Gauge groups and matter spectra in F-theory compactifications on genus-one fibered Calabi-Yau 4-folds without section — hypersurface and double cover constructions

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We investigate gauge theories and matter contents in F-theory compactifications on families of genus-one fibered Calabi–Yau 4-folds lacking a global section. To construct families of genus-one fibered Calabi–Yau 4-folds that lack a global section, we consider two constructions: hypersurfaces in a product of projective spaces, and double covers of a product of projective spaces. We consider specific forms of defining equations for these genus-one fibrations, so that genus-one fibers possess complex multiplications of specific orders. These symmetries enable a detailed analysis of gauge theories. E_6, E_7, and $SU(5)$ gauge groups arise in some models. Discriminant components intersect with one another in the constructed models, and therefore, discriminant components contain matter curves. We deduce potential matter spectra and Yukawa couplings.

1. Introduction

F-theory [1–3] is a framework that extends the type IIB superstring theory to a nonperturbative regime, and the compactification geometries for F-theory are Calabi–Yau manifolds with a torus fibration. In the F-theory approach, the modular parameter of a genus-one curve, as a fiber of a torus fibration, is identified with the axio-dilaton; this formulation enables the axio-dilaton to have $SL_2(\mathbb{Z})$ monodromy. Local F-theory models have been mainly discussed in recent studies on F-theory model building [4–7]. However, to deal with the issues of gravity and the early universe including inflation, global geometries of F-theory compactifications need to be considered. We investigate the geometries of F-theory compactifications from the global perspective in this study.

A Calabi–Yau manifold with a torus fibration may or may not admit a global section. F-theory models on Calabi–Yau manifolds with a global section have been studied previously, for example, in [8–22]. In recent years, there has been an increasing interest in F-theory models on Calabi–Yau genus-one fibrations without a global section[1]. Initiated in [25, 26], F-theory compactifications lacking a global section have been discussed in recent studies. See also, for example, [27–37] for recent advances in F-theory models that lack a global section. It was argued in [26] that, by considering the Jacobian fibrations, the F-theory models on Calabi–Yau genus-one fibrations without a global section can be related to the geometry of Calabi–Yau elliptic fibrations with a section.

In this note, we construct genus-one fibered Calabi–Yau 4-folds without a global section, and we use these spaces as compactification geometries for F-theory to investigate F-theory models without a section. We consider two constructions: hypersurfaces in a product of projective spaces, and double covers of a product of projective spaces, to construct genus-one fibered Calabi–Yau 4-folds without a rational section. In these constructions, we consider Calabi–Yau 4-folds whose discriminant components intersect with

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[1][23, 24] discussed F-theory compactifications without a global section.

one another. Therefore, a component contains matter curves. Matter[2] with non-trivial chirality arises in F-theory models considered in this note. We discuss gauge theories and matter contents in F-theory compactified on such Calabi–Yau 4-folds. In the two constructions of genus-one fibered Calabi–Yau 4-folds without a section, we particularly focus on the families given by specific equations. The specific equations of genus-one fibered Calabi–Yau 4-folds that we choose enable a detailed investigation of the gauge theories in F-theory models.

In this note, we take a direct approach to deduce physical information directly from the defining equations of the constructed genus-one fibered Calabi–Yau 4-folds without a section. We consider two families of hypersurfaces in a product of projective spaces, which we refer to as "Fermat-type hypersurfaces" and "hypersurfaces in Hesse form"[3]; one family of double covers of a product of projective spaces given by equations of a specific form. Among the families of genus-one fibered Calabi–Yau 4-folds without a global section that we consider in this study, genus-one fibers of Fermat-type hypersurfaces and double covers of a product of projective spaces (given by equations of specific forms) possess particular symmetries; these symmetries of genus-one fibers strictly limit possible monodromies around the singular fibers. Consequently, these symmetries greatly constrain possible non-Abelian gauge groups that can form on the 7-branes. We deduce the non-Abelian gauge symmetries arising on the 7-branes in F-theory models, and utilizing these constraints imposed by the symmetries of genus-one fibers, we perform a consistency check of our results.[4]

Concretely, we consider multidegree (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified over a multidegree (4,4,4,4) 3-fold. We find that, in F-theory compactifications on Fermat-type (3,2,2,2) hypersurfaces, generically $SU(3)$ gauge symmetries arise on the 7-branes, and when the 7-branes coincide, $SU(3)$ symmetries on the 7-branes collide and are enhanced to E_6 symmetry. Only gauge symmetries of type $SU(N)$ arise on the 7-branes in F-theory compactifications on (3,2,2,2) hypersurfaces in Hesse form. In F-theory compactifications on double covers

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[2]See, for example, [38–42] for the correspondence of the singularities of Calabi–Yau manifolds and the associated matter contents. Matter arising from the structure of divisor is discussed in [43, 44]. For discussion of the deformation and the resolution of singularities of manifolds, see, for example, [45]. For analysis of matter in four-dimensional (4d) F-theory with flux, see, e.g., [4, 5].

[3]Similar conventions of terms were made for K3 hypersurfaces in [36].

[4]Similar consistency checks of non-Abelian gauge symmetries that form on the 7-branes can be found in [36, 37].

of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified over a multidegree (4,4,4,4) 3-fold (given by equations of specific form), generically $SU(2)$ gauge symmetries arise on the 7-branes. When the 7-branes coincide, $SU(2)$ gauge symmetries collide and are enhanced to $SO(7)$ symmetry; when more 7-branes coincide, gauge symmetries are enhanced further to E_7 symmetry.

We compute the Jacobian fibrations of the families of genus-one fibered Calabi–Yau 4-folds without a global section. We determine the Mordell–Weil groups of the Jacobian fibrations of specific members of the family of Fermat-type hypersurfaces, and the family of double covers. In particular, we deduce that F-theory compactifications on these specific members do not have a $U(1)$ gauge symmetry.

We also discuss potential matter contents and potential Yukawa couplings. As will be discussed in Section 4, when we consider algebraic 2-cycles as candidates for four-form fluxes[5], we need to consider *intrinsic* algebraic 2-cycles[6]. We need to compute their self-intersections to see if they can cancel the tadpole; however, it is technically difficult to compute the self-intersection of an intrinsic algebraic 2-cycle in the geometry of Calabi–Yau 4-folds that we consider in this note. We only deduce the potential matter contents, and potential Yukawa couplings. We compute the Euler characteristics of the constructed Calabi–Yau 4-folds, to derive constraints imposed on the self-intersection of a four-form flux to cancel the tadpole.

The outline of this note is as follows: In Section 2, we introduce the two constructions of genus-one fibered Calabi–Yau 4-folds without a section. The constructions use hypersurfaces in a product of projective spaces, and double covers of a product of projective spaces; to perform a detailed study of gauge theories, we only consider families given by specific equations in these constructions. We determine the discriminant loci and their components. We describe the forms of the discriminant components. In Section 3, we deduce the non-Abelian gauge symmetries arising on the 7-branes in F-theory compactifications on the families of genus-one fibered Calabi–Yau 4-folds lacking a global section, as introduced in Section 2. We choose the defining equations of Fermat-type Calabi–Yau hypersurfaces, and Calabi–Yau 4-folds constructed as double covers, so that genus-one fibers possess complex multiplications of specific orders. These particular symmetries constrain possible non-Abelian gauge groups that can form on 7-branes. We confirm that the non-Abelian gauge groups that we deduce are in agreement with

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[5]Four-form flux and a generated superpotential were studied in [46]. See, for example, [34, 47–61] for recent progress of four-form flux in F-theory.

[6]We explain what we mean by the term "intrinsic algebraic cycles" in Section 4.2.

these constraints. This gives a consistency check of our solutions. In Section 4, we consider the existence of a consistent four-form flux. We compute the Euler characteristics of Calabi–Yau 4-folds, to derive conditions for the self-intersections of four-form fluxes to cancel the tadpole. In Section 5, we determine the potential matter spectra, and potential Yukawa couplings. In Section 6, we state our conclusions.

2. Genus-one fibered Calabi–Yau 4-folds without a global section, and discriminant loci

In this section, we construct genus-one fibered Calabi–Yau 4-folds that lack a global section. We consider the following two constructions:

- multidegree (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
- double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a multidegree (4,4, 4,4) 3-fold.

These two constructions have the trivial canonical bundles $K = 0$, and they are therefore Calabi–Yau 4-folds. Furthermore, natural projections onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ give genus-one fibrations, so they are genus-one fibered. Additionally, they have natural projections onto $\mathbb{P}^1 \times \mathbb{P}^1$, which give K3 fibrations.

For each of these two constructions, we only consider families given by specific equations, whose symmetries allow for a detailed investigation of gauge theories. Gauge theories in F-theory on the families of Calabi–Yau 4-folds will be discussed in Section 3. In this section, we introduce the families of genus-one fibered Calabi–Yau 4-folds given by specific equations. We show that they do not admit a global section. We determine the discriminant loci of the families of Calabi–Yau 4-folds, and we describe the forms of the discriminant components.

2.1. Multidegree $(3, 2, 2, 2)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

2.1.1. Two types of equations for $(3, 2, 2, 2)$ hypersurfaces. Multidegree (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are Calabi–Yau 4-folds. A fiber of the natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a degree 3 hypersurface in $\mathbb{P}^2$, which is a genus-one curve; therefore, (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ are genus-one fibration over the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. A fiber of a natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ is a bidegree (3,2) hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$, which is a genus-one fibered K3 surface, and therefore, projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ gives a K3 fibration.

In this note, we particularly focus on two families of (3,2,2,2) hypersurfaces given by the following two types of equations:

(1) $(t - \alpha_1)(t - \alpha_2)fX^3 + (t - \alpha_3)(t - \alpha_4)gY^3 + (t - \alpha_5)(t - \alpha_6)hZ^3 = 0$

(2) $(t - \beta_1)(t - \beta_2)aX^3 + (t - \beta_3)(t - \beta_4)bY^3 + (t - \beta_5)(t - \beta_6)cZ^3$
$$- 3(t - \beta_7)(t - \beta_8)dXYZ = 0.$$

$[X : Y : Z]$ is homogeneous coordinates on $\mathbb{P}^2$, and t is the inhomogeneous coordinate on the first $\mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. α_i $(i = 1, \ldots, 6)$ and β_j $(j = 1, \ldots, 8)$ are points in this first $\mathbb{P}^1$. f, g, h and a, b, c, d are bidegree $(2,2)$ polynomials on $\mathbb{P}^1 \times \mathbb{P}^1$, where the $\mathbb{P}^1$'s in the product $\mathbb{P}^1 \times \mathbb{P}^1$ are the last two $\mathbb{P}^1$'s in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We refer to the family of hypersurfaces given by the first type equation (1) as *Fermat-type* hypersurfaces, and we refer to the family of hypersurfaces given by the second type equation (2) as hypersurfaces in *Hesse form*.

For Fermat-type hypersurface (1), a K3 fiber of the projection onto the product $\mathbb{P}^1 \times \mathbb{P}^1$ of the second and third $\mathbb{P}^1$'s is described by the following equation:

(3) $\quad (t - \alpha_1)(t - \alpha_2)\, X^3 + (t - \alpha_3)(t - \alpha_4)\, Y^3 + (t - \alpha_5)(t - \alpha_6)\, Z^3 = 0$

This is Fermat-type K3 hypersurface, which is discussed in [36]. Similarly, for the hypersurface in Hesse form (2), a K3 fiber of the projection onto the product $\mathbb{P}^1 \times \mathbb{P}^1$ of the second and third $\mathbb{P}^1$'s is given by the following equation:

(4) $\quad (t - \beta_1)(t - \beta_2)\, X^3 + (t - \beta_3)(t - \beta_4)\, Y^3$
$$+ (t - \beta_5)(t - \beta_6)\, Z^3 - 3(t - \beta_7)(t - \beta_8)\, XYZ = 0.$$

This is K3 hypersurface in Hesse form, which is discussed in [36].

In [36], it was shown that Fermat-type K3 hypersurfaces (3) and K3 hypersurfaces in Hesse form (4) are genus-one fibered, but their generic members lack a global section to the fibration. If Fermat-type (3,2,2,2) Calabi–Yau hypersurfaces (1) admit a rational section, it restricts as a global section to the K3 fiber. This means that Fermat-type K3 hypersurfaces (3) admit a global section, which is a contradiction. Similar reasoning applies to (3,2,2,2) Calabi–Yau hypersurfaces in Hesse form (2). We therefore conclude that

Fermat-type (3,2,2,2) Calabi–Yau hypersurfaces (1) and Calabi–Yau hypersurfaces in Hesse form (2) are genus-one fibered, but they lack a rational section.

2.1.2. Discriminant locus and forms of discriminant components of Fermat-type (3, 2, 2, 2) hypersurfaces.

We determine the discriminant locus, and the forms of the discriminant components of Fermat-type (3,2,2,2) hypersurface

$$(5) \quad (t - \alpha_1)(t - \alpha_2)fX^3 + (t - \alpha_3)(t - \alpha_4)gY^3 + (t - \alpha_5)(t - \alpha_6)hZ^3 = 0.$$

A genus-one fibered Calabi–Yau 4-fold and its Jacobian fibration have identical discriminant loci. We deduce the discriminant components of Fermat-type (3,2,2,2) Calabi–Yau hypersurface (5) by studying the Jacobian fibration.

The Jacobian fibration of Fermat-type hypersurface (5) is given by the following equation:

$$(6) \qquad\qquad X^3 + Y^3 + \Pi_{i=1}^6 (t - \alpha_i) \cdot fgh \cdot Z^3 = 0.$$

The Jacobian fibration (6) transforms into the following Weierstrass form [62]

$$(7) \qquad\qquad y^2 = x^3 - 2^4 \cdot 3^3 \cdot \Pi_{i=1}^6 (t - \alpha_i)^2 \cdot f^2 g^2 h^2.$$

Therefore, the discriminant of the Jacobian fibration (6) is given by the following equation:

$$(8) \qquad\qquad \Delta \sim \Pi_{i=1}^6 (t - \alpha_i)^4 \cdot f^4 g^4 h^4.$$

The discriminant locus of the Jacobian (6), which is given by $\Delta = 0$, is identical to the discriminant locus of the Fermat-type hypersurface (5).

Therefore, the loci given by the following equations in the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ describe the discriminant locus of the Fermat-type hypersurface (5):

$$(9) \qquad\qquad\qquad \begin{aligned} t &= \alpha_i \quad (i = 1, \ldots, 6) \\ f &= 0 \\ g &= 0 \\ h &= 0. \end{aligned}$$

Each equation in (9) gives a discriminant component. We use the following notations to denote the discriminant components:

$$
\begin{aligned}
A_i &:= \{t = \alpha_i\} \quad (i = 1, \ldots, 6) \\
B_1 &:= \{f = 0\} \\
B_2 &:= \{g = 0\} \\
B_3 &:= \{h = 0\}.
\end{aligned}
\tag{10}
$$

We require that

$$
B_1 \cap B_2 \cap B_3 = \phi
\tag{11}
$$

to ensure that the Calabi–Yau condition is unbroken.

Component A_i, $i = 1, \ldots, 6$, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The bidegree (2,2) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is a curve of genus 1 [7], i.e., an elliptic curve Σ_1, and therefore, component B_i, $i = 1, 2, 3$, is isomorphic to $\mathbb{P}^1 \times \Sigma_1$.

Next, we determine the intersections of discriminant components; in other words, we find the forms of matter curves that discriminant components contain. When $\alpha_i \neq \alpha_j$, A_i and A_j are parallel. Intersection $A_i \cap B_j$ is a genus-one curve Σ_1. Two bidegree (2,2) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ meet at 8 points[8], and therefore, $B_i \cap B_j$, $i \neq j$, is a sum of parallel 8 rational curves $\mathbb{P}^1$. We summarize the forms of discriminant components and their intersections in Table 1 below.

2.1.3. Discriminant locus and forms of discriminant components of $(3, 2, 2, 2)$ hypersurfaces in Hesse form. We determine the discriminant locus and the forms of the discriminant components of (3,2,2,2) hypersurface in Hesse form

$$
\begin{aligned}
(t - \beta_1)(t - \beta_2)aX^3 &+ (t - \beta_3)(t - \beta_4)bY^3 \\
&+ (t - \beta_5)(t - \beta_6)cZ^3 - 3(t - \beta_7)(t - \beta_8)dXYZ = 0.
\end{aligned}
\tag{12}
$$

We require that all four polynomials $\{a, b, c, d\}$ do not have simultaneous zero, to preserve the Calabi–Yau condition. We also assume that $\beta_7, \beta_8 \neq \beta_i$, $i = 1, \ldots, 6$.

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[7]A nonsingular curve of bidegree (a, b) in $\mathbb{P}^1 \times \mathbb{P}^1$ is a curve of genus $(a - 1)(b - 1)$.

[8]Two curves of bidegrees (a, b) and (c, d) in $\mathbb{P}^1 \times \mathbb{P}^1$ meet at $ad + bc$ points.

Component	Topology
A_i	$\mathbb{P}^1 \times \mathbb{P}^1$
B_i	$\mathbb{P}^1 \times \Sigma_1$
Intersections	
$A_i \cap B_j$	Σ_1
$B_i \cap B_j$	parallel 8 $\mathbb{P}^1$'s

Table 1: Discriminant components of Fermat-type hypersurface, and their intersections.

We use the following notations

$$
(13) \qquad
\begin{aligned}
A &:= (t - \beta_1)(t - \beta_2)a \\
B &:= (t - \beta_3)(t - \beta_4)b \\
C &:= (t - \beta_5)(t - \beta_6)c \\
D &:= (t - \beta_7)(t - \beta_8)d,
\end{aligned}
$$

and the notation

$$
(14) \qquad
\begin{aligned}
F_{Hesse} :=\ & (t - \beta_1)(t - \beta_2)aX^3 + (t - \beta_3)(t - \beta_4)bY^3 \\
& + (t - \beta_5)(t - \beta_6)cZ^3 - 3(t - \beta_7)(t - \beta_8)dXYZ.
\end{aligned}
$$

Genus-one fiber degenerates exactly when the equations

$$
(15) \qquad \partial_X F_{Hesse} = \partial_Y F_{Hesse} = \partial_Z F_{Hesse} = 0
$$

have a solution for $[X : Y : Z] \in \mathbb{P}^2$.

From this and by comparing degrees, we obtain the discriminant of the equation (12), as follows:

$$
(16) \qquad \Delta = ABC(ABC - D^3)^3
$$

The discriminant (16) may be rewritten explicitly as

$$
(17) \qquad \Delta = \Pi_{i=1}^6 (t - \beta_i) \cdot abc \cdot \left[\Pi_{i=1}^6 (t - \beta_i) \cdot abc - (t - \beta_7)^3 (t - \beta_8)^3 d^3 \right]^3.
$$

We use the notation

$$(18) \qquad e := \Pi_{i=1}^{6}(t - \beta_i) \cdot abc - (t - \beta_7)^3 (t - \beta_8)^3 d^3$$

for simplicity. The vanishing of the discriminant $\Delta = 0$ describes the discriminant locus. Therefore, the following equations describe the discriminant components:

$$
\begin{aligned}
(19) \qquad t &= \beta_i \quad (i = 1, \ldots, 6) \\
a &= 0 \\
b &= 0 \\
c &= 0 \\
e &= 0.
\end{aligned}
$$

We use the following notations to denote the discriminant components:

$$
\begin{aligned}
(20) \qquad A_i &:= \{t = \beta_i\} \quad (i = 1, \ldots, 6) \\
B_1 &:= \{a = 0\} \\
B_2 &:= \{b = 0\} \\
B_3 &:= \{c = 0\} \\
B_4 &:= \{e = 0\}.
\end{aligned}
$$

Component A_i is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The bidegree $(2,2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is a genus-one curve Σ_1, and therefore, components B_1, B_2 and B_3 are isomorphic to $\mathbb{P}^1 \times \Sigma_1$. B_4 is some complicated complex surface. We do not discuss the form of B_4.

When $\beta_i \ne \beta_j$, components A_i and A_j are parallel. Intersection $A_i \cap B_j$, $i = 1, \ldots, 6$, $j = 1, \ldots, 4$, is isomorphic to Σ_1. $B_i \cap B_j$, $i, j = 1, 2, 3$, $i \ne j$, is a sum of 8 disjoint rational curves $\mathbb{P}^1$. $B_i \cap B_4$, $i = 1, 2, 3$, is a union of 8 $\mathbb{P}^1$'s and 2 Σ_1's. The forms of the discriminant components and their intersections are shown in Table 2 below.

2.2. Double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified along a multidegree $(4, 4, 4, 4)$ 3-fold

2.2.1. Equations for double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
Double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified along a multidegree $(4,4,4,4)$ 3-fold are Calabi–Yau 4-folds. A fiber of the natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of $\mathbb{P}^1$ branched along 4 points, which is a genus-one curve.

Component	Topology
A_i	$\mathbb{P}^1 \times \mathbb{P}^1$
B_i $(i = 1, 2, 3)$	$\mathbb{P}^1 \times \Sigma_1$
Intersections	
$A_i \cap B_j$ $(j = 1, \dots, 4)$	Σ_1
$B_i \cap B_j$ $(i, j = 1, 2, 3, i \neq j)$	disjoint 8 $\mathbb{P}^1$'s
$B_i \cap B_4$ $(i = 1, 2, 3)$	union of 8 $\mathbb{P}^1$'s and 2 Σ_1's

Table 2: Discriminant components of hypersurfaces in Hesse form, and their intersections. Form of component B_4 is omitted.

Therefore, projection onto $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a genus-one fibration. Additionally, a fiber of natural projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a (4,4) curve, which is a genus-one fibered K3 surface; projection onto $\mathbb{P}^1 \times \mathbb{P}^1$ gives a K3 fibration.

In this note, we focus on the family of double covers given by the following type of equation:

$$(21) \qquad \tau^2 = f \cdot a(t) \cdot x^4 + g \cdot b(t).$$

x is the inhomogeneous coordinate on the first $\mathbb{P}^1$ in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and t is the inhomogeneous coordinate on the second $\mathbb{P}^1$. a and b are degree 4 polynomials in the variable t. f and g are bidegree (4,4) polynomials on $\mathbb{P}^1 \times \mathbb{P}^1$, where the $\mathbb{P}^1$'s in the product $\mathbb{P}^1 \times \mathbb{P}^1$ are the last two $\mathbb{P}^1$'s in the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. By splitting the polynomials a and b into linear factors, the equation (21) may be rewritten as:

$$(22) \qquad \tau^2 = f \cdot \Pi_{i=1}^{4}(t - \alpha_i) \cdot x^4 + g \cdot \Pi_{j=5}^{8}(t - \alpha_j).$$

The fiber of the projection onto the product of the third and the fourth $\mathbb{P}^1$'s in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by the following equation:

$$(23) \qquad \tau^2 = \Pi_{i=1}^{4}(t - \alpha_i) \cdot x^4 + \Pi_{j=5}^{8}(t - \alpha_j).$$

This is a genus-one fibered K3 surface discussed in [37], and it was shown in [37] that this K3 surface does not admit a global section. Therefore, by a similar argument as that stated in Section 2.1.1 we conclude that the double covers (21) do not have a rational section.

2.2.2. Discriminant locus and forms of discriminant components of double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We determine the discriminant locus, and the forms of the discriminant components of double cover (21). The Jacobian fibration of double cover (21) is given by the following equation [63]:

$$(24) \qquad \tau^2 = \frac{1}{4}x^3 - fg \cdot \Pi_{i=1}^8 (t - \alpha_i) \cdot x.$$

The discriminant of the Jacobian fibration (24) is given by

$$(25) \qquad \Delta \sim f^3 g^3 \cdot \Pi_{i=1}^8 (t - \alpha_i)^3.$$

The condition $\Delta = 0$ describes the discriminant locus of the Jacobian (24). This is identical to the discriminant locus of double cover (21).

Therefore, the discriminant locus in the base $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is described by the following equations:

$$(26) \qquad \begin{aligned} t &= \alpha_i \quad (i = 1, \ldots, 8) \\ f &= 0 \\ g &= 0. \end{aligned}$$

Each equation in (26) gives a discriminant component. We use the following notations to denote the discriminant components:

$$(27) \qquad \begin{aligned} A_i &:= \{t = \alpha_i\} \quad (i = 1, \ldots, 8) \\ B_1 &:= \{f = 0\} \\ B_2 &:= \{g = 0\}. \end{aligned}$$

Discriminant component A_i, $i = 1, \ldots, 8$, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The bidegree (4,4) curve in $\mathbb{P}^1 \times \mathbb{P}^1$ is a genus 9 curve Σ_9, and therefore, component B_i, $i = 1, 2$, is isomorphic to $\mathbb{P}^1 \times \Sigma_9$.

We determine the forms of the intersections of components. When $\alpha_i \neq \alpha_j$, A_i and A_j are parallel. $A_i \cap B_j$, $i = 1, \ldots, 8$, $j = 1, 2$, is isomorphic to genus 9 curve Σ_9. Two bidegree (4,4) curves in $\mathbb{P}^1 \times \mathbb{P}^1$ meet at 32 points, and therefore, $B_1 \cap B_2$ is the disjoint sum of 32 rational curves $\mathbb{P}^1$. The

forms of the discriminant components and their intersections are shown in Table 3 below.

Component	Topology
A_i	$\mathbb{P}^1 \times \mathbb{P}^1$
B_i	$\mathbb{P}^1 \times \Sigma_9$
Intersections	
$A_i \cap B_j$	Σ_9
$B_1 \cap B_2$	disjoint 32 $\mathbb{P}^1$'s

Table 3: Discriminant components of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and their intersections.

3. Gauge symmetries

We deduce the non-Abelian gauge symmetries that form on the 7-branes in F-theory compactifications on genus-one fibered Calabi–Yau 4-folds lacking a global section, which we constructed in Section 2. Genus-one fibers of the Fermat-type Calabi–Yau hypersurfaces (1) and double covers (21) possess complex multiplications of specific orders. These greatly limit the possible monodromies around the singular fibers, and as a result, possible types of singular fibers are also restricted. These strictly constrain the possible non-Abelian gauge groups that can form on the 7-branes. Using this fact, we check the consistency of solutions of non-Abelian gauge groups in Section 3.4. Some F-theory models that do not have $U(1)$ gauge symmetry are discussed in Section 3.5.

3.1. Non-Abelian gauge groups and singular fibers

When a Calabi–Yau 4-fold has a genus-one fibration, the structures of singular fibers[9] along the codimension one locus in the base are in essence

---

[9]See [64–72] for discussion of elliptic surfaces, elliptic fibration, and singular fibers. [62] discusses elliptic curves and the Jacobian. [73–75] discuss elliptic fibrations of 3-folds.
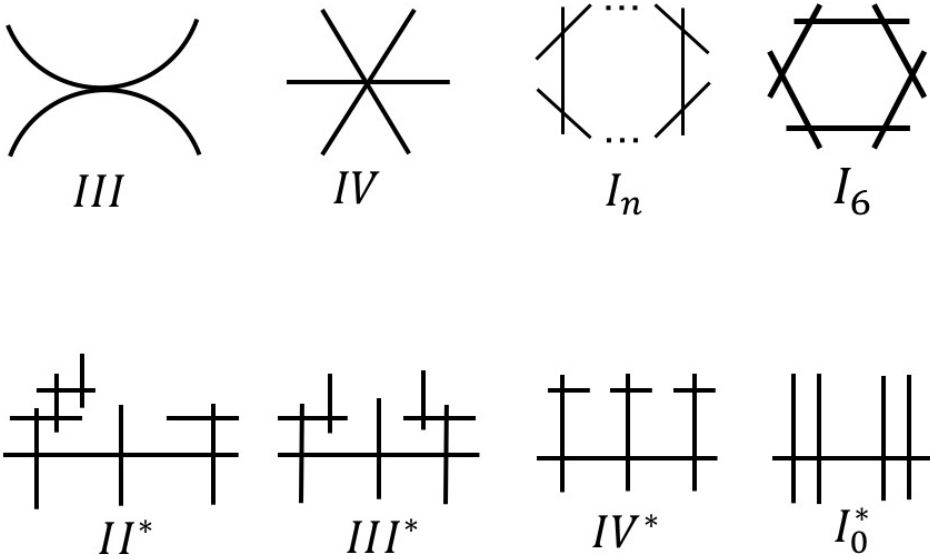
Figure 1: Singular fibers.

the same as those of singular fibers of elliptic surfaces. Therefore, Kodaira's classification [64, 65] applies to singular fibers on discriminant components. According to Kodaira's classification, the types of singular fibers fall into two classes: i) six types II, III, IV, II^*, III^*, and IV^*; and ii) two infinite series I_n ($n \geq 1$) and I_m^* ($m \geq 0$).

Fibers of type I_1 and II are rational curves $\mathbb{P}^1$ with one singularity (II is a rational curve with a cusp, and I_1 is a rational curve with a node); fibers of the other types are unions of smooth $\mathbb{P}^1$'s intersecting in specific ways. Type III fiber is a union of two rational curves tangential to each other at one point, and type IV fiber is a union of three rational curves meeting at one point. For each fiber type I_n, n rational curves intersect to form an n-gon. Figure 1 shows images of the singular fibers. Each line in the image represents a rational curve $\mathbb{P}^1$. Two rational curve components in a singular fiber intersect only when two lines in an image intersect.

Non-Abelian gauge group that forms on the 7-branes is determined by the singular fiber type over a discriminant component. The correspondence between the non-Abelian gauge symmetries on the 7-branes and the fiber types is discussed in [3, 38]. The correspondences of the types of singular fibers and the singularity types are presented in Table 4 below.

fiber type	singularity type
$I_n\ (n \geq 2)$	A_{n-1}
$I_m^*\ (m \geq 0)$	D_{4+m}
III	A_1
IV	A_2
II^*	E_8
III^*	E_7
IV^*	E_6

Table 4: Correspondence between singular fiber types and singularity types.

3.2. Non-Abelian gauge groups in F-theory on $(3, 2, 2, 2)$ hypersurfaces

We deduce non-Abelian gauge symmetries in F-theory compactification on (3,2,2,2) hypersurfaces.

3.2.1. Fermat-type $(3, 2, 2, 2)$ hypersurfaces. We deduce the non-Abelian gauge symmetries that form on the 7-branes in F-theory compactifications on the Fermat-type (3,2,2,2) Calabi–Yau hypersurfaces
(28)
$$(t - \alpha_1)(t - \alpha_2)fX^3 + (t - \alpha_3)(t - \alpha_4)gY^3 + (t - \alpha_5)(t - \alpha_6)hZ^3 = 0.$$

As stated in Section 2.1.2, the Jacobian fibration of Fermat-type hypersurface (28) is given by the following equation:

(29)
$$X^3 + Y^3 + \Pi_{i=1}^6 (t - \alpha_i) \cdot fgh \cdot Z^3 = 0.$$

The Jacobian fibration (29) transforms into the following Weierstrass form:

(30)
$$y^2 = x^3 - 2^4 \cdot 3^3 \cdot \Pi_{i=1}^6 (t - \alpha_i)^2 \cdot f^2 g^2 h^2,$$

and the discriminant of the Jacobian fibration (29) is given by the following equation:

(31)
$$\Delta \sim \Pi_{i=1}^6 (t - \alpha_i)^4 \cdot f^4 g^4 h^4.$$

Fermat-type (3,2,2,2) hypersurface (28) and the Jacobian fibration (29) have the identical singular fiber types over the same discriminant loci, thus the result of singular fibers for the Jacobian fibration (29) gives identical singular fibers of Fermat-type hypersurface (28).

We determine the types of singular fibers of the Jacobian fibration (29) from the Weierstrass form (30) and the discriminant (31). We show the correspondence of the singular fiber types and the vanishing orders of the coefficients of the Weierstrass form in Table 5. We find that, when α_i ($i =$

Fiber type	Ord(a_4)	Ord(a_6)	Ord(Δ)
I_0	≥ 0	≥ 0	0
I_n ($n \geq 1$)	0	0	n
II	≥ 1	1	2
III	1	≥ 2	3
IV	≥ 2	2	4
I_0^*	2	≥ 3	6
	≥ 2	3	6
I_m^* ($m \geq 1$)	2	3	$6 + m$
IV^*	≥ 3	4	8
III^*	3	≥ 5	9
II^*	≥ 4	5	10

Table 5: Correspondence of the types of singular fibers and the vanishing orders of coefficients a_4, a_6, and the discriminant Δ, of the Weierstrass form $y^2 = x^3 + a_4 x + a_6$.

$1, \ldots, 6$) are mutually distinct, the singular fiber on component A_i is of type IV. The polynomial

$$(32) \qquad y^2 + 2^4 \cdot 3^3 \cdot \Pi_{i=1}^6 (t - \alpha_i)^2 \cdot f^2 g^2 h^2$$

splits into linear factors as

$$(33) \ (y + 2^2 \cdot 3\sqrt{3}i \cdot \Pi_{i=1}^6 (t - \alpha_i) \cdot fgh)(y - 2^2 \cdot 3\sqrt{3}i \cdot \Pi_{i=1}^6 (t - \alpha_i) \cdot fgh).$$

Thus, we find that type IV fiber on component A_i is of split type [38]. Therefore, $SU(3)$ gauge symmetry arises on 7-branes wrapped on component A_i. When the multiplicity of α_i is 2, (i.e. when there is one $j \neq i$ such that $\alpha_i = \alpha_j$), 7-branes wrapped on components A_i and A_j coincide, and the fiber type is enhanced to IV^*. Since polynomial (32) splits into linear factors as (33), we find that type IV^* fiber on component A_i is split. The corresponding gauge group on 7-branes is enhanced to E_6. To preserve Calabi–Yau condition, the multiplicity cannot be greater than 2. Type of singular fibers on component B_i is IV, and we see that they are of split type from factorization (33); $SU(3)$ gauge symmetry arises on 7-branes wrapped on component B_i. The results are summarized in Table 6 below.

Component	Fiber type	non-Abel. Gauge Group
A_i	IV	$SU(3)$
	IV^*	E_6
B_i	IV	$SU(3)$

Table 6: Types of singular fibers and corresponding non-Abelian gauge groups on discriminant components of Fermat-type hypersurface.

3.2.2. $(3, 2, 2, 2)$ hypersurfaces in Hesse form.

We determine the types of singular fibers of (3,2,2,2) hypersurfaces in Hesse form

$$(34) \quad (t - \beta_1)(t - \beta_2)aX^3 + (t - \beta_3)(t - \beta_4)bY^3 \\ + (t - \beta_5)(t - \beta_6)cZ^3 - 3(t - \beta_7)(t - \beta_8)dXYZ = 0$$

by computing the singular fibers of the Jacobian fibration. As we saw in Section 2.1.3, the equation for (3,2,2,2) hypersurface in Hesse form (34) has the following discriminant:

$$(35) \quad \Delta = \Pi_{i=1}^{6}(t - \beta_i) \cdot abc \cdot e^3.$$

In (35), we have used the notation

$$(36) \quad e = \Pi_{i=1}^{6}(t - \beta_i) \cdot abc - (t - \beta_7)^3(t - \beta_8)^3 d^3$$

for simplicity.

The Jacobian fibration of (3,2,2,2) hypersurface in Hesse form (34) is given as:

$$(37) \qquad X^3 + Y^3 + \Pi_{i=1}^6 (t - \beta_i) \cdot abc \cdot Z^3 - 3(t - \beta_7)(t - \beta_8) dXYZ = 0.$$

The discriminant of the Jacobian fibration (37) of (3,2,2,2) hypersurface in Hesse form (34) is also given by (35).

As in Section 2.1.3, we use the following notations:

$$(38) \qquad\qquad\qquad A := (t - \beta_1)(t - \beta_2)a$$
$$B := (t - \beta_3)(t - \beta_4)b$$
$$C := (t - \beta_5)(t - \beta_6)c$$
$$D := (t - \beta_7)(t - \beta_8)d,$$

Using the notations (38), (3,2,2,2) hypersurface in Hesse form (34) may be rewritten as:

$$(39) \qquad\qquad AX^3 + BY^3 + CZ^3 - 3D \cdot XYZ = 0.$$

The Jacobian fibration (37) of (3,2,2,2) hypersurface in Hesse form (34) may be rewritten as:

$$(40) \qquad\qquad X^3 + Y^3 + ABCZ^3 - 3D \cdot XYZ = 0.$$

Using the notations (38), both the discriminants of (3,2,2,2) hypersurface in Hesse form (39) and the Jacobian fibration (40) are given as follows:

$$(41) \qquad\qquad \Delta = ABC(ABC - D^3)^3.$$

Jacobian fibration (40) transforms into the general Weierstrass form as

$$(42) \qquad\qquad y^2 - 3Dxy + (ABC - D^3)y = x^3.$$

We complete the square in y as $\tilde{y} = y + \frac{1}{2}(-3Dx + ABC - D^3)$, and complete the cube in x as $\tilde{x} = x + \frac{3}{4}D^2$ to obtain the following Weierstrass form:

$$(43) \qquad \tilde{y}^2 = \tilde{x}^3 - \left(\frac{3}{2}ABCD + \frac{3}{16}D^4 \right) \tilde{x}$$
$$+ \left(\frac{1}{4}(ABC)^2 + \frac{5}{8}ABCD^3 - \frac{1}{32}D^6 \right).$$

We deduce from Weierstrass form (43) that type of singular fibers over each discriminant component is I_n for some $n \geq 1$. Therefore, the types of

singular fibers can be determined by studying the orders of the zeros of the discriminant (35). In the general Weierstrass form (42), polynomial

$$(44) \qquad\qquad\qquad y^2 - 3Dxy$$

can be factored as

$$(45) \qquad\qquad\qquad y(y - 3Dx).$$

Thus, we conclude that singular fibers on component B_4 are of split type.

Under the translation in x and y that replaces x with $x - D^2$, and y with $y - D^3$, the general Weierstrass form (42) transforms into another general Weierstrass form:

$$(46) \qquad y^2 - 3Dxy + ABCy = x^3 - 3D^2x^2 + ABCD^3.$$

Polynomial

$$(47) \qquad\qquad\qquad y^2 - 3Dxy + 3D^2x^2$$

splits into linear factors as

$$(48) \qquad \left(y - \frac{1}{2}(3 - i\sqrt{3})Dx\right)\left(y - \frac{1}{2}(3 + i\sqrt{3})Dx\right).$$

Therefore, we deduce that singular fibers on components A_i, $i = 1, 2, \ldots, 6$, are split. Non-Abelian gauge groups that form on the 7-branes wrapped on components A_i, $i = 1, 2, \ldots, 6$, and component B_4, are of the form $SU(N)$.

When β_i's are mutually distinct, the fiber type on component A_i is I_1, and non-Abelian gauge symmetry does not form on the 7-brane wrapped on A_i. As the multiplicity of β_i increases, more 7-branes become coincident, and the non-Abelian gauge group becomes further enhanced. The maximum enhancement occurs when all β_i, $i = 1, \ldots, 6$, are equal, and all six 7-branes wrapped on A_i coincide. The fiber type on component A_1 for this case is I_6, and $SU(6)$ gauge symmetry arises on the 7-branes wrapped on A_1. In Section 5, we compute the potential matter spectra for this most enhanced situation.

Singular fibers on component B_i, $i = 1, 2, 3$, have type I_1; a non-Abelian gauge group does not form on the 7-brane wrapped on component B_i, $i = 1, 2, 3$. Singular fibers on component B_4 have type I_3, and $SU(3)$ gauge group arises on 7-branes wrapped on B_4. Results are summarized in Table 7.

Component	Fiber type	non-Abel. Gauge Group
	I_1	None.
	I_2	$SU(2)$
A_i	I_3	$SU(3)$
	I_4	$SU(4)$
	I_5	$SU(5)$
	I_6	$SU(6)$
$B_{1,2,3}$	I_1	None.
B_4	I_3	$SU(3)$

Table 7: Types of singular fibers and corresponding non-Abelian gauge groups on discriminant components of hypersurface in Hesse form.

3.3. Non-Abelian gauge groups in F-theory on double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

We deduce the non-Abelian gauge groups in F-theory compactifications on double covers

$$(49) \qquad \tau^2 = f \cdot \Pi_{i=1}^4 (t - \alpha_i) \cdot x^4 + g \cdot \Pi_{j=5}^8 (t - \alpha_j).$$

As stated in Section 2.2.2, the Jacobian fibration of double cover (49) is given by the following equation:

$$(50) \qquad \tau^2 = \frac{1}{4} x^3 - fg \cdot \Pi_{i=1}^8 (t - \alpha_i) \cdot x.$$

The discriminant of the Jacobian fibration (50) is given by

$$(51) \qquad \Delta \sim f^3 g^3 \cdot \Pi_{i=1}^8 (t - \alpha_i)^3.$$

We determine the types of singular fibers of double cover (49) by computing the types of singular fibers of the Jacobian fibration (50). When α_i's are mutually distinct, the singular fiber on component A_i has type III; the $SU(2)$ gauge group arises on the 7-branes wrapped on component A_i for this

case. When the multiplicity of α_i is 2, say there is $j \neq i$ such that $\alpha_i = \alpha_j$, then the 7-branes wrapped on components A_i and A_j become coincident, and singular fiber on component A_i has type I_0^*. The polynomial

$$(52) \qquad\qquad x^3 - fg \cdot x$$

splits into the quadratic factor and the linear factor as

$$(53) \qquad\qquad x(x^2 - fg)$$

for generic polynomials f, g. Therefore, we conclude that I_0^* fiber on component A_i is semi-split; the non-Abelian gauge symmetry on the 7-branes wrapped on component A_i becomes enhanced to $SO(7)$. When the multiplicity of α_i is 3, the singular fiber on component A_i has type III^*, and the gauge symmetry on component A_i is further enhanced to E_7. To preserve the Calabi–Yau condition, no further enhancement is possible. The singular fibers on component B_i is of type III; the $SU(2)$ gauge group arises on 7-branes wrapped on component B_i. The results are displayed in Table 8 below.

Component	Fiber type	non-Abel. Gauge Group
	III	$SU(2)$
A_i	I_0^*	$SO(7)$
	III^*	E_7
B_i	III	$SU(2)$

Table 8: Types of singular fibers and corresponding non-Abelian gauge groups on discriminant components of double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (49).

3.4. Consistency check by monodromy

We consider monodromies around singular fibers to perform a consistency check of solutions of non-Abelian gauge groups, which we obtained in Sections 3.2.1, 3.3. Genus-one fibers of Fermat-type (3,2,2,2) hypersurfaces (28)

and double covers (49) possess particular symmetries, and as a result, these symmetries strictly constrain monodromies around singular fibers. We confirm that the non-Abelian gauge symmetries obtained by us in agreement with these restrictions.

3.4.1. Monodromy and J-invariant.

Genus-one fibers of Fermat-type (3,2,2,2) hypersurfaces (28) and double covers (49) have constant j-invariants; they are constant over the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Concretely, generic genus-one fiber of the Fermat-type (3,2,2,2) hypersurface is the Fermat curve[10], whose j-invariant is known to be 0. Therefore, the j-invariant of singular fibers is forced to be 0.

Smooth genus-one fiber of double cover (49) is invariant under the map:

$$(54) \qquad\qquad\qquad x \to e^{2\pi \mathrm{i}/4} x,$$

whose order is 4. This is a complex multiplication of order 4, and therefore, the generic genus-one fiber has the j-invariant 1728. This forces the j-invariant of singular fibers to be 1728.

Each fiber type has a specific monodromy and j-invariant. We display the monodromy and their orders in $SL_2(\mathbb{Z})$, and the j-invariant, for each fiber type in Table 9 below. "Finite" in the table means that the j-invariant of fiber type I_0^* can take any finite value in $\mathbb{C}$. Results in Table 9 were derived in [64, 65][11].

3.4.2. Fermat-type $(3, 2, 2, 2)$ hypersurfaces.

As we saw in Section 3.4.1, singular fibers of the Fermat-type (3,2,2,2) hypersurface have j-invariant 0. As can be seen in Table 9, the fiber types with j-invariant 0 are only II, IV, I_0^*, IV^*, and II^*. (j-invariant of type I_0^* fiber can take the value 0.) Fiber types on discriminant components that we obtained in Section 3.2.1 are IV, IV^*, which is in agreement with constraint imposed by the j-invariant. Monodromies of order 3 characterize non-Abelian gauge symmetries arising on 7-branes in F-theory compactifications on Fermat-type (3,2,2,2) hypersurfaces.

3.4.3. Double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

As we saw in Section 3.4.1, singular fibers of double cover (49) have j-invariant 1728. According to Table 9, fiber types with j-invariant 1728 are only III, I_0^*, and III^*. This

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[10]The Fermat curve possesses complex multiplication of order 3.

[11]Euler numbers of fiber types were obtained in [65], and they have an interpretation as the number of 7-branes wrapped on.

Fiber Type	j-invariant	Monodromy	order of Monodromy	# of 7-branes (Euler number)
I_0^*	finite	$-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2	6
I_b	∞	$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	infinite	b
I_b^*	∞	$-\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$	infinite	$b+6$
II	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	6	2
II^*	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	6	10
III	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	4	3
III^*	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	4	9
IV	0	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	3	4
IV^*	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	3	8

Table 9: Fiber types, their j-invariants, monodromies, and the associated numbers of 7-branes.

agrees with the fiber types that we obtained in Section 3.3 on discriminant components of double covers. Monodromies of order 2 and 4 characterize non-Abelian gauge symmetries on 7-branes in F-theory compactification on double covers (49).

3.5. F-theory models without $U(1)$ symmetry

We specify the Mordell–Weil groups of the Jacobian fibrations of special genus-one fibered Calabi–Yau 4-folds. We find that the Mordell–Weil groups of Jacobian fibrations of the special genus-one fibered Calabi–Yau 4-folds

that we consider here have the rank 0, therefore, we deduce that F-theory compactifications on these special genus-one fibered Calabi–Yau 4-folds do not have a $U(1)$ gauge symmetry.

3.5.1. Special Fermat-type $(3, 2, 2, 2)$ hypersurface.

We particularly consider the following special Fermat-type (3,2,2,2) hypersurface:

$$(55) \qquad (t - \alpha_1)^2 f X^3 + (t - \alpha_2)^2 g Y^3 + (t - \alpha_3)^2 h Z^3 = 0.$$

The Jacobian fibration of this special Fermat-type hypersurface (55) is given by the following equation:

$$(56) \qquad X^3 + Y^3 + (t - \alpha_1)^2 (t - \alpha_2)^2 (t - \alpha_3)^2 \cdot fgh \cdot Z^3 = 0.$$

The projection onto the last two $\mathbb{P}^1$'s in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ gives a K3 fibration, and picking a point in the base surface $\mathbb{P}^1 \times \mathbb{P}^1$ gives a specialization to this K3 fiber. The K3 fiber of the Jacobian fibration (56) is given by the following equation:

$$(57) \qquad X^3 + Y^3 + (t - \alpha_1)^2 (t - \alpha_2)^2 (t - \alpha_3)^2 Z^3 = 0.$$

This is the Jacobian fibration of the Fermat-type K3 hypersurface (3), which is discussed in [36], with reducible fiber type E_6^3. According to Table 2 in [76], extremal K3 surface[12] with reducible fiber type E_6^3 is uniquely determined, and its transcendental lattice has the intersection matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The Mordell–Weil group of this extremal K3 surface is determined in [76, 77] to be $\mathbb{Z}_3$.

By considering the specialization of the Jacobian fibration (56) to its K3 fiber (57), we find that the Mordell–Weil group of the Jacobian (56) is isomorphic to that of its K3 fiber (57), which is $\mathbb{Z}_3$. This shows that the Mordell–Weil group of the Jacobian fibration (56) is isomorphic to $\mathbb{Z}_3$. Thus, we conclude that the global structure of the non-Abelian gauge group in F-theory compactified on the special Fermat-type (3,2,2,2) hypersurface (55)

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[12]Extremal K3 surface is an elliptic K3 surface with a section having the Picard number 20, with the Mordell–Weil rank 0.

is given by the following:

$$
(58) \qquad E_6^3 \times SU(3)^3/\mathbb{Z}_3.
$$

In particular, the Mordell–Weil group of the Jacobian fibration (56) has rank 0, therefore, F-theory compactified on the Fermat-type (3,2,2,2) hypersurface (55) does not have a $U(1)$ gauge symmetry.

3.5.2. Special double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Next, we consider the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the following equation:

$$
(59) \qquad \tau^2 = f \cdot (t - \alpha_1)^3 (t - \alpha_2) \cdot x^4 + g \cdot (t - \alpha_2)(t - \alpha_3)^3.
$$

The Jacobian fibration of this double cover is given by:

$$
(60) \qquad \tau^2 = \frac{1}{4} x^3 - fg \cdot (t - \alpha_1)^3 (t - \alpha_2)^2 (t - \alpha_3)^3 \cdot x.
$$

The K3 fiber of the Jacobian fibration (60) is given by the equation:

$$
(61) \qquad \tau^2 = \frac{1}{4} x^3 - (t - \alpha_1)^3 (t - \alpha_2)^2 (t - \alpha_3)^3 \cdot x.
$$

K3 surface (61) is extremal K3 with the reducible fiber type $E_7^2 D_4$. As discussed in [37], the Mordell–Weil group of this extremal K3 surface (61) is $\mathbb{Z}_2$ [76, 77].

As per reasoning similar to the argument in Section 3.5.1, we consider the specialization of the Jacobian fibration of the double cover (60) to its K3 fiber (61) and find that the Mordell–Weil group of the Jacobian (60) is isomorphic to that of its K3 fiber (61). Therefore, we conclude that the Mordell–Weil group of the Jacobian fibration (60) is isomorphic to $\mathbb{Z}_2$. Thus, we deduce that the global structure of the non-Abelian gauge group in F-theory compactification on the special double cover (59) is given by the following:

$$
(62) \qquad E_7^2 \times SO(7) \times SU(2)^2/\mathbb{Z}_2.
$$

The Mordell–Weil group of the Jacobian (60) has rank 0, therefore, it follows that F-theory compactification on the double cover (59) does not have a $U(1)$ gauge symmetry.

4. Discussion of consistent four-form flux and Euler characteristics of Calabi–Yau 4-folds

4.1. Review of conditions on four-form flux

We briefly review physical conditions imposed on four-form flux G_4 of genus-one fibered Calabi–Yau 4-fold Y. The quantization condition [78] imposed on four-form flux is given by the following equation:

$$(63) \qquad G_4 + \frac{1}{2}c_2(Y) \in H^4(Y, \mathbb{Z}).$$

In particular, when the second Chern class $c_2(Y)$ is even, the term $\frac{1}{2}c_2(Y)$ is irrelevant. To preserve supersymmetry in 4d theory, the following conditions need to be imposed [79] on four-form flux:

$$(64) \qquad G_4 \in H^{2,2}(Y)$$

$$(65) \qquad G_4 \wedge J = 0.$$

J in the condition (65) represents a Kähler form.

Furthermore, to ensure that the 4d effective theory has Lorentz symmetry, four-form flux is required to have one leg in the fiber [80]. When genus-one fibration admits a global section, this condition is given by the following equations:

$$(66) \qquad G_4 \cdot \tilde{p}^{-1}(C) \cdot \tilde{p}^{-1}(C') = 0$$
$$(67) \qquad G_4 \cdot S_0 \cdot \tilde{p}^{-1}(C) = 0$$

for any $C, C' \in H^{1,1}(B_3)$. B_3 denotes base 3-fold. In the equations (66) and (67), $\tilde{p}$ denotes the projection from elliptically fibered Calabi–Yau 4-fold Y onto base 3-fold B_3. In the equation (67), S_0 denotes a rational zero section.

Generalization of the conditions (66) and (67) to genus-one fibration without a section was proposed in [34]; the generalized equations are as follows:

$$(68) \qquad G_4 \cdot p^{-1}(C) \cdot p^{-1}(C') = 0$$
$$(69) \qquad G_4 \cdot \hat{N} \cdot p^{-1}(C) = 0$$

for any $C, C' \in H^{1,1}(B_3)$. In the equations (68) and (69), p denotes the projection from genus-one fibered Calabi–Yau 4-fold Y onto base 3-fold B_3.

$\hat{N}$ is some appropriate sum of an n-section N that Calabi–Yau genus-one fibration Y possesses and exceptional divisors.

The condition to cancel the tadpole, including 3-branes, is given as follows [81, 82]:

$$(70) \qquad \frac{\chi(Y)}{24} = \frac{1}{2} G_4 \cdot G_4 + N_3.$$

N_3 denotes the number of 3-branes minus anti 3-branes, and the stability of compactification requires $N_3 \geq 0$.

4.2. Intrinsic algebraic 2-cycles as candidates for four-form fluxes

We use algebraic 2-cycles as candidates for four-form fluxes. With this choice, the condition (64) is satisfied.

We refer to algebraic 2-cycles of (3,2,2,2) hypersurfaces as the *intrinsic* algebraic 2-cycles of (3,2,2,2) hypersurfaces in this study, when they do not belong to the algebraic 2-cycles obtained as the restrictions of algebraic cycles in the ambient space $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Similarly, we refer to the algebraic 2-cycles of double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as the intrinsic algebraic 2-cycles, when they do not belong to the algebraic 2-cycles obtained as the pullbacks of algebraic cycles of the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We show that the nonintrinsic algebraic 2-cycles of a (3,2,2,2) hypersurface, namely the algebraic 2-cycles obtained as the restrictions of algebraic cycles in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, do not yield consistent four-form fluxes. This can be shown as follows: an algebraic 2-cycle obtained as the restriction of an algebraic cycle in the product $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given as follows:

$$(71) \qquad (\alpha_1 \, x^2 + \alpha_2 \, xy + \alpha_3 \, xz + \alpha_4 \, xw + \alpha_5 \, yz + \alpha_6 \, yw + \alpha_7 \, zw)|_Y.$$

We used $|_Y$ to denote the restriction to Calabi–Yau (3,2,2,2) hypersurface Y. α_i, $i = 1, \ldots, 7$, are the coefficients. We apply the condition (68). For the pair (y, z) in the base 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the condition (68) requires that

$$
\begin{aligned}
(72) \qquad & (\alpha_1 \, x^2 + \alpha_2 \, xy + \alpha_3 \, xz + \alpha_4 \, xw + \alpha_5 \, yz + \alpha_6 \, yw + \alpha_7 \, zw) \cdot yz|_Y \\
& = (\alpha_1 \, x^2 yz + \alpha_4 \, xyzw)(3x + 2y + 2z + 2w) \\
& = (2\alpha_1 + 3\alpha_4) \, x^2 yzw = 0.
\end{aligned}
$$

Therefore, we obtain:

$$(73) \qquad 2\alpha_1 + 3\alpha_4 = 0.$$

Similarly, by applying the condition (68) to the pairs (y, w) and (z, w), we obtain:

$$(74) \qquad 2\alpha_1 + 3\alpha_3 = 0$$
$$2\alpha_1 + 3\alpha_2 = 0.$$

Thus, we find that the algebraic 2-cycle (71) should be of the following form:

$$(75) \quad \left(\alpha_1 \, x^2 - \frac{2}{3}\alpha_1 \, xy - \frac{2}{3}\alpha_1 \, xz - \frac{2}{3}\alpha_1 \, xw + \alpha_5 \, yz + \alpha_6 \, yw + \alpha_7 \, zw \right) |_Y.$$

A Kähler form J can be expressed as follows:

$$(76) \qquad J = a \, x + b \, y + c \, z + d \, w,$$

where coefficients a, b, c, d are strictly positive:

$$(77) \qquad a, \, b, \, c, \, d > 0.$$

By applying the condition (65), we obtain:

$$
\begin{aligned}
(78) \quad & \left(\alpha_1 \, x^2 - \frac{2}{3}\alpha_1 \, xy - \frac{2}{3}\alpha_1 \, xz - \frac{2}{3}\alpha_1 \, xw + \alpha_5 \, yz + \alpha_6 \, yw + \alpha_7 \, zw \right) \\
& \times (a \, x + b \, y + c \, z + d \, w)|_Y \\
= & \left(\alpha_1 \, x^2 - \frac{2}{3}\alpha_1 \, xy - \frac{2}{3}\alpha_1 \, xz - \frac{2}{3}\alpha_1 \, xw + \alpha_5 \, yz + \alpha_6 \, yw + \alpha_7 \, zw \right) \\
& \times (a \, x + b \, y + c \, z + d \, w)(3x + 2y + 2z + 2w) \\
= & \, a \left(3\alpha_5 - \frac{8}{3}\alpha_1 \right) x^2 yz + a \left(3\alpha_6 - \frac{8}{3}\alpha_1 \right) x^2 yw \\
& + a \left(3\alpha_7 - \frac{8}{3}\alpha_1 \right) x^2 zw + \left[2a(\alpha_5 + \alpha_6 + \alpha_7) + b \left(3\alpha_7 - \frac{8}{3}\alpha_1 \right) \right. \\
& \left. + c \left(3\alpha_6 - \frac{8}{3}\alpha_1 \right) + d \left(3\alpha_5 - \frac{8}{3}\alpha_1 \right) \right] xyzw \\
= & \, 0.
\end{aligned}
$$

Thus, we obtain the following conditions on coefficients:

(79)
$$\alpha_5 = \frac{8}{9}\alpha_1$$
$$\alpha_6 = \frac{8}{9}\alpha_1$$
$$\alpha_7 = \frac{8}{9}\alpha_1.$$

Therefore, the algebraic 2-cycle (71) should be of the following form:

(80)
$$\left(\alpha_1\, x^2 - \frac{2}{3}\alpha_1\, xy - \frac{2}{3}\alpha_1\, xz - \frac{2}{3}\alpha_1\, xw \right.$$
$$\left. + \frac{8}{9}\alpha_1\, yz + \frac{8}{9}\alpha_1\, yw + \frac{8}{9}\alpha_1\, zw \right)|_Y.$$

With the conditions (79), the equation (78) reduces to

(81)
$$\frac{16a}{3}\alpha_1\, xyzw = 0.$$

Thus, we find that that the conditions (65) and (68) require that

(82)
$$\alpha_1 = 0.$$

This means that the algebraic 2-cycle (80) vanishes. Thus, we conclude that the conditions (65) and (68) rule out all algebraic 2-cycles obtained as the restrictions of algebraic cycles in the ambient space $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to the (3,2,2,2) hypersurface.

A similar argument as that stated previously shows that nonintrinsic algebraic 2-cycles of a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ do not yield consistent four-form flux.

In Calabi–Yau 4-folds that we constructed, however, it is considerably difficult to explicitly describe intrinsic algebraic 2-cycles. Consequently, it is difficult to compute the self-intersections of intrinsic algebraic 2-cycles in constructed Calabi–Yau 4-folds, and owing to this, it is difficult to determine whether the tadpole can be cancelled using intrinsic algebraic 2-cycles. We do not discuss whether a consistent four-form flux exists. In Section 4.3 below, we compute the Euler characteristics of Calabi–Yau 4-folds, to derive conditions on the self-intersection of four-form flux to cancel the tadpole.

4.3. Euler characteristics and self-intersection of four-form flux to cancel tadpole

4.3.1. Multidegree $(3, 2, 2, 2)$ hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
We compute the Euler characteristic of a multidegree $(3,2,2,2)$ hypersurface Y in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We have the following exact sequence of bundles:

$$(83) \qquad 0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|_Y \longrightarrow \mathcal{N}_Y \longrightarrow 0.$$

$\mathcal{T}_Y$ is the tangent bundle of a genus-one fibered Calabi–Yau multidegree $(3,2,2,2)$ hypersurface Y, and this naturally embeds into the tangent bundle $\mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$ of the ambient space $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $|_Y$ means the restriction to Y. $\mathcal{N}_Y$ is the resultant normal bundle. We have

$$(84) \qquad\qquad\qquad \mathcal{N}_Y \cong \mathcal{O}(3, 2, 2, 2).$$

From the exact sequence (83), we obtain

$$(85) \qquad\qquad\qquad c(\mathcal{T}_Y) = \frac{c(\mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_Y}{c(\mathcal{N}_Y)}.$$

We have

$$(86) \qquad c(\mathcal{T}_{\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_Y = (1 + 3x + 3x^2)(1 + 2y)(1 + 2z)(1 + 2w)|_Y,$$

and

$$(87) \qquad\qquad\qquad c(\mathcal{N}_Y) = 1 + 3x + 2y + 2z + 2w.$$

From equations (85), (86), and (87), we can compute $c(\mathcal{T}_Y)$. The top Chern class of $c(\mathcal{T}_Y)$ gives the Euler characteristic of $(3,2,2,2)$ Calabi–Yau hypersurface Y. Therefore, we find that

$$(88) \qquad\qquad\qquad \chi(Y) = 1584,$$

and

$$(89) \qquad\qquad\qquad \frac{\chi(Y)}{24} = 66.$$

We also obtain the second Chern class $c_2(Y)$ from (85):

$$(90) \qquad c_2(Y) = (3x^2 + 6xy + 6xz + 6xw + 4yz + 4zw + 4wy)|_Y.$$

From this, we see that the second Chern class $c_2(Y)$ is not even.

From (88), we obtain the net number of 3-branes N_3 needed to cancel the tadpole as:

$$(91) \qquad N_3 = \frac{\chi(Y)}{24} - \frac{1}{2} G_4 \cdot G_4 = 66 - \frac{1}{2} G_4 \cdot G_4.$$

This must be a non-negative integer, and we therefore obtain a numerical bound on the self-intersection of a four-form flux G_4:

$$(92) \qquad 132 \geq G_4 \cdot G_4.$$

Notice that the result (88) of the Euler characteristic is valid for both the Fermat-type hypersurface and the hypersurface in Hesse form.

4.3.2. Double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified along a multi-degree $(4, 4, 4, 4)$ 3-fold. We compute the Euler characteristic of double cover Y of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a $(4,4,4,4)$ 3-fold B. The Euler characteristic $\chi(Y)$ of a double cover Y is given by

$$(93) \qquad \chi(Y) = 2 \cdot \chi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) - \chi(B).$$

We have

$$(94) \qquad \chi(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 2^4 = 16,$$

therefore

$$(95) \qquad \chi(Y) = 32 - \chi(B).$$

We use the exact sequence:

$$(96) \qquad 0 \longrightarrow \mathcal{T}_B \longrightarrow \mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}|_B \longrightarrow \mathcal{N}_B \longrightarrow 0$$

to obtain the equality

$$(97) \qquad c(\mathcal{T}_B) = \frac{c(\mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_B}{c(\mathcal{N}_B)}.$$

$$(98) \qquad \mathcal{N}_B \cong \mathcal{O}(4, 4, 4, 4),$$

therefore

$$(99) \qquad c(\mathcal{N}_B) = 1 + 4x + 4y + 4z + 4w.$$

We have

(100) $\qquad c(\mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1})|_B = (1 + 2x)(1 + 2y)(1 + 2z)(1 + 2w)|_B.$

From the equality (97), we can compute $c(B)$. The top Chern class of $c(B)$ gives the Euler characteristic $\chi(B)$. Therefore, we deduce that

(101) $\qquad\qquad\qquad\qquad \chi(B) = -3712.$

We finally obtain the Euler characteristic $\chi(Y)$:

(102) $\qquad\qquad \chi(Y) = 32 - \chi(B) = 32 - (-3712) = 3744.$

This is divisible by 24:

(103) $\qquad\qquad\qquad\qquad \dfrac{\chi(Y)}{24} = 156.$

The net number of 3-branes N_3 needed to cancel the tadpole is

(104) $\qquad\qquad N_3 = \dfrac{\chi(Y)}{24} - \dfrac{1}{2} G_4 \cdot G_4 = 156 - \dfrac{1}{2} G_4 \cdot G_4.$

N_3 must be a non-negative integer, and therefore, a bound on the self-intersection of four-form flux G_4 that we obtain is

(105) $\qquad\qquad\qquad\qquad 312 \geq G_4 \cdot G_4.$

5. Matter spectra and Yukawa couplings

We discuss matter fields arising on discriminant components and along matter curves in F-theory compactifications on constructed genus-one fibered Calabi–Yau 4-folds. As discussed in [5], suppose gauge group G on 7-branes breaks to a subgroup Γ such that

(106) $\qquad\qquad\qquad\qquad \Gamma \times H \subset G$

is maximal. This corresponds to the deformation of singularity associated with gauge group G, and consequently, matter fields arise on 7-branes [39]. When $\Gamma \times H$ has a representation (τ, T), matter fields arise in representation

τ of Γ, and its generation is given by [5]

$$(107) \qquad n_\tau - n_{\tau^*} = -\int_S c_1(S)c_1(\mathcal{T}).$$

S denotes a component of the discriminant locus on which 7-branes are wrapped, and $\mathcal{T}$ denotes a bundle transforming in representation T of H. We consider the case in which H is $U(1)$. Let $\mathcal{L}$ be a supersymmetric line bundle on component S.

We discuss matter contents in F-theory compactifications on families of $(3,2,2,2)$ hypersurfaces in Hesse form and double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a multidegree $(4,4,4,4)$ 3-fold below. We focus on specific discriminant components whose forms are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Supersymmetric line bundles on these components are isomorphic to $\mathcal{O}(a,b)$ for some integers a and b, $a, b \in \mathbb{Z}$; for line bundles $\mathcal{O}(a,b)$ to be supersymmetric, the integers a and b are subject to the condition $ab < 0$ [5].

As discussed in [5], Yukawa couplings arise from the following three cases:

- interaction of three matter fields on a bulk component

- interaction of a field on a bulk component and two matter fields localized along a matter curve, and

- triple intersection of three matter curves meeting in a point

Components we consider below have forms isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which is a Hirzebruch surface. Therefore, Yukawa coupling does not arise from the first case [5]. We consider Yukawa couplings arising from the second case.

As stated in Section 4, the existence of a consistent four-form flux is undetermined for Calabi–Yau genus-one fibrations constructed in this note. We can only say that matter contents and Yukawa couplings that we obtain below *could* arise.

5.1. Matter spectra for $(3, 2, 2, 2)$ hypersurfaces in Hesse form

We compute matter spectra in F-theory compactifications on $(3,2,2,2)$ hypersurfaces in Hesse Form. We focus on component A_1, and we consider the extreme case in which all six components $\{A_i\}_{i=1}^6$ are coincident. A_1 is abbreviated to A below. In this case, singular fibers on the bulk A have type I_6, and the $SU(6)$ gauge group arises on the 7-branes wrapped on A. The form of A is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

When $SU(6)$ breaks to $SU(5)$ with

$$(108) \qquad\qquad SU(6) \supset SU(5) \times U(1),$$

the adjoint $\mathbf{35}$ of $SU(6)$ decomposes as [83]:

$$(109) \qquad\qquad \mathbf{35} = \mathbf{24}_0 + \mathbf{5}_6 + \overline{\mathbf{5}}_{-6} + \mathbf{1}_0.$$

Therefore, matter fields $\mathbf{5}$ (could) arise on the bulk A. The generation of matter fields $\mathbf{5}$ on the bulk A is given by:

$$(110) \qquad\qquad n_{\mathbf{5}} - n_{\overline{\mathbf{5}}} = -\int_A c_1(A)c_1(\mathcal{L}^6) = -12(a+b).$$

$A \cap B_i = \Sigma_1$, $i = 1, 2, 3, 4$, and therefore, the bulk A contains four matter curves Σ_1, which are genus-one curves. When the supersymmetric line bundle $\mathcal{L}$ is turned on, $\mathbf{20}$ of $SU(6)$ along matter curve Σ_1 decomposes as

$$(111) \qquad\qquad \mathbf{20} = \mathbf{10}_{-3} + \overline{\mathbf{10}}_3.$$

Therefore, the mater fields $\mathbf{10}$ could localize along a matter curve Σ_1.

Since matter curve $A \cap B_i = \Sigma_1$ is a bidegree $(2,2)$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$, the restriction $\mathcal{L}_{\Sigma_1}$ of the line bundle $\mathcal{L} \cong \mathcal{O}(a,b)$ to matter curve $A \cap B_i = \Sigma_1$ is

$$(112) \qquad\qquad \mathcal{L}_{\Sigma_1} \cong \mathcal{O}_{\Sigma_1}(V)$$

for some divisor V with $\deg V = 2(a+b)$. We have

$$(113) \qquad\qquad n_{\mathbf{10}} = h^0(K_{\Sigma_1}^{1/2} \otimes \mathcal{L}_{\Sigma_1}^{-3}) = h^0(\mathcal{O}_{\Sigma_1}(-3V)).$$

Similarly, we have

$$(114) \qquad\qquad n_{\overline{\mathbf{10}}} = h^0(3V).$$

By the Riemann–Roch theorem,

$$(115) \qquad\qquad n_{\mathbf{10}} - n_{\overline{\mathbf{10}}} = \deg(-3V) = -6(a+b).$$

Therefore, when $a + b > 0$ mater fields $\overline{\mathbf{5}}_{-6}$ arise on the bulk A, and matter fields $\overline{\mathbf{10}}_3$ localize along matter curve Σ_1. For this case, Yukawa

coupling that arises is

$$(116) \qquad \overline{\mathbf{5}}_{-6} \cdot \overline{\mathbf{10}}_3 \cdot \overline{\mathbf{10}}_3.$$

When $a + b < 0$, matter fields $\mathbf{5}_6$ arise on the bulk A, and matter fields $\mathbf{10}_{-3}$ localize along matter curve Σ_1. Yukawa coupling for this case is

$$(117) \qquad \mathbf{5}_6 \cdot \mathbf{10}_{-3} \cdot \mathbf{10}_{-3}.$$

The results are shown in Table 10 below.

(3,2,2,2) Calabi–Yau hypersurface in Hesse form has a 3-section, therefore F-theory compactification on it has a discrete $\mathbb{Z}_3$ symmetry [28, 33, 84]. Thus, massless fields are charged under a discrete $\mathbb{Z}_3$ symmetry; Yukawa coupling has to be invariant under the action of $\mathbb{Z}_3$ [29]. We confirm that Yukawa couplings (116) and (117) indeed satisfy this requirement.

Gauge Group	$a + b$	Matter on A	# Gen. on A	Matter on Σ_1	# Gen. on Σ_1	Yukawa
$SU(6)$	> 0	$\overline{\mathbf{5}}$	$12(a + b)$	$\overline{\mathbf{10}}$	$6(a + b)$	$\overline{\mathbf{5}} \cdot \overline{\mathbf{10}} \cdot \overline{\mathbf{10}}$
	< 0	$\mathbf{5}$	$-12(a + b)$	$\mathbf{10}$	$-6(a + b)$	$\mathbf{5} \cdot \mathbf{10} \cdot \mathbf{10}$

Table 10: Potential matter spectra for hypersurface in Hesse form.

5.2. Matter spectra for double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a multidegree $(4, 4, 4, 4)$ 3-fold

We compute matter spectra in F-theory compactifications on double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a multidegree (4,4,4,4) 3-fold (21).

When A_1 is not coincident with any other A_i, $i \neq 1$, singular fibers on A_1 have type III, and $SU(2)$ gauge groups arise on the 7-branes wrapped on A_1. For this situation, matter does not arise on the 7-branes wrapped on A_1.

When A_1 is coincident with another A_i, say $A_1 = A_2$, $SO(7)$ gauge group arises on the 7-branes wrapped on A_1. A_1 is abbreviated to A. When gauge group $SO(7)$ breaks to $USp(4)$ under

$$(118) \qquad SO(7) \supset USp(4) \times U(1),$$

$\mathbf{21}$ of $SO(7)$ decomposes as

$$(119) \qquad \mathbf{21} = \mathbf{10}_0 + \mathbf{5}_2 + \mathbf{5}_{-2} + \mathbf{1}_0.$$

Therefore, matter fields $\mathbf{5}$ (could) arise on the bulk A. The generations of $\mathbf{5}$ on the bulk A is given by:

$$(120) \qquad n_{\mathbf{5}_2} - n_{\mathbf{5}_{-2}} = -\int_A c_1(A)c_1(\mathcal{L}^2) = -4(a+b).$$

$A \cap B_i = \Sigma_9$, $i = 1, 2$, and therefore, the bulk A contains 2 matter curves[13] Σ_9 of genus 9. $\mathbf{8}$ of $SO(7)$ decomposes under (118) as

$$(121) \qquad \mathbf{8} = \mathbf{4}_1 + \mathbf{4}_{-1}.$$

Therefore, the matter fields $\mathbf{4}$ (could) localize along matter curves Σ_9. Since f and g are bidegree (4,4) polynomials, the restriction $\mathcal{L}_{\Sigma_9}$ of the line bundle $\mathcal{L}$ to the matter curve Σ_9 has degree $4(a+b)$. The degree of the canonical bundle K_{Σ_9} is $2g - 2 = 16$. Let W be the divisor associated with the line bundle $K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}$, so that $\mathcal{O}_{\Sigma_9}(W) = K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}$. The degree of W is $8 + 4(a+b)$. Now, by the Riemann–Roch theorem,

$$
\begin{aligned}
(122) \qquad n_{\mathbf{4}_1} - n_{\mathbf{4}_{-1}} &= h^0(W) - h^0(K_{\Sigma_9} - W) \\
&= \deg W + 1 - 9 \\
&= 4(a+b).
\end{aligned}
$$

Therefore, we have

$$(123) \qquad n_{\mathbf{5}_2} - n_{\mathbf{5}_{-2}} = -(n_{\mathbf{4}_1} - n_{\mathbf{4}_{-1}}).$$

When $a + b > 0$, matter fields $\mathbf{5}_{-2}$ arise on the bulk A, and matter fields $\mathbf{4}_1$ localize along matter curves Σ_9. Yukawa coupling that arises is

$$(124) \qquad \mathbf{5}_{-2} \cdot \mathbf{4}_1 \cdot \mathbf{4}_1.$$

When $a + b < 0$, matters $\mathbf{5}_2$ arise on the bulk A, and matter fields $\mathbf{4}_{-1}$ localise along matter curves Σ_9. Yukawa coupling for this case is

$$(125) \qquad \mathbf{5}_2 \cdot \mathbf{4}_{-1} \cdot \mathbf{4}_{-1}.$$

Next, we consider the case in which component A_1 is coincident with two other components. Then, singular fiber on A_1 are enhanced to type

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[13]There are only two matter curves Σ_9, $A \cap B_1$ and $A \cap B_2$, in component A; triple intersection of matter curves in bulk A does not occur for double covers (21).

III^*, and E_7 gauge group arises on the 7-branes wrapped on A_1. We again abbreviate component A_1 to A. When E_7 breaks to E_6 under

$$(126) \qquad E_7 \supset E_6 \times U(1),$$

$\mathbf{133}$ of E_7 decomposes as

$$(127) \qquad \mathbf{133} = \mathbf{78}_0 + \mathbf{27}_2 + \overline{\mathbf{27}}_{-2} + \mathbf{1}_0.$$

Therefore, matter fields $\mathbf{27}$ (could) arise on component A. The generations of $\mathbf{27}$ on the bulk A is given by:

$$(128) \qquad n_{\mathbf{27}} - n_{\overline{\mathbf{27}}} = - \int_A c_1(A) c_1(\mathcal{L}^2) = -4(a+b).$$

Bulk A contains two matter curves Σ_9 of genus 9, $A \cap B_i = \Sigma_9$, $i = 1, 2$. $\mathbf{56}$ of E_7 decomposes under (126) as

$$(129) \qquad \mathbf{56} = \mathbf{27}_{-1} + \overline{\mathbf{27}}_1 + \mathbf{1}_3 + \mathbf{1}_{-3}.$$

Therefore, matter fields $\mathbf{27}$ localize along the matter curves Σ_9. The restriction $\mathcal{L}_{\Sigma_9}$ of the line bundle $\mathcal{L}$ to matter curve Σ_9 has degree $4(a+b)$. Let W be the divisor associated with the line bundle $K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}^{-1}$, so that $\mathcal{O}_{\Sigma_9}(W) = K_{\Sigma_9}^{1/2} \otimes \mathcal{L}_{\Sigma_9}^{-1}$. By applying the Riemann–Roch theorem, we find that the generation of $\mathbf{27}$ along matter curve Σ_9 is given by:

$$(130) \qquad n_{\mathbf{27}} - n_{\overline{\mathbf{27}}} = h^0(W) - h^0(K_{\Sigma_9} - W) = -4(a+b).$$

When $a + b > 0$, matter fields $\overline{\mathbf{27}}$ arise on the bulk A, and along matter curves Σ_9. Yukawa coupling that arises is

$$(131) \qquad \overline{\mathbf{27}}_{-2} \cdot \overline{\mathbf{27}}_1 \cdot \overline{\mathbf{27}}_1.$$

When $a + b < 0$, matter fields $\mathbf{27}$ arise on the bulk A, and along matter curves Σ_9. Yukawa coupling for this case is

$$(132) \qquad \mathbf{27}_2 \cdot \mathbf{27}_{-1} \cdot \mathbf{27}_{-1}.$$

Double cover (21) has a bisection, and F-theory compactification on it has a discrete $\mathbb{Z}_2$ symmetry [26, 84]. Massless fields are charged under a discrete $\mathbb{Z}_2$ symmetry, and Yukawa coupling has to be invariant under the $\mathbb{Z}_2$

action. We confirm that Yukawa couplings (124), (125), (131), (132) satisfy this requirement.

The results are shown in Table 11 below.

Gauge Group	$a + b$	Matter on A	# Gen. on A	Matter on Σ_9	# Gen. on Σ_9	Yukawa
E_7	> 0	$\overline{\mathbf{27}}$	$4(a+b)$	$\overline{\mathbf{27}}$	$4(a+b)$	$\overline{\mathbf{27}} \cdot \overline{\mathbf{27}} \cdot \overline{\mathbf{27}}$
	< 0	$\mathbf{27}$	$-4(a+b)$	$\mathbf{27}$	$-4(a+b)$	$\mathbf{27} \cdot \mathbf{27} \cdot \mathbf{27}$
$SO(7)$	> 0	$\mathbf{5}$	$4(a+b)$	$\mathbf{4}$	$4(a+b)$	$\mathbf{5} \cdot \mathbf{4} \cdot \mathbf{4}$
	< 0	$\mathbf{5}$	$-4(a+b)$	$\mathbf{4}$	$-4(a+b)$	$\mathbf{5} \cdot \mathbf{4} \cdot \mathbf{4}$

Table 11: Potential matter spectra for double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (21).

6. Conclusions

We considered (3,2,2,2) hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, and double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ramified over a (4,4,4,4) 3-fold, to construct genus-one fibered Calabi–Yau 4-folds. By considering specific types of equations, we constructed two families of (3,2,2,2) hypersurfaces, namely Fermat-type hypersurfaces and hypersurfaces in Hesse form. For double covers, we considered a family described by specific types of equations:

$$(133) \qquad\qquad \tau^2 = f \cdot a(t) \cdot x^4 + g \cdot b(t).$$

We showed that these three families of genus-one fibered Calabi–Yau 4-folds lack a global section. Genus-one fibers of Fermat-type (3,2,2,2) hypersurfaces and double covers (133) possess complex multiplication of specific orders, 3 and 4, respectively, and these symmetries enabled a detailed study of the gauge theories in F-theory compactifications.

We determined the discriminant loci of these families, and we specified the forms of the discriminant components and their intersections. In particular, discriminant components contain matter curves.

$SU(3)$ gauge groups generically arise on 7-branes wrapped on discriminant components in F-theory compactifications on Fermat-type (3,2,2,2) hypersurfaces; when 7-branes coincide, the gauge symmetry is enhanced to E_6. Only gauge groups of the form $SU(N)$ arise on 7-branes in F-theory compactifications on (3,2,2,2) hypersurfaces in Hesse form. $SU(2)$ gauge groups generically arise on 7-branes in F-theory compactifications on double covers of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (133). When 7-branes coincide, the $SU(2)$ gauge

group is enhanced to $SO(7)$; when more 7-branes coincide, gauge group is enhanced to E_7.

We specified the Mordell–Weil groups of Jacobian fibrations of specific Fermat-type hypersurfaces and specific double covers. They are $\mathbb{Z}_3$ and $\mathbb{Z}_2$, such the Mordell–Weil groups have the rank 0, and F-theory compactifications on these specific Calabi–Yau genus-one fibrations do not have $U(1)$ gauge symmetry.

We computed the potential matter spectra and potential Yukawa couplings on specific components. We did not discuss the existence of a consistent four-form flux in this note. We computed the Euler characteristics of Calabi–Yau 4-folds constructed in this note, in order to derive the conditions imposed on four-form fluxes to cancel the tadpole.

Acknowledgments

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