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4d $\mathcal{N} = 2$ SCFT and singularity theory Part III: Rigid singularity

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Dedicated to Professor Herbert Blaine Lawson, Jr.
on the occasion of his 77th birthday

We classify three fold isolated quotient Gorenstein singularity C^3/G. These singularities are rigid, i.e. there is no non-trivial deformation, and we conjecture that they define 4d $\mathcal{N} = 2$ SCFTs which do not have a Coulomb branch.

1. Introduction

Four dimensional (4d) $\mathcal{N} = 2$ superconformal field theory (SCFT) can be defined using type IIB string theory on following background

$$R^{1,3} \times X; \tag{1}$$

Here X is conjectured to be an isolated rational Gorenstein singularity [XY] with a good C^* action, and we take string coupling $g_s \to 0$ and go to infrared limit [SV, GKP]. These rational Gorenstein singularities naturally appear in the degeneration limit of compact Calabi-Yau three manifolds, and in fact general definition of Calabi-Yau variety allows such singularity [G].

4d $\mathcal{N} = 2$ SCFT has a $SU(2)_R \times U(1)_R$ R symmetry, and there are two kinds of half-BPS operators $E_{r,(0,0)}$ and $\hat{B}_1$ [DO]. The Coulomb branch deformations are described as follows [ALLM]:

1) Deformation using half-BPS operator $E_{r,(0,0)}$:

$$\delta S = \lambda \int d^4x dQ^4 E_{r,(0,0)} + c.c. \tag{2}$$

2) Deformation using half-BPS operator $\hat{B}_1$:

$$\delta S = m \int d^4 x Q^2 \hat{B}_1 + c.c. \tag{3}$$

3) We can also turn on expectation value of operator $E_{r,(0,0)}$: $u_r = \langle E_{r,(0,0)} \rangle$.

A central question of understanding 4d $\mathcal{N}=2$ SCFT is to understand the low energy physics for general deformations parameterized by $S=(\lambda, m, u_r)$. The low energy physics is best captured by the Seiberg-Witten geometry [SW]. Usually Seiberg-Witten geometry is described by a family of Rieman surfaces fibered over space S, and it is conjectured in [XY] that more general Coulomb branch geometry can be captured by the **mini-versal** deformation of certain kind of three fold singularity X [GLS]. Roughly speaking, a deformation is a flat morphism $\pi : Y \to S$, with $\pi^{-1}(0)$ isomorphic to the singularity X, and a mini-versal deformation essentially captures all the deformations. Here S is identified with the parameter space (λ, m, u_r) of our (generalized) Coulomb branch.

Therefore the study of 4d $\mathcal{N}=2$ SCFT and its Coulomb branch solution are reduced to the study of singularity X and its mini-versal deformation. We have classified such X which can be described by complete intersection [XY, YY1, CX], and the physical aspects of these 4d $\mathcal{N}=2$ SCFTs are studied in [XY1, XY2, XY3, XYY]. All the complete intersection examples studied in [XY, YY1, CX] have non-trivial mini-versal deformation and therefore non-trivial Coulomb branch.

The purpose of this note is to study non-complete intersection rational Gorenstein singularities. An interesting class of such singularities are quotient singularity C^3/G with G a finite subgroup of $SL(3)$. One of main results of this paper is the classification of the three dimensional isolated Gorenstein quotient singularity.

We then would like to study mini-versal deformation of these singularities, and a surprising theorem by Schlessinger [S] shows that all such singularities are rigid, i.e. they have no non-trivial deformation[1]. Therefore the corresponding 4d theory has no Coulomb branch[2]. We call such theories rigid $\mathcal{N}=2$ theories. It would be very interesting to study more properties of these theories.

[1]See [V] for example of rigid compact Calabi-Yau manifolds.

[2]Free hypermultiplets do have a Coulomb branch as we can turn on mass deformation.

2. Three-fold singularity and 4d $\mathcal{N} = 2$ SCFT

Let's discuss more about the interpretation of $\mathcal{N} = 2$ SCFT defined using three fold rational Gorenstein singularity (they are also called canonical singularity [R]). There are two special ways of smoothing a singularity: crepant resolution [R] and mini-versal deformation [GLS]. For the singularities we are interested, we have following facts:

- Every isolated singularity has a mini-versal deformation [GLS], however, the deformation might be trivial. A class of examples are the quotient singularity considered in this paper.
- Every three fold canonical singularity has a crepant resolution $f : Y \rightarrow X$ such that Y is Q-factorial terminal. [3] [K]. There is no crepant resolution for Q-factorial terminal singularity. An example of Q-factorial terminal singularity is the hypersurface singularity: $x^2 + y^2 + z^2 + w^{2k+1} = 0$ [R]. The quotient singularity considered in this paper has a crepant resolution with Y smooth as can be seen using toric method.

Now let's try to interpret the appearance of SCFT using the smoothing of singularity:

- If our singularity admits non-trivial deformation and the smooth manifold has three cycles (such as the hypersurface singularity), the low energy effective theory includes massless vector multiplet from compactifying self-dual RR four form, and we also have massive BPS states from D3 brane wrapping three cycles. These massive BPS states are in general mutually non-local. In the singular limit, the massive BPS states become massless, and it is expected that one get a SCFT [APSW].
- If our singularity admits non-trivial crepant resolution, and the smooth manifold has two cycles and four cycles. One can have massless hypermultiplets using various NS-NS and RR two forms, and one also have tensile strings from wrapping D3 branes on two cycles (or D5 branes on four cycles). In the singular limit, one get tensionless string and it is expected that one get a SCFT [W].

The SCFT considered in [XY, WX] can be interpreted using the deformation of singularity, while the SCFT considered in this paper can be interpreted using crepant resolution.

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[3] A Q-factorial variety means that every Weil divisor on it is Q-Cartier, i.e., some multiple of it is a Cartier divisor.

The Coulomb branch of a 4d theory is described by the deformation, while the Higgs branch is described by the crepant resolution. The exact Coulomb branch physics is described by the classical geometry of the deformation. The exact Higgs branch is difficult to compute, but we can count its dimension by computing the dimension of Mori cone[4] associated with the crepant resolution. The number of abelian flavor symmetry is given by the rank of local class group of the singularity.

Example 1. Let's consider a 3d singularity defined by equation $x^2 + y^2 + z^2 + w^{2k+1} = 0$, and the corresponding $\mathcal{N} = 2$ SCFT is (A_1, A_{2k}) Argyres-Douglas theory. The Coulomb branch is identified with the base of mini-versal deformation from which one can compute the Coulomb branch spectrum. There is no Higgs branch, and this agrees with the fact that there is no crepant resolution for the singularity.

Example 2. Let's consider the singularity $x^2 + y^2 + z^2 + w^{2k} = 0$, and the corresponding $\mathcal{N} = 2$ SCFT is the (A_1, A_{2k-1}) Argyres-Douglas theory. The Coulomb branch is identified with the base of mini-versal deformation from which one can compute the Coulomb branch spectrum. There is a one dimensional Higgs branch, and this agrees with the fact that there is a crepant resolution whose Mori cone has dimension one!

3. Classification of rigid quotient singularity

Let G be a finite subgroup of $GL(3, \mathbb{C})$ and it acts on $\mathbb{C}^3$ in a natural may. Cartan [Car] has studied the quotient variety $\mathbb{C}^3/G$ and proved that the singularities of $\mathbb{C}^3/G$ are normal. So the dimension of the singular set of $\mathbb{C}^3/G$ is either 0 or 1. In this article we are interested in the case that $\mathbb{C}^3/G$ has a Gorenstein isolated singularity. By a theorem of Khinich [Kh] and Watanabe [Wa], we know that

Theorem 3.1. *([Kh] and [Wa]) Let G be a finite subgroup of $GL(3, \mathbb{C})$. Then $\mathbb{C}^3/G$ is Gorenstein if and only if G is a subgroup of $SL(3, \mathbb{C})$.*

Let G' be another finite subgroup of $GL(3, \mathbb{C})$. We say G is linear equivalent to G' if there exists $g \in GL(3, \mathbb{C})$ such that $G = gGg^{-1}$. It's obvious that $\mathbb{C}^3/G \cong \mathbb{C}^3/G'$ if G is linear equivalent to G'. Yau and Yu [YY2] tell us that

[4]Mori cone describes the space of complete curves, which will generate free hypermultiplets.

Theorem 3.2. *([YY2]) Let G be a finite subgroup of $SL(3,\mathbb{C})$, then $\mathbb{C}^3/G$ has a Gorenstein isolated singularity if and only if G is linear equivalent to a diagonal abelian subgroup (i.e. any element in this subgroup is a diagonal matrix) and 1 is not an eigenvalue of g for every nontrivial element g in G.*

In this section, we will find out all subgroups $G \subseteq SL(3,\mathbb{C})$ which satisfy the condition in Theorem 3.2, i.e. all the subgroups which corresponds to a three-dimensional Gorenstein isolated quotient singularity. In fact, we prove that

Theorem 3.3. *Let G be a finite subgroup of $SL(3,\mathbb{C})$. Then $\mathbb{C}^3/G$ has a Gorenstein isolated singularity if and only if G is linear equivalent to a cyclic subgroup which is generated by a diagonal matrix*

$$\begin{pmatrix} \zeta(1/n) & 0 & 0 \\ 0 & \zeta(p/n) & 0 \\ 0 & 0 & \zeta(q/n) \end{pmatrix}$$

where $\zeta(*) = e^{2\pi\sqrt{-1}*}$ and p, q, n are positive integers such that p, q are coprime with n and $1 + p + q = n$.

Before the proof, we first introduce some notations.

(1) Let g be a monomial in $\mathbb{C}[x_1, \ldots, x_n]$. Denote by $\mathrm{Supp}(g)$ the set consists of variables involved in g. For example, if $g = x_1x_2$, then $\mathrm{Supp}(g) = \{x_1, x_2\}$.

(2) We denote by $\langle a, b, c\rangle$ the 3×3 diagonal matrix whose diagonal elements are a, b, c. Similarly we denote by $\langle a, b\rangle$ the 2×2 diagonal matrix whose diagonal elements are a, b.

(3) Let $\zeta(q) = e^{2\pi\sqrt{-1}q}$ for any real number q.

(4) If A is a matrix, we denote its (i, j)-entry by $A[i, j]$.

Proof of Theorem 3.3. We first prove the sufficiency. If G is generated by a diagonal matrix $\langle \zeta(1/n), \zeta(p/n), \zeta(q/n)\rangle$, where p, q, n are positive integers, p, q are coprime with n and $1 + p + q = n$, then each element $g \in G$ can be written as $\langle \zeta(k/n), \zeta(kp/n), \zeta(kq/n)\rangle$ for some integer k. If 1 is an eigenvalue of g, since $1, p, q$ are coprime with n, we have $k \equiv 0 \pmod{n}$, which follows that g is the unit matrix. By Theorem 3.2, $\mathbb{C}^3/G$ has a Gorenstein isolated singularity.

Next we prove the necessity. If $\mathbb{C}^3/G$ has an isolated singularity, then by Theorem 3.2, 1 is not an eigenvalue of g for every nontrivial element g in G and we may suppose that G is a diagonal abelian subgroup. By the fundamental theorem for finite abelian groups, G is the direct sum of cyclic groups:

$$G = \oplus_{i=1}^{m} \oplus_{j=1}^{r_i} G_{ij}$$

where G_{ij} is a cyclic group whose order is $p_i^{n_{ij}}$, $p_1, p_2, \ldots, p_m$ are distinct prime numbers and

$$1 \le n_{i1} \le n_{i2} \le \cdots \le n_{ir_i}, \quad i = 1, \ldots, m.$$

G_{ij} is generated by a diagonal matrix

$$g_{ij} = \langle \zeta(a_{ij}/p_i^{n_{ij}}), \zeta(b_{ij}/p_i^{n_{ij}}), \zeta(c_{ij}/p_i^{n_{ij}}) \rangle$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, r_i$, and $a_{ij} + b_{ij} + c_{ij} \equiv 0 \pmod{p_i^{n_{ij}}}$. Since $g_{ij}^t \ne I$ (I is the unit matrix) for $1 \le t < p_i^{n_{ij}}$, 1 is not an eigenvalue of g_{ij}^t, hence $ta_{ij}, tb_{ij}, tc_{ij} \not\equiv 0 \pmod{p_i^{n_{ij}}}$ for $1 \le t < p_i^{n_{ij}}$. Thus a_{ij}, b_{ij}, c_{ij} are coprime with p_i for $i = 1, \ldots, m$ and $j = 1, \ldots, r_i$.

We claim that $r_i = 1$ for $i = 1, \ldots, m$. Assume the opposite that $r_1 > 1$, and for convenience in the sequel we will denote p_1 by p and denote $G_{1i}, g_{1i}, a_{1i}, b_{1i}, c_{1i}, n_{1i}$ by $G_i, g_i, a_i, b_i, c_i, n_i$ respectively, so G_1 is generated by

$$g_1 = \langle \zeta(a_1/p^{n_1}), \zeta(b_1/p^{n_1}), \zeta(c_1/p^{n_1}) \rangle$$

and G_2 is generated by

$$g_2 = \langle \zeta(a_2/p^{n_2}), \zeta(b_2/p^{n_2}), \zeta(c_2/p^{n_2}) \rangle.$$

Since a_2 is coprime with p, there exist a integer s such that $p^{n_1} \mid (a_1 + sa_2)$, hence $p^{n_2} \mid (p^{n_2-n_1}a_1 + p^{n_2-n_1}sa_2)$. Let $s' = p^{n_2-n_1}s$ then

$$\zeta(a_1/p^{n_1})\zeta(a_2/p^{n_2})^{s'} = \zeta((p^{n_2-n_1}a_1 + s'a_2)/p^{n_2}) = 1,$$

thus 1 is an eigenvalue of $g_1 g_2^{s'}$. It follows that $g_1 g_2^{s'} = I$. That leads to contradiction with $G_1 \cap G_2 = \{I\}$. Thus $r_1 = 1$. Similarly we have $r_i = 1$ for $i = 1, 2, \ldots, m$ and thus G is generated by matrices

$$g_i = \langle \zeta(a_i/p_i^{n_i}), \zeta(b_i/p_i^{n_i}), \zeta(c_i/p_i^{n_i}) \rangle$$

for $i = 1, \ldots, m$, where $p_1, p_2, \ldots, p_m$ are distinct primes, a_i, b_i, c_i are coprime with p_i and $a_i + b_i + c_i \equiv 0 \pmod{p_i^{n_i}}$.

Since a_i is coprime with p_i, there exists a integer s_i such that $0 \le s_i < p_i^{n_i}$ and $a_i s_i + b_i \equiv 0 \pmod{p_i^{n_i}}$. Using the fact $p_i's$ are pairwise distinct prime and Chinese remainder theorem, there exist a integer k such that $k \equiv s_i \pmod{p_i^{n_i}}$, hence $a_i k + b_i \equiv 0 \pmod{p_i^{n_i}}$. Let $n = \prod_{i=1}^{m} p_i^{n_i}$. Next we prove that G is generated by a matrix

$$g = \langle \zeta(1/n), \zeta((n-k)/n), \zeta((k-1)/n) \rangle.$$

Let $k_i = a_i \prod_{j \neq i} p_j^{n_j}$, then k_i is coprime with p_i for $i = 1, 2, \dots m$. Then we have $g[1,1]^{k_i} = g_i[1,1]$ (as we have mentioned above $g[a,b]$ (resp. $g_i[a,b]$) means the (a,b)-entry of g (resp. g_i)). And since

$$g[2,2] = g[1,1]^{-k}, \quad g_i[2,2] = g_i[1,1]^{-k}$$
$$g[3,3] = g[1,1]^{-1} g[2,2]^{-1}, \quad g_i[3,3] = g_i[1,1]^{-1} g_i[2,2]^{-1},$$

we have $g[2,2]^{k_i} = g_i[2,2]$ and $g[3,3]^{k_i} = g_i[3,3]$, which implies that $g^{k_i} = g_i$. Since k_i is coprime with p_i for each i, then the greatest common divisor of $n, k_1, k_2, \dots, k_n$ is 1, thus there exist t_i such that $t_1 k_1 + t_2 k_2 + \cdots t_m k_m \equiv 1$ (mod n). Hence $\prod_{i=1}^{m} g_i^{t_i} = \prod_{i=1}^{m} g^{t_i k_i} = g$ (because $g^n = 1$), which implies $g \in G$. Hence G is generated by the matrix g.

Finally we only need to prove $n - k$ and $k - 1$ is coprime with n. If $n - k$ is not coprime with n, then there exists $0 < r < n$ such that $n \mid (n-k)r$. Then g^r has eigenvalue 1 but g^r is not the unit matrix, which leads to contradiction. Similarly we can prove that $k - 1$ is coprime with n and the proof is complete. □

Minimal generators of the invariant ring and their relations:

A polynomial $f \in \mathbb{C}[x,y,z]$ is called an invariant polynomial of $G \subseteq SL(3, \mathbb{C})$ if $f(g(p)) = f(p)$ for any element $g \in G$ and any point $p \in \mathbb{C}^3$. Denote by S^G the subalgebra of $\mathbb{C}[x,y,z]$ that consists of all invariants of G. Then the quotient variety $\mathbb{C}^3/G$ is isomorphic to the algebraic variety $\mathrm{Spec}(S^G)$. If $\{f_1, \dots, f_k\}$ is a minimal set of homogeneous polynomials which generated S^G (as a $\mathbb{C}-$algebra), then we call $f_i^{\prime}s$ minimal generators of S^G. Geometrically, k, the number of minimal generators of S^G, is the minimal embedding dimension of $\mathbb{C}^3/G$.

Consider the following ring homomorphism

$$\begin{aligned} \phi : \mathbb{C}[y_1, \dots, y_k] &\to S^G \\ y_i \quad &\mapsto f_i \end{aligned}$$

where $f_1, \dots, f_k$ are minimal generators of S^G. Let K be the kernel of ϕ, then the generators of K are called the relations of minimal generators

$f_1, \ldots, f_k$. Geometrically, these relations are the equations which define the affine variety $\mathrm{Spec}(S^G)$ as a subvariety of $\mathbb{C}^k$. Associate to $y_1, y_2, \ldots, y_k$ a weight system $(w_1, w_2, \ldots, w_k)$, where

$$w_i = \deg f_i \tag{4}$$

for $i = 1, 2, \ldots, k$. With respect to this weight system, K is a weighted homogeneous ideal of $\mathbb{C}[y_1, \ldots, y_k]$, so $\mathbb{C}^3/G$ has a weighted homogeneous singularity.

Denote by $H_{n,p}$ the subgroup of $SL(3, \mathbb{C})$ generated by the matrix

$$g_{n,p} = \langle \zeta(1/n), \zeta(p/n), \zeta((n-p-1)/n) \rangle$$

where p and $n-p-1$ are coprime with p. By Theorem 3.3 we know that $\mathbb{C}^3/G$ defines a three-dimensional Gorenstein isolated singularity. A polynomial $f \in \mathbb{C}[x, y, z]$ is an invariant polynomial of $H_{n,p}$ if each term $x^a y^b z^c$ in f satisfies

$$a + pb + (n-p-1)c \equiv 0 \pmod{n}.$$

Denote by $S_{n,p}$ the subalgebra of $\mathbb{C}[x, y, z]$ that consists of all invariants of $H_{n,p}$. Then $\mathbb{C}^3/H_{n,p}$ is isomorphic to the algebraic variety $\mathrm{Spec}(S_{n,p})$. Next we will determine a set of minimal generators of $S_{n,p}$ and find out their relations for all n, p such that p and $n-p-1$ are coprime with n.

First let's recall a result of Riemenschneider [R] about two-dimensional cyclic quotient singularities.

For any positive integer k, k can be written as

$$k = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_e}}}}$$

where $a_i's$ are positive integers. It's called the continued fraction expansion of k, and is denoted by

$$k = [a_1, a_2, \ldots, a_e].$$

We call e the length of the continued fraction expansion of k, which is denoted by $l(k)$.

Theorem 3.4. *([R]) Let $G = G_{n,p}$ be the subgroup of $SL(2, \mathbb{C})$, generated by $\begin{pmatrix} \zeta(1/n) & 0 \\ 0 & \zeta(p/n) \end{pmatrix}$. The continue fraction of $n/(n-p)$ is $[a_1, a_2, \ldots, a_e]$.*

Then a set of minimal generators of the invariant ring $\mathbb{C}[u,v]^G$ *is* $\{f_k = u^{i_k}v^{j_k}\}_{k=0}^{e+1}$, *where* i_k, j_k *are determined as follows:*

$$\begin{array}{llll} i_0 = n, & i_1 = n-p, & i_{k+1} = a_k i_k - i_{k-1} & \textit{for } 1 \le k \le e \\ j_0 = 0, & j_1 = 1, & j_{k+1} = a_k j_k - j_{k-1} & \textit{for } 1 \le k \le e \end{array} \tag{5}$$

The relations of $\mathbb{C}[u,v]^G$ *are*

$$f_{i-1}f_{j+1} = f_i f_j \prod_{k=i}^{j} f_k^{a_k-2} \tag{6}$$

for $0 < i < j < e+1$.

Remark 3.1. It is not hard to see that $i_{e+1} = 0$ and $j_{e+1} = n$. So $f_0 = u^n$ and $f_{e+1} = v^n$

Let's see a example.

Example 1. *Let* $G = G_{3,1}$, *then* $3/(3-1) = [2,2]$ *and* $e = 2$. *We have*

$$\begin{array}{llll} i_0 = 3, & i_1 = 2, & i_2 = 2i_1 - i_0 = 1 & i_3 = 2i_2 - i_1 = 0 \\ j_0 = 0, & j_1 = 1, & j_2 = 2j_1 - j_0 = 2 & j_3 = 2j_2 - j_1 = 3 \end{array}$$

Thus $\mathbb{C}[u,v]^G$ *is generated by*

$$\{f_0 = u^3, f_1 = u^2v, f_2 = uv^2, f_3 = v^3\}$$

And the relations are

$$\{f_0f_2 = f_1^2, \quad f_0f_3 = f_1f_2, \quad f_1f_3 = f_2^2\}.$$

Now come back to the three-dimensional case. Consider the subring $S_{n,p} \cap \mathbb{C}[x,y]$ of $S_{n,p}$. Since $S_{n,p} \cap \mathbb{C}[x,y]$ consists of all monomial $x^a y^b$ such that $a + pb \equiv 0 \pmod n$, we have

$$S_{n,p} \cap \mathbb{C}[x,y] = \mathbb{C}[x,y]^{G_{n,p}},$$

where $G_{n,p}$ is the subgroup of $SL(2,\mathbb{C})$ which is generated by $\langle \zeta(1/n), \zeta(p/n) \rangle$. Using Theorem 3.4, we know that $S_{n,p} \cap \mathbb{C}[x,y]$ is generated by $\{f_{1,k} = x^{i_{1,k}} y^{j_{1,k}}\}_{k=0}^{e_1+1}$, where e_1 is the length of the continue fraction $n/(n-p)$, and

$i_{1,k}$,$j_{1,k}$ is defined as equations (5) in Theorem 3.4. And the relations of $\{f_{1,k} = x^{i_{1,k}} y^{j_{1,k}}\}_{k=0}^{e_1+1}$ are

$$f_{1,i-1} f_{1,j+1} = f_{1,i} f_{1,j} \prod_{k=i}^{j} f_{1,k}^{a_{1,k}-2}, \tag{7}$$

for $0 < i < j < e_1 + 1$, where $[a_{1,1}, a_{1,2}, \ldots, a_{1,e_1}]$ is the continue fraction of $n/(n-p)$. Denote the set $\{f_{1,k} = x^{i_{1,k}} y^{j_{1,k}}\}_{k=1}^{e_1}$ by $A_{xy}(n,p)$, then $S_{n,p} \cap \mathbb{C}[x,y]$ is generated by $A_{xy}(n,p) \cup \{x^n, y^n\}$. And we denote the set of relations (7) by $R_{xy}(n,p)$. Similarly, $S_{n,p} \cap \mathbb{C}[x,z] = \mathbb{C}[x,z]^{G_{n,n-p-1}}$, and we denote the set of its minimal generators by $\{x^n, z^n\} \cup A_{xz}(n, n-p-1) = \{x^n, z^n, f_{2,1} = x^{i_{2,1}} z^{j_{2,1}}, f_{2,2} = x^{i_{2,2}} z^{j_{2,2}}, \ldots, f_{2,e_2} = x^{i_{2,e_2}} z^{j_{2,e_2}}\}$ and denote the set of relations by $R_{xz}(n, n-p-1)$. Next we consider $S_{n,p} \cap \mathbb{C}[y,z]$. Obviously $S_{n,p} \cap \mathbb{C}[y,z] = \mathbb{C}[y,z]^G$ where G is the subgroup of $SL(2,\mathbb{C})$ generated by $\langle \zeta(p), \zeta(n-p-1) \rangle$. Since p is coprime with n, there exists q such that $pq \equiv 1 \pmod{n}$ and q is coprime with n. We have $q(n-p-1) \equiv r \pmod{n}$ for some positive integer r less than n. Hence

$$\langle \zeta(p/n), \zeta((n-p-1)/n) \rangle^q = \langle \zeta(1/n), \zeta(r/n) \rangle$$

and

$$\langle \zeta(p/n), \zeta((n-p-1)/n) \rangle = \langle \zeta(1/n), \zeta(r/n) \rangle^p.$$

Hence G is generated by $\langle \zeta(1/n), \zeta(r/n) \rangle$. As before, we denote the set of minimal generator of $\mathbb{C}[y,z]^{G_{n,r}}$ by $\{y^n, z^n\} \cup A_{yz}(n,r) = \{y^n, z^n, f_{3,1} = y^{i_{3,1}} z^{j_{3,1}}, f_{3,2} = y^{i_{3,2}} z^{j_{3,2}}, \ldots, f_{3,e_3} = y^{i_{3,e_3}} z^{j_{3,e_3}}\}$ and the set of their relations by $R_{yz}(n,r)$. Obviously $xyz \in S_{n,p}$, and our following theorem will prove that $\{g_1 = x^n, g_2 = y^n, g_3 = z^n, g_4 = xyz\} \cup A_{xy}(n,p) \cup A_{xz}(n, n-p-1) \cup A_{yz}(n,r)$ is a set of minimal generators of $S_{n,p}$. These generators (exclude g_4) form a triangle as the following picture

$$\begin{array}{cccccc}
g_1 & & & & & \\
f_{1,1} & f_{2,1} & & & & \\
f_{1,2} & & f_{2,2} & & & \\
\cdots & & & \cdots & & \\
f_{1,e_1} & & & & f_{2,e_2} & \\
g_2 & f_{3,1} & f_{3,2} & \cdots & f_{3,e_3} & g_3
\end{array} \tag{8}$$

We call $\{g_1, f_{1,1}, f_{1,2}, \ldots, f_{1,e_1}, g_2\}, \{g_1, f_{2,1}, \ldots, f_{2,e_2}, g_3\}$ and $\{g_2, f_{3,1}, \ldots, f_{3,e_3}, g_3\}$ the first, second and third side of the triangle (8) respectively. Relations of generators which lie on the same side of the above triangle have been known, now we need to explore relations of generators which are on different sides. Obverse that if we take two generators f and g which lie on different sides, for example $g = g_1$ and $f = f_{3,1}$, then $g_4 = xyz \mid fg$. We introduce the definition "basic form" of a element in $S_{n,p}$. For any monomial $h = x^a y^b z^c \in S_{n,p}$, without loss of generality, we may assume that $c = \min\{a, b, c\}$. Since $g_4 = xyz \in S_{n,p}$, we have $x^{a-c}y^{b-c} \in S_{n,p} \cap \mathbb{C}[x, y]$, which follows that $x^{a-c}y^{b-c}$ can be generated by $\{g_1 = x^n, g_2 = y^n\} \cup A_{xy}(n, p)$. Hence

$$h = g_4^c \widetilde{h}(g_1, g_2, f_{1,1}, \ldots, f_{1,e_1}) \quad \text{in} \quad \mathbb{C}[x, y, z],$$

where $\widetilde{h}(g_1, g_2, f_{1,1}, \ldots, f_{1,e_1})$ is a polynomial in $g_1, g_2, f_{1,1}, \ldots, f_{1,e_1}$. We call $g_4^c \widetilde{h}(g_1, g_2, f_{1,1}, \ldots, f_{1,e_1})$ a basic form of h, and denote it by $B(h)$. In other two cases ($a = \min\{a, b, c\}$ and $b = \min\{a, b, c\}$) we can define $B(h)$ in a similar way. Let's see an example for basic forms.

Example 2. *Let $n = 3$ and $p = 1$. Then $S_{n,p} \cap \mathbb{C}[x, y] = \mathbb{C}[x, y]^{G_{3,1}}$, which is generated by $\{g_1 = x^3, g_2 = y^3\} \cup A_{xy}(3, 1)$. From Example 1 we know that*

$$A_{xy}(3, 1) = \{f_{1,1} = x^2 y, f_{1,2} = xy^2\}.$$

and

$$R_{xy}(3, 1) = \{g_1 f_{1,2} = f_{1,1}^2, \quad g_1 g_2 = f_{1,1} f_{1,2} \quad f_{1,1} g_2 = f_{1,2}^2\}.$$

Let $f = x^4 y^4 z \in S_{n,p}$, then $f = g_4 \cdot x^3 y^3$. $x^3 y^3 \in C[x, y]^{G_{3,1}}$ and it can be written as $f_{1,1} f_{1,2}$. Hence $B(f) = g_4 f_{1,1} f_{1,2}$ is basic form of f.

Now we can prove the main theorem of this section.

Theorem 3.5. *Using the notation above,*

$$\begin{aligned} &\{g_1 = x^n, g_2 = y^n, g_3 = z^n, g_4 = xyz\} \\ &\quad \cup A_{xy}(n, p) \cup A_{xz}(n, n-p-1) \cup A_{yz}(n, r) \end{aligned} \tag{9}$$

are minimal generators of the invariant ring $S_{n,p}$. And their relations are

$R_{xy}(n, p) \cup R_{xz}(n, n-p-1) \cup R_{yz}(n, r)$
$\cup \{gf - B(gf) \mid$ *generators g, f do not lie on the same side of triangle (8)*$\}$

where $B(gf)$ is a basic form of gf. More explicitly, the relations are

$$
\begin{aligned}
(10)\qquad & R_{xy}(n,p)\cup R_{xz}(n,n-p-1)\cup R_{yz}(n,r)\\
&\cup\{g_1f-B(g_1f)\mid f\in A_{yz}(n,r)\}\\
&\cup\{g_2f-B(g_2f)\mid f\in A_{xz}(n,n-p-1)\}\\
&\cup\{g_3f-B(g_3f)\mid f\in A_{xy}(n,p)\}\\
&\cup\{fg-B(fg)\mid f\in A_{xy}(n,p),\ g\in A_{xz}(n,n-p-1)\}\\
&\cup\{fg-B(fg)\mid f\in A_{xy}(n,p),\ g\in A_{yz}(n,r)\}\\
&\cup\{fg-B(fg)\mid f\in A_{xz}(n,n-p-1),\ g\in A_{yz}(n,r)\}.
\end{aligned}
$$

Remark 3.2. It's easy to see that $\deg(gf)=\deg(B(gf))$ with respect to the weight system (4) for any

$$f,g\in\{g_1,\dots,g_4,f_{1,1},\dots,f_{1,e_1},f_{2,1},\dots,f_{2,e_2},f_{3,1},\dots,f_{3,e_3}\}.$$

Hence equations in (10) are weighted homogeneous.

Proof. For any element $f\in S_{n,p}$, from its basic form $B(f)$, we know that f can be generated by (9). Hence (9) generate $S_{n,p}$. Theorem 3.4 tells us that $A_{xy}(n,p)\cup\{x^n,y^n\}$ are minimal generators of $S_{n,p}\cap\mathbb{C}[x,y]$, hence each element in $A_{xy}(n,p)\cup\{x^n,y^n\}$ can not be generated by other elements in $A_{xy}(n,p)\cup\{x^n,y^n\}$. Similarly each element in $A_{xz}(n,n-p-1)\cup\{x^n,z^n\}$ (resp. $A_{yz}(n,r)\cup\{y^n,z^n\}$) can not be generated by other elements in $A_{xz}(n,n-p-1)\cup\{x^n,z^n\}$ (resp. $A_{yz}(n,r)\cup\{y^n,z^n\}$). And it's clear that xyz can not be generated by other elements in (9). Hence (9) are minimal generators.

Consider ring homomorphism

$$
\begin{gathered}
\phi:\mathbb{C}[g_1,\dots,g_4,f_{1,1},\dots,f_{1,e_1},f_{2,1},\dots,f_{2,e_2},f_{3,1},\dots,f_{3,e_3}]\to S_{n,p}\\
g_1\mapsto x^n\quad g_2\mapsto y^n\quad g_3\mapsto z^n\quad g_4\mapsto xyz\\
f_{1,k}\mapsto x^{i_{1,k}}y^{j_{1,k}}\quad f_{2,k}\mapsto x^{i_{2,k}}y^{j_{2,k}}\quad f_{3,k}\mapsto x^{i_{3,k}}y^{j_{3,k}}
\end{gathered}
$$

Denote the kernel of ϕ by $K_{n,p}$. We will prove that $K_{n,p}$ is generated by (10) as an ideal of $\mathbb{C}[g_1,\dots,g_4,f_{1,1},\dots,f_{1,e_1},f_{2,1},\dots,f_{2,e_2},f_{3,1},\dots,f_{3,e_3}]$.

First let's prove a claim.

Claim 3.1. *Let $P=\mathbb{C}[g_1,\dots,g_4,f_{1,1},\dots,f_{1,e_1},f_{2,1},\dots,f_{2,e_2},f_{3,1},\dots,f_{3,e_3}]$. For any monomial F in P, there exists a non-negative integer k and a monomial H in P such that*

(1) *$F - g_4^k H$ is generated by (10);*

(2) *H is independent of g_4;*

(3) *elements in Supp(H) lie on a side of trangle (8) and Supp(H) contains at most one vertex of that side. (here Supp(H) means the set consists of variables which appear in H). More explicitly, this condition requires that H satisfies one of the following conditions:*

(i) *Supp$(H) \subseteq \{g_1, f_{1,1}, \dots, f_{1,e_1}\}$;*

(ii) *Supp$(H) \subseteq \{g_1, f_{2,1}, \dots, f_{2,e_2}\}$;*

(iii) *Supp$(H) \subseteq \{g_2, f_{1,1}, \dots, f_{1,e_1}\}$;*

(iv) *Supp$(H) \subseteq \{g_2, f_{3,1}, \dots, f_{3,e_3}\}$;*

(v) *Supp$(H) \subseteq \{g_3, f_{2,1}, \dots, f_{2,e_2}\}$;*

(vi) *Supp$(H) \subseteq \{g_3, f_{3,1}, \dots, f_{3,e_3}\}$.*

Proof of Claim 3.1. We prove this claim by induction on the weighted degree of F (with respect to the weight system (4)). Without loss of generality, we may assume that F is independent of g_4 (if $F = g_4^k F'$, we can replace F by F'). There are following three cases:

(a) There exist $g, f \in Supp(F)$ such that f, g do not lie on the same side of the triangle (8), then F can be written as

$$F = gfF' = B(gf)F' + (gf - B(gf))F'.$$

Because $\deg(gf) = \deg(B(gf))$ we have $\deg(F) = \deg(B(gf)F')$. Since $(gf - B(gf))F'$ is generated by (10), we only need prove the claim for $B(gf)F'$. By the definition of $B(gf)$, we know that $g_4 \mid B(gf)$. Hence $B(gf)F'$ can be written as $g_4 F''$, then $\deg F'' < \deg B(gf)F' = \deg F$. By inductive assumption, we know the claim holds for F'', which implies that the claim holds for $g_4 F'' = B(gf)F'$.

(b) $g_1 g_2 g_3 \mid F$. Write $F = g_1 g_2 g_3 F'$. Since $g_1 g_2 = f_{1,1} f_{1,e_1} \prod_{k=1}^{e_1} f_{1,k}^{a_{1,k}-2} \in R_{x,y}(n,p)$, we only need to prove the claim for $f_{1,1} f_{1,e_1} \prod_{k=1}^{e_1} f_{1,k}^{a_{1,k}-2} g_3 F'$, and this has already been treated in case (a).

(c) Elements in Supp(F) lie on the same side of the triangle (8). Without loss of generality, we may assume that F is a monomial on variables $g_1, g_2, f_{1,1}, \dots, f_{1,e_1}$. If $g_1 g_2 \mid F$, write $F = g_1^s g_2^t F'$, where F' is independent of g_1, g_2 and we may suppose that $s \le t$. Since $g_1 g_2 - f_{1,1} f_{1,e_1} \prod_{k=1}^{e_1} f_{1,k}^{a_{1,k}-2} \in$

$R_{xy}(n,p)$, we have

$$F - \left(f_{1,1}f_{1,e_1}\prod_{k=1}^{e_1} f_{1,k}^{a_{1,k}-2}\right)^s g_2^{t-s}F'$$

can be generated by (10). Let $H = (f_{1,1}f_{1,e_1}\prod_{k=1}^{e_1} f_{1,k}^{a_{1,k}-2})^s g_2^{t-s}F'$, then the claim holds. $\square$

Now come back to the proof of Theorem 3.5. For any $F(g_1,\ldots,g_4,f_{1,1},\ldots f_{1,e_1},f_{2,1},\ldots,f_{2,e_2},f_{3,1},\ldots,f_{3,e_3}) \in K_{n,p}$, where F is a polynomial in $4+e_1+e_2+e_3$ variables, we have $\phi(F)=0$. By Claim 3.1, we may assume that $F = F_0 + g_4F_1 + g_4^2F_2 + \cdots + g_4^mF_m$, where F_i is independent of g_4 and each term of F_i satisfies the condition (3) in Claim 3.1. Hence $xyz \nmid \phi(F_i)$ unless $\phi(F_i)=0$. Since $\phi(F)=0$, then we have

$$\phi(F_0) + xyz\phi(F_1) + (xyz)^2\phi(F_2) + \cdots + (xyz)^m\phi(F_m) = 0$$

in $S_{n,p}$. Since $xyz \nmid \phi(F_i)$ unless $\phi(F_i)=0$, we have $\phi(F_i)=0$ for $i=0,1,\ldots,m$. Now we only need to prove each F_i can be generated by (10). Since each term of F_i satisfies the condition (3) in Claim 3.1 and is independent of g_4, we can write

$$F_i = H_1 + H_2 + H_3 + c_0 + c_1g_1^{k_1} + c_2g_2^{k_2} + c_3g_3^{k_3}$$

where H_j is a polynomial such that each term t in H_j satisfies that

(1) elements in $\mathrm{Supp}(t)$ lie on the j-th side of the triangle (8),

(2) $\mathrm{Supp}(t) \cap \{f_{j,1},\ldots,f_{j,e_j}\} \neq \emptyset$,

for $j=1,2,3$. Then we have $xy \mid \phi(H_1)$, $xz \mid \phi(H_2)$ and $yz \mid \phi(H_3)$ and we have $\phi(H_1) \in \mathbb{C}[x,y]$, $\phi(H_2) \in \mathbb{C}[x,z]$, $\phi(H_3) \in \mathbb{C}[y,z]$. Since

$$\phi(F_i) = \phi(H_1) + \phi(H_2) + \phi(H_3) + c_0 + c_1x^{k_1n} + c_2y^{k_2n} + c_3z^{k_3n} = 0,$$

and $xy \mid \phi(H_1)$ and $\phi(H_2) \in \mathbb{C}[x,z]$ and $\phi(H_3) \in \mathbb{C}[y,z]$, we get $\phi(H_1)=0$. Using Theorem 3.3, we get H_1 is generated by $R_{xy}(n,p)$. Similarly $\phi(H_2) = \phi(H_3) = 0$ and H_2 (resp. H_3) can be generated by $R_{xz}(n,n-p-1)$ (resp. $R_{yz}(n,r)$). Hence $c_0 + c_1x^{k_1n} + c_2y^{k_2n} + c_3z^{k_3n} = 0$, which implies that $c_0 = c_1 = c_2 = c_3 = 0$. Hence $F_i = H_1 + H_2 + H_3$ can be generated by (10), and the proof is complete. $\square$

The following corollary tells us that the minimal embedding dimension of a three-dimensional Gorenstein isolated quotient singularity $\mathbb{C}^3/G$ is no less than 10.

Corollary 3.1. *The minimal embedding dimension d of $\mathbb{C}^3/H_{n,k}$ is*

$$4 + l(n/(n-p)) + l(n/(p+1)) + l(n/(n-r)) \geq 10,$$

where $l(k)$ means the length of the continue fraction for a positive integer k.

Proof. Since the minimal embedding dimension d of $\mathbb{C}^3/H_{n,k}$ is equal to the number of minimal generators, using Theorem 3.5, we have $d = 4 + l(n/(n-p)) + l(n/(p+1)) + l(n/(n-r))$. And since $p, n-p-1$ and r are coprime with n, we have $l(n/(n-p)), l(n/(p+1)), l(n/(n-r) \geq 2$. Hence

$$d \geq 10.$$

□

Remark 3.3. [CX] proves that the minimal embedding dimension of a three-dimensional rational isolated complete intersection singularity is at most 5. Hence a three-dimensional Gorenstein isolated quotient singularity must be non-complete intersection.

Example 3. *Let $H = H_{3,1}$ be the subgroup of $SL(3,\mathbb{C})$ generated by*

$$\langle \zeta(1/3), \zeta(1/3), \zeta(1/3) \rangle.$$

As in Example 2,

$$g_1 = x^3, \quad g_2 = y^3, \quad g_3 = z^3, \quad g_4 = xyz, \quad f_{1,1} = x^2y,$$
$$f_{1,2} = xy^2, \quad f_{2,1} = x^2z, \quad f_{2,2} = xz^2, \quad f_{3,1} = y^2z, \quad f_{3,2} = yz^2$$

are minimal generators of $S_{3,1}$. And their relations are

$$R_{xy}(3,1) = \{g_1f_{1,2} = f_{1,1}^2,\ g_1g_2 = f_{1,1}f_{1,2},\ f_{1,1}g_2 = f_{1,2}^2\},$$
$$R_{xz}(3,1) = \{g_1f_{2,2} = f_{2,1}^2,\ g_1g_3 = f_{2,1}f_{2,2},\ f_{2,1}g_3 = f_{2,2}^2\},$$
$$R_{yz}(3,1) = \{g_2f_{3,2} = f_{3,1}^2,\ g_2g_3 = f_{3,1}f_{3,2},\ f_{3,1}g_3 = f_{3,2}^2\}$$

and

$$
\begin{aligned}
&\{gf - B(gf) \mid \textit{generators } g, f \textit{ do not lie on the same side of triangle (8)}\}\\
&= \{g_1 f_{3,1} = g_4 f_{1,1},\ g_1 f_{3,2} = g_4 f_{2,1},\ g_2 f_{2,1} = g_4 f_{1,2},\ g_2 f_{2,2} = g_4 f_{3,1},\\
&\quad g_3 f_{1,1} = g_4 f_{2,2},\ g_3 f_{1,2} = g_4 f_{3,2},\ f_{1,1} f_{2,1} = g_1 g_4,\ f_{1,1} f_{2,2} = g_4 f_{2,1},\\
&\quad f_{1,1} f_{3,1} = g_4 f_{1,2},\ f_{1,1} f_{3,2} = g_4^2,\ f_{1,2} f_{2,1} = g_4 f_{1,1},\ f_{1,2} f_{2,2} = g_4^2,\\
&\quad f_{1,2} f_{3,1} = g_2 g_4,\ f_{1,2} f_{3,2} = g_4 f_{3,1},\ f_{2,1} f_{3,1} = g_4^2,\ f_{2,1} f_{3,2} = g_4 f_{2,2},\\
&\quad f_{2,2} f_{3,1} = g_4 f_{3,2},\ f_{2,2} f_{3,2} = g_3 g_4\}.
\end{aligned}
$$

4. Toric geometry perspective

The cyclic quotient singularity is toric and we can use toric method to understand the examples studied above. Let's first review briefly the toric singularity, for more details, see [CLS]. We start with a three dimensional standard lattice N, and its dual lattice M. A convex cone σ in N_R is defined by a set of lattice points v_ρ:

$$\sigma = \{r_1 v_1 + \cdots + r_n v_n,\ r_i \geq 0\}. \tag{11}$$

The dual cone is defined as

$$\sigma^\vee = \{m \cdot v_\rho \geq 0,\ m \in M_R\}. \tag{12}$$

The toric singularity is defined as $Spec(\sigma^\vee \cap M)$. We have following facts:

- The Gorenstein condition implies that there is a lattice vector $m_0 \in M$ such that $m_0 \cdot v_\rho = 1$ for any vector v_ρ. We can choose coordinate such that $v_\rho = (p_\rho, q_\rho, 1)$, so a Gorenstein toric singularity is defined by a convex lattice polygon P.
- The isolated singularity implies that there is no internal lattice points on boundary edges of P.
- We are interested in the case where there is no flavor symmetry, and this implies that the local class group of the singularity is trivial. This implies that P is a triangle.

So we need to classify triangle P with no lattice points on the boundary edges. Now we can put one vertex at origin using translational invariance, and we can also put another vertex at point $(1, 0)$. The third vertex can be constrained so that its coordinate is (a, b) with $a > 0, b > 0$. The constraints on (a,b) so that there is no lattice point on boundary edges are

$$(a,b)=1,\ (a-1,b)=1 \tag{13}$$

Here (p,q) means the maximal common divisor of p *and* q. See figure. 1 for the example.

Now let's compare our result with theorem 3.3, where the defining data also involves two positive integers n,p such that $(n,p)=1$ and $(n,n-p-1)=1$, with $0<p<n$. With some computation, one can see that the classification from toric perspective is the same as that from the quotient singularity point of view.

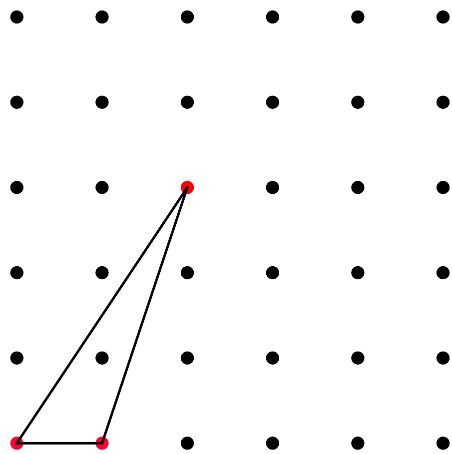

Figure 1: Isolated toric Gorenstein singularity with trivial class group is defined by a lattice triangle with no lattice points on the boundary.

Finally, we would like to point out that the deformation theory of isolated Gorenstein toric singularity has been studied in [AL], and above singularity is indeed rigid. The crepant resolution of the singularity is found from the unimodular lattice triangulation of P, from which we can read off the Higgs branch dimension.

5. Discussion

The singularities studied in this paper has trivial mini-versal deformation, and the underlying four dimensional $\mathcal{N}=2$ SCFT has no Coulomb branch (including mass deformation). The singularity admits non-trivial crepant resolution, and so it should have non-trivial Higgs branch. For example,

C^3/Z_3 singularity has a crepant resolution with one exceptional divisor which is nothing but a CP^2. There is one compact curve on resolved geometry and we expect the Higgs branch to be one dimensional. This theory should have no flavor symmetry, since otherwise one can turn on mass deformation and then have non-trivial Coulomb branch. This fact is verified from toric point of view as the local class group is trivial. While there are many 4d $\mathcal{N}=2$ theories admitting no Higgs branch, to our knowledge we do not know any example admitting no Coulomb branch.

From Higgs branch point view, the SCFT point is nontrivial as there are already massless degree of freedom in the deformed theory. The question is whether they are just free hypermultiplets. We used tensionless string argument to argue that the theory is interacting. Another reasoning is that if the theory is free, we should see the flavor symmetry and the mass deformation which are all absent in the geometry. Given these reasonings, we tend to believe that the theory is interacting. We believe that examples presented in this paper can help us better understand the space of 4d $\mathcal{N}=2$ SCFTs.

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