

Ring objects in the equivariant derived Satake category arising from Coulomb branches

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Appendix B by Gus Lonergan

This is the second companion paper of [Part II]. We consider the morphism from the variety of triples introduced in [Part II] to the affine Grassmannian. The direct image of the dualizing complex is a ring object in the equivariant derived category on the affine Grassmannian (equivariant derived Satake category). We show that various constructions in [Part II] work for an arbitrary commutative ring object.

The second purpose of this paper is to study Coulomb branches associated with star shaped quivers, which are expected to be conjectural Higgs branches of $3d$ Sicilian theories in type A by [BTX10].

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1. Introduction

This is the second companion paper of [Part II], where we give a mathematical definition of the Coulomb branch \mathcal{M}_C of a $3d$ SUSY gauge theory associated with a complex reductive group G and its symplectic representation \mathbf{M} of a form $\mathbf{N} \oplus \mathbf{N}^*$. Recall that \mathcal{M}_C is defined as an affine algebraic variety whose coordinate ring is the equivariant Borel Moore homology group $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ of a certain space \mathcal{R} , called the variety of triples. The product is given by the convolution. Here $G_{\mathcal{O}}$ is the $\mathbb{C}[[z]]$ -valued points of G .

By its definition, we have a projection $\pi: \mathcal{R} \rightarrow \mathrm{Gr}_G$, where Gr_G is the affine Grassmannian for G . Therefore we have a natural object \mathcal{A} in an appropriate Ind-completion $D_G(\mathrm{Gr}_G)$ of the derived $G_{\mathcal{O}}$ -equivariant constructible category on Gr_G defined by $\pi_* \omega_{\mathcal{R}}[-2 \dim \mathbf{N}_{\mathcal{O}}]$, where $\omega_{\mathcal{R}}$ is the dualizing complex on \mathcal{R} . We can recover $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ as $H_{G_{\mathcal{O}}}^*(\mathrm{Gr}_G, \mathcal{A})$. Moreover the construction of the convolution product gives us a homomorphism $\mathfrak{m}: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}$, where \star is the convolution product on $D_G(\mathrm{Gr}_G)$. It is an associative multiplication on \mathcal{A} . Then we have an induced multiplication on $H_{G_{\mathcal{O}}}^*(\mathrm{Gr}_G, \mathcal{A})$ from \mathfrak{m} , which is the same as the product on $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ defined in [Part II]. We also prove that it is a commutative object in $D_G(\mathrm{Gr}_G)$, and hence the induced multiplication on $H_{G_{\mathcal{O}}}^*(\mathrm{Gr}_G, \mathcal{A})$ is commutative. It is the second proof of the commutativity of the product on $H_*^{G_{\mathcal{O}}}(\mathcal{R})$, which is more conceptual than the first computational proof in [Part II].

In view of the original proposal in [Nak16], we expect that this construction can be generalized to the case when \mathbf{M} is not necessarily of the form $\mathbf{N} \oplus \mathbf{N}^*$.

Anyhow if we have a commutative ring object \mathcal{A} in $D_G(\mathrm{Gr}_G)$, we get a commutative ring structure on $H_{G_{\mathcal{O}}}^*(\mathrm{Gr}_G, \mathcal{A})$, and hence the ‘Coulomb branch’ as its spectrum.

Our reformulation of the definition of the Coulomb branch via $(\mathcal{A}, \mathfrak{m})$ reminds us a construction of the nilpotent cone and its Springer resolution via a perverse sheaf \mathcal{A}_R [ABG04]. Here \mathcal{A}_R is a perverse sheaf corresponding to the regular representation $\mathbb{C}[G^{\vee}]$ of the Langlands dual group G^{\vee} under the geometric Satake correspondence, and hence is a commutative ring object in $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$. Let us call it the *regular sheaf*. It is given by $\bigoplus_{\lambda} (V_{G^{\vee}}^{\lambda})^{\vee} \otimes_{\mathbb{C}} \mathrm{IC}(\overline{\mathrm{Gr}}_G^{\lambda})$, where $(V_{G^{\vee}}^{\lambda})^{\vee}$ is the dual of the irreducible representation of G^{\vee} with the highest weight λ and $\overline{\mathrm{Gr}}_G^{\lambda}$ is the closure of the $G_{\mathcal{O}}$ -orbit of z^{λ} in Gr_G . We prove that \mathcal{A}_R is realised as a variant of the above \mathcal{A} for a quiver gauge theory in type A . (We consider the framed quiver gauge theory of type A_{N-1} with $\dim V = (N-1, N-2, \dots, 1)$, $\dim W = (N, 0, \dots, 0)$

and consider the pushforward to $\mathrm{Gr}_{\mathrm{PGL}(N)}$. See §2(v) for more detail.) This construction might be generalized to type BCD , once we can generalize our definition to the case when \mathbf{M} is not necessarily of a form $\mathbf{N} \oplus \mathbf{N}^*$ (cotangent type). However we do *not* expect \mathcal{A}_R arises in a similar way for exceptional types. Hence we have more examples of commutative ring objects in $D_G(\mathrm{Gr}_G)$ than our construction.

Once we have a collection $\{\mathcal{A}_i\}$ of commutative ring objects in $D_G(\mathrm{Gr}_G)$, we can construct a new commutative ring object as $i_{\Delta}^!(\boxtimes \mathcal{A}_i)$, where $i_{\Delta} : \mathrm{Gr}_G \rightarrow \prod_i \mathrm{Gr}_G$ is the diagonal embedding. We call this the *gluing construction*. It is motivated by [CHMZ14a]. (See [Nak16, 5(i)] for a quick review and links to other physics literature.)

The second purpose of this paper is to study Coulomb branches associated with a star shaped quiver. It is regarded as an example of the gluing construction of a ring objects from those for legs. It is expected that Coulomb branches of star shaped quiver gauge theories are conjectural Higgs branches of 3d Sicilian theories in type A [BTX10]. (See [Nak16, 3(iii)] for a review for a mathematician.) Expected properties of these Higgs branches are listed in [MT12]. Recently Ginzburg-Kazhdan [GK] construct holomorphic symplectic varieties satisfying (most of) these properties for any type. The construction of \mathcal{A}_R as \mathcal{A} implies that Coulomb branches of star shaped quiver gauge theories are isomorphic to Ginzburg-Kazhdan varieties in type A via [Bap15]. We check two among the remaining properties, which identify Ginzburg-Kazhdan varieties of type A_1, A_2 with $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and the minimal nilpotent orbit of E_6 respectively.

We do *not* expect Ginzburg-Kazhdan varieties for exceptional groups are Coulomb branches of gauge theories. This is compatible with physicists' expectation that 3d Sicilian theories are *not* lagrangian theories. Nevertheless 3d Sicilian theories are accepted as well-defined quantum field theories. And there are many such examples. It is compatible with our observation that

- 1) We have examples of ring objects on $D_G(\mathrm{Gr}_G)$, which may not arise from any pair (G, \mathbf{N}) .
- 2) We have manipulations on ring objects, such as the gluing construction and hamiltonian reduction (see §5(viii) for the latter).

We thus hope that ring objects are useful to study non-lagrangian theories.

There is an Appendix A, which discusses a result of independent interest. We construct a complex reductive group hamiltonian action on the Coulomb branch of a framed quiver gauge theory by integrating hamiltonian vector

fields of functions introduced in [Quiver, Appendix B]. This extends a torus action constructed in [Part II, §3(v)] by grading on $H_*^{G\circ}(\mathcal{R})$. The regular sheaf \mathcal{A}_R has the G^\vee -action, which is identified with this group action for the framed quiver gauge theory mentioned above.

The other parts of the paper are organized as follows. In §2 we show that $\pi_*\omega_{\mathcal{R}}[-2\dim\mathbf{N}_{\mathcal{O}}]$ and its cousin for gauge theory with a flavor symmetry group are ring objects. We observe that $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A})$ is a commutative ring for a commutative ring object \mathcal{A} in $D_G(\mathrm{Gr}_G)$, where $\mathbf{1}_{\mathrm{Gr}_G}$ is the skyscraper sheaf at the base point in Gr_G . Considering skyscraper sheaves at other points, we construct line bundles over a partial resolution of $\mathrm{Spec}\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A})$. We follow [ABG04] for these constructions. The gluing construction is explained in §2(viii). In §3 we give a proof of commutativity of \mathfrak{m} . The idea is well-known: we use Beilinson-Drinfeld Grassmannian to deform a situation where the product is manifestly symmetric. Then we use nearby cycle functors and dual specialization homomorphisms. In §4 we show that the regular sheaf \mathcal{A}_R arises as a pushforward in a framed quiver gauge theory in type A . In §5 we study Coulomb branches associated with star shaped quivers. Since §§4, 5 depend crucially on the construction in Appendix A, the authors recommend the reader to go to Appendix A before visiting §§4, 5.

In Appendix B written by Gus Lonergan, we give another proof of the commutativity of the convolution product. This proof is more direct than the proof in the main text. A key ingredient is a global version of the convolution diagram for the variety of triples \mathcal{R} .

Notation

We basically follow the notation in [Part II] and [Quiver]. The Weyl group is denoted by \mathbb{W} in order to distinguish a vector space W used for a quiver.

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2. Complexes on the affine Grassmannian

In this section we interpret the convolution product $*$ in terms of a complex on the affine Grassmannian. Our goal is to construct a commutative ring object in $D_G(\mathrm{Gr}_G)$, an appropriate Ind-completion of the $G_{\mathcal{O}}$ -equivariant derived constructible category on Gr_G . Here the multiplication is given by the product \star appearing in geometric Satake correspondence [MV07].

The construction of this section, except §2(viii), is motivated by the work of Arkhipov, Bezrukavnikov and Ginzburg [ABG04], where the nilpotent cone \mathcal{N} of the Langlands dual group is constructed from the regular sheaf \mathcal{A}_R on Gr_G .

The construction of §2(viii) is motivated by [CHMZ14a], as we have mentioned already in Introduction.

2(i). Categorical generalities

Let X be a scheme of finite type over \mathbb{C} . Then we denote by $D(X)$ the ind-completion of the bounded derived category of constructible sheaves on X ; same definition applies to the equivariant derived category $D_G(X)$ where G is a (pro)algebraic group acting on X . It is obvious that for a G -equivariant morphism $f: X \rightarrow Y$ the derived direct images $f_!, f_*: D_G(X) \rightarrow D_G(Y)$ are well-defined. The same thing is true for the inverse images $f^!, f^*: D_G(Y) \rightarrow D_G(X)$.

Assume that G has finitely many orbits on X . Then a morphism $\mathcal{F} \rightarrow \mathcal{G}$ in $D_G(X)$ is an isomorphism if and only if it is an isomorphism on all $!$ -stalks (the assumption that G acts with finitely many orbits is needed in order to guarantee that there is an open dense subset of X on which both \mathcal{F} and \mathcal{G} are locally constant).

Let now X be an ind-scheme which is a filtered inductive limit of schemes of finite type over \mathbb{C} with respect to closed embeddings. For simplicity we shall assume that X is just the union of closed subschemes $X_0 \subset X_1 \subset \dots$ where each X_i is a scheme of finite type over \mathbb{C} and each inclusion $X_i \subset X_{i+1}$ is a closed embedding; we denote this embedding by σ_i . We shall call such ind-schemes *good*. Assume that a (pro)algebraic group G acts on each X_i and this action commutes with σ_i 's. Then we shall say that X is a good G -scheme.

For a good G -ind-scheme X we define the category $D_G(X)$ whose objects are systems $(\mathcal{F}_i, \kappa_i)_{i=0}^\infty$ where

- $\mathcal{F}_i \in D_G(X_i)$
- $\kappa_i: \sigma_i^! \mathcal{F}_{i+1} \rightarrow \mathcal{F}_i$ is an isomorphism.

A morphism $\alpha: (\mathcal{F}_i, \kappa_i) \rightarrow (\mathcal{F}'_i, \kappa'_i)$ is collection of morphisms $\mathcal{F}_i \rightarrow \mathcal{F}'_i$ for each i which commute with the κ_i 's. It is easy to see that $D_G(X)$ is a triangulated category. Assume that G acts with finitely many orbits on each X_i ; in this case we shall say that X is a very good G -ind-scheme. Then again a morphism $\mathcal{F} \rightarrow \mathcal{G}$ in $D_G(X)$ is an isomorphism if and only if it is an isomorphism on all $!$ -stalks.

Let X, Y be two good G -ind-schemes and let $f: X \rightarrow Y$ be a G -equivariant morphism. Then we can define the functor $f_*: D_G(X) \rightarrow D_G(Y)$ (but a priori not the functor $f_!$). It is defined in the following way. Given an object $(\mathcal{F}_i, \kappa_i)$ of $D_G(X)$ we need to define an object (\mathcal{G}_j, η_j) of $D_G(Y)$. Let $Z_j = f^{-1}(Y_j)$. This is again a good G -ind-scheme – it is the inductive limit of $Z_{i,j} = X_i \cap f^{-1}(Y_j)$. Let $\mathcal{F}_{i,j}$ denote the $!$ -restriction of \mathcal{F}_i to $Z_{i,j}$. Let also $f_{i,j}: Z_{i,j} \rightarrow Y_j$ denote the natural morphism. Since $(\sigma_i^!)$ is right adjoint to $(\sigma_i)_!$, the isomorphism κ_i gives rise to a map $(\sigma_i)_! \mathcal{F}_i = (\sigma_i)_* \mathcal{F}_i \rightarrow \mathcal{F}_{i+1}$; $!$ -restricting this to Z_j we get a morphism $(\sigma_i)_* \mathcal{F}_{i,j} \rightarrow \mathcal{F}_{i+1,j}$ which gives rise to a natural map $(f_{i,j})_* \mathcal{F}_{i,j} \rightarrow (f_{i+1,j})_* \mathcal{F}_{i+1,j}$. Hence the inductive limit of $(f_{i,j})_* \mathcal{F}_{i,j}$'s (with respect to i) makes sense and we denote it by \mathcal{G}_j . The construction of isomorphisms η_j between the $!$ -restriction of \mathcal{G}_{j+1} and \mathcal{G}_j is immediate from the usual base change.

In what follows we are going apply it for example to X being Gr_G for some reductive group G . In this case we can talk about the equivariant derived category $D_{G_{\mathcal{O}}}(\text{Gr}_G)$ which as before we shall simply denote by $D_G(\text{Gr}_G)$ (a priori it depends on a choice of X_i 's above; to simplify the discussion we are going to make this choice, although it is not difficult to see that the resulting category is independent of that choice); it is clear that (for any choice of X_i 's) Gr_G is a very good $G_{\mathcal{O}}$ -ind-scheme. The above general discussion also shows that given two objects $\mathcal{F}, \mathcal{G} \in D_G(\text{Gr}_G)$ we can define their convolution $\mathcal{F} \star \mathcal{G} \in D_G(\text{Gr}_G)$.

2(ii). Pushforward to the affine Grassmannian

Let \mathbf{N} be a finite dimensional representation of a complex reductive group G . Let \mathcal{R} be the variety of triples as in [Part II], and $\omega_{\mathcal{R}}$ its dualizing complex.

Proposition 2.1. *Let $\pi: \mathcal{R} \rightarrow \mathrm{Gr}_G$ be the projection and*

$$\mathcal{A} \stackrel{\mathrm{def.}}{=} \pi_* \omega_{\mathcal{R}}[-2 \dim \mathbf{N}_{\mathcal{O}}] \in D_G(\mathrm{Gr}_G).$$

(1) *There exists a natural multiplication homomorphism*

$$m: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A},$$

where the left hand side is the convolution product of \mathcal{A} with itself given by the diagram [Part II, (3.1)].

(2) *Let $\mathbf{1}_{\mathrm{Gr}_G}$ denote the skyscraper sheaf at the base point in Gr_G . Recall that it is the unit element in $D_G(\mathrm{Gr}_G)$, i.e., we have natural isomorphisms $\mathbf{1}_{\mathrm{Gr}_G} \star \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \star \mathbf{1}_{\mathrm{Gr}_G}$. We have a homomorphism $1: \mathbf{1}_{\mathrm{Gr}_G} \rightarrow \mathcal{A}$ such that*

$$\mathcal{A} \cong \mathcal{A} \star \mathbf{1}_{\mathrm{Gr}_G} \xrightarrow{\mathrm{id} \star 1} \mathcal{A} \star \mathcal{A} \xrightarrow{m} \mathcal{A}, \quad \mathcal{A} \cong \mathbf{1}_{\mathrm{Gr}_G} \star \mathcal{A} \xrightarrow{1 \star \mathrm{id}} \mathcal{A} \star \mathcal{A} \xrightarrow{m} \mathcal{A}$$

are both $\mathrm{id}_{\mathcal{A}}$.

(3) *Under the natural associativity isomorphism $\mathcal{A} \star (\mathcal{A} \star \mathcal{A}) \cong (\mathcal{A} \star \mathcal{A}) \star \mathcal{A}$, we have*

$$m \circ (m \star \mathrm{id}) = m \circ (\mathrm{id} \star m).$$

(4) *The product on $H_{G_{\mathcal{O}}}^*(\mathrm{Gr}_G, \mathcal{A}) \cong H_*^{G_{\mathcal{O}}}(\mathcal{R})$ induced by m is the same as the convolution product $*$.*

(5) (1)~(4) *remain true for the $G_{\mathcal{O}} \rtimes \mathbb{C}^\times$ -equivariant setting.*

The product in (4) is defined as follows: Let

$$x, y \in H_{G_{\mathcal{O}}}^*(\mathcal{A}) = \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}).$$

Then $x \star y \in \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{C}_{\mathrm{Gr}_G} \star \mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A} \star \mathcal{A})$. We have a natural homomorphism $\mathbb{C}_{\mathrm{Gr}_G} \rightarrow \mathbb{C}_{\mathrm{Gr}_G} \star \mathbb{C}_{\mathrm{Gr}_G}$ from the adjunction homomorphism $\mathbb{C}_{\mathrm{Gr}_G} \rightarrow m_* m^* \mathbb{C}_{\mathrm{Gr}_G}$. Therefore we combine it with $m: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}$, we get $x \star y \in \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A})$.

Proof. Let us combine two diagrams [Part II, (3.1), (3.2)]:

$$(2.2) \quad \begin{array}{ccccccc} \mathcal{R} \times \mathcal{R} & \xleftarrow{\tilde{p}} & p^{-1}(\mathcal{R} \times \mathcal{R}) & \xrightarrow{\tilde{q}} & q(p^{-1}(\mathcal{R} \times \mathcal{R})) & \xrightarrow{\tilde{m}} & \mathcal{R} \\ i \times \text{id}_{\mathcal{R}} \downarrow & & i' \downarrow & & \downarrow \bar{i} & & \downarrow i \\ \mathcal{T} \times \mathcal{R} & \xleftarrow{p} & G_{\mathcal{K}} \times \mathcal{R} & \xrightarrow{q} & G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{R} & \xrightarrow{m} & \mathcal{T} \\ \pi \times \pi \downarrow & & \text{id}_{G_{\mathcal{K}}} \times \pi \downarrow & & \downarrow \bar{\pi} & & \downarrow \pi \\ \text{Gr}_G \times \text{Gr}_G & \xleftarrow{\bar{p}} & G_{\mathcal{K}} \times \text{Gr}_G & \xrightarrow{\bar{q}} & \text{Gr}_G \tilde{\times} \text{Gr}_G & \xrightarrow{\bar{m}} & \text{Gr}_G, \end{array}$$

where we have changed the notation for morphisms in the bottom row putting ‘bar’. We also denote $\pi \circ i$ simply by π for brevity.

The restriction with support homomorphism [Part II, (3.7)] induces

$$\begin{aligned} \mathcal{A} \boxtimes \mathcal{A} &= (\pi \times \pi)_*(\omega_{\mathcal{R} \times \mathcal{R}})[-4 \dim \mathbf{N}_{\mathcal{O}}] \\ &\rightarrow (\pi \times \pi)_* \tilde{p}_*(\omega_{p^{-1}(\mathcal{R} \times \mathcal{R})}[-2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]) \\ &\cong \bar{p}_*(\text{id}_{G_{\mathcal{K}}} \times \pi)_* i'_*(\omega_{p^{-1}(\mathcal{R} \times \mathcal{R})}[-2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]). \end{aligned}$$

By adjunction, we get

$$\bar{p}^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow (\text{id}_{G_{\mathcal{K}}} \times \pi)_* i'_*(\omega_{p^{-1}(\mathcal{R} \times \mathcal{R})}[-2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]).$$

Since \tilde{q} is the quotient by $G_{\mathcal{O}}$, the right hand side is

$$\begin{aligned} &(\text{id}_{G_{\mathcal{K}}} \times \pi)_* i'_* \tilde{q}^*(\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]) \\ &\cong \tilde{q}^* \bar{\pi}_* \bar{i}_*(\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathbf{N}_{\mathcal{O}}]). \end{aligned}$$

Applying $(\tilde{q}^*)^{-1}$, we get a homomorphism

$$(2.3) \quad \mathcal{A} \tilde{\boxtimes} \mathcal{A} = (\tilde{q}^*)^{-1} \bar{p}^*(\mathcal{A} \boxtimes \mathcal{A}) \rightarrow \bar{\pi}_* \bar{i}_*(\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[-2 \dim \mathbf{N}_{\mathcal{O}}]).$$

We further apply \bar{m}_* :

$$\mathcal{A} \star \mathcal{A} = \bar{m}_*(\mathcal{A} \tilde{\boxtimes} \mathcal{A}) \rightarrow \pi_* i_* \tilde{m}_*(\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[2 \dim \mathbf{N}_{\mathcal{O}}]).$$

The left hand side is nothing but the convolution product $\mathcal{A} \star \mathcal{A}$ defined by the diagram [Part II, (3.1)].

Since \tilde{m} is proper, we have a natural homomorphism

$$\tilde{m}_*(\omega_{q(p^{-1}(\mathcal{R} \times \mathcal{R}))}[2 \dim \mathbf{N}_{\mathcal{O}}]) \rightarrow \omega_{\mathcal{R}}[2 \dim \mathbf{N}_{\mathcal{O}}].$$

Thus we obtain the homomorphism in (1).

Proofs of (2),(3) are already given in the proof of [Part II, Thm. 3.10]. Note that the associativity isomorphism is given by the Gr_G -version of the big square diagram appearing in the proof of [Part II, Thm. 3.10]. See [MV07, Prop. 4.6].

Taking hypercohomology groups, one can check (4). We omit the detail. □

Remarks 2.4. (1) By [BF08, Thm. 5], $\mathcal{A} \in D_{G_{\mathcal{O}} \times \mathbb{C}^\times}(\mathrm{Gr}_G)$ corresponds to a certain differential graded Harish-Chandra bimodule of G^\vee . We do not know anything about it except the example just below.

(2) Let us denote by \mathcal{A}_R the regular sheaf, i.e., the perverse sheaf corresponding to the regular representation $\mathbb{C}[G^\vee]$ of the Langlands dual group G^\vee under the geometric Satake correspondence. It was denoted by \mathcal{R} in [ABG04], but it conflicts with our notation for the space \mathcal{R} . It is endowed with a natural morphism $\mathbf{m}: \mathcal{A}_R \star \mathcal{A}_R \rightarrow \mathcal{A}_R$ with properties listed in Proposition 2.1. The nilpotent cone \mathcal{N} of G^\vee and its Springer resolution $\tilde{\mathcal{N}}$ were constructed from \mathcal{A}_R in [ABG04]. Since it is more natural to compare \mathcal{A}_R with \mathcal{A} arising in the framework of a flavor symmetry group, more detail will be given §2(v). Finally, the dg-Harish-Chandra bimodule corresponding to \mathcal{A}_R is the ring $U_{\hbar}^{\square} \rtimes \mathbb{C}[G^\vee]$ of \hbar -differential operators on G^\vee .

The construction in this and the subsequent subsections shows that it is enough to have \mathcal{A} with $\mathbf{m}: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}$, i.e., a ring object in $D_G(\mathrm{Gr}_G)$ to define the Coulomb branch \mathcal{M}_C . For example, \mathcal{A}_R . Since \mathcal{A}_R for an exceptional group is unlikely to arise from any gauge theory (G, \mathbf{N}) , it is interesting to find other recipes to construct such an $(\mathcal{A}, \mathbf{m})$. We give one example of such a recipe in §2(viii) below.

2(iii). Commutativity

In this subsection we forget the loop rotation.

Let $\Theta: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A} \star \mathcal{A}$ be the commutativity constraint of the convolution product. Its construction, following [MV07, §5] and also [Gai01], will be recalled in §3(i).

Theorem 2.5. *We have $\mathbf{m} \circ \Theta \cong \mathbf{m}$ as homomorphism $\mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}$.*

It means that $(\mathcal{A}, \mathbf{m})$ is a *commutative* ring object in $(D_G(\mathrm{Gr}_G), \star)$. We give a proof in §3.

Our proof is indirect. We construct another multiplication $\mathbf{m}^\psi: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}$ using nearby cycle functors and dual specialization. We have $\mathbf{m}^\psi \circ \Theta \cong$

\mathfrak{m}^ψ . Therefore $(\mathcal{A}, \mathfrak{m}^\psi)$ is commutative, but we cannot check \mathfrak{m}^ψ is associative directly.

Next we show $\mathfrak{m}^\psi = \mathfrak{m}$ for $\mathbf{N} = 0$. This implies that $\mathfrak{m}^\psi = \mathfrak{m}$ holds after the fixed point localization for general \mathbf{N} . We do not have torsion where \mathfrak{m} and \mathfrak{m}^ψ live, hence this is enough.

2(iv). A complex on the affine Grassmannian of the flavor symmetry group

We suppose that \mathbf{N} is a representation of a larger group \tilde{G} containing G as a normal subgroup as in [Part II, §3(viii), §3(ix)]. Let $G_F = \tilde{G}/G$. We are going to construct a ring object in $D_{G_F}(\mathrm{Gr}_{G_F})$.

Let us denote $\mathcal{T}_{\tilde{G}, \mathbf{N}}, \mathcal{R}_{\tilde{G}, \mathbf{N}}$ by $\tilde{\mathcal{T}}, \tilde{\mathcal{R}}$ respectively for short as before. Composing $\tilde{\mathcal{T}} \rightarrow \mathrm{Gr}_{\tilde{G}}$ or $\tilde{\mathcal{R}} \rightarrow \mathrm{Gr}_{\tilde{G}}$ with the morphism $\mathrm{Gr}_{\tilde{G}} \rightarrow \mathrm{Gr}_{G_F}$, we have

$$(2.6) \quad \tilde{\pi}: \tilde{\mathcal{T}} \text{ or } \tilde{\mathcal{R}} \rightarrow \mathrm{Gr}_{G_F}.$$

Let us denote the fiber over $\lambda \in \mathrm{Gr}_{G_F}$ by $\tilde{\mathcal{R}}^\lambda$, where λ is a coweight of G_F regarded as a point in Gr_{G_F} . (In [Part II, §3(ix)] it was denoted by λ_F .)

As in Proposition 2.1, we consider a pushforward of the dualizing sheaf $\omega_{\tilde{\mathcal{R}}}$. Here we consider the dualizing sheaf of the larger space $\tilde{\mathcal{R}}$, and take the pushforward $\mathcal{A} \stackrel{\text{def.}}{=} \tilde{\pi}_* \omega_{\tilde{\mathcal{R}}}[-2 \dim \mathbf{N}_\mathcal{O}]$ to Gr_{G_F} . We consider it as an object in $D_{\tilde{G}}(\mathrm{Gr}_{G_F})$, an appropriate Ind-completion of the $\tilde{G}_\mathcal{O}$ -equivariant derived constructible category of Gr_{G_F} . We also have $Q_{\tilde{\pi}*} \omega_{\tilde{\mathcal{R}}}[-2 \dim \mathbf{N}_\mathcal{O}] = Q_{\mathrm{id}*} \mathcal{A}$, which is a $(G_F)_\mathcal{O}$ -equivariant object on Gr_{G_F} . Here ‘id’ is the identity of Gr_{G_F} and the general pushforward functor $Q_{\mathrm{id}*}, Q_{\tilde{\pi}*}$ changes the equivariance group from $\tilde{G}_\mathcal{O}$ to $(G_F)_\mathcal{O}$. See [BL94, §6].

In the same way as in Proposition 2.1, we have natural homomorphisms

$$(2.7) \quad \mathfrak{m}: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}, \quad \mathfrak{m}: Q_{\mathrm{id}*} \mathcal{A} \star Q_{\mathrm{id}*} \mathcal{A} \rightarrow Q_{\mathrm{id}*} \mathcal{A},$$

that satisfy the unit and associativity properties. It also satisfies the commutativity.

Let us give a small remark for the construction of the homomorphisms: When we define the convolution product $\mathcal{A} \star \mathcal{A}$, we use $(q^*)^{-1}$ for Gr_{G_F} . For this, we only need the $(G_F)_\mathcal{O}$ -equivariant structure, therefore we can replace the second factor \mathcal{A} by $Q_{\mathrm{id}*} \mathcal{A}$. However in the definition of the first homomorphism \mathfrak{m} , we need to go back to the space $\tilde{\mathcal{R}}$, hence we need the

\tilde{G}_O -equivariant structure. The second homomorphism \mathfrak{m} is induced from the first by applying $Q_{\text{id}*}$ and using the smooth base change.

Let $\mathbf{1}_{\text{Gr}_{G_F}}$ be the skyscraper sheaf at the base point in Gr_{G_F} . As in [ABG04, §7.2] we have an algebra structure on $\text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^*(\mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A})$: Let $x \in \text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^i(\mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A})$, $y \in \text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^j(\mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A})$. We consider $x \star y \in \text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^{i+j}(\mathbf{1}_{\text{Gr}_{G_F}} \star \mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A} \star \mathcal{A})$. We compose $\mathbf{1}: \mathbf{1}_{\text{Gr}_{G_F}} \cong \mathbf{1}_{\text{Gr}_{G_F}} \star \mathbf{1}_{\text{Gr}_{G_F}}$ and $\mathfrak{m}: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}$, we get $\mathfrak{m}(x \star y)\mathbf{1} \in \text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^{i+j}(\mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A})$.

Note that ext-groups in D_{G_F} and $D_{\tilde{G}_F}$ are isomorphic:

$$\text{Ext}_{D_{G_F}(\text{Gr}_{G_F})}^*(\mathbf{1}_{\text{Gr}_{G_F}}, Q_{\text{id}*}\mathcal{A}) \cong \text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^*(\mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A}),$$

where the right hand side is regarded as $H_{G_F}^*(\text{pt})$ -module via $H_{G_F}^*(\text{pt}) \rightarrow H_{\tilde{G}}^*(\text{pt})$. See [BL94, §13.5]. Thus the difference between \mathcal{A} and $Q_{\text{id}*}\mathcal{A}$ is not essential, we omit $Q_{\text{id}*}$ hereafter.

Since the fiber of $\tilde{\pi}: \tilde{\mathcal{R}} \rightarrow \text{Gr}_{G_F}$ at the base point is our original \mathcal{R} , we have a natural isomorphism

$$(2.8) \quad \text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^*(\mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A}) \cong H_*^{\tilde{G}_O}(\mathcal{R})$$

of $H_{\tilde{G}}^*(\text{pt})$ -modules.

The definition of the multiplication on $\text{Ext}_{D_{\tilde{G}}(\text{Gr}_{G_F})}^*(\mathbf{1}_{\text{Gr}_{G_F}}, \mathcal{A})$ uses \tilde{G} (or G_F) equivariance, as we use the descent $(q^*)^{-1}$.¹ On the other hand, the multiplication on the right hand side given in [Part II, Prop. 3.22] descends to $H_*^{\tilde{G}_O}(\mathcal{R})$. In fact, we will see that a simple modification of the definition gives a multiplication on the left hand side with the group changed from \tilde{G} to G in §2(vi).

Lemma 2.9. *The isomorphism (2.8) respects the multiplication. The same is true for $\tilde{G}_O \times \mathbb{C}^\times$ -equivariant groups.*

Proof. Let us consider a modification of the commutative diagram (2.2):

$$(2.10) \quad \begin{array}{ccccccc} \tilde{\mathcal{T}} \times \tilde{\mathcal{R}} & \xleftarrow{p} & \tilde{G}_{\mathcal{K}} \times \tilde{\mathcal{R}} & \xrightarrow{q} & \tilde{G}_{\mathcal{K}} \times_{\tilde{G}_O} \tilde{\mathcal{R}} & \xrightarrow{m} & \tilde{\mathcal{T}} \\ \tilde{\pi} \times \tilde{\pi} \downarrow & & \xi \times \tilde{\pi} \downarrow & & \downarrow \tilde{\pi} & & \downarrow \tilde{\pi} \\ \text{Gr}_{G_F} \times \text{Gr}_{G_F} & \xleftarrow{\bar{p}} & (G_F)_{\mathcal{K}} \times \text{Gr}_{G_F} & \xrightarrow{\bar{q}} & \text{Gr}_{G_F} \tilde{\times} \text{Gr}_{G_F} & \xrightarrow{\bar{m}} & \text{Gr}_{G_F}, \end{array}$$

¹We thank Roman Bezrukavnikov for a clarification of this point.

where $\xi: \tilde{G}_{\mathcal{K}} \rightarrow (G_F)_{\mathcal{K}}$ is a morphism induced from $\tilde{G} \rightarrow G_F$, and all other maps are given by replacing G, \mathcal{R}, \dots by $\tilde{G}, \tilde{\mathcal{R}}, \dots$, and composing $\text{Gr}_{\tilde{G}} \rightarrow \text{Gr}_{G_F}$, etc. We omit the first row for brevity.

Let $[1_{G_F}]$ denote the base point in Gr_{G_F} . We take the inverse images of $[1_{G_F}] \times [1_{G_F}]$, $(G_F)_{\mathcal{O}} \times [1_{G_F}]$, $[1_{G_F}] \star [1_{G_F}]$, $[1_{G_F}]$ in the first row. They are $\mathcal{T} \times \mathcal{R}, \tilde{G}_{\mathcal{K}}^{\mathcal{O}} \times \mathcal{R}, \tilde{G}_{\mathcal{K}}^{\mathcal{O}} \times_{\tilde{G}_{\mathcal{O}}} \mathcal{R}, \mathcal{T}$ respectively. Here $\tilde{G}_{\mathcal{K}}^{\mathcal{O}} = \xi^{-1}((G_F)_{\mathcal{O}})$ is the group introduced in [Part II, §3(viii)]. Thus we recover the diagram [Part II, (3.23)]. Now the assertion is easy to check, and hence we omit the detail. \square

2(v). An alternative construction of a regular sheaf

Consider a quiver gauge theory of type A_{N-1} with $\dim V = (N - 1, N - 2, \dots, 1)$, $\dim W = (N, 0, \dots, 0)$ with $G = \text{GL}(V) = \prod_{i=1}^{N-1} \text{GL}(i)$, $\tilde{G} = (\text{GL}(V) \times \text{GL}(W))/Z$, where $Z \cong \mathbb{C}^{\times}$ is the diagonal central subgroup. We have $G_F = \text{PGL}(W) = \text{PGL}(N)$ and apply the above construction to define \mathcal{A} . It is a complex on $\text{Gr}_{\text{PGL}(N)}$. We also know that the Coulomb branch \mathcal{M}_C of this quiver gauge theory is the nilpotent cone in $\mathfrak{sl}(N)$. (We know that \mathcal{M}_C is a transversal slice in the affine Grassmannian by [Quiver, §3] for a quiver gauge theory of type ADE. And in this case the transversal slice in the affine Grassmannian is the nilpotent cone by [Lus81]. See also [MV03].) Recall \mathcal{A}_R in Remark 2.4(2). We take $G = \text{PGL}(N)$. Then $\text{Ext}_{D(\text{Gr}_{\text{PGL}(N)})}^*(\mathbf{1}_{\text{Gr}_{\text{PGL}(N)}}, \mathcal{A}_R)$ gives also the nilpotent cone [ABG04, 7.3.1]. This is not a coincidence. We have

Theorem 2.11. *\mathcal{A}_R and \mathcal{A} are isomorphic as ring objects in*

$$D_{\text{PGL}(N)}(\text{Gr}_{\text{PGL}(N)}).$$

The proof will be given in §4.

2(vi). Line bundles via homology groups of fibers

We now return back to a general situation: we are given a commutative ring object in $D_G(\text{Gr}_G)$, i.e., we are given $\mathcal{A} \in D_G(\text{Gr}_G)$ with $1: \mathbf{1}_{\text{Gr}_G} \rightarrow \mathcal{A}$, $\mathfrak{m}: \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}$ satisfying the unit and associativity properties in Proposition 2.1(2) and the commutativity as in Theorem 2.5. The object constructed in §2(ii), as well as the object \mathcal{A} or $Q_{\text{id} \star \mathcal{A}}$ in §2(iv) is an example when we regard G_F as G . In fact, the latter is our primary example.

Let $D(\mathrm{Gr}_G)$ denote an appropriate Ind-completion of the constructible derived category on Gr_G (without $G_{\mathcal{O}}$ -equivariance structure). Let $\mathrm{For} : D_G(\mathrm{Gr}_G) \rightarrow D(\mathrm{Gr}_G)$ be the forgetful functor.

Remark 2.12. In the setting of §2(iv), we could consider $D_G(\mathrm{Gr}_{G_F})$, an appropriate Ind-completion of the $G_{\mathcal{O}}$ -equivariant constructible derived category on Gr_{G_F} . Note that $G_{\mathcal{O}}$ acts trivially on Gr_{G_F} . Let $\mathrm{Res}_{G_{\mathcal{O}}, \tilde{G}_{\mathcal{O}}}$ be the restriction functor $D_{\tilde{G}}(\mathrm{Gr}_{G_F}) \rightarrow D_G(\mathrm{Gr}_{G_F})$ restricting the group action from $\tilde{G}_{\mathcal{O}}$ to $G_{\mathcal{O}}$. Then we could consider $\mathcal{A}^{\mathrm{res}} = \mathrm{Res}_{G_{\mathcal{O}}, \tilde{G}_{\mathcal{O}}} \mathcal{A} \in D_G(\mathrm{Gr}_{G_F})$. This allows us to consider $\mathrm{Ext}_{D_G(\mathrm{Gr}_{G_F})}^*(\mathbf{1}_{\mathrm{Gr}_{G_F}}, \mathcal{A}^{\mathrm{res}})$, but the difference between this Ext group and $\mathrm{Ext}_{D(\mathrm{Gr}_{G_F})}^*(\mathbf{1}_{\mathrm{Gr}_{G_F}}, \mathrm{For} Q_{\mathrm{id}} * \mathcal{A})$ is not essential as we have remarked above. Therefore we do not keep two groups G, G_F , and just consider the above situation for brevity of the notation.

Let $\mathcal{A}^{\mathrm{for}} \stackrel{\mathrm{def.}}{=} \mathrm{For} \mathcal{A}$. Note that $\mathcal{A}^{\mathrm{for}} \star \mathcal{A}^{\mathrm{for}}$ is not defined as we do not have $(q^*)^{-1}$ for non $G_{\mathcal{O}}$ -equivariant objects. However we still have $\mathrm{For} m : \mathrm{For}(\mathcal{A} \star \mathcal{A}) \rightarrow \mathcal{A}^{\mathrm{for}} = \mathrm{For} \mathcal{A}$.

Viewing a coweight λ of G as a point in Gr_G , we denote the embedding by $i_\lambda : \{\lambda\} \rightarrow \mathrm{Gr}_G$.

Recall $m : \mathrm{Gr}_G \star \mathrm{Gr}_G \rightarrow \mathrm{Gr}_G$. For a coweight χ , let $\mathrm{Gr}_\chi^2 \stackrel{\mathrm{def.}}{=} m^{-1}(\chi)$ and denote the embedding $\mathrm{Gr}_\chi^2 \rightarrow \mathrm{Gr}_G \star \mathrm{Gr}_G$ by j_χ . We have the base change $i_\chi^! m_* = m_* j_\chi^!$.

Recall $\mathcal{A} \star \mathcal{A} = m_*(q^*)^{-1} p^*(\mathcal{A} \boxtimes \mathcal{A})$. Let us set $\mathcal{A} \tilde{\boxtimes} \mathcal{A} = (q^*)^{-1} p^*(\mathcal{A} \boxtimes \mathcal{A})$. As the forgetful functor commutes with m_* , we have

$$\mathrm{For}(\mathcal{A} \star \mathcal{A}) = m_* \mathrm{For}(\mathcal{A} \tilde{\boxtimes} \mathcal{A}).$$

We have

$$(2.13) \quad m_* j_\chi^! \mathrm{For}(\mathcal{A} \tilde{\boxtimes} \mathcal{A}) = i_\chi^! m_* \mathrm{For}(\mathcal{A} \tilde{\boxtimes} \mathcal{A}) = i_\chi^! \mathrm{For}(\mathcal{A} \star \mathcal{A}) \xrightarrow{i_\chi^! \mathrm{For} m} i_\chi^! \mathcal{A}^{\mathrm{for}}.$$

Claim. The embedding $\{\lambda\} \times \{\mu\} \rightarrow \mathrm{Gr}_{\lambda+\mu}^2$ induces a natural homomorphism

$$(2.14) \quad H^*(i_\lambda^! \mathcal{A}^{\mathrm{for}}) \otimes H^*(i_\mu^! \mathcal{A}^{\mathrm{for}}) \rightarrow H^*(j_{\lambda+\mu}^! \mathrm{For}(\mathcal{A} \tilde{\boxtimes} \mathcal{A})).$$

Proof. Let us regard λ as an element in $G_{\mathcal{K}}$ and denote the embedding $\{\lambda\} \rightarrow G_{\mathcal{K}}$ by \tilde{i}_λ . The morphism $q(\tilde{i}_\lambda \times i_\mu) : \{\lambda\} \times \{\mu\} \rightarrow \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ factors through $\mathrm{Gr}_{\lambda+\mu}^2$. Let us write the embedding $k_{\lambda,\mu} : \{\lambda\} \times \{\mu\} \rightarrow \mathrm{Gr}_{\lambda+\mu}^2$.

We note $k_{\lambda,\mu}^! j_{\lambda+\mu}^! q^! = (\tilde{i}_\lambda \times i_\mu)^! q^! = (\tilde{i}_\lambda \times i_\mu)^! q^*[2 \dim G_{\mathcal{O}}]$. Since the forgetful functor commutes with pull back homomorphisms [BL94, §3.4], we get

$$\begin{aligned}
 (2.15) \quad & k_{\lambda,\mu}^! j_{\lambda+\mu}^! \text{For}(\mathcal{A} \boxtimes \mathcal{A}) \\
 &= (\tilde{i}_\lambda \times i_\mu)^! p^*(\mathcal{A}^{\text{for}} \boxtimes \mathcal{A}^{\text{for}})[2 \dim G_{\mathcal{O}}] \\
 &= (\tilde{i}_\lambda \times i_\mu)^! p^!(\mathcal{A}^{\text{for}} \boxtimes \mathcal{A}^{\text{for}}) = (i_\lambda \times i_\mu)^!(\mathcal{A}^{\text{for}} \boxtimes \mathcal{A}^{\text{for}}).
 \end{aligned}$$

Since $k_{\lambda,\mu}$ is proper, we have a homomorphism $k_{\lambda,\mu*} k_{\lambda,\mu}^! = k_{\lambda,\mu!} k_{\lambda,\mu}^! \rightarrow \text{id}$. Now the assertion is clear. \square

Combining (2.13) with $\chi = \lambda + \mu$ and (2.14), we obtain a multiplication

$$(2.16) \quad H^*(i_\lambda^! \mathcal{A}^{\text{for}}) \otimes H^*(i_\mu^! \mathcal{A}^{\text{for}}) \rightarrow H^*(i_{\lambda+\mu}^! \mathcal{A}^{\text{for}}).$$

Remarks 2.17. (1) Note that the embedding i_λ is $T_{\mathcal{O}}$ -equivariant. Therefore we can use the restriction functor $\text{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}$ from $G_{\mathcal{O}}$ to $T_{\mathcal{O}}$ instead of the forgetful functor For . Then the same construction gives a multiplication

$$(2.18) \quad H_{T_{\mathcal{O}}}^*(i_\lambda^! \text{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}} \mathcal{A}) \otimes H_{T_{\mathcal{O}}}^*(i_\mu^! \text{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}} \mathcal{A}) \rightarrow H_{T_{\mathcal{O}}}^*(i_{\lambda+\mu}^! \text{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}} \mathcal{A}).$$

(2) Suppose $G = T$. Then $\text{Gr}_T = \bigsqcup_{\lambda \in Y} \{\lambda\}$, hence

$$H_{T_{\mathcal{O}}}^*(\text{Gr}_T, \mathcal{A}) = \bigoplus_{\lambda \in Y} H_{T_{\mathcal{O}}}^*(i_\lambda^! \mathcal{A}).$$

The multiplication explained after Proposition 2.1 is Y -graded, hence gives $H_{T_{\mathcal{O}}}^*(i_\lambda^! \mathcal{A}) \otimes H_{T_{\mathcal{O}}}^*(i_\mu^! \mathcal{A}) \rightarrow H_{T_{\mathcal{O}}}^*(i_{\lambda+\mu}^! \mathcal{A})$. It is clear that this multiplication is same as (2.18).

Suppose $\lambda = \mu = 0$. We have a commutative diagram

$$\begin{array}{ccc}
 (2.19) \quad \text{Ext}_{D_G(\text{Gr}_G)}^*(\mathbf{1}_{\text{Gr}_G}, \mathcal{A}) \otimes \text{Ext}_{D_G(\text{Gr}_G)}^*(\mathbf{1}_{\text{Gr}_G}, \mathcal{A}) & \xrightarrow{\mathfrak{m}(\bullet\bullet\bullet)\mathbf{1}} & \text{Ext}_{D_G(\text{Gr}_G)}^*(\mathbf{1}_{\text{Gr}_G}, \mathcal{A}) \\
 \text{For} \downarrow & & \downarrow \text{For} \\
 H^*(i_0^! \mathcal{A}^{\text{for}}) \otimes H^*(i_0^! \mathcal{A}^{\text{for}}) & \xrightarrow{(2.16)} & H^*(i_0^! \mathcal{A}^{\text{for}})
 \end{array}$$

via the isomorphism $\text{Ext}_{D(\text{Gr}_G)}(\mathbf{1}_{\text{Gr}_G}, \mathcal{A}^{\text{for}}) \cong H^*(i_0^! \mathcal{A}^{\text{for}})$.

In fact, the only place we need to check is the commutativity of

$$\begin{array}{ccc}
 \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}) \otimes \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}) & \xrightarrow{(q^*)^{-1}p^*} & \mathrm{Ext}_{D_G(\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G)}^*(\mathbb{C}_{[1_G] \star [1_G]}, \mathcal{A} \boxtimes \mathcal{A}) \\
 \mathrm{For} \downarrow & & \downarrow \mathrm{For} \\
 H^*(i_0^! \mathcal{A}^{\mathrm{for}}) \otimes H^*(i_0^! \mathcal{A}^{\mathrm{for}}) & \xrightarrow{(2.16)} & H^*(j_0^! \mathrm{For}(\mathcal{A} \boxtimes \mathcal{A})),
 \end{array}$$

where the right vertical arrow is defined as the embedding of $[1_G] \star [1_G]$ into $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ factors through Gr_0^2 . This commutativity is clear from (2.15).

In the setting of the previous subsection, the upper row of (2.19) is the same as the multiplication on $H_*^{\tilde{G}^\circ}(\mathcal{R})$ by Lemma 2.9, hence the lower row is also the same as $*$ on $H_*^{G^\circ}(\mathcal{R})$. In this sense the multiplication in (2.16) is a generalization of $*$.

Thus $\bigoplus H^*(i_\lambda^! \mathcal{A}^{\mathrm{for}})$ is an algebra graded by the coweight lattice of G . For $\lambda = 0$, we have a subalgebra $H^*(i_0^! \mathcal{A}^{\mathrm{for}})$, which is isomorphic to $H_*^{G^\circ}(\mathcal{R})$ in the setting of the previous subsection. One can also take a direct sum over *dominant* coweights λ of G .

For a fixed coweight λ , we consider the direct sum of $H^*(i_{n\lambda}^! \mathcal{A}^{\mathrm{for}})$ with degrees $n\lambda$ ($n \in \mathbb{Z}_{\geq 0}$). It is an algebra graded by $\mathbb{Z}_{\geq 0}$. The associated Proj

$$\mathrm{Proj} \left(\bigoplus_{n \geq 0} H^*(i_{n\lambda}^! \mathcal{A}^{\mathrm{for}}) \right)$$

has a natural projective morphism to $\mathrm{Spec}(H^*(i_0^! \mathcal{A}^{\mathrm{for}}))$. We have a natural line bundle $\mathcal{O}(1)$ on $\mathrm{Proj}(\bigoplus_{n \geq 0} H^*(i_{n\lambda}^! \mathcal{A}^{\mathrm{for}}))$ such that $H^*(i_{n\lambda}^! \mathcal{A}^{\mathrm{for}})$ is identified with the space of sections of $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$. Under some circumstances we expect $\mathrm{Proj}(\bigoplus_{n \geq 0} H^*(i_{n\lambda}^! \mathcal{A}^{\mathrm{for}}))$ is a (partial) resolution of $\mathrm{Spec}(H^*(i_0^! \mathcal{A}^{\mathrm{for}}))$.

In the example in Remark 2.4, \mathcal{A}_R gives the Springer resolution of the nilpotent cone \mathcal{N} of G^\vee , the Langlands dual group of G . See [ABG04, 8.5.2].

See [Nak16, §5.1] (and also [Part II, Remark 3.26]) for a physical origin of this construction.

Remark 2.20. In view of Remark 2.17(2), the construction in [Part II, §3(ix)] and the above construction is the same for \mathcal{A} in §2(iv). Here the construction in [Part II, §3(ix)] is as follows: Let us suppose $G \triangleleft \tilde{G}$ as in §2(iv) and further assume $G_F = \tilde{G}/G$ is a torus. Let us write $T_F = G_F$. The Coulomb branch $\mathcal{M}_C(\tilde{G}, \mathbf{N})$ for the larger group \tilde{G} has an action of $\pi_1(T_F)^\wedge = T_F^\vee$, and [Part II, Prop. 3.18] says that $\mathcal{M}(G, \mathbf{N})$ is the Hamiltonian reduction of $\mathcal{M}(\tilde{G}, \mathbf{N})$ by T_F^\vee . Let us denote the moment map by $\mu_{T_F^\vee}$.

The hamiltonian reduction more precisely means the affine algebro-geometric quotient $\mu_{T_F^\vee}^{-1}(0) // T_F^\vee$. If we have a cocharacter λ_F of T_F , we view it as a character of T_F^\vee and consider the GIT quotient $\mu_{T_F^\vee}^{-1}(0) //_{\lambda_F} T_F^\vee$.

2(vii). Wakimoto sheaves

The original definition of the multiplication (2.16) in [ABG04] was given by Wakimoto sheaves, and the above definition is taken from the proof of [ABG04, Thm. 8.5.2]. Although it is unnecessary, let us review the construction for the sake of the reader.

Let I be the Iwahori subgroup of $G_{\mathcal{K}}$ and let $\text{Fl}_G = G_{\mathcal{K}}/I$ be the affine flag variety. We have a smooth proper morphism $\varpi: \text{Fl}_G \rightarrow \text{Gr}_G$ of ind-schemes. Let \mathcal{W}_λ be the Wakimoto sheaf on Fl_G for G corresponding to a coweight λ . See [ABG04, §8] for the definition (due to Mirković). By [ABG04, §8.4], we have a ‘multiplication’

$$(2.21) \quad \mathbf{E}_\lambda \otimes \mathbf{E}_\mu \rightarrow \mathbf{E}_{\lambda+\mu}, \quad \mathbf{E}_\lambda = \text{Ext}_{D_I(\text{Gr}_G)}^*(\mathbf{1}_{\text{Gr}_G}, \mathcal{W}_\lambda \star \mathcal{A}),$$

where \star is the convolution product on I -equivariant complexes on Fl_G and Gr_G : Let $x \in \mathbf{E}_\lambda, y \in \mathbf{E}_\mu$. We consider the composite

$$\begin{aligned} y \cdot x: \mathbf{1}_{\text{Gr}_G} &\xrightarrow{y} \mathcal{W}_\mu \star \mathcal{A} = \mathcal{W}_\mu \star \mathbf{1}_{\text{Gr}_G} \star \mathcal{A} \xrightarrow{\mathcal{W}_\mu \star x \star \mathcal{A}} \mathcal{W}_\mu \star \mathcal{W}_\lambda \star \mathcal{A} \star \mathcal{A} \\ &= \mathcal{W}_{\lambda+\mu} \star \mathcal{A} \star \mathcal{A} \xrightarrow{m} \mathcal{W}_{\lambda+\mu} \star \mathcal{A}. \end{aligned}$$

Note that $\mathcal{W}_\mu \star x$ is well-defined as x is an I -equivariant homomorphism, and hence $\mathcal{W}_\mu \boxtimes x$ descends for the morphism q .

We have an isomorphism $\mathbf{E}_\lambda \cong H_{T_G}^*(i_\lambda^! \text{Res}_{T_G, G_G} \mathcal{A})$ (see [ABG04, (8.7.2)]), and the above multiplication is the same as (2.18).

2(viii). Gluing construction

One of motivations of [CHMZ14a] extending the monopole formula from the Hilbert series of the coordinate ring of the Coulomb branch \mathcal{M}_G to the character of the space of sections of a line bundle (see [Part II, Remark 3.26]) is to write down the Hilbert series of a complicated Coulomb branch from simpler ones. We use the machinery prepared in earlier subsections to introduce the corresponding construction at the level of commutative ring objects in $D_G(\text{Gr}_G)$.

The setting in [CHMZ14a] is as follows. Suppose that we have a finite collection $\{(G_i, \mathbf{N}_i)\}$ ($i = 1, 2, \dots$) of gauge theories sharing the common

flavor symmetry group, i.e., \mathbf{N}_i is a representation of a larger group \tilde{G}_i containing G_i as a normal subgroup with $G_F = \tilde{G}_i/G_i$, independent of i . Then we define G as the fiber product of \tilde{G}_i over G_F , and $\mathbf{N} = \bigoplus \mathbf{N}_i$. The monopole formula for the Hilbert series of the Coulomb branch of (G, \mathbf{N}) is given by extended monopole formula for (G_i, \mathbf{N}_i) . See also [Nak16, §5(i)] for a review.

An example is a star shaped quiver gauge theory, which is the 3d mirror of the Sicilian theory of type A_{N-1} , reviewed in [Nak16, §3(iii)]. See Figure 1. We have three copies of type A_{N-1} quiver gauge theory with $\dim V = (N - 1, N - 2, \dots, 1)$, $\dim W = (N, 0, \dots, 0)$ as in §2(v). We divide the group $\mathrm{GL}(V) = \prod \mathrm{GL}(V_i)$ by the diagonal central subgroup Z and take it as the gauge group. The common flavor symmetry group is $G_F = \mathrm{PGL}(N)$.

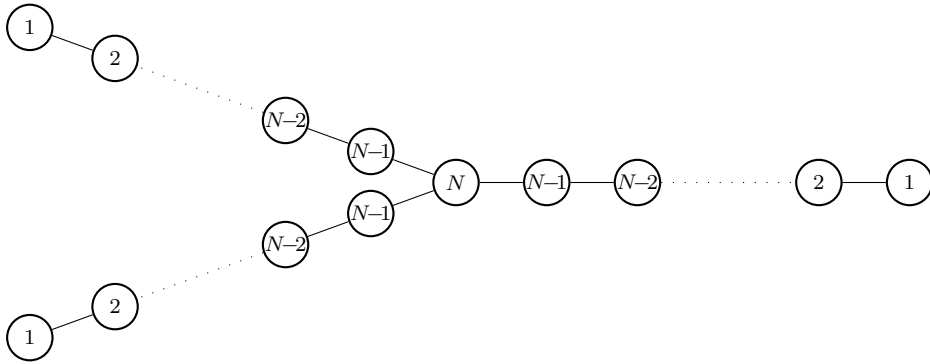


Figure 1: A star shaped quiver gauge theory.

The variety $\mathcal{R}_{G, \mathbf{N}}$ is the fiber product of $\mathcal{R}_{\tilde{G}_i, \mathbf{N}_i}$ over Gr_{G_F} . Let us denote the natural projections $\mathcal{R}_{G, \mathbf{N}} \rightarrow \mathrm{Gr}_{G_F}$ and $\mathcal{R}_{\tilde{G}_i, \mathbf{N}_i} \rightarrow \mathrm{Gr}_{G_F}$ by π and π_i respectively. Then

$$\pi_* \omega_{\mathcal{R}_{G, \mathbf{N}}}[-2 \dim \mathbf{N}_{\mathcal{O}}] = i_{\Delta}^! \left(\boxtimes \pi_{i*} \omega_{\mathcal{R}_{\tilde{G}_i, \mathbf{N}_i}}[-2 \dim(\mathbf{N}_i)_{\mathcal{O}}] \right),$$

where $i_{\Delta}: \mathrm{Gr}_{G_F} \rightarrow \prod_i \mathrm{Gr}_{G_F}$ is the diagonal embedding. Note that

$$\pi_{i*} \omega_{\mathcal{R}_{\tilde{G}_i, \mathbf{N}_i}}[-2 \dim(\mathbf{N}_i)_{\mathcal{O}}]$$

is the commutative ring object in $D_{G_F}(\mathrm{Gr}_{G_F})$, considered in §2(iv).

Motivated by the above example, we consider the following setting. (We use the convention in §2(vi), i.e., replace G_F by G .) Suppose that we have a finite collection $\{A_i\}$ of commutative ring objects in $D_G(\mathrm{Gr}_G)$. Let $i_{\Delta}: \mathrm{Gr}_G \rightarrow \prod_i \mathrm{Gr}_G$ be the diagonal embedding. Then the following is clear:

Proposition 2.22. $\mathcal{A} \stackrel{\text{def.}}{=} i_{\Delta}^! (\boxtimes \mathcal{A}_i)$ is a commutative ring object in $D_G(\text{Gr}_G)$. In particular, we can consider the affine scheme $\text{Spec } H_{G_{\circ}}^*(\text{Gr}_G, \mathcal{A})$.

In fact, we have $\boxtimes m: (\boxtimes \mathcal{A}_i) \star (\boxtimes \mathcal{A}_i) = \boxtimes (\mathcal{A}_i \star \mathcal{A}_i) \rightarrow \boxtimes \mathcal{A}_i$ from $m: \mathcal{A}_i \star \mathcal{A}_i \rightarrow \mathcal{A}_i$. Then we apply $i_{\Delta}^!$. We claim that there is a natural homomorphism

$$(2.23) \quad i_{\Delta}^! (\boxtimes \mathcal{A}_i) \star i_{\Delta}^! (\boxtimes \mathcal{A}_i) \rightarrow i_{\Delta}^! (\boxtimes (\mathcal{A}_i \star \mathcal{A}_i)),$$

hence its composition with $i_{\Delta}^! (\boxtimes m)$ gives the desired multiplication homomorphism of $i_{\Delta}^! (\boxtimes \mathcal{A}_i)$. We prove the claim by comparing the convolution diagrams [Part II, (3.1)] for Gr_G and $\prod_i \text{Gr}_G$. Since p, q are smooth, p^*, q^* commute with $i_{\Delta}^!$. The last part of the convolution diagram for G and $\prod_i G$ is

$$\begin{array}{ccc} \text{Gr}_G \tilde{\times} \text{Gr}_G & \xrightarrow{m} & \text{Gr}_G \\ i_{\Delta}^! \downarrow & & \downarrow i_{\Delta} \\ \prod_i \text{Gr}_G \tilde{\times} \text{Gr}_G = \text{Gr}_{\prod_i G} \tilde{\times} \text{Gr}_{\prod_i G} & \xrightarrow{\prod_i m} & \text{Gr}_{\prod_i G} = \prod_i \text{Gr}_G, \end{array}$$

where we denote the diagonal embedding of the left column by i'_{Δ} to distinguish it from the right column. Let $\boxtimes (\mathcal{A}_i \tilde{\boxtimes} \mathcal{A}_i)$ denote the complex on $\text{Gr}_{\prod_i G} \tilde{\times} \text{Gr}_{\prod_i G}$ obtained in the course of the convolution product for $\prod_i G$. We define the homomorphism as

$$\begin{aligned} m_* i_{\Delta}^! (\boxtimes (\mathcal{A}_i \tilde{\boxtimes} \mathcal{A}_i)) &= m_* \bigotimes^! (\mathcal{A}_i \tilde{\boxtimes} \mathcal{A}_i) \rightarrow \bigotimes^! m_* (\mathcal{A}_i \tilde{\boxtimes} \mathcal{A}_i) \\ &= i_{\Delta}^! \left(\prod_i m \right)_* \boxtimes (\mathcal{A}_i \tilde{\boxtimes} \mathcal{A}_i) \end{aligned}$$

by the natural homomorphism [KS90, (2.6.24) or the dual of (2.6.22)].

See §5 for an application of the gluing construction.

3. Proof of commutativity

We denote Gr_G by Gr for brevity in this section. In this section we closely follow [MV07, §5], [Gai01] and [BeiDr, §5.3].

3(i). Commutativity constraint

Let us give a definition of the commutativity constraint Θ .

Let us choose a smooth curve X . We define Gr_X the moduli space of triples $(x, \mathcal{P}, \varphi)$ of a point $x \in X$, a G -bundle \mathcal{P} on X and its trivialization φ over $X \setminus \{x\}$. We also have a group scheme $G_{X, \mathcal{O}}$, the global analog of $G_{\mathcal{O}}$.

More generally, we introduce an ind-scheme Gr_{X^n} as the moduli space of $(x_1, \dots, x_n, \mathcal{P}, \varphi)$ of n ordered points in X , a G -bundle \mathcal{P} on X and its trivialization φ over $X \setminus \bigcup\{x_i\}$. We also have $G_{X^n, \mathcal{O}}$, which is the moduli space of $(x_1, \dots, x_n, \mathcal{P}, \kappa_{x_1, \dots, x_n})$ where $(x_1, \dots, x_n) \in X^n$, \mathcal{P} the trivial G -bundle on X , and κ_{x_1, \dots, x_n} is a trivialization of \mathcal{P} on $\hat{X}_{x_1, \dots, x_n}$.

Then we define the convolution product of $\mathcal{A}, \mathcal{B} \in D_{G_{X, \mathcal{O}}}(\mathrm{Gr}_X)$ as before, using the global version of the diagram [Part II, (3.1)]:

$$(3.1) \quad \mathrm{Gr}_X \times \mathrm{Gr}_X \xleftarrow{p_X} \widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X} \xrightarrow{q_X} \mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X \xrightarrow{m_X} \mathrm{Gr}_{X^2}.$$

Here $\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}$ is the moduli space of $(x_1, x_2, \mathcal{P}_1, \varphi_1, \kappa, \mathcal{P}_2, \varphi_2)$, a pair of points $(x_1, x_2) \in X^2$, two G -bundles $\mathcal{P}_1, \mathcal{P}_2$ and their trivializations φ_i over $X \setminus \{x_i\}$ together with a trivialization κ of \mathcal{P}_1 on the formal neighborhood of x_2 . The twisted product $\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X$ is the moduli space of $(x_1, x_2, \mathcal{P}_1, \varphi_1, \mathcal{P}, \eta)$ as above, but $\eta: \mathcal{P}_1|_{X \setminus x_2} \cong \mathcal{P}|_{X \setminus x_2}$ instead of φ_2 and κ . The morphism q_X is given by defining \mathcal{P} as the gluing of $\mathcal{P}_1|_{X \setminus x_2}$ and $\mathcal{P}_2|_{\hat{X}_{x_2}}$ by $\varphi_2^{-1} \circ \kappa$ over $(X \setminus x_2) \cap \hat{X}_{x_2} = \hat{X}_{x_2} \setminus x_2$. (When $X = D$, the formal disk, \mathcal{P} and \mathcal{P}_2 are isomorphic. Hence this construction was omitted before.) The definitions of morphisms p_X, m_X are as before, and are omitted. (See [MV07, §5].) Note that p_X is a $G_{X, \mathcal{O}}$ -torsor by the action changing κ . The second projection q_X is also a $G_{X, \mathcal{O}}$ -torsor by the action changing κ and φ_2 simultaneously.

The diagram (3.1) gives a $G_{X^2, \mathcal{O}}$ -equivariant object defined on Gr_{X^2} by $\mathcal{A}_X \star_X \mathcal{B}_X \stackrel{\text{def.}}{=} m_{X*} (q_X^*)^{-1} p_X^* (\mathcal{A}_X \boxtimes \mathcal{B}_X)$ for $\mathcal{A}_X, \mathcal{B}_X \in D_{G_{X, \mathcal{O}}}(\mathrm{Gr}_X)$.

We take $X = \mathbb{A}^1$. We have $\mathrm{Gr}_X \cong X \times \mathrm{Gr}$ thanks to a choice of a global coordinate on \mathbb{A}^1 . In particular, we have a projection $\tau: \mathrm{Gr}_X \rightarrow \mathrm{Gr}$. For an object $\mathcal{A} \in D_G(\mathrm{Gr})$, we can attach $\mathcal{A}_X \in D_{G_{X, \mathcal{O}}}(\mathrm{Gr}_X)$ by $\tau^* \mathcal{A}[1]$. In fact, we can do more generally if we use the $\mathrm{Aut}(\mathcal{O})$ -bundle over X parametrizing all choices of local coordinates and consider $\mathrm{Aut}(\mathcal{O})$ -equivariant objects as in [BeiDr, Gai01].

Let Δ denote the diagonal in X^2 and U denote the complement $X^2 \setminus \Delta$. The restrictions of Gr_{X^2} to Δ and U are isomorphic to Gr_X and $(\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U$ respectively. In fact, the restriction to Δ is obvious. For a given $(x_1, x_2, \mathcal{P}, \varphi)$ with $x_1 \neq x_2$, we define \mathcal{P}_i by gluing $\mathcal{P}_i|_{X \setminus x_i} = (X \setminus x_i) \times G$

and $\mathcal{P}_i|_{X \setminus x_{3-i}} = \mathcal{P}|_{X \setminus x_{3-i}}$ by φ on $X \setminus \{x_1, x_2\}$. Hence we have the diagram

$$(3.2) \quad \begin{array}{ccccc} \mathrm{Gr}_X & \xrightarrow{\iota} & \mathrm{Gr}_{X^2} & \xleftarrow{J} & (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \longrightarrow & X^2 & \longleftarrow & U. \end{array}$$

We consider the nearby cycle functor

$$\psi_{\mathrm{Gr}_{X^2}} : D_{(G_{X,\circ} \times G_{X,\circ})|_U}((\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U) \rightarrow D_{G_{X,\circ}}(\mathrm{Gr}_X).$$

See [KS90, §8.6], where we change the source domain to objects defined on $(\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U$, and shift by -1 , following the convention in [Gai01].

Then an argument in [Gai01, Prop. 6] shows there is a natural isomorphism

$$(3.3) \quad \psi_{\mathrm{Gr}_{X^2}}((\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U) \cong (\mathcal{A} \star \mathcal{B})_X.$$

We have the isomorphism $(\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U \cong (\mathcal{B}_X \boxtimes \mathcal{A}_X)|_U$ exchanging the factors. Therefore together with (3.3) it gives us an isomorphism $\mathcal{A} \star \mathcal{B} \cong \mathcal{B} \star \mathcal{A}$. This is the definition of the commutativity constraint Θ used in Theorem 2.5.

Let us briefly explain how (3.3) is constructed. For a later purpose, we give a slightly different explanation from [Gai01].

By the definition of the nearby cycle functor, we have a natural homomorphism

$$(3.4) \quad \mathrm{ps} : \psi_{\mathrm{Gr}_{X^2}}((\mathcal{A}_X \star_X \mathcal{B}_X)|_U) \rightarrow \iota^!(\mathcal{A}_X \star_X \mathcal{B}_X)[1].$$

It is the *dual* of the specialization homomorphism. See [KS90, (8.6.7)]. We restrict the diagram (3.1) to the diagonal to see that

$$(3.5) \quad (\mathcal{A} \star \mathcal{B})_X \cong \iota^!(\mathcal{A}_X \star_X \mathcal{B}_X)[1].$$

Therefore we need to check

Claim.

(3.6a) We have a natural isomorphism $(\mathcal{A}_X \star_X \mathcal{B}_X)|_U \cong (\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U$.

(3.6b) ps in (3.4) is an isomorphism.

Proof. Let us denote the restrictions of p_X, q_X, m_X to inverse images of U by p_U, q_U, m_U respectively.

Over U , we have a natural commutative diagram
 (3.7)

$$\begin{array}{ccccccc}
 (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U & \xleftarrow{p_U} & \widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}|_U & \xrightarrow{q_U} & \mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X|_U & \xrightarrow[m_U]{\cong} & \mathrm{Gr}_{X^2}|_U \\
 \parallel & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
 (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U & \xleftarrow{} & G_{X,\mathcal{O}} \times_X (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U & \longrightarrow & (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U & \xlongequal{\quad} & (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U
 \end{array}$$

where $\mathrm{Gr}_X \times \mathrm{Gr}_X \rightarrow X$ in the bottom middle term is through the projection $X \times X \rightarrow X$ to the second factor. Here the second vertical isomorphism is given by regarding κ as a trivialization of the trivial bundle over \hat{X}_{x_2} via the trivialization $\varphi_1: \mathcal{P}_1|_{\hat{X}_{x_2}} \xrightarrow{\cong} \hat{X}_{x_2} \times G$. The third vertical isomorphism is given by considering η as a trivialization of \mathcal{P} . The lower left arrow is given by forgetting $G_{X,\mathcal{O}}$. The lower right arrow is given by the action of $G_{X,\mathcal{O}}$ on the second factor of $\mathrm{Gr}_X \times \mathrm{Gr}_X$. Since we are considering equivariant objects, we have a canonical isomorphism $(q_U^*)^{-1}p_U^*((\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U) \cong (\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U$. We now apply m_{U*} and observe that $m_{U*}(q_U^*)^{-1}p_U^*((\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U) = (\mathcal{A}_X \star_X \mathcal{B}_X)|_U$. Thus we have checked (a).

Let us turn to the assertion (b). The idea is to consider nearby cycle functors for four spaces in (3.1).

Let us start with m_X . Since nearby cycle functors commute with proper morphisms, we have

$$\psi_{\mathrm{Gr}_{X^2}}(m_{U*}(\mathcal{A}_X \tilde{\boxtimes} \mathcal{B}_X)|_U) \cong m_{\Delta*} \psi_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}((\mathcal{A}_X \tilde{\boxtimes} \mathcal{B}_X)|_U),$$

where m_Δ is the restriction of m to Δ .

Next consider p_X and q_X . They are both smooth ([MV07, p.114]), and hence commute with nearby cycle functors. Therefore

$$\begin{aligned}
 \psi_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}((\mathcal{A}_X \tilde{\boxtimes} \mathcal{B}_X)|_U) &\cong (q_\Delta^*)^{-1} \psi_{\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}}(p_U^*(\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U) \\
 &\cong (q_\Delta^*)^{-1} p_\Delta^* \psi_{\mathrm{Gr}_X \times \mathrm{Gr}_X}((\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U),
 \end{aligned}$$

where p_Δ, q_Δ are restrictions of p_X, q_X to Δ . Hence

$$\psi_{\mathrm{Gr}_{X^2}}((\mathcal{A}_X \star_X \mathcal{B}_X)|_U) \cong m_{\Delta*} (q_\Delta^*)^{-1} p_\Delta^* \psi_{\mathrm{Gr}_X \times \mathrm{Gr}_X}((\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U).$$

Now $\mathrm{Gr}_X \times \mathrm{Gr}_X = X \times X \times \mathrm{Gr} \times \mathrm{Gr}$, hence $\psi_{\mathrm{Gr}_X \times \mathrm{Gr}_X}((\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U)$ is just $(\mathcal{A} \boxtimes \mathcal{B})_X$. More precisely, the isomorphism is given by the dual specialization homomorphism

$$\mathrm{ps}: \psi_{\mathrm{Gr}_X \times \mathrm{Gr}_X}((\mathcal{A}_X \boxtimes \mathcal{B}_X)|_U) \xrightarrow{\cong} \iota^!(\mathcal{A}_X \boxtimes \mathcal{B}_X)[1] = (\mathcal{A} \boxtimes \mathcal{B})_X,$$

thanks to vanishing of the vanishing cycle functor $\varphi_{\text{Gr}_X \times \text{Gr}_X}(\mathcal{A}_X \boxtimes \mathcal{B}_X)$. Thus

$$\psi_{\text{Gr}_{X^2}}(\mathcal{A}_X \star_X \mathcal{B}_X)|_U \xrightarrow[\cong]{m_{\Delta^*}(q_{\Delta}^*)^{-1}p_{\Delta}^* \text{ps}} m_{\Delta^*}(q_{\Delta}^*)^{-1}p_{\Delta}^*((\mathcal{A} \boxtimes \mathcal{B})_X).$$

Notice that the restriction of (3.1) to the diagonal is just the product of X and the diagram [Part II, (3.1)]. Therefore the right hand side is $(\mathcal{A} \star \mathcal{B})_X$. Now one can check that dual specialization homomorphisms commute with proper pushforward and smooth pull-backs so that they are compatible with the commutation of nearby cycle functors. (See the argument in the proof of Lemma 3.10 below.) Therefore $m_{\Delta^*}(q_{\Delta}^*)^{-1}p_{\Delta}^* \text{ps}$ is equal to ps over Gr_{X^2} . Thus (b) is checked. \square

3(ii). Factorization version of \mathcal{R}

We define a global version of the variety of triples \mathcal{R} in this subsection.

Let us assume that we are given a smooth connected curve X , an algebraic group G and a representation \mathbf{N} of G and a finite set I . Consider a functor $\text{Schemes}/\mathbb{C} \rightarrow \text{Sets}$ which sends a scheme S to the following data: 1) A map $f: S \rightarrow X^I$. We shall think about f as a collection of maps $f_i: S \rightarrow X$ for $i \in I$ and we denote by Γ the union of graphs of f_i – this is a closed subscheme of $S \times X$.

2) A G -bundle \mathcal{P} on $S \times X$.

3) A trivivialization φ of \mathcal{P} over $S \times X \setminus \Gamma$.

4) A section s of the associated bundle $\mathcal{P}_{\mathbf{N}}$ over the formal neighbourhood of Γ in $S \times X$ and a section s' of the trivial \mathbf{N} - bundle over the same formal neighbourhood which are equal on the “formal punctured neighbourhood” (this makes sense because of 3). These notions (formal neighbourhood, formal punctured neighbourhood) are explained in [KV04].

Now we claim that this functor is representable by an ind-scheme. Moreover, this ind-scheme has a natural closed embedding into $\text{Gr}_{X^I} \times_{X^I} \mathcal{J}_{\mathbf{N}, X^I}$ where

a) Gr_{X^I} is the factorization (a.k.a. Beilinson-Drinfeld) Grassmannian over X^I

b) $\mathcal{J}_{\mathbf{N}, X^I}$ is the Kapranov-Vasserot factorization version of the \mathbf{N} -jet space over X^I .

Indeed it is enough to construct this closed embedding (as a closed subfunctor of an ind-scheme is also an ind-scheme). But an S -point of $\text{Gr}_{X^I, G, BD}$ is precisely the data of 1), 2), 3) and an S -point of $\mathcal{J}_{\mathbf{N}, X^I}$ is the data of 1), 2) and s' from 4). Since s is obviously uniquely determined by all the data

and since the existence of s is a closed condition on the other data we get the above closed embedding.

Let us denote the above ind-scheme by \mathcal{R}_{X^I} . Then obviously from 1) we get a morphism $\pi_{X^I} : \mathcal{R}_{X^I} \rightarrow X^I$ and it is clear that the restriction of \mathcal{R}_{X^I} to the complement U of all the diagonals in X^I is naturally isomorphic to the similar restriction of $(\mathcal{R}^{(1)})^I$.

On the other hand, assume that we are given a surjective morphism $I \rightarrow J$ of finite sets. Such a morphism defines a closed embedding $X^J \hookrightarrow X^I$ (as a partial diagonal) and it follows that the restriction of \mathcal{R}_{X^I} to X^J is naturally isomorphic to $\mathcal{R}^{(J)}$.

Similarly, we can define a factorization version of the bundle \mathcal{T} over Gr . By definition an S -point of \mathcal{T}_{X^I} is a quadruple $(f, \mathcal{P}, \varphi, s)$ as above (i.e. no s').² We claim again that this functor is representable by an ind-scheme. For this it is enough to show that the morphism $\mathcal{T}_{X^I} \rightarrow \text{Gr}_{X^I}$ (which corresponds to forgetting s) is representable. This can be done by a word-by-word repetition of the proof of the fact that the factorization version of the jet scheme is representable by a scheme (cf. again Section 3 of [KV04]).

In what follows we shall only need the above spaces when $I = \{1, 2\}$.

3(iii). Definition of another multiplication

We consider the space \mathcal{R}_{X^2} , its dualizing complex $\omega_{\mathcal{R}_{X^2}}$ and the pushforward $\pi_* \omega_{\mathcal{R}_{X^2}}$. Its restriction to U is isomorphic to $(\pi_* \omega_{\mathcal{R}_X} \boxtimes \pi_* \omega_{\mathcal{R}_X})|_U$ under j in (3.2). We consider two dual specialization homomorphisms

$$(3.8) \quad \begin{array}{ccc} \psi_{\text{Gr}_{X^2}}(\pi_* \omega_{\mathcal{R}_{X^2}}|_U) & \xrightarrow{\text{ps}} & \iota^! \pi_* \omega_{\mathcal{R}_{X^2}}[1] \\ \cong \uparrow & & \uparrow \text{---} \\ \psi_{\text{Gr}_{X^2}}((\pi_* \omega_{\mathcal{R}_X} \star_X \pi_* \omega_{\mathcal{R}_X})|_U) & \xrightarrow[\cong]{\text{ps}} & \iota^!(\pi_* \omega_{\mathcal{R}_X} \star_X \pi_* \omega_{\mathcal{R}_X})[1] \end{array}$$

where $\iota: \text{Gr}_X \rightarrow \text{Gr}_{X^2}$ is the inclusion, and the vertical arrow is given by $\pi_* \omega_{\mathcal{R}_{X^2}}|_U \cong (\pi_* \omega_{\mathcal{R}_X} \boxtimes \pi_* \omega_{\mathcal{R}_X})|_U \cong (\pi_* \omega_{\mathcal{R}_X} \star_X \pi_* \omega_{\mathcal{R}_X})|_U$. (See (3.6a).) The lower homomorphism is an isomorphism thanks to (3.6b). Note that $\pi_* \omega_{\mathcal{R}_X}$ is $(\pi_* \omega_{\mathcal{R}})_X[1]$. Therefore the right bottom term is $(\pi_* \omega_{\mathcal{R}} \star \pi_* \omega_{\mathcal{R}})_X[2]$ by (3.5). Note also $\iota^! \pi_* \omega_{\mathcal{R}_{X^2}}[1] = \pi_* \omega_{\mathcal{R}_X}[1] = (\pi_* \omega_{\mathcal{R}})_X[2]$. Therefore we obtain

²Note that if we instead only choose s' and do not choose s then the resulting functor is represented by $\text{Gr}_{X^I} \times_{X^I} \mathcal{J}_{\mathbf{N}, X^I}$.

a homomorphism

$$(3.9) \quad \mathfrak{m}^\psi : \mathcal{A} \star \mathcal{A} \rightarrow \mathcal{A}, \quad \mathcal{A} = \pi_* \omega_{\mathcal{R}}[-2 \dim \mathbf{N}_{\mathcal{O}}],$$

by specializing the dotted arrow at a point in X .

The degree shift should be checked by going back to finite dimensional approximation of \mathcal{R} . We have shifts by $\dim \mathbf{N}_{\mathcal{O}}/z^{d_1} \mathbf{N}_{\mathcal{O}}$ and $\dim \mathbf{N}_{\mathcal{O}}/z^{d_2} \mathbf{N}_{\mathcal{O}}$ for two factors in $\psi_{\mathrm{Gr}_{X^2}}((\pi_* \omega_{\mathcal{R}_X} \boxtimes \pi_* \omega_{\mathcal{R}_X})|_U)$. Then we have a shift

$$\dim \mathbf{N}_{\mathcal{O}}/z^{d_1+d_2} \mathbf{N}_{\mathcal{O}}$$

for $\pi_* \omega_{\mathcal{R}_X}$.

Now our goal is to check two properties:

- (i) $\mathfrak{m}^\psi = \mathfrak{m}$,
- (ii) \mathfrak{m}^ψ is invariant under the exchange of factors of $\mathcal{A}_X \star \mathcal{A}_X$. (More precisely, exchange after going back to $(\mathcal{A} \boxtimes \mathcal{A})|_U$.)

The property (ii) is clear as the diagram (3.2) is invariant under the exchange of two factors of $X^2 = X \times X$.

We will check (i) for $\mathbf{N} = 0$ in the next subsection. We have a difficulty to check (i) directly for general \mathbf{N} , so we will argue indirectly by reduction to the case $\mathbf{N} = 0$.

3(iv). The case $\mathbf{N} = 0$

We first consider the case $\mathbf{N} = 0$.

We consider the dual specialization homomorphism for $\omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}$:

$$\mathrm{ps}_! : \psi_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}((\omega_{\mathrm{Gr}_X} \boxtimes \omega_{\mathrm{Gr}_X}|_U) \rightarrow \iota'_! \omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}[1] = \omega_{X \times \mathrm{Gr} \tilde{\times} \mathrm{Gr}}[1])$$

where $\iota' : X \times \mathrm{Gr} \tilde{\times} \mathrm{Gr} = (\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X)|_{\Delta} \rightarrow \mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X$ is the embedding from the definition of the nearby cycle functor. Here we have used $\omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}|_U \cong (\omega_{\mathrm{Gr}_X} \boxtimes \omega_{\mathrm{Gr}_X}|_U)$ from (3.7).

The following two assertions identify (3.9) with the pull-back of \mathfrak{m} under (3.3), hence we obtain the property (i) for $\mathbf{N} = 0$.

Lemma 3.10. (1) $\mathrm{ps}_!$ is equal to the composition of $m_{\Delta^*} \mathrm{ps}_!$, and the natural morphism $m_{\Delta^*} m_{\Delta}^! = m_{\Delta}^! m_{\Delta}^! \rightarrow \mathrm{id}$.

(2) The homomorphism $\mathrm{ps}_!$ coincides with the homomorphism $\omega_{\mathrm{Gr}} \tilde{\boxtimes} \omega_{\mathrm{Gr}} \rightarrow \omega_{\mathrm{Gr} \tilde{\times} \mathrm{Gr}}$ constructed in (2.3), pull-backed by $\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X \rightarrow \mathrm{Gr} \tilde{\times} \mathrm{Gr}$. It is an isomorphism.

Proof. (1) Recall the definition of the nearby cycle functor and the dual specialization morphism ([KS90, §8.6]). We have $f: \mathrm{Gr}_{X^2} \rightarrow X^2 = \mathbb{C}^2 \rightarrow \mathbb{C}$, where the second map is $(x_1, x_2) \mapsto x_1 - x_2$. We then consider $p: \widetilde{\mathbb{C}}^\times \rightarrow \mathbb{C}$, the composition of the universal covering $\widetilde{\mathbb{C}}^\times \rightarrow \mathbb{C}^\times$ and the inclusion $\mathbb{C}^\times \rightarrow \mathbb{C}$. We then pull back p by f to get $\tilde{p}: \widetilde{\mathrm{Gr}}_{X^2}^\times \rightarrow \mathrm{Gr}_{X^2}$. Then

$$\psi_{\mathrm{Gr}_{X^2}}((\omega_{\mathrm{Gr}_X} \boxtimes \omega_{\mathrm{Gr}_X})|_U) = \iota^* \tilde{p}_* \tilde{p}^* \omega_{\mathrm{Gr}_{X^2}}[-1] \cong \iota^* \mathcal{H}om(f^* p_! \mathbb{C}_{\widetilde{\mathbb{C}}^\times}, \omega_{\mathrm{Gr}_{X^2}}[-1])$$

and ps is defined from $\mathbb{C}_{\{0\}} \rightarrow p_! \mathbb{C}_{\widetilde{\mathbb{C}}^\times}[2]$.

Let us write the identification

$$(\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X)|_U \xrightarrow{\cong} \mathrm{Gr}_{X^2}|_U \cong (\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U$$

explicitly as m_U , the restriction of m_X to U . The commutativity of

$$\begin{array}{ccc} \psi_{\mathrm{Gr}_{X^2}}(m_U^* \omega_{(\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X)|_U}) & \xrightarrow{\mathrm{ps}} & \iota^! m_{X^*} \omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}[1] \\ \cong \downarrow & & \downarrow \cong \\ m_{\Delta^*} \psi_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}(\omega_{(\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X)|_U}) & \xrightarrow{m_{\Delta^*} \mathrm{ps}_!} & m_{\Delta^*} \iota_! \omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}[1] \end{array}$$

is clear as both vertical arrows are given by base change and adjunction. This property has been already used in the construction of the commutativity constraint above. Next the commutativity of

$$\begin{array}{ccc} \psi_{\mathrm{Gr}_{X^2}}(m_U^* \omega_{(\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X)|_U}) & \xrightarrow{\mathrm{ps}} & \iota^! m_{X^*} \omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}[1] \\ \psi_{\mathrm{Gr}_{X^2}}(m_U^! m_U^* \rightarrow \mathrm{id}) \downarrow \cong & & \downarrow m_{X^*} m_X^! \rightarrow \mathrm{id} \\ \psi_{\mathrm{Gr}_{X^2}}((\omega_{\mathrm{Gr}_X} \boxtimes \omega_{\mathrm{Gr}_X})|_U) & \xrightarrow{\mathrm{ps}} & \iota^! \omega_{\mathrm{Gr}_{X^2}}[1] \end{array}$$

is also clear from the definition. Therefore we get the assertion.

(2) The following diagram is commutative:

$$\begin{array}{ccc} q_{\Delta^*}^* \psi_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}(\omega_{(\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X)|_U}) & \xrightarrow{q_{\Delta^*}^* \mathrm{ps}_!} & q_{\Delta^*}^* \iota_! \omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}[1] \\ \cong \downarrow & & \downarrow \cong \\ \psi_{\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}}(q_U^* \omega_{(\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X)|_U}) & \xrightarrow{\mathrm{ps}_!} & \iota_{\mathcal{K}}^! q_X^* \omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}[1], \end{array}$$

where $\iota_{\mathcal{K}}: G_{\mathcal{K}} \times \mathrm{Gr}_X \rightarrow \widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}$ is the inclusion given by $\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}|_{\Delta} \cong G_{\mathcal{K}} \times \mathrm{Gr}_X$, and $\mathrm{ps}_!$ is the dual specialization for $\omega_{\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}} = q_X^* \omega_{\mathrm{Gr}_X \tilde{\times} \mathrm{Gr}_X}$.

In fact, the left vertical arrow is given by the composition of

$$\begin{aligned} q_\Delta^* \psi_{\mathrm{Gr}_X \times \tilde{\mathrm{Gr}}_X}(\omega_{\mathrm{Gr}_X \times \tilde{\mathrm{Gr}}_X}|_U) &\cong q_\Delta^* \iota_i^* \mathcal{H}om(m_X^* f^* p! \mathbb{C}_{\tilde{\mathcal{C}}^\times}, \omega_{\mathrm{Gr}_X \times \tilde{\mathrm{Gr}}_X}[-1]) \\ &\cong \iota_\eta^* q_X^* \mathcal{H}om(m_X^* f^* p! \mathbb{C}_{\tilde{\mathcal{C}}^\times}, \omega_{\mathrm{Gr}_X \times \tilde{\mathrm{Gr}}_X}[-1]) \\ &\cong \iota_\eta^* \mathcal{H}om(q_X^* m_X^* f^* p! \mathbb{C}_{\tilde{\mathcal{C}}^\times}, q_X^* \omega_{\mathrm{Gr}_X \times \tilde{\mathrm{Gr}}_X}[-1]) \cong \psi_{\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}}(q_U^* \omega_{\mathrm{Gr}_X \times \tilde{\mathrm{Gr}}_X}|_U) \end{aligned}$$

where we have used [KS90, Prop. 3.1.13] and $q_X^! = q_X^*[2 \dim G_{\mathcal{O}}]$ for the third isomorphism. Now we apply $\mathbb{C}_{\{0\}} \rightarrow p! \mathbb{C}_{\tilde{\mathcal{C}}^\times}[2]$.

In the same way, we have another commutative diagram

$$\begin{array}{ccc} \psi_{\widetilde{\mathrm{Gr}_X \times \mathrm{Gr}_X}}(p_U^* \omega_{(\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U}) & \xrightarrow{\mathrm{ps}_\eta} & \iota_\eta^! p_X^* \omega_{\mathrm{Gr}_X \times \mathrm{Gr}_X}[1] \\ \cong \downarrow & & \downarrow \cong \\ p_\Delta^* \psi_{\mathrm{Gr}_X \times \mathrm{Gr}_X}(\omega_{(\mathrm{Gr}_X \times \mathrm{Gr}_X)|_U}) & \xrightarrow{p_\Delta^* \mathrm{ps}_\eta} & p_\Delta^* \iota_\eta^! \omega_{\mathrm{Gr}_X \times \mathrm{Gr}_X}[1], \end{array}$$

where $\iota_\eta: X \times \mathrm{Gr} \times \mathrm{Gr} \rightarrow \mathrm{Gr}_X \times \mathrm{Gr}_X$ is the embedding given by $\mathrm{Gr}_X \times \mathrm{Gr}_X|_\Delta \cong X \times \mathrm{Gr} \times \mathrm{Gr}$. Now the assertion is proved. Note $p_\Delta^* \mathrm{ps}_\eta$ is an isomorphism, hence so is ps_η . \square

Remark 3.11. We have a difficulty to generalize the argument in §3(iv) to $\mathbf{N} \neq 0$ since we lack an \mathcal{R} -version of $\mathrm{Gr}_X \times \tilde{\mathrm{Gr}}_X$, as a well-defined ind-scheme. This difficulty will be overcome in Appendix B written by Gus Lonergan.

3(v). Completion of the proof

Let $\mathbf{z}_{X^2}: \mathrm{Gr}_{X^2} \rightarrow \mathcal{T}_{X^2}$ be the factorization version of the embedding $\mathbf{z}: \mathrm{Gr} \rightarrow \mathcal{T}$ discussed in [Part II, §5(iv)]. It factors as $\mathbf{z}_{X^2} = i \circ \tilde{\mathbf{z}}_{X^2}$, where $\tilde{\mathbf{z}}_{X^2}: \mathrm{Gr}_{X^2} \rightarrow \mathcal{R}_{X^2}$, and $i: \mathcal{R}_{X^2} \rightarrow \mathcal{T}_{X^2}$ is the embedding. Since $\mathcal{T}_{X^2} \rightarrow \mathrm{Gr}_{X^2}$ is a vector bundle, we have

$$\mathbf{z}_{X^2}^* \omega_{\mathcal{T}_{X^2}} \rightarrow \omega_{\mathrm{Gr}_{X^2}}[2 \dim \mathbf{N}_{\mathcal{O}}],$$

and also

$$\omega_{\mathcal{R}_{X^2}}[-2 \dim \mathbf{N}_{\mathcal{O}}] \rightarrow \tilde{\mathbf{z}}_{X^2}^* \omega_{\mathrm{Gr}_{X^2}}$$

by the pull-back with support. We apply π_* to obtain

$$\mathbf{z}_{X^2}^*: \pi_* \omega_{\mathcal{R}_{X^2}}[-2 \dim \mathbf{N}_{\mathcal{O}}] \rightarrow \omega_{\mathrm{Gr}_{X^2}}.$$

We now apply the nearby cycle functor $\psi_{\mathrm{Gr}_{X^2}}$ and the dual specialization homomorphisms:

$$\begin{array}{ccc}
 \psi_{\mathrm{Gr}_{X^2}}(\pi_*(\omega_{\mathcal{R}_{X^2}}[-2 \dim \mathbf{N}_{\mathcal{O}}])|_U) & \xrightarrow{\text{ps}} & \pi_*\omega_{\mathcal{R}_X}[-2 \dim \mathbf{N}_{\mathcal{O}} + 1] \\
 \downarrow \mathbf{z}_{X^2}^* & & \downarrow \mathbf{z}^* \\
 \psi_{\mathrm{Gr}_{X^2}}(\omega_{\mathrm{Gr}_{X^2}}|_U) & \xrightarrow[\text{ps}]{} & \omega_{\mathrm{Gr}_X}[1].
 \end{array}$$

This is a commutative diagram by the argument in the proof of Lemma 3.10. Removing the unnecessary factor X , we get

$$(3.12) \quad \begin{array}{ccc}
 \mathcal{A} \star \mathcal{A} & \xrightarrow{\mathfrak{m}_{\mathcal{R}}^\psi} & \mathcal{A} \\
 \downarrow \mathbf{z}^* \star \mathbf{z}^* = \mathbf{z}_{X^2}^* & & \downarrow \mathbf{z}^* \\
 \omega_{\mathrm{Gr}} \star \omega_{\mathrm{Gr}} & \xrightarrow[\mathfrak{m}_{\mathrm{Gr}}^\psi]{} & \omega_{\mathrm{Gr}},
 \end{array}$$

where we put the superscript ψ to emphasize that the multiplication is defined via nearby cycle functors. Since the restriction of $\mathbf{z}_{X^2}^*$ to U is $\mathbf{z}^* \boxtimes \mathbf{z}^*$, the left vertical arrow is equal to $\mathbf{z}^* \star \mathbf{z}^*$.

Let us view \mathfrak{m} , \mathfrak{m}^ψ as elements of $\mathrm{Ext}_{D_G(\mathrm{Gr})}^*(\mathcal{A} \star \mathcal{A}, \mathcal{A})$. It is a module over $H_G^*(\mathrm{pt})$. We consider the restriction functor $\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}$ from the $G_{\mathcal{O}}$ -equivariant derived category to the $T_{\mathcal{O}}$ -equivariant one. Then we have

$$\mathrm{Ext}_{D_G(\mathrm{Gr})}^*(\mathcal{A} \star \mathcal{A}, \mathcal{A}) \rightarrow \mathrm{Ext}_{D_T(\mathrm{Gr})}^*(\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A} \star \mathcal{A}), \mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A})),$$

and the latter is a module over $H_T^*(\mathrm{pt}) = \mathbb{C}[t]$.

We have

- (i)' \mathfrak{m}^ψ and \mathfrak{m} are equal in $\mathrm{Ext}_{D_T(\mathrm{Gr})}^*(\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A} \star \mathcal{A}), \mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A})) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)$. Hence \mathfrak{m} is commutative up to an element which vanishes in $\mathrm{Ext}_{D_T(\mathrm{Gr})}^*(\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A} \star \mathcal{A}), \mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A})) \otimes_{\mathbb{C}[t]} \mathbb{C}(t)$.

This statement recovers (i), as $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathcal{A} \star \mathcal{A}, \mathcal{A})$ is the Weyl group invariant part of $\mathrm{Ext}_{D_T(\mathrm{Gr}_G)}^*(\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A} \star \mathcal{A}), \mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A}))$, and $\mathrm{Ext}_{D_T(\mathrm{Gr}_G)}^*(\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A} \star \mathcal{A}), \mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{A}))$ is a free $\mathbb{C}[t]$ -module: More generally, for $\mathcal{F}, \mathcal{G} \in D_G(\mathrm{Gr})$, $\mathrm{Ext}_{D_T(\mathrm{Gr})}^*(\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{F}), \mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}(\mathcal{G}))$ is a free $\mathbb{C}[t]$ -module. Indeed, by devissage it reduces to the case of irreducible perverse \mathcal{F}, \mathcal{G} where it is well known, see e.g. [Gin91].

Let us suppress $\mathrm{Res}_{T_{\mathcal{O}}, G_{\mathcal{O}}}$ hereafter.

Let us consider the commutative diagram (3.12). We have the corresponding diagram for $\mathfrak{m}_{\mathcal{R}}$, the multiplication constructed in Proposition 2.1,

where the lower arrow is \mathfrak{m}_{Gr} , cf. [Part II, Lemma 5.11]. We compose \mathbf{z}^* to get

$$\text{Ext}_{D_G(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \mathcal{A}) \xrightarrow{\mathbf{z}^*} \text{Ext}_{D_G(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \omega_{\text{Gr}}).$$

The commutativity of the diagram and $\mathfrak{m}_{\text{Gr}} = \mathfrak{m}_{\text{Gr}}^\psi$ (§3(iv)) imply that $\mathbf{z}^* \mathfrak{m}_{\mathcal{R}}^\psi$ and $\mathbf{z}^* \mathfrak{m}_{\mathcal{R}}$ are equal in $\text{Ext}_{D_G(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \omega_{\text{Gr}})$. Therefore it is enough to check that

$$\text{Ext}_{D_T(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \mathcal{A}) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t}) \xrightarrow{\mathbf{z}^*} \text{Ext}_{D_T(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \omega_{\text{Gr}}) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t})$$

is an isomorphism. The argument is almost the same as the one in the proof of [Part II, Lemma 5.13].

By the definition of \mathbf{z}^* , it factors through

$$\text{Ext}_{D_T(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \pi_* \omega_{\mathcal{T}}[-2 \dim \mathbf{N}_{\mathcal{O}}]) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t}).$$

Since $\mathcal{T} \rightarrow \text{Gr}$ is a vector bundle of rank $2 \dim \mathbf{N}_{\mathcal{O}}$, we have $\pi_* \omega_{\mathcal{T}}[-2 \dim \mathbf{N}_{\mathcal{O}}] \cong \omega_{\text{Gr}}$. Therefore it is enough to check that the morphism

$$\text{Ext}_{D_T(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \mathcal{A}) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t}) \xrightarrow{i_*} \text{Ext}_{D_T(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \pi_* \omega_{\mathcal{T}}) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t})$$

given by the closed embedding $i: \mathcal{R} \rightarrow \mathcal{T}$ is an isomorphism. Let $j: \mathcal{T} \setminus \mathcal{R} \rightarrow \mathcal{T}$ be the inclusion of the complement. We have the distinguished triangle $i_! i^! \omega_{\mathcal{T}} \rightarrow \omega_{\mathcal{T}} \rightarrow j_* j^* \omega_{\mathcal{T}}$. From the associated long exact sequence, it is enough to show that $\text{Ext}_{D_T(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \pi_* j_* j^* \omega_{\mathcal{T}}) \otimes_{\mathbb{C}[\mathfrak{t}]} \mathbb{C}(\mathfrak{t})$ vanishes. But $\text{Ext}_{D_T(\text{Gr})}^*(\mathcal{A} \star \mathcal{A}, \pi_* j_* j^* \omega_{\mathcal{T}}) = \text{Ext}_{D_T(\mathcal{T} \setminus \mathcal{R})}^*(j^* \pi^*(\mathcal{A} \star \mathcal{A}), \omega_{\mathcal{T} \setminus \mathcal{R}})$ is an equivariant cohomology group over $\mathcal{T} \setminus \mathcal{R}$ which does not contain T -fixed points by [Part II, Lemma 5.1]. Therefore it is torsion and vanishes once we take a tensor product with $\mathbb{C}(\mathfrak{t})$.

4. Proof of Theorem 2.11

In this section we prove Theorem 2.11.³

During the proof, the \mathbb{C}^\times -action on the Coulomb branch will play an important role. The \mathbb{C}^\times -action is given by the homological grading, shifted

³The second named author thanks Roman Bezrukavnikov for his numerous explanations about the Andersen-Jantzen sheaves on Kleinian surfaces and nilpotent cones.

according to the convention in [Part II, Remark 2.8(2)]. Then the monopole formula in [Part II, Prop. 2.7] is modified to

$$(4.1) \quad P_t^{\text{mod}}(\mathcal{R}) = \sum_{\lambda} t^{2\Delta(\lambda)} P_G(t; \lambda).$$

As mentioned in [Part II, Remark 2.8(2)], this modification is harmless as the difference $d_{\lambda} - 2\langle \rho, \lambda \rangle - \Delta(\lambda)$ depends only on connected components of \mathcal{R} . Nevertheless we will see that this convention is a correct choice.

4(i). Characters of global sections of line bundles on Kleinian surfaces

Recall that \mathcal{S}_N is the hypersurface in \mathbb{A}^3 given by the equation $zy = w^N$. It is the categorical quotient $\mathbb{A}^2 // (\mathbb{Z}/N\mathbb{Z})$ where $\zeta \in \mathbb{Z}/N\mathbb{Z}$ takes $(u, v) \in \mathbb{A}^2$ to $(\zeta u, \zeta^{-1}v)$. We consider the following action of $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ on \mathbb{A}^2 : $(x, t) \cdot (u, v) = (t^{-1}x^{-1}u, t^{-1}xv)$. This action descends to \mathcal{S}_N . The action of the second \mathbb{C}^{\times} $t \cdot (u, v) = (t^{-1}u, t^{-1}v)$ is a restriction of an $SU(2)$ -action on $\mathbb{A}^2 = \mathbb{R}^4$ rotating hyper-Kähler structures. Hence it is natural in view of [Part II, Remark 2.8].

We are interested in characters of certain $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ -equivariant sheaves on \mathcal{S}_N . The tautological characters of $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ will be denoted by x and t . We denote by $\pi: \tilde{\mathcal{S}}_N \rightarrow \mathcal{S}_N$ the minimal resolution of \mathcal{S}_N . The action of $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ lifts to $\tilde{\mathcal{S}}_N$. We recall the well known facts about the $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ -fixed points in $\tilde{\mathcal{S}}_N$.

We will denote these points by p_0, \dots, p_{N-1} , so that the exceptional divisor $E \subset \tilde{\mathcal{S}}_N$ consists of projective lines E_1, \dots, E_{N-1} , and E_r contains p_{r-1}, p_r . The character of the tangent space $T_{p_r} \tilde{\mathcal{S}}_N$ is $t^{N-2r-2}x^{-N} + t^{2r-N}x^N$. The Picard group $\text{Pic}(\tilde{\mathcal{S}}_N)$ is canonically identified with the weight lattice of $SL(N)$. Namely, $\tilde{\mathcal{S}}_N \subset T^*\mathcal{B}$ is the preimage of a subregular (Slodowy) slice in the Springer resolution $T^*\mathcal{B} \rightarrow \mathcal{N}$ of the nilpotent cone for $SL(N)$. For an $SL(N)$ -weight λ the corresponding line bundle \mathcal{L}_{λ} on $\tilde{\mathcal{S}}_N$ is the restriction to $\tilde{\mathcal{S}}_N \subset T^*\mathcal{B}$ of the pullback to $T^*\mathcal{B}$ of the line bundle $\mathcal{O}(\lambda)$ on the flag variety \mathcal{B} . The line bundle \mathcal{L}_{ω_i} corresponding to the fundamental weight ω_i , $1 \leq i \leq N - 1$, admits a natural $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ -equivariant structure such that the character of its fiber at p_r is $t^{N-i}x^{i-N}$ provided $0 \leq r \leq i - 1$, and $t^i x^i$ provided $i \leq r \leq N - 1$. This is defined so that $\Gamma(\tilde{\mathcal{S}}_N, \mathcal{L}_{\omega_i})$ is the space of semi-invariants $\mathbb{C}[\mathbb{A}^2]^{x^i}$ where $\chi_i(\zeta) = \zeta^i$. Under the above identification, $\mathcal{O}_{\tilde{\mathcal{S}}_N}(-E_r)$ is nothing but \mathcal{L}_{α_r} where α_r is the r -th simple root of $SL(N)$.

We will write a dominant $SL(N)$ -weight λ as a partition $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ defined up to simultaneous shift of all λ_i . In other words, $\lambda = \sum_{i=1}^{N-1} (\lambda_i - \lambda_{i+1}) \omega_i$. Then \mathcal{L}_λ admits a natural $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant structure (as a tensor product of fundamental line bundles) such that the character of its fiber at p_r , $0 \leq r \leq N - 1$, is $t^{\sum_{i=1}^N |\lambda_i - \lambda_{r+1}|} x^{\sum_{i=1}^N (\lambda_i - \lambda_{r+1})}$. If λ is not necessarily dominant, we get the character $t^{\sum_{i=1}^r (\lambda_i - \lambda_{r+1}) + \sum_{i=r+2}^N (\lambda_{r+1} - \lambda_i)} x^{\sum_{i=1}^N (\lambda_i - \lambda_{r+1})}$.

Lemma 4.2. *For dominant λ , the character of $\Gamma(\tilde{\mathcal{S}}_N, \mathcal{L}_\lambda)$ equals*

$$\sum_{m \in \mathbb{Z}} x^{\sum_{i=1}^N (\lambda_i - m)} t^{\sum_{i=1}^N |\lambda_i - m|} (1 + t^2 + t^4 + \dots).$$

Proof. We compute the above expression as

$$\begin{aligned} & \frac{x^{\sum_{i=1}^N \lambda_i}}{1 - t^2} \left(\sum_{m=\lambda_1}^{\infty} x^{-Nm} t^{-\sum_{i=1}^N (\lambda_i - m)} + \sum_{m=\lambda_2}^{\lambda_1 - 1} x^{-Nm} t^{\lambda_1 - m - \sum_{i=2}^N (\lambda_i - m)} \right. \\ & \quad \left. + \dots + \sum_{m=-\infty}^{\lambda_N - 1} x^{-Nm} t^{\sum_{i=1}^N (\lambda_i - m)} \right) \\ &= \frac{x^{\sum_{i=1}^N \lambda_i}}{1 - t^2} \left(\frac{x^{-N\lambda_1} t^{-\sum_{i>1} (\lambda_i - \lambda_1)}}{1 - x^{-N} t^N} - \frac{x^{-N\lambda_1} t^{-\sum_{i>1} (\lambda_i - \lambda_1)}}{1 - x^{-N} t^{N-2}} \right. \\ & \quad \left. + \frac{x^{-N\lambda_2} t^{\lambda_1 - \lambda_2 - \sum_{i>2} (\lambda_i - \lambda_2)}}{1 - x^{-N} t^{N-2}} + \dots - \frac{x^{-N\lambda_N} t^{\sum_{i<N} \lambda_i - \lambda_N}}{1 - x^{-N} t^{-N}} \right). \end{aligned}$$

We combine $(2r - 1)$ th and $(2r)$ th terms ($1 \leq r \leq N$) to get

$$\begin{aligned} & \frac{x^{\sum_i (\lambda_i - \lambda_r)} t^{\sum_{i<r} (\lambda_i - \lambda_r) - \sum_{i>r} (\lambda_i - \lambda_r)}}{1 - t^2} \left(\frac{1}{1 - x^{-N} t^{N-2r+2}} - \frac{1}{1 - x^{-N} t^{N-2r}} \right) \\ &= \frac{x^{\sum_i (\lambda_i - \lambda_r)} t^{\sum_{i<r} (\lambda_i - \lambda_r) - \sum_{i>r} (\lambda_i - \lambda_r)}}{(1 - x^{-N} t^{N-2r+2})(1 - x^{-N} t^{2r-N})}. \end{aligned}$$

This is the contribution of p_{r-1} to the Lefschetz fixed point formula to the Euler characteristic of \mathcal{L}_λ . (The denominator is $\Lambda_{-1} T_{p_{r-1}}^* \tilde{\mathcal{S}}_N$, and the numerator is $(\mathcal{L}_\lambda)_{p_{r-1}}$.) Since λ is dominant, higher cohomology vanishes. Hence this is the character of $\Gamma(\tilde{\mathcal{S}}_N, \mathcal{L}_\lambda)$. \square

4(ii). Pushforwards of line bundles on Kleinian surfaces

For dominant λ we denote by \mathcal{F}_λ the torsion free sheaf $R\pi_* \mathcal{L}_\lambda = \pi_* \mathcal{L}_\lambda$ on \mathcal{S}_N . We also set $\bar{\lambda} = \omega_{|\lambda| \pmod N}$ where $\omega_0 := 0$.

Lemma 4.3. *For dominant weight λ let \mathcal{F} be a $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant torsion-free sheaf on \mathcal{S}_N such that the character of $\Gamma(\mathcal{S}_N, \mathcal{F})$ coincides with the character of $\Gamma(\mathcal{S}_N, \mathcal{F}_\lambda)$. Then*

(a) *The restriction $\mathcal{F}|_{\mathcal{S}_N^\circ}$ is a line bundle, isomorphic to $\mathcal{F}_{\bar{\lambda}}|_{\mathcal{S}_N^\circ}$. Here $\mathcal{S}_N^\circ := \mathcal{S}_N \setminus \{0\}$.*

(b) *An isomorphism in (a) is defined uniquely up to multiplication by a scalar, even if one forgets the $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariance.*

(c) *The composition of isomorphisms $\mathcal{F}|_{\mathcal{S}_N^\circ} \simeq \mathcal{F}_{\bar{\lambda}}|_{\mathcal{S}_N^\circ} \simeq \mathcal{F}_\lambda|_{\mathcal{S}_N^\circ}$ gives an isomorphism $\mathcal{F}|_{\mathcal{S}_N^\circ} \xrightarrow{\sim} \mathcal{F}_\lambda|_{\mathcal{S}_N^\circ}$ which extends to an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{F}_\lambda$.*

Proof. An automorphism of a line bundle on \mathcal{S}_N° is given by multiplication by an invertible function on \mathcal{S}_N° . Any invertible function on \mathcal{S}_N° is constant. Indeed, it lifts to $\mathbb{A}^2 \setminus \{0\}$ where all the invertible functions are constant. Hence uniqueness in (b).

A torsion free sheaf \mathcal{F} is locally free on the complement of \mathcal{S}_N to finitely many points. Due to the $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariance, \mathcal{F} is locally free on \mathcal{S}_N° . Let us denote by $j: \mathcal{S}_N^\circ \hookrightarrow \mathcal{S}_N$ the open embedding. Then $\mathcal{F} \hookrightarrow j_*(\mathcal{F}|_{\mathcal{S}_N^\circ})$. Since we know the character of $\Gamma(\mathcal{S}_N, \mathcal{F})$, we conclude that \mathcal{F} is generically of rank one, i.e. $\mathcal{F}|_{\mathcal{S}_N^\circ}$ is a line bundle. Now $\text{Pic}(\mathcal{S}_N^\circ) = \mathbb{Z}/N\mathbb{Z}$, and any line bundle on \mathcal{S}_N° is isomorphic to $\mathcal{F}_{\bar{\mu}}|_{\mathcal{S}_N^\circ}$ for $\bar{\mu} \in \{0, \omega_1, \dots, \omega_{N-1}\}$. Thus $\mathcal{F} \hookrightarrow j_*(\mathcal{F}_{\bar{\mu}}|_{\mathcal{S}_N^\circ})$ (if we disregard the $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant structure). But any two $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant structures on the line bundle $\mathcal{F}_{\bar{\mu}}|_{\mathcal{S}_N^\circ}$ are isomorphic up to twist by a character χ of $\mathbb{C}^\times \times \mathbb{C}^\times$. So we have a $\mathbb{C}^\times \times \mathbb{C}^\times$ -equivariant embedding $\mathcal{F} \otimes \chi \hookrightarrow j_*(\mathcal{F}_{\bar{\mu}}|_{\mathcal{S}_N^\circ})$. We claim that $\bar{\mu}$ is congruent to λ modulo the root lattice, that is $\bar{\mu} = \omega_{|\lambda| \pmod N} = \bar{\lambda}$. Indeed, we take a sufficiently negative m in the formula of Lemma 4.2 for the character of $\Gamma(\mathcal{S}_N, \mathcal{F})$, so that $\lambda_i - m > 0$ for any i . Then $\sum_i (\lambda_i - m) = -Nm + \sum_i \lambda_i$, and so $|\lambda| \pmod N$ is determined from the character of $\Gamma(\mathcal{S}_N, \mathcal{F})$.

However, $\mathcal{F}_{\bar{\lambda}} \xrightarrow{\sim} j_*(\mathcal{F}_{\bar{\lambda}}|_{\mathcal{S}_N^\circ})$ (see Lemma 4.6 below and restrict to a sub-regular slice $\mathcal{S}_N \subset \mathcal{N}$). Thus we have $\mathcal{F} \otimes \chi \hookrightarrow \mathcal{F}_{\bar{\lambda}} \hookrightarrow \mathcal{F}_\lambda$, and we have to check that the images of $\mathcal{F} \otimes \chi$ and \mathcal{F}_λ inside $\mathcal{F}_{\bar{\lambda}}$ coincide, and $\chi = 1$. But the character of (global sections of) $\mathcal{F}_{\bar{\lambda}}$ is multiplicity free, and the characters of $\mathcal{F}_{\lambda_1} \otimes \chi_1, \mathcal{F}_{\lambda_2} \otimes \chi_2$ coincide if and only if $\lambda_1 = \lambda_2, \chi_1 = \chi_2$, so the equality of characters of \mathcal{F} and \mathcal{F}_λ guarantees $\chi = 1$ and the coincidence of the images of \mathcal{F} and \mathcal{F}_λ in $\mathcal{F}_{\bar{\lambda}}$. \square

**4(iii). Line bundles on Kleinian surfaces via
homology groups of fibers**

Recall the setup of §2(iv) and §2(vi). We consider the quiver gauge theory of type A_1 with $\dim V = 1$, $\dim W = N$ with $G = \mathrm{GL}(V) = \mathbb{C}^\times$, $\tilde{G} = \mathrm{GL}(V) \times \mathrm{GL}(W)/Z$, $G_F = \mathrm{PGL}(W) = \mathrm{PGL}(N)$, and varieties of triples \mathcal{R} , $\tilde{\mathcal{R}}$ for (G, \mathbf{N}) , (\tilde{G}, \mathbf{N}) and the corresponding complex \mathcal{A} on $\mathrm{Gr}_{\mathrm{PGL}(N)}$. See §2(iv). We are interested in its costalks at the points $\lambda \in \mathrm{Gr}_{\mathrm{PGL}(N)}$ where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N)$ is a dominant coweight of $\mathrm{PGL}(N)$. According to (2.16), the costalk $i_\lambda^! \mathcal{A}^{\mathrm{for}}$ forms a module over the algebra $i_0^! \mathcal{A}^{\mathrm{for}}$. The algebra $i_0^! \mathcal{A}^{\mathrm{for}}$ is nothing but the Coulomb branch $H_*^{G^\circ}(\mathcal{R}) \simeq \mathbb{C}[\mathcal{S}_N]$ where $\mathbf{N} = \mathrm{Hom}(W, V)$ by [Part II, §4(iv)]. The costalk $i_\lambda^! \mathcal{A}^{\mathrm{for}}$ is nothing but $H_*^{G^\circ}(\tilde{\mathcal{R}}^\lambda)$ where $\tilde{\mathcal{R}}^\lambda$ is the fiber of $\tilde{\pi}: \tilde{\mathcal{R}} \rightarrow \mathrm{Gr}_{G_F} = \mathrm{Gr}_{\mathrm{PGL}(N)}$ over $\lambda \in \mathrm{Gr}_{G_F}$, see (2.6).

Lemma 4.4. *The $i_0^! \mathcal{A}^{\mathrm{for}}$ -module $i_\lambda^! \mathcal{A}^{\mathrm{for}}$ is torsion free.*

Proof. Both $i_0^! \mathcal{A}^{\mathrm{for}}$ and $i_\lambda^! \mathcal{A}^{\mathrm{for}}$ are free $H_G^*(\mathrm{pt})$ -modules. So if $i_\lambda^! \mathcal{A}^{\mathrm{for}}$ had torsion, then it would still have torsion after the base change to $H_T^*(\mathrm{pt})$ and localization to the generic point of $H_T^*(\mathrm{pt})$. However, this is impossible since after this localization, $i_\lambda^! \mathcal{A}^{\mathrm{for}}$ becomes a free (rank 1) $i_0^! \mathcal{A}^{\mathrm{for}}$ -module by the Localization Theorem. \square

Recall that $H_*^{G^\circ}(\mathcal{R})$ has an additional grading induced from $\pi_0(\mathrm{Gr}_G) = \pi_1(G) = \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$ compatible with the convolution product (see [Part II, §3(v)]). We also have an additional grading on $H_*^{G^\circ}(\tilde{\mathcal{R}}^\lambda)$ compatible with the $H_*^{G^\circ}(\mathcal{R})$ -module structure from $\pi_0(\mathrm{Gr}_{\tilde{G}}) = \pi_1(\tilde{G})$ in the same way. We choose $\pi_0(\mathrm{Gr}_{\tilde{G}}) \rightarrow \mathbb{Z}$ so that the connected component of $\tilde{\mathcal{R}}^\lambda$ corresponding to the m -th component of Gr_G goes to $\sum_{i=1}^N (\lambda_i - m)$. This is well-defined as it is invariant under simultaneous shift of all λ_i and m .

Proposition 4.5. *Under the identification $i_0^! \mathcal{A}^{\mathrm{for}} = H_*^{G^\circ}(\mathcal{R}) \simeq \mathbb{C}[\mathcal{S}_N]$, the $i_0^! \mathcal{A}^{\mathrm{for}}$ -module $i_\lambda^! \mathcal{A}^{\mathrm{for}} = H_*^{G^\circ}(\tilde{\mathcal{R}}^\lambda)$ is isomorphic to the $\mathbb{C}[\mathcal{S}_N]$ -module $\Gamma(\mathcal{S}_N, \mathcal{F}_\lambda)$. More precisely,*

- (a) *The localization of $i_\lambda^! \mathcal{A}^{\mathrm{for}}$ to \mathcal{S}_N° is a line bundle isomorphic to $\mathcal{F}_\lambda|_{\mathcal{S}_N^\circ}$.*
- (b) *An isomorphism in (a) is defined uniquely up to multiplication by a scalar.*
- (c) *An isomorphism in (a) extends to an isomorphism $i_\lambda^! \mathcal{A}^{\mathrm{for}} \xrightarrow{\sim} \Gamma(\mathcal{S}_N, \mathcal{F}_\lambda)$.*

Proof. By the monopole formula of [Part II, Prop. 2.7] with the convention [Part II, Remark 2.8(2)], the Hilbert series of the bigraded module $H_*^{G^\circ}(\tilde{\mathcal{R}}^\lambda)$

is $\sum_{m \in \mathbb{Z}} x^{\sum_{i=1}^N (\lambda_i - m)} t^{\sum_{i=1}^N |\lambda_i - m|} (1 + t^2 + t^4 + \dots)$. By Lemma 4.4, $H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda)$ is a torsion-free $H_*^{G \circ}(\mathcal{R})$ -module. Comparing its Hilbert series with the formula of Lemma 4.2 and applying the criterion of Lemma 4.3 we obtain the desired result. \square

Let us write down the isomorphism more concretely when λ is the n -th fundamental coweight ω_n .

Recall w, y, z are identified with elements in $H_*^{G \circ}(\mathcal{R})$ as follows (see [Part II, §4(iv)]):

- w is the generator of $H_G^*(\text{pt})$.
- y is the fundamental class of the fiber $\pi^{-1}(1)$, where $\pi: \mathcal{R} \rightarrow \text{Gr}_G \simeq \mathbb{Z}$.
- z is the fundamental class of the fiber $\pi^{-1}(-1)$.

The space $\Gamma(\tilde{\mathcal{S}}_N, \mathcal{L}_{\omega_n})$ of sections of the line bundle corresponding to ω_n is identified with the space of semi-invariants $\mathbb{C}[\mathbb{A}^2]^{\chi_n}$ where $\chi_n(\zeta) = \zeta^n$. It has a linear basis

$$u^n z^m w^k, \quad v^{N-n} y^m w^k \quad (m, k \in \mathbb{Z}_{\geq 0}),$$

where $w = uv, z = u^N, y = v^N$.

Let us consider a coweight $(m, \underbrace{1, \dots, 1}_n, \underbrace{0, \dots, 0}_{N-n})$ ($m \in \mathbb{Z}$) of \tilde{G} , where the first m is a coweight of G . Let r^m denote the fundamental class of the corresponding fiber for the projection $\tilde{\mathcal{R}} \rightarrow \text{Gr}_{\tilde{G}}$. Note that the pairing between the coweight above and weights of $\text{Hom}(W, V)$ are $m - 1, \dots, m - 1$ (n times) and m, \dots, m ($N - n$ times). Thus we have n negative terms if $m = 0$, $N - n$ positive terms if $m = 1$, all negative or all positive otherwise. Therefore

$$y r^m = \begin{cases} r^{m+1} & \text{if } m > 0, \\ w^n r^{m+1} & \text{if } m = 0, \\ w^N r^{m+1} & \text{if } m < 0, \end{cases} \quad z r^m = \begin{cases} r^{m-1} & \text{if } m \leq 0, \\ w^{N-n} r^{m-1} & \text{if } m = 1, \\ w^N r^{m-1} & \text{if } m > 1, \end{cases}$$

by [Part II, §4]. (Note that we can replace \tilde{G} by $\text{GL}(V) \times T(W)/Z$ where $T(W) \subset \text{GL}(W)$ is a maximal torus of $\text{GL}(W)$ as in [Part II, §3(ix)]. Hence we can use computation in [Part II, §4].) Now we get an isomorphism

$i_{\omega_n}^! \mathcal{A}^{\text{for}} \xrightarrow{\cong} \mathbb{C}[\mathbb{A}^2]^{\chi_n}$ of $\mathbb{C}[\mathcal{M}_C] = \mathbb{C}[y, z, w]/(yz = w^N)$ -modules by setting

$$w^k r^m = \begin{cases} v^{N-n} y^{m-1} w^k & \text{if } m > 0, \\ u^n z^{-m} w^k & \text{if } m \leq 0. \end{cases}$$

4(iv). Andersen-Jantzen sheaves on a nilpotent cone

We denote by \mathcal{N} the nilpotent cone of \mathfrak{sl}_N . We denote by \mathcal{B} the flag variety of \mathfrak{sl}_N , and by $T^*\mathcal{B}$ its cotangent bundle. We denote by $\pi: T^*\mathcal{B} \rightarrow \mathcal{N}$ the Springer resolution. We denote by $j: \mathbb{O}_{\text{reg}} \hookrightarrow \mathcal{N}$ the embedding of the regular nilpotent orbit. For a dominant weight $\lambda = (\lambda_1 \geq \dots \geq \lambda_N)$ we denote by $\mathcal{O}(\lambda)$ the line bundle on $T^*\mathcal{B}$ obtained by the pullback of the corresponding line bundle on \mathcal{B} . It is known that $\mathcal{J}_\lambda := \pi_* \mathcal{O}(\lambda) = R\pi_* \mathcal{O}(\lambda)$ is a torsion-free sheaf on \mathcal{N} (an *Andersen-Jantzen sheaf*, see e.g. [BK05, Thm. 5.2.1]).

Lemma 4.6. *For $\bar{\lambda} \in \{0, \omega_1, \dots, \omega_{N-1}\}$ we have $\mathcal{J}_{\bar{\lambda}} = j_*(\mathcal{J}_{\bar{\lambda}}|_{\mathbb{O}_{\text{reg}}})$.*

Proof. We have to check that $\mathcal{J}_{\bar{\lambda}}$ is Cohen-Macaulay. It follows from the fact that its Grothendieck-Serre dual $R\pi_*(\mathcal{O}(-\bar{\lambda}))$ has no higher cohomology by [BK05, Thm. 5.2.1]. □

Recall that according to [Lus81], \mathcal{N} is isomorphic to the transversal slice $\overline{\mathcal{W}}_0^{N\omega_1}$ in the affine Grassmannian $\text{Gr}_{\text{GL}(N)}$. Recall the factorization morphism $\Pi := \pi_{N\omega_1^*} \circ s_\mu^{N\omega_1}: \mathcal{N} = \overline{\mathcal{W}}_0^{N\omega_1} \rightarrow \mathbb{A}^{N\omega_1^*}$ of [Quiver, Lemma 2.7] (it is also called the *Gelfand-Tsetlin integrable system*).

Lemma 4.7. *The morphism $\Pi \circ \pi: T^*\mathcal{B} \rightarrow \mathbb{A}^{N\omega_1^*}$ is flat.*

Proof. It suffices to prove that all the fibers of $\Pi \circ \pi$ have the same dimension $N(N - 1)/2$. We recall the proof of [Quiver, Lemma 2.7]. There the dimension estimate on the fibers of Π followed from the semismallness of the convolution morphism \mathbf{q} . Under the identification $\mathcal{N} = \overline{\mathcal{W}}_0^{N\omega_1}$, the Springer resolution $\pi: T^*\mathcal{B} \rightarrow \mathcal{N}$ corresponds to the iterated convolution morphism $\mathbf{m}: \text{Gr}_{\text{GL}(N)}^{\omega_1} \widetilde{\times} \dots \widetilde{\times} \text{Gr}_{\text{GL}(N)}^{\omega_1} \rightarrow \text{Gr}_{\text{GL}(N)}$ restricted to the slice $\overline{\mathcal{W}}_0^{N\omega_1} \subset \text{Gr}_{\text{GL}(N)}$. Now the convolution morphism \mathbf{m} is semismall, and moreover, its composition with \mathbf{q} is semismall as well, so the proof of [Quiver, Lemma 2.7] goes through in the present situation as well. □

4(v). Andersen-Jantzen sheaves via homology groups of fibers

We change the setup of §4(iii) to that of §2(v). According to (2.16), the costalk $i_\lambda^! \mathcal{A}^{\text{for}}$ forms a module over the algebra $i_0^! \mathcal{A}^{\text{for}}$. The algebra $i_0^! \mathcal{A}^{\text{for}}$ is nothing but the Coulomb branch $H_*^{G \circ}(\mathcal{R}) \simeq \mathbb{C}[\mathcal{N}]$. The costalk $i_\lambda^! \mathcal{A}^{\text{for}}$ is nothing but $H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda)$ where $\tilde{\pi}: \tilde{\mathcal{R}} \rightarrow \text{Gr}_{G_F} = \text{Gr}_{\text{PGL}(N)}$ and $\tilde{\mathcal{R}}^\lambda = \tilde{\pi}^{-1}(\lambda)$, see (2.6).

We have the $L^{\text{bal}} = \text{PGL}(N)$ -action on the Coulomb branch $H_*^{G \circ}(\mathcal{R})$ by Proposition A.3 and example A.6. By example A.5 it coincides with the standard action on \mathcal{N} .

Theorem 4.8. *Under the identification $i_0^! \mathcal{A}^{\text{for}} = H_*^{G \circ}(\mathcal{R}) \simeq \mathbb{C}[\mathcal{N}]$, the $i_0^! \mathcal{A}^{\text{for}}$ -module $i_\lambda^! \mathcal{A}^{\text{for}} = H_*^{G \circ}(\tilde{\mathcal{R}}^\lambda)$ is isomorphic to the $\mathbb{C}[\mathcal{N}]$ -module $\Gamma(\mathcal{N}, \mathcal{J}_\lambda)$.*

Proof. The $\mathbb{C}[\mathcal{N}]$ -module $i_\lambda^! \mathcal{A}^{\text{for}}$ is torsion free generically of rank 1, see Lemma 4.4. By Proposition A.3 and example A.6, we have an action of $\tilde{L}^{\text{bal}} = \text{SL}(N)$ on $i_\lambda^! \mathcal{A}^{\text{for}}$. The $i_0^! \mathcal{A}^{\text{for}}$ -module $i_\lambda^! \mathcal{A}^{\text{for}}$ is $\text{SL}(N)$ -equivariant (under the natural projection $\text{SL}(N) \rightarrow \text{PGL}(N)$). Hence, the restriction of the associated coherent sheaf $(i_\lambda^! \mathcal{A}^{\text{for}})_{\text{loc}}$ to $\mathbb{O}_{\text{reg}} \subset \mathcal{N}$ is a line bundle. Now $\text{Pic}(\mathbb{O}_{\text{reg}}) = \mathbb{Z}/N\mathbb{Z}$, and any line bundle on \mathbb{O}_{reg} is isomorphic to $\mathcal{J}_{\bar{\mu}}|_{\mathbb{O}_{\text{reg}}}$ for $\bar{\mu} \in \{0, \omega_1, \dots, \omega_{N-1}\}$. Thus we obtain an embedding

$$i_\lambda^! \mathcal{A}^{\text{for}} \hookrightarrow \Gamma(\mathcal{N}, j_*(\mathcal{J}_{\bar{\mu}}|_{\mathbb{O}_{\text{reg}}})) = \Gamma(\mathcal{N}, \mathcal{J}_{\bar{\mu}}).$$

We claim that $\bar{\mu} = \omega_{|\lambda| \pmod{N}} = \bar{\lambda}$. Indeed, $\text{SL}(N)$ -module $i_\lambda^! \mathcal{A}^{\text{for}}$ has the same central character as $V^{\bar{\lambda}}$.

Thus we obtain an embedding $i_\lambda^! \mathcal{A}^{\text{for}} \hookrightarrow \Gamma(\mathcal{N}, \mathcal{J}_{\bar{\lambda}})$. Similarly, we have an embedding $\Gamma(\mathcal{N}, \mathcal{J}_\lambda) \hookrightarrow \Gamma(\mathcal{N}, \mathcal{J}_{\bar{\lambda}})$. In other words, denoting $i_\lambda^! \mathcal{A}^{\text{for}}|_{\mathbb{O}_{\text{reg}}}$ the restriction of $(i_\lambda^! \mathcal{A}^{\text{for}})_{\text{loc}}$ to \mathbb{O}_{reg} , we obtain an isomorphism of line bundles $i_\lambda^! \mathcal{A}^{\text{for}}|_{\mathbb{O}_{\text{reg}}} \simeq \mathcal{J}_{\bar{\lambda}}|_{\mathbb{O}_{\text{reg}}}$. Note that this isomorphism is defined uniquely up to a scalar multiplication since the automorphism group of any line bundle on \mathbb{O}_{reg} is $\Gamma(\mathbb{O}_{\text{reg}}, \mathcal{O}^\times) = \mathbb{C}^\times$. Indeed, an invertible function on \mathbb{O}_{reg} extends to a regular function on \mathcal{N} due to normality of \mathcal{N} . This extended function is still invertible since otherwise its zero divisor would intersect \mathbb{O}_{reg} . Its lift to $T^*\mathcal{B}$ is invertible and hence constant on each fiber of $T^*\mathcal{B} \rightarrow \mathcal{B}$. So it is lifted from \mathcal{B} and hence constant.

We will show that the above isomorphism extends to \mathcal{N} . To this end we use the factorization morphism $\Pi: \mathcal{N} \rightarrow \mathbb{A}^{N\omega_1^*} = \mathfrak{t}(V)/\mathbb{W}$ as in [Part II,

Thm. 5.26] and [Part II, Remark 5.27], where $\mathfrak{t}(V)$ is a Cartan subalgebra of $\mathfrak{g} = \mathfrak{gl}(V)$, and \mathbb{W} is the Weyl group of $(\mathfrak{gl}(V), \mathfrak{t}(V))$. The condition $\Pi_* \mathcal{J}_\lambda = \Pi_* \pi_* \mathcal{O}(\lambda) \xrightarrow{\sim} j_* \Pi_* \pi_* \mathcal{O}(\lambda)|_{T^* \mathcal{B}^\bullet} = j_* \Pi_* \mathcal{J}_\lambda|_{\mathcal{N}^\bullet}$ of [Part II, Remark 5.27] is satisfied since the complement of $T^* \mathcal{B}^\bullet$ in $T^* \mathcal{B}$ is of codimension 2 by Lemma 4.7. So it suffices to check the regularity of our rational isomorphism after the base change $\mathfrak{t}(V) \rightarrow \mathfrak{t}(V)/\mathbb{W}$ and localizations at general points of the root hyperplanes. Moreover, since we already know that our isomorphism is regular at \mathbb{O}_{reg} , it remains to check the regularity at the localizations at general points of the coordinate hyperplanes $w_{1,r} = 0$, $r = 1, \dots, N - 1$, cf. the proof of [Quiver, Thm. 3.10]. By an application of the Localization Theorem, just as in *loc. cit.*, the comparison reduces to Proposition 4.5. Namely, let t be a general point of the hyperplane $w_{1,r} = 0$, and let x be a point of the subregular nilpotent orbit above t . Then there is a slice $\mathcal{S}_N \subset \mathcal{N}$ through x such that the isomorphism of $\mathcal{J}_\lambda|_{\mathbb{O}_{\text{reg}}}$ and $i_\lambda^! \mathcal{A}^{\text{for}}|_{\mathbb{O}_{\text{reg}}}$ restricted to \mathcal{S}_N extends to the localization $(\mathcal{S}_N)_t$ (by Proposition 4.5). Due to the $\text{SL}(N)$ -equivariance, the pullback of the above isomorphism to $\text{SL}(N) \times \mathcal{S}_N \xrightarrow{\text{act}} \mathbb{O}_{\text{reg}}$ extends to $(\text{SL}(N) \times \mathcal{S}_N)_t$. By the faithfully flat descent, the above isomorphism extends to \mathcal{N}_t , and hence to the whole of \mathcal{N} . \square

4(vi). Modified homological grading

Let us write down the modified monopole formula (4.1) in our case explicitly. (This appeared first in [CHMZ14a, (3.9)].) It is

$$(4.9) \quad P_t^{\text{mod}}(i_\lambda^! \mathcal{A}^{\text{for}}) = \sum_{\vec{\lambda}} t^{2\Delta(\vec{\lambda})} P_{\text{GL}(V)}(t, \vec{\lambda})$$

(the sum over the dominant coweights $\vec{\lambda} = (\lambda^1, \dots, \lambda^{N-1})$ of $\text{GL}(N - 1) \times \dots \times \text{GL}(1)$), where

$$2\Delta(\lambda, \lambda^1, \dots, \lambda^{N-1}) := \sum_{j=1}^{N-1} \sum_{i, i'} |\lambda_i^{j-1} - \lambda_{i'}^j| - 2 \sum_{j=1}^{N-1} \sum_{i < i'} |\lambda_i^j - \lambda_{i'}^j|,$$

and we set for convenience $\lambda^0 := \lambda$. We also set $n(\lambda) = \sum_{i=1}^N (i - 1)\lambda_i$. Then $\dim \text{Gr}_{\text{PGL}(N)}^\lambda = \langle 2\rho_{\text{PGL}(N)}^\vee, \lambda \rangle = (N - 1)|\lambda| - 2n(\lambda)$.

Lemma 4.10. *$i_\lambda^! \mathcal{A}^{\text{for}}$ lives in (modified) degrees $\geq \dim \text{Gr}_{\text{PGL}(N)}^\lambda$, and its component of this degree has the same dimension as the irreducible $\text{SL}(N)$ -module V^λ .*

Proof. We have to compute

$$t^{2n(\lambda)-(N-1)|\lambda|} \sum_{\vec{\lambda}} t^{2\Delta(\vec{\lambda})} P_{\text{GL}(V)}(t, \vec{\lambda})|_{t=0} = t^{2n(\lambda)-(N-1)|\lambda|} \sum_{\vec{\lambda}} t^{2\Delta(\vec{\lambda})}|_{t=0}.$$

One checks that this is the sum of 1's over the set of $(N - 1)$ -tuples $\vec{\lambda}$ which interlace, i.e. $\lambda_i^j \geq \lambda_i^{j+1} \geq \lambda_{i+1}^j$, $0 \leq j \leq N - 2$, $1 \leq i \leq N - j - 1$ (recall that $\lambda^0 = \lambda$).⁴ In other words, this is the cardinality of the set of Gelfand-Tsetlin patterns of shape λ , that is $\dim V^\lambda$. \square

Remark 4.11. Characters of $\Gamma(\mathcal{N}, \mathcal{J}_\lambda)$ are given by Hall-Littlewood polynomials by computation of Euler characteristic [Hes80, Bry89] and the vanishing theorem [Bro93]. Therefore (4.9) gives a combinatorial expression of Hall-Littlewood polynomials. We asked several people (including mathoverflow [Nak17]) whether it is known or not. But we could not find earlier appearance. In view of the argument in the special case $t = 0$ in Lemma 4.10, there should be a purely combinatorial proof.

4(vii). Modified grading of Andersen-Jantzen modules

We have the dilatation action of \mathbb{C}^\times on $T^*\mathcal{B}$ and the natural \mathbb{C}^\times -equivariant structure on $\mathcal{O}(\lambda)$; hence a grading on $\Gamma(T^*\mathcal{B}, \mathcal{O}(\lambda)) = \Gamma(\mathcal{N}, \mathcal{J}_\lambda)$ starting in degree 0 with $\Gamma(\mathcal{B}, \mathcal{O}(\lambda)) = (V^\lambda)^\vee$. We modify the grading by doubling all the degrees and shifting it by $(N - 1)|\lambda| - 2n(\lambda)$. From now on we consider $\Gamma(\mathcal{N}, \mathcal{J}_\lambda)$ with this modified grading only.

Theorem 4.12. *The isomorphism of $\mathbb{C}[\mathcal{N}]$ -modules $i_\lambda^! \mathcal{A}^{\text{for}} \simeq \Gamma(\mathcal{N}, \mathcal{J}_\lambda)$ of Theorem 4.8 is a graded isomorphism.*

Proof. For $\lambda = 0$ the claim is nothing but [Quiver, Remark 3.13]. Clearly, $\Gamma(\mathcal{N}, \mathcal{J}_\lambda)$ is a graded $\text{SL}(N) \times \mathbb{C}[\mathcal{N}]$ -module; $i_\lambda^! \mathcal{A}^{\text{for}}$ is also a graded $\text{SL}(N) \times \mathbb{C}[\mathcal{N}]$ -module by construction of Appendix A (see example A.6). Both embeddings $i_\lambda^! \mathcal{A}^{\text{for}} \hookrightarrow \Gamma(\mathcal{N}, \mathcal{J}_{\bar{\lambda}})$ and $\Gamma(\mathcal{N}, \mathcal{J}_\lambda) \hookrightarrow \Gamma(\mathcal{N}, \mathcal{J}_{\bar{\lambda}})$ are compatible with the gradings up to a shift since the structure of a $\text{SL}(N)$ -equivariant line bundle on $\mathcal{J}_{\bar{\lambda}}|_{\mathbb{O}_{\text{reg}}}$ extends to a $\text{SL}(N) \times \mathbb{C}^\times$ -equivariant structure uniquely up to tensoring with a character of \mathbb{C}^\times . The above shifts match because both the grading of $i_\lambda^! \mathcal{A}^{\text{for}}$ and the grading of $\Gamma(\mathcal{N}, \mathcal{J}_\lambda)$ start in the same degree $(N - 1)|\lambda| - 2n(\lambda)$. \square

⁴We learned this observation in [Gor17].

4(viii). The regular sheaf

By Lemma 4.10, the complex \mathcal{A} belongs to ${}^pD_{\text{PGL}(N)}^{\geq 0}(\text{Gr}_{\text{PGL}(N)})$, and its 0th perverse cohomology ${}^pH^0(\mathcal{A})$ is isomorphic to

$$\bigoplus_{\lambda} (V^{\lambda})^{\vee} \otimes \text{IC}(\overline{\text{Gr}}_{\text{PGL}(N)}^{\lambda}) =: \mathcal{A}_R.$$

Theorem 4.13. *The natural morphism $\sigma: \mathcal{A}_R = {}^pH^0(\mathcal{A}) \rightarrow \mathcal{A}$ is an isomorphism of ring objects.*

Proof. First we prove that σ is an isomorphism disregarding the ring structure. We have to check $\tau_{>0}\mathcal{A} = \text{Cone}(\sigma) = 0$. Note that all the costalks of $\text{IC}(\overline{\text{Gr}}_{\text{PGL}(N)}^{\lambda})$ live in the degrees of the same parity as $|\lambda|$, see [Lus83]. We will call this phenomenon *parity vanishing*. The parity vanishing for \mathcal{A} also holds true (on a given connected component of $\text{Gr}_{\text{PGL}(N)}$, all the costalks of \mathcal{A}^{for} live in the same parity as all the costalks of any IC sheaf on this component, see (4.9)). This implies that ${}^pH^{\text{odd}}(\mathcal{A}) = 0$, and hence ${}^pH^{\text{odd}}(\tau_{>0}\mathcal{A}) = 0$. Now the Hilbert series of $i_{\lambda}^! \mathcal{A}^{\text{for}}$ and $i_{\lambda}^! \mathcal{A}_R$ coincide by Theorem 4.12 and the comparison of [Bry89] and [Lus83]. Hence if σ were not an isomorphism, its costalk σ_{λ} would have both kernel and cokernel for some λ . Thus, $\text{Cone}(\sigma)$ would have a costalk of wrong parity at λ . This would contradict the parity vanishing for $\tau_{>0}\mathcal{A} = \text{Cone}(\sigma)$. We conclude that σ is an isomorphism.

Now we compare the ring structures. Since both \mathcal{A} and \mathcal{A}_R are perverse, it suffices to check that the fiber functor $H^{\bullet}(\sigma)$ induces an isomorphism of the rings $H^{\bullet}(\text{Gr}_{\text{PGL}(N)}, \mathcal{A}_R)$ and $H^{\bullet}(\text{Gr}_{\text{PGL}(N)}, \mathcal{A}^{\text{for}})$. It is enough to check the assertion for $\text{GL}(N)$ instead of $\text{PGL}(N)$, as $\text{Gr}_{\text{GL}(N)}$ is the union of copies of $\text{Gr}_{\text{PGL}(N)}$. We have $H^{\bullet}(\text{Gr}_{\text{GL}(N)}, \mathcal{A}_R) \simeq \mathbb{C}[\text{GL}(N)]$ by geometric Satake equivalence. On the other hand, the cohomology $H^{\bullet}(\text{Gr}_{\text{GL}(N)}, \mathcal{A}^{\text{for}})$ is the quotient of the equivariant cohomology $H_{\text{GL}(N)}^{\bullet}(\text{Gr}_{\text{GL}(N)}, \mathcal{A})$ modulo the augmentation ideal of $H_{\text{GL}(N)}^{\bullet}(\text{pt})$. And

$$H_{\text{GL}(N)}^{\bullet}(\text{Gr}_{\text{GL}(N)}, \mathcal{A}) = H_*^{(\text{GL}(V) \times \text{GL}(W))_{\circ}}(\mathcal{R}_{\text{GL}(V) \times \text{GL}(W), \mathbf{N}})$$

where $(\text{GL}(V) \times \text{GL}(W), \mathbf{N})$ is the quiver gauge theory obtained from $(\text{GL}(V), \mathbf{N})$ by turning $\text{GL}(W)$ to a gauge group. By [Quiver, Thm. 3.1],

$$H_*^{(\text{GL}(V) \times \text{GL}(W))_{\circ}}(\mathcal{R}_{\text{GL}(V) \times \text{GL}(W), \mathbf{N}}) \simeq \mathbb{C}[\check{Z}_{\text{PGL}(N+1)}^{\alpha}]$$

where $\alpha = N\alpha_1 + (N - 1)\alpha_2 + \dots + \alpha_N$. By [BP08, Thm. 1], $\mathring{Z}_{\mathrm{PGL}(N+1)}^\alpha \simeq \mathrm{GL}(W) \times W$, and its projection to $\mathrm{Spec} H_{\mathrm{GL}(N)}^\bullet(\mathrm{pt})$ is nothing but the projection of $\mathrm{GL}(W) \times W$ to W . Hence the zero fiber of this projection is isomorphic to $\mathrm{GL}(W) = \mathrm{GL}(N)$. \square

5. Mirrors of Sicilian theories

In the first half of this section, we study examples of Coulomb branches \mathcal{M}_C of star shaped quiver gauge theories as in Figure 1. As explained at the end of Introduction, they are conjectural Higgs branches of Sicilian theories.

Let us briefly review [MT12] on expected properties of Higgs branches of Sicilian theories. It is conjectured that there exists a functor from the category of 2-bordisms to a category \mathcal{HS} of holomorphic symplectic varieties with Hamiltonian group actions. For the latter, objects are complex algebraic semisimple groups. A homomorphism from G to G' is a holomorphic symplectic variety X with a \mathbb{C}^\times -action scaling the symplectic form with weight 2 together with hamiltonian $G \times G'$ action commuting with the \mathbb{C}^\times -action. For $X \in \mathrm{Hom}(G', G)$, $Y \in \mathrm{Hom}(G, G'')$, their composition $Y \circ X \in \mathrm{Hom}(G', G'')$ is given by the symplectic reduction of $Y \times X$ by the diagonal G -action. The identity $\in \mathrm{Hom}(G, G)$ is the cotangent bundle T^*G with the left and right multiplication of G .

Let us fix a complex semisimple group G . Physicists associate a 3d Sicilian theory to G and a Riemann surface with boundary, and consider its Higgs branch. It depends only on the topology of the Riemann surface, and gives a functor as above. We associate S^1 with G , and a cylinder with T^*G . Since T^*G is the identity in \mathcal{HS} , it is one of requirements.

Physical argument shows that the variety associated with a disk is $G \times S$, where S is the Kostant slice to the regular nilpotent orbit.

Let $W \equiv W_G$ be the variety associated with S^2 three disks removed. This is a fundamental piece as other varieties are obtained by reductions of products of its copies. It has an action of $\mathfrak{S}_3 \times G^3$. It is expected that

- $W = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ if $G = \mathrm{SL}(2)$.
- W is the minimal nilpotent orbit of E_6 if $G = \mathrm{SL}(3)$.

For other groups, W is unknown.

Recently Ginzburg and Kazhdan [GK] construct a functor, and check most of properties, in particular show that the gluing of Riemann surfaces corresponds to the hamiltonian reduction with respect to the diagonal action. Via a result of [Bap15] their symplectic variety associated with S^2

minus b disks is defined as

$$W^b \stackrel{\text{def.}}{=} \text{Spec } H_{G_{\mathcal{O}}}^*(\text{Gr}_G, i_{\Delta}^!(\boxtimes_{k=1}^b (\mathcal{A}_R)_k)),$$

where $(\mathcal{A}_R)_k$ is a copy of the regular sheaf on Gr_G . Here the complex symplectic group taken as the object of the target category is G^{\vee} , the Langlands dual group. (E.g., $b = 2$ gives T^*G^{\vee} .)

By Theorem 2.11 together with §2(viii) we immediately get the following:

Theorem 5.1. *The symplectic variety W^b of Ginzburg-Kazhdan for $G^{\vee} = \text{SL}(N)$ is isomorphic to the Coulomb branch of the star shaped quiver gauge theory in Figure 1 with b legs instead of 3.*

More precisely, as we divide $\text{GL}(V) \times \text{GL}(W)$ by \mathbb{C}^{\times} in §2(v), we also divide $\text{GL}(V)$ for the star shaped quiver gauge theory also by the diagonal central subgroup $Z \cong \mathbb{C}^{\times}$. If we replace the central $\text{GL}(N)$ by $\text{SL}(N)$ instead of taking the quotient by Z , we get W for $G^{\vee} = \text{PGL}(N)$.⁵

We could consider the Coulomb branch for more general quiver gauge theory associated with a Riemann surface with boundary as in [Nak16, 3(iii) Figure 5] (namely we have b legs, as well as g loops at the central vertex), which is the Higgs branch of a $3d$ Sicilian theory, obtained by compactifying $6d$ $\mathcal{N} = (2, 0)$ theory of type A by $S^1 \times$ (punctured Riemann surface). Ginzburg-Kazhdan construction is also generalized. See §5(x). We conjecture that Theorem 5.1 is generalized.

Conjecture 5.2. Let $W^{g,b} \stackrel{\text{def.}}{=} \text{Spec } H_{G_{\mathcal{O}}}^*(\text{Gr}_G, \mathcal{A}^b \otimes^! \mathcal{B}^g)$ as in §5(x) for $G^{\vee} = \text{SL}(N)$. It is isomorphic to the Coulomb branch of the gauge theory associated with the quiver [Nak16, 3(iii) Figure 5].

By §2(viii) it is enough to show that the complex $\mathcal{B} = \mathcal{B}^{g=1}$ introduced in §5(x) is isomorphic to the object $\mathcal{A} = \pi_* \omega_{\mathcal{R}}[-2 \dim \mathbf{N}_{\mathcal{O}}]$ associated with $(G, \mathbf{N}) = (\text{PGL}(N), \mathfrak{pgl}(N))$. We conjecture that this is true for general G and its adjoint representation $\mathbf{N} = \mathfrak{g}$. Note that we prove that $\text{Spec } H_{G_{\mathcal{O}}}^*(\text{Gr}_G, \mathcal{B}) = (T^{\vee} \times \mathfrak{t})/W$ (using Losev’s result in §5(xiii)), which coincides with the Coulomb branch for the adjoint representation. (See [Part II, §3(x)(b)].)

The remainder of this section is as follows. In the first five subsections, we study examples of \mathcal{M}_C , in particular check two cases $\text{SL}(2)$, $\text{SL}(3)$ above. These are basically applications of [Quiver] and Appendix A, and we will

⁵We thank Yuji Tachikawa for an explanation of this procedure.

not use the sheaf \mathcal{A} . In the subsequent five subsections, we show the gluing property and also $W^{b=2} = T^*G^\vee$. They were shown in [GK], but we give proofs for completeness. They are direct consequences of [BF08]. In §§5(xi), 5(xii) we explain similarities between the gluing property and hamiltonian reduction.

Let us use the following notation as in [Quiver]. Let Q be a quiver with sets I, Ω of vertices and arrows respectively. We take an I -graded vector space $V = \bigoplus V_i$ with dimension vector $\alpha = (\dim V_i)_{i \in I}$. We set $\mathrm{GL}(V) = \prod \mathrm{GL}(V_i)$, $\mathbf{N} \equiv \mathbf{N}^\alpha = \bigoplus \mathrm{Hom}(V_i, V_j)$, where the sum is over the arrows $i \xrightarrow{h} j \in \Omega$. We also take the diagonal central subgroup $Z = \mathbb{C}^\times \subset \mathrm{GL}(V)$ and set $\mathrm{PGL}(V) = \mathrm{GL}(V)/Z$. We consider the complex $\mathcal{R} \equiv \mathcal{R}_{\mathrm{PGL}(V), \mathbf{N}}$ and the corresponding Coulomb branch $\mathcal{M}_C = \mathrm{Spec}(H_*^{\mathrm{PGL}(V)^\circ}(\mathcal{R}_{\mathrm{PGL}(V), \mathbf{N}}))$.

Remark 5.3. Consider the regular sheaf \mathcal{A}_R on Gr_G . In type A , it arises as the ring object associated with a quiver gauge theory by Theorem 2.11. Using Sp/SO quiver as in [Nak16, Appendix A.2], we can conjecture that \mathcal{A}_R for classical groups is constructed in a similar way, once we can generalize our definition to the case when \mathbf{M} is not necessarily of cotangent type. For exceptional groups, we do not expect that \mathcal{A}_R appears in this way, as argued in [Nak16, §3(i)]. Nevertheless it is expected that \mathcal{A}_R arises from the $3d \mathcal{N} = 4$ quantum field theory $T(G)$, which was introduced in Gaiotto-Witten [GW09]. This theory is not a usual gauge theory nor a lagrangian theory for an exceptional group, hence is difficult to understand from a mathematical point of view. But it has a $G \times G^\vee$ -symmetry, and its Higgs/Coulomb branches are nilpotent cones \mathcal{N} and \mathcal{N}^\vee of G and G^\vee respectively. The Sicilian theory $S_{G^\vee}(g, b)$ associated with b punctured genus g Riemann surface C considered above is constructed from $T(G)$ by ‘gauging’ quantum field theories up to $3d$ mirror:

$$(5.4) \quad S_{G^\vee}(g, b) \overset{3d \text{ mirror}}{\longleftrightarrow} T[G]^b \times \mathrm{Hyp}(\mathfrak{g} \oplus \mathfrak{g}^*)^g \not\# G_{\mathrm{diag}},$$

where we use the notation $\not\#$ for the gauging in [Tac]. (See also [Tac18].) This observation was given in [BTX10]. Note that we ignore the parameter τ in [Tac, §2.6]. The deformation parameter, which corresponds to the complex structure of C , is not relevant to Higgs branches as complex symplectic varieties. Hence we can safely write $S_{G^\vee}(g, b)$ instead of $S_{G^\vee}(C)$, and understand that the Higgs branch of $S_{G^\vee}(g, b)$ is the Coulomb branch of the right hand side. A similarity between (5.4) and the definition $W^{g,b} = \mathrm{Spec} H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{A}^b \otimes^! \mathcal{B}^g)$ is clear. We identify $T[G]$ with

\mathcal{A} , $\text{Hyp}(\mathfrak{g} \oplus \mathfrak{g}^*)$ with \mathcal{B} , and $\#G_{\text{diag}}$ with taking $H_{G_{\circ}}^*(\text{Gr}_G, \bullet)$ after the !-restriction to the diagonal subgroup. See §5(xi) for a further discussion. We thank Davide Gaiotto and Yuji Tachikawa for this remark.

5(i). Cylinder

Consider the two legs star shaped quiver gauge theory instead of three legs in Figure 1. It is a quiver gauge theory of type A_{2N-1} with $\dim V = (1, 2, \dots, N-1, N, N-1, \dots, 2, 1)$. We first consider the Coulomb branch for $\text{GL}(V)$. By [Quiver, §3(i)], $\mathcal{M}_C(\text{GL}(V), \mathbf{N})$ is the moduli space $\check{Z}_{\text{PGL}(2N)}^\alpha$ of based maps from \mathbb{P}^1 to a flag variety of type A_{2N-1} with degree $\alpha = \dim V$. By [BP08, Thm. 7.2], it is isomorphic to $T^* \text{GL}(N)$.

Note that $\alpha = 2\omega_N$. By [Quiver, Remark 3.12], we have an isomorphism $\check{Z}_{\text{PGL}(2N)}^\alpha \xrightarrow{\sim} S_\alpha \cap \overline{\mathcal{W}}_0^\alpha$ and the natural action of $\text{Stab}_{\text{PGL}(2N)}(\alpha) = \text{PGL}(N, N) := \text{GL}(N) \times \text{GL}(N) / \mathbb{C}^\times$ on $\check{Z}_{\text{SL}(2N)}^\alpha$, where \mathbb{C}^\times is the diagonal central subgroup of $\text{GL}(N) \times \text{GL}(N)$. It coincides with the natural action of $\text{GL}(N) \times \text{GL}(N)$ on $T^* \text{GL}(N)$ through the quotient homomorphism $\text{GL}(N) \times \text{GL}(N) \rightarrow \text{PGL}(N, N)$. By example A.5 this action coincides with the one given in Appendix A. More precisely the $\text{PGL}(N, N)$ action on $\overline{\mathcal{W}}_0^\alpha$ coincides with the one given in Appendix A, and the embedding $S_\alpha \cap \overline{\mathcal{W}}_0^\alpha \rightarrow \overline{\mathcal{W}}_0^\alpha$ is equivariant for both actions, as it is given by [Quiver, Remark 3.11] as Coulomb branches.

By [Quiver, Remark 3.5] $H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{GL}(V), \mathbf{N}}) \rightarrow H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}})$ is nothing but the restriction to the level set $F_\alpha^{-1}(1)$, where F_α is the boundary function (see [Quiver, §2(i)]). In this particular case, we have $F_\alpha = c \det$ for $c \in \mathbb{C}^\times$: all the invertible regular function on \check{Z}^α are of the form cF_α^k , $k \in \mathbb{Z}$, $c \in \mathbb{C}^\times$ [BDF16, Lemma 5.4]. Now by degree reasons, $\det = cF_\alpha$. Therefore

$$H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}}) \cong \mathfrak{gl}(N) \times \text{SL}(N).$$

Moreover, the morphism

$$H_*^{\text{PGL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}}) \rightarrow H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}})$$

is nothing but the projection $\mathfrak{gl}(N) \times \text{SL}(N) \rightarrow \text{Lie PGL}(N) \times \text{SL}(N)$. Identifying $\text{Lie PGL}(N)$ with $\mathfrak{sl}(N)^*$ via the Killing form, we get

$$\mathcal{M}_C(\text{PGL}(V), \mathbf{N}) \cong T^* \text{SL}(N).$$

This is the symplectic variety associated with a cylinder as expected.

Let us check how the action in Appendix A is affected by the replacement $\mathrm{GL}(V) \rightarrow \mathrm{PGL}(V)$. The semisimple Lie algebra $\mathfrak{g}_{\mathrm{ss}}^{\mathrm{bal}}$ remains the same: the variety $\mathcal{R}_{\mathrm{PGL}(V), \mathbf{N}}$ is obtained from $\mathcal{R}_{\mathrm{GL}(V), \mathbf{N}}$ by identifying isomorphic connected components. Therefore the construction of [Quiver, Appendix B] applies. On the other hand $\pi_1(\mathrm{GL}(V)) \cong \bigoplus_{i \in Q_0} \mathbb{Z} \alpha_i$ is replaced by $\pi_1(\mathrm{PGL}(V)) \cong \pi_1(\mathrm{GL}(V)) / \mathbb{Z}(\sum_{i \in Q_0} \dim V_i \alpha_i)$. The root datum is $R^{\mathrm{bal}} \subset \pi_1(\mathrm{PGL}(V))$, $R^{\mathrm{bal}^\vee} \subset \pi_1(\mathrm{PGL}(V))^\vee$. Thus $\mathrm{PGL}(N, N)$ is replaced by its subgroup $\mathrm{PGL}(N, N)' \stackrel{\mathrm{def.}}{=} \{[g_1, g_2] \mid \det g_1 = \det g_2\}$. We have $\mathrm{SL}(N) \times \mathrm{SL}(N) \twoheadrightarrow \mathrm{PGL}(N, N)'$ with kernel $\mathbb{Z}/N\mathbb{Z}$, the diagonal central subgroup. The standard action on $T^* \mathrm{SL}(N)$ coincides with the one given in Appendix A.

On the other hand, if we replace the central $\mathrm{GL}(N)$ by $\mathrm{SL}(N)$, the corresponding Coulomb branch is the hamiltonian reduction of $T^* \mathrm{GL}(N)$ with respect to the \mathbb{C}^\times -action corresponding to $\pi_1(\mathrm{GL}(N)) \cong \mathbb{Z}$. (See [Part II, Prop. 3.18].) In this case \mathbb{C}^\times -action is the scalar multiplication on $T^* \mathrm{GL}(N)$ ([Quiver, Remark 3.2]), hence the reduction is $T^* \mathrm{PGL}(N)$ as expected.

5(ii). Disk

The variety for the disk is calculated as for the cylinder. We consider a quiver gauge theory of type A with $\dim V = (1, 2, \dots, N)$. As we remarked in the proof of Theorem 4.13, the Coulomb branch is $\check{Z}_{\mathrm{PGL}(N+1)}^\alpha \simeq \mathrm{GL}(N) \times \mathbb{C}^N$, where \mathbb{C}^N is identified with the Kostant slice for $\mathrm{GL}(N)$, and $\alpha = (N + 1)\omega_N$. By [Quiver, Remark 3.12], we have an isomorphism $\check{Z}_{\mathrm{PGL}(N+1)}^\alpha \xrightarrow{\sim} S_\alpha \cap \overline{W}_0^\alpha$ and the natural action of $\mathrm{Stab}_{\mathrm{PGL}(N+1)}(\alpha) = \mathrm{GL}(N)$ on $\check{Z}_{\mathrm{PGL}(N+1)}^\alpha$ coinciding with the natural action of $\mathrm{GL}(N)$ on $\mathrm{GL}(N) \times \mathbb{C}^N$ (trivial on \mathbb{C}^N and by left shifts on $\mathrm{GL}(N)$). By example A.5 this action coincides with the one given in Appendix A.

The modification to cases $\mathrm{SL}(N)$, $\mathrm{PGL}(N)$ are similar to the above.

5(iii). S^2 with three punctures for $\mathrm{SL}(2)$

We next consider the Higgs branch of the Sicilian theory of type $\mathrm{SL}(2)$ associated with S^2 with three punctures. The mirror quiver gauge theory is of type D_4 .

We consider the D_4 quiver with the central vertex 1 and other vertices 2, 3, 4. We orient the edges from the central vertex. We take $V_1 = \mathbb{C}^2$, $V_2 =$

$V_3 = V_4 = \mathbb{C}$. The diagonal central subgroup $Z = \mathbb{C}^\times \subset \mathrm{GL}(V)$ acts trivially on $\mathbf{N} = \bigoplus_{i=2}^4 \mathrm{Hom}(V_1, V_i)$, so the action of $\mathrm{GL}(V)$ factors through $\mathrm{PGL}(V) := \mathrm{GL}(V)/Z$. We will prove $\mathcal{M}_C(\mathrm{PGL}(V), \mathbf{N}) \simeq \mathbb{A}^8$.

According to [Quiver, Thm. 3.1], $\mathcal{M}_C(\mathrm{GL}(V), \mathbf{N}) \simeq \mathring{Z}^\alpha$, the moduli space of degree α based maps from \mathbb{P}^1 to the flag variety \mathcal{B} of the simply connected group $G = \mathrm{Spin}(8)$ of type D_4 . Here $\alpha = 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ is the highest coroot. Note that $\alpha = \omega_1$ is a fundamental coweight. We also consider the transversal slice $s_0^\alpha: \overline{\mathcal{W}}_{G,0}^\alpha \rightarrow Z^\alpha$ (see [Quiver, §2(ii)]; note that $-w_0 = \mathrm{Id}$ for our G). It is the moduli space of the data $(\mathcal{P}_{\mathrm{triv}} \xrightarrow{\sigma} \mathcal{P})$ where $\mathcal{P}_{\mathrm{triv}}$ is the trivialized G -bundle on \mathbb{P}^1 , and σ is an isomorphism on $\mathbb{P}^1 \setminus \{0\}$ with a trivial G -bundle \mathcal{P} possessing a degree α pole at $0 \in \mathbb{P}^1$. We consider an open subset $U \subset \overline{\mathcal{W}}_{G,0}^\alpha$ formed by the data $(\mathcal{P}_{\mathrm{triv}} \xrightarrow{\sigma} \mathcal{P})$ such that the transformation of the (unique) degree 0 complete flag in \mathcal{P} with value B_- at $\infty \in \mathbb{P}^1$ viewed as a generalized B -structure in $\mathcal{P}_{\mathrm{triv}}$ acquires no defect at $0 \in \mathbb{P}^1$. We have $s_0^\alpha: U \xrightarrow{\sim} \mathring{Z}^\alpha$. We also have another open subset $U' \subset \overline{\mathcal{W}}_{G,0}^\alpha$ formed by the data $(\mathcal{P}_{\mathrm{triv}} \xrightarrow{\sigma} \mathcal{P})$ such that the transformation of the (unique) degree 0 complete flag in $\mathcal{P}_{\mathrm{triv}}$ with value B_- at $\infty \in \mathbb{P}^1$ viewed as a generalized B -structure in \mathcal{P} acquires no defect at $0 \in \mathbb{P}^1$. This open subset U' is nothing but the intersection of $\overline{\mathcal{W}}_{G,0}^\alpha$ with the semiinfinite orbit $T_{-\alpha} \subset \mathrm{Gr}_G$. Since the trivialization of \mathcal{P} at $\infty \in \mathbb{P}^1$ (arising from σ) uniquely extends to the trivialization of \mathcal{P} over the whole of \mathbb{P}^1 , we obtain an involution $\iota: \overline{\mathcal{W}}_{G,0}^\alpha \xrightarrow{\sim} \overline{\mathcal{W}}_{G,0}^\alpha$ reversing the roles of \mathcal{P} and $\mathcal{P}_{\mathrm{triv}}$ and replacing σ by σ^{-1} . We have $\iota: U \xrightarrow{\sim} U'$. Thus we obtain an isomorphism $s_0^\alpha \circ \iota: U' \xrightarrow{\sim} \mathring{Z}^\alpha$.

Since α is the highest coroot, $\overline{\mathcal{W}}_{G,0}^\alpha$ is isomorphic to the minimal nilpotent orbit closure $\mathcal{N}_{\min} = 0 \sqcup \mathcal{O}_{\min} \subset \mathfrak{g}$, see [BeiDr, 4.5.12, page 182] or [MOV05, Lemma 2.10] for a published account. The projectivization of \mathcal{N}_{\min} is the partial flag variety $\mathcal{B}_\alpha = G/P_{234}$: the quotient with respect to a submaximal parabolic subgroup. Thus we have a \mathbb{C}^\times -bundle $p: \mathcal{O}_{\min} \rightarrow \mathcal{B}_\alpha$. The big Bruhat cell (the open B_- -orbit) $C \subset \mathcal{B}_\alpha$ is the free orbit of the unipotent radical U_{234}^- of P_{234}^- . Via the exponential map, $U_{234}^- \simeq \mathfrak{u}_{234}^-$, the nilpotent radical of the Lie algebra of P_{234}^- . For the Levi subgroup $L_{234} \subset P_{234}^-$ we have $[L_{234}, L_{234}] \simeq \mathrm{SL}(2)_2 \times \mathrm{SL}(2)_3 \times \mathrm{SL}(2)_4$, and \mathfrak{u}_{234}^- as a $[L_{234}, L_{234}]$ -module is isomorphic to $\mathbb{C}_2^2 \otimes \mathbb{C}_3^2 \otimes \mathbb{C}_4^2 \oplus \mathbb{C}_2^1 \otimes \mathbb{C}_3^1 \otimes \mathbb{C}_4^1$. Note that the center $Z(\mathrm{SL}(2)_2 \times \mathrm{SL}(2)_3 \times \mathrm{SL}(2)_4) = (\mathbb{Z}/2\mathbb{Z})^3$ has a natural projection onto $\mathbb{Z}/2\mathbb{Z}$ (the sum of coordinates). The kernel K of this projection, as a subgroup of $\mathrm{Spin}(8)$, coincides with the center $Z(\mathrm{Spin}(8))$. The action of $\mathrm{SL}(2)_2 \times \mathrm{SL}(2)_3 \times \mathrm{SL}(2)_4$ on \mathfrak{u}_{234}^- factors through the action of $\overline{L}_{234} := (\mathrm{SL}(2)_2 \times \mathrm{SL}(2)_3 \times \mathrm{SL}(2)_4)/K$.

Finally, we have $U' = p^{-1}(C)$. Thus we obtain a projection (a \mathbb{C}^\times -bundle) $p \circ (s_0^\alpha \circ \iota)^{-1}: \mathring{Z}^\alpha \rightarrow C$. This action of \mathbb{C}^\times is nothing but the composition of the natural T -action (Cartan torus $T = B \cap B_-$) with the cocharacter $\alpha_1: \mathbb{C}^\times \rightarrow T$. The boundary equation $F_\alpha: \mathring{Z}^\alpha \rightarrow \mathbb{C}^\times$ has weight 1 with respect to $\mathbb{C}^\times \xrightarrow{\alpha_1} T$ [BF14, Prop. 4.4]. It follows that $(p \circ (s_0^\alpha \circ \iota)^{-1}, F_\alpha): \mathring{Z}^\alpha \xrightarrow{\sim} C \times \mathbb{C}^\times$. The action of \bar{L}_{234} on $\mathring{Z}^\alpha = C \times \mathbb{C}^\times$ is via the above action on C . Note that \bar{L}_{234} is nothing but $[L^{\text{bal}}, L^{\text{bal}}] \subset \text{PSO}(8)$ of Appendix A, and the action of \bar{L}_{234} on $\mathcal{M}_C(\text{GL}(V), \mathbf{N})$ coincides with the action constructed in Appendix A by example A.5.

Now the surjection

$$\mathbb{C}[\mathring{Z}^\alpha] = \mathbb{C}[\mathcal{M}_C(\text{GL}(V), \mathbf{N})] = H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{GL}(V), \mathbf{N}}) \rightarrow H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}})$$

is nothing but the restriction to the level set $F_\alpha^{-1}(1)$ (see [Quiver, Remark 3.5]), hence $\text{Spec } H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}}) \simeq C$. Furthermore, the embedding

$$\begin{aligned} \mathbb{C}[\mathcal{M}_C(\text{PGL}(V), \mathbf{N})] &= H_*^{\text{PGL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}}) \\ &\hookrightarrow H_*^{\text{GL}(V)\circ}(\mathcal{R}_{\text{PGL}(V), \mathbf{N}}) = \mathbb{C}[F_\alpha^{-1}(1)] \end{aligned}$$

is nothing but the embedding of the ring of functions invariant with respect to the translations action of \mathbb{G}_a on \mathring{Z}^α . Here we view \mathbb{G}_a as a subgroup of automorphisms of \mathbb{P}^1 preserving $\infty \in \mathbb{P}^1$; its action on \mathring{Z}^α preserves the boundary equation F^α and its level set $F_\alpha^{-1}(1)$. In terms of the identification $F_\alpha^{-1}(1) \simeq C \simeq \mathfrak{u}_{\bar{2}34} \simeq \mathbb{C}_2^2 \otimes \mathbb{C}_3^2 \otimes \mathbb{C}_4^2 \oplus \mathbb{C}$, the action of \mathbb{G}_a is nothing but the action of the last summand \mathbb{C} , and hence $\mathcal{M}_C(\text{PGL}(V), \mathbf{N}) = F_\alpha^{-1}(1)/\mathbb{G}_a \simeq \mathbb{C}_2^2 \otimes \mathbb{C}_3^2 \otimes \mathbb{C}_4^2$.

The above action of $\bar{L}_{234} = [L^{\text{bal}}, L^{\text{bal}}]$ on $\mathcal{M}_C(\text{GL}(V), \mathbf{N})$ induces its action on $\mathcal{M}_C(\text{PGL}(V), \mathbf{N})$. One can also see that \bar{L}_{234} is the reductive group corresponding to the root datum $R^{\text{bal}} \subset \pi_1(\text{PGL}(V))$, $R^{\text{bal}^\vee} \subset \pi_1(\text{PGL}(V))^\vee$ via $\pi_1(\text{GL}(V)) \rightarrow \pi_1(\text{PGL}(V))$ as in §5(i).

Remark 5.5. Let us give another argument, which the third named author was taught by Amihay Hanany.

Let us consider functions $E_1^{(1)}, F_1^{(1)}$ for the middle vertex 1 by Remark A.7. Since $\langle \alpha, \alpha_1 \rangle = 1$, we have the action of \mathbb{G}_a^2 by integrating hamiltonian vector fields $H_{E_1^{(1)}}, H_{F_1^{(1)}}$. We combine it with the action of $\text{SL}(2)_2 \times \text{SL}(2)_3 \times \text{SL}(2)_4$. Let us consider the Lie subalgebra of $\mathbb{C}[\mathcal{M}_C(\text{PGL}(V), \mathbf{N})]$ generated by $E_i^{(1)}, F_i^{(1)}$ ($i = 1, 2, 3, 4$). Viewing $(\text{PGL}(V), \mathbf{N})$ as a framed quiver gauge theory of type A_3 with $\dim V = 1 \ 2 \ 1$, $\dim W = 0 \ 1 \ 0$, we see

$\mathcal{R}_{\mathrm{GL}(V),\mathbb{N}}$ by identifying isomorphic connected components. Therefore the construction of [Quiver, Appendix B] applies. The computation of Poisson brackets $\{H_i^{(1)}, E_0^{(1)}\}, \{H_i^{(1)}, F_0^{(1)}\}$ remains unchanged by the replacement $\mathrm{GL}(V) \rightarrow \mathrm{PGL}(V)$, hence we conclude that $E_0^{(1)}, F_0^{(1)}$ are root vectors corresponding to the highest weight of $\mathfrak{l} = \mathfrak{e}_6$. It also follows that $E_0^{(1)}, F_0^{(1)}$ together with $E_6^{(1)}, F_6^{(1)}$ generate an additional $\mathfrak{sl}(3)$, and $\mathrm{SL}(3)$.

We have the \mathfrak{S}_3 -action on \mathcal{M}_C induced by permutation of three legs. From the above consideration, it is clear that it corresponds to \mathfrak{S}_3 of automorphisms of \mathfrak{e}_6 exchanging root subspaces corresponding to the highest weight and two remaining special vertices. (See [Kac90, Thm. 8.6] for the detail of the construction of automorphisms.)

5(v). Torus with one puncture for $\mathrm{SL}(3)$

We consider the Higgs branch of the Sicilian theory of type $\mathrm{SL}(3)$ associated with a torus with one puncture. According to Conjecture 5.2 the mirror quiver gauge theory is $1 \rightarrow 2 \rightarrow 3 \hookrightarrow$, where numbers are dimensions (and we use them also for indices of vertices). Note that we have an edge loop at the vertex 3. Let us denote the Coulomb branch of this quiver gauge theory by \mathcal{M} . The following result is informed to the third-named author by Amihay Hanany. (It is based on an earlier observation in [GR12, §2.1], [CHMZ14b, (3.3.2)].)

Proposition 5.6. *\mathcal{M} is isomorphic to the subregular orbit closure $\overline{\mathcal{O}}_{\mathrm{subreg}} \subset \mathfrak{g}_2$.*

Proof. Let us first construct the action of G_2 .

We consider operators $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ ($i = 1, 2, 3$) as in Lemma A.2. The vertices 1, 2 are balanced, while 3 is *not*. But we still have $\deg E_3^{(1)}, F_3^{(1)} = 1$ by [Quiver, (A.4)]. Let us consider the Lie subalgebra \mathfrak{g} of $\mathbb{C}[\mathcal{M}]$ generated by these operators. By Appendix A(ii) $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ ($i = 1, 2$) define the Lie algebra $\mathfrak{sl}(3)$, and the corresponding hamiltonian vector fields are integrated to an $\mathrm{SL}(3)$ -action on \mathcal{M} .

The proof of commutation relations $\{E_i^{(1)}, F_3^{(1)}\} = 0 = \{F_i^{(1)}, E_3^{(1)}\}, \{H_1^{(1)}, E_3^{(1)}\} = 0 = \{H_1^{(1)}, F_3^{(1)}\}, \{H_2^{(1)}, E_3^{(1)}\} = -E_3^{(1)}, \{H_2^{(1)}, F_3^{(1)}\} = F_3^{(1)}$ in [Quiver, Appendix B] remains to work even though 3 has an edge loop. Similarly to [KN18, Lemma 6.8] we calculate $\{E_3^{(1)}, F_3^{(1)}\} = 2(3w_{2,1} + 3w_{2,2} - 2w_{3,1} - 2w_{3,2} - 2w_{3,3})$. Since $H_1^{(1)} = 2w_{1,1} - w_{2,1} - w_{2,2}$ and $H_2^{(1)} = 2w_{2,1} + 2w_{2,2} - w_{1,1} - w_{3,1} - w_{3,2} - w_{3,3}$, we conclude that $\{E_3^{(1)}, F_3^{(1)}\} = 4H_2^{(1)} +$

$2H_1^{(1)}$. Hence \mathfrak{g} is simple of type G_2 with Cartan subalgebra \mathfrak{h} spanned by $H_1^{(1)}$ and $H_2^{(1)}$, and $\mathfrak{sl}(3)$ is spanned by \mathfrak{h} and the long roots. Note that $E_3^{(1)}$ and $F_3^{(1)}$ generate the fundamental representations V and V^\vee of $\mathfrak{sl}(3)$, and $\mathfrak{g} = \mathfrak{sl}(3) \oplus V \oplus V^\vee$.

More concretely,

$$\begin{aligned} F_3^{(1)} &= u_{3,1} + u_{3,2} + u_{3,3}, \quad \text{and} \\ E_3^{(1)} &= (w_{3,1} - w_{2,1})(w_{3,1} - w_{2,2})u_{3,1}^{-1} + (w_{3,2} - w_{2,1})(w_{3,2} - w_{2,2})u_{3,2}^{-1} \\ &\quad + (w_{3,3} - w_{2,1})(w_{3,3} - w_{2,2})u_{3,3}^{-1}, \quad \text{and} \\ E_2^{(1)} &= (w_{2,1} - w_{1,1})(w_{2,2} - w_{2,1})^{-1}u_{2,1}^{-1} \\ &\quad + (w_{2,2} - w_{1,1})(w_{2,1} - w_{2,2})^{-1}u_{2,2}^{-1}, \quad \text{and} \\ F_2^{(1)} &= (w_{2,1} - w_{3,1})(w_{2,1} - w_{3,2})(w_{2,1} - w_{3,3})(w_{2,2} - w_{2,1})^{-1}u_{2,1} \\ &\quad + (w_{2,2} - w_{3,1})(w_{2,2} - w_{3,2})(w_{2,2} - w_{3,3})(w_{2,1} - w_{2,2})^{-1}u_{2,2}, \end{aligned}$$

while $F_1^{(1)} = (w_{1,1} - w_{2,1})(w_{1,1} - w_{2,2})u_{1,1}$, and $E_1^{(1)} = u_{1,1}^{-1}$.

Now consider an auxiliary quiver gauge theory of type D_4 with 1-dimensional framing at the middle vertex numbered by 2, and the outer vertices numbered by (3, 1), (3, 2), (3, 3). We take $\dim V_{3,1}'' = \dim V_{3,2}'' = \dim V_{3,3}'' = 1$, $\dim V_2'' = 2$ (and $\dim W_2'' = 1$). We know that $\mathcal{M}_C(\mathrm{GL}(V''), \mathbf{N}'') = \mathcal{N}_{\min}$ is the closure of the minimal orbit in \mathfrak{so}_8 . On the other hand, we consider a quiver gauge theory of the affine type D_4 with the extra vertex numbered by 1, $\dim V_1 = 1$, and all the other dimensions as before, but no framing. We denote the corresponding graded vector space by $V' = V'' \oplus V_1$, and the corresponding representation of $\mathrm{GL}(V')$ by \mathbf{N}' . Then $\mathcal{M}_C(\mathrm{GL}(V''), \mathbf{N}'') = \mathcal{M}_C(\mathrm{PGL}(V'), \mathbf{N}')$. We have an embedding

$$\begin{aligned} &\mathbb{C}[\mathcal{M}_C(\mathrm{PGL}(V'), \mathbf{N}')] \\ \hookrightarrow &\mathbb{C}(w_{1,1}, w_{2,1}, w_{2,2}, w_{3,1}, w_{3,2}, w_{3,3}, u_{1,1}, u_{2,1}, u_{2,2}, u_{3,1}, u_{3,2}, u_{3,3})^{S_2} \end{aligned}$$

where the symmetric group S_2 acts by permuting $(w_{2,1}, u_{2,1})$ and $(w_{2,2}, u_{2,2})$. Also we have an embedding

$$\mathbb{C}[\mathcal{M}] \hookrightarrow \mathbb{C}(w_{1,1}, w_{2,1}, w_{2,2}, w_{3,1}, w_{3,2}, w_{3,3}, u_{1,1}, u_{2,1}, u_{2,2}, u_{3,1}, u_{3,2}, u_{3,3})^{S_2 \times S_3}$$

where the symmetric group S_3 acts by permuting $(w_{3,1}, u_{3,1})$, $(w_{3,2}, u_{3,2})$, and $(w_{3,3}, u_{3,3})$. By inspection of [Quiver, (A.3), (A.5), and Thm. B.18] we check $F_2^{(1)} = {}'F_2^{(1)}$, $E_2^{(1)} = {}'E_2^{(1)}$, $F_3^{(1)} = {}'F_{3,1}^{(1)} + {}'F_{3,2}^{(1)} + {}'F_{3,3}^{(1)}$, $E_3^{(1)} = {}'E_{3,1}^{(1)} + {}'E_{3,2}^{(1)} + {}'E_{3,3}^{(1)}$ where $'E, 'F$ refer to the generators of \mathfrak{so}_8 in $\mathbb{C}[\mathcal{M}_C(\mathrm{PGL}(V'), \mathbf{N}')]$,

while E, F refer to the generators of \mathfrak{g}_2 in the previous paragraph. Since the projection $\mathcal{M}_C(\mathrm{PGL}(V'), \mathbf{N}') = \mathcal{N}_{\min} \rightarrow \mathfrak{so}_8$ is an embedding, we conclude that the projection $\mathcal{M} \rightarrow \mathfrak{g}_2$ is generically an embedding. Hence the differential of the G_2 -action on \mathcal{M} is generically surjective, so \mathcal{M} has an open G_2 -orbit $\tilde{\mathbb{O}} \subset \mathcal{M}$ which is a nonramified cover of its image adjoint orbit $\mathbb{O} \subset \mathfrak{g}_2$.

Now the monopole formula for \mathcal{M} gives degrees in \mathbb{N} . Indeed, the contribution of a dominant coweight $\lambda = (\lambda_{1,1}, \lambda_{2,1} \geq \lambda_{2,2}, \lambda_{3,1} \geq \lambda_{3,2} \geq \lambda_{3,3})$ of $(\mathrm{GL}(1) \times \mathrm{GL}(2) \times \mathrm{GL}(3))/Z$ equals $\Delta(\lambda) = \lambda_{2,2} - \lambda_{2,1} + \frac{1}{2}|\lambda_{2,1} - \lambda_{1,1}| + \frac{1}{2}|\lambda_{2,2} - \lambda_{1,1}| + \frac{1}{2} \sum_{r,s} |\lambda_{2,r} - \lambda_{3,s}|$ which is easily seen to be nonnegative and integral. We conclude that \mathcal{M} is conical, and hence its image $\overline{\mathbb{O}} \subset \mathfrak{g}_2$ is conical as well. It follows that \mathbb{O} is a nilpotent orbit. But \mathfrak{g}_2 has a unique 10-dimensional nilpotent orbit: the subregular one. Hence $\mathbb{O} = \mathbb{O}_{\mathrm{subreg}}$.

Now we have to identify the cover $\tilde{\mathbb{O}} \rightarrow \mathbb{O}$. It is known that the universal cover of $\mathbb{O}_{\mathrm{subreg}}$ is an open piece of the minimal nilpotent orbit $\mathbb{O}_{\min} \subset \mathfrak{so}_8$, see e.g. [BK94], and the Galois group of this cover is S_3 . Moreover, the degree 1 functions on the universal cover constitute the Lie algebra \mathfrak{so}_8 . It follows that if the cover $\tilde{\mathbb{O}} \rightarrow \mathbb{O}$ corresponds to a subgroup $\pi_1 \subset S_3$, then the degree 1 functions on $\tilde{\mathbb{O}}$ constitute $\mathfrak{so}_8^{\pi_1}$. Since we know that the degree 1 functions on \mathcal{M} constitute $\mathfrak{g}_2 = \mathfrak{so}_8^{S_3}$, we conclude that $\pi_1 = S_3$, so that $\tilde{\mathbb{O}} \xrightarrow{\sim} \mathbb{O}_{\mathrm{subreg}}$. Finally, the normality property of the orbit closure $\overline{\mathbb{O}}_{\mathrm{subreg}}$ guarantees that $\mathcal{M} \xrightarrow{\sim} \overline{\mathbb{O}}_{\mathrm{subreg}}$. \square

Note that a torus with one puncture is obtained from S^2 with three punctures by gluing two punctures. We have computed the Higgs branch associated with the latter in §5(iv). The Higgs branch is the closure $\mathcal{N}_{\min}(\mathfrak{e}_6)$ of the minimal nilpotent orbit of \mathfrak{e}_6 . Therefore the Higgs branch \mathcal{M} for a torus with one puncture is the Hamiltonian reduction $\mathcal{N}_{\min}(\mathfrak{e}_6) // \Delta_{\mathrm{SL}(3)}$ with respect to the diagonal $\mathrm{SL}(3)$ in $\mathrm{SL}(3) \times \mathrm{SL}(3)$ corresponding to two legs which are glued. Therefore we have an action of the centralizer of $\Delta_{\mathrm{SL}(3)}$ in E_6 , which is G_2 . (See e.g., [Rub08, §3.2] and the references therein.) Combining with Proposition 5.6, we should have $\mathcal{N}_{\min}(\mathfrak{e}_6) // \Delta_{\mathrm{SL}(3)} \cong \overline{\mathbb{O}}_{\mathrm{subreg}}(\mathfrak{g}_2)$, the closure of the subregular nilpotent orbit of \mathfrak{g}_2 . We do not have a proof of this statement, though it might be known to an expert.

5(vi). Recollections on derived Satake equivalence

We consider a reductive group G with Langlands dual group G^\vee and its Lie algebra \mathfrak{g}^\vee . We have a commutative ring object $\mathcal{A}_R = \bigoplus_\lambda \mathrm{IC}(\overline{\mathrm{Gr}}^\lambda) \otimes (V^\lambda)^* \in D_G(\mathrm{Gr}_G)$.

Let $e, h, f \in \mathfrak{g}^\vee$ be a principal \mathfrak{sl}_2 -triple such that f is lower triangular, and e is upper triangular. We consider the Kostant slice $e + \mathfrak{z}(f)$ to the regular nilpotent orbit. Let Σ be the image of $e + \mathfrak{z}(f)$ under a G^\vee -invariant isomorphism $\mathfrak{g}^\vee \simeq (\mathfrak{g}^\vee)^*$. Let Υ be the image of $e + \mathfrak{b}_-^\vee$ (Borel subalgebra) under a G^\vee -invariant isomorphism $\mathfrak{g}^\vee \simeq (\mathfrak{g}^\vee)^*$. We have canonical isomorphisms $\Sigma = \mathfrak{t}/\mathbb{W} = \Upsilon/U_-^\vee$ (unipotent subgroup) by the compositions $\Sigma \hookrightarrow (\mathfrak{g}^\vee)^* \rightarrow (\mathfrak{g}^\vee)^*/G^\vee \cong \mathfrak{t}/\mathbb{W}$ and $\Sigma \hookrightarrow \Upsilon \rightarrow \Upsilon/U_-^\vee$.

According to [BF08, Thm. 5], there is an equivalence of monoidal triangulated categories $\Psi: D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee)) \rightarrow D_G(\mathrm{Gr}_G)$. Recall that $D_G(\mathrm{Gr}_G)$ stands for the Ind-completion of the bounded derived equivariant constructible category on Gr_G . Accordingly, $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ stands for the Ind-completion of the triangulated category $D_{\mathrm{perf}}^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ formed by the G^\vee -equivariant perfect dg-modules over $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$: the graded symmetric algebra of \mathfrak{g}^\vee where any element of \mathfrak{g}^\vee is assigned degree 2 (with trivial differential). The monoidal structure on $D_G(\mathrm{Gr}_G)$ is given by the convolution \star , and the monoidal structure on $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ is $M_1, M_2 \mapsto M_1 \otimes_{\mathrm{Sym}^\square(\mathfrak{g}^\vee)} M_2$. The algebra $\mathbb{C}[\Sigma] = \mathrm{Sym}(\mathfrak{g}^\vee)^{G^\vee}$ acts on $\mathrm{Ext}_{D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))}^*(M_1, M_2)$, and this action is compatible with the action of $\mathbb{C}[\Sigma] = H_G^*(\mathrm{pt})$ on

$$\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\Psi(M_1), \Psi(M_2)).$$

Since $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ is the homotopy category of a dg-category, we have $\mathrm{RHom}_{D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))}: D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee)) \times D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee)) \rightarrow D(\mathrm{Vect})$.

The functor Ψ^{-1} is uniquely characterized by the property that

$$\Psi^{-1}(\mathrm{IC}(\overline{\mathrm{Gr}}^\lambda)) = \mathrm{Sym}^\square(\mathfrak{g}^\vee) \otimes V^\lambda.$$

For $\mathcal{F} \in D_G(\mathrm{Gr}_G)$ we have [BF08, Thm. 2]:

$$(5.7) \quad H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{F}) = \kappa^l(\Psi^{-1}(\mathcal{F})) := H^*(\Psi^{-1}(\mathcal{F}) \otimes_{\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square)$$

in the following sense. The eigenvalues of $-h$ from the above \mathfrak{sl}_2 -triple define a grading of \mathfrak{g}^\vee , hence a grading of $(\mathfrak{g}^\vee)^*$ and a grading of $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$. Thus \mathfrak{g}^\vee , $(\mathfrak{g}^\vee)^*$ and $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$ acquire a bigrading such that the total degree of e is zero. We consider $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$ with the new grading given by the total degree and denote it $\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)$. Now the projection $\mathrm{Sym}(\mathfrak{g}^\vee) \rightarrow \mathbb{C}[\Sigma]$ is compatible with the new grading, and induces a grading on $\mathbb{C}[\Sigma]$ denoted $\mathbb{C}[\Sigma]^\square$. Finally, we consider both $\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)$ and $\mathbb{C}[\Sigma]^\square$ as dg-algebras with trivial differential (and zero components of odd degrees). Note that $\Psi^{-1}(\mathcal{F})$ is still a dg-module over $\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)$ due to its G^\vee -equivariance.

Let \mathbf{D} stand for the duality

$$M \mapsto \mathrm{RHom}_{\mathrm{Sym}^{\square}(\mathfrak{g}^{\vee})}(M, \mathrm{Sym}^{\square}(\mathfrak{g}^{\vee}))$$

in $D^{G^{\vee}}(\mathrm{Sym}^{\square}(\mathfrak{g}^{\vee}))$. Let \mathbb{D} stand for the Verdier duality in $D_G(\mathrm{Gr}_G)$. We denote by $\mathfrak{C}_{G^{\vee}}$ the autoequivalence of $D^{G^{\vee}}(\mathrm{Sym}^{\square}(\mathfrak{g}^{\vee}))$ induced by the canonical outer automorphism of G^{\vee} interchanging conjugacy classes of g and g^{-1} (the Chevalley involution). We denote by \mathcal{C}_G the autoequivalence of $D_G(\mathrm{Gr}_G)$ induced by $g \mapsto g^{-1}$, $G((z)) \rightarrow G((z))$. Then $\mathcal{C}_G \circ \Psi = \Psi \circ \mathfrak{C}_{G^{\vee}}$. According to [BF08, Lemma 14], we have $\Psi \circ \mathfrak{C}_{G^{\vee}} \circ \mathbf{D} = \mathbb{D} \circ \Psi$ and $\Psi \circ \mathbf{D} = \mathcal{C}_G \circ \mathbb{D} \circ \Psi$.

The following lemma is well known. (See [Gin95, §2.4]. Also the proof of [BF08, Lemma 14] depends on it.) Let us give its proof for completeness. Recall $\mathbf{1}_{\mathrm{Gr}_G}$ denotes the skyscraper sheaf at the base point in Gr_G .

Lemma 5.8. *$\mathcal{C}_G \circ \mathbb{D}$ is the rigidity for $(D_G(\mathrm{Gr}_G), \star)$. That is, for any $\mathcal{F}_1, \mathcal{F}_2 \in D_G(\mathrm{Gr}_G)$ we have a canonical isomorphism*

$$\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{F}_1 \star \mathcal{F}_2) \xrightarrow{\sim} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{C}_G \circ \mathbb{D}\mathcal{F}_1, \mathcal{F}_2).$$

Proof. For any group H , the convolution operation $\mathcal{F}_1 \star \mathcal{F}_2 = m_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ on $D(H)$ has rigidity $\mathcal{F} \mapsto \mathcal{C}_H \circ \mathbb{D}\mathcal{F}$ where \mathcal{C}_H is induced by the automorphism $h \mapsto h^{-1}$, $H \rightarrow H$. Namely,

$$\begin{aligned} \mathrm{RHom}(\mathbf{1}_H, \mathcal{F}_1 \star \mathcal{F}_2) &= i_e^!(\mathcal{F}_1 \star \mathcal{F}_2) = \mathrm{RHom}(\mathbb{C}_H, \nabla^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)) \\ &= \mathrm{RHom}(\mathbb{C}_H, \mathcal{C}_H\mathcal{F}_1 \otimes^! \mathcal{F}_2) = \mathrm{RHom}(\mathcal{C}_H \circ \mathbb{D}\mathcal{F}_1, \mathcal{F}_2), \end{aligned}$$

where $\nabla: H \hookrightarrow H \times H$ is the antidiagonal embedding $h \mapsto (h^{-1}, h)$.

We apply this to the category of $G_{\mathcal{O}}$ -left-right equivariant sheaves on $H = G_{\mathcal{K}}$.

More formally, let us use the six operations for constructible derived categories on Artin stacks. There is a reference [LO08] for $\overline{\mathbb{Q}}_l$ -coefficients. We choose an isomorphism $\overline{\mathbb{Q}}_l \cong \mathbb{C}$ and use it for complex coefficients. Our stack is $\mathcal{X} := G_{\mathcal{O}} \backslash \mathrm{Gr}_G$. It is the moduli stack of pairs $\mathcal{P}_1, \mathcal{P}_2$ of G -bundles on the formal disc D equipped with an isomorphism $\eta: \mathcal{P}_1|_{D^*} \xrightarrow{\sim} \mathcal{P}_2|_{D^*}$. There is an involution $\mathbf{i}: \mathcal{X} \rightarrow \mathcal{X}$ induced by the inversion $g \mapsto g^{-1}$ of $G_{\mathcal{K}}$. In modular terms, $\mathbf{i}(\mathcal{P}_1, \mathcal{P}_2, \eta) = (\mathcal{P}_2, \mathcal{P}_1, \eta^{-1})$. Recall that Gr_G is the moduli space of G -bundles \mathcal{P} on D equipped with an isomorphism $\sigma: \mathcal{P}_{\mathrm{triv}}|_{D^*} \xrightarrow{\sim} \mathcal{P}|_{D^*}$. We have a projection $\mathrm{pr}_2: \mathrm{Gr}_G \rightarrow \mathcal{X}$ sending (\mathcal{P}, σ) to $(\mathcal{P}_{\mathrm{triv}}, \mathcal{P}, \sigma)$. Similarly, we define ${}_{G} \mathrm{Gr}$ as the moduli space of G -bundles \mathcal{P} on D equipped with an isomorphism $\tau: \mathcal{P}|_{D^*} \xrightarrow{\sim} \mathcal{P}_{\mathrm{triv}}|_{D^*}$. We have a projection $\mathrm{pr}_1: {}_{G} \mathrm{Gr} \rightarrow \mathcal{X}$

sending (\mathcal{P}, τ) to $(\mathcal{P}, \mathcal{P}_{\text{triv}}, \tau)$. We have isomorphisms $\mathbf{i}: \text{Gr}_G \xrightarrow{\sim} {}_G\text{r}G$ and $\mathbf{i}: {}_G\text{r}G \xrightarrow{\sim} \text{Gr}_G$ sending σ to $\tau = \sigma^{-1}$ and τ to $\sigma = \tau^{-1}$. Obviously, $\mathbf{i} \text{pr}_1 = \text{pr}_2 \mathbf{i}$ and $\mathbf{i} \text{pr}_2 = \text{pr}_1 \mathbf{i}$. There is a morphism $\mathbf{m}: {}_G\text{r}G \times \text{Gr}_G \rightarrow \mathcal{X}$ induced by the multiplication in $G_{\mathcal{K}}$. In modular terms, $\mathbf{m}(\mathcal{P}_1, \tau; \mathcal{P}_2, \sigma) = (\mathcal{P}_1, \mathcal{P}_2, \sigma \circ \tau)$. The convolution on $D(\mathcal{X})$ is defined as $\mathcal{F}_1 \star \mathcal{F}_2 := \mathbf{m}_*(\text{pr}_1^* \mathcal{F}_1 \boxtimes \text{pr}_2^* \mathcal{F}_2)$. The unit object $\mathbf{1}$ is $\mathbb{C}_{G_{\mathcal{O}} \setminus G_{\mathcal{R}}}$. We claim that the rigidity is $\mathbf{i}^* \circ \mathbb{D}$. Indeed,

$$\begin{aligned} \text{RHom}_{\mathcal{X}}(\mathbf{1}, \mathcal{F}_1 \star \mathcal{F}_2) &= i_0^!(\mathcal{F}_1 \star \mathcal{F}_2) \\ &= \text{RHom}_{G_{\mathcal{O}} \setminus G_{\mathcal{R}}}(\mathbb{C}_{G_{\mathcal{R}}}, \nabla^!(\text{pr}_1^* \mathcal{F}_1 \boxtimes \text{pr}_2^* \mathcal{F}_2)) \\ &= \text{RHom}_{G_{\mathcal{O}} \setminus G_{\mathcal{R}}}(\mathbb{C}_{G_{\mathcal{R}}}, \mathbf{i}^* \text{pr}_1^* \mathcal{F}_1 \otimes^! \text{pr}_2^* \mathcal{F}_2) \\ &= \text{RHom}_{G_{\mathcal{O}} \setminus G_{\mathcal{R}}}(\mathbf{i}^* \circ \mathbb{D} \text{pr}_1^* \mathcal{F}_1, \text{pr}_2^* \mathcal{F}_2) \\ &= \text{RHom}_{\mathcal{X}}(\mathbf{i}^* \circ \mathbb{D} \mathcal{F}_1, \mathcal{F}_2), \end{aligned}$$

where $\nabla: \text{Gr}_G \rightarrow {}_G\text{r}G \times \text{Gr}_G$ is (\mathbf{i}, id) . □

5(vii). Regular sheaf and derived Satake equivalence

Under the equivalence Ψ^{-1} , the ring object $\mathcal{A}_R \in D_G(\text{Gr}_G)$ corresponds to the G^\vee -equivariant free $\text{Sym}^\square(\mathfrak{g}^\vee)$ -module $\mathbb{C}[G^\vee] \otimes \text{Sym}^\square(\mathfrak{g}^\vee)$ which will be denoted $\mathbb{C}[T^*G^\vee]^\square$ for short. The G^\vee -action comes from the *left* action of G^\vee on $T^*G^\vee = G^\vee \times (\mathfrak{g}^\vee)^*$, $g_1(g_2, \xi) = (g_1g_2, \xi)$. And the action of $\text{Sym}^\square(\mathfrak{g}^\vee)$ on $\mathbb{C}[T^*G^\vee]^\square$ comes from the morphism $\mu_l: T^*G^\vee \rightarrow (\mathfrak{g}^\vee)^*$, $(g, \xi) \mapsto \text{Ad}_g \xi$ (the moment map of the left action). Recall that \mathcal{A}_R is equipped with an action of G^\vee . Under the equivalence Ψ^{-1} , this action goes to the action on $\mathbb{C}[T^*G^\vee]^\square$ coming from the *right* action of G^\vee on $T^*G^\vee = G^\vee \times (\mathfrak{g}^\vee)^*$, $g_1 \cdot (g_2, \xi) = (g_2g_1^{-1}, \text{Ad}_{g_1} \xi)$. For this reason the action of G^\vee on \mathcal{A}_R will be called the *right* action. The moment map of the right G^\vee -action on T^*G^\vee is $\mu_r: T^*G^\vee \rightarrow (\mathfrak{g}^\vee)^*$, $(g, \xi) \mapsto \xi$.

Also note that $\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R)$ is a formal dg-algebra (since e.g. $\mathbb{C}[T^*G^\vee]^\square$ is a free $\text{Sym}^\square(\mathfrak{g}^\vee)$ -module), so μ_r gives rise to a G^\vee -equivariant morphism of dg-algebras

$$(5.9) \quad \text{Sym}^\square(\mathfrak{g}^\vee) \rightarrow \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R).$$

Altogether we have the action of $G^\vee \ltimes \text{Sym}^\square(\mathfrak{g}^\vee)$ on \mathcal{A}_R that will be called the *right* action.

Remark 5.10. For $\mathcal{F}_1, \mathcal{F}_2 \in D_G(\text{Gr}_G)$ we distinguish $\text{Ext}_{D_G(\text{Gr}_G)}^*(\mathcal{F}_1, \mathcal{F}_2)$ and $\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{F}_1, \mathcal{F}_2)$. They are isomorphic in $D(\text{Vect})$, the derived

category of vector spaces, which is equivalent to Vect^{gr} , the category of graded vector spaces. But when we consider additional structures, such as a dg-algebra structure or a structure of a dg-module over $G^\vee \times \text{Sym}^\square(\mathfrak{g}^\vee)$, they are not isomorphic. We thus understand

$$\text{Ext}_{D_G(\text{Gr}_G)}^*(\mathcal{F}_1, \mathcal{F}_2) = H^*(\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{F}_1, \mathcal{F}_2)).$$

Definition 5.11. The morphism (5.9) $\text{Sym}^\square(\mathfrak{g}^\vee) \rightarrow \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R)$ induces, for any $\mathcal{F} \in D_G(\text{Gr}_G)$, the composed morphism

$$\text{Sym}^\square(\mathfrak{g}^\vee) \rightarrow \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R) \rightarrow \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R \otimes^! \mathcal{F}, \mathcal{A}_R \otimes^! \mathcal{F})$$

of dg-algebras. Also, the morphism (5.9) induces, for any $\mathcal{F} \in D_G(\text{Gr}_G)$, the composed morphism

$$\begin{aligned} & \text{Sym}^\square(\mathfrak{g}^\vee) \otimes \text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F}) \\ & \rightarrow \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R) \otimes \text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F}) \\ & \rightarrow \text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F}) \end{aligned}$$

of complexes of vector spaces. This morphism is G^\vee -equivariant for the G^\vee -action on $\text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F})$ induced by the right G^\vee -action on \mathcal{A}_R . Thus, the complex $\text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F})$ acquires the structure of an object of $D^{G^\vee}(\text{Sym}^\square(\mathfrak{g}^\vee))$, and $\text{Ext}_{D_G(\text{Gr}_G)}^*(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R \otimes^! \bullet)$ gets upgraded to the functor

$$\text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R \otimes^! \bullet): D_G(\text{Gr}_G) \rightarrow D^{G^\vee}(\text{Sym}^\square(\mathfrak{g}^\vee)).$$

Similarly, the dg-modules

$$\begin{aligned} & \text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{D}\mathcal{A}_R, \mathcal{F}), \\ & \text{RHom}_{D_G(\text{Gr}_G)}(\mathbf{1}_{\text{Gr}_G}, \mathbb{D}\mathcal{A}_R \star \mathcal{F}), \\ & \text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \mathcal{F}), \text{ etc.} \end{aligned}$$

all acquire the structures of objects of $D^{G^\vee}(\text{Sym}^\square(\mathfrak{g}^\vee))$, and $\text{Ext}_{D_G(\text{Gr}_G)}^*(\mathcal{A}_R, \bullet)$ gets upgraded to the functor

$$\text{RHom}_{D_G(\text{Gr}_G)}(\mathcal{A}_R, \bullet): D_G(\text{Gr}_G) \rightarrow D^{G^\vee}(\text{Sym}^\square(\mathfrak{g}^\vee)).$$

Lemma 5.12. *From Definition 5.11 we obtain an action of $(G^\vee)^b \times \text{Sym}^\square(\mathfrak{g}^\vee)^{\otimes b}$ on $\text{RHom}_{D_G(\text{Gr}_G)}(\mathbb{C}_{\text{Gr}_G}, \mathcal{A}_R^{\otimes b})$. The resulting dg-module over $(G^\vee)^b \times \text{Sym}^\square(\mathfrak{g}^\vee)^{\otimes b}$ is formal.*

Proof. Using [BF08, Prop. 5] we reformulate the claim for $D_G(\mathrm{Gr}_G)$ replaced by $D_{G_{\overline{\mathbb{F}}_q}}(\mathrm{Gr}_{G, \overline{\mathbb{F}}_q})$. Then \mathcal{A}_R is pointwise pure (meaning that all its costalks are pure with respect to the Frobenius action). Hence $\mathcal{A}_R^{\otimes^1 b}$ is also pointwise pure. Then the Cousin spectral sequence for the Schubert stratification of Gr_G shows that $H_{G_{\mathcal{O}, \overline{\mathbb{F}}_q}}^*(\mathrm{Gr}_{G, \overline{\mathbb{F}}_q}, \mathcal{A}_R^{\otimes^1 b})$ is pure. Also, $\mathrm{Ext}_{D_{G_{\overline{\mathbb{F}}_q}}(\mathrm{Gr}_{G, \overline{\mathbb{F}}_q})}^*(\mathcal{A}_R, \mathcal{A}_R)$ is pure. Now the argument of [BF08, §6.5] proves that the dg-algebra $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R)$ is formal as well as its b -th tensor power, and the dg-module $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R^{\otimes^1 b})$ over $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{A}_R, \mathcal{A}_R)^{\otimes b}$ is formal. \square

The Kostant-Whittaker (hamiltonian) reduction of T^*G^\vee with respect to the right action is $T^*G^\vee //_{U_-, \psi} := \mu_r^{-1}(\Upsilon)/U_-^\vee$. (We use $//$ for a hamiltonian reduction in order to avoid a conflict with $//$ for a GIT quotient.) At the level of dg-modules, $\kappa^r(\mathbb{C}[T^*G^\vee]^\square) := (\mathbb{C}[\mu_r^{-1}(\Upsilon)]^\square)^{U_-^\vee} = \mathbb{C}[\mu_r^{-1}(\Sigma)]^\square := \mathbb{C}[T^*G^\vee]^\square \otimes_{\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square$, tensor product with respect to the action of the right copy of $\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)$.⁶ (We have an isomorphism $U_-^\vee \times \Sigma \xrightarrow{\sim} \Upsilon$ given by the action of U_-^\vee on Υ . Hence $\mathbb{C}[\Upsilon]^{U_-^\vee} \cong \mathbb{C}[\Sigma]$. Moreover, for any U_-^\vee -equivariant sheaf \mathcal{F} on Υ , we have $\mathcal{F}^{U_-^\vee} = \mathcal{F}|_\Sigma$. Similarly, for a U_-^\vee -equivariant dg-module M over $\mathbb{C}[\Upsilon]^\square$ we have $M^{U_-^\vee} = M \otimes_{\mathbb{C}[\Upsilon]^\square} \mathbb{C}[\Sigma]^\square$.) This is an object of $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ (with respect to the residual *left* action of G^\vee) corresponding under the equivalence Ψ^{-1} to the dualizing complex ω_{Gr_G} [BF08, Prop. 4]. (In fact, [BF08, Prop. 4] is proved for the extra equivariance under the loop rotations.) Instead of $\omega_{\mathrm{Gr}_G} = \Psi(\kappa^r(\Psi^{-1}\mathcal{A}_R))$, we will write $\omega_{\mathrm{Gr}_G} = \kappa^r(\mathcal{A}_R)$ for short.

Under the dualities \mathbf{D}, \mathbb{D} we have $\mathbf{D}\mathbb{C}[T^*G^\vee]^\square = \mathbb{C}[T^*G^\vee]^\square$, while $\mathbb{D}\mathcal{A}_R = \mathcal{C}_G\mathcal{A}_R$.

We define $\Phi := \mathfrak{C}_{G^\vee} \circ \Psi^{-1}: D_G(\mathrm{Gr}_G) \rightarrow D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$. We have $\Phi(\mathcal{A}_R) = \mathbb{C}[T^*G^\vee]^\square$.

Lemma 5.13. (a) *Let us define a functor $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathcal{A}_R, \bullet): D_G(\mathrm{Gr}_G) \rightarrow D(\mathrm{Vect}) = \mathrm{Vect}^{\mathrm{gr}}$ by*

$$\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathcal{A}_R, \mathcal{F}) := \bigoplus_{\lambda} \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathrm{IC}(\overline{\mathrm{Gr}}^\lambda) \otimes (V^\lambda)^*, \mathcal{F}).$$

⁶Passing to cohomology, we obtain the usual hamiltonian reduction:

$$H^*(\kappa^r(\mathbb{C}[T^*G^\vee]^\square)) = \mathbb{C}[T^*G^\vee] \otimes_{\mathrm{Sym}(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma] =: \kappa^r(\mathbb{C}[T^*G^\vee]).$$

We use the same notation κ^r for the hamiltonian reduction of the usual modules and of dg-modules. It is clear from the context which one is used in what follows.

It is canonically isomorphic to $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \bullet)$ by the rigidity together with $\mathcal{C}_G \mathbb{D} \mathcal{A}_R = \mathcal{A}_R$ (see Lemma 5.8). Then both $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathcal{A}_R, \bullet)$ and $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \bullet)$ are canonically isomorphic to the composition $\mathrm{Forg} \circ \Psi^{-1}$, where Forg is the forgetful functor $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee)) \rightarrow D(\mathrm{Vect}) = \mathrm{Vect}^{\mathrm{gr}}$.

Their upgrades $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{A}_R, \bullet)$ and $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \bullet) : D_G(\mathrm{Gr}_G) \rightarrow D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ (see Definition 5.11) are canonically isomorphic to Ψ^{-1} .

(b) For $\mathcal{F}_1, \mathcal{F}_2 \in D_G(\mathrm{Gr}_G)$, there are canonical isomorphisms

$$\begin{aligned} H_{G_O}^*(\mathrm{Gr}_G, \mathcal{F}_1 \otimes^! \mathcal{F}_2) &= \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{F}_1 \otimes^! \mathcal{F}_2) \\ &\cong \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{D} \mathcal{F}_1, \mathcal{F}_2) \\ &\cong \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{F}_1 \star \mathcal{F}_2) \end{aligned}$$

in $D(\mathrm{Vect}) = \mathrm{Vect}^{\mathrm{gr}}$.

(c) Let us define a functor $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{D} \mathcal{A}_R, \bullet) : D_G(\mathrm{Gr}_G) \rightarrow D(\mathrm{Vect}) = \mathrm{Vect}^{\mathrm{gr}}$ by

$$\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{D} \mathcal{A}_R, \mathcal{F}) := \bigoplus_{\lambda} \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathrm{IC}(\overline{\mathrm{Gr}}^\lambda) \otimes V^\lambda, \mathcal{F}).$$

It is canonically isomorphic to $\mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{A}_R \star \bullet)$ by the rigidity together with $\mathcal{C}_G \mathcal{A}_R = \mathbb{D} \mathcal{A}_R$ (see Lemma 5.8). We have canonical isomorphisms

$$\begin{aligned} \mathrm{Forg} \circ \Phi(\bullet) &\xrightarrow{\sim} \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{A}_R \star \bullet) \\ &\xrightarrow[\text{(b)}]{\sim} \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{D} \mathcal{A}_R, \bullet) \\ &\xrightarrow[\text{(b)}]{\sim} \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \bullet) \end{aligned}$$

of functors $D_G(\mathrm{Gr}_G) \rightarrow D(\mathrm{Vect}) = \mathrm{Vect}^{\mathrm{gr}}$.

The upgraded functors (see Definition 5.11)

$$\begin{aligned} \Phi &\xrightarrow{\sim} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{A}_R \star \bullet) \\ &\xrightarrow{\sim} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{D} \mathcal{A}_R, \bullet) \xrightarrow{\sim} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \bullet) \end{aligned}$$

are isomorphic as functors from $D_G(\mathrm{Gr}_G)$ to $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$.

Proof. (a) We consider $\mathbb{C}[T^*G^\vee]^\square$ as a $G^\vee \times G^\vee$ -equivariant $(\mathrm{Sym}^\square(\mathfrak{g}^\vee), \mathrm{Sym}^\square(\mathfrak{g}^\vee))$ bimodule by the left and right action. We have the canonical

matrix coefficient morphisms

$$(5.14) \quad M \xrightarrow[\varphi_M]{\cong} \mathrm{RHom}_{D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))}(\mathrm{Sym}^\square(\mathfrak{g}^\vee), \mathbb{C}[T^*G^\vee]^\square \otimes M),$$

where $G^\vee \times (\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ acts on $\mathbb{C}[T^*G^\vee]^\square$ by the left action, and the right hand side is regarded as an object in $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ by the residual right action on $\mathbb{C}[T^*G^\vee]^\square$. Here φ_M is defined as follows: Given a G^\vee -module M , we have $\varphi_M: M \otimes M^* \rightarrow \mathbb{C}[G^\vee]$ by $\varphi_M(m \otimes m^*)(g) := \langle gm, m^* \rangle$. It is a morphism of $G^\vee \times G^\vee$ -modules. By swapping M^* to the target, φ_M can be viewed as a morphism $\varphi_M: M \rightarrow \mathrm{Hom}_{G^\vee}(\mathbb{C}, \mathbb{C}[G^\vee] \otimes M)$. The morphism φ_M is an isomorphism. One can think about it as the usual fiber functor Forg on $\mathrm{Rep}(G^\vee)$ being represented by $\mathbb{C}[G^\vee]$. Its inverse φ_M^{-1} is given by the evaluation $\mathbb{C}[G^\vee] \rightarrow \mathbb{C}$ at $g = 1$. We consider this over $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$ to get (5.14). We now apply the derived Satake equivalence. We get

$$\begin{aligned} \Psi^{-1}(\mathcal{F}) &\xrightarrow[\varphi_{\Psi^{-1}(\mathcal{F})}]{\sim} \mathrm{RHom}_{D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))}(\mathrm{Sym}^\square(\mathfrak{g}^\vee), \mathbb{C}[T^*G^\vee]^\square \otimes \Psi^{-1}(\mathcal{F})) \\ &\xrightarrow{\sim} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes \mathcal{F}). \end{aligned}$$

The second isomorphism holds since the construction of Ψ^{-1} actually passes through dg-categories (as opposed to being defined at the level of derived categories), and is compatible with the action of the formal dg-algebra $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$. We have an isomorphism

$$\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{A}_R, \bullet) \cong \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \bullet)$$

by the rigidity plus $\mathcal{A}_R = \mathcal{C}_G \circ \mathbb{D}\mathcal{A}_R$.

(b) The first isomorphism is the rigidity for the monoidal category $(D_G(\mathrm{Gr}_G), \otimes^!)$, while the second is for $(D_G(\mathrm{Gr}_G), \star)$. See Lemma 5.8.

(c) The first isomorphism is a consequence of (a) together with $\mathbb{D}\mathcal{A}_R = \mathcal{C}_G\mathcal{A}_R$. The second and third are nothing but (b). In order to see that the second and third isomorphisms are upgraded to RHom , we observe that quasi-isomorphisms

$$\begin{aligned} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G\mathcal{A}_R \star \bullet) &\xrightarrow{\sim} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{D}\mathcal{A}_R, \bullet) \\ &\xrightarrow{\sim} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \bullet) \end{aligned}$$

are compatible with the action of $G^\vee \times \mathrm{Sym}^\square(\mathfrak{g}^\vee)$. □

Let us suppose $\mathcal{F} \in D_G(\mathrm{Gr}_G)$ is a ring object, i.e. it is equipped with a commutative multiplication homomorphism $\mathfrak{m}_{\mathcal{F}}: \mathcal{F} \star \mathcal{F} \rightarrow \mathcal{F}$. Then

$\Psi^{-1}(\mathcal{F}) \in D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ is also a ring object, i.e. it is equipped with $\Psi^{-1}(\mathfrak{m}_{\mathcal{F}}): \Psi^{-1}(\mathcal{F}) \otimes_{\mathrm{Sym}^\square(\mathfrak{g}^\vee)} \Psi^{-1}(\mathcal{F}) \rightarrow \Psi^{-1}(\mathcal{F})$. The same is true for Φ . On the other hand, $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \mathcal{F}) = \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{A}_R, \mathcal{F})$ in Lemma 5.13(b) is equipped with a multiplication by $\mathfrak{m}_{\mathcal{F}}$ and $\mathfrak{m}_{\mathcal{A}_R}$ (equivalently, a coproduct $\mathcal{C}_G \mathbb{D}\mathfrak{m}_{\mathcal{A}_R}: \mathcal{A}_R \rightarrow \mathcal{A}_R \star \mathcal{A}_R$). Similarly,

$$\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{A}_R \star \mathcal{F}) = \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{D}\mathcal{A}_R, \mathcal{F})$$

in Lemma 5.13(c) is equipped with a multiplication by $\mathfrak{m}_{\mathcal{F}}$ and $\mathcal{C}_G \mathfrak{m}_{\mathcal{A}_R}: \mathcal{C}_G \mathcal{A}_R \star \mathcal{C}_G \mathcal{A}_R \rightarrow \mathcal{C}_G \mathcal{A}_R$ (equivalently, a coproduct $\mathbb{D}\mathfrak{m}_{\mathcal{A}_R}: \mathbb{D}\mathcal{A}_R \rightarrow \mathbb{D}\mathcal{A}_R \star \mathbb{D}\mathcal{A}_R$). Finally, a multiplication on $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F})$ is defined as in Proposition 2.22.

Proposition 5.15. (a) *Multiplications on $\Psi^{-1}(\mathcal{F})$, $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{A}_R, \mathcal{F})$ and $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \mathcal{F})$ are equal under the isomorphism in Lemma 5.13(a).*

(b) *The same is true for*

$$\begin{aligned} &\Phi(\mathcal{F}), \quad \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{A}_R \star \mathcal{F}), \\ &\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{D}\mathcal{A}_R, \mathcal{F}), \quad \text{and} \\ &\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F}), \end{aligned}$$

under the isomorphisms of Lemma 5.13(c).

Proof. (a) The isomorphism

$$\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{A}_R, \mathcal{F}) \cong \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \mathcal{F})$$

is given by the rigidity, and respects the multiplication by definition. Therefore it is enough to check the compatibility under the isomorphism $\Psi^{-1}(\mathcal{F}) \cong \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{A}_R \star \mathcal{F})$. This isomorphism is nothing but φ_M (where $M = \Psi^{-1}(\mathcal{F})$) in (5.14) under the derived Satake equivalence. Therefore it is enough to check that φ_M respects the multiplication when M is an algebra in the category $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$. This is trivial as φ_M^{-1} is given by the evaluation $\mathrm{ev}_1: \mathbb{C}[G^\vee] \rightarrow \mathbb{C}$ at $1 \in G^\vee$.

(b) Applying (a) to $\mathcal{C}_G \mathcal{F}$, we see that the multiplications on $\Phi(\mathcal{F})$ and $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{A}_R \star \mathcal{F}) = \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{D}\mathcal{A}_R, \mathcal{F})$ are equal under the isomorphisms of Lemma 5.13(c). It remains to compare them with the multiplication on $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F})$ defined in Proposition 2.22 as the composition

$$\begin{aligned}
& \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F}) \otimes \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F}) \\
& \rightarrow \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}\left(\mathbb{C}_{\mathrm{Gr}_G}, (\mathcal{A}_R \otimes^! \mathcal{F}) \star (\mathcal{A}_R \otimes^! \mathcal{F})\right) \\
& \xrightarrow{(2.23)} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}\left(\mathbb{C}_{\mathrm{Gr}_G}, (\mathcal{A}_R \star \mathcal{A}_R) \otimes^! (\mathcal{F} \star \mathcal{F})\right) \\
& \xrightarrow{\mathfrak{m}} \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{A}_R \otimes^! \mathcal{F}).
\end{aligned}$$

Note that

$$\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{D}\mathcal{A}_R, \mathcal{F}) \cong \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F})).$$

Recall that the convolution product \star is defined as $m_*(q^*)^{-1}p^*$, see (2.2), where we omit $-$ for brevity. Since p and q are smooth with fibers both $G_{\mathcal{O}}$, we have $(q^*)^{-1}p^* = (q^!)^{-1}p^!$. By [KS90, (2.6.24)] for m_* and [KS90, Prop. 3.1.13] for $p^!$, $q^!$, we have a natural homomorphism

$$(5.16) \quad \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F}) \star \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F}) \rightarrow \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R \star \mathbb{D}\mathcal{A}_R, \mathcal{F} \star \mathcal{F}).$$

Hence we have the multiplication on $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F}))$ by $\mathfrak{m}_{\mathcal{F}}$ and $\mathbb{D}\mathfrak{m}_{\mathcal{A}_R}$. Then the isomorphism

$$\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{D}\mathcal{A}_R, \mathcal{F}) \cong \mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathbb{C}_{\mathrm{Gr}_G}, \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F}))$$

is compatible with the multiplication.

Now our remaining task is to check that the isomorphism $\mathcal{A}_R \otimes^! \mathcal{F} \cong \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F})$ (see [KS90, Prop. 3.4.6]) is compatible with the multiplication. Since the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{A}_R \star \mathcal{A}_R) \otimes^! (\mathcal{F} \star \mathcal{F}) & \xrightarrow{\mathfrak{m}_{\mathcal{A}_R} \otimes^! \mathfrak{m}_{\mathcal{F}}} & \mathcal{A}_R \otimes^! \mathcal{F} \\
\wr \downarrow & & \wr \downarrow \\
\mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R \star \mathbb{D}\mathcal{A}_R, \mathcal{F} \star \mathcal{F}) & \xrightarrow{\mathfrak{m}_{\mathcal{F}} \circ \mathbb{D}\mathfrak{m}_{\mathcal{A}_R}} & \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F}),
\end{array}$$

the proof is reduced to the commutativity of

$$\begin{array}{ccc}
(\mathcal{A}_R \otimes^! \mathcal{F}) \star (\mathcal{A}_R \otimes^! \mathcal{F}) & \xrightarrow{(2.23)} & (\mathcal{A}_R \star \mathcal{A}_R) \otimes^! (\mathcal{F} \star \mathcal{F}) \\
\wr \downarrow & & \wr \downarrow \\
\mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F}) \star \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R, \mathcal{F}) & \xrightarrow{(5.16)} & \mathcal{R}\mathcal{H}om(\mathbb{D}\mathcal{A}_R \star \mathbb{D}\mathcal{A}_R, \mathcal{F} \star \mathcal{F}).
\end{array}$$

Recall that horizontal arrows in (5.17) are defined as composite of homomorphisms for $p^!, q^!, m_*$ under $\star = m_*(q^!)^{-1}p^!$. Thus the commutativity follows from compatibilities of homomorphisms for $p^!, q^!, m_*$ under the isomorphism [KS90, Prop. 3.4.6]. We leave the reader to check the detail. \square

5(viii). Hamiltonian reduction

The right Kostant-Whittaker reduction of $\mathcal{A}_R = \Psi(\mathbb{C}[T^*G^\vee]^\square)$ equipped with G^\vee -action is a particular case of the following construction.

Let \mathcal{G} be a commutative ring object of $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$ equipped with an action of an algebraic group H . Let \mathfrak{h} be the Lie algebra of H . Let $\mathrm{Sym}^\square(\mathfrak{h})$ be the symmetric algebra of \mathfrak{h} equipped with a nonnegative grading (not necessarily the standard one, nor the doubled standard one) and viewed as a dg-algebra with a trivial differential. Let

$$\mu^*: \mathrm{Sym}^\square(\mathfrak{h}) \rightarrow \mathrm{RHom}_{D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))}(\mathcal{G}, \mathcal{G})$$

be an H -equivariant homomorphism of dg-algebras such that the multiplication morphism $m: \mathcal{G} \otimes_{\mathrm{Sym}^\square(\mathfrak{g}^\vee)} \mathcal{G} \rightarrow \mathcal{G}$ is H -equivariant and $\mathrm{Sym}^\square(\mathfrak{h})$ -linear.

In all the examples below the following property holds: after applying the forgetful functor and taking cohomology and their spectrum, $\mathrm{Spec} H^*(\mathrm{Forg} \mathcal{G})$ is equipped with an H -invariant symplectic form, and μ^* is compatible with a moment map $\mu: \mathrm{Spec} H^*(\mathrm{Forg} \mathcal{G}) \rightarrow \mathfrak{h}^*$.

Given an H -invariant subvariety $X \subset \mathfrak{h}^*$ such that the projection $\mathrm{Sym}(\mathfrak{h}) \twoheadrightarrow \mathbb{C}[X]$ is compatible with the grading $\mathrm{Sym}^\square(\mathfrak{h})$ and induces the grading $\mathbb{C}[X]^\square$, we define $\mathcal{G} \mathbb{I} \mathbb{I} (H, X) := (\mathcal{G} \otimes_{\mathrm{Sym}^\square(\mathfrak{h})} \mathbb{C}[X]^\square)^H$. This is a commutative ring object of $D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))$. If $X = \{0\} \subset \mathfrak{h}^*$, we simply write $\mathcal{G} \mathbb{I} \mathbb{I} H$ for $\mathcal{G} \mathbb{I} \mathbb{I} (H, \{0\})$.

5(ix). Leg amputation

Following Proposition 2.22, we consider a commutative ring object $\mathcal{A}^b := i_\Delta^!(\boxtimes_{k=1}^b (\mathcal{A}_R)_k)$ (in particular, the ring object associated with S^2 with three punctures is \mathcal{A}^3 in our present notation). According to Appendix A, \mathcal{A}^b is equipped with an action of $\mathrm{SL}(N)^b = (G^\vee)^b$. More generally, we consider a commutative ring object $\mathcal{A}^b := i_\Delta^!(\boxtimes_{k=1}^b (\mathcal{A}_R)_k)$ on Gr_G equipped with an action of $(G^\vee)^b$ for a reductive flavor group G . We set $W_G^b := \mathrm{Spec} H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{A}^b)$. We conjecture that $H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{A}^b)$ is finitely generated, which we checked so far only in type A . We assume it hereafter. Then W_G^b is

an affine variety with Poisson structure equipped with a hamiltonian action of $(G^\vee)^b$. In particular, $W_{\mathrm{PGL}(N)}^3$ is W of the beginning of current §5.

Also, $W_G^2 = T^*G^\vee$ since $\mathbb{C}[W_G^2] = \mathrm{Forg} \circ \Phi(\mathcal{A}_R) = \mathbb{C}[T^*G^\vee]$, see Lemma 5.13(c).

According to Definition 5.11, we have the action of b copies of $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$ on \mathcal{A}^b . We can consider its Kostant-Whittaker reduction $\kappa_b^r(\mathcal{A}^b) = \mathcal{A}^b \otimes_{\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square$ with respect to the last copy of G^\vee in $(G^\vee)^b$ (cf. §5(viii)). More precisely we apply Ψ to $\kappa_b^r(\Psi^{-1}\mathcal{A}^b) = (\Psi^{-1}\mathcal{A}^b) \otimes_{\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square$.

Lemma 5.18. $\kappa_b^r(\mathcal{A}^b) = \mathcal{A}^{b-1}$.

Proof. We have

$$\begin{aligned} \kappa_b^r(\mathcal{A}^b) &= \kappa_b^r(i_\Delta^!(\boxtimes_{k=1}^b(\mathcal{A}_R)_k)) = \kappa_b^r(i_\Delta^!(\mathcal{A}^{b-1} \boxtimes \mathcal{A}_R)) \\ &= i_\Delta^!(\mathcal{A}^{b-1} \boxtimes \kappa^r(\mathcal{A}_R)) = i_\Delta^!(\mathcal{A}^{b-1} \boxtimes \omega_{\mathrm{Gr}_G}) \\ &= \mathcal{A}^{b-1}. \end{aligned}$$

Indeed, $\kappa_b^r(\bullet) = \bullet \otimes_{\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square$ (with respect to the action of the b -th copy of $\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)$). In the third equality we use that for $\mathcal{F} = \mathcal{A}^{b-1} \in D_G(\mathrm{Gr}_G)$ and $\mathcal{F}' = \mathcal{A}_R \in D_G(\mathrm{Gr}_G)$ with a dg-algebra $A = \mathrm{Sym}^\square(\mathfrak{g}^\vee)$ equipped with a homomorphism to $\mathrm{RHom}_{D_G(\mathrm{Gr}_G)}(\mathcal{F}', \mathcal{F}')$, and a dg-module $M = \mathbb{C}[\Sigma]^\square$ over A , we have $(\mathcal{F} \otimes^! \mathcal{F}') \otimes_A M = \mathcal{F} \otimes^! (\mathcal{F}' \otimes_A M)$ by associativity of tensor product. This equality is compatible with the commutative ring structures by the construction in Proposition 2.22 (the reduction $\kappa_b^r(\mathcal{A}^b)$ carries the induced ring structure by the explanation in §5(viii) since the multiplication $\mathfrak{m}: \mathcal{A}_R \star \mathcal{A}_R \rightarrow \mathcal{A}_R$ is $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$ -linear for the right action of $\mathrm{Sym}^\square(\mathfrak{g}^\vee)$ on $\mathcal{A}_R = \Psi(\mathbb{C}[T^*G]^\square)$.) \square

Corollary 5.19. $\kappa_b(W_G^b) = W_G^{b-1}$.

Proof. We have to check that κ_b commutes with $H_{G^\circ}^*(\mathrm{Gr}_G, \bullet)$. After applying the derived Satake equivalence we have to check that κ_b^r commutes with κ^l . Recall that $\kappa^l(\bullet) = H^*(\bullet \otimes_{\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square)$ (with respect to the action of the left copy of $\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)$), while $\kappa_r^b(\bullet) = \bullet \otimes_{\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square$ (with respect to the action of the b -th right copy of $\mathrm{Sym}_{\mathrm{new}}^\square(\mathfrak{g}^\vee)$). We have

$\Psi^{-1}(\mathcal{A}^b) \in D^{(G^\vee)^{b+1}}(\mathrm{Sym}^\square((\mathfrak{g}^\vee)^{\oplus b+1}))$ (one left structure and b right structures). We assign number 0 to the left structure. Then

$$\begin{aligned} \mathbb{C}[W_G^{b-1}] &= H^* \left((\Psi^{-1}(\mathcal{A}^b) \otimes_{b \mathrm{Sym}^\square_{\mathrm{new}}(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square) \otimes_{0 \mathrm{Sym}^\square_{\mathrm{new}}(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square \right) \\ &= H^* \left((\Psi^{-1}(\mathcal{A}^b) \otimes_{0 \mathrm{Sym}^\square_{\mathrm{new}}(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square) \otimes_{b \mathrm{Sym}^\square_{\mathrm{new}}(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square \right) \\ &= H^* \left((\Psi^{-1}(\mathcal{A}^b) \otimes_{0 \mathrm{Sym}^\square_{\mathrm{new}}(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma]^\square) \right) \otimes_{b \mathrm{Sym}(\mathfrak{g}^\vee)} \mathbb{C}[\Sigma] \\ &= \kappa_b(\mathbb{C}[W_G^b]). \end{aligned}$$

The third equality (commutation of taking cohomology and tensor product with a $b \mathrm{Sym}^\square_{\mathrm{new}}(\mathfrak{g}^\vee)$ -module) is clear for free modules, and then for perfect complexes by devissage, and then for Ind-perfect complexes since cohomology commutes with direct images. \square

5(x). General surfaces for arbitrary reductive groups and fusion

First we study the case of cylinder and give another explanation of the identification $W_G^2 = T^*G^\vee$.

We consider the equivalence

$$\Psi \boxtimes \Psi: D^{G^\vee \times G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee \oplus \mathfrak{g}^\vee)) \rightarrow D_{G \times G}(\mathrm{Gr}_G \times \mathrm{Gr}_G).$$

Under this equivalence, the ring object $\mathcal{A}_R \boxtimes \mathcal{A}_R \in D_{G \times G}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$ corresponds to the $G^\vee \times G^\vee$ -equivariant free $\mathrm{Sym}^\square(\mathfrak{g}^\vee \oplus \mathfrak{g}^\vee)$ -module $\mathbb{C}[G^\vee \times G^\vee] \otimes \mathrm{Sym}^\square(\mathfrak{g}^\vee \oplus \mathfrak{g}^\vee)$ which will be denoted $\mathbb{C}[T^*G^\vee \times T^*G^\vee]^\square$ for short. It is equipped with the *right* action of $G^\vee \times G^\vee$ with the right moment map (μ_r, μ_r) . The hamiltonian reduction with respect to the diagonal right action

$$\begin{aligned} (T^*G^\vee \times T^*G^\vee) // \Delta_{G^\vee} &:= \mathrm{Spec} \left(\mathbb{C}[(\mu_r, \mu_r)^{-1}(\Delta_{(\mathfrak{g}^\vee)^*})]^{\Delta_{G^\vee}} \right) \\ &= (\mu_r, \mu_r)^{-1}(\Delta_{(\mathfrak{g}^\vee)^*}) // \Delta_{G^\vee} \end{aligned}$$

(the categorical quotient is the set-theoretical one, as it is with respect to the free action of G^\vee) is nothing but T^*G^\vee equipped with the residual left action of $G^\vee \times G^\vee: (h_1, h_2)(g, \xi) = (h_2 g h_1^{-1}, \mathrm{Ad}_{h_1} \xi)$, and the equivariant morphism to $(\mathfrak{g}^\vee)^* \oplus (\mathfrak{g}^\vee)^*: (g, \xi) \mapsto (\xi, \mathrm{Ad}_g \xi)$. Note that the natural projection $\mathbb{C}[T^*G^\vee \times T^*G^\vee] \twoheadrightarrow \mathbb{C}[(\mu_r, \mu_r)^{-1}(\Delta_{(\mathfrak{g}^\vee)^*})]$ is compatible with the grading of $\mathbb{C}[T^*G^\vee \times T^*G^\vee]$, and so it induces a grading on the target, to be denoted $\mathbb{C}[(\mu_r, \mu_r)^{-1}(\Delta_{(\mathfrak{g}^\vee)^*})]^\square$. This in turn induces a grading on the Δ_{G^\vee} -invariant subalgebra, to be denoted $\mathbb{C}[T^*G^\vee \times T^*G^\vee // \Delta_{G^\vee}]^\square$. Viewing it as

a $G^\vee \times G^\vee$ -equivariant graded module over $\mathrm{Sym}^\square(\mathfrak{g}^\vee \oplus \mathfrak{g}^\vee)$ (with zero differential) and taking its free resolution, we obtain the same named object of $D^{G^\vee \times G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee \oplus \mathfrak{g}^\vee))$. We will denote $\Psi \boxtimes \Psi(\mathbb{C}[T^*G^\vee \times T^*G^\vee // \Delta_{G^\vee}]^\square)$ by $\mathcal{A}_R \boxtimes \mathcal{A}_R // \Delta_{G^\vee} \in D_{G \times G}(\mathrm{Gr}_G \times \mathrm{Gr}_G)$ for short, cf. §5(viii).

Now $\Psi^{-1}(\mathcal{A}_R \star \mathcal{A}_R) = \mathbb{C}[T^*G^\vee]^\square \otimes_{\mathrm{Sym}^\square(\mathfrak{g}^\vee)} \mathbb{C}[T^*G^\vee]^\square$, and $\Psi^{-1}(\mathrm{IC}(\mathrm{Gr}_G^0)) = \mathrm{Sym}^\square(\mathfrak{g}^\vee)$. Hence, $W_G^2 = \mathrm{Spec} H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{A}^2) = (T^*G^\vee \times T^*G^\vee) // \Delta_{G^\vee} = T^*G^\vee$. The action $G^\vee \times G^\vee$ on W_G^2 is the natural action of $G^\vee \times G^\vee$ on T^*G^\vee : $(h_1, h_2) \cdot (g, \xi) = (h_2 g h_1^{-1}, \mathrm{Ad}_{h_1} \xi)$; in particular, the diagonal action of Δ_{G^\vee} is the adjoint action $h(g, \xi) = (h g h^{-1}, \mathrm{Ad}_h \xi)$.

We denote $\mathcal{A}^2 // \Delta_{G^\vee}$ by $\mathcal{B} \in D_G(\mathrm{Gr}_G)$. We have

$$H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{B}) = H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{A}^2) // \Delta_{G^\vee} = \mathbb{C}[(T^*G^\vee) // \Delta_{G^\vee}] = \mathbb{C}[T^\vee \times \mathfrak{t}]^W.$$

Here the last equality is a multiplicative analog of the isomorphism $(\mathfrak{g}^\vee \times \mathfrak{g}^\vee) // \Delta_{G^\vee} = (\mathfrak{t}^\vee \times \mathfrak{t}^\vee) / W$ [Jos97] due to I. Losev. Its proof is given in §5(xiii) below. More generally, we have

$$\begin{aligned} H_{G^\circ}^*(\mathrm{Gr}_G, i_\Delta^!(\mathcal{A}^b \boxtimes \mathcal{B})) &= \mathbb{C}[W_G^{b+2} // \Delta_{G^\vee}^{b+1, b+2}] \\ &= \mathbb{C}[(\mu_r^{b+1}, \mu_r^{b+2})^{-1} (\Delta_{(\mathfrak{g}^\vee)^*}^{b+1, b+2})]^{\Delta_{G^\vee}^{b+1, b+2}} \end{aligned}$$

where $\Delta_{G^\vee}^{b+1, b+2}$ stands for the diagonal in the product of the last two copies in $(G^\vee)^{b+2}$.

We denote $\mathcal{B}^g := i_\Delta^!(\boxtimes_{k=1}^g \mathcal{B}_k) \in D_G(\mathrm{Gr}_G)$. Then $\mathrm{Spec} H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{A}^b \otimes^! \mathcal{B}^g)$ is an object of \mathcal{HS} associated with a surface of genus g with b punctures. Now we turn to the study of fusion of surfaces.

Proposition 5.20. *Let $\Delta_{G^\vee}^{b_1, b_2}$ denote the diagonal action of the b_1 -st and b_2 -nd copy of G^\vee on $W_G^{b_1} \times W_G^{b_2}$. Then $W_G^{b_1+b_2-2} = (W_G^{b_1} \times W_G^{b_2}) // \Delta_{G^\vee}^{b_1, b_2}$.*

Proof. We have

$$\begin{aligned} \mathbb{C}[W_G^{b_1+b_2-2}] &= H_{G^\circ}^*(\mathrm{Gr}_G, \mathcal{A}^{b_1-1} \otimes^! \mathcal{A}^{b_2-1}) \\ &= \mathrm{Ext}_{D_G(\mathrm{Gr}_G)}^*(\mathbf{1}_{\mathrm{Gr}_G}, \mathcal{C}_G \mathcal{A}^{b_1-1} \star \mathcal{A}^{b_2-1}) \\ &= \mathrm{Ext}_{D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))}^* \left(\mathrm{Sym}^\square(\mathfrak{g}^\vee), \mathfrak{C}_{G^\vee} \Psi^{-1}(\mathcal{A}^{b_1-1}) \otimes_{\mathrm{Sym}^\square(\mathfrak{g}^\vee)} \Psi^{-1}(\mathcal{A}^{b_2-1}) \right) \\ &= \mathrm{Ext}_{D^{G^\vee}(\mathrm{Sym}^\square(\mathfrak{g}^\vee))}^* \left(\mathrm{Sym}^\square(\mathfrak{g}^\vee), \Phi(\mathcal{A}^{b_1-1}) \otimes_{\mathrm{Sym}^\square(\mathfrak{g}^\vee)} \mathfrak{C}_{G^\vee} \Phi(\mathcal{A}^{b_2-1}) \right), \end{aligned}$$

(the second equality is Lemma 5.13(b)).

Now $\text{Ext}_{D^{G^\vee}(\text{Sym}^\square(\mathfrak{g}^\vee))}^* \left(\text{Sym}^\square(\mathfrak{g}^\vee), \Phi(\mathcal{A}^{b_1-1}) \otimes_{\text{Sym}^\square(\mathfrak{g}^\vee)} \mathfrak{C}_{G^\vee} \Phi(\mathcal{A}^{b_2-1}) \right)$ is the hamiltonian reduction

$$(\Phi(\mathcal{A}^{b_1-1}) \boxtimes \mathfrak{C}_{G^\vee} \Phi(\mathcal{A}^{b_2-1})) // \Delta_{G^\vee}$$

of $\Phi(\mathcal{A}^{b_1-1}) \boxtimes \mathfrak{C}_{G^\vee} \Phi(\mathcal{A}^{b_2-1})$ with respect to the diagonal (left) action of G^\vee . According to Lemma 5.13(c) and Lemma 5.12,

$$\Phi(\mathcal{A}^{b-1}) = H_{G_\circ}^*(\text{Gr}_G, \mathcal{A}_R \otimes^! \mathcal{A}^{b-1}) = H_{G_\circ}^*(\text{Gr}_G, \mathcal{A}^b) = \mathbb{C}[W_G^b],$$

and the left $G^\vee \times \text{Sym}^\square(\mathfrak{g}^\vee)$ -module structure in the LHS coincides with the *right* $G^\vee \times \text{Sym}^\square(\mathfrak{g}^\vee)$ -module structure in the RHS (with respect to the last copy of $G^\vee \times \text{Sym}^\square(\mathfrak{g}^\vee)$). This completes the proof. \square

Remark 5.21. The same argument shows that

$$\begin{aligned} H_{G_\circ}^*(\text{Gr}_G, \mathcal{A}_1 \otimes^! \mathcal{A}_2) &\cong H_{G_\circ}^*(\text{Gr}_G, \mathcal{A}_R \otimes^! \mathcal{A}_1) \\ &\otimes \mathfrak{C}_{G^\vee} H_{G_\circ}^*(\text{Gr}_G, \mathcal{A}_R \otimes^! \mathcal{A}_2) // \Delta_{G^\vee} \end{aligned}$$

for ring objects $\mathcal{A}_1, \mathcal{A}_2$ in $D_G(\text{Gr}_G)$.

The natural action of $H_{G_\circ}^*(\text{pt}) \otimes H_{G_\circ}^*(\text{pt}) = \mathbb{C}[\Sigma]^\square \otimes \mathbb{C}[\Sigma]^\square$ on the RHS factors through the multiplication homomorphism $\mathbb{C}[\Sigma]^\square \otimes \mathbb{C}[\Sigma]^\square \rightarrow \mathbb{C}[\Sigma]^\square$, and the resulting action of $H_{G_\circ}^*(\text{pt}) = \mathbb{C}[\Sigma]^\square$ in the RHS coincides with its natural action in the LHS.

Remark 5.22. The results of §5(vi)–§5(x) have their quantum counterparts if we consider an extra equivariance with respect to the loop rotations. They are based on the equivalence of monoidal triangulated categories $D^{G^\vee}(U_\hbar^\square(\mathfrak{g}^\vee)) \xrightarrow{\sim} D_{G_\circ \times \mathbb{C}^\times}(\text{Gr}_G)$ [BF08, Thm. 5] (recall that $D_G(\text{Gr}_G)$ is a shorthand for the G_\circ -equivariant derived category $D_{G_\circ}(\text{Gr}_G)$). In particular, the regular sheaf $\mathcal{A}_R \in D_{G_\circ \times \mathbb{C}^\times}(\text{Gr}_G)$ corresponds to $U_\hbar^\square(\mathfrak{g}^\vee) \times \mathbb{C}[G^\vee]$. For a loop rotation equivariant ring object $\mathcal{A} \in D_{G_\circ \times \mathbb{C}^\times}(\text{Gr}_G)$ one can consider the loop rotation equivariant cohomology ring $H_{G_\circ \times \mathbb{C}^\times}^*(\text{Gr}_G, \mathcal{A})$. Similarly to Remark 5.21, we have

$$\begin{aligned} H_{G_\circ \times \mathbb{C}^\times}^*(\text{Gr}_G, \mathcal{A}_1 \otimes^! \mathcal{A}_2) &\cong H_{G_\circ \times \mathbb{C}^\times}^*(\text{Gr}_G, \mathcal{A}_R \otimes^! \mathcal{A}_1) \\ &\otimes \mathfrak{C}_{G^\vee} H_{G_\circ \times \mathbb{C}^\times}^*(\text{Gr}_G, \mathcal{A}_R \otimes^! \mathcal{A}_2) // \Delta_{G^\vee} \end{aligned}$$

(*quantum Hamiltonian reduction*) for ring objects $\mathcal{A}_1, \mathcal{A}_2$ in $D_{G_\circ \times \mathbb{C}^\times}(\text{Gr}_G)$.

In particular, we set $\mathbb{C}_\hbar[W_G^b] := H_{G_\circ \times \mathbb{C}^\times}^*(\text{Gr}_G, \mathcal{A}^b)$, a quantization of $\mathbb{C}[W_G^b]$. Then $\mathbb{C}_\hbar[W_G^{b_1+b_2-2}] = (\mathbb{C}_\hbar[W_G^{b_1}] \otimes \mathbb{C}_\hbar[W_G^{b_2}]) // \Delta_{G^\vee}$.

5(xi). Gluing construction vs hamiltonian reduction

Let us slightly change the point of view to our gluing construction §2(viii) so that it formally looks similar to a hamiltonian reduction.

Let \mathcal{A} be a commutative ring object on Gr_G . Let G' be a subgroup of G , which is also reductive. We have an inclusion $i: \text{Gr}_{G'} \rightarrow \text{Gr}_G$. Then

The $!$ -pull back $i^!\mathcal{A}$ is a ring object on $\text{Gr}_{G'}$.

When \mathcal{A} arises as $\pi_*(\omega_{\mathcal{R}}[-2 \dim \mathbf{N}_{\mathcal{O}}])$ from a representation \mathbf{N} , $i^!\mathcal{A}$ is the ring object associated with \mathbf{N} viewed as a representation of G' .

Next suppose we have a homomorphism $G \rightarrow G''$ to another reductive group G'' . We consider the induced morphism $p: \text{Gr}_G \rightarrow \text{Gr}_{G''}$, which is equivariant under the induced group homomorphism $G_{\mathcal{O}} \rightarrow G''_{\mathcal{O}}$. Then

The pushforward $Q_{p*}\mathcal{A}$ is a ring object on $\text{Gr}_{G''}$.

Here Q_{p*} is the general pushforward as in §2(iv). The construction of §2(iv) is an example of the pushforward, where G, G'' here are \tilde{G}, G_F there, and $\mathcal{A} \in D_G(\text{Gr}_G)$ here is the ring object on $D_{\tilde{G}}(\text{Gr}_{\tilde{G}})$ associated with a representation \mathbf{N} of \tilde{G} there. When G'' is the trivial group, the pushforward is nothing but taking the cohomology $H_{G_{\mathcal{O}}}^*(\text{Gr}_G, \mathcal{A})$. In physics terminology this operation corresponds to the *gauging* with respect to the kernel of the homomorphism $G \rightarrow G''$.

Note that this construction is *formally* similar to a hamiltonian reduction: suppose that we have a hamiltonian G space X . We take a hamiltonian reduction $X \mathbin{///} G'$ with respect to a normal subgroup $G' \triangleleft G$. Then $X \mathbin{///} G'$ is a hamiltonian $G'' = G/G'$ space. This is not just an analogy if we consider gauging in quantum field theories: The Higgs branch of a gauge theory associated with (G, \mathbf{N}) is the hamiltonian reduction $\mathbf{N} \oplus \mathbf{N}^* \mathbin{///} G$. (See [Tac] for a review for mathematicians.)

As an example of the similarity, let us consider (5.4) which we regard as a quantum field theory *upgrade* of the definition $W^{g,b} = \text{Spec } H_{G_{\mathcal{O}}}^*(\text{Gr}_G, \mathcal{A}^b \otimes^! \mathcal{B}^g)$. Let us consider the Coulomb branch of the left hand side, which should be equal to the Higgs branch of the right hand side. Under the gauging $\mathbin{///}$, the Higgs branch is replaced by the symplectic reduction as we have just mentioned. Hence we get

$$\mathcal{M}_C(S_{G^\vee}(C)) = \mathcal{N}_G^b \times (\mathfrak{g} \oplus \mathfrak{g}^*)^g \mathbin{///} G_{\text{diag}},$$

where \mathcal{N}_G is the nilpotent cone of G , and $\mathfrak{g} \oplus \mathfrak{g}^*$ is symplectic by the natural pairing. Thus the Coulomb branch $\mathcal{M}_C(S_{G^\vee}(g, b))$ is the ‘additive version’

of the G -character variety on the punctured Riemann surface C , where the monodromy around punctures sit in the regular unipotent orbit. When G is of type A , this is the Higgs branch of the quiver gauge theory associated with the quiver [Nak16, 3(iii) Figure 5]. See the references therein to see why it is an additive version of a G -character variety.

5(xii). Gluing in the Higgs branch side

Let us pursue the analogy between the gluing construction and hamiltonian reduction further. Let us consider a ring object associated with $S_G(g, b)$ in the Coulomb branch side instead of the Higgs branch side. It is the Higgs branch ring object associated with the right hand side of (5.4) after exchanging G and G^\vee . Hence it is

$$\mathcal{A}_{S_G(g,b)} = \boxtimes_{k=1}^b (\mathcal{A}_R)_k \boxtimes \boxtimes_{l=1}^g \text{Sym}(\mathfrak{g}^\vee \oplus (\mathfrak{g}^\vee)^*)_l // \Delta_{G^\vee},$$

where $\text{Sym}(\mathfrak{g}^\vee \oplus (\mathfrak{g}^\vee)^*)$ is considered as a ring object on the affine Grassmannian $\text{Gr}_{\{e\}}$ for the trivial group $\{e\}$ with the diagonal G^\vee -action. Therefore $\mathcal{A}_{S_G(g,b)}$ is a ring object in $D_{G^b}(\text{Gr}_{G^b})$. Since Proposition 5.20 is a consequence of an upgraded equality in quantum field theories (due to Gaiotto [Gai12]), we have the corresponding property also for $\mathcal{A}_{S_G(g,b)}$. It is nothing but the following:

Proposition 5.23.

$$p_* i_{\Delta^{b_1, b_2}}^! (\mathcal{A}_{S_G(g_1, b_1)} \boxtimes \mathcal{A}_{S_G(g_2, b_2)}) = \mathcal{A}_{S_G(g_1 + g_2, b_1 + b_2 - 2)},$$

where (a) $i_{\Delta^{b_1, b_2}} : \text{Gr}_G^{b_1 + b_2 - 1} \rightarrow \text{Gr}_G^{b_1} \times \text{Gr}_G^{b_2}$ is the product of the evident map $\text{Gr}_G^{b_1 + b_2 - 2} \xrightarrow{\cong} \text{Gr}_G^{b_1 - 1} \times \text{Gr}_G^{b_2 - 1}$ and the diagonal embedding $\text{Gr}_G \rightarrow (\text{Gr}_G)^2$ of the last factor to the product of the b_1 st and the b_2 nd factors, and (b) $p : (\text{Gr}_G)^{b_1 + b_2 - 1} \rightarrow (\text{Gr}_G)^{b_1 + b_2 - 2}$ is the projection given by forgetting the last factor.

Proof. Let us identify \mathfrak{g}^\vee and $(\mathfrak{g}^\vee)^*$ by a non-degenerate invariant form. Let us consider

$$\begin{aligned} & (T^*G^\vee)^b \times (\mathfrak{g}^\vee \times \mathfrak{g}^\vee)^g // \Delta_{G^\vee}^r \\ = & \underbrace{T^*G^\vee \times \cdots \times T^*G^\vee}_{b \text{ times}} \times \underbrace{(\mathfrak{g}^\vee \times \mathfrak{g}^\vee) \times \cdots \times (\mathfrak{g}^\vee \times \mathfrak{g}^\vee)}_{g \text{ times}} // \Delta_{G^\vee}^r, \end{aligned}$$

where $\Delta_{G^\vee}^r$ is the diagonal subgroup acting on $T^*G^\vee \times \cdots \times T^*G^\vee$ by the right action, and on $(\mathfrak{g}^\vee \times \mathfrak{g}^\vee) \times \cdots \times (\mathfrak{g}^\vee \times \mathfrak{g}^\vee)$ by the adjoint action. The dg-version of its coordinate ring is $\Psi^{-1}\mathcal{A}_{S_G(g,b)}$. It is isomorphic to $(T^*G^\vee)^{b-1} \times (\mathfrak{g}^\vee \times \mathfrak{g}^\vee)^g$ by

$$\begin{aligned} & [g_1, \xi_1, \dots, g_b, \xi_b, \eta_1, \zeta_1, \dots, \eta_g, \zeta_g] \\ \mapsto & (g'_1, \xi'_1, \dots, g'_{b-1}, \xi'_{b-1}, \eta'_1, \zeta'_1, \dots, \eta'_g, \zeta'_g) \\ & g'_k = g_k g_b^{-1}, \quad \xi'_k = \text{Ad}_{g_b} \xi_k \quad (k = 1, \dots, b-1), \\ & \eta'_l = \text{Ad}_{g_b} \eta_l, \quad \zeta'_l = \text{Ad}_{g_b} \zeta_l \quad (l = 1, \dots, g). \end{aligned}$$

The left $(G^\vee)^b$ -action on $(T^*G^\vee)^b \times (\mathfrak{g} \times \mathfrak{g})^g // \Delta_{G^\vee}^r$ is identified with the left G^\vee -action (and the trivial action on $(\mathfrak{g}^\vee \times \mathfrak{g}^\vee)^g$) for the first $(b-1)$ factors, but the last factor acts by

$$\begin{aligned} (T^*G^\vee)^{b-1} \ni & (g'_1, \xi'_1, \dots, g'_{b-1}, \xi'_{b-1}) \\ \mapsto & (g'_1 h_b^{-1}, \text{Ad}_{h_b} \xi'_1, \dots, g'_{b-1} h_b^{-1}, \text{Ad}_{h_b} \xi'_{b-1}, \text{Ad}_{h_b} \eta'_l, \text{Ad}_{h_b} \zeta'_l) \end{aligned}$$

for $h_b \in G$. The corresponding moment map is also the standard one for the first $(b-1)$ -factors, and the last one is

$$-\xi'_1 - \cdots - \xi'_{b-1} - \sum_{l=1}^g [\eta'_l, \zeta'_l].$$

This is nothing but the restriction to the diagonal subgroup of the product of the right action and the adjoint action.

Now by Lemma 5.13(b) and Lemma 5.12, $p_* i^!_{\Delta} (\mathcal{A}_{S_G(g_1, b_1)} \boxtimes \mathcal{A}_{S_g(g_2, b_2)})$ goes to

$$\text{Ext}^*_{D_{G^\vee}(\text{Sym}(\mathfrak{g}^\vee))} (\text{Sym}^\square(\mathfrak{g}^\vee), \mathfrak{C}_{G^\vee} \Psi^{-1} \mathcal{A}_{S_G(g_1, b_1)} \otimes_{\text{Sym}^\square(\mathfrak{g}^\vee)} \Psi^{-1} \mathcal{A}_{S_G(g_2, b_2)}),$$

under the derived Satake equivalence. Here \mathfrak{g}^\vee is the Lie algebra of the diagonal subgroup in the product of last factors of $(G^\vee)^{b_1}$ and $(G^\vee)^{b_2}$. It is equal to

$$\mathbb{C}[(T^*G^\vee)^{b_1+b_2-2} \times (\mathfrak{g}^\vee \times \mathfrak{g}^\vee)^{g_1+g_2} // \Delta_{G^\vee}^r]^\square$$

by the above computation. This is nothing but $\Psi^{-1} \mathcal{A}_{S_G(g_1+g_2, b_1+b_2-2)}$. □

5(xiii). Hamiltonian reduction of T^*G with respect to the adjoint action

Let G be a connected reductive group over \mathbb{C} and let \mathfrak{g} be its Lie algebra. Consider the adjoint action of G on itself and the induced Hamiltonian action of the cotangent bundle T^*G . Using a non-degenerate invariant form we identify \mathfrak{g} with \mathfrak{g}^* , this gives rise to an identification $T^*G \cong G \times \mathfrak{g}$ (with the diagonal action of G). The moment map $\mu: T^*G \rightarrow \mathfrak{g}$ becomes $(g, x) \mapsto \text{Ad}_g x - x$. It follows that $\mu^{-1}(0) = \{(g, x) \mid \text{Ad}_g x = x\}$. Consider the Hamiltonian reduction $\mu^{-1}(0) // G$ with the reduced scheme structure.

Now consider $T^*T = T \times \mathfrak{t}$. We have a natural morphism of varieties $\psi: (T \times \mathfrak{t})/W \rightarrow \mu^{-1}(0) // G$ induced from $T \times \mathfrak{t} \hookrightarrow G \times \mathfrak{g}$.

Proposition 5.24 (I. Losev). *The morphism $\psi: (T \times \mathfrak{t})/W \rightarrow \mu^{-1}(0) // G$ is an isomorphism of varieties.*

We can consider the analogous situation for the Lie algebras: we have the moment map $\underline{\mu}: \mathfrak{g}^2 \rightarrow \mathfrak{g}, (x, y) \mapsto [x, y]$. In this situation, a direct analog of Proposition 5.24 is known thanks to [Jos97]: we have $\mathfrak{t}^2/W \xrightarrow{\sim} \underline{\mu}^{-1}(0) // G$. In particular, the variety $\underline{\mu}^{-1}(0) // G$ is normal.

Proof. The proof is in several steps.

Step 1. Let us show that ψ is a bijection. The variety $\mu^{-1}(0) // G$ parameterizes the closed G -orbits in $\mu^{-1}(0) = \{(g, x) \mid \text{Ad}_g x = x\}$. It follows easily from the Hilbert-Mumford theorem that the orbit $G(g, x)$ is closed if and only if both g, x are semisimple. Also any G -orbit of semisimple commuting elements (g, x) intersects $T \times \mathfrak{t}$ in a single W -orbit. The claim in the beginning of the step follows.

Step 2. We claim that it is enough to show that $\mu^{-1}(0) // G$ is a normal algebraic variety. Indeed, any bijective morphism to a normal variety is an isomorphism. The normality of $\mu^{-1}(0) // G$ will follow if we check that the formal neighborhood of every point in $\mu^{-1}(0) // G$ is normal. In order to do that we will describe the formal neighborhood using a version of a slice theorem for Hamiltonian actions on affine symplectic varieties, see, e.g., [Los06] (in that paper complex analytic neighborhoods were considered, but the result carries over to the formal neighborhood in a straightforward way).

Step 3. Let us recall the slice theorem. Let Y be a smooth affine symplectic variety equipped with a Hamiltonian action (with moment map μ) of a reductive group G and let $y \in Y$ be a point with closed G -orbit. Let us write H for the stabilizer of y in G . The normal space $T_y Y / T_y G y$ can be decomposed as $\mathfrak{h}^\perp \oplus V$, where V is a symplectic vector space with H acting

on V by linear symplectomorphisms. Then the formal neighborhood of Gy in Y is G -equivariantly isomorphic to the formal neighborhood of the zero section in $G \times^H (\mathfrak{h}^\perp \oplus V)$. An isomorphism can be chosen to be compatible with symplectic forms and moment maps. In particular, the moment map $\mu': G \times^H (\mathfrak{h}^\perp \oplus V) \rightarrow \mathfrak{g}$ is the unique G -equivariant map that on the fiber $\mathfrak{h}^\perp \oplus V$ over $1H$ is given by $\mu(z, v) = z + \mu_H(v)$, where $\mu_H: V \rightarrow \mathfrak{h}$ is the standard moment map for a linear symplectic action. In particular, we see that the formal neighborhood of Gy in $\mu^{-1}(0)//G$ is isomorphic to the formal neighborhood of 0 in $\mu_H^{-1}(0)//H$.

Step 4. Consider $Y = G \times \mathfrak{g}$ and $y = (g, 0)$ for a semisimple element $g \in G$. We can identify $T_y Gy$ with $\{\text{Ad}_g x - x \mid x \in \mathfrak{g}\}$ so $T_y Y / T_y Gy = \mathfrak{h} \oplus \mathfrak{g}$ and $H = Z_G(g)$. We conclude that $V \cong \mathfrak{h} \oplus \mathfrak{h}$ with diagonal action of H . By [Jos97], we see that $\mu_H^{-1}(0)//H$ is normal. So the formal neighborhood of $G(g, 0)$ in $\mu^{-1}(0)//G$ is normal, equivalently, $G(g, 0)$ is a normal point.

Step 5. To finish the proof note that \mathbb{C}^\times acts on $\mu^{-1}(0)//G$, the action is induced from the dilation action on \mathfrak{g} . This action contracts $\mu^{-1}(0)//G$ to $G//G$. Since the points in the latter are normal, $\mu^{-1}(0)//G$ is a normal algebraic variety. \square

Appendix A. Group action on the Coulomb branch

In this section we give a proof of the expected property [Nak16, §4(iii)(d)], using an idea of Namikawa [Nam18]. See also [CHLZ17].⁷

A(i). The degree 1 subspace

Let us consider the \mathbb{C}^\times -action on the Coulomb branch \mathcal{M}_C given by $\Delta(\lambda)$ as in [Part II, Remark 2.8(2)]. Recall that the \mathbb{C}^\times -action is shifted from one given by the homological degree by a hamiltonian action. In particular, the Poisson bracket $\{ , \}$ is of degree -1 as in [Part II, §3(vi)].

Consider the subspace \mathfrak{l} of degree 1 elements in $\mathbb{C}[\mathcal{M}_C]$. It forms a Lie subalgebra under the Poisson bracket $\{ , \}$. Then $\mathbb{C}[\mathcal{M}_C]$ can be considered as a representation of this Lie algebra \mathfrak{l} by the Poisson bracket: $\{f, \bullet\}$ ($f \in \mathfrak{l}$). If we restrict it to the regular locus of \mathcal{M}_C , it is nothing but the hamiltonian vector field H_f associated with $f \in \mathfrak{l}$ by the symplectic form. The action preserves the Poisson bracket and the degree. In more geometric term, H_f preserves the symplectic form and commutes with the \mathbb{C}^\times -action.

⁷The third named author thanks Amihay Hanany for his explanation of the idea to use the Lie algebra of degree 1 subspace.

Remark A.1. Namikawa [Nam18] shows that \mathcal{M}_C is the closure of a nilpotent orbit if $\mathbb{C}[\mathcal{M}_C]$ is generated by \mathfrak{l} , under the assumption that \mathcal{M}_C has symplectic singularities. In this case \mathfrak{l} is the Lie algebra of $\text{Aut}^{\mathbb{C}^\times}(\mathcal{M}_C, \omega)$, the group of \mathbb{C}^\times -equivariant symplectic automorphisms of \mathcal{M}_C . We conjecture that this statement is true for general \mathcal{M}_C . Namikawa’s argument works in much more general cases without the assumption that the coordinate ring is generated by \mathfrak{l} .⁸ But we are not sure as we do not know \mathcal{M}_C has symplectic singularities, and the \mathbb{C}^\times -action is not conical in general. These seem essential in Namikawa’s argument.

A(ii). Balanced vertices in quiver gauge theories

Let us take a quiver $Q = (Q_0, Q_1)$ and two Q_0 -graded vector spaces $V = \bigoplus V_i$, $W = \bigoplus W_i$. We consider the associated quiver gauge theory $(\text{GL}(V), \mathbf{N})$ as in [Nak16, §2(iv)] and [Quiver, §3], i.e.,

$$\text{GL}(V) = \prod_{i \in Q_0} \text{GL}(V_i), \quad \mathbf{N} = \bigoplus_{h \in Q_1} \text{Hom}(V_{o(h)}, V_{i(h)}) \oplus \bigoplus_{i \in Q_0} \text{Hom}(W_i, V_i).$$

In order to treat a group action on a line bundle in [Part II, §3(ix)] and §2(vi), we also consider a larger symmetry group $\tilde{G} = \text{GL}(V) \times \text{GL}(W)/\mathbb{C}^\times$ with $\tilde{G}/\text{GL}(V) = \text{PGL}(W) = \prod_{i \in Q_0} \text{GL}(W_i)/\mathbb{C}^\times$, where both \mathbb{C}^\times are diagonal scalar subgroups.

Recall $\mathbb{C}[\mathcal{M}_C]$ has a grading parametrized by $\pi_1(\text{GL}(V))$ ([Part II, §3(v)]). In our situation, we have $\pi_1(\text{GL}(V)) = \bigoplus \pi_1(\text{GL}(V_i)) \cong \mathbb{Z}^{Q_0}$. For the larger symmetry group, we have $\pi_1(\tilde{G}) \cong \mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{\{i \in Q_0 | W_i \neq 0\}} / \mathbb{Z}$, where \mathbb{Z} is embedded into $\mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{\{i \in Q_0 | W_i \neq 0\}}$ by $1 \mapsto (\dim V_i, \dim W_i)$. We have the corresponding action of $\pi_1(\text{GL}(V))^\wedge \cong (\mathbb{C}^\times)^{Q_0}$ on \mathcal{M}_C and $\pi_1(\tilde{G})^\wedge \cong (\mathbb{C}^\times)^{\#Q_0 + \#\{i | W_i \neq 0\} - 1}$ (modulo finite groups) on a line bundle in [Part II, §3(ix)], §2(vi). Here $()^\wedge$ is the Pontryagin dual. We will not be interested in the action of finite groups, hence we replace $\pi_1(\tilde{G})$ by its free part $\pi_1(\tilde{G})_{\text{fr}}$ hereafter. We have the corresponding space $H_{\text{GL}(V)}^2(\text{pt})$ (or $H_{\tilde{G}}^2(\text{pt})$ for $\pi_1(\tilde{G})^\wedge$), which consists of degree 1 elements.

A vertex i is *balanced* if there is no edge loop at i , and the corresponding coweight μ satisfies $\langle \mu, \alpha_i \rangle = 0$, i.e., $2 \dim V_i = \dim W_i + \sum_j a_{ij} \dim V_j$, where a_{ij} is the number of edges (either in Q_1 or its opposite) between i and j . We consider the subquiver Q^{bal} of Q consisting of balanced vertices and edges among them. By a well-known result (e.g. [Kac90, Thm. 4.3]),

⁸The third named author thanks Yoshinori Namikawa for explanation.

Q^{bal} is a union of finite *ADE* quivers, unless Q^{bal} is a union of connected components of Q of affine type with $W = 0$ on them. We suppose it is not the latter case.

We consider elements $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ from the shifted Yangian considered in [Quiver, Appendix B]. Looking at relations therein, we see that their Poisson brackets satisfy the relations of \mathfrak{sl}_2 , as $H_i^{(p)} = 0$ ($p < 0$), $H_i^{(0)} = 1$ as $\langle \mu, \alpha_i \rangle = 0$. Moreover if both i and j are balanced, $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}, E_j^{(1)}, F_j^{(1)}, H_j^{(1)}$ satisfy the relations of \mathfrak{sl}_3 or $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ according to whether i and j are connected in the quiver or not. We then have the corresponding semisimple Lie algebra $\mathfrak{l}_{\text{ss}}^{\text{bal}}$ generated by $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ ($i \in Q_0^{\text{bal}}$).

From the definition of $H_i(z)$ in [Quiver, Appendix B], $H_i^{(1)}$ is the coefficient of z^{-1} in $Z_i(z) \prod_j W_j(z)^{a_{ij}} / W_i(z)^2$, where $Z_i(z) = \prod_{k:i_k=i} (z - z_k)$, $W_i(z) = \prod_r (z - w_{i,r})$. (Note that we set $\hbar = 0$.) Here $z_k, w_{i,r}$ are equivariant variables for $\prod \text{GL}(W_i)$ and $\text{GL}(V_i)$ respectively. Therefore $H_i^{(1)}$ is $-\sum_{k:i_k=i} z_k - \sum_{j,s} a_{ij} w_{j,s} + 2 \sum_r w_{i,r}$. This is nothing but $-c_1(W_i) - \sum_j a_{ij} c_1(V_j) + 2c_1(V_i)$ if we regard V_i, W_i as representations of $\text{GL}(V_i), \text{GL}(W_i)$ respectively. Now we apply [Part II, Lemma 3.20]. The Poisson bracket $\{H_i^{(1)}, \bullet\}$ is given by $\gamma_i^W + \sum_j a_{ij} \gamma_j - 2\gamma_i$ on the component with grading $\gamma = (\gamma_j, \gamma_j^W) \in \mathbb{Z}^{Q_0} \oplus \mathbb{Z}^{Q_0} / \mathbb{Z}$. In particular, the action of $H_i^{(1)}$ is lifted to $\pi_1(\text{GL}(V))^\wedge \cong (\mathbb{C}^\times)^{Q_0}$ for \mathcal{M}_C , and to $\pi_1(\tilde{G})_{\text{fr}}^\wedge$ for line bundles.

Lemma A.2. $E_i^{(1)}, F_i^{(1)}, H_i^{(1)}$ ($i \in Q_0^{\text{bal}}$) are of degree 1.

Therefore the Lie algebra \mathfrak{l} in the previous subsection at least contains the semisimple Lie algebra $\mathfrak{l}_{\text{ss}}^{\text{bal}}$ above.

Note that $H_i^{(1)}$ is in $H_{\text{GL}(V)}^2(\text{pt})$ or $H_{\tilde{G}}^2(\text{pt})$ when we consider the larger group \tilde{G} . Let $\mathfrak{l}^{\text{bal}}$ (resp. $\tilde{\mathfrak{l}}^{\text{bal}}$) be the Lie subalgebra of \mathfrak{l} generated by $\mathfrak{l}_{\text{ss}}^{\text{bal}}$ and $H_{\text{GL}(V)}^2(\text{pt})$ (resp. $H_{\tilde{G}}^2(\text{pt})$).

Proof. We have already checked the assertion for $H_i^{(1)}$.

Looking at the definition of the homomorphism in [Quiver, Thm. B.18], we see that $E_i^{(1)}, F_i^{(1)}$ are fundamental classes $[\mathcal{R}_{\pm \varpi_{i,1}}]$ up to sign. By the formula for $\Delta(\pm \varpi_{i,1})$ in [Quiver, (A.4)], their degree is 1 as i is a balanced vertex. \square

Recall $\{H_i^{(1)}, \bullet\}$ defines an element of $\pi_1(\tilde{G})_{\text{fr}}^\vee$ by $\gamma \mapsto \gamma_i^W + \sum_j a_{ij} \gamma_j - 2\gamma_i$. More generally H_α corresponding to a root α of $\mathfrak{l}_{\text{ss}}^{\text{bal}}$ defines an element of $\pi_1(\tilde{G})_{\text{fr}}^\vee$. Thus we regard naturally coroots $\in R^{\text{bal}\vee}$ of $\mathfrak{l}_{\text{ss}}^{\text{bal}}$ as elements in

$\pi_1(\tilde{G})_{\text{fr}}^\vee$. If we disregard the flavor symmetry, we consider the restriction to $\gamma_j^W = 0$, hence we still have $R^{\text{bal}\vee} \subset \pi_1(G)^\vee$.

On the other hand, $\pi_1(\prod_{i \in Q_0^{\text{bal}}} \text{GL}(V_i)) \cong \mathbb{Z}^{Q_0^{\text{bal}}}$ is naturally identified with the root lattice of the Lie algebra $\mathfrak{t}_{\text{ss}}^{\text{bal}}$ by sending the i -th coordinate vector to the i -th simple root α_i . Thus we consider roots $\in R^{\text{bal}}$ as elements of $\pi_1(\prod_{i \in Q_0^{\text{bal}}} \text{GL}(V_i)) \subset \pi_1(\text{GL}(V)) \subset \pi_1(\tilde{G})_{\text{fr}}$.

We regard $R^{\text{bal}} \subset \pi_1(\text{GL}(V))$, $R^{\text{bal}\vee} \subset \pi_1(\text{GL}(V))^\vee$ as a root datum, and consider the corresponding reductive group L^{bal} . For line bundles, we consider $R^{\text{bal}} \subset \pi_1(\tilde{G})_{\text{fr}}$, $R^{\text{bal}\vee} \subset \pi_1(\tilde{G})_{\text{fr}}^\vee$. We denote the corresponding reductive group by \tilde{L}^{bal} .

Proposition A.3. *The actions of $\{E_i^{(1)}, \bullet\}$, $\{F_i^{(1)}, \bullet\}$ are locally nilpotent. Hence the action of $\mathfrak{t}^{\text{bal}}$ (resp. $\mathfrak{l}^{\text{bal}}$) is lifted to L^{bal} (resp. \tilde{L}^{bal}).*

Proof. When the \mathbb{C}^\times -action on \mathcal{M}_C is conical, this is clear as subspaces of $\mathbb{C}[\mathcal{M}_C]$ with given degree are finite dimensional, and $E_i^{(1)}$, $F_i^{(1)}$ preserve them. In order to deal with general cases, we modify the argument.

Consider a closed subvariety $\mathcal{R}_{\leq \lambda}$ as in [Part II, §2(i)]. Since $\text{GL}(V) = \prod_{j \in Q_0} \text{GL}(V_j)$, we can modify it by imposing the constraint at $j \neq i$, but not on i . Let us denote the resulted closed subvariety by $\mathcal{R}_{\preceq \lambda}$. It is still true that $H_*^{\text{GL}(V)^\circ}(\mathcal{R})$ is the limit of $H_*^{\text{GL}(V)^\circ}(\mathcal{R}_{\preceq \lambda})$. Operators $\{E_i^{(1)}, \bullet\}$, $\{F_i^{(1)}, \bullet\}$ are well-defined on $H_*^{\text{GL}(V)^\circ}(\mathcal{R}_{\preceq \lambda})$, as we do not impose the constraint at i .

Let $H_*^{\text{GL}(V)^\circ}(\mathcal{R}_{\preceq \lambda})[d]$ denote the subspace of $H_*^{\text{GL}(V)^\circ}(\mathcal{R}_{\preceq \lambda})$ of degree d elements. It is enough to show that it is finite dimensional, as $\{E_i^{(1)}, \bullet\}$, $\{F_i^{(1)}, \bullet\}$ preserve this subspace.

Suppose that an element in $H_*^{\text{GL}(V)^\circ}(\mathcal{R}_{\preceq \lambda})[d]$ is contained in $H_*^{\text{GL}(V)^\circ}(\mathcal{R}_{\leq \mu})[d]$. If we decompose μ as (μ^j) according to $j \in Q_0$, the component μ^j with $j \neq i$ is less than or equal to the component λ^j of λ by the definition of \preceq .

In order to bound the remaining component μ^i , Let us look at the formula of $\Delta(\mu)$:

$$\Delta(\mu) = - \sum_{\alpha \in \Delta^+} |\langle \alpha, \mu \rangle| + \frac{1}{2} \sum_{\chi} |\langle \chi, \mu \rangle| \dim \mathbf{N}(\chi).$$

We have $\Delta(\mu) \leq d$ by our assumption. Let us look at terms involving μ^i :

$$(A.4) \quad - \sum_{a < b} |\mu_a^i - \mu_b^i| + \frac{1}{2} \left(\sum_j a_{ij} \sum_{a,b} |\mu_a^i - \mu_b^j| + \dim W_i \sum_a |\mu_a^i| \right),$$

where we write $\mu^j = (\mu_1^j, \mu_2^j, \dots)$. This is bounded by a constant from above, as we have bounds on μ_a^j ($j \neq i$). Since μ_b^j is bounded, the middle term can be replaced by $\sum_j a_{ij} \dim V_j \sum_a |\mu_a^i|$. Now by the assumption $2 \dim V_i = \dim W_i + \sum a_{ij} \dim V_j$, the first term can be absorbed in the middle and last term, so that we still have a bound on $\sum_a |\mu_a^i|$. Thus μ is bounded by a constant depending on λ and d . Hence $H_*^{\text{GL}(V)^\circ}(\mathcal{R}_{\leq \lambda})[d]$ is finite dimensional.

This argument works also for the case of \tilde{L}^{bal} . □

Note that the comoment map of the L^{bal} -action on \mathcal{M}_C is the natural homomorphism

$$\mathbb{C}[\mathfrak{t}^{\text{bal}*}] = \text{Sym}(\mathfrak{t}^{\text{bal}}) \rightarrow \mathbb{C}[\mathcal{M}_C]$$

by the definition of the action.

Example A.5 (cf. [Quiver, Remark 3.12]). Consider a framed quiver gauge theory of type ADE . Let us define two coweights

$$\lambda = \sum_{i \in Q_0} \dim W_i \omega_i, \quad \mu = \lambda - \sum_{i \in Q_0} \dim V_i \alpha_i.$$

Then the Coulomb branch is the generalized slice $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$ for the adjoint group G of type Q , where $\lambda^* = -w_0(\lambda)$, $\mu^* = -w_0(\mu)$ ([Quiver, Thm. 3.10]). Then the group L^{bal} , acting on \mathcal{M}_C is $\text{Stab}_G(\mu^*)$, as $\pi_1(\text{GL}(V))$ is the weight lattice of G . The action is the standard one, at least when μ is dominant. The following argument is explained to the authors by Joel Kamnitzer:

First consider the case $\mu = 0$. Then $\overline{\mathcal{W}}_0^{\lambda^*}$ is the intersection of $\overline{\text{Gr}}_G^\lambda$ and $\text{Gr}_{G,0} = G_1[[t^{-1}]]$, where $G_1[[t^{-1}]]$ is the first congruence subgroup of $G[[t^{-1}]]$ as in [KWWY14]. Then the assertion follows from a computation of Poisson brackets on $\mathbb{C}[G_1[[t^{-1}]]]$ in [KWWY14, Prop. 2.13]. For a general dominant μ , we replace $\text{Gr}_{G,0}$ by Gr_{G,μ^*} the orbit of $G_1[[t^{-1}]]$ through μ^* . But $\mathbb{C}[\text{Gr}_{G,\mu^*}]$ is a Poisson subalgebra of $\mathbb{C}[\text{Gr}_{G,0}]$ preserved by the action of $\text{Stab}_G(\mu^*)$ (see [KWWY14, a paragraph before Lemma 2.19]). Hence the assertion follows from the $\mu = 0$ case.

Example A.6. Let us consider the quiver gauge theory in §2(v). All vertices are balanced in this case. We have

$$\pi_1(\tilde{G}) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{N \text{ times}} / (N, N - 1, \dots, 2, 1)\mathbb{Z} \cong \mathbb{Z}^{N-1},$$

where the isomorphism is given by $[\lambda_1, \dots, \lambda_N] \mapsto (\lambda_1 - N\lambda_N, \dots, \lambda_{N-1} - 2\lambda_N)$. We have an exact sequence

$$0 \rightarrow \pi_1(G) \cong \mathbb{Z}^{N-1} \rightarrow \pi_1(\tilde{G}) \cong \mathbb{Z}^{N-1} \rightarrow \pi_1(\mathrm{PGL}(N)) \cong \mathbb{Z}/N\mathbb{Z} \rightarrow 0,$$

where the first inclusion is given by $\lambda_1 = 0$, and the last projection is $[\lambda_1, \dots, \lambda_N] \mapsto \lambda_1 \bmod N$. It is clear that $\pi_1(G)$ is the weight lattice of $\mathrm{PGL}(N)$, while $\pi_1(\tilde{G})$ is that of $\mathrm{SL}(N)$. Therefore $L^{\mathrm{bal}} = \mathrm{PGL}(N)$, but $\tilde{L}^{\mathrm{bal}} = \mathrm{SL}(N)$.

Remark A.7. Let us consider the case $\langle \mu, \alpha_i \rangle = -1$ instead of 0. Then $H_i^{(p)} = 0$ ($p \leq 0$), $H_i^{(1)} = 1$. Thus we have $\{E_i^{(1)}, F_i^{(1)}\} = 1$. (Note also $\Delta(\pm\varpi_{i,1}) = 1/2$ by [Quiver, (A.4)].) Looking at the argument in the proof of Proposition A.3, we see that we only need a bound $\dim V_i - 1 \leq \dim W_i + \sum a_{ij} \dim V_j$ to derive a bound on $\sum_a |\mu_a^i|$ from (A.4). In particular, the proof of Proposition A.3 works in the case $\langle \mu, \alpha_i \rangle = -1$, hence $\{E_i^{(1)}, \bullet\}$, $\{F_i^{(1)}, \bullet\}$ are locally nilpotent, and the corresponding hamiltonian vector fields $H_{E_i^{(1)}}, H_{F_i^{(1)}}$ are integrable. Moreover $[H_{E_i^{(1)}}, H_{F_i^{(1)}}] = 0$ as $\{E_i^{(1)}, F_i^{(1)}\} = 1$. Therefore we have an action of \mathbb{G}_a^2 . Let $\Phi = (F_i^{(1)}, -E_i^{(1)}): \mathcal{M}_C \rightarrow \mathbb{A}^2$. Then Φ is \mathbb{G}_a^2 -equivariant, and the action map $\mathbb{G}_a^2 \times \Phi^{-1}(0) \cong \mathcal{M}_C$ is an isomorphism. See also [Nam18, Thm.(i)].

Suppose further that j is balanced, i.e., $\langle \mu, \alpha_j \rangle = 0$. Commutation relations in [Quiver, Appendix B] imply that

$$\begin{aligned} \{F_j^{(1)}, E_i^{(1)}\} &= 0, & \{H_j^{(1)}, E_i^{(1)}\} &= -(\alpha_i \cdot \alpha_j)E_i^{(1)} \\ \text{and } \{E_j^{(1)}, F_i^{(1)}\} &= 0, & \{H_j^{(1)}, F_i^{(1)}\} &= (\alpha_i \cdot \alpha_j)F_i^{(1)}. \end{aligned}$$

Thus $E_i^{(1)}$ (resp. $F_i^{(1)}$) is a lowest (resp. highest) weight vector of an $\mathfrak{sl}(2)_j = \langle E_j^{(1)}, F_j^{(1)}, H_j^{(1)} \rangle$ module with the highest (resp. lowest) weight $\mp(\alpha_i \cdot \alpha_j)$.

Appendix B. A global convolution diagram for the variety of triples

GUS LONERGAN

The aim of this appendix is to give another proof of the commutativity of the Coulomb branch by constructing a global convolution diagram for \mathcal{R} . This is a direct generalization of the traditional proof of the case $\mathbf{N} = 0$, which uses the Beilinson-Drinfeld global convolution diagram for Gr_G .

B(i). Preliminaries on arc-spaces and loop-spaces

(a). In this section, we recall certain standard constructions and facts of [BeiDr, Chapters 4–5].

(b). Let X be a smooth complex curve and let S be a finite set. Given a commutative ring R and an R -point x of X^S , we denote the coordinates of x by x_s ($s \in S$), and write $\Delta_S(x)$ for the formal neighborhood of the union of the graphs of x_s ($s \in S$). For notational simplicity, we frequently remove commas and braces from S , and also drop the part (x) , when it is clear which point we refer to. So for example the expression:

$$\Delta_{\{1,2\}}(x)$$

becomes:

$$\Delta_{12}.$$

(c). Now fix an affine algebraic group A over \mathbb{C} . Consider the following functor from commutative rings to groups over X^S :

$$A_S(R) := \{(x, f) | x \in X^S(R), f: \Delta_S \rightarrow A\}.$$

Then A_S is represented by the limit of a projective system of smooth affine group schemes over X^S :

$$A_S = \varprojlim (\cdots \rightarrow (A_S)_2 \rightarrow (A_S)_1)$$

such that each transition morphism is a smooth homomorphism. In particular, A_S is a formally smooth affine group scheme (of countably infinite type) over X^S , but this is not so important for us. Recall that in the definition of the Coulomb branch as a convolution algebra formal homological shifts such as

$$[2 \dim A(\mathcal{O})]$$

appear (for $A = G, N$). Similarly, in the global situation formal homological shifts such as

$$[2 \dim A_S]$$

will appear⁹. For example, in the case where the underlying space is A_S , for each d we have $\omega_{(A_S)_d} \cong \mathbb{C}_{(A_S)_d}[2 \dim(A_S)_d]$. These complexes are compatible in the natural way under $!$ -pullbacks along the transition morphisms.

⁹Only for S of cardinality 1 or 2; but it clarifies the picture and simplifies the exposition to work more generally at this point.

We thus consider ω_{A_S} as the formal homological shift

$$\omega_{A_S} \cong \mathbb{C}_{A_S}[2 \dim A_S],$$

where both sides are to be understood by evaluating on smooth quotients of A_S and ‘piecing together’ using $!$ -pullbacks. Likewise we have a formal expression

$$\omega_{A_S}[-2 \dim A_S] \cong \mathbb{C}_{A_S}$$

where both sides are to be understood by evaluating on the smooth quotients $(A_S)_d$ of A_S and ‘piecing together’ using $*$ -pullbacks.

(d). Let $\theta: S' \rightarrow S$ be a morphism of finite sets. It induces a map $X^S \rightarrow X^{S'}$. Given an R -point x of X^S , this map determines an R -point x' of $X^{S'}$, and an embedding $\Delta_{S'}(x') \rightarrow \Delta_S(x)$. Hence by restriction along this embedding we obtain a map

$$p^\theta: A_S \rightarrow A_{S'}.$$

This induces a homomorphism

$$q^\theta: A_S \rightarrow A_{S'} \times_{X^{S'}} X^S$$

over the base X^S . If θ is surjective, then q^θ is an isomorphism. If θ is injective, then q^θ seems strange at first sight. For instance if θ' is a section of θ then q^θ is an isomorphism over the resulting copy of $X^{S'} \subset X^S$, whereas over a typical point of X^S , q^θ takes the form of a projection map

$$A(\mathcal{O})^S \rightarrow A(\mathcal{O})^{S'}.$$

However, this is misleading: q^θ is *pro-smooth* when θ is injective. What we mean by this is that the projective systems of smooth affine group schemes over X^S with smooth transition morphisms

$$((A_S)_d)_{d \in \mathbf{N}}$$

underlying A_S may be taken, simultaneously for all S , to be compatible with all q^θ , i.e. so that q^θ is the limit of a morphism

$$(q_d^\theta: (A_S)_d \rightarrow (A_{S'})_d \times_{X^{S'}} X^S)_{d \in \mathbf{N}}$$

of projective systems, where each map q_d^θ is a smooth homomorphism over X^S . Thus, it makes sense to write (and is true that):

$$(q^\theta)^* \omega_{A_{S'} \times_{X^{S'}}(X^S)}[-2 \dim A_{S'} - 2(|S| - |S'|)] = \omega_{A_S}[-2 \dim A_S],$$

et cetera, where the formula should be understood as a statement about complexes on smooth quotients over X^S , compatible under $*$ -pullbacks.

(e). Example. Consider the case $S = \{1\}$. Then A_1 is a Zariski-locally trivial $A(\mathcal{O})$ -bundle over X . Then, the formal homological shift $[2 \dim A(\mathcal{O})]$ also makes sense in this context, and we have $[2 \dim A(\mathcal{O})] = [2 \dim A_1 - 2]$.

(f). Example. Consider for instance the case $A = \mathbb{C}$ and $S = \{1, 2\}$. Then A_{12} should be thought of as a deformation of the first following projective system into the second:

$$(\mathbb{C}[[t]]/t^{2d})_d \rightsquigarrow (\mathbb{C}[[t]]/t^d \times \mathbb{C}[[t]]/t^d)_d$$

while A_1 should be thought of as a trivial deformation:

$$(\mathbb{C}[[t]]/t^d)_d \rightsquigarrow (\mathbb{C}[[t]]/t^d)_d$$

and we have the morphism of deformations of projective systems:

$$\begin{array}{ccc} (\mathbb{C}[[t]]/t^{2d})_d & \rightsquigarrow & (\mathbb{C}[[t]]/t^d \times \mathbb{C}[[t]]/t^d)_d \\ \downarrow & & \downarrow \\ (\mathbb{C}[[t]]/t^d)_d & \rightsquigarrow & (\mathbb{C}[[t]]/t^d)_d \end{array}$$

where the first downward arrow is the quotient map, and the second downward arrow is the projection map (to the first factor), both of which halve dimension in the d^{th} approximation. It just happens that the limit of the first downward arrow is an isomorphism, while the limit of the second downward arrow is a non-trivial projection.

(g). From now on, we assume θ is an injection, and identify S' with its image under θ . Now, in addition to the formal neighborhood Δ_S we have the punctured formal neighborhood

$$\Delta_S^{S'}(x) := \Delta_S(x) - \cup_{s \in S'} x_s$$

where in this formula we conflate the point x_s with its graph. The general notational paradigm¹⁰ here is that subscripts determine discs and superscripts determine punctures. Consider the functor

$$A_S^{S'}(R) := \{(x, f) | x \in X^S(R), f: \Delta_S^{S'}(x) \rightarrow A\}.$$

Then $A_S^{S'}$ is represented by an ind-scheme, formally smooth over X^S . It is a group in ind-schemes (over X^S), but not an inductive limit of groups.

¹⁰Warning: this doesn't apply to X !

Nonetheless, it is an *ind-locally nice, reasonable* ind-scheme in the sense of [Dri06], meaning that it is a direct limit of closed embeddings with finitely generated ideals:

$$(A_S^{S'})^1 \rightarrow (A_S^{S'})^2 \rightarrow \dots$$

of schemes over X^S , each of which is locally nice, meaning that Zariski-locally¹¹ it is the product of a finite-type scheme with an affine space (of countable dimension). We shall call such an ind-scheme *reasonably nice*. The subgroup A_S may be taken as the first subscheme $(A_S^{S'})^1$ in this inductive structure. The left- and right-regular actions of the subgroup A_S preserve the inductive structure, meaning that each $(A_S^{S'})^c$ has an action on both sides by A_S over X^S , even though it is not itself a group. Moreover the quotient $(A_S^{S'})^c/A_S$ is of finite-type over X^S , and flat, although generally quite singular. The result is that the quotient

$$A_S^{S'}/A_S$$

has the structure of ind-finite-type flat ind-scheme over X^S .

Lemma B.1. 1) $A_S^{S'}/A_S$ is ind-projective if and only if A is reductive.
 2) $A_S^{S'}/A_S$ is reduced if and only if A has non no-trivial characters.

Remark B.2. A_S^S is the Beilinson-Drinfeld grassmannian (on $|S|$ points).

(h). For any chain of inclusions $S'' \xrightarrow{\theta'} S' \xrightarrow{\theta} S$ we have natural maps

$$\begin{aligned} p^\theta &: A_S^{S''} \rightarrow A_{S'}^{S''}, \\ q^\theta &: A_S^{S''} \rightarrow A_{S'}^{S''} \times_{X^{S'}} X^S, \end{aligned}$$

defined as in §(d). Then $q^\theta: A_S^{S''} \rightarrow A_{S'}^{S''} \times_{X^{S'}} X^S$ has as a subgroup $q^\theta: A_S \rightarrow A_{S'} \times_{X^{S'}} X^S$, and the resulting map

$$A_S^{S''}/A_S \rightarrow (A_{S'}^{S''}/A_{S'}) \times_{X^{S'}} X^S$$

is an isomorphism.

(i). Warning. Observe that $A_S^{S''}$ is an ind- A_S -torsor over the ind-scheme $(A_{S'}^{S''}/A_{S'}) \times_{X^{S'}} X^S$, and the homomorphism $q^\theta: A_S^{S''} \rightarrow A_{S'}^{S''} \times_{X^{S'}} X^S$ is surjective. It is tempting therefore to try to view $A_S^{S''}$ as being in some

¹¹In [Dri06] this is relaxed to ‘Nisnevich-locally’.

sense a torsor over $A_{S'}^{S''} \times_{X^{S'}} X^S$ for some group $\ker q^\theta$. However, the kernel of the projective system

$$((A_S)_d \rightarrow (A_{S'})_d \times_{X^{S'}} X^S)_{d \in \mathbf{N}}$$

of §(d) is not Mittag-Leffler. We are not sure how to overcome this issue, so do not attempt to take this point of view.

B(ii). Global convolution diagram for \mathcal{R}

(a). For a finite set S , we put

$$\mathcal{T}_S^{S'}(R) = \{(x, \mathcal{E}, f, \tilde{v})\} / \sim$$

where $x \in X^S(R)$, \mathcal{E} is a principal G -bundle on Δ_S , f is a trivialization of \mathcal{E} on $\Delta_S^{S'}$, and \tilde{v} is an \mathbf{N} -section of \mathcal{E} , taken up to equivalence. This is the same as the balanced product

$$\mathcal{T}_S^{S'} = G_S^{S'} \times_{\frac{X^S}{G_S}} \mathbf{N}_S.$$

Thus, $\mathcal{T}_S^{S'}$ is represented by a reasonably nice ind-scheme with an ind-pro-smooth map to the Beilinson-Drinfeld grassmannian $G_S^{S'}/G_S$. In particular it is formally smooth. Multiplication gives us a map

$$\mathcal{T}_S^{S'} \rightarrow \mathbf{N}_S^{S'}$$

and we define $\mathcal{R}_S^{S'}$ to be the fiber product

$$\mathcal{R}_S^{S'} := \mathcal{T}_S^{S'} \times_{\mathbf{N}_S^{S'}} \mathbf{N}_S.$$

Over any closed X^S -subscheme of $G_S^{S'}/G_S$, the embedding $\mathcal{R}_S^{S'} \rightarrow \mathcal{T}_S^{S'}$ has finite codimension. Therefore $\mathcal{R}_S^{S'}$ is also a reasonably nice ind-scheme, mapping to $G_S^{S'}/G_S$, and of ind-finite codimension in $\mathcal{T}_S^{S'}$. Note that $\mathcal{R}_S^{S'}$ is *not* formally smooth, and in particular the map $\mathcal{R}_S^{S'} \rightarrow G_S^{S'}/G_S$ is no longer ind-pro-smooth. As a functor we have

$$\mathcal{R}_S^{S'}(R) = \{(x, \mathcal{E}, f, v)\} / \sim$$

where x, \mathcal{E}, f are as in $\mathcal{T}_S^{S'}$, and v is an \mathbf{N} -section of \mathcal{E} such that $f(v)$ extends¹² to Δ_S . We define the shifted dualizing complex on $\mathcal{T}_S^{S'}, \mathcal{R}_S^{S'}$ as for \mathcal{T}, \mathcal{R} . Namely:

- 1) On each closed subscheme $(\mathcal{T}_S^{S'})^c$ of $(\mathcal{T}_S^{S'})^c$, pro-smooth over $(G_S^{S'}/G_S)^c$ we set

$$\omega_{(\mathcal{T}_S^{S'})^c}[-2 \dim \mathbf{N}_S + 2|S|]$$

to be the pullback of the dualizing complex of $(G_S^{S'}/G_S)^c$, i.e. the collection of its pullbacks to each formally smooth quotient $(\mathcal{T}_S^{S'})_d^c$ of $(\mathcal{T}_S^{S'})^c$ smooth over $(G_S^{S'}/G_S)^c$, compatible under $*$ -pullback;

- 2) Since $\mathcal{T}_S^{S'}$ is a reasonably nice ind-scheme, we can apply the $!$ -pullback to such a collection of complexes on $(\mathcal{T}_S^{S'})^c$, and obtain one on $(\mathcal{T}_S^{S'})^{c-1}$. In this way, the collections $\omega_{(\mathcal{T}_S^{S'})^c}[-2 \dim \mathbf{N}_S + 2|S|]$ are compatible under $!$ -pullbacks. The resulting compatible collection is called $\omega_{\mathcal{T}_S^{S'}}[-2 \dim \mathbf{N}_S + 2|S|]$.
- 3) Using the ind-finite codimensionality of the embedding $i: \mathcal{R}_S^{S'} \rightarrow \mathcal{T}_S^{S'}$, we form a $!$ -compatible collection of $*$ -compatible collections of complexes

$$\omega_{\mathcal{R}_S^{S'}}[-2 \dim \mathbf{N}_S + 2|S|] := i^! \omega_{\mathcal{T}_S^{S'}}[-2 \dim \mathbf{N}_S + 2|S|].$$

(b). We will apply the abbreviations of §B(i)(b) to our spaces \mathcal{R}, \mathcal{T} etc. so that for instance

$$\mathcal{R}_{\{1,2\}}^{\{2\}}$$

becomes

$$\mathcal{R}_{12}^2.$$

We will also write X^S as $\prod_{s \in S} X_s$, e.g. $X^{\{1,2\}} = X_1 \times X_2$. The obvious starting point for the global convolution diagram is $\mathcal{R}_1^1 \times \mathcal{R}_2^2$, a Zariski-locally trivial \mathcal{R} -bundle over $X_1 \times X_2$. Consider the following space:

$$\mathcal{R}_{1+2}(R) = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2)\} / \sim$$

where x_1, x_2 are R -points of X , each \mathcal{E}_i a principal G -bundle on Δ_{12} , f_i is a trivialization of \mathcal{E}_i on Δ_{12}^i , and v_i is an \mathbf{N} -section of \mathcal{E}_i such that $f_i(v_i)$

¹²It is a priori defined only on $\Delta_S^{S'}$. The extension is necessarily unique.

extends to Δ_{12} . It is constructed as

$$\mathcal{R}_{1+2} = \mathcal{R}_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2,$$

a reasonably nice ind-scheme over $X_1 \times X_2$. It is of ind-finite codimension in the formally smooth reasonably nice ind-scheme

$$\mathcal{T}_{1+2} = \mathcal{T}_{12}^1 \times_{X_1 \times X_2} \mathcal{T}_{12}^2 = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, \tilde{v}_1, \tilde{v}_2)\} / \sim .$$

There is a map

$$\alpha: \mathcal{R}_{1+2} \rightarrow \mathcal{R}_1^1 \times \mathcal{R}_2^2$$

given by restricting \mathcal{E}_i, f_i, v_i to $\Delta_i \subset \Delta_{12}$. Over the diagonal $X_0 \subset X_1 \times X_2$, this map α is an isomorphism. But on the complement U of the diagonal, we have a canonical isomorphism

$$\mathcal{R}_{1+2}|_U = (\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U \times_U (\mathbf{N}_1 \times \mathbf{N}_2)|_U$$

and α is just the projection. Nonetheless, α is *ind-pro-smooth*. Indeed, it is the product over $X_1 \times X_2$ of maps

$$\begin{aligned} \mathcal{R}_{12}^1 &\rightarrow \mathcal{R}_1^1 \times X_2, \\ \mathcal{R}_{12}^2 &\rightarrow \mathcal{R}_2^2 \times X_1; \end{aligned}$$

so it suffices to see that the former is ind-pro-smooth. But note that we can write

$$\mathcal{T}_1^1 \times X_2 = G_1^1 \times_{G_1}^{X_1} \mathbf{N}_1 \times X_2 = G_{12}^1 \times_{G_{12}}^{X_1 \times X_2} \mathbf{N}_1$$

where G_{12} acts on \mathbf{N}_1 via the homomorphism $G_{12} \rightarrow G_1$. Then, the natural map

$$\mathcal{T}_{12}^1 \rightarrow \mathcal{T}_1^1 \times X_2$$

is that associated to the pro-smooth map $\mathbf{N}_{12} \rightarrow \mathbf{N}_1$, so is ind-pro-smooth. The fact that the diagram

$$\begin{array}{ccc} \mathcal{R}_{12}^1 & \rightarrow & \mathcal{R}_1^1 \times X_2 \\ \downarrow & & \downarrow \\ \mathcal{T}_{12}^1 & \rightarrow & \mathcal{T}_1^1 \times X_2 \end{array}$$

is Cartesian gives the result. We have:

$$(B.3) \quad \alpha^* \omega_{\mathcal{R}_1^1 \times \mathcal{R}_2^2}[-2 \dim \mathbf{N}_1 \times \mathbf{N}_2] \cong \omega_{\mathcal{R}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}].$$

Note that $\mathcal{R}_1^1 \times \mathcal{R}_2^2, \mathcal{T}_1^1 \times \mathcal{T}_2^2$ are acted on factor-wise by $G_1 \times G_2$, which receives the factor-wise map from $G_{12} \times_{X_1 \times X_2} G_{12}$. This latter group also acts in the natural way on $\mathcal{R}_{1+2}, \mathcal{T}_{1+2}$, and the diagram

$$\begin{array}{ccc} \mathcal{R}_{1+2} & \rightarrow & \mathcal{R}_1^1 \times \mathcal{R}_2^2 \\ \downarrow & & \downarrow \\ \mathcal{T}_{1+2} & \rightarrow & \mathcal{T}_1^1 \times \mathcal{T}_2^2 \end{array}$$

is $G_{12} \times_{X_1 \times X_2} G_{12}$ -equivariant. This action preserves the inductive structure of the diagram, and also the locally nice structure of each closed piece, which allows us to view the appropriately shifted dualizing complex on each space as $G_{12} \times_{X_1 \times X_2} G_{12}$ -equivariant. We may thus define the shifted equivariant Borel-Moore homologies:

$$\begin{aligned} & H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2), \\ & H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_1^1 \times \mathcal{R}_2^2), \\ & H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_{1+2}), \end{aligned}$$

as the colimits of the equivariant cohomologies of the appropriately shifted dualizing complexes on the various finite-dimensional approximations. We have maps

$$\begin{aligned} H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) &\rightarrow H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) \\ &\rightarrow H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_{1+2}). \end{aligned}$$

The first map is the restriction of the equivariant structure, while the second is induced by α^* , using (B.3). This is the first step of our global convolution story.

(c). Let’s define the remaining parts of the global convolution diagram. We set

$$\tilde{\mathcal{R}}_{1+2} = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2, g_1)\} / \sim$$

where $x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_1, v_2$ are as in \mathcal{R}_{1+2} , and g_1 is a trivialization of \mathcal{E}_1 (on Δ_{12}) required to satisfy:

$$g_1 v_1 = f_2 v_2.$$

Note that v_1 is determined by the rest of the data as $v_1 = g_1^{-1} f_2 v_2$. That is, $\widetilde{\mathcal{R}}_{1+2}$ is related to

$$\widetilde{\mathcal{T}}_{1+2} := \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, f_2, v_2, g_1)\} / \sim = G_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2$$

by the Cartesian square

$$\begin{array}{ccc} \widetilde{\mathcal{R}}_{1+2} & \rightarrow & \widetilde{\mathcal{T}}_{1+2} \\ \downarrow & & \downarrow \\ \mathcal{R}_{12}^1 & \rightarrow & \mathcal{T}_{12}^1 \end{array}$$

where the rightmost downward arrow is the composition

$$\widetilde{\mathcal{T}}_{1+2} = G_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2 \rightarrow G_{12}^1 \times_{X_1 \times X_2} \mathbf{N}_{12} \rightarrow G_{12}^1 \frac{\times_{X_1 \times X_2}}{G_{12}} \mathbf{N}_{12} = \mathcal{T}_{12}^1.$$

We have factor-wise actions of $G_{12} \times_{X_1 \times X_2} G_{12}$ on $\widetilde{\mathcal{R}}_{1+2}, \widetilde{\mathcal{T}}_{1+2}$, such that the Cartesian diagram

$$(B.4) \quad \begin{array}{ccc} \widetilde{\mathcal{R}}_{1+2} & \xrightarrow{\beta} & \mathcal{R}_{1+2} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{T}}_{1+2} & \xrightarrow{b} & \mathcal{T}_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2 \end{array}$$

is equivariant. In terms of points, the left-hand G_{12} acts by changing the trivialization f_1 , while the right-hand factor acts by changing simultaneously the trivializations g_1, f_2 ; β is the map which simply forgets g_1 . The right-hand G_{12} acts freely, and the quotient space is

$$\overline{\mathcal{R}}_{1+2} = \{((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, g_1^{-1} f_2, v_1, v_2)\} / \sim$$

where $x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, f_1, v_1$ are as in \mathcal{R}_{1+2} , while $g_1^{-1} f_2$ is an isomorphism from \mathcal{E}_2 to \mathcal{E}_1 over Δ_{12}^2 , and v_2 is an \mathbf{N} -section of \mathcal{E}_2 such that $g_1^{-1} f_2 v_2$ extends to Δ_{12} and is equal to v_1 there (again v_1 is determined by the rest of the data). We write

$$\gamma: \widetilde{\mathcal{R}}_{1+2} \rightarrow \overline{\mathcal{R}}_{1+2}$$

for the projection. It is ind-pro-smooth. Finally, we have a natural map

$$\begin{array}{ccc} \delta: \overline{\mathcal{R}}_{1+2} & \rightarrow & \mathcal{R}_{12}^{12} = \{((x_1, x_2), \mathcal{E}, f, v)\} / \sim \\ ((x_1, x_2), \mathcal{E}_1, \mathcal{E}_2, f_1, g_1^{-1} f_2, v_1, v_2) & \mapsto & ((x_1, x_2), \mathcal{E}_2, f_1 g_1^{-1} f_2, v_2). \end{array}$$

Note that δ factors as $\delta = \delta' \delta''$ where $\delta'': \overline{\mathcal{R}}_{1+2} \rightarrow \bullet$ is an ind-closed embedding of finite codimension and $\delta': \bullet \rightarrow \mathcal{R}_{12}^{12}$ is defined by the Cartesian

square

$$\begin{array}{ccc}
 \bullet & & \xrightarrow{\delta'} \mathcal{R}_{12}^{12} \\
 \downarrow & & \downarrow \\
 G_{12}^1 \times_{G_{12}}^{X_1 \times X_2} G_{12}^2 / G_{12} & \xrightarrow{d} & G_{12}^{12} / G_{12}
 \end{array}$$

where the bottom row is simply the top row for $\mathbf{N} = 0$, and the vertical maps forget v_1, v_2, v . It is well-known that d is ind-projective; this fact shows up already in [MV07] and essentially follows from Lemma B.1. It follows that δ is also ind-projective, meaning that in each piece of the inductive structure, δ is Zariski-locally of the form

$$Y \times \mathbb{A} \xrightarrow{f \times \text{id}} Z \times \mathbb{A}$$

for $f: Y \rightarrow Z$ a projective map between schemes of finite type, and \mathbb{A} some affine space of countable dimension. In fact, δ is an isomorphism over U , while over the diagonal its fibers are products of closed subvarieties of affine Grassmannians. Furthermore, δ is G_{12} -equivariant.

(d). The global convolution diagram is

$$\mathcal{R}_1^1 \times \mathcal{R}_2^2 \xleftarrow{\alpha} \mathcal{R}_{1+2} \xleftarrow{\beta} \tilde{\mathcal{R}}_{1+2} \xrightarrow{\gamma} \overline{\mathcal{R}}_{1+2} \xrightarrow{\delta} \mathcal{R}_{12}^{12}.$$

As we have explained, α, β are $G_{12} \times_{X_1 \times X_2} G_{12}$ -equivariant, γ is the quotient map by the free action of the right-hand G_{12} , and δ is equivariant for the remaining copy of G_{12} . We have already explained how α defines a map

$$\alpha^* : H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) \rightarrow H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_{1+2}).$$

Everything else works out essentially as in the main paper, as we now indicate. First, recall the $G_{12} \times_{X_1 \times X_2} G_{12}$ -equivariant Cartesian diagram (B.4):

$$\begin{array}{ccc}
 \tilde{\mathcal{R}}_{1+2} & \xrightarrow{\beta} & \mathcal{R}_{1+2} \\
 \downarrow & & \downarrow \\
 \tilde{\mathcal{T}}_{1+2} & \xrightarrow{b} & \mathcal{T}_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2
 \end{array}$$

and recall that $\tilde{\mathcal{T}}_{1+2}$ is nothing other than $G_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2$. Thus we may write

$$\begin{aligned}
 b &= \text{pr}_1 b \times_{X_1 \times X_2} \text{pr}_2 b \\
 \text{pr}_1 b &= \psi \phi \\
 \text{pr}_2 b &= \text{pr}_2
 \end{aligned}$$

where we have factored $\text{pr}_1 b$ as

$$G_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2 \xrightarrow{\phi} G_{12}^1 \times_{X_1 \times X_2} \mathbf{N}_{12} \xrightarrow{\psi} \mathcal{T}_{12}^1.$$

It follows that

$$b^* \omega_{\mathcal{T}_{12}^1 \times_{X_1 \times X_2} \mathcal{R}_{12}^2}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}] \cong \omega_{\tilde{\mathcal{T}}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}]$$

and hence by base change we have have a map

$$(B.5) \quad \beta^* \omega_{\mathcal{R}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}] \rightarrow \omega_{\tilde{\mathcal{R}}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}].$$

This map is equivariant, and it therefore determines a ‘pullback with support’ map:

$$\beta^* : H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_{1+2}) \rightarrow H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\tilde{\mathcal{R}}_{1+2}).$$

Since it is a G_{12} -torsor, γ induces an isomorphism

$$\gamma^* : H_{*-2 \dim \mathbf{N}_{12}}^{G_{12}}(\overline{\mathcal{R}}_{1+2}) \xrightarrow{\sim} H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\tilde{\mathcal{R}}_{1+2}).$$

Finally since it is ind-proper and equivariant, δ induces a map

$$\delta_* : H_{*-2 \dim \mathbf{N}_{12}}^{G_{12}}(\overline{\mathcal{R}}_{1+2}) \rightarrow H_{*-2 \dim \mathbf{N}_{12}}^{G_{12}}(\mathcal{R}_{12}^{12}).$$

(e). Recall that (dual) specialization maps commute with pullbacks along smooth maps and pushforwards along proper maps, and are compatible with equivariance with respect to smooth group schemes. Therefore, since every space in sight is a reasonably nice ind-scheme and the groups G_S are pro-smooth over X^S , we have (dual) specialization maps to the diagonal $X_0 \subset X_1 \times X_2$:

$$\begin{aligned} s_1 : H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) &\rightarrow H_{*-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0}^{G_0 \times_{X_0} G_0}(\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0) \\ s_2 : H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) &\rightarrow H_{*-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0}^{G_0 \times_{X_0} G_0}(\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0) \\ s_3 : H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\mathcal{R}_{1+2}) &\rightarrow H_{*-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0}^{G_0 \times_{X_0} G_0}(\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0) \\ s_4 : H_{*-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12}}^{G_{12} \times_{X_1 \times X_2} G_{12}}(\tilde{\mathcal{R}}_{1+2}) &\rightarrow H_{*-2 \dim \mathbf{N}_0 \times_{X_0} G_0}^{G_0 \times_{X_0} G_0}(\tilde{\mathcal{R}}_0) \\ s_5 : H_{*-2 \dim \mathbf{N}_{12}}^{G_{12}}(\overline{\mathcal{R}}_{1+2}) &\rightarrow H_{*-2 \dim \mathbf{N}_0}^{G_0}(\overline{\mathcal{R}}_0) \\ s_6 : H_{*-2 \dim \mathbf{N}_{12}}^{G_{12}}(\mathcal{R}_{12}^{12}) &\rightarrow H_{*-2 \dim \mathbf{N}_0}^{G_0}(\mathcal{R}_0^0). \end{aligned}$$

Here $\tilde{\mathcal{R}}_0, \overline{\mathcal{R}}_0$ are respectively locally trivial $p^{-1}(\mathcal{R} \times \mathcal{R}), q(p^{-1}(\mathcal{R} \times \mathcal{R}))$ -bundles over X_0 in the notations of diagram (3.2). In fact, the restriction

of the convolution diagram to X_0 induces the following maps between the targets of the specialization maps:

$$\begin{aligned} H_{*-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0}^{G_0 \times_{X_0} G_0}(\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0) &\xrightarrow{\text{id}} H_{*-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0}^{G_0 \times_{X_0} G_0}(\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0) \\ &\xrightarrow{\text{id}} H_{*-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0}^{G_0 \times_{X_0} G_0}(\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0) \\ &\xrightarrow{\beta_0^*} H_{*-2 \dim \mathbf{N}_0 \times_{X_0} G_0}^{G_0 \times_{X_0} G_0}(\tilde{\mathcal{R}}_0) \\ &\xrightarrow{(\gamma_0^*)^{-1}} H_{*-2 \dim \mathbf{N}_0 \times_{X_0} G_0}^{G_0 \times_{X_0} G_0}(\tilde{\mathcal{R}}_0) \\ &\xrightarrow{(\delta_0)_*} H_{*-2 \dim \mathbf{N}_0}^{G_0}(\mathcal{R}_0^0) \end{aligned}$$

I claim that the maps $\alpha^*, \beta^*, (\gamma^*)^{-1}, \delta_*$ are intertwined with $\text{id}, \beta_0^*, (\gamma_0^*)^{-1}, (\delta_0)_*$ by the (dual) specialization maps. For $\alpha^*, (\gamma^*)^{-1}$ it is a consequence of ind-pro-smoothness of α, γ (and also pro-smoothness of G_{12}). For δ_* it is a consequence of ind-properness. For β^* , it is because the map

$$(\beta_0)^* \omega_{\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0}[-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0] \rightarrow \omega_{\tilde{\mathcal{R}}_0}[-2 \dim \mathbf{N}_0 \times_{X_0} G_0]$$

defined using the Cartesian square:

$$\begin{array}{ccc} \tilde{\mathcal{R}}_0 & \xrightarrow{\beta_0} & \mathcal{R}_0^0 \\ \downarrow & & \downarrow \\ \tilde{\mathcal{T}}_0^0 & \xrightarrow{b_0} & \mathcal{T}_0^0 \times_{X_0} \mathcal{R}_0^0 \end{array}$$

obtained by restricting diagram (B.4) to X_0 , factors as:

$$\begin{aligned} &(\beta_0)^* \omega_{\mathcal{R}_0^0 \times_{X_0} \mathcal{R}_0^0}[-2 \dim \mathbf{N}_0 \times_{X_0} \mathbf{N}_0] \\ &\cong (\beta_0)^* i_1^! \omega_{\mathcal{R}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12} + 2] \\ &\xrightarrow{\text{can}} i_2^! \beta^* \omega_{\mathcal{R}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} \mathbf{N}_{12} + 2] \\ &\xrightarrow{i_2^![2](\text{B.5})} i^! \omega_{\tilde{\mathcal{R}}_{1+2}}[-2 \dim \mathbf{N}_{12} \times_{X_1 \times X_2} G_{12} + 2] \\ &\cong \omega_{\tilde{\mathcal{R}}_0}[-2 \dim \mathbf{N}_0 \times_{X_0} G_0]. \end{aligned}$$

Here *can* is the canonical map arising from the base change isomorphism, (B.5) denotes the map of (B.5), and i_1, i_2 denote the appropriate inclusions of the diagonal subspaces. The consequence is the following formula:

$$\begin{aligned} s_6 \delta_* (\gamma^*)^{-1} \beta^* \alpha^* &= (\delta_0)_* (\gamma_0^*)^{-1} \beta_0^* s_1 : H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) \\ &\rightarrow H_{*-2 \dim \mathbf{N}_0}^{G_0}(\mathcal{R}_0^0). \end{aligned}$$

(f). Now each (dual) specialization map s_n factors as $s'_n j_n^*$ where j_n^* is the restriction map to the equivariant Borel-Moore homology of the part lying over U , and s'_n is some other map. Furthermore, the restriction of the convolution diagram to U induces the following maps between the targets of the restriction maps:

$$\begin{aligned}
 & H_{*-2 \dim(\mathbf{N}_1 \times \mathbf{N}_2)}^{(G_1 \times G_2)|_U} ((\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U) \\
 \rightarrow & H_{*-2 \dim(\mathbf{N}_1 \times \mathbf{N}_2)}^{(G_1 \times G_2)|_U \times_U (G_1 \times G_2)|_U} ((\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U) \\
 \rightarrow & H_{*-2 \dim(\mathbf{N}_1 \times \mathbf{N}_2)}^{(G_1 \times G_2)|_U \times_U (G_1 \times G_2)|_U} ((\mathcal{R}_1^1 \times \mathbf{N}_2)|_U \times_U (\mathbf{N}_1 \times \mathcal{R}_2^2)|_U) \\
 \rightarrow & H_{*-2 \dim((\mathbf{N}_1 \times_{X_1} G_1) \times (G_2 \times_{X_2} G_2))|_U} ((\tilde{\mathcal{R}}_1 \times (G_2 \times_{X_2} \mathcal{R}_2^2))|_U) \\
 \rightarrow & H_{*-2 \dim(\mathbf{N}_1 \times \mathbf{N}_2)}^{(G_1 \times G_2)|_U} ((\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U) \\
 \rightarrow & H_{*-2 \dim(\mathbf{N}_1 \times \mathbf{N}_2)}^{(G_1 \times G_2)|_U} ((\mathcal{R}_1^1 \times \mathcal{R}_2^2)|_U)
 \end{aligned}$$

Let us explain what each map does:

- 1) The first map views any $(G_1 \times G_2)|_U$ -equivariant class as also equivariant for the trivial actions of the left-hand copy of G_2 , and the right-hand copy of G_1 , in $(G_1 \times G_2)|_U \times_U (G_1 \times G_2)|_U$.
- 2) The second map pulls this back along the $(\mathbf{N}_2 \times \mathbf{N}_1)|_U$ -bundle map (i.e. multiplies fiberwise by the equivariant fundamental class of $\mathbf{N}(\mathcal{O}) \times \mathbf{N}(\mathcal{O})$).
- 3) The third map starts by rewriting $(\mathcal{R}_1^1 \times \mathbf{N}_2)|_U \times_U (\mathbf{N}_1 \times \mathcal{R}_2^2)|_U$ as $((\mathcal{R}_1^1 \times_{X_1} \mathbf{N}_1) \times (\mathbf{N}_2 \times_{X_2} \mathcal{R}_2^2))|_U$, and rewriting the action of $(G_1 \times G_2)|_U \times_U (G_1 \times G_2)|_U$ as one of $((G_1 \times_{X_1} G_1) \times (G_2 \times_{X_2} G_2))|_U$. By definition, $\tilde{\mathcal{R}}_1$ is the locally trivial $p^{-1}(\mathcal{R} \times \mathcal{R})$ -bundle on X_1 given as

$$\tilde{\mathcal{R}}_1 = \mathbf{N}_1 \times_{\mathbf{N}_1^1} (G_1^1 \times_{X_1} \mathbf{N}_1).$$

The $G_1 \times_{X_1} G_1$ -equivariant map from here to $\mathcal{R}_1^1 \times_{X_1} \mathbf{N}_1$ is given as the product (over X_1) of the quotient by the right-hand copy of G_1 with the projection to the right-hand copy of \mathbf{N}_1 . The ‘pullback with support’ map

$$H_{*-2 \dim \mathbf{N}_1 \times_{X_1} \mathbf{N}_1}^{G_1 \times_{X_1} G_1} (\mathcal{R}_1^1 \times_{X_1} \mathbf{N}_1) \rightarrow H_{*-2 \dim \mathbf{N}_1 \times_{X_1} G_1}^{G_1 \times_{X_1} G_1} (\tilde{\mathcal{R}}_1)$$

corresponds to the composition of usual ‘pullback with support’ (spread out over X_1) with multiplication by $H_{G_1}^*(X_1)$ under the identification

$$H_{*-2 \dim \mathbf{N}_1 \times_{X_1} \mathbf{N}_1}^{G_1 \times_{X_1} G_1}(\mathcal{R}_1^1 \times_{X_1} \mathbf{N}_1) = H_{*-2 \dim \mathbf{N}_1}^{G_1}(\mathcal{R}_1^1) \otimes_{H^*(X_1)} H_{G_1}^*(X_1).$$

Meanwhile, the ‘pullback with support’ (actually, here no support is required) map

$$H_{*-2 \dim \mathbf{N}_2 \times_{X_2} \mathbf{N}_2}^{G_2 \times_{X_2} G_2}(\mathbf{N}_2 \times_{X_2} \mathcal{R}_2^2) \rightarrow H_{*-2 \dim G_2 \times_{X_2} \mathbf{N}_2}^{G_2 \times_{X_2} G_2}(G_2 \times_{X_2} \mathcal{R}_2^2)$$

is isomorphic simply to the multiplication map

$$H_{G_2}^*(X_2) \otimes_{H^*(X_2)} H_{*-2 \dim \mathbf{N}_2}^{G_2}(\mathcal{R}_2^2) \rightarrow H_{*-2 \dim \mathbf{N}_2}^{G_2}(\mathcal{R}_2^2).$$

4) The fourth map is the isomorphism, and the fifth is the identity.

The result is that the composition of all these maps is the identity. On the other hand, since the restriction maps j_n^* intertwine these maps with the corresponding maps on the $X_1 \times X_2$ level, we have the following:

$$\begin{aligned} (\delta_0)_*(\gamma_0^*)^{-1} \beta_0^* s_1 &= s_6 \delta_*(\gamma^*)^{-1} \beta^* \alpha^* \\ &= s'_6 j_6^* \delta_*(\gamma^*)^{-1} \beta^* \alpha^* \\ &= s'_6 j_1^*. \end{aligned}$$

(g). Finally, note that this last map $s'_6 j_1^*$ is symmetric with respect to the automorphism τ of $H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2)$ induced by the degree 2 automorphisms of $G_1 \times G_2$, $\mathcal{R}_1^1 \times \mathcal{R}_2^2$ which switch the factors (and also exchange 1 with 2). Therefore, $(\delta_0)_*(\gamma_0^*)^{-1} \beta_0^* s_1$ has the same property. But, taking $X = \mathbb{C}$, we identify the domain

$$H_{*-2 \dim \mathbf{N}_1 \times \mathbf{N}_2}^{G_1 \times G_2}(\mathcal{R}_1^1 \times \mathcal{R}_2^2) = H_{*-2 \dim \mathbf{N}(\mathcal{O})}^{G(\mathcal{O})}(\mathcal{R}) \otimes H_{*-2 \dim \mathbf{N}(\mathcal{O})}^{G(\mathcal{O})}(\mathcal{R})$$

and the target

$$H_{*-2 \dim \mathbf{N}_0}^{G_0}(\mathcal{R}_0^0) = H_{*-2 \dim \mathbf{N}(\mathcal{O})}^{G(\mathcal{O})}(\mathcal{R}).$$

The map $(\delta_0)_*(\gamma_0^*)^{-1} \beta_0^* s_1$ is the usual convolution (s_1 is an isomorphism) while τ is the standard twist. Therefore, the Coulomb branch is commutative as claimed.

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