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Topological T-duality for stacks using a Gysin sequence

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In this paper we study the topological T-dual of spaces with a non-free circle action mainly using the stack theory method of Bunke and co-workers [3]. We first compare three formalisms for obtaining the Topological T-dual of a semi-free S^1-space in a simple example. Then, we calculate the T-dual of general KK-monopole backgrounds using the stack theory method. We define the dyonic coordinate for these backgrounds. We introduce an approach to Topological T-duality using classifying spaces which simultaneously generalizes the methods of Bunke et al [3] and Mathai and Wu [10]. Then, we define a cohomology Gysin sequence for prinicpal bundles of stacks and describe an application to Topological T-duality for stacks. We apply the above to calculate the Topological T-dual of a general compact three-manifold with an *arbitrary* smooth circle action. We point out a possible application of these T-duals to higher-dimensional black holes.

1. Introduction

Topological T-duality is a recent theory inspired by the theory of T-duality in String Theory. Principal circle (and torus) bundles with a class in H^3 of the total space of the bundle possess an unusual symmetry, namely, from this data it is possible to naturally construct *T-dual* bundles which are also principal circle bundles over the same base with a class in H^3 of the total space of the T-dual bundle (See Ref. [1, 2] for examples).

In Refs. [3, 4], the authors generalize Topological T-duality to principal bundles of topological stacks (see Refs. [3, 5, 6, 8] for an introduction) $\mathcal{E} \to \mathcal{Y}$ with an S^1-gerbe $\mathcal{G}$ on the stack $\mathcal{E}$. The authors show that in such a situation, the T-dual exists and is also a principal bundle of stacks together with a gerbe on it. It was argued in Ref. [4] that since the T-dual exists for principal bundles of stacks, it should be possible to compute the T-dual of spaces with circle group actions which are not necessarily free. In this paper we study the Topological T-dual of several classes of spaces with a non-free circle action.

A brief outline of the paper is as follows: In Ref. [9] the T-duals of some semi-free S^1-spaces were derived using C^*-algebraic techniques. In Ref. [10] a general formalism using the Borel construction was used to derive the T-dual of any semi-free S^1-space. Neither of these constructions used stack theory, and it would be interesting to compare the T-duals obtained using all three theories. In Sec. (2) below, we attempt to do this in a simple example.

We calculate Topological T-duals of semi-free spaces for the examples of Ref. [9] (these are the Kaluza-Klein monopole backgrounds of string theory) using the methods of Ref. [3] in Sec. (3). We also comment on the results obtained.

A phenomenon seen in backgrounds with Kaluza-Klein monopoles is the dyonic coordinate (See Refs. [11, 12] for details). A model for this using C^*-algebraic methods was developed in Ref. [9]. We argue in Sec. (4) that this phenomenon may also be obtained completely independently of the C^*-algebraic formalism in the stack theory formalism of Ref. [3] using the results from Ref. [13].

Classifying spaces for stacks were introduced in Ref. [5]. In Sec. (5) we introduce a formalism based on classifying spaces for stacks which simultaneously generalizes the methods of Bunke et al [2] and Mathai and Wu [10] to determine the T-dual of a given principal bundle of stacks. Using this we prove an interesting property of Topological T-duals calculated using any of the above methods.

In Ref. [14], the authors give a Gysin sequence for a S^1-stack. Very roughly, this is an exact sequence of stack cohomology groups which is derived by taking classifying spaces for $\mathcal{E}$ and $\mathcal{Y}$, obtaining an ordinary principal bundle, and then using the ordinary (homology) Gysin sequence. In Sec. (6) we derive a cohomology Gysin sequence based on this argument. We argue that this may be used to determine the T-dual principal bundle of stacks just as the ordinary Gysin sequence may be used to determine the T-dual of a principal bundle of spaces. In Sec. (6) we develop this argument in more detail and calculate a few T-duals. We prove that the T-dual of a semi-free space obtained using this method is the one obtained by Mathai and Wu in Ref. [10].

In Sec. (3) we had determined the T-dual of a three-manifold, the KK-monopole spacetime. In Sec. (7) we apply the arguments in Secs. (5, 6) to determine the T-dual of a compact three-manifold with an *arbitrary* circle-action. Many of these spaces are three-manifolds with a non-free circle action.

We conclude with a few remarks on this paper in Sec. (8).

Throughout this paper we restrict ourselves to circle actions on stacks and stacks which are principal circle bundles over a stack. We us the formalism and notation of topological stack theory developed in Noohi (Refs. [5, 6, 21]) and in Heinloth (Ref. [8]) for the rest of this paper. In addition, in the following, if we use the formalism or notations from other papers in the stack theory literature in any section, we mention those papers there. We have tried to ensure that the usage of all stack theory notation in this paper is consistent with the notation of Noohi.

2. Comparision of the three formalisms

We use the notations and definitions given in the review by Heinloth in Ref. [8] in this paper. In particular, if X is a topological space then $\underline{X}$ is X viewed as a *stack* using the Yoneda Lemma. Further let G be a topological group acting continuously on a topological space X, then $[X/G]$ is the quotient stack of the *topological space* X by this action as in Heinloth, Ref. [8], Example (2.5).

Three completely different formalisms have been proposed to calculate the topological T-dual of a space with a circle action: The formalism of Mathai and Rosenberg based on continuous-trace algebras and the crossed product in Ref. [1], the formalism of Bunke et al using methods from algebraic topology in Ref. [2–4], and the formalism of Mathai and Wu using methods from equivariant cohmology in Ref. [10].

To prevent confusion in the rest of this paper we refer to the formalism of Mathai and Rosenberg in Ref. [1] as the *C^*-algebraic formalism* of Topological T-duality. We also refer to the formalism of Bunke et al in Ref. [2–4] as the *stack formalism* of Topological T-duality. Throughout this paper we will also use the formalism of Mathai and Wu in Ref. [10] as needed. In this section we propose to compare the T-duals calculated using these three formalisms in some simple examples.

We use the notation of Ref. [8] and refer to the stack $\underline{\text{pt}}$ as the canonical stack associated to the singleton space $\text{pt} = \{*\}$ by the Yoneda lemma.

Consider a space which is a point $\underline{\text{pt}}$ with an S^1-action which fixes that point. One reason to study this space is that this space is extremely simple and yet shows the difference between the three formalisms. In addition, this space (see proof of Lemma (3.7) below) is stack homotopy equivalent to a more interesting space (the cone over the three sphere with a circle action) which we also study in Sec. (3) below.

It is interesting to compare the T-duals obtained for this space using the formalisms of Refs. [1, 3, 10]. We have the following theorem:

Theorem 2.1. *Consider a point $\underline{\text{pt}}$ with an S^1-action which fixes the point. Then,*

1) *The Toplogical T-dual of the space $\underline{\text{pt}}$ with the above circle action in the formalism of Mathai and Rosenberg (see Ref. [1]) is $\mathbb{R}$ with quotient space the point.*
2) *The Topological T-dual of the principal bundle of stacks $\underline{\text{pt}} \to [\underline{\text{pt}}/S^1]$ in the formalism of Bunke et al. (see Refs. [2–4]) is the principal bundle of stacks $[\text{pt}/S^1] \times S^1 \to [\text{pt}/S^1]$ with a gerbe on the total space corresponding to H-flux.*
3) *The Topological T-dual of the space $\underline{\text{pt}}$ in the formalism of Mathai-Wu (see Ref. [10]) is $BS^1 \times S^1 \times \text{pt}$ with H-flux.*

Proof. Suppose one considers a point with a S^1-action which fixes the point. The quotient is still a point.

1) In the C^*-algebraic formalism, to this geometry we would assign the C^*-algebra of compact operators $\mathcal{K}$ (since the spectrum of $\mathcal{K}$ is the point). The trivial action of S^1 on $\mathcal{K}$ lifts to a trivial action α of $\mathbb{R}$ on $\mathcal{K}$. Also, any other action of $\mathbb{R}$ on $\mathcal{K}$ is exterior equivalent to this trivial one. The T-dual would be the spectrum of the crossed product

$\mathcal{K} \underset{\alpha}{\rtimes} \mathbb{R} \simeq C_0(\mathbb{R})$ i. e. $\mathbb{R}$ where $\simeq$ denotes Morita equivalence (See Ref. [17]). The group action on the T-dual C^*-algebra $C_0(\mathbb{R})$ is induced from the translation action of $\mathbb{R}$ on itself (See Ref. [17] for details). The quotient would be the point as expected.

2) In Ref. [3], Sec. (4.2, 4.3), Prop. (4.3), the T-dual of a stack is obtained by the following procedure: One passes to the geometric realization of the simplicial space of the groupoid associated with that stack. This gives an ordinary principal bundle with H-flux from the principal bundle of stacks one one began with. This principal bundle may be T-dualized in the normal way [3].

   Consider the principal bundle of stacks $\underline{\mathrm{pt}} \to [\mathrm{pt}/S^1]$ and the atlas $\mathrm{pt} \to [\mathrm{pt}/S^1]$. Let $Y = \mathrm{pt} \underset{[\mathrm{pt}/S^1]}{\times} \underline{\mathrm{pt}}$ be an atlas for $\underline{\mathrm{pt}}$ (See Ref. [3] Sec. (4.2)). We have a commutative square

$$\begin{array}{ccc} Y & \longrightarrow & \underline{\mathrm{pt}} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \longrightarrow & [\mathrm{pt}/S^1]. \end{array}$$

   The groupoid associated to $\underline{\mathrm{pt}} \to [\mathrm{pt}/S^1]$ is $\mathrm{pt} \times S^1 \rightrightarrows \mathrm{pt}$ as there is a canonical isomorphism $Y \simeq (\mathrm{pt} \times S^1)$ since $(\mathrm{pt} \times S^1)$ is the canonical bundle over pt (See Heinloth Ex. 2.5 and following). Similarly the groupoid associated to the atlas $Y = \left(\mathrm{pt} \underset{[\mathrm{pt}/S^1]}{\times} \underline{\mathrm{pt}}\right) \to \underline{\mathrm{pt}}$ is $Y \underset{\underline{\mathrm{pt}}}{\times} Y \rightrightarrows \mathrm{pt}$. Since the fiber product of Y with itself over pt is $\mathrm{pt} \times (S^1)^2$ the associated groupoid would be $\mathrm{pt} \times (S^1)^2 \to \mathrm{pt}$.

   It is clear that the iterated fiber product of Y with itself n times would be isomorphic to $\mathrm{pt} \times (S^1)^n$. The total space of the associated simplicial bundle would then (by definition of EG) be ES^1 and the base would (by the construction above) be BS^1. Therefore the T-dual of the simplicial bundle would be $BS^1 \times S^1$ with a gerbe on total space (See Refs. [2, 3]). This corresponds to the T-dual bundle $[\mathrm{pt}/S^1] \times S^1 \to [\mathrm{pt}/S^1]$ with a gerbe on the total space of the bundle corresponding to the H-flux.

3) In the formalism of Mathai and Wu (See Ref. [10]), the original space would be replaced by $ES^1 \times \mathrm{pt}$ as a principal circle bundle over $BS^1 \times \mathrm{pt}$ and the T-dual would be $BS^1 \times S^1 \times \mathrm{pt}$ as a principal circle bundle over $BS^1 \times \mathrm{pt}$ with H-flux. The T-dual obtained here namely $BS^1 \times$

$S^1 \times \mathrm{pt}$ should be compared with the T-dual $[\mathrm{pt}/S^1] \times S^1$ obtained in Part (2).

□

Thus, the formalisms of Bunke-Schick and Mathai-Wu give similar answers here for this example and the C^*-algebraic formalism gives a different one. This difference is probably due to the fact that in Ref. [1], an S^1-action on a principal bundle lifts to an $\mathbb{R}$-action (with $\mathbb{Z}$-stabilizers) on the C^*-dynamical system associated to that space while in Ref. [10], the S^1-action remains an S^1-action. That is, in the C^*-algebraic formalism a circle action is viewed as an $\mathbb{R}$-action with $\mathbb{Z}$-stabilizers while in the other formalisms circle action is viewed only as a circle action.

In Topological T-duality, it is expected that the original and T-dual spaces have the same K-theory up to a degree shift. It was shown by Mathai and Rosenberg in Ref. [1] using the Connes-Thom isomorphism, that the C^*-algebraic T-dual will have this property. Similarly, the Topological T-duals calculated from the other two formalisms will also have this property. This was demonstrated by Bunke and co-workeres in Refs. [2–4] for the stack theory formalism. For the formalism of Topological T-duality using equivariant cohomology proposed by Mathai and Wu this was proved in in Ref. [10].

3. T-dual of Kaluza-Klein monopole backgrounds

We now restrict our attention to spaces with semi-free circle actions. We may further restrict ourselves to spaces which contain Kaluza-Klein monopoles (KK-monopoles)[1]. Away from fixed points such spaces are equivariantly S^1-homeomorphic to the total space of a principal circle bundle. Since the Topological T-dual of a principal circle bundle is well-known, it is enough to determine the T-dual of a neighbourhood of the fixed point (by the discussion before Thm. (3.5) and by Thm. (3.5) below).

In a neighbourhood of a fixed point four-manifolds with a semi-free circle action are equivariantly S^1-homeomorphic to $\mathbb{R}^4$ with an orthogonal S^1-action. The associated topological stacks are the stacks $[CS^3/\mathbb{Z}_k]$. In Ref. [9], we computed the T-duals of these semi-free spaces using the C^*-algebraic approach of Mathai and Rosenberg in Ref. [1].

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[1] In String Theory, the Taub-NUT metric (see Refs.[12]) describes a space with one KK-monopole. For more details see Ref. [9] and references therein.

Stacks of the form $E = [CS^3/\mathbb{Z}_l]$ with $E/S^1 = CS^2$ for $l \geq 2$, and $E = CS^3$ and $E/S^1 = CS^2$ for $l = 1$ are the stacks associated to the total spaces of KK-monopoles of charge $l \in \mathbb{N}, l > 0$.

In Thm. (3.4) below we compute the T-dual of these spaces. For $l = 1$ the associated principal bundle of stacks would be $\underline{CS^3} \to [CS^3/S^1]$. For $l \geq 2$, the associated principal bundle of stacks would be $[CS^3/\mathbb{Z}_l] \to [CS^3/S^1]$. We consider the T-dual of spaces containing multiple KK-monopoles in Thm. (3.5) and Cor. (3.6) below.

Consider a spacetime which is a KK-monopole spacetime with charge 1. This corresponds to T-dualizing the space CS^3 with its natural circle action. Physically, the T-dual would have a source of H-flux over the set in the base corresponding to the image of the singular fiber and the H-flux would be undefined at the location of the source[2]. In the C^*-formalism of Topological T-duality the C^*-algebra describing the background loses the continuous-trace property exactly on this locus. In Ref. [9], it was argued that this is a model for a space with a *source* of H-flux.

In the formalisms of Refs. [3, 10], however, the T-dual H-flux would be everywhere defined, that is, there would be no *source* of H-flux present. In particular, for these two theories, the following would hold: The T-dual of a single NS5-brane with a background of k-units of (sourceless) H-flux would be indistinguishable from the T-dual of a space with $(k+1)$-units of sourceless H-flux. We will see this for the stack theory method when we calculate the T-dual of these spaces in Thms. (3.4,3.5) and in Cor. (3.6) below. We discuss this matter in more detail after Cor. (3.8) below.

Before we calculate the T-dual of the above spaces, we need some preliminary notation and results. Let $\mathcal{E} \to \mathcal{Y}$ be a principal S^1-bundle of stacks over $\mathcal{Y}$. It is clear from the axioms of a principal S^1-bundle (see Ref. [8] after Remark (2.14)) that the topological stack $\mathcal{E}$ is a space with a left S^1-action (in the sense of Ref. [14], Sec. (3)). In Sec. (2) above, we had used the definition of a quotient stack $[X/G]$ of a *topological space* X with an action of a topological group G. In this Section we use in addition the notation and notion of a quotient stack of an *arbitrary topological stack* by the action of a topological group G from the work of Ginot and Noohi, Ref. [14]. In Ref. [14] Sec. (4.3) the authors define a *quotient stack* $[G\backslash\mathcal{X}]$ of a group action G on an arbitrary *topological stack* $\mathcal{X}$. We assume the reader is familiar with these ideas and will refer to them freely in what follows.

Since the spaces we study in this paper are not orbispaces we need to make a remark about the existence of T-dual stacks for the spaces under

[2]See Ref. [9] for a detailed discussion.

study. In this paper, we will use the method of Ref. [3] to calculate the T-dual of a stack associated to a semi-free space. In the method of Ref. [3], if we restrict ourselves to $U(1)$-bundles over such stacks, the associated simplicial bundles, being circle bundles, may always be T-dualized. There is always a 'T-duality diamond' of Ref. [2] (see Diagram (2.14) in Lemma (2.13) of that reference) for the associated simplicial bundles, since these are only circle bundles. This gives a diagram of the form Diagram (4.1.2) in Ref. [4] for the associated *stacks*. Hence, for such a stack, the T-dual exists in the sense of Def. (4.1.4) of Ref. [4], even if the base is not an orbispace.

In the following, the stacks $\mathcal{E}$ we dualize are not Seifert fibered spaces as in Ref. [3], but, due to the above, the principal bundles $p : \mathcal{E} \to \mathcal{B}$ may be completed into a diagram of the form of Diagram (4,1.2) of Ref. [4], and hence for these stacks the T-dual exists by Ref. [4], Def. (4.1.4).

We now prove some preliminary results about principal bundles of stacks and spaces with a circle action which will be useful later. We use the notion for the quotient stack $[G\backslash\mathcal{E}]$ of a *topological stack* $\mathcal{E}$ by the action of a topological group G from Ref. [14] in the following.

Lemma 3.1.

1) *Let E be a space with a circle action. Then, $\underline{E} \to [E/S^1]$ is a principal bundle of topological stacks.*
2) *Let $p : \mathcal{E} \to \mathcal{Y}$ be a principal S^1-bundle of topological stacks. Then $\mathcal{E}$ is a stack with a left S^1-action in the sense of Ref. [14] and $\mathcal{Y} \simeq \left[S^1\backslash\mathcal{E}\right]$.*
3) *For a space E with a circle action, $[S^1\backslash\underline{E}] \simeq [E/S^1]$.*

Proof.

1) E is a space with a S^1-action and satisfies the conditions for a principal S^1-bundle described after Remark (2.14) in Ref. [8]. This is equivalent to the definition of a principal bundle of stacks using atlases[3] by the Claim before Example (2.15) in Ref. [8].
2) That $\mathcal{E}$ is a stack with a left S^1-action follows from the definition of a principal bundle of stacks (See Ref. [8]).

   From Ginot and Noohi (Ref. [14]), Prop. (4.8), $\left[S^1\backslash\mathcal{E}\right]$ is a stack. Let $\tilde{p} : \mathcal{E} \to \left[S^1\backslash\mathcal{E}\right]$ be the natural map defined in Sec. (4.1) of Ref. [14].

[3]See Ref. [8], Def. (2.11)

Let $T \to \mathcal{Y}$ be an atlas for $\mathcal{Y}$. By definition of a principal bundle of stacks, (see Ref. [8]), we have a commutative square

$$\begin{array}{ccc} P & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{Y} \end{array} \tag{3.1}$$

where P is a principal bundle over T and an atlas for $\mathcal{E}$. As noted above, we have a natural map $\tilde{p} : \mathcal{E} \to [S^1 \backslash \mathcal{E}]$. By the second part of the proof of Prop. (4.8) of Ref. [14], P is also an atlas for $[S^1 \backslash \mathcal{E}]$. By Ref. [14], Sec. (4.1) the stack $[S^1 \backslash \mathcal{E}]$ is the stackification of the prestack $\lfloor S^1 \backslash \mathcal{E} \rfloor$. Then, we have

$$\begin{aligned} \lfloor S^1 \backslash \mathcal{E} \rfloor(P) &\simeq S^1 \backslash (\mathcal{E}(P)), \text{(by definition of } \lfloor S^1 \backslash \mathcal{E} \rfloor\text{, see Sec. (4.1) of Ref. [14])}, \\ &\simeq \mathcal{Y}(P), \text{(By definition, see Sec. (4.1) of Ref. [14])}, \\ &\simeq \mathcal{Y}(P/S^1) \simeq \mathcal{Y}(T), \text{(because the Diagram (3.1) commutes)}. \end{aligned}$$

Hence, the stackification of $\mathcal{Y}$ is isomorphic to the stackification of $\lfloor S^1 \backslash \mathcal{E} \rfloor$. Since $\mathcal{Y}$ is a stack, this implies that $\mathcal{Y} \simeq [S^1 \backslash \mathcal{E}]$.

3) This follows from Parts (1) and (2) above.

□

Lemma 3.2. *Let $\mathcal{X}$ be a stack with a S^1-action in the sense of Ginot et. al (See Ref. [14], Def. (3.1)). Let $q : \mathcal{X} \to [S^1 \backslash \mathcal{X}]$ be the quotient map of Ref. [14], Sec. (3.2). Then $[S^1 \backslash \mathcal{X}]$ is a topological stack and $q : \mathcal{X} \to [S^1 \backslash \mathcal{X}]$ is a principal bundle of stacks in the sense of Ref. [8].*

Proof. By definition, we have an action μ of S^1 on the stack X. By Prop. (4.9) of Ref. [14], $[S^1 \backslash \mathcal{X}]$ is also a topological stack. By Prop. (4.8) of the same reference, the map q is representable. It can be checked that the conditions for a principal bundle of stacks given after Remark (2.14) in Ref. [8] are satisfied with $act = \mu, p = q$. □

Corollary 3.3. *$p : \mathcal{E} \to \mathcal{Y}$ is a principal bundle of stacks iff $\mathcal{E}$ is a stack with a left S^1-action in the sense of Ref. [14] and $\mathcal{Y} \simeq [S^1 \backslash \mathcal{E}]$.*

Proof. This follows from Lemma (3.1), Part (2), and Lemma (3.2) above. □

In Thm. (2.1) (2) we determined the Topological T-dual of the principal bundle of stacks $\underline{\mathrm{pt}} \to [\mathrm{pt}/S^1]$. The calculation for the Topological T-dual of

CS^3 is nearly similar to that for the Topological T-dual of $\underline{\text{pt}} \to [\text{pt}/S^1]$ for the following two reasons: Firstly, the space CS^3 is equivariantly homotopy equivalent to its vertex pt (see proof of Lemma (3.7 below). Secondly, CS^3 and CS^2 are contractible and are homeomorphic to $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. As are result, any principal bundle over CS^3 is trivial. Due to this, we can take the above proof and replace $\underline{\text{pt}}$ with CS^3 and $[\text{pt}/S^1]$ with CS^2 and obtain a working proof.

Theorem 3.4.

1) *The Topological T-dual of the principal bundle of stacks*

$$\underline{CS^3} \to [CS^3/S^1] \quad \textit{(associated to a KK-monopole)}$$

in the formalism of [3] is the principal bundle of stacks $[CS^3/S^1] \times S^1 \to [CS^3/S^1]$ *with a gerbe on the stack* $[CS^3/S^1] \times S^1$.

2) *Consider the* S^1*-action on a point which fixes the point. This gives a principal bundle of stacks* $\underline{\text{pt}} \to [\text{pt}/S^1]$. *Consider the subgroup* $\mathbb{Z}_k \hookrightarrow S^1$ *for any natural number* $k > 1$. *Then,* $[\text{pt}/\mathbb{Z}_k] \to [\text{pt}/S^1]$ *is also a principal bundle of stacks. This bundle with no H-flux has as T-dual the principal bundle of stacks* $[\text{pt}/S^1] \times S^1 \to [\text{pt}/S^1]$ *with H-flux of k units.*

3) *For k any natural number larger than* 1, *there is a principal bundle of stacks* $[CS^3/\mathbb{Z}_k] \to [CS^3/S^1]$ *(corresponding to a KK-monopole of charge k). The Topological T-dual of this bundle with no H-flux, in the formalism of [3], is the principal bundle of stacks* $([CS^3/S^1] \times S^1) \to [CS^3/S^1]$ *with k units of H-flux.*

Proof.

1) Consider the principal bundle of stacks $\underline{CS^3} \to \left[CS^3/S^1\right]$. The space $\mathbb{R}^4 \simeq CS^3$ is an atlas for the stack $[CS^3/S^1]$. Let $Y = CS^3 \underset{[CS^3/S^1]}{\times} \underline{CS^3}$ be an atlas for $\underline{CS^3}$ (See Ref. [3] Sec. (4.2)) induced by the atlas for the stack $[CS^3/S^1]$. We have a commutative square

$$\begin{array}{ccc} Y & \longrightarrow & \underline{CS^3} \\ \downarrow & & \downarrow \\ CS^3 & \longrightarrow & [CS^3/S^1]. \end{array}$$

The atlas associated to $\underline{CS^3}$ is $CS^3 \times S^1 \to \underline{CS^3}$: There is a canonical isomorphism $Y \simeq (CS^3 \times S^1)$, since, due to the contractibility of $CS^3 \simeq \mathbb{R}^4$, $(CS^3 \times S^1)$ is the canonical bundle over CS^3 (See Heinloth Ex. 2.5 and following).

The groupoid associated to the atlas $Y = \left(CS^3 \underset{[CS^3/S^1]}{\times} \underline{CS^3}\right) \to \underline{CS^3}$ is $Y \underset{\underline{CS^3}}{\times} Y \rightrightarrows CS^3$. The fiber product $Y \underset{\underline{CS^3}}{\times} Y$ is $CS^3 \times (S^1)^2$ by definition. Therefore, the groupoid is $CS^3 \times (S^1)^2 \rightrightarrows CS^3$. It is clear from the definition that the iterated fiber product of Y with itself n times is $CS^3 \times (S^1)^n$.

The associated simplicial space in each degree would be $CS^3 \times (S^1)^n$. The simplicial space is thus the fiber product of $CS^3 \to CS^2 \sim *$ with $* \times (S^1)^n \to *$. Therefore by Ref. [15], Cor. (11.6), the simplicial space is the fiber product $(A \underset{*}{\times} ES^1)$ where A is the geometric realisation of the simplicial space which is CS^3 in each degree. Since CS^3 is contractible, the space $(A \underset{*}{\times} ES^1)$ would be homotopy equivalent to ES^1.

We had noted above that the total space of the bundle is the simplicial space which in each degree is $CS^3 \times (S^1)^n$. Hence, the base of the simplical bundle would be the simplicial space which in each degree > 1 is $CS^2 \times (S^1)^{n-1}$ and at degree 1 is $CS^2 \times \mathrm{pt}$. Let B be the simplicial space which is CS^2 in each degree. By the above argument the base is the fiber product $(B \underset{*}{\times} BS^1)$. Since B is contractible, the base has the homotopy type of BS^1. Thus, the simplicial bundle associated to the bundle of stacks $\underline{CS^3} \to [CS^3/S^1]$ is $(A \underset{*}{\times} ES^1) \to (B \underset{*}{\times} BS^1)$.

Therefore the T-dual of the simplicial bundle would be $(B \underset{*}{\times} BS^1) \times S^1$ with a gerbe on total space (See Refs. [2, 3]). The class in H^3 of the total space associated to the gerbe would correspond to the generator of $H^3((B \underset{*}{\times} BS^1) \times S^1, \mathbb{Z}) \simeq H^3(BS^1 \times S^1, \mathbb{Z})$. It is clear from the above that this is the simplicial bundle associated to $[CS^2/S^1] \times S^1$. Also, the H-flux on the total space of the bundle is 1. Therefore the T-dual stack has a gerbe on it.

2) Consider the S^1-action on a point which fixes the point. This gives a S^1-bundle of stacks $\underline{\mathrm{pt}} \to [\mathrm{pt}/S^1]$. Then $\mathbb{Z}_k \hookrightarrow S^1$ acts on $\underline{\mathrm{pt}}$ as well. The S^1-action on $\underline{\mathrm{pt}}$ gives an action of S^1 on $[\mathrm{pt}/\mathbb{Z}_k]$. The quotient is still $[\mathrm{pt}/S^1]$. Therefore, $[\mathrm{pt}/\mathbb{Z}_k] \to [\mathrm{pt}/S^1]$ is a principal bundle of stacks by Lemmas (3.1,3.2) above.

The groupoid associated to the stack $[\mathrm{pt}/\mathbb{Z}_k]$ is $\mathbb{Z}_k \rightrightarrows \underline{\mathrm{pt}}$ and the associated simplicial space is $B\mathbb{Z}_k$ (See Ref. [3] Sec. (5.1)). The simplicial space associated to $[\mathrm{pt}/S^1]$ is BS^1. The associated simplicial principal bundle is $B\mathbb{Z}_k \to BS^1$. This is $ES^1/\mathbb{Z}_k \to BS^1$. Thus the T-dual simplicial bundle is $BS^1 \times S^1$ with H-flux k. This is the simplicial bundle associated to the stack $[\mathrm{pt}/S^1] \times S^1$ with a gerbe on the stack.

3) For every $k > 1$, there is an S^1-action on $CS^3/\mathbb{Z}_k$ induced by the S^1-action on the S^1-space CS^3 with quotient CS^3/S^1. Thus there is an S^1-action on $[CS^3/\mathbb{Z}_k]$ such that the quotient under the S^1-action is $[CS^3/S^1]$. Therefore, by Lemmas (3.1,3.2), there is a principal bundle of stacks $[CS^3/\mathbb{Z}_k] \to [CS^3/S^1]$.

The simplicial space associated to $[CS^3/\mathbb{Z}_k]$ is determined in a manner similar to Part (1). Consider the principal bundle of stacks $\underline{CS^3} \to [CS^3/\mathbb{Z}_k]$.

Let W be an atlas for the stack $\underline{CS^3}$ induced by the the atlas CS^3 for the stack $[CS^3/\mathbb{Z}_k]$. Then, W is a principal $\mathbb{Z}_k$-bundle over CS^3. We have a commutative square

$$\begin{array}{ccc} W & \longrightarrow & \underline{CS^3} \\ \downarrow & & \downarrow \\ CS^3 & \longrightarrow & [\underline{CS^3}/\mathbb{Z}_k]. \end{array}$$

There is a canonical isomorphism $W \simeq (CS^3 \times \mathbb{Z}_k)$ by the same argument as in Part (1) with S^1 replaced by $\mathbb{Z}_k$. The groupoid associated to the atlas W is $W \underset{\underline{CS^3}}{\times} W \rightrightarrows CS^3$. We have that $W \underset{\underline{CS^3}}{\times} W \simeq CS^3 \times (\mathbb{Z}_k)^2$. Also, $W \underset{\underline{CS^3}}{\times} \cdots \underset{\underline{CS^3}}{\times} W \simeq CS^3 \times (\mathbb{Z}_k)^n$. Let A be the simplicial space which is CS^3 in each degree. By an argument similar to that in Part (1), the simplicial space associated to $\underline{CS^3}$ by this atlas is then $(A \underset{*}{\times} E\mathbb{Z}_k)$. This has a natural action of $\mathbb{Z}_k$ on each factor. It is a principal $\mathbb{Z}_k$-bundle over the simplicial space associated to $[CS^3/\mathbb{Z}_k]$. (By the above diagram, W is a principal $\mathbb{Z}_k$ bundle over CS^3, the result follows from the definition of the simplicial space associated to a groupoid.) Hence, the simplicial space associated to $[CS^3/\mathbb{Z}_k]$ by this atlas is $(A/\mathbb{Z}_k \underset{*}{\times} B\mathbb{Z}_k)$. It is also a principal circle bundle over the simplicial space associated to $[CS^3/S^1]$.

By Part (1), the simplicial space associated to $[CS^3/S^1]$ is $(B \underset{*}{\times} BS^1)$. Therefore the simplicial circle bundle associated to the principal bundle of stacks $[CS^3/\mathbb{Z}_k] \to [CS^3/S^1]$ is $(A/\mathbb{Z}_k \underset{*}{\times} B\mathbb{Z}_k) \to (B \underset{*}{\times} BS^1)$.

Thus, the T-dual would be $(B \underset{*}{\times} BS^1) \times S^1 \to (B \underset{*}{\times} BS^1)$ with H-flux. This is the principal bundle associated to $([CS^3/S^1] \times S^1) \to [CS^3/S^1]$ with a gerbe on it.

□

It is interesting to observe that if a property holds for T-duality pairs in the sense of Ref. [2], it can be applied to the pair consisting of the simplicial bundle and H-flux. Sometimes, this has interesting consequences for spaces with a non-free S^1-action. We present two examples below using this property: In the first we calculate the T-dual of a semi-free space with countably many isolated fixed sets. In the second example done in the next Section, we develop a model for the dyonic coordinate of Ref. [9] using the stack-theoretic approach.

Note the following property of T-dual principal S^1-bundles: Let $p : E \to B$ be a principal S^1-bundle. Let $h \in H^3(E, \mathbb{Z})$ be an H-flux. Let $W_1, \ldots, W_k$ be open subsets of B such that $W_1 \cup \cdots \cup W_k = B$. Let $E_i = p^{-1}(W_i)$ be the induced open cover of E. Let h_i be the H-flux restricted to E_i. Let $q : E^\# \to B$ be the T-dual of the principal bundle $E \to B$ with H-flux $h^\#$. Let $E_i^\# = q^{-1}(W_i)$ and let $h_i^\# = h^\#|_{E_i}$. Then we claim that $(E_i^\#, h_i^\#)$ is the T-dual of (E_i, h_i). This follows from the existence of a classifying space for a pair and the properties of the T-duality map on it in Ref. [2]: If R is the classifying space of pairs of Ref. [2], $T : R \to R$ the T-duality map and $\phi : B \to R$ the classifying map for the pair (E, h), we have, on restriction to W_i, $(T \circ \phi)|_{W_i} = T \circ (\phi|_{W_i})$. However, by definition, $(T \circ \phi)|_{W_i}$ classifies $(E_i^\#, h_i^\#)$ while $\phi|_{W_i}$ classifies $(E_i, h_i) = (E|_{W_i}, h|_{E_i})$.

It is natural to ask if the above can be extended to T-dualize semi-free spaces with many isolated fixed point sets. To do this we need to be able to glue such spaces to other spaces with a circle action and to calculate the T-dual. We show that a result similar to the above holds for semi-free spaces:

Theorem 3.5.

1) *Let E be a semi-free S^1-space with at most countably many isolated fixed sets of the S^1-action $F_1, \ldots, F_k, \ldots$. Suppose we are given disjoint neighbourhoods U_i of F_i. Let $\underline{V_i} = [U_i/S^1]$. Then the T-dual of E is*

determined by the T-dual of the U_i together with the T-dual of the principal circle bundle $P = (E - \bigcup_{l=1}^{i} U_l)$.

2) *The T-dual of a semi-free space E with at most countably isolated fixed sets is the principal bundle of stacks $[E/S^1] \times S^1 \to [E/S^1]$ with H-flux. There will be H-flux on the T-dual coming from an NS5-brane if the T-duals of any of the U_i (see previous part) possess H-flux.*

Proof.

1) Let E be a semi-free S^1-space with at most countably many isolated fixed sets of the S^1-action $F_1, \dots, F_k, \dots$. Let $W = E/S^1$ and let $U_l, l = 1, \dots, k, \dots$ be open disjoint subsets of E, such that each $F_l \subsetneq U_l$. Since the fixed sets are isolated, we may always assume that $U_l \cap U_k = \phi$ for every $l \neq k$. Then, by the classification theorem for spaces with finitely many orbit types (see proof of Cor. (3.6)), $P = (E - \bigcup_{l=1}^{k} U_l)$ is a principal circle bundle over $V = (W - \bigcup_{l=1}^{k} V_l)$ where $V_l = (U_l/S^1)$. Also, E is determined by P and the gluing data for the U_l. Since there is no H-flux on E, these data determine the T-dual of E by Thm. (7.1) item (4) below.

2) Given the data of the previous part, for every i, suppose we are given atlases V_i for $\underline{V_i}$ and induced atlases Q_i for $\underline{U_i}$ in the sense of Ref. [3]. Let SV be the simplical space associated to V, and, for every i, SV_i the simplical space associated to V_i by the above atlases. Similarly, let SP be the simplicial space associated to P and, for every i, SU_i the simplicial space associated to U_i by the above atlases.

   Consider the principal bundle of stacks $\underline{E} \to \underline{W}$. Consider the atlas $X = V \cup_i V_i$ for W. Then $V_i \cap V_j = \phi$ for all $i \neq j$. Also, $V_i \cap V$ need not be empty, but, $V_i \cap V \subseteq V_i$. Now $X \underset{\underline{W}}{\times} X \simeq V \cup V_i \cup (V \cap V_i)$. Similarly, for the same reason, the n-fold fiber product $X \underset{\underline{W}}{\times} \cdots \underset{\underline{W}}{\times} X \simeq V \cup_i V_i \cup (V \cap V_i)$. However, since $(V \cap V_i) \subseteq V_i$, $X \underset{\underline{W}}{\times} X$ may always be written as $V \cup_i V_i$. Also, in the associated simplicial space, V is always glued to each V_i while V_i glue to themselves. Then, the simplicial space associated to X is $SV \cup_f SV_1 \cup_{g_1} \cdots \cup_{g_k} SV_k \cup \cdots$ for some gluing maps f, g_i.

   Consider the atlas $Y = P \cup_i Q_i$ for $\underline{E}$. Here also, we have that $Q_i \cap Q_j = \phi$ for all $i \neq j$. Also, $Q \cap Q_i \subseteq Q_i$ for every i. This implies that $Y \underset{\underline{E}}{\times} Y$ may always be written as $P \cup_i Q_i$ by the intersection property of P and Q_i described above. Similarly the n-fold fiber product $Y \underset{E}{\times} \cdots \underset{E}{\times} Y$ may always be written as $P \cup_i Q_i$ by the intersection property described above. Also, in the associated simplicial space, P is always glued to each P_i while the P_i glue to themselves. Then, the simplicial space associated to Y is $SP \cup_{f'} SQ_1 \cup_{g'_1} \cdots \cup_{g'_k} SQ_k \cdots$ for some gluing maps f', g'_i.

Therefore we have the associated principal bundle $(SP \cup_{f'} SQ_1 \cup_{g'_1} \cdots \cup_{g'_k} SQ_k \cdots) \to (SV \cup_f SV_1 \cup_{g_1} \cdots \cup_{g_k} SV_k \cdots)$ where f', f, g'_i, g are defined above.

By the remark before this theorem, the T-dual will be

$$E^{\#} = ((SP \cup_f SQ_1 \cup_{g_1} \cdots \cup_{g_k} SQ_k \cdots) \times S^1)$$

as a principal bundle over

$$B^{\#} = (SV \cup_f SV_1 \cup_{g_1} \cdots \cup_{g_k} SV_k \cdots).$$

Note that this is the principal simplicial bundle associated to $([E/S^1] \times S^1) \to [E/S^1]$. There will be nonzero H-flux on $E^{\#}$ due to the fact that the original bundle E had nontrivial topology. There will be additional H-flux on $E^{\#}$ due to NS5-branes if there is nonzero H-flux on the T-dual of any of the bundles SQ_i when T-dualized by themselves: By the remark before this Theorem, the H-flux on the total space of the simplicial bundle associated to $E^{\#}$ must restrict to this H-flux on the subspace $SV_i \times S^1$.

□

This Theorem lets us determine the T-dual of any semi-free space with countably many isolated fixed sets. This covers most of the semi-free spaces that would occur in a physical context. In particular, we may now determine the T-dual of a space with at most countably many isolated KK-monopoles.

Corollary 3.6. *Let E be a semi-free S^1-space with at most countably many Kaluza-Klein monopoles $p_1, \ldots, p_k, \ldots$. Then, the T-dual is a trivial principal bundle glued to spaces of the form $([CS^3/S^1] \times S^1)$. There is H-flux present on the T-dual.*

Proof. This is an elementary application of Thm. (3.5). In E, since the KK-monopoles which are the fixed points of the S^1-action are isolated, it is possible to enclose each one in an open set homeomorphic to a ball CS^3. Thus, as topological spaces, each U_i is equivariantly homeomorphic to CS^3.

The atlases U_i and V_i may be chosen as in Thm. (3.4). This construction is always possible by the classification theorem for spaces with finitely many orbit types since there are only two orbit types (fixed points and free orbits) and the fixed points are at most countably many and isolated (See Ref. [16] Chap. V Sec. (5)).

Given this, the T-dual may be found. Note that SV_i are simplicial bundles associated to spaces of the form $[CS^3/S^1] \times S^1$ with H-flux. Thus, the T-dual of E is a stack which is a trivial principal bundle glued to stacks of the form $([CS^3/S^1] \times S^1)$.

There is H-flux present on the T-dual. There will be nonzero H-flux on the T-dual bundle due to the fact that the original bundle had nontrivial topology. There will be additional H-flux on this bundle due to NS5-branes if there is nonzero H-flux on the T-dual of any of the U_i. This is because these will then contribute to a nonzero H-flux on the associated simplicial bundle: By Thm. (3.4), there is nonzero H-flux on the T-dual of any of the $SU_i \subseteq E$, i. e. there is a H-flux on $(SV_i \times S^1) \subseteq E^{\#}$. By the remark before Thm. (3.5), the H-flux on $(SV_i \times S^1)$ is the restriction of the H-flux on the T-dual to $SV_i \times S^1$. However, by the above, this restriction is nonzero hence the T-dual H-flux cannot be zero. □

Note that unlike the C^*-algebraic case ([9]) there is no *source* of H-flux on the T-dual. However, the T-dual does possess H-flux. We make this precise in the following:

Lemma 3.7. *The stack* $[CS^3/S^1]$ *is not equivalent to stack* $[\underline{CS^2}]$ *associated to the space* CS^2.

Proof. The space CS^2 is the coarse moduli space of the stack $[CS^3/S^1]$ (see below) and, as spaces, $CS^3/S^1 \simeq CS^2$.

We show that the stack $\underline{CS^2}$ has different stack cohomology groups to the stack $[CS^3/S^1]$ hence they cannot be equivalent.

The stack $\underline{CS^2}$ is the stack associated to the contractible space CS^2 and hence its stack cohomology groups with $\mathbb{Z}-$coefficients are $H^0 = \mathbb{Z}, H^i = 0, i > 0$.

The stack $[CS^3/S^1]$ is homotopy equivalent to the stack $[*/S^1] \simeq \underline{B}S^1$. This can be seen by considering the equivariant inclusion $* \hookrightarrow CS^3$ which includes the vertex of the cone into the cone. This gives rise to an inclusion of stacks (by Ref. [8], Example (2.9)) $i : [*/S^1] \hookrightarrow [CS^3/S^1]$.

Define an S^1-equivariant homotopy between the identity $1 : CS^3 \to CS^3$ and the projection map $CS^3 \to *$ by, for example, $H : CS^3 \times I \to CS^3$ where the map $H((p,t),s) = (p, \phi(t-s))$, where $\phi(x) = x$ if x is positive and $\phi(x) = 0$ if x is zero or negative.

It is clear that this is S^1-equivariant and descends to a homotopy $H : [CS^3/S^1] \times I \to [CS^3/S^1]$. This homotopy is a homotopy equivalence between $[CS^3/S^1]$ and $[*/S^1]$. Hence, the cohomology of $[CS^3/S^1]$ is the same as the cohomology of $[*/S^1]$ by Sec. (17) of Ref. [5] and Sec. (11) of Ref. [21]. □

Corollary 3.8.

1) *The T-dual $E^{\#}$ of a semi-free S^1-space E with at most countably many KK-monopoles is the principal bundle of stacks $[E/S^1] \times S^1 \to [E/S^1]$ with H-flux.*

2) *$E^{\#}$ is a topological stack which is not equivalent to a topological space if and only if the S^1-action on E has fixed sets.*

3) *The natural map $\phi_{mod} : \underline{E^{\#}} \to \underline{E}^{\#}_{mod}$ is an equivalence iff the S^1-action on $\underline{E}$ has no fixed sets.*

Proof.

1) It follows from the proof of Thm. (3.5) that the simplicial bundle associated to the T-dual stack is the trivial bundle over the base with H-flux. This is the simplicial bundle associated to the principal bundle of stacks $[\underline{E}/S^1] \times S^1 \to [\underline{E}/S^1]$ with H-flux.

2) First note the following: If $\underline{X}$ is a stack equivalent to a topological space X, then, restricting the equivalence to a substack shows that every substack of $\underline{X}$ is equivalent to a topological space. Suppose the action had no fixed sets, i. e. none of the U_i was present in E, then, from the proof of Thm. (3.5) E would be a topological space and so would $E^{\#}$. Now suppose the S^1-action on E had fixed sets. Then one of the U_i would be present in E, then, from the same proof, the T-dual would contain substacks of the form $[U_i/S^1] \times S^1$. Here, by the classification theorem for spaces with finitely many orbit types (see proof of Cor. (3.6) and by the proof of Thms. (3.4,3.5)), each of these would be equivalent to the stack $[CS^3/S^1] \times S^1$. By Lemma (3.7) above, this stack is not equivalent to a topological space. As a result, the T-dual could not be equivalent to a topological space.

3) Suppose the S^1-action on E had fixed points and the map $\phi_{mod} : \underline{E^{\#}} \to \underline{E^{\#}}_{mod}$ induced by ϕ_{mod} was an equivalence of stacks. Then, by the

proof of the previous part choosing suitable neighbourhoods of the fixed points will give an inclusion of stacks $[CS^3/S^1] \to \underline{E^\#}$. Composing with ϕ_{mod} would imply that the map $[CS^3/S^1] \times S^1 \to CS^2 \times S^1$ would be an equivalence of stacks. Since by Lemma (3.7) above, the stacks CS^2 and $[CS^3/S^1]$ are *not* equivalent, the stacks $CS^2 \times S^1$ and $[CS^3/S^1] \times S^1$ aren't equivalent either. Thus the map ϕ_{mod} can't be an equivalence of stacks.

Conversely, suppose the S^1-action on E had no fixed points. Then, by the previous part of the theorem, the T-dual stack would be equivalent to a space and so ϕ_{mod} would give an equivalence $\overline{\phi_{mod}} : \underline{E^\#} \to \underline{E^\#_{mod}}$.

□

Consider the T-dual $[CS^3/S^1] \times S^1]$ of $\underline{CS^3}$: The coarse moduli space of $[CS^3/S^1]$ is CS^2 (See Ref. [5] Example (4.13)), $[CS^3/S^1]$ is the quotient stack of the transformation groupoid $((CS^3 \times S^1) \rightrightarrows CS^3))$. However, since the topological space CS^2 is *contractible* $H^3(CS^2 \times S^1, \mathbb{Z}) = 0$, so there can be no H-flux on $CS^2 \times S^1$. By the above, however, the *stack* $[CS^3/S^1] \times S^1$ possesses H-flux. This is because the simplicial space associated to $[CS^3/S^1]$ (see proof of Thm. (3.4)) is nontrivial and homotopy equivalent to BS^1. Hence, the *stack* cohomology group $H^3([CS^3/S^1] \times S^1, \mathbb{Z})$ is nontrivial.

This nontrivial *stack* cohomology in degree three corresponds, by the above, to a nontrivial $H-$flux on the T-dual $[CS^3/S^1] \times S^1$. Since the H-flux on the T-dual stack would vanish (see Cor. 3.6) if there were no fixed points of the S^1-action on the original space, *presumably this H-flux is the flux generated by the T-dual NS5-brane.* Note that this also happens for the T-dual of $[CS^3/\mathbb{Z}_k]$ for $k > 1$ since the T-duals are the same as the case above only the H-flux changes.

This should also happen in the example in Cor. (3.6) above: The T-dual is a principal bundle P glued to copies of $([CS^3/S^1] \times S^1)$. By the proof of Ref. ([2]), the T-dual is a topological stack. As a space, the coarse moduli space of the T-dual will be $P \times S^1$ glued to $CS^2 \times S^1$. Also, by Cor. (3.8) the T-dual coarse moduli space will be a trivial principal circle bundle. The CS^2 factor is contractible and the resulting space cannot have nonzero H-flux coming from an NS5-brane. (The space will have H-flux only due to the H-flux on $P^\#$). However, the T-dual *stack* does have H-flux coming from this source.

Note that in all these T-duals (see also Cor. (3.8)) above, the reason the T-dual has a nontrivial H-flux is due to the fact that the stack cohomology groups of $[CS^3/S^1]$ are *different* from those of the coarse moduli space $\underline{CS^2}$.

Also, it is interesting to note that the physical T-dual spacetime is the coarse moduli space of the stack. It would be interesting to see whether this is true in other examples of Topological T-duality. We calculate a few more examples of T-duals of three-manifolds in Sec. (7) below.

4. The dyonic coordinate

In String Theory backgrounds which contain KK-monopoles possess a dyonic coordinate. (See Ref. [11] for details. See also Ref. [12]). Roughly speaking, a large gauge transformation of the B-field on a KK-monopole background under T-duality corresponds to a rotation of the T-dual NS5-brane around its circle fiber. A model for this was constructed for KK-monopole backgrounds using C^*-algebraic methods in Ref. [9].

Large gauge transformations of a gerbe on a space X are given by a class in $H^2(X, \mathbb{Z})$ (See Ref. [13]). We would like to understand the behaviour of these classes under Topological T-duality for *semi-free* spaces. As we have argued earlier, the S^1-spaces underlying KK-monopole spacetimes are semi-free spaces.

Using the results of Ref. [13], we show below that for these semi-free spaces X, an automorphism of a trivial gerbe on $\underline{X}$ gives a class in $H^2(X^\#, \mathbb{Z})$ under Topological T-duality.

Theorem 4.1.

1) *Consider the principal bundle of stacks* $[\mathrm{pt}/\mathbb{Z}_k] \to [\mathrm{pt}/S^1]$ *for* $k = 2$. *Consider a trivial gerbe on this stack. Each cyclic subgroup of the group of automorphisms of the gerbe on* $[\mathrm{pt}/\mathbb{Z}_k]$ *gives rise to a cyclic subgroup of* $H^2([\mathrm{pt}/S^1] \times S^1, \mathbb{Z})$. *For* $k = 2$, *this may be calculated explicitly.*
2) *Consider the principal bundle of stacks corresponding to a* KK*-monopole of charge* k. *Consider a trivial gerbe on the total space of the principal bundle. Each cyclic subgroup of automorphisms of the trivial gerbe on the* KK*-monopole of charge* $k > 1$ *gives rise to a cyclic subgroup of the (second) cohomology of the T-dual* $H^2([CS^3/S^1] \times S^1, \mathbb{Z})$.

Proof.

1) Consider a cyclic subgroup of the group of automorphisms of the trivial gerbe on $[\mathrm{pt}/\mathbb{Z}_k]$. It is enough to prove the result for the generator of this subgroup. An automorphism of the trivial gerbe on $[\mathrm{pt}/\mathbb{Z}_k]$ gives rise to a class in $H^2([\mathrm{pt}/\mathbb{Z}_k], \mathbb{Z})$. Consider the proof of Part (2) of Thm. (3.4). The simplicial bundle associated to the principal bundle

of stacks $[\mathrm{pt}/\mathbb{Z}_k] \to [\mathrm{pt}/S^1]$ is $p : B\mathbb{Z}_k \to BS^1$. Since we have a class in $H^2([\mathrm{pt}/\mathbb{Z}_k], \mathbb{Z})$, we obtain a cohomology class on the simplicial space associated to this stack (See Ref. [8], the proof of Prop. (4.7)). In turn this gives a cohomology class on its geometric realization $B\mathbb{Z}_k$ in $H^2(B\mathbb{Z}_k, \mathbb{Z})$.

By the argument in Ref. [13], Thm. (6.3), this class gives rise to a natural class in the second cohomology group of the T-dual bundle $q : BS^1 \times S^1 \to BS^1$. For all natural numbers $k > 1$, $H^2(B\mathbb{Z}_k, \mathbb{Z}) \simeq \mathbb{Z}/k$. Under T-duality an element of $H^2(B\mathbb{Z}_k, \mathbb{Z})$ gives a class of the form $kq^*(a) \simeq k(a \times 1) \in H^2(BS^1 \times S^1, \mathbb{Z})$ for some unknown integer k (where a is the generator of $H^2(BS^1, \mathbb{Z})$ (See Ref. [13], Thm. (6.3)). Thus, an automorphism of the gerbe on $[\mathrm{pt}/\mathbb{Z}_k]$ with this characteristic class gives rise to a cohomology class on the T-dual stack $[\mathrm{pt}/S^1] \times S^1$.

2) This part of the proof is very similar to the previous part. Consider a cyclic subgroup of the group of automorphisms of the trivial gerbe on $[CS^3/\mathbb{Z}_k]$. It is enough to prove the result for the generator of this subgroup. The proof is similar to the proof for the previous part, with the principal bundle changed. An automorphism of the trivial gerbe on $[CS^3/\mathbb{Z}_k]$ gives rise to a class in $H^2([CS^3/\mathbb{Z}_k], \mathbb{Z})$. Consider the proof of Part (3) of Thm. (3.4). The simplicial bundle associated to the principal bundle of stacks $[CS^3/\mathbb{Z}_k] \to [CS^3/S^1]$ is $(A \underset{*}{\times} B\mathbb{Z}_k) \to (B \underset{*}{\times} BS^1)$ (where A and B are defined in Thm. (3.4). Let $P_k = (A \underset{*}{\times} B\mathbb{Z}_k)$ and $W = (B \underset{*}{\times} BS^1)$

Since we have a class in $H^2([CS^3/\mathbb{Z}_k], \mathbb{Z})$, this gives a cohomology class on the simplicial space associated to this stack (See Ref. [8], the proof of Prop. (4.7)) and this, in turn, gives a cohomology class on its geometric realization, that is, a class in $H^2(P_k, \mathbb{Z})$. The T-dual bundle is $W \times S^1 \to W$ with H-flux, and, by Ref. [13], a class in $H^2(P_k, \mathbb{Z})$ under Topological T-duality naturally gives rise to a class in $H^2(W \times S^1, \mathbb{Z})$.

□

Thus a class in $H^2(X, \mathbb{Z})$ naturally gives rise to a class in $H^2(X^\#, \mathbb{Z})$. For a KK-monopole background, a large gauge transformation of the B-field gives rise to a class in $H^2(X, \mathbb{Z})$. By the above this induces a class in $H^2(X^\#, \mathbb{Z})$. For the analogy with the dyonic coordinate to be complete, the induced class in $H^2(X^\#, \mathbb{Z})$ should be viewed as an automorphism of the T-dual semi-free space $X_k^\#$ which rotates each fiber through 2π. However, it is not clear how to prove this in the stack picture.

A similar construction was made in the C^*-algebraic picture of topological T-duality in Ref. [9] where such a rotation did correspond to a *nontrivial spectrum-fixing* automorphism of the T-dual C^*-algebra: The T-dual automorphism obtained there was of this type.

5. Topological T-duality using classifying spaces

In this section we prove that the stack theory method of T-duality of Bunke and coworkers (see Ref. [3, 4]) and the method of T-duality of Mathai and Wu using equivariant cohomology are connected by the notion of a *classifying space of stacks*. In addition we demonstrate that if both methods can be applied to a given space, both will give the same T-dual. The perspective gained by using classifying spaces will help in later sections when we try to T-dualize three-manifolds with an arbitrary circle action.

In Ref. [4] item (1.2.3), Bunke and coworkers argued that the method of Ref. [3] may be generalized to arbitrary S^1-spaces: Briefly, given a space Y with an arbitrary S^1-action, one may consder the quotient map $q : Y \to V$ where $V \simeq Y/S^1$. It is usually impossible to argue about the T-dual of Y since V may be very singular if Y is not a principal circle bundle.

By passing to the stacks we may replace Y with the associated stack $\mathcal{Y} \equiv \underline{Y}$ using the Yoneda lemma. In addition, we may replace the quotient by the natural map of stacks $q : \mathcal{Y} \to \mathcal{V} \simeq [Y/S^1]$. Here the base Y which may be singular has been replaced by the quotient stack $[Y/S^1]$.

Even though the original map of spaces q need not be a principal S^1-bundle of *spaces* (since the S^1-action need not be free), the map of *stacks* $q : \mathcal{Y} \to \mathcal{V}$ is a principal S^1-bundle of *stacks*. This principal circle bundle of stacks may be T-dualized using the arguments given by Bunke and coworkers in Ref. [3] and yields a stack as a T-dual for the space Y. (We have also discussed this method of Bunke and coworkers in Sec. (2) above.)

Another way to T-dualize arbitrary S^1-spaces was proposed by Mathai and Wu in Ref. [10]: To a space with an arbitrary S^1-action Y, we may associate a space $Y \times ES^1$ with the product S^1-action. The S^1-action on this space is free even if the action on Y is not. This space is actually a principal bundle over the space $(Y \times ES^1)/S^1$. The latter space is called the Borel construction.

To a space Y with an arbitrary S^1-action and H-flux H, Mathai and Wu argue that there is associated a nonsingular correspondence space $\hat{Y}$ which is an equivariant circle bundle over Y such that there is a commutative diagram of spaces:

$$
\begin{array}{ccc}
Y & \longleftarrow & \hat{Y} \\
\downarrow & & \downarrow \\
Y/S^1 & \longleftarrow & \hat{Y}/S^1.
\end{array} \tag{5.1}
$$

Here the quotients Y/S^1 and $\hat{Y}/S^1$ might be singular. The authors argue that this diagram may be replaced by another diagram containing nonsingular spaces obtained from the Borel construction above:

$$
\begin{array}{ccc}
Y \times ES^1 & \longleftarrow & \hat{Y} \times ES^1 \\
\downarrow & & \downarrow \\
Y_{S^1} & \longleftarrow & \hat{Y}_{S^1}
\end{array} \tag{5.2}
$$

where $Y_{S^1} \equiv Y \times_{S^1} ES^1 \simeq (Y \times ES^1)/S^1$ (and similarly for $\hat{Y}_S^1$).

In Thm. (1) in Ref. [10], the authors argue that in the situation of Eq. (5.1), the Topological T-dual of Y (with H-flux H) is the (possibly singular) space $\hat{Y}/S^1$. In the case when this space is singular, the authors use the term 'Topological T-dual' to refer to the (nonsingular) correspondence space $\hat{Y}$ - see Ref. [10], Thm. (1) and following. If the T-dual exists, the twisted equivariant cohomology of $\hat{Y}$ gives the twisted cohomology of the T-dual. When the T-dual is singular, the twisted equivariant cohomology of $\hat{Y}$ may still be used, that is, from Ref. [10]:

$$H^{\bullet+1}(\hat{Y}_{S^1}, \hat{H}) \simeq H^{\bullet+1}_{S^1}(\hat{Y}, \hat{H})$$

where $H^*(X, H)$ is the twisted cohomology of X twisted by the class $H \in H^3(X, \mathbb{Z})$ and $H^*_{S^1}(X, H)$ is the twisted equivariant cohomology of X twisted by the class $H \in H^3(X, \mathbb{Z})$ viewed as a three-form $H-$flux H on X.

Similarly, the space $\hat{Y}/S^1$ may be recovered from the nonsingular space $\hat{Y}_{S^1}$ above when the quotient exists. For the purposes of calculation, one may work with the nonsingular diagram Eq. (5.2) above.

The nonsingular diagram in Eq. (5.2) above is actually a Topological T-duality diamond in the sense of Ref. [1] since each of the fibrations $Y \times ES^1 \to Y_{S^1}$ and $\hat{Y}_{S^1} \to Y_{S^1}$ are actually principal circle bundles. The principal circle bundle $\hat{Y}_{S^1} \to Y_{S^1}$ may be identified with the T-dual of the principal circle bundle $Y \times ES^1 \to Y_{S^1}$.

Thus, the Topological T-dual of an arbitrary S^1-space Y proposed by Mathai and Wu may be obtained from the principal bundle of spaces $Y \times$

$ES^1 \to Y_{S^1}$ and the ordinary Topological T-dual (in the sense of Ref. [2]) of this principal circle bundle.

We argue here that both these seemingly different prescriptions for Topological T-duality can be understood from the notion of a *classifying space* associated to a stack.

The following Theorem is Thm. (2.1) from Ref. [18], Sec. (2.2).):

Theorem 5.1. *For every topological stack $\mathcal{X}$, there exists a topological space X together with a morphism $\phi : X \to \mathcal{X}$ which has the property that, for every morphism $T \to \mathcal{X}$ from a topological space T, the pullback $T \times_{\mathcal{X}} X \to T$ is a weak homotopy equivalence.*

A topological space with the above property is termed as a **classifying space** for $\mathcal{X}$. By Ref. [21], every stack $\mathcal{V}$ possesses a classifying space $\Theta(\mathcal{V})$ which is an atlas $\phi : \Theta(\mathcal{V}) \to \mathcal{V}$ possessing the above property.

There are at least two classifying spaces naturally associated to a given stack, the Haefliger-Milnor classifying space and the simplicial classifying space [4]. (By Ref. [21], all classifying spaces for the same stack are weak homotopy equivalent, hence the classifying space is actually a functor Θ : **Stacks** $\to (\mathbf{Top})_{w.e.}$. Here $(\mathbf{Top})_{w.e.}$ is the homotopy category of topological spaces, i.e. the category of topological spaces localized at the weak homotopy equivalences.)

Before we study Topological T-duals, we need to discuss some facts about classifying spaces and principal bundles of stacks.

Let $p : \mathcal{Y} \to \mathcal{V}$ be a principal circle bundle of stacks. To determine the Topological T-dual of $\mathcal{Y}$, we need a gerbe $\mathcal{G}$ on $\mathcal{Y}$ for the $H-$flux.

We pull back the classifying space of $\mathcal{V}$, $V = \Theta(\mathcal{V})$ along p to a classifying space Y for the stack $\mathcal{Y}$, that is, we pull back $\mathcal{V}$ along p to $\Theta(\mathcal{Y})$. Hence, we obtain the following (2-commutative) diagram

$$\begin{array}{ccc} Y \simeq V \times_{\mathcal{V}} \mathcal{Y} & \xrightarrow{p^*(\phi)} & \mathcal{Y} \\ \downarrow & & \downarrow p \\ V = \Theta(\mathcal{V}) & \xrightarrow[\phi]{} & \mathcal{V}. \end{array}$$

This may always be done since p is a principal bundle of stacks. It can be shown (see Ref. [14], proof of Thm. (8.4)) that $Y \to V$ is a principal circle bundle of *spaces* if p is a principal circle bundle of *stacks*.

[4]See Ref. [21] especially Secs. (4), (5) and (6).

The above presumes a *natural* way to choose the classifying space $\Theta(\mathcal{V})$ for a given stack $\mathcal{V}$. Both the approaches to Topological T-duality for stacks require a choice of a classifying space $\Theta(\mathcal{V})$ not just a weak equivalance class of classifying spaces. There are at least two classifying spaces naturally associated to a given stack $\mathcal{V}$, the Haefliger-Milnor classifying space and the Simplicial classifying space (see Ref. [21] Sec. (4)).

The Topological T-duality for stacks of Bunke et al uses the Simplicial classifying space for the stack $\mathcal{V}$. We consider the S^1-space Y as a *stack* $\mathcal{Y}$ with an S^1-action. Similarly we consider the stack $\mathcal{V} \simeq [Y/S^1]$ and the natural map $q : \mathcal{Y} \to \mathcal{V}$ as in Sec. (3) above.

If we have a groupoid presentation $[Y/R]$ for the stack $\mathcal{Y}$ and a similar presentation $[V/R]$ for the stack $\mathcal{V}$, we may use the Simplicial classifying spaces $|\mathcal{Y}|$ and $|\mathcal{V}|$ associated to $\mathcal{Y}$ and $\mathcal{V}$ respectively (see Refs. [3, 21]). We obtain the natural (2-commutative) square:

$$\begin{array}{ccc} |\mathcal{Y}| & \longrightarrow & [Y/R] \simeq \mathcal{Y} \\ \downarrow & & \downarrow p \\ |\mathcal{V}| & \longrightarrow & [V/S] \simeq \mathcal{V}. \end{array}$$

In addition the gerbe $\mathcal{G}$ on $\mathcal{Y}$ induces a $H-$flux H on $|\mathcal{Y}|$ as in Ref. [3]. The stack T-dual of Bunke and coworkers (Ref. [3]) is the stack associated to the simplicial bundle which is the T-dual of the simplicial circle bundle $|\mathcal{Y}| \to |\mathcal{V}|$ with $H-$flux H on $|\mathcal{Y}|$.

We argue that the Topological T-duality of Mathai and Wu for spaces with arbitrary S^1-actions uses a similar prescription but for the the Haefliger-Milnor classifying space.

Let Y be a non-free S^1-space. Let $\mathcal{Y}$ be the underlying stack. Consider the stack $\mathcal{V} = [Y/S^1]$. This has a natural presentation as the transformation groupoid $\mathbb{V} = [(Y \times S^1) \rightrightarrows Y]$. This stack has a natural[5] classifying space $B\mathbb{V}$ (the Haefliger-Milnor Classifying Space) associated to this groupoid which is given by the Borel construction $Y \times_{S^1} ES^1$. The principal bundle of stacks $p : \mathcal{Y} \to \mathcal{V} \simeq [Y/S^1]$ gives a principal bundle of spaces $E\mathbb{V} \simeq$

[5]See Ref. [21], Sec. (4.3), also the proof of Thm. (6.3) for the definition and properties of classifying spaces.

$(Y \times ES^1) \to B\mathbb{V} \simeq (Y \times_{S^1} ES^1)$ by pullback and a 2-cartesian square[6]

$$\begin{array}{ccc} E\mathbb{V} = (Y \times ES^1) & \xrightarrow{f} & \mathcal{Y} \equiv Y \\ {\scriptstyle q}\downarrow & & \downarrow \\ B\mathbb{V} = (Y \times_{S^1} ES^1) & \xrightarrow{\phi} & \mathcal{V} \simeq [Y/S^1] \end{array} \tag{5.3}$$

where the space in each row is a classifying space[7]. Also, the map ϕ is *natural* (See Ref. [21], before Sec. (4.2)). (In general, the Haefliger-Milnor classifying space of a stack depends on the groupoid presentation of that stack. For a quotient stack there is a natural choice of the associated groupoid and hence a natural choice of the associated classifying space.)

The bundle of classifying spaces obtained in Eq. (5.3) above is the same as the principal circle bundle in the first column of the diagram Eq. (5.2). We had remarked above that the T-dual of Mathai and Wu could be obtained from this principal circle bundle by the usual Topological T-duality for circle bundles (see, for example, Ref. [2]).

Hence, from the work of Mathai and Wu (Ref. [10]) (when the T-dual space $\hat{Y}/S^1$ is well-defined), the Topological T-dual of Mathai and Wu can be obtained from the the Topological T-dual of the associated principal circle bundle over the Haefliger-Milnor classifying space above. When $\hat{Y}/S^1$ is *not* well defined in the prescription of Mathai and Wu, the formalism of this section gives a *stack* for the T-dual, the stack associated to the T-dual of the principal bundle of classifying spaces $Y \times ES^1 \to Y \times_{S^1} ES^1$. In this case, the T-dual obtained here is the *stack* $[\hat{Y}/S^1]$ where $\hat{Y}$ is Mathai and Wu's T-dual. Mathai and Wu's T-dual can be naturally recovered as the classifying space $\hat{Y}$ of this T-dual stack $[\hat{Y}/S^1]$.

For either of these prescriptions of Topologial T-duality, then, for a given stack $\mathcal{Y}$ with S^1-action (which may be the stack associated to a space), we consider $\mathcal{Y} \to \mathcal{V} \simeq [\mathcal{Y}/S^1]$. The Topological T-dual of $\mathcal{Y}$ is the stack associated to the T-dual of some natural principal bundle of classifying spaces $p : Y \to V$. That is, we find the T-dual bundle $p^{\#} : Y^{\#} \to V$ of the above bundle p and find a stack $\mathcal{Y}^{\#}$ such that $Y^{\#} = \Theta(\mathcal{Y}^{\#})$.

It is interesting to ask whether the T-dual stack depends on this choice of classifying space. We show that this is not the case. We first need a lemma:

Lemma 5.2. *Let $p_i : Y_i \to V_i, i = 1, 2$ be principal circle bundles over V_i. Let $f : V_1 \to V_2$ be a map which is also a weak homotopy equivalence. Suppose*

[6]See Ref. [21] Sec. (4.3) and cartesian square after Lemma (4.1).
[7]See Prop. (6.1) of Ref. [21].

that f induces a pullback square:

$$
\begin{array}{ccc}
Y_1 & \xrightarrow[\tilde{f}]{} & Y_2 \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
V_1 & \xrightarrow[f]{} & V_2.
\end{array}
\tag{5.4}
$$

Then, the set of pairs over V_1 and the set of pairs over V_2 are isomorphic by an isomorphism B_f which commutes with Topological T-duality.

Proof. Let $Y_i \to V_i$ be principal circle bundles over V_i such that f induces a pullback square:

$$
\begin{array}{ccc}
Y_1 & \xrightarrow[\tilde{f}]{} & Y_2 \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
V_1 & \xrightarrow[f]{} & V_2.
\end{array}
$$

Let $H_i \in H^3(Y_i, \mathbb{Z})$ be the H-fluxes on the Y_i. By definition, (p_i, H_i) are 'pairs' (in the sense of Ref. [2]) over $V_i, i = 1, 2$. Since f is a weak homotopy equivalence, f induces isomorphisms $f_i : \pi_i(V_1) \to \pi_i(V_2), \forall i \geq 0$ by definition. This implies that all the cohomology groups of the V_i are isomorphic. In particular, every principal bundle over V_2 induces a bundle with the same characteristic class over V_1 by pullback. Also, every bundle over V_1 is the pullback of a bundle over V_2 by f^*.

In addition, the above commutative diagram induces a natural isomorphism of Leray-Serre spectral sequences for the spaces Y_i, so that their cohomologies are also isomorphic (see Prop. (1.12) of [24]).

This implies that every H-flux H_2 on $Y_2 \to V_2$, pulls back by the induced map on $H^3(Y_2, \mathbb{Z})$ to one and only one H-flux on $Y_1 \to V_1$. Thus there is a natural bijection $B_f : P(V_2) \to P(V_1)$.

Replacing the above square by the T-dual square

$$
\begin{array}{ccc}
Y_1^{\#} & \xrightarrow[\tilde{f}^{\#}]{} & Y_2^{\#} \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\
V_1 & \xrightarrow[f]{} & V_2.
\end{array}
$$

and applying the above argument shows that there is a natural dual isomorphism which induces isomorphisms on the total spaces of $Y_i^{\#}$.

This isomorphism induces a bijection (also denoted B) of dual pairs in a manner similar to the above.

Further, we need to show that the bijections B_f above commute with the action of Topological T-duality $B_f \circ T \circ P(V_2) = T \circ B_f \circ P(V_2)$. First note that the morphisms $\tilde{f}, \tilde{f}^{\#}$ are induced from f just by pullback.

Given the data in the statement of the theorem and in diagram in Eq. (5.4) above, we can view the pairs on each of the V_i's as a map $h_i : V_i \to R$, where R is the classifying space of pairs of Bunke et al (see Ref. [2] above). The above diagram may be obtained from the pullback of the universal bundle on R along the commutative diagram associated to the equation $h_1 = h_2 \circ f$.

If one composes the maps h_i with the T-duality transformation $T : R \to R$, the diagram at the end of the previous paragraph extends to a more complicated commutative diagram. Examining the pullback of a pair over V_2 along the two obvious routes will show that the image of the T-dual of a pair is the T-dual of the image of that pair for any pair.

This proves the result. □

Theorem 5.3. *Let $p : \mathcal{Y} \to \mathcal{V}$ be a principal circle bundle of stacks. Let $\mathcal{G} \to \mathcal{Y}$ be a circle gerbe on Y. The T-dual stack exists and doesn't depend on the choice of the classifying space for $\mathcal{V}$.*

Proof. The proof depends on Lemma (5.2) above. In addition, we need the notion of a 'pair' from Ref. [3] here. Let $p : \mathcal{Y} \to \mathcal{V}$ be a principal circle bundle of stacks. Let $\mathcal{G} \to \mathcal{Y}$ be a gerbe on $\mathcal{Y}$ with band S^1. We consider the pair $(p, \mathcal{G})$.

By Lemma (5.2) above, there is a bijection between pairs over one classifying space for $\mathcal{V}$ and pairs over another classifying space for $\mathcal{V}$. In particular, this implies that the T-dual pair is uniquely specified for any classifying space for $\mathcal{V}$ once it is specified for one choice of a classifying space for $\mathcal{V}$. By the work of Bunke and coworkers in Ref. [3], Thm. (1.1) and Def. (1.2), the stack T-dual pair exists and is unique if we use simplicial classifying spaces.

Note that any two classifying spaces are weakly homotopy equivalent (see Ref. [21]). Now, if we change the classifying space, by Lemma (5.2) above, the set of pairs over the two classifying spaces are isomorphic and the result doesn't depend on the choice of weak homotopy equivalence. The above proof of Bunke and coworkers can be used to ensure the existence of T-dual pairs no matter which classifying spaces were used. □

6. The Gysin sequence

The results of Bunke et al [4] are for principal bundles over orbispaces. Here, the base stack is Deligne-Mumford, i.e., locally a quotient of a topological stack by the action of a discrete group (see Ref. [8]). We wish to study principal bundles over spaces which do not have this property, for example, the circle actions on the three-manifolds Y in the next section. Here, the base stack would be $[Y/S^1]$ which need not be a Deligne-Mumford. (Note that the method of Mathai and Wu does not have such a restriction.)

In this section, we introduce another method of T-duality, the Gysin Sequence method, which also gives T-duals agreeing with either of the two methods above. This method is the extension of the Gysin sequence method of Topological T-duality for principal circle bundles to principal circle bundles of stacks (see Ref. [19] and references therein for details). We will use this method later in the paper.

Given a pair $([p], H)$ consisting of a principal circle bundle $p : E \to W$, with a H-flux $H \in H^3(E, \mathbb{Z})$ as in Ref. [2], we may compute the characteristic class of the T-dual using the Gysin Sequence as in Ref. [20], Sec. (2.4).

For the above principal circle bundle, we have the following sequence (Gysin sequence):

$$\cdots \to H^2(E,\mathbb{Z}) \xrightarrow{p_!} H^1(W,\mathbb{Z}) \xrightarrow{\cup[p]} H^3(W,\mathbb{Z}) \xrightarrow{p^*} H^3(E,\mathbb{Z}) \xrightarrow{p!} H^2(W,\mathbb{Z}) \xrightarrow{\cup[p]} H^4(W,\mathbb{Z}) \to \cdots$$

The above data for a pair defines a class H in $H^3((E, \mathbb{Z})$. The image of this under $p_!$ gives a class $[\hat{p}]$ in $H^2(W, \mathbb{Z})$. By exactness, $[\hat{p}] \cup [p] = 0$. Let $\hat{p} : \hat{E} \to W$ be the circle bundle over W with characteristic class $[\hat{p}]$. Writing the Gysin sequence for $\hat{p}$, since $[p] \cup [\hat{p}] = [\hat{p}] \cup [p] = 0$, by exactness there exists $\hat{H} \in H^3(\hat{E}, \mathbb{Z})$ such that $p_!(\hat{H}) = [p]$. This is the characteristic class of the H-flux $\hat{H}$ of the T-dual bundle. Since $[p]$ is in the kernel of $\cup[\hat{p}]$, by exactness this element is unique up to an element of $\hat{p}(H^3(W, \mathbb{Z}))$.

Is it possible to T-dualize principal circle bundles of stacks using the above method? For this to be possible a cohomology Gysin sequence for a principal circle bundle of stacks $p : \mathcal{E} \to \mathcal{F}$ is needed. Ref. [14] constructs a homology Gysin sequence from a principal circle bunde of stacks. However, for the above construction we need a cohomology Gysin sequence which we construct below.

We also show that for stacks $\mathcal{E}$ which are the stacks associated to a semi-free S^1-space E, a 'T-dual' stack may be constructed using a stack analogue of the Gysin sequence argument above. We show that that this T-dual stack naturally is the stack associated to the T-dual of E which would be obtained in the formalism of Mathai and Wu (see Ref. [10]). We show at the end of this section that the stack T-dual defined in this section agrees with the T-dual of Bunke et al. in the examples of Thm. (3.4). This would be expected due to Thm. (5.3) above.

We now give a few defnitions which will be needed for what follows. These are from Secs. (3,4,6,7,8) of the paper by Behrend, Ginot, Noohi and Xu Ref. [18].

First, note that it is possible to define a vector bundle, an orientation on a vector bundle and a metric on it for a topological stack (We give the definition from Ref. [18] below.)

Definition 6.1. Let $\mathcal{X}$ be a topological stack. A real **vector bundle** on $\mathcal{X}$ is a representable morphism of stacks $\mathcal{C} \to \mathcal{X}$ which makes $\mathcal{C}$ a vector space object relative to $\mathcal{X}$. That is, we have an addition morphism $\mathcal{C} \times_{\mathcal{X}} \mathcal{C} \to \mathcal{C}$ and an $\mathbb{R}$-action $\mathbb{R} \times \mathcal{C} \to \mathcal{C}$, both relative to $\mathcal{X}$, which satisfy the usual axioms. A complex vector bundle is defined analogously.

Just as for principal circle bundles, specifying a vector bundle $\mathcal{C} \to \mathcal{X}$ is equivalent to specifying a vector bundle $E_U \simeq U \times_{\mathcal{X}} \mathcal{C} \to U$ for every $f : U \to X$ with U a topological space.

Given a vector bundle over a stack we may define an orientation and a metric on it as in Ref. [18], Secs. (3,4).

We now define two properties of morphisms of stacks which are very useful for the development of the Gysin sequence for stacks. (These definitions are Def. (6.1-6.2) of Ref. [18].)

Definition 6.2. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of topological stacks and $\mathcal{C} \to \mathcal{Y}$ a metrizable vector bundle over $\mathcal{Y}$. A lifting $i : \mathcal{X} \to \mathcal{C}$ of f is called **bounded** if there is a choice of metric on $\mathcal{C}$ such that i factors through the unit disk bundle of $\mathcal{C}$.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ of topological stacks is called **bounded proper** if there exists a metrizable orientable vector bundle $\mathcal{C} \to \mathcal{Y}$ on $\mathcal{Y}$ and a bounded lifting i as above such that i is a closed embedding.

A bounder proper morphism $f : \mathcal{X} \to \mathcal{Y}$ is called **strongly proper** if every orientable metrizable vector bundle $\mathcal{G}$ on X is a direct summand of $f^*(\mathcal{G}')$ for some orientable metrizable vector bundle $\mathcal{G}'$ on $\mathcal{Y}$.

To a morphism $f : \mathcal{X} \to \mathcal{Y}$ of topological stacks, we associate a category $C(f)$ as follows: The objects of $C(f)$ are morphisms $a : \mathcal{L} \to X$ such that $fa : \mathcal{L} \to \mathcal{Y}$ is bounded proper. A morphism in $C(f)$ between $a : \mathcal{L} \to \mathcal{X}$ and $b : \mathcal{W} \to \mathcal{X}$ is a homotopy class (relative to $\mathcal{X}$) of morphisms $g : \mathcal{L} \to \mathcal{W}$ over $\mathcal{X}$.

It can be proved that $C(f)$ is cofiltered (see Ref. [18], Sec. (7.1)). We choose, for each object $a : \mathcal{L} \to \mathcal{X}$, a vector bundle $\mathcal{C} \to \mathcal{Y}$ through which fa factors as in the above definition of bounded proper morphisms.

The **bivariant singular homology** of an arbitrary morphism $f : \mathcal{X} \to \mathcal{Y}$ is the $\mathbb{Z}$-graded abelian group

$$H^\bullet(\mathcal{X} \xrightarrow{f} \mathcal{Y}) = \varinjlim_{C(f)} H^{\bullet + \mathrm{rk}\, \mathcal{C}}(\mathcal{C}, \mathcal{C} - \mathcal{K}).$$

Here, any morphism $\phi : \mathcal{K} = \mathcal{L} \to \mathcal{L}'$ in $C(f)$ gives as natural graded pushforward homomorphism $\phi_* : H^{\bullet+m}(\mathcal{C}, \mathcal{C} - \mathcal{L}) \to H^{\bullet+n}(\mathcal{C}', \mathcal{C}' - \mathcal{L}')$ where $m = \mathrm{rk}\mathcal{C}$ and $n = \mathrm{rk}\mathcal{C}'$ as shown in Ref. [18], Sec. (7.1).

We now come to the definition of an **adequate** morphism of stacks. This will be repeatedly used below. (This is Def. (7.4) of Ref. [18].)

Definition 6.3. A morphism $f : \mathcal{X} \to \mathcal{Y}$ of topological stacks is called **adequate** if in the cofiltered category $C(f)$, the subcategory consisting of $a : \mathcal{L} \to X$ such that $fa : \mathcal{L} \to \mathcal{Y}$ is strongly proper is cofinal.

We refer the reader to Sec. (7) of Ref. [18] for the Bivariant theory of topological stacks including the product of bivariant classes (defined after Ex. (7.5) in Ref. [18]) as these are needed to understand the construction of the cohomology Gysin sequence below.

We define below normally nonsingular morphisms of topological stacks. (This is Def. (8.15) of Ref. [18].)

Definition 6.4. We say that a representable morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks is **normally nonsingular**, (**nns** for short), if there exist vector bundles $\mathcal{N}$ and $\mathcal{C}$ over the stacks $\mathcal{X}$ and $\mathcal{Y}$, respectively, and a commutative diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow[i]{} & \mathcal{C} \\ \uparrow{\scriptstyle s} & & \downarrow{\scriptstyle p} \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y} \end{array} \tag{6.1}$$

where s is the zero section of the vector bundle $\mathcal{N}$, i is an open embedding, and $\mathcal{C}$ is oriented.

We refer the reader to Sec. (8.3) of Ref. [18] for the theory of normally nonsingular morphisms of stacks.

We define orientation in the sense of the bivariant theory of stacks of a map of stacks $f : \mathcal{X} \to \mathcal{Y}$ below. The definitions below are Defs. (8.20-21) of Ref. [18].

Definition 6.5. Let $f : \mathcal{X} \to \mathcal{Y}$ be a strongly proper morphism.A bivariant class $\theta \in H(\mathcal{X} \stackrel{f}{\to} \mathcal{Y})$, not necessarily homogenous, is called a **strong orientation** if for every $g : \mathcal{Z} \to \mathcal{X}$, multiplication by θ is an isomorphism $H(\mathcal{Z} \stackrel{g \circ f}{\to} \mathcal{Y}) \stackrel{\sim}{\to} H(\mathcal{X} \stackrel{f}{\to} \mathcal{Y})$.

A strongly proper morphism $f : \mathcal{X} \to \mathcal{Y}$ of topological stacks is called **strongly oriented,** if it is normally nonsingular and it is endowed with a strong orientation $\theta_f \in H^c(f)$, where $c =$ codim f. A topological stack $\mathcal{X}$ is called strongly oriented if the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is strongly oriented. In this case, we define $dim\mathcal{X} := codim\Delta$.

We refer the reader to Defs. (8.20-21) onwards of Ref. [18] for the further development of the above, this will be needed below.

We assume the reader is familiar with the construction and basic properties of the Gysin map associated to a bivariant class in Ref. [14], Sec. (9.1-9.2) and the construction of G-equivariant Gysin maps in Sec. (9.3) of the same paper. In addition we assume the development of the transfer map and the homology Gysin sequence for S^1-stacks based on this in Sec. (8) of Ref. [18] (We briefly outline some of these ideas from Refs. [14, 18] in the proof below.).

Lemma 6.6. *Let $q : \mathcal{E} \to \mathcal{Y}$ be a principal S^1-bundle of stacks. There is a cohomology Gysin sequence for this bundle if q adequate.*

Proof. By Ref. [18], Prop. (8.4), every S^1-stack $\mathcal{E}$ possesses a homology stack Gysin sequence

$$(6.2) \qquad \cdots \to H_i(\mathcal{Y}) \stackrel{q_*}{\to} H_i^{S^1}(\mathcal{Y}) \stackrel{\cap c}{\to} H_{i-2}^{S^1}(\mathcal{Y}) \stackrel{T}{\to} H_{i-1}\mathcal{Y} \stackrel{q_*}{\to} H_{i-1}^{S^1}(\mathcal{Y}) \to \cdots$$

where T is the homology transfer map (also denoted T^{S^1}). We are looking for a cohomology stack Gysin sequence. The reason this might not exist is that it is not clear that a cohomology transfer map T_{S^1} can be defined for arbitrary f.

We briefly review the derivation of the cohomology transfer map from Ref. [18] here: The transfer map T_{S^1} is obtained from the definition of $\theta_!$ of Ref. [18], Sec. (9.1), after Eq. (9.1).

To define $\theta_!$, the authors of Ref. [18], consider the 2-cartesian square (this is Eq. (9.1) of Ref. [18])

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow[f']{} & \mathcal{Y}' \\ \downarrow & & \downarrow u \\ \mathcal{X} & \xrightarrow[f]{} & \mathcal{Y}. \end{array} \tag{6.3}$$

For this diagram of stacks the authors define the cohomology Gysin homomorphism as $\theta_!(b) = f'_*(b \cdot u^*(\theta))$, for $b \in H^j(\mathcal{X}') = H^j(\mathcal{X}' \stackrel{\mathrm{id}}{\to} \mathcal{X}')$. However, this requires that the map f' is adequate or the above product need not be defined (by Ref. [18], after Ex. (7.5)).

The cohomology Gysin map T_{S^1} is obtained by applying the above to the 2-cartesian square

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow[q]{} & [S^1\backslash\mathcal{E}] \\ \downarrow & & \downarrow u \\ * & \xrightarrow[q]{} & [S^1\backslash *]. \end{array} \tag{6.4}$$

The transfer map in cohomology T_{S^1} is $\theta_!$ for the diagram in Eq. (6.3) calculated for the diagram in Eq. (6.4). Hence, by the above, for this map to be well-defined we require that q be adequate. Now, by Lemma (3.1) above, considering the principal circle bundle $q : \mathcal{E} \to [S^1\backslash\mathcal{E}]$ is exactly equivalent to considering the principal circle bundle $q : \mathcal{E} \to \mathcal{Y}$ since by the results there $\mathcal{Y} \simeq [S^1\backslash\mathcal{E}]$ always. Hence, it is enough to assume that $q : \mathcal{E} \to \mathcal{Y}$ is adequate.

The proof of Prop. (8.4) of Ref. [14] may now be followed with H_* replaced by H^*. It is clear that the full proof goes through. Let $Z \to [S^1\backslash\mathcal{E}]$ be a classifying space for $[S^1\backslash\mathcal{E}]$ and $Y \to \mathcal{E}$ be the classifying space for $\mathcal{E}$ obtained by pullback along q. Let c be the Euler class of disk bundle associated to the principal bundle $Y \to Z$, then we obtain the following Gysin Sequence:

$$\cdots \to H^{i-1}_{S^1}(\mathcal{E}) \stackrel{q^*}{\to} H^{i-1}(\mathcal{E}) \stackrel{T_{S^1}}{\to} H^{i-2}_{S^1}(\mathcal{E}) \stackrel{\cup c}{\to} H^i_{S^1}(\mathcal{E}) \stackrel{q^*}{\to} H^i(\mathcal{E}) \to \cdots \tag{6.5}$$

from the cohomology Gysin sequence associated to $Y \to Z$ under the identifications $H^i(Z) \simeq H^i([S^1\backslash\mathcal{E}]) \simeq H^i_{S^1}(\mathcal{E})$, and $H^i(Y) \simeq H^i(\mathcal{E})$ exactly as in Ref. [14], Prop. (8.4). □

We now prove that under some conditions the quotient map p associated to a principal bundle of stacks $p : \mathcal{E} \to [S^1\backslash\mathcal{E}]$ is adequate.

We need some theorems from Ref. [18]. Recall from Sec. (3) above that the semi-free spaces we consider are all total spaces of KK-monopoles. In particular, they are all oriented orbifolds.

They are all strongly oriented by the following proposition: (This is Prop. (8.35) of Ref. [18])

Theorem 6.7. *Let $\mathcal{X}$ be a paracompact orbifold whose tangent bundle is oriented. Then the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is strongly oriented and, in particular, $\mathcal{X}$ is naturally oriented.*

In addition we need the following Proposition (Prop. (8.32) from Ref. [18]). Let G be a compact Lie group and X and Y smooth G-manifolds, with $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/G]$ the corresponding quotient stacks.

Theorem 6.8. *Let X, Y be as above. Assume further that X and Y are oriented and that the G-actions are orientation preserving. Then, every normally nonsingular diagram for $f : \mathcal{X} \to \mathcal{Y}$ is naturally oriented. In particular, when f is strongly proper, we have a strong orientation class $\theta_f \in H^c(f), c = \dim Y - \dim X$. Furthermore, this class is independent of the choice of the normally nonsingular diagram.*

We may now prove the above:

Lemma 6.9. *Let $\mathcal{E}$ be a stack associated to a semi-free space. Let $\mathcal{Y} = [S^1\backslash\mathcal{E}]$ and let $p : \mathcal{E} \to \mathcal{Y}$ be the quotient map. Let $\mathcal{V}$ be the associated vector bundle to the principal bundle $p : \mathcal{E} \to \mathcal{Y}$. Suppose $\mathcal{V}$ is metrizable. Further, suppose $\mathcal{Y}$ is strongly oriented. Then p is strongly oriented. Also, p is adequate.*

Proof. Firstly, $\mathcal{E}$ is strongly oriented due to Prop. (6.7) above. Secondly, $\mathcal{Y}$ is strongly oriented by assumption. Thirdly, we will argue that p is strongly proper and normally nonsingular. Hence, $p : \mathcal{E} \to \mathcal{Y}$ has a strong orientation class by Prop. (6.8).

By Lemma (3.1) above, $p : \mathcal{E} \to \mathcal{Y}$ is a principal S^1-bundle. By Ref. [14], it is also representable (By Prop. (4.7) of Ref. [14]). Let $q : \mathcal{V} \to \mathcal{Y}$ be the associated vector bundle. $\mathcal{V}$ is metrizable by assumption. Let $s : \mathcal{E} \to \mathcal{V}$ be the embedding of $\mathcal{E}$ as the unit sphere bundle in $\mathcal{V}$. Then, the following diagram commutes

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{id} & \mathcal{V} \\ \uparrow{\scriptstyle s} & & \downarrow{\scriptstyle q} \\ \mathcal{E} & \xrightarrow{p} & \mathcal{Y}. \end{array} \tag{6.6}$$

This is the required normally nonsingular diagram for p.

It is clear that p is bounded proper. It remains to prove that p is strongly proper. Let $w : \mathcal{C} \to \mathcal{E}$ be an orientable metrizable vector bundle on $\mathcal{E}$. Choose a classifying space $Y \to \mathcal{Y}$ and pull back Y to $\mathcal{E}$ along p to obtain a classifying space $E \to \mathcal{E}$. This gives a principal bundle $E \to Y$. Pulling $\mathcal{V}$ back to Y we obtain a bundle $\mathcal{V} \underset{\mathcal{Y}}{\times} Y \to Y$. The pullback of this bundle along the map $E \to Y$ is trivial since the bundle has the same Euler class as the bundle of stacks $\mathcal{V} \to \mathcal{Y}$ (this follows from the definition of the Euler class, see Behrend et al. Ref. [18] Ex. (8.26)). It follows that $p^*(\mathcal{V})$ is a trivial bundle. Hence, taking charts, there is an integer $n > 0$ such that $\mathcal{C}$ is a direct summand of $(p^*(\mathcal{V}))^n \simeq p^*(\mathcal{V}^n)$. Now, $\mathcal{V}$ is the vector bundle associated to $\mathcal{E}$, and, since p is orientable, so is $\mathcal{V}$. Further, $\mathcal{V}$ is metrizable by assumption and hence, so is $\mathcal{V}^n$. Hence, p is strongly proper. By the above p is normally nonsingular. Hence, p has a strong orientation class by Prop. (6.8) above. By Ex. (7.5) (1) of Ref. [18], p is adequate. □

We had noted above that the Gysin sequence can be used to calculate the characteristic class and $H-$flux of the T-dual principal bundle of stacks. It is interesting to note that the T-dual data doesn't depend on the choice of classifying space made in this calculation.

Thus we may use the above Gysin sequence to obtain a T-dual by analogy with the usual Gysin sequence argument for Topological T-duality: Given a principal bundle of stacks $p : \mathcal{E} \to \mathcal{Y}$ with Euler class $[p] \in H^2(\mathcal{Y})$ and a gerbe $\mathcal{G} \to \mathcal{E}$ with characteristic class $[H] \in H^3(\mathcal{E})$ (see Ref. [8] Sec. (5), Prop. (5.8)), we can define the T-dual to be the principal bundle of stacks $p^{\#} : \mathcal{E}^{\#} \to \mathcal{Y}$ whose Euler class is $T_{S^1}([H])$ by the Gysin sequence method above. The T-dual H-flux would then be a gerbe on $\mathcal{E}^{\#}$ with characteristic class $[H^{\#}]$ such that $T_{S^1}([H^{\#}]) = [p]$.

To show that this T-dual exists and is unique, we use Thm. (5.3) above. We may pick a classifying space $Y \to \mathcal{Y}$. By definition, the class $T_{S^1}([H]) \in H^2(\mathcal{Y})$ determines a class in $H^2(Y, \mathbb{Z})$ since the two cohomology groups are naturally isomorphic. This class in turn determines, up to an isomorphism of pairs, a dual principal circle bundle of spaces $E^{\#} \to Y$ with dual $H-$flux $H^{\#}$. By the work of Bunke and coworkers (see Ref. [3]), this uniquely determines up to isomorphism of pairs in the sense of Ref. [3], the T-dual principal circle bundle of *stacks* $p^{\#} : \mathcal{E}^{\#} \to \mathcal{Y}$ and T-dual gerbe $\mathcal{G} \to \mathcal{E}^{\#}$ with characteristic class $H^{\#} \in H^2(E^{\#}, S^1)$. By Thm. (5.3), this T-dual doesn't depend on the choice of the classifying space.

It is interesting to note that the Gysin sequence method above can be used to obtain the T-dual in both the method of Mathai and Wu (see Ref. [10]) and Bunke and coworkers (see Ref. [3]) just by changing the choice of classifying space in the calculation.

We would like to apply the above to some concrete examples. We apply these to the KK-monopole spaces in Sec. (3) above. For all the examples of KK-monopoles in Thm. (3.4) above, the total space is the stack $[CS^3/\mathbb{Z}_k]$ associated to an oriented orbifold and hence strongly oriented by Prop. (8.35) of Ref. [18]. From Thm. (3.4) above, the quotient stack by the S^1-action is always $[CS^3/S^1]$ and any vector bundle $\mathcal{V}$ over $[CS^3/S^1]$ is metrizable by Ex. (3.3) of Ref. [18]. Also $[CS^3/S^1]$ is strongly oriented by Prop. (8.33) and Def. (8.21) of Ref. [18]. Hence, in all the examples of KK-monopoles calculated above, the bundle maps $p_k : [CS^3/\mathbb{Z}_k] \to [CS^3/S^1]$ and $p : \underline{CS^3} \to [CS^3/S^1]$ are adequate. Thus, we may use the Gysin sequence argument above to obtain the T-dual.

Corollary 6.10. *For every $k > 1$, k a natural number, the T-dual of the principal bundle of stacks $p_k : [CS^3/\mathbb{Z}_k] \to [CS^3/S^1]$ using the Gysin Sequence method above is the principal bundle of stacks $q : ([CS^3/S^1] \times S^1) \to [CS^3/S^1]$ with H-flux.*

Proof. The Borel construction $CS^3 \times_{S^1} ES^1$ is a classifying space for the stack $[CS^3/S^1]$. Also, the Borel construction gives $(CS^3/S^1) \times B\mathbb{Z}_k$ as the classifying space for the stack $[CS^3/\mathbb{Z}_k]$. By definition, the principal bundle of classifying spaces associated to the principal bundle of stacks p_k above is the principal circle bundle $(CS^3/S^1) \times B\mathbb{Z}_k \to (CS^3/S^1) \times BS^1$. Using the Gysin Sequence for this principal bundle of spaces, since there is no $H-$flux on $(CS^/S^1) \times B\mathbb{Z}_k$, the T-dual would directly be the principal circle bundle $q : ((CS^3/S^1) \times BS^1) \times S^1 \to ((CS^3/S^1) \times BS^1)$ with H-flux. Now, by the above, $((CS^3/S^1) \times BS^1)$ is the classifying space associated to the stack $[CS^3/S^1]$. From the proof of Thm. (3.4) above, the principal circle bundle q above is the bundle of classifying spaces associated to the principal bundle of stacks $q : ([CS^3/S^1] \times S^1) \to [CS^3/S^1]$. The fact that there is a H-flux present on the bundle of classifying spaces implies that there is a gerbe on this T-dual bundle of stacks. □

It is also interesting to note that if one could find another natural classifying space apart from the simplicial and Haefliger-Milnor classifying spaces above (perhaps for a restricted class of stacks), one could use the above Gysin sequence argument to define a procedure to calculate the T-dual entirely using those classifying spaces. In particular, due to the above we could always argue that the stack theory T-dual calculated using principal circle bundles over the new classifying spaces would always agree with the stack theory T-duals calculated using the simplicial classifying space. This might be more convenient in some specialized applications.

7. The T-dual of a three-manifold with S^1-action

In Sec. (3) above, we had studied the T-dual of the space CS^3 with its natural semi-free circle action. In that section we had noted that the topological space associated to the manifold which was the spacetime background was homeomorphic to $CS^3 \times \mathbb{R}$. Note that CS^3 is also the underlying topological space of a three-manifold with circle action. It is interesting to ask what would happen if CS^3 were replaced by an arbitrary compact three-manifold since circle actions on compact three-manifolds are completely classified (see Ref. [22, 25]).

In this section we calculate T-duals of arbitrary compact three-manifolds with an arbitrary circle action. These may be turned into four-manifolds by adding a $\mathbb{R}$-factor as for CS^3 above. (For example, one could view the result as a warped product of $\mathbb{R}$ with a spatial slice). However, it is necessary to check whether the resulting space can be the underlying topological space of a supersymmetric Type II string theory background. Whether these are valid string theoretic backgrounds or not, it is still possible to calculate the Toplogical T-dual of these spaces. It has been observed that when the space being T-dualized is the underlying topological space of a valid supersymmetric Type II background, the Topological T-dual is homeomorphic to the underlying topological space of the physical T-dual spacetime background.

In addition to the above, one may view three-manifolds with a circle action as the event horizons of five-dimensional axisymmetric black hole spacetimes with a $U(1) \times U(1)$ symmetry. We discuss this matter at the end of this section.

In addition, at the end of this section we argue that there is an isomorphism in the twisted K-theory and twisted cohomology of a given three-manifold with circle action and its T-dual. The argument is slightly difficult since the base is not an orbifold.

Unlike all the previous examples, the spaces studied in this section are not semi-free spaces, however, there is no problem in handling these with either of the above formalisms: Consider an arbitrary 3-manifold W with a smooth circle action without H-flux. By Ref. [14], Sec. (11), the associated stack $\underline{W}$ to W is a principal S^1-bundle of stacks $\underline{W} \to [W/S^1]$ over the quotient stack[8] $[W/S^1]$ (compare with Lemma (3.1) in Sec. (3) above).

We show that it is possible to T-dualize this bundle using the methods outlined above for any H-flux on the 3-manifold W. We begin with an elementary observation:

There is a classification of smooth S^1-actions on three-manifolds (see Ref. [22]) which states that any three-manifold with S^1-action is an equivariant connected sum of 'simpler' three-manifolds with possibly non-free S^1-actions.

In the following we need the notion of a homotopy colimit of two spaces X, Y. (We use Ref. [23] Ex. (2.2) in the following). Suppose X, Y are two spaces and A is a subset of both X, Y. We have a diagram $D : X \overset{f}{\leftarrow} A \overset{g}{\to} Y$. A **homotopy colimit** of D is a pushout of this diagram in the category of spaces:

$$\begin{array}{ccc} A & \xrightarrow[g]{} & Y \\ \downarrow f & & \downarrow \\ X & \longrightarrow & \text{hocolim(D)} \end{array}$$

The space hocolim(D) is actually the adjunction space $X \coprod_f (A \times I) \coprod_g Y$ where $f(a) \sim (a, 0)$ and $(a, 1) \sim g(a)$ for $a \in A$.

If f is the inclusion and g is any map, then hocolim(D) is the space X glued to the mapping cylinder of the map $g : A \subset X \to Y$. Let colim(D) be the colimit of the diagram D. It can be shown, (see Ref. [23], Sec. (2)) that there is a natural map hocolim$(D) \to$ colim (D) obtained by collapsing the mapping cylinder M_f. It can also be shown (see Ref. [23], Prop. (13.10)) that hocolim(D) is weakly homotopy equivalent to colim$(D) \simeq X \cup_g Y$.

We can also define an equivariant version of this construction and then colim(D) is the equivariant gluing of two spaces X, Y along a map $f : A \to A$ where $A \subset X, A \subset Y$ is an invariant set (see Ref. [16]).

[8] See Prop. (11.4) of Ref. [14].

We would like to determine the Topological T-dual of an equivariant connected sum of two spaces. We first study the behaviour of the equivariant join of two spaces under Topological T-duality.

Theorem 7.1.

1) *Let X be a space with a S^1-action. Consider the principal bundle of stacks $p : \underline{X} \to [X/S^1]$. Let $\phi : V \to [X/S^1]$ be a classifying space for $[X/S^1]$ and $Y \to \underline{X}$ the classifying space for X obtained by pulling back ϕ along p. Suppose $V \simeq V_A \cup V_B$, and consider the (2)-commutative diagram*

$$\begin{array}{ccc} Y & \longrightarrow & \underline{X} \\ \downarrow & & \downarrow p \\ V \simeq V_A \cup V_B & \xrightarrow[\phi]{} & [X/S^1]. \end{array}$$

Then, we have that $Y \simeq Y_A \cup Y_B$ with Y_i the pullback of V_i along $p|_{V_i}$ for $i = A, B$ respectively.

2) *Let X and Y be spaces with a S^1-action. Let $A \subset X$ be an invariant set (i.e. $z.A \subseteq A, \forall z \in S^1$). Let A be identified via the identity map with an invariant subset $A \subset Y$. Then, the stack T-dual of the equivariant attaching of $X \cup_A Y$ of X to Y along A with $H-$flux H is the equivariant attaching of the stack T-dual of X to the stack T-dual of Y along the T-dual of A with dual $H-$flux $H^\#$.*

3) *Let A be the total space of a principal circle bundle. Let $f : A \to A$ be a map equivariant under the S^1-action on A. Let the Topological T-dual of A be $A^\#$. Let M_f be the equivariant mapping cylinder of f. Then, the Topological T-dual of M_f with $H-$flux H is $M_f^\#$ with $H-$flux $H^\#$ where $f^\# : A^\# \to A^\#$ is an equivariant map.*

4) *Let X and Y be as in Item (2) above. Let $f : A \to A$ be an equivariant map. Then, the T-dual of $X \cup_f Y$, the equivariant gluing of X to Y along A using f, with $H-$flux H is $X^\# \cup_{f^\#} Y^\#$ with T-dual $H-$flux $H^\#$ for some map $f^\#$ which can be computed.*

Proof.

1) Let $\mathcal{V} \simeq [X/S^1]$. By definition

$$Y \simeq V \times_{\mathcal{V}} X \simeq (V_A \cup V_B) \times_{\mathcal{V}} X.$$

We would like to prove that this is equal to $(V_A \times_{\mathcal{V}} X) \cup (V_B \times_V X)$. This trivially follows from the gluing property of the fiber product of stacks (see Ref. [8] for example).

By definition of the fibered product of stacks (see Ref. [8] for example), for any topological space U we have that:

$$
\begin{aligned}
(V_A \cup V_B) \times_{\mathcal{V}} X(U) \simeq &< (f, f', \psi) \mid f : U \to (V_A \cup V_B), \\
&\qquad f' : U \to X \text{ s. t. } \psi : p \circ f' \Rightarrow \phi \circ f > \\
&\subseteq (V_A \times_{\mathcal{V}} X)(U) \cup (V_B \times_{\mathcal{V}} X)(U)
\end{aligned}
$$

where the second line follows by composing f in the first line with projections to the $V_i, i = A, B$ and using the definition of the fibered products $(V_i \times_V X), i = A, B$.

Also, we have that maps $f : U \to U_A$ and $f' : U \to X$ together with 2-morphisms $\psi : p \circ f' \Rightarrow \phi_A \circ f$ give maps $U \to U_A \cup U_B$ by composing f and f' with the obvious inclusions. These maps satisfy the condition $p \circ f' \Rightarrow \phi \circ f$ since $\phi_i = \phi|_{V_i}, i = A, B$.

Hence, the reverse inclusion is true as well and the result follows.

2) This is actually an argument about the classifying space associated to a stack. As was discussed above, the Toplogical T-dual is computed using the classifying space associated with a given stack: To a principal bundle of stacks we associate a bundle in which each space is the associated classifying space. By the previous part of this theorem the bundle of classifying spaces associated to $X \cup_A Y$ is obtained by equivariantly attaching the classifying space bundles associated to X and Y along A.

Also note that, by definition, $A \subseteq X$ is a union of (possibly nonfree) circle orbits. Thus, its T-dual $A^{\#}$ is well-defined. Thus, we have an inclusion of substacks which are the total spaces of principal bundles of stacks

$$
\begin{array}{ccc}
\underline{A} & \xrightarrow[i]{} & \underline{X} \\
\Big\downarrow{\scriptstyle p_A} & & \Big\downarrow{\scriptstyle p_X} \\
[A/S^1] & \xrightarrow[i]{} & [X/S^1].
\end{array}
$$

Passing to classifying spaces we obtain a inclusion of spaces which are total spaces of principal bundles of classifying spaces of stacks

$$\begin{array}{ccc} Y_A & \xrightarrow[i]{} & Y_X \\ \downarrow{\scriptstyle p_A} & & \downarrow{\scriptstyle p_X} \\ V_A & \xrightarrow[i]{} & V_X. \end{array}$$

Also, we have that $p_A = p_X|_{V_A}$. Since this is a bundle of *spaces*, semi-locality of Topological T-duality for spaces (as shown in Sec. (3), Thm. (3.4) above) proves that the T-dual $A^{\#}$ of A is a substack of the T-dual $X^{\#}$ of X. By interchanging the roles of X and Y we see that the same is true of $A^{\#} \subseteq Y^{\#}$.

This implies that the T-dual of $X \cup_A Y$ is $X^{\#} \cup_{A^{\#}} Y^{\#}$.

3) By definition, $M_f = (A \times I)$ with $A \times \{1\}$ glued to A via the map f. If the circle action on A only has free orbits, it is clear that the circle action on M_f must have only free orbits.

Let $A/S^1 = B$. Then, M_f is the total space of a principal circle bundle over a base V. V is the mapping cylinder of the map $f_B : B \to B$ induced by the equivariant map $f : A \to A$. Hence, the T-dual of M_f, say Y, must be a principal circle bundle $\pi : Y \to V$ since M_f is one.

Let H be a $H-$flux on M_f and let $U \subseteq I$. We are T-dualizing along the circle orbits in $A \hookrightarrow M_f$. By the semi-locality of Topological T-duality, the T-dual of $A \times U \subseteq M_f$ with $H-$flux the restriction of H to $A \times U$ is $A^{\#} \times U$ with some T-dual $H-$flux which is the restriction of $H^{\#}$ to $A^{\#} \times U$. Hence just by semi-locality of Topological T-duality, the T-dual of M_f is $A^{\#} \times [0, 1)$ glued to $A^{\#} \times \{1\}$ by some map $f^{\#}$. Hence there is a natural surjective map $i_1 : A^{\#} \times I \to Y$ given by the quotient map. Also we have the natural inclusion $i_2 : A^{\#} \to Y$ which sends a to the equivalence class of $(a, 1)$ in the above quotient.

It is clear that for every $t \in [0, 1)$, $f^{\#}(x, t) = (x, 1)$ where $(x, t) \in A^{\#} \times [0, 1) \subseteq Y$. Note that $f^{\#}$ is S^1-equivariant by definition.

We now use the universal property of a mapping cylinder to prove that Y is the mapping cylinder of $f^{\#}$.

Given a map $f^{\#} : A^{\#} \to A^{\#}$, the mapping cylinder $M_{f^{\#}}$ is a pushout

$$\begin{array}{ccc} A^{\#} & \xrightarrow[f^{\#}]{} & A^{\#} \\ \downarrow & & \downarrow \\ A^{\#} \times I & \longrightarrow & M_{f^{\#}}. \end{array}$$

The mapping cylinder has the universal property that for any space Z, and a mapping $g_1 : A^\# \times I \to Z$, $g_2 : A^\# \to Z$, such that $g_1(x,1) = g_2(f^\#(x))$ for all $x \in A^\#$, there is a unique $k : M_{f^\#} \to Z$ such that the composition $A \times I \to M_{f^\#} \to Z$ equals g_1 and the composition $A \to M_{f^\#} \to Z$ equals g_2.

Suppose Z is any space and we are given g_1, g_2 as above. We note that by the above, we already have inclusions $i_1 : A^\# \times I \to Y$ and $i_2 : A^\# \to Y$. Also $i_1(a,1) = i_2(f^\#(a))$ by definition of $f^\#$ above.

Given the above, define $k : Y \to Z$ by $k((a,t)) = g_1(a,t)$ for all $t \in [0,1]$. It is clear that k is well defined. By uniqueness of $M_{f^\#}$, this implies that $Y \simeq M_{f^\#}$ is a mapping cylinder.

4) Let X and Y be S^1-spaces and A an invariant subset of X and Y. Let $f : A \to A$ be an equivariant map.

Consider the given pair $(X \cup_f Y, H)$. We try to compute its T-dual by replacing the space $X \cup_f Y$ by a weakly homotopy equivalent space.

Let D be the diagram of S^1-spaces and equivariant maps $X \leftarrow A \xrightarrow{f} Y$. We replace $X \cup_f Y$ by equivariant homotopy colimit of D, $N = \mathrm{hocolim}(D)$. N is weakly equivariantly homotopy equivalent to $X \cup_f Y$. (See discussion before this theorem) by a natural map $h : N \to X \cup_f Y$. Hence, by Lemma (5.2) above, the given pair $(X \cup_f Y, H)$ induces a unique pair (N, H').

Note that the space N contains X, Y and M_f as subspaces. In addition, M_f is the total space of a principal circle bundle since f is equivariant. Since N is made by equivariantly gluing X, Y and M_f, N is a principal bundle over a base space B.

The T-dual of the pair (N, H) is another pair $(N^\#, H^\#)$. By semi-locality of Topological T-duality, the space $N^\#$ is also a principal circle bundle over B, since it contains the T-duals $X^\#, Y^\#$ and $(M_f)^\#$ of X, Y, M_f respectively. By Item (3) above there is a map $f^\# : A^\# \to A^\#$ such that $(M_f)^\# \simeq M_{f^\#}$

Let $D^\#$ be the diagram obtained from D by T-dualizing each space in the diagram. By the semi-locality of Topological T-duality, the inclusion $X \leftarrow A$ is replaced by the inclusion $X^\# \leftarrow A^\#$. In addition, the arrow $f : A \to Y$ is replaced by the arrow $f^\# : A^\# \to Y^\#$.

Thus, the diagram $D^\#$ would be $X^\# \leftarrow A^\# \xrightarrow{f^\#} Y^\#$. By the discussion above and also by Item (3) above, $N^\#$ is homeomorphic to the T-dual homotopy colimit $\mathrm{hocolim}(D^\#)$.

By the discussion before this theorem, this space $N^\#$ is weakly homotopy equivalent to $X^\# \cup_{f^\#} Y^\#$ by a natural map $h^\# : N^\# \to$

$X^\# \cup_{f^\#} Y^\#$. Hence, the pairs on them are in bijection and the pair $(N^\#, H^\#)$ corresponds to a unique pair $(X^\# \cup_{f^\#} Y^\#, H_1^\#)$. This is the T-dual of $(X \cup_f Y, H)$ by the second part of Lemma (5.2) above. □

Corollary 7.2.

1) *Let W_1, W_2 be two 3- manifolds with a smooth S^1-action. Let $W_1 \# W_2$ denote the equivariant connected sum of W_1 and W_2. Then, the Topological T-dual of $W_1 \# W_2$ with arbitrary $H-$flux H is the equivariant connected sum of the T-duals of W_1 and W_2 with a H-flux on the W_i which is the restriction of H to $W_i, i = 1, 2$.*

2) *Let X be a CW-complex with circle action, and let C_0X be the equivariant cone on X with vertex x_0. Then the topological T-dual of C_0X is the stack which is the T-dual of X glued to the stack $\{x_0\} \times S^1$ with a source of H-flux on $\{x_0\} \times S^1$.*

Proof.

1) This follows from the Pt. (2) of Thm. (7.1) above as the equivariant connected sum of two spaces is a special case of the equivariant attaching of two spaces.

2) This follows from Pt. (2) of Thm. (7.1) above as C_0X is equivariantly homeomorphic to $(X \times [0, 1]) \cup_{\{1\}\times X} x_0$.

□

Note that in Thm. (7.1) and Cor. (7.2) above, *we could have replaced some of the spaces with orbispaces* as in the work of Bunke and coworkers (see Ref. [3]) and used the simplicial classifying space for orbispaces. *This would not change any of the above results except that the final answer might not be a space but might be a stack.* Thus the above results trivially extend to T-dualizing the gluing of spaces and manifolds with circle actions to orbispaces with circle actions. In what follows we will glue orbispaces and spaces as freely as needed.

Note that in the second part of the Corollary above, if we take X be a three-manifold, C_0X will be a three-dimensional CW-complex with S^1-action. In particular, taking $X = S^3, S^3/\mathbb{Z}_k$ will recover the results of Sec. (3) above.

By Refs. [22, 25, 26], any compact three-manifold M with S^1-action has the following types of circle orbits only: Free orbits, isolated fixed points, fixed point fibers, exceptional orbits and special exceptional orbits. (The last four orbit types have a stabilizer).

We review each of these below:

- Free orbits are orbits which have the identity as a stabilizer. It can be proved (see Ref. [22]) that the quotient map $q : M \to M/S^1$ restricted to the image of the free orbits is actually a principal circle bundle.

  (It is strange (see Ref. [25]) that this classification of orbit types is also true for three-manifolds which only locally have a S^1-action, i.e. for $O(2)$-spaces.)

- Fixed points have a neighbourhood homeomorphic to the 3-disk with an orthogonal S^1-action. Fixed point fibers have neighbourhoods isomorphic to a solid torus $\mathbb{D}^2 \times S^1$ with the S^1 action:

$$z \cdot (\rho e^{i\theta}, e^{i\psi}) = (z\rho e^{i\theta}, e^{i\psi}), z \in S^1, \theta \in [0, 2\pi]. \tag{7.1}$$

  Here the set $\{(0, e^{i\psi})\}$ is a circle of fixed points.

- Exceptional orbits have a neghbourhood equivariantly homeomorphic to $\mathbb{D}^2 \times S^1$ with the action:

$$z \cdot (\rho e^{i\theta}, e^{i\psi}) = (z^\nu \rho e^{i\theta}, z^\mu e^{i\psi}), z \in S^1. \tag{7.2}$$

  where μ and ν are relatively prime and $0 < \nu < \mu$. The exceptional orbit itself is the set $\{0\} \times S^1$.

- Special exceptional orbits have a neighbourhood equivariantly homeomorphic to $\mathbb{D}^2 \times_{\mathbb{Z}_2} S^1$ (see Ref. [16] for a definition) where $\mathbb{Z}_2$ acts on $\mathbb{D}^2$ by reflection.

The three-manifold M above can be obtained by equivariantly attaching (see Ref. [22]) the above spaces to a principal circle bundle over a two-manifold.

More precisely, given a compact three-manifold M with S^1-action, if there are h fixed point orbits and t special exceptional orbits, we can construct M by taking an equivariant connected sum of a principal circle bundle with the above S^1-spaces (see Ref. [22]) as folows:

Consider the fibration $q : M \to M/S^1$. Pick open nbds of the image of the fixed point sets in M/S^1. Pick an open set W in M/S^1 which does not intersect the images of the fixed sets in M/S^1. Restricted to W, M is a principal circle bundle.

If there are no exceptional orbits, the compact three-manifold M is isomorphic to $(\Sigma_{g,h,t} \times S^1)/\simeq$, where $\Sigma_{g,h,t}$ is a compact oriented surface of index g, with $h + t$ boundary components, $h \geq 1$.+

The equivalence relation $\simeq$ is as follows: For each of the h boundary circles, the (trivial) circle bundle over it is collapsed onto the base circle via the projection map. This corresponds to equivariantly attaching[9] the principal bundle $M|_W$ above to a torus neigbhourhood of the fixed point orbits with the circle action in Eq. (7.1). The h boundary circles are the fixed point orbits.

Over each of the remaining t circles, the fibers of the corresponding (trivial) circle bundle are quotiented out by the antipodal action. This corresponds to equivariantly attaching to the space obtained in the previous paragraph the above neighbourhood $\mathbb{D}^2 \times_{\mathbb{Z}_2} S^1$ of a special exceptional orbit.

By Ref. [22], $M_{g,h,t}$ is isomorphic to an equivariant connected sum of a principal S^1-bundle over a 2-manifold with boundary together with t copies of $\mathbb{RP}^2 \times S^1$. The 2-manifold with boundary may be written as an equivariant connected sum of 3-sphere with handles

$$M_{g,h,t} \simeq S^3 \#(S^2 \times S^1)_1 \# \cdots (S^2 \times S^1)_{2g+h-1} \\ \#(\mathbb{RP}^2 \times S^1)_1 \# \cdots \#(\mathbb{RP}^2 \times S^1)_t.$$

If, in addition to the above, we have exceptional orbits we label these by $I \subset \mathbb{Z}^+$. (If the three-manifold is compact, $|I|$ is finite.) Let (μ_i, ν_i) be the invariants of the exceptional orbit associated to $i \in I$.

Recall that a Lens Space $L(p, q)$ is a quotient of a three-sphere $S^3 = \{(z, w)||z|^2 + |w|^2 = 1\}$ by an action of $\mathbb{Z}_p$, $\mu \cdot (z, w) = (\mu z, \mu^q w)$. By writing the three-sphere as a glue of two solid tori, we may (see Ref. [22], Sec. (7) for details) view $L(p, q)$ as obtained by gluing two solid tori together. The natural circle action on S^3 given by the Hopf fibration descends to an action of the circle group on $L(p, q)$. Due to the above decomposition of $L(p, q)$ as a gluing of two solid tori, we obtain an equivariant decomposition of $L(p, q)$ as a glue of two solid tori one with a circle group action as in Eq. (7.1) above and one with a circle group action as in Eq. (7.2) above. Thus, the circle group action on $L(p, q)$ has a circle of fixed points and one exceptional orbit with invariant (p, q).

We pick a toroidal neighbourhood of each exceptional orbit such that the S^1-action on the three-manifold restricted to this neighbourhood is equivariantly homeomorphic to the given toroidal neighbourhood of the exceptional

[9]See Ref. [16]

orbit in a Lens Space $L(\mu, \nu)$ where (μ, ν) are the invariants of the exceptional orbit described in Ref. [22], Sec. (5), Lemma (5). Let the i'th Lens Space in the above have invariants μ_i, ν_i. Then Ref. [22], Thm. (4) shows that M is the equivariant connected sum

$$M \simeq M_{g,h,t} \# L(\mu_1, \nu_1) \# L(\mu_2, \nu_2) \# \cdots \# L(\mu_k, \nu_k). \tag{7.3}$$

We need to determine the T-dual of $L(p, q)$ above. First we determine the T-dual of a solid torus with the action of Eq. (7.2) on it.

Lemma 7.3. *Consider the solid torus $\mathbb{D}^2 \times S^1$ with the circle action*

$$z \cdot (\rho e^{i\theta}, e^{i\psi}) = (z^{\nu} \rho e^{i\theta}, z^{\mu} e^{i\psi}),$$

where $z \in S^1$, ν, μ are relatively prime and $0 < \nu < \mu$. (This is the circle action in Eq. (7.2) above.) The T-dual of the above space with any $H-$flux H may be calculated using the methods of Ref. [3].

Proof. We divide the proof into two cases: Consider the solid torus $\mathbb{D}^2 \times S^1$ with the circle action in Eq. (7.2) above

$$z \cdot (\rho e^{i\theta}, e^{i\psi}) = (z^{\nu} \rho e^{i\theta}, z^{\mu} e^{i\psi}),$$

where $z \in S^1$, ν, μ are relatively prime and $0 < \nu < \mu$. This is the Lens Space $L(p, q)$ above. We argue that $L(p, q)$ is the total space of a principal circle bundle E over a orbifold B. Thus it may be T-dualized by the methods of Ref. [3], Sec. (5.2).

We consider the disk with the $\mathbb{Z}_\mu$ action which rotates each point on the disk by $2\pi/\mu$. We consider the orbifold quotient B of the disk by the given $\mathbb{Z}_{|\mu|}$action. Note that B is homeomorphic to a slice around the exceptional orbit $\{0\} \times S^1 \subseteq L(p, q)$. Hence the Lens Space $L(p, q)$ above is the total space of a circle bundle over B.

Following Ref. [3], Sec. (5.2), we view the base orbifold B above as a groupoid quotient $B = [\mathcal{G}^1/G^0]$, where the object space is $G^0 := \tilde{\bar{U}}$: Here, $\tilde{\bar{U}}$ is an orbifold chart of the disk and we require that $\tilde{\bar{U}} \to \bar{U}$ is the $n-$fold covering map $z \to z^{\mu}$ of the disc U in complex plane. We require that the following diagram commutes

$$\begin{array}{ccc} \tilde{\bar{U}} & \xrightarrow[f]{} & \bar{U} \\ \downarrow \simeq & & \downarrow \simeq \\ \tilde{\mathbb{D}}^2 \subseteq \mathbb{C} & \xrightarrow[z \mapsto z^{|\mu|}]{} & \mathbb{D}^2 \subseteq \mathbb{C}. \end{array}$$

Note that there is a natural action of $\mathbb{Z}_{|\mu|}$ on $\tilde{\mathbb{D}^2}$ which permutes the fiber of the map $z \mapsto z^{|\mu|}$. This lifts to a natural action on $\tilde{\bar{U}}$ in the above diagram. We view $\tilde{\bar{U}}$ as an *orbifold chart* of the base orbifold B. We fix the arrow space $\mathcal{G}^1$ by requiring that groupoid $\mathcal{G}$ is the transformation groupoid $\mathbb{Z}_{|\mu} \times \tilde{\bar{U}} \Rightarrow \tilde{\bar{U}}$ of the of the natural $\mathbb{Z}_{|\mu|}$ action on $\tilde{\bar{U}}$.

We view E a groupid quotient $[\mathcal{E}/\mathcal{G}^1]$ where $\mathcal{E} \to \mathcal{G}$ is an equivariant S^1-bundle of groupoids. $\mathcal{E}$ is specified by an equivariant S^1-bundle $\mathcal{E} \to \mathcal{G}^0$, together with an action $\mathcal{G}^1 \times_{G^0} \mathcal{E} \to \mathcal{E}$. We set $\mathcal{E} := S^1 \times \mathcal{G}^0$ and we let $Z_{|\mu|}$ act on the fiber over any point in $\tilde{\bar{U}}$ with character $\chi([q]) = \exp(2\pi i \frac{pq}{|\mu|})$, $[q] \in Z_{|\mu|}$.

By Ref. [22], (Sec. (5), after Eq. (5.1)), this is exactly the circle action on the torus neighbourhood of the exceptional orbit in Eq. (7.2) above.

We have a pullback square

$$\begin{array}{ccc} E & \longrightarrow & [U(1)/\mathbb{Z}_{|\mu|}] \\ \downarrow p & & \downarrow \\ B & \longrightarrow & [*/\mathbb{Z}_{|\mu|}] \end{array}$$

where the horizontal maps are equivariant homotopy equivalences. The vertical maps are the bundle projections. (The principal bundle in the second column is the one in Ref. [3], Sec. (5.1).) The stack cohomology of B now follows from the fact that $H^*([*/\mathbb{Z}_{|\mu|}], \mathbb{Z}) \simeq H^*(B\mathbb{Z}_{|\mu|}, \mathbb{Z})$. By the proof of Ref. [3], Sec. (5.1): $H^0(B,\mathbb{Z}) = \mathbb{Z}$, $H^{2l-1}(B,\mathbb{Z}) = 0$ and $H^{2l}(B,\mathbb{Z}) = \mathbb{Z}/|\mu|\mathbb{Z}$ for $l = 1, 2, \ldots$. Further, $c_1(E) = [q]$.

Also, the Gysin sequence for $E \to B$ is the same as the Gysin sequence for the principal bundle of stacks $[U(1)/\mathbb{Z}_{|\mu|}] \to [*/\mathbb{Z}_{|\mu|}]$ calculated in Ref. [3], Sec. (5.1). This Gysin sequence is

$$0 \to H^3(E,\mathbb{Z}) \xrightarrow{\pi_!} \mathbb{Z}/|\mu|\mathbb{Z} \xrightarrow{[q]} \mathbb{Z}/|\mu|\mathbb{Z} \to \cdots$$

and by exactness, $\pi_!$ is injective. Hence $H^3(E, \mathbb{Z})$ is isomorphic to the kernel of the cup product of a class in $\mathbb{Z}/|\mu|\mathbb{Z}$ with $[q]$. Thus,

$$H^3(E, \mathbb{Z}) \simeq \{[s] \in \mathbb{Z}/|\mu|\mathbb{Z} \text{ such that } \mu \text{ divides } sq\} \subset \mathbb{Z}/|\mu|\mathbb{Z}.$$

We pick a $H-$flux $[s]$ in this group.

We note that the collapse map $B \to [*/\mathbb{Z}_{|\mu|}]$ is a homotopy equivalence. By Lemma (5.2) above, this implies that there is map from the T-dual of E to the T-dual of $[U(1)/\mathbb{Z}_{|\mu|}]$.

Since E is a pullback, if we pullback to a small contractible open subset of the base which does not contain the singular point at $(0, 0)$, it is clear that the bundle over this set must be trivial. Hence, the T-dual of $E \to B$ restricted to a small open set in B which does not contain the singular point at $(0, 0)$, is trivial. By the semi-locality of Topological T-duality, this implies that the T-dual is a trivial circle bundle over B *away from* $(0, 0)$. The topology of the T-dual is only from the topology of the fiber over the singular point at $(0, 0)$.

Suppose $(|\mu|, q) = 1$. By Ref. [3], this implies that the $H-$flux on E is zero. By Ref. [3], the T-dual is $[U(1)/\mathbb{Z}_{|\mu|}]$ with a *noneffective* $\mathbb{Z}_{|\mu|}-$action.

In this situation, the T-dual of E is $\mathbb{D}^2 \times S^1$ with the translation S^1-action away from the singular fiber $S^1 \times \{0\}$. The fiber at $(0, 0)$ is $[U(1)/\mathbb{Z}_{|\mu|}]$ with a noneffective $\mathbb{Z}_{|\mu|}$ action. As in the last paragraph of Ref. [3], Sec. (5.1)) this is not equivalent to a space.

We may argue similarly for $h \neq 0$. The topology of the T-dual in this case is obtained from the Gysin sequence for the T-dual bundle $\hat{p} : \hat{E} \to B$ in Ref. [3], Sec. (5.1.6). By Lemma (5.2) above, the T-dual of the bundle p is pulled back from the T-dual $\hat{q}$ of the bundle $q : [U(1)/\mathbb{Z}_{|\mu|}] \to [*/\mathbb{Z}_{|\mu|}]$ and every pair on $\hat{p}$ is pulled back from a pair on $\hat{q}$.

By Ref. [2], Lemma (2.1.2), we have $c_1(\hat{E}) = -\pi_!(h)$. Thus, $\hat{p} : \hat{E} \to B$ with the above $H-$flux has characteristic class $c_1(\hat{E}) = [-s] \in \mathbb{Z}/|\mu|\mathbb{Z}$ since the $H-$flux on $p : E \to B$ (see above) was $[s]$.

The Gysin sequence for the T-dual is

$$0 \to H^3(\hat{E}, \mathbb{Z}) \stackrel{\hat{\pi}_!}{\to} \mathbb{Z}/|\mu|\mathbb{Z} \stackrel{[-s]}{\to} \mathbb{Z}/|\mu|\mathbb{Z} \to \cdots,$$

By exactness, $\hat{\pi}_!$ is injective. Hence $H^3(\hat{E}, \mathbb{Z})$ is isomorphic to the kernel of the cup product of a class in $\mathbb{Z}/|\mu|\mathbb{Z}$ with $[-s]$. Thus, it is isomorphic to $\{[r] \in \mathbb{Z}/|\mu|\mathbb{Z}|\mu|sr\} \subset \mathbb{Z}/|\mu|\mathbb{Z}$. From the Gysin sequence for the T-dual bundle, $c_1(E) = [q] = -\hat{\pi}_!(\hat{h})$. However, by the above $\hat{\pi}_!$ is injective. Hence, the T-dual $H-$flux is $\hat{h} = [-q]$.

The T-dual is a S^1-stack with the above topology, characteristic class and $H-$flux. □

Then we determine the T-dual of $L(p,q)$ above.

Lemma 7.4. *The T-dual of $L(p,q)$ with any H-flux can be calculated using the above.*

Proof. We divide the proof into two cases: First, from Ref. [22], Secs. (7,8), $L(p,q)$ is obtained by equivariantly gluing a solid torus with the circle action equivariantly homeomorphic to Eq. (7.2) above to a solid torus with circle action equivariantly homeomorphic to Eq. (7.1) above. The boundary two-tori of each of the solid tori above are glued using a map from $\mathbb{T}^2 \to \mathbb{T}^2$ of degree (p,q) above.

We use the convention of Ref. [22] to describe these spaces. The spaces above are viewed as S^1-spaces with quotient space susbsets the unit disk in the plane $\mathbb{D}^2 \subseteq \mathbb{R}^2$. The first space is a semi-free space over an annulus $[1,1/2] \times S^1$ with $\{1\} \times S^1$ a circle of fixed points of the circle action and every other circle orbit free. The circle group action on this space is the action in Eq. (7.1) above.

This space in the previous paragraph is glued to a principal circle bundle over a disk orbifold, i.e. a space of the form $\mathbb{D}^2/\mathbb{Z}_{|\mu|}$. Here, the disk of radius $\{1/2\}$ in the plane is taken to be $\mathbb{D}^2/\mathbb{Z}_{|\mu|}$ by an orbifold chart and there is an orbifold point at $(0,0)$. The circle action on the total space of the bundle is the action in Eq. (7.2) above.

The two spaces above are glued to each other along the common torus boundary which is the set $\mathbb{T}^2 \to \{1/2\} \times S^1$ by a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ of degree 2.

1) **Zero $H-$flux:** By Thm. (7.1) the T-dual of the above is the T-dual of each of the two solid tori above glued together by a T-dual map.

   The circle action in Eq. (7.1) above has an annulus as a quotient space. The T-dual of the action in Eq. (7.1) is a trivial circle bundle over the annulus $[1,1/2] \times S^1$ with a source of $H-$flux at $\{1\} \times S^1$ This is because the circle action is semi-free.

   The T-dual of the action in Eq. (7.2) was calculated in the Lemma (7.3) above. By that Lemma, the T-dual with zero $H-$flux is $\mathbb{D}^2_{1/2} \times S^1$ with the fiber $S^1 \times \{0\}$ equivariantly $2-$equivalent to $[U(1)/\mathbb{Z}_{|\mu|}]$ with a *noneffective* $\mathbb{Z}_{|\mu|}-$action. The above two spaces are glued together along the common boundary $\{1/2\} \times S^1 \times S^1$.

This implies that the T-dual is the gluing of the above two spaces together on the $\mathbb{T}^2$ which is their common boundary by a map $f : \mathbb{T}^2 \to \mathbb{T}^2$.

2) **Nonzero $H-$flux:** By Lemma (7.3) above, the T-dual of the above space with $H-$flux is a S^1-bundle of stacks $p : \hat{E} \to B$ with $H-$flux.
This bundle has characteristic class $c_1(\hat{E}) = [-s] \in \mathbb{Z}/|\mu|\mathbb{Z}$ since the $H-$flux on $p : E \to B$ (see above) was $[s]$.
Hence $H^3(\hat{E}, \mathbb{Z})$ is isomorphic to $\{[r] \in \mathbb{Z}/|\mu|\mathbb{Z}|\mu|sr\} \subset \mathbb{Z}/|\mu|\mathbb{Z}$. From the Gysin sequence for the T-dual bundle, $c_1(E) = [q] = -\hat{\pi}_!(\hat{h})$. However, by the above $\hat{\pi}_!$ is injective. Hence, the T-dual $H-$flux is $\hat{h} = [-q]$. The T-dual is the S^1-bundle of stacks with the above topology, characteristic class and $H-$flux.

□

The above facts together with Cor. (7.2) above, let us determine the T-dual of any S^1-action on a three-manifold without H-flux.

Theorem 7.5. *Let W be a compact 3-manifold with smooth circle action.*

1) *The Topological T-dual of the stack $\underline{W}$ without H-flux is an equivariant connected sum of some number copies of T-duals of the following spaces with a particular gluing. (The S^1-actions on these spaces are given in the proof below and in Eqs. (7.1, 7.2) above.)*
   a) S^3,
   b) $S^2 \times S^1$,
   c) $L(\mu, \nu)$,
   d) $\mathbb{RP}^2 \times S^1$,
   e) $\mathbb{D}^2 \times S^1$ *with the action in Eq. (7.1) above.*
   The T-duals of these spaces are listed below.
2) *The Topological T-dual of the stack $\underline{W}$ with H-flux $\mathcal{H}$ is an equivariant connected sum of the T-duals of the spaces in the previous item* with *$H-$flux. These T-duals are listed in the proof below.*

Proof.

1) By the above, W is equivariantly homeomorphic to an equivariant connected sum $W \simeq M_{g,h,t}\#(\mu_1, \nu_1)\# \cdots \#(\mu_k, \nu_k)$ where $M_{g,h,t}$ was defined above.

By Cor. (7.2) above, the T-dual of W is the connected sum of the T-duals of $M_{g,h,t}$ with the T-duals of the $L(\mu_i, \nu_i)$. However, by the above $M_{g,h,t} \simeq S^3 \#(S^2 \times S^1)\# \cdots \#(S^2 \times S^1)\#(\mathbb{RP}^2 \times S^1)\# \cdots \#(\mathbb{RP}^2 \times S^1)$.

The T-duals of each of the stacks in the above connected sum without H-flux are well known, since they are actually the total spaces of principal bundles, usually trivial ones. Since there is no $H-$flux, the T-dual will be a trivial bundle. In partcular, the T-dual of S^3 will be $S^2 \times S^1$ with $H-$flux the generator of $H^3(S^2 \times S^1, \mathbb{Z}) = \mathbb{Z}$. None of the other spaces in the connected sum for $M_{g,h,t}$ will T-dualize. Hence the T-dual of $M_{g,h,t}$ is a connected sum of copies of $S^2 \times S^1$ and $\mathbb{RP}^2 \times S^1$ only with $H-$flux supported on some of the $S^2 \times S^1$ factors. The T-dual of W is connected sum of the above with the T-dual of the lens spaces. Thus, the only problem in computing the T-dual of a three-manifold with circle action is the computation of the T-dual of the lens spaces $L(\mu, \nu)$ with and without $H-$flux.

The T-dual of the Lens Space $L(\mu, \nu)$ without $H-$flux was calculated in Lemma (7.4) Item (1) above. The T-dual consists of the T-dual of the solid torus with the circle action in Eq. (7.1) above glued to the T-dual of the solid torus with the circle action in Eq. (7.2) above. Since there is no $H-$flux, the T-dual is the space described in Item (1) of that lemma and consists of a principal circle bundle of stacks whose base is a disk orbifold whose orbifold chart is the closed unit disk $\mathbb{D}^2 \subseteq \mathbb{R}^2$. The total space is obtained by gluing a trivial bundle over the annulus $[1, 1/2] \times S^1$ with $H-$flux supported at $\{1\} \times S^1$ (and no other $H-$flux) to a trivial circle bundle over the disk of radius $1/2$ around $(0, 0) \in \mathbb{R}^2$ with the fiber over $(0, 0)$ possessing a *noneffective* $\mathbb{Z}_{|\mu|}-$action. The point $(0, 0)$ is an orbifold point.

2) By the above, W is equivariantly homeomorphic to an equivariant connected sum

$$W \simeq M_{g,h,t}\#(\mu_1, \nu_1)\# \cdots \#(\mu_k, \nu_k)$$

where

$$M_{g,h,t} \simeq S^3\#(S^2 \times S^1)\# \cdots \#(S^2 \times S^1)\#(\mathbb{RP}^2 \times S^1)\# \cdots \#(\mathbb{RP}^2 \times S^1)$$

as defined above.

Let H_i be the restriction of H to the i'th factor of the above equivariant connected sum. By the above, the T-dual of W with $H-$ flux is the equivariant connected sum of the T-duals of each of the above spaces with the restriction of the $H-$flux. The T-dual of each of the above factors is well known (see Ref. [19]) except the T-dual of $L(\mu,\nu)$ which was worked out in Lemma (7.4) above. Thus, the T-dual of W is a connected sum of the following spaces:

- **The T-dual of** $p : S^3 \to S^2$ **with** $H-$**flux** H_1 : By Ref. [19], the T-dual is the principal circle bundle $q : S^3/\mathbb{Z}_k \to S^2$ with 1 unit of $H-$flux where k is the value of $p_!(H_1) \in \mathbb{Z} \simeq H^2(S^2,\mathbb{Z})$.
- **The T-dual of** $p : S^2 \times S^1$ **with** $H-$**flux** H_2 : By Ref. [19], the T-dual is the circle bundle $q : S^3/\mathbb{Z}_k \to S^2$ with no $H-$flux where k is the value of $p_!(H_1) \in \mathbb{Z} \simeq H^2(S^2,\mathbb{Z})$.
- **The T-dual of** $\mathbb{RP}^2 \times S^1 \to \mathbb{RP}^2$ **with** $H-$**flux** H_3 : The cohomology of $\mathbb{RP}^2 \times S^1$ is $H^0(\mathbb{RP}^2 \times S^1,\mathbb{Z}) = \mathbb{Z}$, and $H^3(\mathbb{RP}^2 \times S^1,\mathbb{Z}) = \mathbb{Z}_2$. Hence, H_3 can only be zero or nonzero. If the $H-$flux is zero, there is no T-duality. If the $H-$flux is nonzero ($1 \in \mathbb{Z}_2$), the result is the only nontrivial bundle E over $\mathbb{RP}^2$, (recall $H^2(\mathbb{RP}^2,\mathbb{Z}) \simeq \mathbb{Z}_2$) $q : E \to \mathbb{RP}^2$. This bundle was calculated in Ref. [19], Sec. (4.4). The cohomology of the total space of the bundle is

$$H^0(E,\mathbb{Z}) = \mathbb{Z}, H^1(E,\mathbb{Z}) = \mathbb{Z}, H^2(E,\mathbb{Z}) = 0, H^3(E,\mathbb{Z}) = \mathbb{Z}_2.$$

  and the resulting bundle has trivial $H-$flux.
- **The T-dual of** $L(\mu,\nu)$ **with** $H-$**flux** H_4 : This was calculated in Lemma (7.4) above. By Lemma (7.3) above, the T-dual of the above space with $H-$flux is a S^1-bundle of stacks $p : \hat{E} \to B$ with $H-$flux.
  This bundle has characteristic class $c_1(\hat{E}) = [-s] \in \mathbb{Z}/|\mu|\mathbb{Z}$ since the $H-$flux on $p : E \to B$ (see above) was $[s]$.
  Hence $H^3(\hat{E},\mathbb{Z})$ is isomorphic to $\{[r] \in \mathbb{Z}/|\mu|\mathbb{Z}|\mu|sr\} \subset \mathbb{Z}/|\mu|\mathbb{Z}$. From the Gysin sequence for the T-dual bundle, $c_1(E) = [q] = -\hat{\pi}_!(\hat{h})$. However, by Lemma (7.4) above, $\hat{\pi}_!$ is injective. Hence, the T-dual $H-$flux is $\hat{h} = [-q]$. The T-dual is the S^1-bundle of stacks with the above topology, characteristic class and $H-$flux.

The above facts together with Cor. (7.2) above, let us determine the T-dual of any S^1-action on a three-manifold without H-flux. □

It would be interesting to relate the twisted K-theories of the two sides of the duality. However, since the base is not a Deligne-Mumford stack, the results of Bunke et al concerning the isomorphisms of twisted $K-$theory in Ref. [4] will not apply. We would need the results of Mathai and Wu (see Ref. [10], Appendix and Thm. (1) for details). We can show using these that the twisted $K-$theory of W is isomorphic to the twisted equivariant $K-$theory of the corrrespondence space. When the T-dual exists in the sense of Mathai-Wu, this is the twisted equivariant $K-$theory of the T-dual. In addition, the twisted cohomology of W is isomorphic to the equivariant twisted cohomology of the correspondence space. Similarly to the above, when the T-dual exists, this is the twisted cohomology of the T-dual.

Thus, if there are no exceptional orbits, the twisted $K-$theory and twisted cohomology of W are isomorphic (with a degree shift) to the twisted $K-$theory and twisted cohomology of the T-dual.

If there are exceptional orbits, the correspondence space remains a space, but the T-dual spacel is in general a stack. If one could define twisted cohomology and twisted $K-$theory for stacks along the lines of Ref. [14], Def. (5.1), we could extend the above isomorphism to include this case.

In this section we assumed that W was compact. The case of noncompact W can be handled, we might have to take the equivariant connected sum of an infinite number of lens spaces as in Ref. [22].

In this section we have calculated the Topological T-dual of an arbitrary compact three-manifold with circle action. It would be interesting to relate this to the known T-duals in String Theory. Since the T-duals of KK-monopole spacetimes agree with the corresponding T-duals in String Theory (see Ref. [9] for details), one would expect these T-duals to match as well.

There appear to be two ways to connect three-manifolds with String Theory backgrounds: As was argued at the beginning of this section, one could view the three-manifold as a spacelike slice of a four manifold. One could then T-dualize the four-manifold by viewing it as a String Theory background in the usual ways, for example by viewing it as a compactification. It would be interesting to see if the Topological T-dual matches with the String Theoretic T-dual in this case.

There might be another way to connect the Topological T-dual of three-manifolds with string theory: We could view the three-manifold as the event horizon of a higher-dimensional black hole (see Ref. [28] for details).

It is well known (Hawking's Theorem) that a stationary axisymmetric four dimensional spacetime always possesses a $U(1)$ isometry. The fixed point sets of the isometries are stratified by dimension. The event horizon of the black hole is the highest dimension fixed point set. For a black hole in a four-dimensional spacetime this is two-dimensional.

A recent topic in String Theory is the study of higher-dimensional black holes. Under certain conditions (see Ref. [28]), it can be shown that there exist five-dimensional black hole spacetimes with isometry groups containing the group $U(1) \times U(1)$. For such spacetimes it can proved that the event horizon is a three-manifold with a circle action. The classification of three-manifolds with circle actions has been used to study these spacetimes in Ref. [29] (see, for example, the results in Sec. (2) of that paper). Such spacetimes are valid backgrounds for String Theory possibly after adding D-branes. It would be interesting to see the effect of T-duality on these spacetimes. Presumably, the Topological T-dual of a three-manifold with circle action above would be related to these T-duals.

An example of a higher dimensional black hole with the above properties is the spinning black ring on Taub-NUT space. This space has a $U(1)-$isometry along the isometry direction x^5 (in the notation of Ref. [30]). The geometry of this background in terms of D-branes has been discussed in Ref. [30].

8. Final remarks

In this paper we have studied the Topological T-dual of spaces containing KK-monopoles. In Secs. (2,3) of the paper above, we have explicitly calculated the T-dual of several spaces with KK-monopoles. In Sec. (4) we have attempted to model the 'dyonic coordinate' associated with KK-monopoles within the stack theory formalism. In Ref. [9] we had obtained a model for the dyonic coordinate using the C^*-algebraic formalism of Topological T-duality. It is interesting that the same phenomenon appears in two completely independent approaches to Topological T-duality.

The formalism of Bunke et al. [2–4] may be applied to a KK-monopole by passing to the associated simplicial bundle and taking the Topological T-dual of that simplicial bundle. When the above stack is the stack associated to the total space of a principal circle bundle, the formalism of Bunke and co workers in Ref. [3] would give the same answer as the C^*-algebraic formalism. However, if the stack was the stack associated to a space with a non-free circle action we show by an explicit example that the two formalisms dol not give the same T-dual. This is because the formalism of Bunke and coworkers 'regularizes' the neighbourhood of the fixed point when it passes from the given stack to its associated simplicial bundle.

Also, as has been argued above, the C^*-algebraic T-dual lifts the S^1-action on the space to a $\mathbb{R}$-action on the C^*-algebra. The formalism of Bunke et al. does not lift the S^1-action to a $\mathbb{R}$-action. This causes a difference in T-duals when fixed points are encountered. The formalism of Mathai-Wu [10] also does not lift the S^1-action to a $\mathbb{R}$-action.

In addition, a connection between the topology of the T-dual and the coarse moduli space of the T-dual stack was demonstrated in Sec. (3) after

Cor. (3.6) . There we observed that in the example of the $KK-$monopole, the physical T-dual spacetime was the coarse moduli space of the T-dual stack. It would be interesting to investigate this further. In particular, if the coarse moduli space of the T-dual is different from the T-dual stack, the T-dual stack possesses nontrivial inertia groups. This appearance of the coarse moduli space is extremely interesting, and seems to be connected to the nontrivial stabilizers of the non-free orbits of the original stack.

In Sec. (5) we argued that the methods of Bunke et al and Mathai and Wu were connected by the notion of a classifying space of a stack: They differed only in the choice of a classifying space. It would be interesting to see if a new prescriptions for Topological T-duality similar to the formalisms of Bunke et al and Mathai and Wu could be made by other natural choices for a classifying space for other families of stacks apart from the ones in this section. Since the notion of a classifying space for a topological stack has been studied in detail (see Refs. [21]) there is a chance that this might be possible.

Under certain conditions on the stack quotient map $p : \mathcal{E} \to \mathcal{Y}$ we dervie a stack cohomology Gysin sequence. We then prove that this Gysin sequence may also be used to determine the T-dual of a principal circle bundle of stacks.

It is interesting to note that the calculation of Topological T-duals for $U(1)$-gerbes on a principal $\mathbb{T}^n$-bundle over an arbitrary topological groupoid using the theory of crossed products of groupoid C^*-algebras has also been done by Daenzer in Ref. [27]. The formalism can T-dualize non-free group actions. It would be interesting to compare the results of that formalism with the results of this paper.

In Sec. (7) we discuss the T-dual of an arbitrary compact three-manifold without boundary with a circle action. If there are no exceptional orbits, with zero $H-$flux, the T-dual is a connected sum of $S^2 \times S^1$ and $\mathbb{RP}^2 \times S^1$ with $H-$flux supported on some of the $S^2 \times S^1$ factors. If there are no exceptional orbits, with nonzero $H-$flux, the T-dual is copies of $S^3/\mathbb{Z}_k$ with $H-$flux glued to copies $S^3/\mathbb{Z}_m$ without $H-$flux and to $E \to \mathbb{RP}^2$ (the only nontrivial bundle over $\mathbb{RP}^2$). If there are exceptional orbits present, however, we have to glue in copies of the stacks described in Thm. (7.5) above, one for each exceptional orbit.

Some connections between string theory backgrounds and the above examples were noted in this section as well. Among other things, it was noted that the event horizon of a five dimensional black hole background with a $U(1) \times U(1)-$isometry is a three-manifold with $U(1)-$isometry. It would be interesting if Topological T-duality was able to describe the T-dual of these String/M-Theory backgrounds.

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