\begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \quad \text{such that } k_0,\, k_1 \in \mathbb{Z}$$

and that the reduction function is:

$$v^S\colon \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \mapsto \left\langle \frac{k_1}{3} \right\rangle$$

The S-extended I-function is:

$$I^S(t,x,z) = ze^{(u_1t_1+u_2t_2)/z} \sum_{k_0=0}^{\infty}\sum_{k_1=0}^{\infty} \frac{x_0^{k_0}x_1^{k_1}}{z^{k_0+k_1}k_0!k_1!}\mathbf{1}_{\langle\frac{k_1}{3}\rangle} \times \prod_{\substack{\langle b\rangle=\langle -\frac{k_1}{3}\rangle \\ -\frac{k_1}{3}<b\leq 0}} (u_1+bz)(u_2+bz).$$

This is homogeneous of degree 1 if we set $\deg t_1 = \deg t_2 = 0$, $\deg x_0 = \deg z = 1$, and $\deg x_1 = \frac{1}{3}$. Theorem 7 gives that $I^S(x,-z) \in \mathcal{L}_{\mathcal{X}}$, and we have:

$$I^S(t,x,z) = z + t_1u_1 + t_2u_2 + x_0\mathbf{1}_0 + x_1\mathbf{1}_{\frac{1}{3}} + O(z^{-1})$$

Thus condition S-$\sharp$ holds, and we obtain an expression for the J-function of $\mathcal{X}$:

$$J_{\mathcal{X}}\big(t_1u_1 + t_2u_2 + x_0\mathbf{1}_0 + x_1\mathbf{1}_{\frac{1}{3}}, z\big) = I^S(t,x,z)$$

3.3. Example 3: $\mathbb{P}(1,1,3)$

This is the toric Deligne–Mumford stack $\mathcal{X}$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where:

$$\rho = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} : \mathbb{Z}^3 \to N = \mathbb{Z}^2 + \frac{1}{3}(1,1)\mathbb{Z}$$

and Σ is the complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}^2$ with rays given by the columns of ρ. We identify $\mathrm{Box}(\mathbf{\Sigma}) = \big\{0, \frac{1}{3}(1,1), \frac{2}{3}(1,1)\big\}$ with the set $\big\{0, \frac{1}{3}, \frac{2}{3}\big\}$ via the map κ that sends x to $x(1,1)$. We consider the S-extended I-function where $S = \big\{0, \frac{1}{3}\big\}$ and S maps to N_{Σ} via κ. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} : \mathbb{Z}^{3+2} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^3$ is identified as a subset of $\mathbb{Q}^{3+2}$ via the inclusion:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix}$$

The S-extended Mori cone is the positive octant. We see that $\Lambda^S \subset \mathbb{L}^S_{\mathbb{Q}}$ is the lattice of vectors:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \quad \text{such that } l,\, k_0,\, k_1 \in \mathbb{Z}$$

and that the reduction function is:

$$v^S\colon \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto \left\langle -\frac{l}{3} + \frac{k_1}{3} \right\rangle$$

Let us identify the Novikov ring $\mathbf{\Lambda}$ with $\mathbb{C}[\![Q]\!]$ via the map that sends $d \in H_2(X;\mathbb{Z})$ to $Q^{\int_d c_1(\mathcal{O}(3))}$. The S-extended I-function is:

$$\begin{aligned} I^S(t,x,z) &= z e^{(u_1 t_1 + u_2 t_2 + u_3 t_3)/z} \\ &\quad \times \sum_{l=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{Q^l x_0^{k_0} x_1^{k_1} e^{(t_1+t_2+3t_3)l}}{z^{k_0+k_1} k_0! k_1!} \\ &\quad \times \frac{\prod_{\substack{\langle b \rangle = \langle \frac{l}{3} - \frac{k_1}{3} \rangle \\ b \leq 0}} (u_1 + bz)(u_2 + bz)}{\prod_{\substack{\langle b \rangle = \langle \frac{l}{3} - \frac{k_1}{3} \rangle \\ b \leq \frac{l}{3} - \frac{k_1}{3}}} (u_1 + bz)(u_2 + bz)} \frac{\mathbf{1}_{\langle -\frac{l}{3} + \frac{k_1}{3} \rangle}}{\prod_{0 < b \leq l} (u_3 + bz)} \end{aligned}$$

This is homogeneous of degree 1 if we set $\deg t_1 = \deg t_2 = \deg t_3 = 0$, $\deg x_0 = \deg z = 1$, $\deg x_1 = \frac{1}{3}$, and $\deg Q = \frac{5}{3}$. Theorem 7 gives that $I^S(x,t,-z) \in \mathcal{L}$, and we have:

$$I^S(t,x,z) = z + t_1 u_1 + t_2 u_2 + t_3 u_3 + x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{\frac{1}{3}} + O(z^{-1})$$

Thus condition S-$\sharp$ holds, and we obtain an expression for the J-function of $\mathcal{X}$:

$$J_{\mathcal{X}}\big(z + t_1 u_1 + t_2 u_2 + t_3 u_3 + x_0 \mathbf{1}_0 + x_1 \mathbf{1}_{\frac{1}{3}}, z\big) = I^S(t,x,z)$$

Remark 12. Condition $\sharp$ holds for any weighted projective space, but condition S-$\sharp$ fails in general. Indeed we chose $S = \left\{0, \frac{1}{3}\right\}$ here rather than $S = \mathrm{Box}(\mathbf{\Sigma}) = \left\{0, \frac{1}{3}, \frac{2}{3}\right\}$ because with the latter choice condition S-$\sharp$ fails and we do not obtain a closed form expression for $J_{\mathcal{X}}$. This is what we meant in [13, Remark 34].

Remark 13. The non-equivariant limit of our S-extended I-function, in the notation of [17], is:

$$(5)\quad ze^{(t_1+t_2+3t_3)P/z}\sum_{l=0}^{\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{Q^l x_0^{k_0}x_1^{k_1}e^{(t_1+t_2+3t_3)l}}{z^{k_0+k_1}k_0!k_1!} \times \frac{\prod_{\substack{\langle b\rangle=\langle\frac{l}{3}-\frac{k_1}{3}\rangle\\ b\leq 0}}(P+bz)^2}{\prod_{\substack{\langle b\rangle=\langle\frac{l}{3}-\frac{k_1}{3}\rangle\\ b\leq\frac{l}{3}-\frac{k_1}{3}}}(P+bz)^2}\frac{\mathbf{1}_{\langle-\frac{l}{3}+\frac{k_1}{3}\rangle}}{\prod_{0<b\leq l}(3P+bz)}$$

Theorem 7 implies that this lies on the Lagrangian submanifold $\mathcal{L}^{\mathrm{non}}$ for non-equivariant Gromov–Witten theory of $\mathcal{X}$ and, since (5) takes the form

$$z+(t_1+t_2+3t_3)P+x_0\mathbf{1}_0+x_1\mathbf{1}_{\frac{1}{3}}+O(z^{-1})$$

we see that this determines the non-equivariant J-function $J_{\mathcal{X}}(t,x,z)$ for $t=t_1P+x_0\mathbf{1}_0+x_1\mathbf{1}_{\frac{1}{3}}$. Theorem 7 thus determines the orbifold quantum product $\star_t$, for t as above, in a straightforward way. This improves on the results of [17], which determine $J_{\mathcal{X}}(t,x,z)$ for t in the small quantum cohomology locus $H^2(\mathcal{X})\subset H^{\bullet}_{\mathrm{CR}}(\mathcal{X})$ and thus determine the small quantum orbifold cohomology ring of $\mathcal{X}$.

Remark 14. To determine the full big quantum orbifold cohomology ring of $\mathcal{X}$ (equivariant or non-equivariant) from Theorem 7 is more involved. One needs to take $S=\left\{0,\frac{1}{3},\frac{2}{3}\right\}$, so in particular condition S-$\sharp$ fails, and then compute the big J-function $J_{\mathcal{X}}(t,z)$ by Birkhoff factorization, as in §3.8 below. We do not know a closed-form expression for the structure constants.

Remark 15. Note that:

$$I^S_{\mathbb{P}(1,1,3)}(0,x,z)|_{Q=0}=I^S_{\frac{1}{3}(1,1)}(0,x,z)$$

and that, as discussed in Remark 13, $I^S(0,0,z)$ essentially coincides, after passing to the non-equivariant limit and changing notation for degrees (replacing d by $\frac{d}{3}$), with the *small I-function* of $\mathbb{P}(1,1,3)$ as written in [17].

3.4. Example 4: $\mathbb{P}(2,2)$

This is the toric Deligne–Mumford stack $\mathcal{X}$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where:

$$\rho = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{Z}^2 \to N = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

and Σ is the fan in $N_{\mathbb{Q}} \cong \mathbb{Q}$ with rays given by -1 and 1. We identify $\mathrm{Box}(\mathbf{\Sigma}) = \{(0,0),(0,1)\}$ with the set $\{0, \frac{1}{2}\}$ via the map κ that sends 0 to $(0,0)$ and $\frac{1}{2}$ to $(0,1)$. We consider the S-extended I-function where $S = \{(0,0),(0,1),(-1,1),(1,0)\}$ and $S \to N_\Sigma$ is the canonical inclusion. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} : \mathbb{Z}^{2+4} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^5$ is identified as a subset of $\mathbb{Q}^{2+4}$ via the inclusion:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

The S-extended Mori cone is the positive orthant. We see that $\Lambda^S \subset \mathbb{L}^S_{\mathbb{Q}}$ is the lattice of vectors:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} \quad \text{such that } l,\ k_0,\ k_1,\ k_2,\ k_3 \in \mathbb{Z}$$

and that the reduction function is:

$$v^S : \begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} \mapsto \left\langle \frac{l + k_1 + k_2 + k_3}{2} \right\rangle$$

Let us identify the Novikov ring $\mathbf{\Lambda}$ with $\mathbb{C}[\![Q]\!]$ via the map that sends $d \in H_2(X;\mathbb{Z})$ to $Q^{\int_d c_1(\mathcal{O}(2))}$. The S-extended I-function is:

$$I^S(t,x,z) = ze^{(u_1t_1+u_2t_2)/z} \sum_{(l,k_0,\ldots,k_3)\in\mathbb{N}^5} \frac{Q^l x_0^{k_0}x_1^{k_1}x_2^{k_2}x_3^{k_3}e^{(t_1+t_2)l}}{z^{k_0+k_1+k_2+k_3}k_0!k_1!k_2!k_3!} \times \frac{\prod_{b\leq 0}(u_1+bz)}{\prod_{b\leq l-k_2}(u_1+bz)} \frac{\prod_{b\leq 0}(u_2+bz)}{\prod_{b\leq l-k_3}(u_2+bz)} \mathbf{1}_{\left\langle \frac{l+k_1+k_2+k_3}{2}\right\rangle}$$

This is homogeneous of degree 1 if we set $\deg t_1 = \deg t_2 = \deg x_2 = \deg x_3 = 0$, $\deg x_0 = \deg x_1 = \deg z = 1$, and $\deg Q = 2$. Theorem 7 gives that $I^S(x,t,-z) \in \mathcal{L}$, and straightforward calculation gives:

$$I^S(t,x,z) = z\mathbf{1}_0 + \tau(x,t) + O(z^{-1})$$

where:

$$\tau(t,x) = x_0\mathbf{1}_0 + \big(t_1 + \tfrac{1}{2}\log(1-x_2^2)\big)u_1\mathbf{1}_0 + \big(t_2 + \tfrac{1}{2}\log(1-x_3^2)\big)u_2\mathbf{1}_0 + x_1\mathbf{1}_{\frac{1}{2}} + \frac{1}{2}\log\left(\frac{1+x_2}{1-x_2}\right)u_1\mathbf{1}_{\frac{1}{2}} + \frac{1}{2}\log\left(\frac{1+x_3}{1-x_3}\right)u_2\mathbf{1}_{\frac{1}{2}} \tag{6}$$

Thus $J_{\mathcal{X}}\big(\tau(t,x),z\big) = I^S(t,x,z)$. We can invert the mirror map $(x,t) \mapsto \tau(x,t)$ in closed form: if $\tau(x,t) = a_0\mathbf{1}_0 + a_1u_1\mathbf{1}_0 + a_2u_2\mathbf{1}_0 + b_0\mathbf{1}_{\frac{1}{2}} + b_1u_1\mathbf{1}_{\frac{1}{2}} + b_2u_2\mathbf{1}_{\frac{1}{2}}$ then:

$$\begin{aligned} &x_0 = a_0 && x_1 = b_0 && x_2 = \tanh b_1 \\ &x_3 = \tanh b_2 && t_1 = a_1 - \log\operatorname{sech} b_1 && t_2 = a_2 - \log\operatorname{sech} b_2 \end{aligned} \tag{7}$$

This gives a closed-form expression for the J-function $J_{\mathcal{X}}(\tau,z)$.

Remark 16. It is instructive to consider the specialisations of $I^S(t,x,z)$ to $Q = x_2 = x_3 = 0$ and to $x_0 = x_1 = x_2 = x_3 = 0$. Note that $\mathbb{P}(2,2)$ satisfies condition $\sharp$ but not condition S-$\sharp$.

3.5. Example 5: $\mathbb{P}^1 \times B\boldsymbol{\mu}_2$

This is the toric Deligne–Mumford stack $\mathcal{X}$ associated to the stacky fan $\mathbf{\Sigma} = (N,\Sigma,\rho)$, where:

$$\rho = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} : \mathbb{Z}^2 \to N = \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

and Σ is the fan in $N_{\mathbb{Q}} \cong \mathbb{Q}$ with rays given by -1 and 1. We identify $\mathrm{Box}(\boldsymbol{\Sigma}) = \{(0,0),(0,1)\}$ with the set $\{0, \frac{1}{2}\}$ via the map κ that sends 0 to $(0,0)$ and $\frac{1}{2}$ to $(0,1)$. We consider the S-extended I-function where $S = \{(0,0),(0,1),(-1,1),(1,1)\}$ and $S \to N_\Sigma$ is the canonical inclusion. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} : \mathbb{Z}^{2+4} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^5$ is identified as a subset of $\mathbb{Q}^{2+4}$ via the inclusion:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

The S-extended Mori cone is the positive orthant. We see that $\Lambda^S \subset \mathbb{L}^S_{\mathbb{Q}}$ is the lattice of vectors:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} \quad \text{such that } l,\ k_0,\ k_1,\ k_2,\ k_3 \in \mathbb{Z}$$

and that the reduction function is:

$$v^S : \begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} \mapsto \left\langle \frac{k_1 + k_2 + k_3}{2} \right\rangle.$$

Let us identify the Novikov ring $\boldsymbol{\Lambda}$ with $\mathbb{C}[\![Q]\!]$ via the map that sends $d \in H_2(X;\mathbb{Z})$ to $Q^{\int_d c_1(\mathcal{O}_{\mathbb{P}^1}(1))}$. The S-extended I-function is:

$$\begin{aligned} I^S(t,x,z) = z e^{(u_1 t_1 + u_2 t_2)/z} & \sum_{(l,k_0,\ldots,k_3) \in \mathbb{N}^5} \frac{Q^l x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3} e^{(t_1+t_2)l}}{z^{k_0+k_1+k_2+k_3} k_0! k_1! k_2! k_3!} \\ & \times \frac{\prod_{b \le 0}(u_1 + bz)}{\prod_{b \le l - k_2}(u_1 + bz)} \frac{\prod_{b \le 0}(u_2 + bz)}{\prod_{b \le l - k_3}(u_2 + bz)} \mathbf{1}_{\left\langle \frac{k_1+k_2+k_3}{2} \right\rangle} \end{aligned}$$

Except for the difference in reduction function, this coincides with the S-extended I-function for $\mathbb{P}(2,2)$ in §3.4. Once again, Theorem 7 gives that $I^S(x,t,-z) \in \mathcal{L}$, and:

$$I^S(t,x,z) = z\mathbf{1}_0 + \tau(x,t) + O(z^{-1})$$

with $\tau(x,t)$ as in (6). Thus $J_{\mathcal{X}}\big(\tau(t,x),z\big) = I^S(t,x,z)$. Inverting the mirror map (7) gives a closed-form expression for the J-function $J_{\mathcal{X}}(\tau,z)$.

3.6. Example 6: $\mathbb{P}_{2,2}$

This is the unique Deligne–Mumford stack with coarse moduli space equal to $\mathbb{P}^1$, isotropy group μ_2 at $0 \in \mathbb{P}^1$, isotropy group μ_2 at $\infty \in \mathbb{P}^1$, and no other non-trivial isotropy groups. It is the toric Deligne–Mumford stack $\mathcal{X}$ associated to the stacky fan $\mathbf{\Sigma} = (N,\Sigma,\rho)$, where:

$$\rho = \begin{pmatrix} -1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to N = \mathbb{Z} + \frac{1}{2}\mathbb{Z}.$$

and Σ is the fan in $N_{\mathbb{Q}} \cong \mathbb{Q}$ with rays given by -1 and 1. We identify $\mathrm{Box}(\mathbf{\Sigma})$ with the set $\big\{(0,0), \big(\frac{1}{2},0\big), \big(0,\frac{1}{2}\big)\big\}$ via the map ρ. We consider the S-extended I-function where $S = \mathrm{Box}(\mathbf{\Sigma})$ and $S \to N_\Sigma$ is the canonical inclusion. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} -1 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} : \mathbb{Z}^{2+3} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^4$ is identified as a subset of $\mathbb{Q}^{2+3}$ via the inclusion:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \end{pmatrix}$$

The S-extended Mori cone is the positive orthant. We see that $\Lambda^S \subset \mathbb{L}^S_{\mathbb{Q}}$ is the subset (*not* sublattice) of vectors:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \end{pmatrix} \quad \text{such that } l,\ k_0,\ k_1,\ k_2 \in \mathbb{Z} \text{ and at least one of } l-k_1,\ l-k_2 \text{ is even}$$

and that the reduction function is:

$$v^S\colon \begin{pmatrix} l \\ k_0 \\ k_1 \\ k_2 \end{pmatrix} \mapsto \Big(\big\langle \tfrac{k_1-l}{2}\big\rangle, \big\langle \tfrac{k_2-l}{2}\big\rangle\Big).$$

Let us identify the Novikov ring $\mathbf{\Lambda}$ with $\mathbb{C}[\![Q]\!]$ via the map that sends $d \in H_2(X;\mathbb{Z})$ to $Q^{\int_d c_1(\mathcal{O}_X(1))}$. The S-extended I-function is:

$$\begin{aligned} I^S(t,x,z) = ze^{(u_1t_1+u_2t_2)/z} &\sum_{(l,k_0,k_1,k_2)\in\Lambda^S} \frac{Q^l x_0^{k_0} x_1^{k_1} x_2^{k_2} e^{(t_1+t_2)l}}{z^{k_0+k_1+k_2} k_0! k_1! k_2!} \\ &\times \frac{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{2}\rangle \\ b\leq 0}} (u_1+bz)}{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{2}\rangle \\ b\leq \frac{l-k_1}{2}}} (u_1+bz)} \frac{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_2}{2}\rangle \\ b\leq 0}} (u_2+bz)}{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_2}{2}\rangle \\ b\leq \frac{l-k_2}{2}}} (u_2+bz)} \mathbf{1}_{(\langle\frac{k_1-l}{2}\rangle, \langle\frac{k_2-l}{2}\rangle)} \end{aligned}$$

This is homogeneous of degree 1 if we set $\deg t_1 = \deg t_2 = 0$, $\deg x_0 = \deg Q = \deg z = 1$, and $\deg x_1 = \deg x_2 = \frac{1}{2}$. Theorem 7 gives that $I^S(x,t,-z) \in \mathcal{L}$, and since:

$$\begin{aligned} I^S(x,t,z) = z\mathbf{1}_{(0,0)} &+ t_1u_1\mathbf{1}_{(0,0)} + t_2u_2\mathbf{1}_{(0,0)} \\ &+ x_0\mathbf{1}_{(0,0)} + x_1\mathbf{1}_{(\frac{1}{2},0)} + x_2\mathbf{1}_{(0,\frac{1}{2})} + O(z^{-1}) \end{aligned}$$

we conclude that:

$$J_{\mathcal{X}}\big(t_1u_1\mathbf{1}_{(0,0)} + t_2u_2\mathbf{1}_{(0,0)} + x_0\mathbf{1}_{(0,0)} + x_1\mathbf{1}_{(\frac{1}{2},0)} + x_2\mathbf{1}_{(0,\frac{1}{2})}, z\big) = I^S(x,t,z)$$

3.7. Example 7: a toric surface

We have already seen examples (in §3.4 and §3.5) where condition S-$\sharp$ fails. We now give the simplest example of a *Fano* toric stack such that condition $\sharp$ fails. Consider the toric Deligne–Mumford stack $\mathcal{X}$ associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where:

$$\rho = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & -2 \end{pmatrix} : \mathbb{Z}^2 \to N = \mathbb{Z}^2$$

and Σ is the complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}^2$ with rays given by the columns of ρ. We identify $\mathrm{Box}(\mathbf{\Sigma}) = \big\{(0,0),(0,-1)\big\}$ with the set $\big\{0, \frac{1}{2}\big\}$ via the map

κ that sends x to $(0, -2x)$. We identify $\mathbb{L}_{\mathbb{Q}} \cong \mathbb{Q}^2$ as a subset of $\mathbb{Q}^4$ via the inclusion:

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$$

The Mori cone $\mathrm{NE}(\mathcal{X})$ is the cone of vectors

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \in \mathbb{R}^2 \quad \text{such that } l_1 \geq 0 \text{ and } 3l_1 + 2l_2 \geq 0.$$

We see that $\Lambda^{\varnothing} \subset \mathbb{L}_{\mathbb{Q}}$ is the lattice of vectors:

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \in \mathrm{NE}\,\mathcal{X} \quad \text{such that } l_1 \in \mathbb{Z} \text{ and } l_2 \in \frac{1}{2}\mathbb{Z}$$

and that the reduction function is:

$$v^S \colon \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \mapsto \langle -l_2 \rangle$$

Let us write the element of the Novikov ring corresponding to $(l_1, l_2) \in \Lambda^{\varnothing}$ as $Q^{(l_1,l_2)}$. The I-function (that is, the S-extended I-function with $S = \varnothing$) is:

$$\begin{aligned} I(t,x,z) = {} & z e^{(u_1 t_1 + u_2 t_2 + u_3 t_3 + u_4 t_4)/z} \\ & \times \sum_{\substack{(l_1,l_2) \in \mathbb{Z} \times \frac{1}{2}\mathbb{Z}: \\ l_1 \geq 0, 3l_1 + 2l_2 \geq 0}} \frac{Q^{(l_1,l_2)} e^{(t_1 + 3t_2 + t_3)l_1} e^{(2t_2 + t_4)l_2}}{\prod_{\substack{\langle b \rangle = 0 \\ 0 < b \leq l_1}} (u_1 + bz)(u_3 + bz)} \\ & \qquad \times \frac{\mathbf{1}_{\langle -l_2 \rangle}}{\prod_{\substack{\langle b \rangle = 0 \\ 0 < b \leq 3l_1 + 2l_2}} (u_2 + bz)} \frac{\prod_{\substack{\langle b \rangle = \langle l_2 \rangle \\ b \leq 0}} (u_4 + bz)}{\prod_{\substack{\langle b \rangle = \langle l_2 \rangle \\ b \leq l_2}} (u_4 + bz)} \end{aligned}$$

This is homogeneous of degree 1 if we set $\deg t_1 = \deg t_2 = \deg t_3 = \deg t_4 = 0$, $\deg z = 1$, and $\deg Q^{(l_1,l_2)} = 5l_1 + 3l_2$. We therefore have:

$$\begin{aligned} I(t,x,z) = {} & z\mathbf{1}_0 + t_1 u_1 \mathbf{1}_0 + t_2 u_2 \mathbf{1}_0 + t_3 u_3 \mathbf{1}_0 + t_4 u_4 \mathbf{1}_0 \\ & - \tfrac{1}{2} Q^{(1,-\frac{3}{2})} e^{t_1 + t_3 - \frac{3}{2} t_4} \mathbf{1}_{\frac{1}{2}} + O(z^{-1}) \end{aligned}$$

and condition $\sharp$ fails.

Remark 17. The coarse moduli space X of $\mathcal{X}$ is the ruled surface $\mathbb{F}_3$. Let A and B denote the natural divisors on $\mathbb{F}_3$, with A the fibre and B the negative section. Then $\mathcal{X}$ can be interpreted as the moduli stack of square roots of B [5, §2], [1, Appendix B]. The stack $\mathcal{X}$ contains a substack $\{x_4 = 0\}$ supported on B and isomorphic to $\mathbb{P}(2,2)$. In this context it is natural to identify the *integral* Chow group $\mathrm{CH}(\mathcal{X}, \mathbb{Z})$ with the subring of $\mathrm{CH}^\bullet(X, \mathbb{Q})$ multiplicatively generated by A and $B/2$; the cycle class of $\mathbb{P}(2,2) \subset \mathcal{X}$ is $B/2$. This gives an interpretation of the degrees (l_1, l_2) occurring in the definition of $I(t, x, z)$.

3.8. Example 8: $\mathbb{P}^2$

There is a well-known closed formula [20] for the small J-function of $\mathcal{X} = \mathbb{P}^2$, that is, for the J-function $J_{\mathcal{X}}(t, z)$ with $t \in H^2(\mathcal{X})$. We now show how to use an S-extended I-function to obtain arbitrarily many terms of the Taylor expansion of the big J-function of $\mathcal{X}$, that is, of the J-function $J_{\mathcal{X}}(t, z)$ with $t \in H^\bullet(\mathcal{X})$. We use the Birkhoff factorization procedure described in [15, §8]. We will compute the non-equivariant version of the J-function, as the equivariant calculation is significantly more involved.

The variety $\mathcal{X}$ is the toric Deligne–Mumford stack associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where:

$$\rho = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} : \mathbb{Z}^3 \to N = \mathbb{Z}^2$$

and Σ is the complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}^2$ with rays given by the columns of ρ. We have $\mathrm{Box}(\mathbf{\Sigma}) = \{0\}$. Consider the S-extended I-function where $S = \big\{(0,0), (0,-1)\big\}$ and the map $S \to N_\Sigma$ is the canonical inclusion. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 \end{pmatrix} : \mathbb{Z}^{3+2} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^3$ is identified as a subset of $\mathbb{Q}^{3+2}$ via the inclusion:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix}$$

The S-extended Mori cone is the positive octant. We see that $\Lambda^S \subset \mathbb{L}_{\mathbb{Q}}^S$ is the lattice of vectors:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \quad \text{such that } l,\ k_0,\ k_1 \in \mathbb{Z}$$

The reduction function v^S is trivial. Let $P \in H^2(\mathcal{X})$ denote the first Chern class of $\mathcal{O}(1)$, and identify the Novikov ring $\mathbf{\Lambda}$ with $\mathbb{C}[\![Q]\!]$ via the map that sends $d \in H_2(\mathcal{X};\mathbb{Z})$ to $Q^{\int_d P}$. The non-equivariant limit of the S-extended I-function is:

$$\begin{aligned} I^S_{\text{non}}(t,x,z) = ze^{(t_1+t_2+t_3)P/z} \sum_{(l,k_0,k_1)\in\mathbb{N}^3} & \frac{Q^l x_0^{k_0} x_1^{k_1} e^{(t_1+t_2+t_3)l}}{z^{k_0+k_1}k_0!k_1!} \\ & \times \frac{\prod_{b\le 0}(P+bz)^2}{\prod_{b\le l-k_1}(P+bz)^2} \frac{1}{\prod_{0<b\le l}(P+bz)} \end{aligned}$$

This takes values in the non-equivariant cohomology ring $H^\bullet(\mathcal{X};\mathbb{C}) = \mathbb{C}[P]/(P^3)$. It is homogeneous of degree 1 if we set $\deg t_1 = \deg t_2 = \deg t_3 = 0$, $\deg x_0 = \deg z = 1$, $\deg x_1 = -1$, and $\deg Q = 3$. Note that, unlike the other examples in this paper, in this case the I-function contains arbitrarily large positive powers of z; this reflects the fact that some of the variables have negative degree.

We have:

$$\begin{aligned} I^S_{\text{non}}(t,x,z) = z + \tfrac{1}{2}zx_1^2P^2 + x_0 + (t_1+t_2+t_3)P + x_1P^2 \\ + \frac{1}{2}x_0x_1^2P^2 + O(z^{-1}) + O(x_1^3) \end{aligned}$$

The non-equivariant version of the mirror theorem for toric Deligne–Mumford stacks [13, Corollary 32] gives that $I^S(t,x,-z) \in \mathcal{L}^{\text{non}}$, where $\mathcal{L}^{\text{non}}$ is the Givental cone for non-equivariant Gromov–Witten theory (see e.g. [12, §3]). Set $t_2 = t_3 = 0$. Condition S-$\sharp$ holds modulo x_1^2, so:

$$J_{\mathcal{X}}(x_0 + t_1P + x_1P^2, z) + O(x_1^2) = I^S_{\text{non}}(t,x,z) + O(x_1^2)$$

The following elements lie in $T_{I^S_{\text{non}}(t,x,z)}\mathcal{L}^{\text{non}}$:

$$\frac{\partial I^S_{\text{non}}}{\partial x_0} = e^{Pt_1/z}e^{x_0/z}\Big(1 + O(z^{-2}) + O(x_1)\Big)$$

$$\frac{\partial I^S_{\mathrm{non}}}{\partial t_1} = e^{Pt_1/z}e^{x_0/z}\Big(P + O(z^{-2}) + O(x_1)\Big)$$
$$\frac{\partial I^S_{\mathrm{non}}}{\partial x_1} = e^{Pt_1/z}e^{x_0/z}\Big(P^2 + z^{-1}Qe^{t_1} + O(z^{-2}) + O(x_1)\Big)$$

We have that:

$$I^S_{\mathrm{non}}(t,x,z) = e^{Pt_1/z}e^{x_0/z}\Big(z + \tfrac{1}{2}zx_1^2P^2 + x_1P^2 + O(z^{-1}) + O(x_1^3)\Big)$$

and general properties of $\mathcal{L}^{\mathrm{non}}$ guarantee [23] [13, Appendix B] that:

$$\begin{aligned}I^S_{\mathrm{non}}(t,x,-z) &+ C_0(t,x,z)z\frac{\partial I^S_{\mathrm{non}}}{\partial x_0}(t,x,-z)\\ &+ C_1(t,x,z)z\frac{\partial I^S_{\mathrm{non}}}{\partial t_1}(t,x,-z) + C_2(t,x,z)z\frac{\partial I^S_{\mathrm{non}}}{\partial x_1}(t,x,-z) \in \mathcal{L}^{\mathrm{non}}\end{aligned}$$

for any C_0, C_1, C_2 depending polynomially on z. But:

$$\begin{aligned}&I^S_{\mathrm{non}}(t,x,z) - \tfrac{1}{2}zx_1^2\frac{\partial I^S_{\mathrm{non}}}{\partial x_1}\\ &= e^{Pt_1/z}e^{x_0/z}\Big(z + x_1P^2 - \tfrac{1}{2}x_1^2Qe^{t_1} + O(z^{-1}) + O(x_1^3)\Big)\\ &= z + x_0 - \tfrac{1}{2}x_1^2Qe^{t_1} + t_1P + x_1P^2 + O(z^{-1}) + O(x_1^3)\end{aligned}$$

and thus:

$$J_{\mathcal{X}}(\tau,z) + O(x_1^3) = I^S_{\mathrm{non}}(t,x,z) - \tfrac{1}{2}x_1^2z\frac{\partial I^S_{\mathrm{non}}}{\partial x_1} + O(x_1^3)$$

where:

$$\tau(x_0,t_1,x_1) = \big(x_0 - \tfrac{1}{2}x_1^2Qe^{t_1}\big)1 + t_1P + x_1P^2 + O(x_1^3)$$

Inverting the mirror map $(x_0,t_1,x_1) \mapsto \tau$ gives a closed-form expression for the big J-function $J_{\mathcal{X}}(a_0 + a_1P + a_2P^2, z)$ to order 2 in a_2. One can repeat this procedure to compute the big J-function $J_{\mathcal{X}}(a_0 + a_1P + a_2P^2, z)$ to arbitrarily high order in a_2.

Remark 18. The extended I-function is closely related to Barannikov's big quantum cohomology mirror for $\mathbb{P}^n$ [2]: it satisfies the Picard–Fuchs differential equation for the mirror oscillatory integrals [26, Example 4.15]. "Big" mirror symmetry for $\mathbb{P}^2$ has been also studied via tropical geometry [25] and via quasimap theory [10, 32].

4. Twisted *I*-functions

Let $\mathcal{X}$ be the toric Deligne–Mumford stack defined by an S-extended stacky fan $\mathbf{\Sigma}$, as in §2.3. Suppose that the coarse moduli space of $\mathcal{X}$ is semi-projective. Let $\varepsilon_1, \ldots, \varepsilon_r \in (\mathbb{L}^S)^\vee$. The canonical inclusion $i\colon \mathbb{L} \to \mathbb{L}^S$ induces $i^\vee\colon (\mathbb{L}^S)^\vee \to \mathbb{L}^\vee = \mathrm{Pic}(\mathcal{X})$, and so the classes $\varepsilon_1, \ldots, \varepsilon_r$ define line bundles $\mathcal{E}_1, \ldots, \mathcal{E}_r$ over $\mathcal{X}$ via $i^\vee$. Let $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r$, and let $\mathbf{c}$ denote the invertible multiplicative characteristic class

$$\mathbf{c}(-) = \exp\bigl(\textstyle\sum_{k=0}^\infty s_k \operatorname{ch}_k(-)\bigr) \tag{8}$$

where $s_0, s_1, \ldots$ are parameters. We consider the Gromov–Witten theory of $\mathcal{X}$ twisted, in the sense of [12, 15, 37], by the vector bundle $\mathcal{E}$ and the characteristic class $\mathbf{c}$. Let $\mathcal{L}^{\mathrm{tw}}$ denote Givental's Lagrangian cone for $(\mathbf{c}, \mathcal{E})$-twisted Gromov–Witten theory, as in [12, §3].

Let $D_i \in H^2(\mathcal{X}; \mathbb{C})$, denote the (non-equivariant) class Poincaré dual to the ith toric divisor for $1 \leq i \leq n$, and the zero class for $n < i \leq n + |S|$. Let $E_j \in H^2(\mathcal{X}; \mathbb{C})$, $1 \leq j \leq r$, denote the (non-equivariant) first Chern class of $\mathcal{E}_j$. Let $\mathcal{L}^{\mathrm{non}}$ denote Givental's Lagrangian cone for the non-equivariant Gromov–Witten theory of $\mathcal{X}$, as in [12, §3], and let $I^S_{\mathrm{non}}(t, x, z)$ denote the non-equivariant limit of the S-extended $\mathbb{T}$-equivariant I-function $I^S(t, x, z)$.

Notation 19. Given parameters $s_0, s_1, s_2, \ldots$ as above, write $s(x) := \sum_{k=0}^\infty s_k \frac{x^k}{k!}$.

Definition 20 (cf. [12, §4]). Given $b \in \mathrm{Box}(\mathbf{\Sigma})$ and $\lambda \in \Lambda E^S_b$, define the *modification factor*:

$$M_{\lambda,b}(z) := \prod_{j=1}^r \frac{\prod_{\substack{a : \langle a \rangle = \langle \varepsilon_j \cdot \lambda \rangle \\ a \leq \varepsilon_j \cdot \lambda}} \exp\bigl(s(E_j + az)\bigr)}{\prod_{\substack{a : \langle a \rangle = \langle \varepsilon_j \cdot \lambda \rangle \\ a \leq 0}} \exp\bigl(s(E_j + az)\bigr)}$$

Definition 21. The *S-extended $(\mathbf{c}, \mathcal{E})$-twisted I-function of $\mathcal{X}$* is:

$$I^S_{\mathbf{c},\mathcal{E}}(t, x, z) = \sum_{b \in \mathrm{Box}(\mathbf{\Sigma})} \sum_{\lambda \in \Lambda E^S_b} \widetilde{Q}^\lambda I_{\lambda,b}(z) M_{\lambda,b}(z) y^b$$

where $I^S_{\mathrm{non}}(t, x, z) = \sum_{b \in \mathrm{Box}(\mathbf{\Sigma})} \sum_{\lambda \in \Lambda E^S_b} \widetilde{Q}^\lambda I_{\lambda,b}(z) y^b$.

Theorem 22. *With hypotheses and notation as above, we have that* $I^S_{\mathbf{c},\mathcal{E}}(t, x, -z) \in \mathcal{L}^{\mathrm{tw}}$.

Proof. Recall from the proof of Theorem 4.8 in [12] that:

$$G_y(x,z) := \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} s_{m+l-1}\frac{B_m(y)}{m!}\frac{x^l}{l!}z^{m-1}$$

satisfies:

$$\begin{aligned} G_y(x,z) &= G_0(x+yz,z) \\ G_0(x+z,z) &= G_0(x,z)+s(x) \end{aligned} \tag{9}$$

Here $B_m(y)$ is the mth Bernoulli polynomial, and s_{-1} is defined to be zero. Thus:

$$\begin{aligned} M_{\lambda,b}(-z) &= \prod_{j=1}^{r}\exp\left(\sum_{\substack{a:\langle a\rangle=\langle\epsilon_j\cdot\lambda\rangle\\ a\leq\epsilon_j\cdot\lambda}} s(E_j-az) - \sum_{\substack{a:\langle a\rangle=\langle\epsilon_j\cdot\lambda\rangle\\ a\leq 0}} s(E_j-az)\right) \\ &= \prod_{j=1}^{r}\exp\Big(G_0\big(E_j+\langle-\varepsilon_j\cdot\lambda\rangle\, z,z\big)-G_0\big(E_j-(\varepsilon_j\cdot\lambda)z,z\big)\Big) \\ &= \prod_{j=1}^{r}\exp\Big(G_{\langle-\varepsilon_j\cdot\lambda\rangle}(E_j,z)-G_0\big(E_j-(\varepsilon_j\cdot\lambda)z,z\big)\Big) \end{aligned}$$

where for the last two equalities we used (9).

Let $b\in\mathrm{Box}(\mathbf{\Sigma})$, and let $f(b,j)\in[0,1)$ be the rational number such that if $(x,g)\in I\mathcal{X}_b$, then g acts on the fiber of $\mathcal{E}_j$ over $x\in\mathcal{X}$ by multiplication by $\exp\big(2\pi\sqrt{-1}f(b,j)\big)$. Note that if $\lambda\in\Lambda E_b^S$ then $f(b,j)=\langle-\varepsilon_j\cdot\lambda\rangle$. Tseng has proven [37] that the operators:

$$\Delta_j := \bigoplus_{b\in\mathrm{Box}(\mathbf{\Sigma})}\exp\Big(G_{f(b,j)}(E_j,z)\Big)$$

and $\Delta := \prod_{j=1}^{r}\Delta_j$ satisfy $\Delta(\mathcal{L}^{\mathrm{non}})=\mathcal{L}^{\mathrm{tw}}$; the direct sum in the definition of Δ_j here reflects the decomposition

$$H^\bullet_{\mathrm{CR}}(\mathcal{X};\mathbb{C}) = \bigoplus_{b\in\mathrm{Box}(\mathbf{\Sigma})} H^{\bullet-\mathrm{age}(b)}(I\mathcal{X}_b;\mathbb{C}).$$

To show that $I^S_{\mathbf{c},\mathcal{E}}(t,x,-z)$ lies in $\mathcal{L}^{\mathrm{tw}}$, therefore, it suffices to show that:

$$\sum_{b\in\mathrm{Box}(\mathbf{\Sigma})}\sum_{\lambda\in\Lambda E_b^S}\widetilde{Q}^\lambda I_{\lambda,b}\prod_{j=1}^{r}\exp\Big(-G_0\big(E_j-(\varepsilon_j\cdot\lambda)z,z\big)\Big)y^b \tag{10}$$

lies in the cone $\mathcal{L}^{\mathrm{non}}$ for the untwisted theory. Recall the splitting $\mathbb{L}^S \otimes \mathbb{Q} \cong (\mathbb{L} \otimes \mathbb{Q}) \oplus \mathbb{Q}^m$ from (4). Under this splitting, $\varepsilon_j \in (\mathbb{L}^S)^\vee$ induces $E_j \in \mathbb{L}^\vee \otimes \mathbb{Q} \cong H_2(\mathcal{X};\mathbb{Q})$ and $(f_{j1}, \ldots, f_{jm}) \in \mathbb{Q}^m$. Choose $e_{ji} \in \mathbb{Q}$ such that $E_j = \sum_{i=1}^n e_{ji} D_i$ and define the differential operator ∇_{ε_j} by

$$\nabla_{\varepsilon_j} = e_{j1}\frac{\partial}{\partial t_1} + \cdots + e_{jn}\frac{\partial}{\partial t_n} + f_{j1}x_1\frac{\partial}{\partial x_1} + \cdots + f_{jm}x_m\frac{\partial}{\partial x_m}$$

Then (10) is:

$$\prod_{j=1}^{r} \exp\Big(-G_0\big(-z\nabla_{\varepsilon_j}, z\big)\Big) I^S_{\mathrm{non}}(t,x,-z) \tag{11}$$

and we know by the non-equivariant version of the mirror theorem for toric Deligne–Mumford stacks [13, Corollary 32], [8] that $I^S_{\mathrm{non}}(t,x,-z) \in \mathcal{L}^{\mathrm{non}}$. Arguing as in the proof of [12, Theorem 4.8] now shows that:

$$\prod_{j=1}^{r} \exp\Big(-G_0\big(-z\nabla_{\varepsilon_j}, z\big)\Big) I^S_{\mathrm{non}}(t,x,-z) \in \mathcal{L}^{\mathrm{non}}$$

as required. □

Remark 23. Theorem 22, roughly speaking, states that a certain hypergeometric modification of the untwisted I-function I^S_{non} lies on the twisted cone $\mathcal{L}^{\mathrm{tw}}$. The proof of Theorem 22 is essentially the same as the proof of Theorem 4.8 in [12], where we showed that a hypergeometric modification of the untwisted J-function $J_{\mathcal{X}}$ lies on $\mathcal{L}^{\mathrm{tw}}$. The essential properties of the J-function $J_{\mathcal{X}}$ used there are that $J_{\mathcal{X}}(t,-z) \in \mathcal{L}^{\mathrm{non}}$ (which holds by definition) and the Divisor Equation [17, Lemma 4.7(3)]. The essential properties of the I-function I^S_{non} used here are that $I^S_{\mathrm{non}}(x,t,-z) \in \mathcal{L}^{\mathrm{non}}$ (our mirror theorem) and that $\nabla_{\varepsilon_j} I^S_{\mathrm{non}}(t,x,z) = \big(E_j + (\varepsilon_j \cdot \lambda)z\big) I^S_{\mathrm{non}}(x,t,z)$. This latter property, which is a version of the Divisor Equation for the I-function, allows us to replace (10) by (11).

Remark 24. With a little extra effort — modifying the formal setup in [12] to include equivariant parameters — one could prove the $\mathbb{T}$-equivariant analog of Theorem 22 in exactly the same way. We omit this here, however, as we know of no applications of these results. In current applications one either treats toric complete intersections by taking $\mathbf{c}$ to be the S^1-equivariant Euler class $\mathbf{e}$ (see §5), in which case the $\mathbb{T}$-action on the ambient space is irrelevant as it does not preserve the complete intersection, or one treats non-compact geometries by taking $\mathbf{c} = \mathbf{e}^{-1}$ to be the S^1-equivariant inverse Euler class. The latter case can be treated directly using Theorem 7, as the total space of $\mathcal{E}$ is itself a toric Deligne–Mumford stack.

5. A mirror theorem for toric complete intersection stacks

We now describe how, under appropriate hypotheses on the vector bundle $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_r$ and the toric Deligne–Mumford stack $\mathcal{X}$, Theorem 22 gives a mirror theorem for the complete intersection stack $\mathcal{Y}$ cut out by a generic section of $\mathcal{E}$. Suppose that the vector bundle $\mathcal{E}$ is *convex*, that is, for every genus-zero stable map $f\colon (\mathcal{C}, x_1, \ldots, x_n) \to \mathcal{X}$ from a pointed orbicurve $(\mathcal{C}, x_1, \ldots, x_n)$, one has $H^1(\mathcal{C}, f^*\mathcal{E}) = 0$. This is a very restrictive assumption: it is equivalent to requiring that:

(positivity): $c_1(\mathcal{E}_j) \cdot d \geq 0$ for every degree d of a genus-zero stable map; and

(coarseness): $\mathcal{E}_j$ is the pull-back of a line bundle on the coarse moduli space of $\mathcal{X}$;

hold for all $j = 1, \ldots, r$. See [14] for a detailed discussion of convexity for vector bundles on orbifolds. Under these conditions, we may choose $\varepsilon_j \in (\mathbb{L}^S)^\vee$ in §4 so that it vanishes on the vectors in (3), i.e. ε_j corresponds to (the class of $\mathcal{E}_j, 0$) under the splitting $(\mathbb{L}^S)^\vee \otimes \mathbb{Q} \cong (\mathbb{L}^\vee \otimes \mathbb{Q}) \oplus \mathbb{Q}^m = (\mathrm{Pic}(\mathcal{X}) \otimes \mathbb{Q}) \oplus \mathbb{Q}^m$ induced by (4). In fact, the coarseness implies that the pairings of ε_j with the vectors in (3) lie in $\mathbb{Z}$, and in view of the exact sequence:

$$0 \longrightarrow (\mathbb{Z}^m)^* \longrightarrow (\mathbb{L}^S)^\vee \longrightarrow \mathbb{L}^\vee \longrightarrow 0$$

one may always shift ε_j by the action of $(\mathbb{Z}^m)^*$ so that these pairings vanish. We take the characteristic class $\mathbf{c}$ in (8) to be the S^1-equivariant Euler class $\mathbf{e}$:

$$\mathbf{e}(\mathcal{V}) = \prod_{v\colon \text{Chern roots of } \mathcal{V}} (\kappa + v)$$

where we consider the fiberwise S^1-action on vector bundles and κ denotes the S^1-equivariant parameter. This corresponds to the choice of parameters

$$s_k = \begin{cases} \log \kappa & \text{for } k = 0 \\ (-1)^{k-1}(k-1)!\kappa^{-k} & \text{for } k \geq 1 \end{cases}$$

With the convexity hypothesis for $\mathcal{E}$ and the choices for ε_j as above, the $(\mathbf{e},\mathcal{E})$-twisted I-function takes the form:

$$\begin{aligned} I^S_{\mathbf{e},\mathcal{E}}(t,x,z) = z e^{\sum_{i=1}^n t_i D_i/z} & \\ \times \sum_{b\in\mathrm{Box}(\mathbf{\Sigma})} \sum_{\lambda\in\Lambda E^S_b} & \widetilde{Q}^\lambda e^{\lambda t} \left(\prod_{i=1}^{n+m} \frac{\prod_{\langle a\rangle=\langle\lambda_i\rangle, a\le 0}(D_i+az)}{\prod_{\langle a\rangle=\langle\lambda_i\rangle, a\le\lambda_i}(D_i+az)} \right) \\ & \times \left(\prod_{i=1}^r \prod_{a=1}^{E_j\cdot d} (\kappa+E_j+az) \right) y^b \end{aligned}$$

where we write $\lambda=(d,k)$ via (4). Note that $E_j\cdot d\in\mathbb{Z}_{\ge 0}$ by the convexity assumption. Let $i^\star\colon H^\bullet_{\mathrm{CR}}(\mathcal{X})\to H^\bullet_{\mathrm{CR}}(\mathcal{Y})$ denote the pullback along the inclusion $i\colon\mathcal{Y}\to\mathcal{X}$, and define:

$$I^S_{\mathcal{Y}}(t,x,z) = \lim_{\kappa\to 0} i^\star I^S_{\mathbf{e},\mathcal{E}}(t,x,z)$$

Theorem 22 implies that $I^S_{\mathbf{e},\mathcal{E}}(t,x,-z)$ lies in the $(\mathbf{e},\mathcal{E})$-twisted cone $\mathcal{L}^{\mathrm{tw}}$. Combining this with functoriality for the virtual fundamental class [35, §2], [33] either as in the argument of [27, Proposition 2.4] or using [11, Theorem 1.1 and Remark 2.2], proves the following Mirror Theorem for complete intersections in toric Deligne–Mumford stacks.

Theorem 25. *Let $\mathcal{X}$ be a toric Deligne–Mumford stack with semi-projective coarse moduli space, and let S be a finite set equipped with a map $S\to N_\Sigma$ as in Theorem 7. Let $\mathcal{E}=\mathcal{E}_1\oplus\cdots\oplus\mathcal{E}_r$ be the sum of convex line bundles over $\mathcal{X}$ and $\mathcal{Y}\subset\mathcal{X}$ be the zero-locus of a transverse section of $\mathcal{E}$. Then $I^S_{\mathcal{Y}}(t,x,-z)$ lies in Givental's Lagrangian submanifold $\mathcal{L}_{\mathcal{Y}}$ for $\mathcal{Y}$.*

Remark 26. We chose ε_j so that it vanishes on the vectors (3): this choice yields the optimal z^{-1}-asymptotics for $I^S_{\mathcal{Y}}$. In order for $I^S_{\mathbf{e},\mathcal{E}}$ to have a well-defined non-equivariant limit $\kappa\to 0$, we only need to assume that ε_j pairs with the vectors (3) non-negatively, and Theorem 25 is still valid under this weaker assumption. When the pairings of ε_j with the vectors (3) are positive, the corresponding I-function $I^S_{\mathcal{Y}}$ has worse z^{-1}-asymptotics, i.e. contains higher powers in z.

Remark 27. Theorem 25 can also be proved by combining the methods of Cheong–Ciocan-Fontanine–Kim [8] with the methods of [9, §7]. This gives a different approach, which relies on virtual localization rather than the quantum Lefschetz theorem.

Suppose that the I-function $I^S_{\mathcal{Y}}$ has the following asymptotics (cf. Condition S-$\sharp$ in §2.6)

$$I^S_{\mathcal{Y}}(t,x,z) = F(t,x)z + G(t,x) + O(z^{-1}) \tag{12}$$

where F is an $H^0(\mathcal{Y})$-valued function and G is an $H^\bullet_{\mathrm{CR}}(\mathcal{Y})$-valued function. This holds, for example, if the following conditions are met:

- $c_1(T\mathcal{X}) - c_1(\mathcal{E})$ is nef, and
- the image of $S \to N_\Sigma$ is contained in $\{b \in N_\Sigma : \mathrm{age}(b) \leq 1\}$.

Define the mirror map by:

$$\tau(t,x) = \frac{G(t,x)}{F(t,x)}$$

Theorem 25 determines the unique point $F(t,x)^{-1} I^S_{\mathcal{Y}}(t,x,-z)$ on $\mathcal{L}_{\mathcal{Y}}$ of the form $-z + \tau + O(z^{-1})$: this is the J-function $J_{\mathcal{Y}}(\tau,-z)$ for $\mathcal{Y}$. Thus we obtain the following mirror theorem.

Corollary 28. *With hypotheses and notation as above, we have:*

$$J_{\mathcal{Y}}(\tau(t,x),z) = \frac{I^S_{\mathcal{Y}}(t,x,z)}{F(t,x)}$$

Remark 29. In general the $(\mathbf{e},\mathcal{E})$-twisted I-function will not satisfy (12), but one can still obtain the $(\mathbf{e},\mathcal{E})$-twisted J-function by Birkhoff factorization as in §3.8. Provided that the bundle $\mathcal{E}$ is convex, this allows the computation of genus-zero Gromov–Witten invariants of $\mathcal{Y}$.

Remark 30. If $\mathcal{E}$ is not convex then the relationship between $(\mathbf{e},\mathcal{E})$-twisted Gromov–Witten invariants of $\mathcal{X}$ and Gromov–Witten invariants of $\mathcal{Y}$ is not well understood [14]. This merits further investigation.

Remark 31. Certain components of $I^S_{\mathcal{Y}}$ can be written as (exponential) periods of the Landau–Ginzburg model mirror to $\mathcal{Y}$: see [27].

5.1. Example 9: a sextic hypersurface in $\mathbb{P}(1,1,1,3,3)$

Let the orbifold $\mathcal{Y}$ be a smooth sextic hypersurface in $\mathcal{X} = \mathbb{P}(1,1,1,3,3)$; this is a Fano 3-fold with canonical singularities. The ambient space $\mathcal{X}$ is the toric Deligne–Mumford stack associated to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \rho)$, where:

$$\rho = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 0 & 1 \end{pmatrix} : \mathbb{Z}^5 \to N = \mathbb{Z}^4$$

and Σ is the complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}^4$ with rays given by the columns $\rho_1, \ldots, \rho_5$ of ρ. We identify $\mathrm{Box}(\mathbf{\Sigma})$ with the set $\left\{0, \frac{1}{3}, \frac{2}{3}\right\}$ via the map $\kappa\colon x \mapsto x(\rho_1 + \rho_2 + \rho_3)$. Consider the S-extended I-function where $S = \left\{0, \frac{1}{3}\right\}$ and $S \to N_\Sigma$ is the map κ. The S-extended fan map is:

$$\rho^S = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 & -1 \\ -3 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} : \mathbb{Z}^{5+2} \to N$$

so that $\mathbb{L}^S_{\mathbb{Q}} \cong \mathbb{Q}^3$ is identified as a subset of $\mathbb{Q}^{5+2}$ via the inclusion:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{3} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix}$$

The S-extended Mori cone is the positive octant. We see that $\Lambda^S \subset \mathbb{L}^S_{\mathbb{Q}}$ is the sublattice of vectors:

$$\begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \qquad \text{such that } l,\ k_0,\ k_1 \in \mathbb{Z}$$

and that the reduction function is:

$$v^S\colon \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto \left\langle \frac{k_1 - l}{3} \right\rangle$$

Let $P \in H^2(\mathcal{X}; \mathbb{Q})$ denote the (non-equivariant) first Chern class of $\mathcal{O}_{\mathcal{X}}(1)$, and identify the Novikov ring $\mathbf{\Lambda}$ with $\mathbb{C}[\![Q]\!]$ via the map that sends $d \in H_2(X; \mathbb{Z})$ to $Q^{\int_d 3P}$. With notation as in §4 we have $D_1 = D_2 = D_3 = P$,

$D_4 = D_5 = 3P$, and so the non-equivariant limit of the S-extended I-function is:

$$I^S_{\mathrm{non}}(t,x,z) = z e^{(t_1+t_2+t_3+3t_4+3t_5)P/z} \\ \times \sum_{(l,k_0,k_1)\in\mathbb{N}^3} \frac{Q^l x_0^{k_0} x_1^{k_1} e^{(t_1+t_2+t_3+3t_4+3t_5)l}}{z^{k_0+k_1} k_0! k_1!} \\ \times \frac{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{3}\rangle \\ b\le 0}} (P+bz)^3}{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{3}\rangle \\ b\le \frac{l-k_1}{3}}} (P+bz)^3} \frac{\mathbf{1}_{\left\langle \frac{k_1-l}{3}\right\rangle}}{\prod_{\substack{\langle b\rangle=0 \\ 1\le b\le l}} (3P+bz)^2}$$

Let $\mathcal{E}\to\mathcal{X}$ be the line bundle corresponding to the element $\varepsilon \in (\mathbb{L}^S)^\vee$ given by:

$$\varepsilon\colon \begin{pmatrix} l \\ k_0 \\ k_1 \end{pmatrix} \mapsto 2l$$

so that $\mathcal{E} = \mathcal{O}(6)$. The S-extended $(\mathbf{e},\mathcal{E})$-twisted I-function of $\mathcal{X}$ is:

$$I^S_{\mathbf{e},\mathcal{E}}(t,x,z) = z e^{(t_1+t_2+t_3+3t_4+3t_5)P/z} \\ \times \sum_{(l,k_0,k_1)\in\mathbb{N}^3} \frac{Q^l x_0^{k_0} x_1^{k_1} e^{(t_1+t_2+t_3+3t_4+3t_5)l}}{z^{k_0+k_1} k_0! k_1!} \\ \times \frac{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{3}\rangle \\ b\le 0}} (P+bz)^3}{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{3}\rangle \\ b\le \frac{l-k_1}{3}}} (P+bz)^3} \frac{\prod_{\substack{\langle b\rangle=0 \\ 1\le b\le 2l}} (\kappa+6P+bz)}{\prod_{\substack{\langle b\rangle=0 \\ 1\le b\le l}} (3P+bz)^2} \mathbf{1}_{\left\langle \frac{k_1-l}{3}\right\rangle}$$

This is homogeneous of degree 1 if we set $\deg t_1 = \deg t_2 = \deg t_3 = \deg t_4 = \deg t_5 = 0$, $\deg z = \deg Q = \deg x_0 = \deg \kappa = 1$, and $\deg x_1 = 0$. We therefore have:

$$I^S_{\mathcal{Y}}(t,x,z) = z + (t_1+t_2+t_3+3t_4+3t_5)P + x_0\mathbf{1}_0 + f(x_1)\mathbf{1}_{\frac{1}{3}} + O(z^{-1})$$

where:

$$f(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{3m+1}}{(3m+1)!} \frac{\Gamma(m+\frac{1}{3})^3}{\Gamma(\frac{1}{3})^3}$$

Let g denote the power series inverse to f, so that $g(x) = x + \frac{x^4}{648} + \cdots$, and set:

$$t_i = \begin{cases} \tau & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \qquad x_i = \begin{cases} \xi_0 & \text{if } i = 0 \\ g(\xi_1) & \text{if } i = 1 \end{cases}$$

Then:

$$I^S_{\mathcal{Y}}(t, x, z) = z + \tau P + \xi_0 \mathbf{1}_0 + \xi_1 \mathbf{1}_{\frac{1}{3}} + O(z^{-1})$$

and Corollary 28 implies that:

$$J_{\mathcal{Y}}\big(\tau P + \xi_0 \mathbf{1}_0 + \xi_1 \mathbf{1}_{\frac{1}{3}}, z\big) = z e^{\tau P/z} \sum_{(l,k_0,k_1)\in\mathbb{N}^3} \frac{Q^l \xi_0^{k_0} g(\xi_1)^{k_1} e^{\tau l}}{z^{k_0+k_1} k_0! k_1!}$$
$$\times \frac{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{3}\rangle \\ b\leq 0}} (P+bz)^3}{\prod_{\substack{\langle b\rangle = \langle \frac{l-k_1}{3}\rangle \\ b\leq \frac{l-k_1}{3}}} (P+bz)^3} \frac{\prod_{\substack{\langle b\rangle = 0 \\ 0\leq b\leq 2l}} (6P+bz)}{\prod_{\substack{\langle b\rangle = 0 \\ 1\leq b\leq l}} (3P+bz)^2} \mathbf{1}_{\langle \frac{k_1-l}{3}\rangle}$$

For example, the coefficient of $\mathbf{1}_0$ in $J_{\mathcal{Y}}\big(\tau P + \xi_0 \mathbf{1}_0 + \xi_1 \mathbf{1}_{\frac{1}{3}}, z\big)$ is:

$$\sum_{l=0}^{\infty} \sum_{k_0=0}^{\infty} \sum_{\substack{k_1: 0\leq k_1\leq l \\ k_1\equiv l \bmod 3}} \frac{Q^l \xi_0^{k_0} g(\xi_1)^{k_1} e^{\tau l}}{z^{k_0+k_1-1} k_0! k_1!} \frac{1}{\left(\frac{l-k_1}{3}\right)!} \frac{(2l)!}{(l!)^2}$$

This is the so-called *quantum period* of $\mathcal{Y}$.

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