Cut-and-join description of generalized Brezin–Gross–Witten model

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We investigate the Brezin–Gross–Witten model, a tau-function of the KdV hierarchy, and its natural one-parameter deformation, the generalized Brezin–Gross–Witten tau-function. In particular, we derive the Virasoro constraints, which completely specify the partition function. We solve them in terms of the cut-and-join operator. The Virasoro constraints lead to the loop equations, which we solve in terms of the correlation functions. Explicit expressions for the coefficients of the tau-function and the free energy are derived, and a compact formula for the genus zero contribution is conjectured. A family of polynomial solutions of the KdV hierarchy, given by the Schur functions, is obtained for the half-integer values of the parameter. The quantum spectral curve and its classical limit are discussed.

1. Introduction

The Brezin–Gross–Witten (BGW) model

$$(1) \qquad Z_{BGW} = \int [dU] \, e^{\frac{1}{\hbar} \text{Tr} \, (A^\dagger U + A U^\dagger)}$$

was introduced in the lattice gauge theory over 35 years ago [1, 2]. Later it was shown that in the weak coupling phase this model satisfies the Virasoro constraints [3]. Moreover, it is a tau-function of the KdV integrable hierarchy and can be described by the generalized Kontsevich model [4].

This makes the BGW model interesting and, in many respects, similar to the Kontsevich-Witten tau-function [5, 6] – one of the most important and beautiful ingredients of the modern mathematical physics. However, unlike the Kontsevich–Witten (KW) tau-function, which generates the intersection numbers of the moduli spaces or Riemann surfaces, and many other matrix models, for which enumerative geometry/combinatorics interpretation is known, similar interpretation of the BGW tau-function is still not available. Using the generalized Kontsevich model description of this tau-function, one can try to identify it with the generating function of the r-spin intersection numbers for $r = -2$. However, corresponding geometrical construction is not available yet, thus, it is impossible to compare the intersection numbers with the correlation functions of the matrix model.

In spite of this absence of geometrical interpretation, the BGW tau-function is known to play (similarly to the KW tau-function) an important role in the topological recursion/Givental decomposition [7–11]. Namely, it appears in decomposition of the complex matrix model [12–14] and, in general, corresponds to the hard walls (see [15] and references therein).

Recently, it was shown that a natural one parametric deformation of the KW tau-function, called the Kontsevich–Penner model, describes open intersection numbers[16–18], a new and extremely interesting set of enumerative geometry invariants, which was introduced in [19, 20]. The matrix integral description allows us to show that their generating function is a tau-function of the modified KP (MKP) hierarchy, and to construct a full family of the Virasoro and W-constraints. This model possess a number of nice properties and, arguably, is even more beautiful and natural then any of its specifications (in particular, the KW tau-function).

Thus, to find a natural interpretation of the BGW tau-function one can try to consider its deformation, analogous to the Kontsevich–Penner deformation of the KW tau-function. It is easy to construct this deformation using

the generalized Kontsevich model representation. In this representation it corresponds to the logarithmic deformation of the potential. This deformed model was introduced in [4] and is given by the matrix integral

$$
(2) \qquad \tau_N \sim \int [d\Phi] \exp\left(\mathrm{Tr}\left(\frac{\Lambda^2 \Phi}{\hbar} + \frac{1}{\hbar \Phi} + (N - M)\log \Phi \right) \right).
$$

From the general properties of the generalized Kontsevich model (GKM) [21] it follows that it is a tau-function of the MKP hierarchy with discrete time N. However, other properties of this model have not been investigated in detail so far. In particular, the Virasoro constraints were not known. The main goal of this paper is to fill this gap and to describe the generalized BGW model (2) and its interesting specifications, in particular the original BGW model.

We show that the tau-function (2) is well definite for any complex (not necessarily integer!) value of N. Moreover, for any given value of N this is a tau-function of the KdV hierarchy. We describe the Kac–Schwarz algebra for this tau-function and derive the Virasoro constraints. Here the difference with the Kontsevich–Penner model is quite transparent: to describe the Kontsevich–Penner model one should introduce higher W-constraints, while the partition function of the generalized BGW model is completely fixed by the Virasoro constraints. Moreover, only the first of them (the string equation) depends on N, thus, on the level of linear constraints, the case of general N is almost as simple as the case with $N = 0$.

Often the Virasoro and W-constraints can be solved in terms of the cut-and-join operator. Corresponding method was introduced in [22] for the Gaussian branch of the Hermitian matrix model and later has been applied to the KW tau-function [23] and to the Kontsevich–Penner model [17, 18]. We solve the Virasoro constraints for the BGW and generalized BGW tau-functions in terms of the cut-and-join operator:

$$
(3) \qquad \tau_N = e^{\hbar \widehat{W}_N} \cdot 1,
$$

where

$$
\begin{aligned}
(4) \qquad \widehat{W}_N ={}& \frac{1}{2} \sum_{k,m=0}^{\infty} (2k+1)(2m+1) t_{2k+1} t_{2m+1} \frac{\partial}{\partial t_{2k+2m+1}} \\
&+ \frac{1}{4} \sum_{k,m=0}^{\infty} (2k+2m+3) t_{2k+2m+3} \frac{\partial^2}{\partial t_{2k+1} \partial t_{2m+1}} \\
&+ \left(\frac{1}{16} - \frac{N^2}{4} \right) t_1.
\end{aligned}
$$

Using this operator we derive the coefficients of expansion of the tau-function and free energy. Here we see that the case of generalized BGW tau-function is much more interesting comparing to the original BGW tau-function. In particular, while for the BGW tau-function the genus zero contribution to the free energy is equal to zero (and higher genera contributions are rational functions of only finite number of times), for general N this is not the case. Namely, for any genus the free energy is a non-trivial function of all times. The results of computations allow us to conjecture a compact expression for the genus zero free energy of the generalized BGW tau-function.

We also derive an equation for the quantum spectral curve of the generalized BGW tau-function,

$$(5) \qquad \left(\hbar^2 x^2 \frac{\partial^2}{\partial x^2} + \hbar^2 x \frac{\partial}{\partial x} - x - \frac{S^2}{4} \right) \Psi_S(x) = 0,$$

where $S = \hbar^{-1} N$. As for other KP/Toda tau-functions, which describe the enumerative geometry invariants, the equation for the quantum spectral curve, up to a conjugation, coincides with one of the Kac–Schwarz operators [17, 24, 25]. In the classical limit we get a genus zero spectral curve with one branch point.

The Virasoro constraints allow us to derive the loop equations and to solve them recursively. The correlation functions are defined on the spectral curve and they are symmetric polynomials in the inverse global coordinate. Thus, corresponding differentials are meromorphic with poles only at the branch point.

For the half-integer values of the parameter N, the generalized BGW tau-function is a polynomial in times. More specifically, it is given by the Schur functions of the dilaton shifted times, labelled by the triangular partitions. We describe this family of the KdV tau-functions (which constitute an infinite MKP tau-function) in detail.

All this allows us to conclude that, as in the case of the Kontsevich–Penner model, the deformed model appears to be more beautiful and natural then the original one. Unfortunately, a unitary integral representation of this deformed model is not known, and we do not expect that this model is directly related to the original lattice gauge models. However, some of our results (in particular, the cut-and-join representation) should be useful for the original BGW model. Moreover, from the Virasoro constraints derived in Section 3.1 it follows that the generalized BGW model describes a model

of the open-closed string theory involving gravity[26], which can be obtained from the unitary matrix model in a double scaling limit [27].

The present paper is organized as follows. In Section 2 we consider the original BGW model and, basically following [4], describe it in terms of the GKM. Section 3 is devoted to the generalized BGW tau-function. In the Appendices we present explicit expressions for expansion of the tau-function and free energy of BGW and generalized BGW tau-functions.

2. Brezin–Gross–Witten model

The partition function of the BGW model [1, 2] is given by an $M \times M$ unitary matrix integral

$$(6) \qquad Z_{BGW} = \int [dU] \, e^{\frac{1}{\hbar} \mathrm{Tr}\,(A^\dagger U + A U^\dagger)}.$$

Here the Haar measure on the unitary group $U(M)$ is normalised by $\int [dU] = 1$ and the parameter $\hbar$ describes the topological expansion (see below). Naively, (6) depends on two external matrices, A and $A^\dagger$, but actually it depends only on their product, more precisely on the square root of it

$$(7) \qquad \Lambda := \left(A^\dagger A\right)^{\frac{1}{2}}.$$

The behaviour of this matrix model is essentially different at large and small values of $\hbar^{-1} \mathrm{Tr}\,\Lambda^{-1}$ and there is a phase transition between these two regimes [1, 2, 28]. In this paper we consider only the so-called Kontsevich (weak coupling) phase, which corresponds to the large values of the eigenvalues of the matrix Λ. Below for simplicity we assume that the matrix Λ is diagonal

$$(8) \qquad \Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_M).$$

2.1. Description in terms of generalized Kontsevich model

As many other important matrix models, the BGW model can be described in terms of the generalized Kontsevich model [21]. Namely, as it was shown

by A. Mironov, A. Morozov and G. W. Semenoff in [4],

$$(9) \qquad Z_{BGW} = \frac{\int [d\Phi] \exp\left(\mathrm{Tr}\left(\frac{\Lambda^2\Phi}{\hbar} + \frac{1}{\hbar\Phi} - M\log\Phi\right)\right)}{\int [d\Phi] \exp\left(\mathrm{Tr}\left(\frac{1}{\hbar\Phi} - M\log\Phi\right)\right)}.$$

In this section we basically follow the approach of [4].

Actually, (9) as well as (6) depends only on the ratio $\Lambda/\hbar$, thus it is convenient to introduce

$$(10) \qquad \tilde{\Lambda} := \frac{\Lambda}{\hbar} = \mathrm{diag}\left(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M\right),$$

and $\tilde{\lambda}_i = \lambda_i/\hbar$.

In (9) we integrate over $M \times M$ normal matrices, that is diagonolizable matrices

$$(11) \qquad \Phi = U \mathrm{diag}\left(\phi_1, \ldots, \phi_M\right) U^\dagger, \qquad \phi_i \in \gamma,$$

where U is unitary and the contour γ runs from $-\infty$ to a small circle enclosing zero, and then returning to $-\infty$. Then the measure of integration can be expressed in terms of U and ϕ_i's in the standard way

$$(12) \qquad [d\Phi] = \Delta(\phi)^2 [dU] \prod_{i=1}^{M} d\phi_i,$$

where

$$(13) \qquad \Delta(\phi) = \prod_{i<j}(\phi_j - \phi_i)$$

is the Vandermonde determinant.

After integration over the unitary matrix U with the help of the HCIZ formula, (9) reduces to

$$(14) \qquad Z_{BGW} = (-1)^{\frac{M(M-1)}{2}} \prod_{j=1}^{M}(j-1)! \, \frac{\det_{i,j=1}^{M}\left(\tilde{\lambda}_j^{M-i} I_{M-i}(2\tilde{\lambda}_j)\right)}{\Delta(\tilde{\lambda}^2)}.$$

Here

$$(15) \qquad I_\nu(x) = \left(\frac{2}{x}\right)^\nu \frac{1}{2\pi i} \int_\gamma e^{\frac{x^2\phi}{4} + \frac{1}{\phi}} \frac{d\phi}{\phi^{\nu+1}}$$

is the modified Bessel function and the normalization of (14) can be easily found from its small x expansion

$$(16) \qquad I_\nu(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu (1 + O(x)).$$

From this eigenvalue integral representation it immediately follows that in the Kontsevich phase

$$(17) \qquad \tau_{BGW}(\Lambda) = \mathcal{C}_{BGW}^{-1} Z_{BGW},$$

where

$$(18) \qquad \mathcal{C}_{BGW} = \frac{e^{2\operatorname{Tr}\tilde\Lambda} \prod_{i=1}^M (j-1)!}{(2\pi)^{\frac{M}{2}} \det\left(\tilde\Lambda \otimes 1 + 1 \otimes \tilde\Lambda\right)^{\frac{1}{2}}},$$

is a tau-function of the KP hierarchy. Indeed,

$$(19) \qquad \tau_{BGW}(\Lambda) = \frac{\det_{i,j=1}^M \Phi_j(\lambda_i)}{\Delta(\lambda)},$$

which defines a tau-function in the Miwa parametrization

$$(20) \qquad t_k = \frac{1}{k} \operatorname{Tr} \Lambda^{-k}.$$

Here Φ_j's are the so called basis vectors, which can be expressed in terms of the modified Bessel functions (15),

$$(21) \qquad \Phi_j(\lambda) = \sqrt{4\pi\tilde\lambda}\, \lambda^{j-1} e^{-2\tilde\lambda} I_{j-1}(2\tilde\lambda)$$
$$= \frac{\sqrt{4\pi\tilde\lambda}}{2\pi i} \hbar^{j-1} e^{-2\tilde\lambda} \int_\gamma e^{\tilde\lambda^2 t + \frac{1}{t}} \frac{dt}{t^j}.$$

We consider only the asymptotic expansion of the modified Bessel function for large values of λ (we assume that $\arg \lambda \neq \pi$)

$$(22) \qquad \Phi_j(\lambda) = \lambda^{j-1} \left(1 + \sum_{k=1}^\infty \frac{(-\hbar)^k}{\lambda^k} \frac{a_k(j)}{16^k\, k!}\right),$$

where

$$(23) \quad a_k(j) = (4(j-1)^2 - 1^2)(4(j-1)^2 - 3^2) \cdots (4(j-1)^2 - (2k-1)^2),$$

thus, $\Phi_j(\lambda)$'s are of the form

$$(24) \qquad\qquad \Phi_j(\lambda) = \lambda^{j-1}(1 + O(\lambda^{-1})).$$

This guarantees that

$$(25) \qquad\qquad \tau_{BGW}(\Lambda) = 1 + O(\lambda_j^{-1}).$$

Vectors (21) are defined for all $j \in \mathbb{Z}$. Vectors for $j \geq 1$ define a point of the big cell of the Sato Grassmannian [29–31][1]

$$(26) \qquad\qquad \mathcal{W}_{BGW} = \langle \Phi_1, \Phi_2, \Phi_3, \dots \rangle.$$

Any such point corresponds to a tau-function of the KP hierarchy, which is a formal series in the times t_k and solves the bilinear identity

$$(27) \qquad\qquad \oint_\infty e^{\xi(\mathbf{t}-\mathbf{t}',z)}\, \tau(\mathbf{t} - [z^{-1}], \hbar)\, \tau(\mathbf{t}' + [z^{-1}], \hbar) dz = 0.$$

Here $\xi(\mathbf{t}, z) = \sum_{k=1}^\infty t_k z^k$ and we use the standard notation

$$(28) \qquad\qquad \mathbf{t} \pm [z^{-1}] = \left\{ t_1 \pm \frac{1}{z}, t_2 \pm \frac{1}{2z^2}, t_3 \pm \frac{1}{3z^3}, \dots \right\}.$$

Thus, the BGW tau-function

$$(29) \qquad\qquad \tau_{BGW}(\mathbf{t}, \hbar)$$

is defined by the point (26), or equivalently, it can be considered as a limit of the ration of determiants (19) as the size of the matrices M tends to infinity. In this limit all the Miwa variables (20) are independent.

In the Sato Grassmannian description the first basis vector plays a special role. It is related to the tau-function by

$$(30) \qquad\qquad \Phi_1(\lambda) = \tau([\lambda^{-1}], \hbar),$$

and is equal to the dual Baker–Akhiezer function at $\mathbf{t} = 0$.

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[1] In this paper we consider only the index (or charge) zero sector of the Sato Grassmannian, thus all points corresponding to the different values of the discrete time are described in the same space. Equivalent description should include a flag of the Sato Grassmannians with different indices.

It is clear that the parameter $\hbar$ is not independent and can be removed by the time variables rescaling

$$(31) \qquad \tau_{BGW}(\mathbf{t}, \hbar) = \tau_{BGW}(\mathbf{t}, 1)\Big|_{t_k = \hbar^k t_k}.$$

Let us stress that the expansion of $\tau_{BGW}(\mathbf{t}, \hbar)$ in $\hbar$ is not the genus expansion, but the topological expansion. More concretely,

$$(32) \qquad \tau_{BGW}(\mathbf{t}, \hbar) = \exp\left(\sum_{g=0}^{\infty}\sum_{n=1}^{\infty} \hbar^{-\chi} \mathcal{F}_{g,n}(\mathbf{t})\right),$$

where $\chi = 2 - 2g - n$ can be considered as the Euler characteristic. Here $\mathcal{F}_{g,n}(\mathbf{t})$ is a genus g contribution to free energy, which is a homogeneous polynomial in times t_k of degree n,

$$(33) \qquad \sum_{k=0}^{\infty} t_k \frac{\partial}{\partial t_k} \mathcal{F}_{g,n}(\mathbf{t}) = n\,\mathcal{F}_{g,n}(\mathbf{t}).$$

To get the genus expansion, one should multiply the times by $\hbar^{-1}$:

$$(34) \qquad \tau_{BGW}(\hbar^{-1}\mathbf{t}, \hbar) = \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(\mathbf{t})\right).$$

$\mathcal{F}_g(\mathbf{t})$ is the genus g contribution to the free energy and

$$(35) \qquad \mathcal{F}_g(\mathbf{t}) = \sum_{n=1}^{\infty} \mathcal{F}_{g,n}(\mathbf{t}).$$

It is known [14, 32] that

$$(36) \qquad \mathcal{F}_0 = 0, \mathcal{F}_1 = -\frac{1}{8}\log\left(1 - \frac{t_1}{2}\right),$$

and for $g > 1$ all $\mathcal{F}_g$ are polynomials in the variables

$$(37) \qquad T_k = \frac{t_k}{(2 - t_1)^k}.$$

Variables T_k are the "moment variables" and expressions for $\mathcal{F}_k(\mathbf{T})$ for small k were obtained in [14, 32]. With the help of the cut-and-join description of Section 2.3 we are able to find expressions for $\mathcal{F}_g(\mathbf{T})$ for $g \leq 30$. See Appendix A for the expressions of $\mathcal{F}_g(\mathbf{T})$ for $g \leq 9$.

2.2. KdV hierarchy and Virasoro constraints

It is well-known that the tau-function $\tau_{BGW}(\mathbf{t}, \hbar)$ does not depend on even times t_{2k} [32]. Thus, it is a tau-function of the 2-reduction of the KP hierarchy, which is the KdV hierarchy [4]. Probably the simplest way to show it is to use the Sato Grassmannian description and the Kac–Schwarz operators [33] as it was done in [4].

The Kac–Schwarz (KS) operators [4, 24, 33–36] are the differential operators in one variable which stabilize the point of the Sato Grassmannian for a given tau-function. For any tau-function the corresponding KS operators constitute an algebra (a subalgebra in $w_{1+\infty}$). Thus, for any KS operator we can use a correspondence between the $w_{1+\infty}$ and $W_{1+\infty}$ algebras [33, 34, 36, 37] to construct an operator from $W_{1+\infty}$, which annihilates the tau-function.

Let us consider the operators

$$(38) \qquad a = \frac{\lambda}{2}\frac{\partial}{\partial\lambda} + \frac{\lambda}{\hbar} - \frac{1}{4}, b = \lambda^2,$$

satisfying the commutation relations

$$(39) \qquad\qquad [a, b] = b.$$

Using the integral representation (21) of the basis vectors it is easy to show [4] that

$$(40) \qquad \begin{aligned} a\,\Phi_j &= (j-1)\Phi_j + \frac{1}{\hbar}\Phi_{j+1}, \\ b\,\Phi_j &= j\hbar\Phi_{j+1} + \Phi_{j+2}, \end{aligned}$$

thus operators a and b stabilize the point (26) of the Sato Grassmannian

$$(41) \qquad \begin{aligned} a\,\mathcal{W}_{BGW} &\subset \mathcal{W}_{BGW}, \\ b\,\mathcal{W}_{BGW} &\subset \mathcal{W}_{BGW}, \end{aligned}$$

and are the KS operators.

However, these two operators do not completely specify the point of the Sato Grassmannian and the tau-function. Thus, they do not generate the KS algebra. Let us find some other KS operators. Integration by parts yields

$$(42) \qquad \frac{1}{b}a\,\Phi_j = \left(\frac{1}{2\lambda}\frac{\partial}{\partial\lambda} + \frac{1}{\hbar\lambda} - \frac{1}{4\lambda^2}\right)\Phi_j = \frac{1}{\hbar}\Phi_{j-1}.$$

The operator $\frac{1}{b}a$ is not a KS operator

$$(43) \qquad \frac{1}{b}a\,\Phi_1 = \frac{1}{\hbar}\Phi_0 \notin \mathcal{W}_{BGW}.$$

However, combining (42) with (40) one obtains

$$(44) \qquad \frac{1}{b}a^2\,\Phi_j = \frac{1}{\hbar}(j-1)\Phi_{j-1} + \frac{1}{\hbar^2}\Phi_j$$

and

$$(45) \qquad c = \frac{1}{b}a^2 = \frac{1}{4}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\hbar}\frac{\partial}{\partial\lambda} + \frac{1}{\hbar^2} + \frac{1}{16\lambda^2}$$

is the KS operator. To the best of our knowledge, this KS operator for the BGW tau-function has never been considered . Operators a, b and c satisfy the commutation relations

$$(46) \qquad [c,a] = c, \quad [c,b] = 2a+1,$$

and (39).

Proposition 2.1. *Operators a and c completely specify the point $\mathcal{W}_{BGW}$ of the Sato Grassmannian.*

Proof. From (45) we see that the operator c acts as

$$(47) \qquad c\,\lambda^k = \frac{1}{\hbar^2}\lambda^k\left(1 + O(\lambda^{-1})\right).$$

Thus, if this is the KS operator for some point of the Sato Grassmannian, then the first basis vector should be the eigenfunction of this operator:

$$(48) \qquad c\,\Phi_1 = \frac{1}{\hbar^2}\Phi_1.$$

From this equation it immediately follows that the solution corresponds to the big cell of the Sato Grassmannian,

$$(49) \qquad \Phi_1 = 1 + O(\lambda^{-1}),$$

and it is unique. All higher basis vectors can be generated from Φ_1 by the operator a. $\qquad\square$

From the correspondence between $w_{1+\infty}$ and its central extension $W_{1+\infty}$ it immediately follows that the KS operators b^k and $b^k a$ correspond to the constraints

$$(50) \qquad \frac{\partial}{\partial t_{2k}} \tau_{BGW} = \nu_k \tau_{BGW}, \quad k \geq 1,$$

and

$$(51) \qquad \left(\frac{1}{2} \widehat{L}_{2k} - \frac{1}{\hbar} \frac{\partial}{\partial t_{2k+1}} \right) \tau_{BGW} = \mu_k \tau_{BGW}, \quad k \geq 0,$$

for some constants ν_k and μ_k. Here

$$(52) \qquad \widehat{L}_m = \frac{1}{2} \sum_{a+b=-m} ab t_a t_b + \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+m}} + \frac{1}{2} \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b}$$

is an operator from the Virasoro subalgebra of the $W_{1+\infty}$ symmetry algebra of the KP hierarchy.

From the commutation relations between the operators in the l.h.s. of (50) and (51) it follows that

$$(53) \qquad \nu_k = \mu_k = 0, \qquad k > 0.$$

However, this argument does not allow us to find μ_0. This fact corresponds to the observation that the KS operators a and b do not completely specify a point of the Sato Grassmannian. From the normalization condition (25) and the constraint (51) with $k = 0$ it follows that this constant is proportional to the first derivative of the tau-function:

$$(54) \qquad \mu_0 = -\frac{1}{\hbar} \frac{\partial}{\partial t_1} \tau_{BGW} \Big|_{\mathbf{t}=0}.$$

This derivative is equal to the coefficient in front of λ^{-1} of the expansion (22) of $\Phi_1(\lambda)$,

$$(55) \qquad \Phi_1(\lambda) = 1 + \frac{\hbar}{16\lambda} + O(\lambda^{-2}),$$

thus

$$(56) \qquad \mu_0 = -\frac{1}{16}.$$

Since the tau-function is independent of the even times, the Virasoro constraints (50) can be represented as

$$(57) \qquad \hbar \widehat{\mathcal{L}}_m \, \tau_{BGW}(\mathbf{t}, \hbar) = \frac{\partial}{\partial t_{2m+1}} \tau_{BGW}(\mathbf{t}, \hbar), \quad m \geq 0,$$

where

$$(58) \qquad \widehat{\mathcal{L}}_m := \frac{1}{2} \sum_{k=0}^{\infty} (2k+1) t_{2k+1} \frac{\partial}{\partial t_{2k+2m+1}}$$
$$+ \frac{1}{4} \sum_{a+b=m-1} \frac{\partial^2}{\partial t_{2a+1} \partial t_{2b+1}} + \frac{1}{16} \delta_{m,0}.$$

These Virasoro constraints for the BGW tau-function were obtained already in [3]. Constraints (57) have a unique solution with the normalisation (25). This solution will be constructed in the next section.

The KS operator c corresponds to the $W_{1+\infty}$ operator

$$(59) \qquad \widehat{W}_c = \frac{1}{4} \widehat{M}_{-2} + \frac{1}{\hbar} \widehat{L}_{-1} - \frac{1}{8} t_2,$$

where

$$(60) \qquad \widehat{M}_k = \frac{1}{3} \sum_{a+b+c=k} {}^{*}_{*} \widehat{J}_a \widehat{J}_b \widehat{J}_c {}^{*}_{*}$$
$$= \frac{1}{3} \sum_{a+b+c=-k} a\,b\,c\,t_a\,t_b\,t_c + \sum_{c-a-b=k} a\,b\,t_a\,t_b \frac{\partial}{\partial t_c}$$
$$+ \sum_{b+c-a=k} a\,t_a \frac{\partial^2}{\partial t_b \partial t_c} + \frac{1}{3} \sum_{a+b+c=k} \frac{\partial^3}{\partial t_a \partial t_b \partial t_c}$$

are the cubic operators from the $W_{1+\infty}$ algebra. Thus, τ_{BGW} is the eigenfunction of the operator $\widehat{W}_c$ and, from the consideration of the corresponding linear constraint at the point $t_k = 0$ for all k we conclude that the eigenvalue is equal to zero:

$$(61) \qquad \widehat{W}_c \, \tau_{BGW} = 0.$$

This equation also allows us to find μ_0. Indeed, from the KdV reduction condition (50) it follows that (61) is equivalent to

$$(62) \qquad \sum_{k=0}^{\infty} (2k+2) t_{2k+2} \left(\widehat{\mathcal{L}}_k - \frac{1}{\hbar} \frac{\partial}{\partial t_{2k+1}} \right) \tau_{BGW} = 0.$$

2.3. Cut-and-join operator

Using the approach introduced in [22] we solve the constraints (57) and construct a simple recursion, which allows us to calculate the coefficients of the $\hbar$-expansion of the tau-function

$$(63) \qquad \tau_{BGW}(\mathbf{t}, \hbar) = 1 + \sum_{k=1}^{\infty} \hbar^k \tau_{BGW}^{(k)}(\mathbf{t}).$$

Namely, we introduce the Euler operator

$$(64) \qquad \widehat{D} := \sum_{k=0}^{\infty} (2k+1) t_{2k+1} \frac{\partial}{\partial t_{2k+1}}.$$

Then, combining the Virasoro constraints (57) we obtain

$$(65) \qquad \hbar \widehat{W}_{BGW}\, \tau_{BGW} = \widehat{D}\, \tau_{BGW},$$

where

$$(66) \quad \widehat{W}_{BGW} = \sum_{k=0}^{\infty} (2k+1) t_{2k+1} \widehat{\mathcal{L}}_k$$

$$= \frac{1}{2} \sum_{k,m=0}^{\infty} (2k+1)(2m+1) t_{2k+1} t_{2m+1} \frac{\partial}{\partial t_{2k+2m+1}}$$

$$+ \frac{1}{4} \sum_{k,m=0}^{\infty} (2k+2m+3) t_{2k+2m+3} \frac{\partial^2}{\partial t_{2k+1} \partial t_{2m+1}} + \frac{t_1}{16}.$$

does not depend on $\hbar$. From (31) it follows that

$$(67) \qquad \widehat{D}\, \tau_{BGW}^{(k)} = k\, \tau_{BGW}^{(k)}.$$

and after substitution of (63) into (65) we get a recursion

$$(68) \qquad \tau_{BGW}^{(k+1)} = \frac{1}{k+1} \widehat{W}_{BGW}\, \tau_{BGW}^{(k)}.$$

Since $\tau_{BGW}^{(0)} = 1$, we have

$$(69) \qquad \tau_{BGW}^{(k)} = \frac{\widehat{W}_{BGW}^k}{k!} \cdot 1.$$

Thus, we proved

Theorem 2.2.

$$(70) \qquad \tau_{BGW} = e^{\hbar \widehat{W}_{BGW}} \cdot 1$$

where the differential operator $\widehat{W}_{BGW}$ is given by (66).

With a few lines of Maple code the author was able to find all $\tau_{BGW}^{(k)}$ for $k \leq 90$. Let us stress that the obtained expressions allow us to find explicitly all correlation functions $\omega_{g,n}$ for $g \leq 30$ and arbitrary n (see below).

3. Generalized Brezin–Gross–Witten model

There exists a deformation of the BGW model, which depends on an additional parameter N (not to be confused with M, the size of the matrices)

$$(71) \qquad Z_N(\Lambda) = \frac{\int [d\Phi] \exp\left(\operatorname{Tr}\left(\frac{\Lambda^2 \Phi}{\hbar} + \frac{1}{\hbar\Phi} + (N - M)\log \Phi\right)\right)}{\int [d\Phi] \exp\left(\operatorname{Tr}\left(\frac{1}{\hbar\Phi} + (N - M)\log \Phi\right)\right)}.$$

For $N = 0$ it obviously coincides with the BGW model (9), and for $N \neq 0$ the unitary integral representation of (71) is not known.

This model was introduced in [4], and in the weak coupling limit (large $\tilde{\Lambda}$) it has very natural integrable properties. Namely, from the general theory of GKM [21], it follows that after a multiplication by a simple quasi-classical prefactor it is a tau-function of the MKP hierarchy, where $N \in \mathbb{Z}$ is the discrete time.

Following the description of the open intersection numbers in terms of the Kontsevich–Penner model, we do not require N to be an integer. It appears that the model (71) is defined perfectly well for an arbitrary $N \in \mathbb{C}$. Moreover, the tau-functions corresponding to the half-integer values of N are particularly interesting: they are polynomials. We call (71) the generalized Brezin–Gross–Witten model. In this section we consider the generalized BGW tau-function in detail.

3.1. MKP hierarchy and Virasoro constraints

After integration over the unitary group (71) reduces to

$$
(72) \qquad Z_N(\Lambda) = (-1)^{\frac{M(M-1)}{2}} \det(\tilde{\Lambda})^{2N}
$$
$$
\times \prod_{j=1}^{M} \Gamma(j-N) \frac{\det_{i,j=1}^{M} \left(\tilde{\lambda}_j^{M-N-i} \, I_{M-N-i}(2\tilde{\lambda}_j) \right)}{\Delta(\tilde{\lambda}^2)},
$$

which satisfies $Z_N(0) = 1$.

From the general theory of GKM it follows that for the large values of the eigenvalues of Λ matrix integral (71) corresponds to the MKP tau-function

$$
(73) \qquad\qquad \tau_N = \mathcal{C}_N^{-1} Z_N,
$$

where

$$
(74) \qquad\qquad \mathcal{C}_N = \frac{e^{2\operatorname{Tr}\tilde{\Lambda}} \det \tilde{\Lambda}^N \prod_{i=1}^{M} \Gamma(j-N)}{(2\pi)^{\frac{M}{2}} \det \left(\tilde{\Lambda} \otimes 1 + 1 \otimes \tilde{\Lambda} \right)^{\frac{1}{2}}}.
$$

Indeed, from (72) and (74) we have

$$
(75) \qquad\qquad \tau_N = \frac{\det_{i,j=1}^{M} \left(\Phi_j^{(N)}(\lambda_i) \right)}{\Delta(\lambda)},
$$

where the basis vectors

$$
(76) \qquad\qquad \Phi_j^{(N)}(\lambda) := \lambda^N \Phi_{j-N}(\lambda),
$$

and Φ_j's were defined in (21). The coefficients of their asymptotic series expansion for the large values of $|\lambda|$ depend only on $j - N$

$$
(77) \qquad\qquad \Phi_j^{(N)}(\lambda) = \lambda^{j-1} \left(1 + \sum_{k=1}^{\infty} \frac{(-\hbar)^k}{\lambda^k} \frac{a_k(j-N)}{16^k \, k!} \right),
$$

where $a_k(j)$ is a polynomial both in k and j given by (23). These basis vectors define a point on the big cell of the Sato Grassmannian

$$
(78) \qquad\qquad \mathcal{W}_N = \left\langle \Phi_1^{(N)}, \Phi_2^{(N)}, \Phi_3^{(N)}, \dots \right\rangle.
$$

The value $N = 0$ corresponds to the original BGW model considered in Section 2:

$$\tau_0 = \tau_{BGW}.\tag{79}$$

From (77) it follows that

$$\tau_N(\mathbf{t}, \hbar) = \tau_N(\mathbf{t}, 1)\Big|_{t_k = \hbar^k t_k}.\tag{80}$$

Let us stress that (77) defines a point of the big cell of the Sato Grassmannian, thus, a KP tau-function for any $N \in \mathbb{C}$. Moreover, it defines an MKP hierarchy, which relates τ_N and τ_{N+n} for any $n \in \mathbb{Z}$, $N \in \mathbb{C}$. The MKP hierarchy can be described by the bilinear identity, satisfied by the tau-function $\tau_N(\mathbf{t}, \hbar)$, namely, in our case,

$$\oint_\infty z^n e^{\xi(\mathbf{t}-\mathbf{t}',z)} \tau_{N+n}(\mathbf{t} - [z^{-1}], \hbar)\, \tau_N(\mathbf{t}' + [z^{-1}], \hbar) dz = 0, \quad N \in \mathbb{C}, \quad n \in \mathbb{N}_0.\tag{81}$$

Here $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ is the set of non-negative integers.

Again, for all N we have[2]

$$a\,\Phi_j^{(N)} = \left(j - 1 - \frac{N}{2}\right)\Phi_j^{(N)} + \frac{1}{\hbar}\Phi_{j+1}^{(N)},$$
$$b\,\Phi_j^{(N)} = (j - N)\hbar\Phi_{j+1}^{(N)} + \Phi_{j+2}^{(N)}.\tag{82}$$

Here the KS operators a and b are given by (38) and do not depend on N. This means, in particular, that they can not uniquely specify the point of the Sato Grassmannian, because they stabilize all points $\mathcal{W}_N$.

Integration by parts yields

$$\frac{1}{b}\left(a - \frac{N}{2}\right)\Phi_j^{(N)} = \frac{1}{\hbar}\Phi_{j-1}^{(N)}.\tag{83}$$

Thus

$$c_N = \frac{1}{b}\left(a^2 - \frac{N^2}{4}\right)\tag{84}$$

---

[2]This expression for the KS operators indicates that the generalized BGW tau-function is closely related to the model, considered in [38].

is the KS operator for τ_N:

$$(85) \qquad c_N \, \Phi_j^{(N)} = \frac{1}{\hbar}(j-1)\Phi_{j-1}^{(N)} + \frac{1}{\hbar^2}\Phi_j^{(N)}$$

and

$$(86) \qquad c_N \, \mathcal{W}_N = \mathcal{W}_N.$$

It satisfies the commutation relations

$$(87) \qquad [c_N, a] = c_N, \quad [c_N, b] = 2a + 1,$$

and, similar to the case $N = 0$ considered in Section 2, we have

Proposition 3.1. *Operators a and c_N completely specify the point $\mathcal{W}_N$ of the Sato Grassmannian.*

Using the Kac–Schwarz description (82) it is easy to show that the tau-function $\tau_N(\mathbf{t}, \hbar)$ satisfies the Virasoro constraints

$$(88) \qquad \hbar \widehat{\mathcal{L}}_m^{(N)} \, \tau_N(\mathbf{t}, \hbar) = \frac{\partial}{\partial t_{2m+1}}\tau_N(\mathbf{t}, \hbar), \quad m \geq 0,$$

where

$$(89) \qquad \widehat{\mathcal{L}}_m^{(N)} = \frac{1}{2}\sum_{k=0}^{\infty}(2k+1)t_{2k+1}\frac{\partial}{\partial t_{2k+2m+1}}$$
$$+ \frac{1}{4}\sum_{a+b=m-1}\frac{\partial^2}{\partial t_{2a+1}\partial t_{2b+1}} + \mu_0\delta_{m,0},$$

and

$$(90) \qquad \mu_0 = \frac{1}{16} - \frac{N^2}{4}.$$

Again, the value of μ_0 can be extracted from the expansion of the first basis vector

$$(91) \qquad \Phi_1^{(N)}(\lambda) = 1 + \hbar\frac{1 - 4N^2}{16\lambda} + O(\lambda^{-2}).$$

In the next section we prove

Theorem 3.2. *There exists a unique (up to normalization) solution of the Virasoro constraints (88).*

This theorem for $\hbar = 1$ was proved in [39], we prove it constructively and describe the solution in terms of the cut-and-join operator. Thus, the generalized BGW tau-function $\tau_N(\mathbf{t}, \hbar)$ is the unique solution of the Virasoro constraints (88) which satisfies the normalisation condition

$$\tag{92} \tau_N(\mathbf{0}, \hbar) = 1.$$

Equation (88) for $m = 0$ is the string equation for the generalized BGW tau-function. From the KS description it follows that this equation completely specifies the KdV tau-function.

Lemma 3.3. *There is only one tau-function of the KdV hierarchy, which satisfies the string equation*

$$\tag{93} \hbar \widehat{\mathcal{L}}_0^{(N)} \tau_N(\mathbf{t}) = \frac{\partial}{\partial t_1} \tau_N(\mathbf{t})$$

and the normalization condition (92).

Alternatively, the Virasoro constraints can be derived from the expansion of the operator $\widehat{W}_{c_N}$, the derivation is completely similar to the one from Section 2.2. In particular, this operator specifies the value of constant μ_0.

3.2. Cut-and-join operator

Similar to the case of the BGW tau-function, considered in Section 2, we can solve the Virasoro constraints for the generalized BGW tau-function in terms of the cut-and-join operator:

Lemma 3.4. *The solution of the Virasoro constraints (88) with the normalization (92) is given by*

$$\tag{94} \tau_N(\mathbf{t}) = e^{\hbar \widehat{W}_N} \cdot 1$$

where

$$(95) \qquad \widehat{W}_N = \frac{1}{2} \sum_{k,m=0}^{\infty} (2k+1)(2m+1)t_{2k+1}t_{2m+1} \frac{\partial}{\partial t_{2k+2m+1}}$$

$$+ \frac{1}{4} \sum_{k,m=0}^{\infty} (2k+2m+3)t_{2k+2m+3} \frac{\partial^2}{\partial t_{2k+1} \partial t_{2m+1}}$$

$$+ \left(\frac{1}{16} - \frac{N^2}{4} \right) t_1$$

$$= \widehat{W}_{BGW} - \frac{N^2}{4} t_1.$$

All other solutions of the Virasoro constraints (88) correspond to the multiplication of (94) by a constant.

Proof. Let us consider an arbitrary series in the time variables t_k

$$(96) \qquad Z(\mathbf{t}) = C + \sum_{k=1}^{\infty} Z^{(k)}(\mathbf{t}).$$

where $Z^{(k)}(\mathbf{t})$ is a homogeneous polynomial of degree k,

$$(97) \qquad \widehat{D} Z^{(k)}(\mathbf{t}) = k\, Z^{(k)}(\mathbf{t}),$$

and C is some constant. Then, if $Z(t)$ solves the Virasoro constraints (88), then

$$(98) \qquad \hbar \widehat{W}_N Z(\mathbf{t}) = \widehat{D} Z(\mathbf{t}).$$

From the comparison of the terms in the r.h.s. and the r.h.s. with the same degree we conclude

$$(99) \qquad \hbar \widehat{W}_N Z^{(k)}(\mathbf{t}) = \widehat{D} Z^{(k+1)}(\mathbf{t}),$$

thus, from (97) it follows that

$$(100) \qquad Z^{(k+1)}(\mathbf{t}) = \frac{\hbar}{k+1} \widehat{W}_N Z^{(k)}(\mathbf{t})$$

or

$$(101) \qquad Z^{(k)}(\mathbf{t}) = \frac{\hbar^k}{k!} \widehat{W}_N^k C.$$

In particular, for $C = 1$ we get the solution (94), which coincides with the generalized BGW tau-function. $\square$

We call (95) the cut-and-join operator for the generalized BGW tau-function. This operator does not belong to the $W_{1+\infty}$ algebra of symmetries of the KP hierarchy, thus, integrability is not obvious from the representation (94).

From the proof of Lemma 95 we see that the coefficients of the topological expansion

$$(102) \qquad \tau_N(\mathbf{t}, \hbar) = 1 + \sum_{k=1}^{\infty} \hbar^k \, \tau_N^{(k)},$$

satisfy the recursion

$$(103) \qquad \tau_N^{(k+1)} = \frac{1}{k+1} \widehat{W}_N \, \tau_N^{(k)}.$$

Using this recursion we calculated $\tau_N^{(k)}$ for $k \leq 60$, expressions for $k \leq 10$ are given in Appendix B. There we introduce

$$(104) \qquad B_k(N) = (-1)^k a_k(N+1)$$
$$= (1 - 4N^2)(3^2 - 4N^2) \cdots ((2k-1)^2 - 4N^2).$$

In Section 3.6 we show that for $k \leq \frac{m(m+1)}{2}$ the polynomials $\tau_N^{(k)}$ are divisible by $B_m(N)$.

From (94) we see that the tau-function is actually a series in N^2 (not in N), thus

$$(105) \qquad \tau_{-N}^{BGW}(\mathbf{t}, \hbar) = \tau_N^{BGW}(\mathbf{t}, \hbar).$$

From this observation and from the explicit expression for $\tau_N^{(1)}$ we conclude

Lemma 3.5.

$$(106) \qquad \tau_N(\mathbf{t}, \hbar) = \tau_{\tilde{N}}(\mathbf{t}, \hbar)$$

if and inly if $\tilde{N} = \pm N$.

In particular, it means that the generalized BGW tau-function is not periodic in the variable N, and it is enough to consider only the values of N with $\Re N \geq 0$.

Operator (66) has a rather natural free field representation. Indeed, let us introduce a bosonic current

$$(107) \qquad \widehat{J}(z) = \sum_{k=1}^{\infty} \left((2k+1)t_{2k+1}z^{2k} + \frac{1}{2z^{2k+2}} \frac{\partial}{\partial t_{2k+1}} \right) + \frac{iN}{2z},$$

then

$$(108) \qquad \widehat{W}_N = \frac{1}{2\pi i} \oint \left(\vdots \frac{1}{3}\widehat{J}(z)^3 + \frac{1}{16z^2}\widehat{J}(z) \vdots \right) z\,dz,$$

where we use the standard normal ordering for the bosonic operators.

To consider the genus expansion we have to rescale the times $t_k \mapsto \hbar^{-1}t_k$. Then we can rewrite (94) as

$$(109) \qquad \tau_N(\hbar^{-1}\mathbf{t}, \hbar) = e^{\frac{1}{\hbar^2}\widehat{W}^{(-1)} + \widehat{W}^{(0)} + \hbar^2\widehat{W}^{(1)}} \cdot 1,$$

where

$$(110) \qquad \begin{aligned} \widehat{W}^{(-1)} &= -\frac{S^2}{4}t_1, \\ \widehat{W}^{(0)} &= \frac{1}{2}\sum_{k,m=0}^{\infty}(2k+1)(2m+1)t_{2k+1}t_{2m+1}\frac{\partial}{\partial t_{2k+2m+1}} + \frac{1}{16}t_1, \\ \widehat{W}^{(1)} &= \frac{1}{4}\sum_{k,m=0}^{\infty}(2k+2m+3)t_{2k+2m+3}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2m+1}}, \end{aligned}$$

and we introduced a new parameter

$$(111) \qquad S = \hbar N.$$

From this representation it follows that after the times rescaling the generalized BGW tau-function has a natural genus expansion

$$(112) \qquad \tau_N(\hbar^{-1}\mathbf{t}, \hbar) = \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2}\mathcal{F}_g(\mathbf{t}, S) \right).$$

From the zeroth equation (88) it immediately follows that

$$(113) \qquad \sum_{k=0}^{\infty}(2k+1)\tilde{t}_{2k+1}\frac{\partial}{\partial t_{2k+1}}\mathcal{F}_g(\mathbf{t}, S) = \frac{S^2}{2}\delta_{g,0} - \frac{1}{8}\delta_{g,1},$$

where the dilaton shift of the time variables is defined by

$$(114) \qquad \tilde{t}_k = t_k - \frac{2}{\hbar}\delta_{k,1}.$$

Thus, up to the genus zero and genus one contributions, we can express all $\mathcal{F}_g(\mathbf{t}, S)$ in terms of the "moment variables"

$$(115) \qquad T_k = \frac{t_k}{(2 - t_1)^k},$$

namely

$$(116) \qquad \mathcal{F}_g(\mathbf{t}, S) = \tilde{\mathcal{F}}_g(\mathbf{T}, S) + \left(\frac{S^2}{2}\delta_{g,0} - \frac{1}{8}\delta_{g,1} \right) \log\left(1 - \frac{t_1}{2} \right).$$

Moreover, from (80) it follows that $\mathcal{F}_g(\mathbf{T}, S)$ are the homogeneous functions of degree $g - 1$

$$(117) \qquad \left(\sum_{m=1}^{\infty} m\, T_{2m+1} \frac{\partial}{\partial T_{2m+1}} - \frac{S}{2} \frac{\partial}{\partial S} \right) \tilde{\mathcal{F}}_g(\mathbf{T}, S) = (g - 1)\, \tilde{\mathcal{F}}_g(\mathbf{T}, S).$$

Thus, the genus g contribution is given by the sum

$$(118) \qquad \tilde{\mathcal{F}}_g(\mathbf{T}, S) = \sum_{k=0}^{\infty} (-1)^k S^{2k} \tilde{\mathcal{F}}_g^{(k)}(\mathbf{T}),$$

where we introduced the polynomials $\tilde{\mathcal{F}}_g^{(k)}(\mathbf{T})$ such that

$$(119) \qquad \sum_{m=1}^{\infty} m\, T_{2m+1} \frac{\partial}{\partial T_{2m+1}} \tilde{\mathcal{F}}_g^{(k)}(\mathbf{T}) = (g + k - 1)\tilde{\mathcal{F}}_g^{(k)}(\mathbf{T}).$$

For $k = 0$ they coincide with the free energies for BGW tau-function, given in Appendix A,

$$(120) \qquad \tilde{\mathcal{F}}_g^{(0)}(\mathbf{T}) = \mathcal{F}_g(\mathbf{T}).$$

Using the recursion (103) we found expressions for $\mathcal{F}_g^{(k)}(\mathbf{T})$ for all $g + k \le 20$. For $k > 0$ and $g + k \le 8$ they are given in Appendix C.

3.3. Quantum spectral curve

The quantum spectral curve[3] for the generalized BGW tau-function, as for many other examples of the generating functions related to the KP/Toda hierarchies [17, 24, 25], can be derived from the Sato Grassmannian description. Actually, the principal specialisation of any KP tau-function coincides with the first basis vector of the corresponding point of the Sato Grassmannian. Often, the KS algebra contains an operator, which annihilates this basis vector, and namely this operator describes the quantum spectral curve.

It follows from (85) that for the generalized BGW tau-function this vector is annihilated by a shifted operator c_N:

$$(121) \qquad \left(c_N - \frac{1}{\hbar^2}\right)\Phi_1^{(N)}(\lambda) = 0.$$

Let us introduce a new variable:

$$(122) \qquad x = \lambda^2.$$

Then the corresponding wave function

$$(123) \qquad \Psi_S(x) := \frac{\hbar}{\sqrt{4\pi}\, x^{\frac{1}{4}}}\, e^{\frac{2\sqrt{x}}{\hbar}}\, \Phi_1^{(S\hbar^{-1})}(\sqrt{x})$$

is the modified Baker function

$$(124) \qquad \Psi_S(x) = I_{S\hbar^{-1}}\left(\frac{2\sqrt{x}}{\hbar}\right).$$

It satisfies the modified Bessel equation

$$(125) \qquad \left(\hbar^2 x^2 \frac{\partial^2}{\partial x^2} + \hbar^2 x \frac{\partial}{\partial x} - x - \frac{S^2}{4}\right)\Psi_S(x) = 0.$$

which is the quantum spectral curve equation for the generalized BGW model. If we introduce the operators

$$(126) \qquad \hat{x} = x, \quad \hat{y} = \hbar \frac{\partial}{\partial x},$$

---

[3]For more details on quantum spectral curves see [40] and references therein.

then we can rewrite the quantum spectral curve equation as

$$(127) \qquad \left(\hat{x}\hat{y}\hat{x}\hat{y} - \hat{x} - \frac{S^2}{4} \right) \Psi_S(x) = 0,$$

which in the classical limit reduces to the curve

$$(128) \qquad x^2 y^2 - x - \frac{S^2}{4} = 0$$

or, equivalentely

$$(129) \qquad y^2 = \frac{1}{x} + \frac{S^2}{4x^2}.$$

This curve admits a rational parametrization:

$$(130) \qquad x = \frac{S^2 (z-1)}{(z-2)^2}, \quad y = \frac{z(z-2)}{2S(z-1)},$$

thus, the spectral curve is of genus zero.

The branch points are the zeros of the differential dx,

$$(131) \qquad dx = -\frac{S^2 z}{(z-2)^3} dz,$$

which do not coincide with the zeros of the differential dy,

$$(132) \qquad dy = \frac{z^2 - 2z + 2}{2 (z-1)^2 S} dz.$$

We see, that on the curve (130) there is only one branch point,

$$(133) \qquad z = 0,$$

which corresponds to

$$(134) \qquad y = 0, \quad x = -\frac{S^2}{4}.$$

For the BGW model, that is for $S = 0$, the quantum spectral curve equation reduces to

$$(135) \qquad (\hat{y}\hat{x}\hat{y} - 1) \Psi_0(x) = 0$$

and in the classical limit

$$(136) \qquad\qquad\qquad y^2 = \frac{1}{x}.$$

In this limit y plays the role of the global rational coordinate. This can be considered as the curve for the r-spin intersection numbers with $r = -2$.

We claim that the Chekhov–Eynard–Orantin topological reduction [9–11] for the spectral curves (129) and (136) should give the expressions for the correlation functions of the generalized BGW and BGW models correspondingly. However, in the next section we will derive the recursion relation for the correlation functions using only the Virasoro constraints (88).

3.4. Correlation functions

The Virasoro constraints can also be reformulated in terms of the correlation functions (multiresolvents). This reformulation leads to the loop equations [41–46].

Sometimes the loop equations can be solved systematically, producing simple recursive relations for the correlation functions [47–50]. Let use define the connected correlation functions

$$(137) \qquad W_{g,n}(x_1, \ldots, x_n) := \widehat{\nabla}(x_1)\widehat{\nabla}(x_2)\cdots\widehat{\nabla}(x_n)\mathcal{F}_g(\mathbf{t}, S)\Big|_{\mathbf{t}=0},$$

where

$$(138) \qquad\qquad\qquad \widehat{\nabla}(x) = \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \frac{\partial}{\partial t_{2k+1}}.$$

Obviously, the correlation functions are symmetric functions of the variables $x_1, \ldots, x_n$ and contain all information about the tau-function.

From the Virasoro constraints (88) it follows that the correlation functions of the generalized BGW tau-function satisfy the loop equations:

(139)

$$
W_{g,m+1}(x, x_1, \ldots, x_m)
$$
$$
= \frac{1}{4} W_{g-1,m+2}(x, x, x_1, \ldots, x_m) + \left(\frac{1}{16x} \delta_{g,1} - \frac{S^2}{4x} \delta_{g,0} \right) \delta_{m,0}
$$
$$
+ \frac{1}{4} \sum_{q+p=g,\, I \cup J=\{1,2,\ldots,m\}} W_{q,m_1+1}(x, x_{i_1}, \ldots, x_{i_{m_1}}) W_{p,m_2+1}(x, x_{j_1}, \ldots, x_{j_{m_2}})
$$
$$
+ \sum_{i=1}^{m} \left(x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \right) \frac{W_{g,m}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m) - W_{g,m}(x_1, \ldots, x_m)}{x - x_i}
$$

for all $m \geq 0$ and $g \geq 0$. This is a simple S-deformation of the loop equations for the BGW tau-function, which were derived in [14].

The simplest case is $g = m = 0$, and in this case (139) gives a quadratic equation for $W_{0,1}$:

(140)
$$
W_{0,1}(x)^2 - 4W_{0,1}(x) - \frac{S^2}{x} = 0
$$

so that

(141)
$$
W_{0,1}(x) = 2 \left(1 - \sqrt{1 + \frac{S^2}{4x}} \right),
$$

or

(142)
$$
W_{0,1}(x) = 2 \sum_{k=0}^{\infty} \left(-\frac{S^2}{8x} \right)^{k+1} \frac{(2k-1)!!}{(k+1)!}.
$$

This allows us to solve recursively the Loopequations (139) for $g + m > 0$,

Lemma 3.6.

(143)

$$W_{g,m+1}(x, x_1, \ldots, x_m) = \frac{1}{\sqrt{1 + \frac{S^2}{4x}}}$$

$$\times \left(\frac{1}{4} \sum_{q+p=g, I \cup J=\{1,2,\ldots,m\}}' W_{q,m_1+1}(x, x_{i_1}, \ldots, x_{i_{m_1}}) W_{p,m_2+1}(x, x_{j_1}, \ldots, x_{j_{m_2}}) \right.$$

$$+ \sum_{i=1}^{m} \left(x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \right) \frac{W_{g,m}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m) - W_{g,m}(x_1, \ldots, x_m)}{x - x_i}$$

$$+ \left. \frac{1}{4} W_{g-1,m+2}(x, x, x_1, \ldots, x_m) + \frac{1}{16x} \delta_{g,1} \delta_{m,0} \right),$$

where we exclude from the sum two terms: with $q = g$, $I = \{1, 2, \ldots, m\}$,
$J = \{\emptyset\}$ and with $p = g$, $I = \{\emptyset\}$, $J = \{1, 2, \ldots, m\}$.

In particular, the genus zero two-point function is

$$(144) \quad W_{0,2}(x_1, x_2) = \frac{1}{\sqrt{1 + \frac{S^2}{4x_1}}} \left(x_2 \frac{\partial}{\partial x_2} + \frac{1}{2} \right) \frac{W_{0,1}(x_1) - W_{0,1}(x_2)}{x_1 - x_2}$$

$$= \frac{1}{2(x_1 - x_2)^2} \left(\frac{S^2 + 2(x_1 + x_2)}{\sqrt{1 + \frac{S^2}{4x_1}} \sqrt{1 + \frac{S^2}{4x_2}}} - 2(x_1 + x_2) \right).$$

It is regular at the coincident points (that is when $x_1 = x_2$ and $y_1 = y_2$),
and has a second order pole when the points are on different sheets above
the same base point (that is when $x_1 = x_2$ and $y_1 = -y_2$),

$$(145) \qquad\qquad W_{0,2}(x_1, x_2) = -\frac{4x_1}{(x_1 - x_2)^2} + \cdots.$$

In genus one we have

$$(146) \qquad W_{1,1}(x) = \frac{1}{\sqrt{1 + \frac{S^2}{4x}}} \left(\frac{1}{4} W_{0,2}(x, x) + \frac{1}{16x} \right) = \frac{1}{2^4 x \left(1 + \frac{S^2}{4x}\right)^{\frac{5}{2}}},$$

or

$$(147) \qquad\qquad W_{1,1}(x) = \frac{1}{2^4 \cdot 3x} \sum_{k=0}^{\infty} \left(-\frac{S^2}{8x} \right)^k \frac{(2k+3)!!}{k!}.$$

On the next level of recursion we have

$$(148) \qquad W_{0,3}(x_1, x_2, x_3) = -\frac{S^2}{8x_1 x_2 x_3 \sqrt{1 + \frac{S^2}{4x_1}} \sqrt{1 + \frac{S^2}{4x_2}} \sqrt{1 + \frac{S^2}{4x_3}}},$$

$$(149) \qquad W_{1,2}(x_1, x_2) = \frac{1}{2^{12} x_1^3 x_2^3 \left(1 + \frac{S^2}{4x_1}\right)^{\frac{7}{2}} \left(1 + \frac{S^2}{4x_2}\right)^{\frac{7}{2}}}$$
$$\times \left(S^8 - 6\left(x_1 + x_2\right) S^6 - 136\, S^4 x_2 x_1 \right.$$
$$\left. - 128\, x_1 x_2 \left(x_2 + x_1\right) S^2 + 128\, x_1{}^2 x_2{}^2 \right),$$

$$(150) \qquad W_{2,1}(x) = \frac{S^4 - 20 S^2 x + 9 x^2}{2^{10} x^4 \left(1 + \frac{S^2}{4x}\right)^{\frac{11}{2}}}.$$

For the stable cases (the cases with $2g + n - 2 > 0$, that is, for all $W_{g,n}$'s except for $W_{0,1}$ and $W_{0,2}$) let use define the differentials forms

$$(151) \qquad \omega_{g,n}(z_1, \ldots, z_n) := S^{2g-2+n} W_{g,n}(x_1, \ldots, x_n) d\sqrt{x_1} \cdots d\sqrt{x_n}$$
$$= S^{2g-2+n} \frac{W_{g,n}(x_1, \ldots, x_n)}{2^n \sqrt{x_1 \cdots x_n}} dx_1 \cdots dx_n.$$

They satisfy the recursion relations which follow from (143) and can be easily found for small g and n. In particular,

$$(152) \qquad \omega_{1,1}(z) = \frac{z-1}{z^4} dz,$$

$$(153) \qquad \omega_{2,1}(z) = \frac{\left(105 - 210\, z + 133\, z^2 - 28\, z^3 + z^4\right)(z-2)^2(z-1)}{z^{10}} dz$$

$$(154) \qquad \omega_{1,2}(z_1, z_2) = \left(54\, z_2{}^2 z_1{}^4 + 24\, z_2{}^3 z_1{}^3 - 14\, z_2{}^3 z_1{}^4 + 54\, z_2{}^4 z_1{}^2 \right.$$
$$- 14\, z_2{}^4 z_1{}^3 + z_2{}^4 z_1{}^4 + 24\, z_1{}^2 z_2{}^2 - 80\, z_1{}^4 z_2$$
$$- 24\, z_1{}^3 z_2{}^2 - 24\, z_1{}^2 z_2{}^3 - 80\, z_1 z_2{}^4$$
$$\left. + 40\, z_2{}^4 + 40\, z_1{}^4 \right) \frac{1}{z_1{}^6 z_2{}^6} dz_1 dz_2$$

$$(155) \quad \omega_{0,3}(z_1, z_2, z_3) = \frac{8}{z_1^2 z_2^2 z_3^2} dz_1 dz_2 dz_3$$

Conjecture 3.7. *All $\omega_{g,n}$ are the meromorphic differentials, defined on the spectral curve (130) and symmetric in z_j's. Moreover, for any g and n*

$$(156) \qquad \frac{z_1^2 \cdots z_n^2\, \omega_{g,n}(z_1, \ldots, z_n)}{dz_1 \cdots dz_n}$$

is a polynomial in each of variables $z_1^{-1}, \ldots, z_n^{-1}$. Thus, $\omega_{g,n}$ have poles of finite degree only at the branch point $z_j = 0$.

It should be simple to prove this conjecture using the Chekhov–Eynard–Orantin topological recursion methods, which are beyond the scope of this paper.

In the limit of $S = 0$ the correlation functions $W_{g,n}$ coincide with the original BGW model. In this case all $W_{g,n}$ are polynomials in x_j^{-1}, see [14].

3.5. Genus zero contribution

Formulas (C1)–(C8) as well as higher terms indicate that the coefficients of the expansion of genus zero free energy are quite simple.

Conjecture 3.8.

$$(157) \quad \left[T_3^{j_1} T_5^{j_2} T_7^{j_3} \cdots \right] \tilde{\mathcal{F}}_0(\mathbf{T}, S) = \frac{(-1)^{m+1} (3j_1 + 5j_3 + 7j_3 + \cdots - 1)!}{2^m (2m+2)!}$$
$$\times \frac{(3!!)^{j_1} (5!!)^{j_2} (7!!)^{j_3} \cdots}{(1!)^{j_1} j_1! (2!)^{j_2} j_2! (3!)^{j_3} j_3! \cdots} S^{2m+2},$$

where

$$(158) \qquad\qquad m = j_1 + 2j_2 + 3j_3 + \cdots .$$

From this conjecture and from the definition of the variables T_k it immediately follows that

$$(159) \quad \mathcal{F}_0(\mathbf{t}, S) = \sum_{\substack{j_0, j_1, j_2, \ldots \\ j_0 + j_1 + \cdots > 0}} A(j_0, j_1, j_2, \ldots)$$
$$\times \frac{(-1)^{m+1} S^{2m+2} (j_0 + 3j_1 + 5j_2 + \cdots - 1)!}{2^{m+j_0+3j_1+5j_2+\cdots} (2m+2)!} t_1^{j_0} t_3^{j_1} t_5^{j_2} \cdots ,$$

where

$$(160) \qquad A(j_0, j_1, j_2, \ldots) = \frac{(1!!)^{j_0} (3!!)^{j_1} (5!!)^{j_2} \cdots}{(0!)^{j_0} j_0! (1!)^{j_1} j_1! (2!)^{j_2} j_2! \cdots}.$$

This expression is consistent with expressions for the correlation functions obtained in Section 3.4. It should help to identify the coefficients of the generalized BGW model with the enumerative geometry invariants. This

conjecture probably can be proved with the help of the Baker–Campbell–Hausdorff analysis of (109).

Let us give for comparison an expression for the genus zero free energy of the Kontsevich–Witten tau-function

$$(161) \qquad \mathcal{F}_0^{KW}(\mathbf{t}, S) = \sum_{j_0, j_1, j_2, \ldots} A(j_0, j_1, j_2, \ldots)$$
$$\times\, m!\, \delta(-j_0 + j_2 + 2j_3 + \cdots + 3)\, t_1^{j_0} t_3^{j_1} t_5^{j_2} \cdots .$$

3.6. Polynomial tau-functions of KdV hierarchy

It appears that for $N - \frac{1}{2} \in \mathbb{Z}$ the generalized BGW tau-function is a polynomial in times. From (105) it follows that it is enough to consider only positive values of N. In this section we assume that

$$(162) \qquad\qquad\qquad l = N - \frac{1}{2} \in \mathbb{N}_0.$$

Then we have

Theorem 3.9. *For the half-integer value of N the generalized BGW tau-function is polynomial in times. Moreover, up to the dilaton shift of the times, it is equal to the the Schur function corresponding to the triangular partition of $\frac{l(l+1)}{2}$,*

$$(163) \qquad\qquad\qquad \lambda(l) = (l, l-1, l-2, \ldots, 1).$$

Namely

$$(164) \qquad\qquad\qquad \tau_{l+\frac{1}{2}}(\mathbf{t}) = C_l\, s_{\lambda(l)}(\tilde{\mathbf{t}})$$

where the dilaton shift is given by (114) and

$$(165) \qquad\qquad\qquad C_l = \frac{(-\hbar)^{\frac{l(l+1)}{2}}}{2^{l^2}} \prod_{k=1}^{l} \frac{(2l - 2k + 1)!}{(l-k)!}.$$

Proof. In this case all sums in the expressions for the basic vectors (77) have only a finite numbers of terms:

(166)

$$
\Phi_j^{(l+\frac{1}{2})}(\lambda) =
\begin{cases}
\lambda^{j-1} + \displaystyle\sum_{k=1}^{l-j+1} (-\hbar)^k \frac{a_k(j-l-\frac{1}{2})}{16^k\, k!}\, \lambda^{j-k-1} & \text{for } j \le l+1, \\[6pt]
\lambda^{j-1} + \displaystyle\sum_{k=1}^{j-l-2} (-\hbar)^k \frac{a_k(j-l-\frac{1}{2})}{16^k\, k!}\, \lambda^{j-k-1} & \text{for } j > l+1.
\end{cases}
$$

Thus, $\Phi_j^{(l+1/2)}$ is not polynomial (contains negative powers of λ) only for $j < \frac{l}{2} + 1$, and in this case the most singular term is proportional to λ^{2j-l-2}. Moreover,

(167)
$$
\Phi_{l+1}^{(l+\frac{1}{2})}(\lambda) = \lambda^l, \quad \Phi_{l+2}^{(l+\frac{1}{2})}(\lambda) = \lambda^{l+1},
$$

thus, from (82) we see that

(168)
$$
\lambda^k \in \mathcal{W}_{l+\frac{1}{2}} \quad \text{for} \quad k \ge l.
$$

Therefore, for any $M \ge l$ a ratio of determinants

(169)
$$
\tau_{l+\frac{1}{2}}(\Lambda) = \frac{\det_{i,j=1}^M \left(\Phi_j^{(l+\frac{1}{2})}(\lambda_i) \right)}{\Delta(\lambda)}
$$

is a symmetric polynomial (not homogeneous!) in the eigenvalues λ_j^{-1} of total degree $\frac{l(l+1)}{2}$. It means that if we put $\deg t_k = k$, $\tau_{l+\frac{1}{2}}$ is a polynomial in times t_k of degree $\frac{l(l+1)}{2}$, for example

$$
\tau_{\frac{1}{2}} = 1,
$$

(170)
$$
\tau_{\frac{3}{2}} = 1 - \hbar \frac{t_1}{2} = -\frac{\hbar}{2}\tilde{t}_1,
$$

$$
\tau_{\frac{5}{2}} = 1 - \frac{3}{2}\hbar t_1 + \frac{3}{4}\hbar^2 t_1{}^2 + \frac{3}{8}\hbar^3 t_3 - \frac{1}{8}\hbar^3 t_1{}^3 = \hbar^3 \left(\frac{3}{8}\tilde{t}_3 - \frac{1}{8}\tilde{t}_1^3 \right).
$$

Let us prove, that the tau-unctions $\tau_{l+\frac{1}{2}}$ actually coincide (up to a constant normalization) with the Schur functions. The shift of the times in the tau-functions corresponds to the action of the multiplication operator on the Sato Grassmannian. Namely, if a given tau-function $\tau(\mathbf{t})$ is described by

the point $\mathcal{W}$ of the Sato Grassmannian, the tau-function

$$(171) \qquad \tilde{\tau}(\mathbf{t}) = \tau(\mathbf{t} + \mathbf{a})$$

corresponds to the point of the Sato Grassmannian, specified by

$$(172) \qquad \widetilde{\mathcal{W}} = e^{\sum \frac{a_k \lambda^k}{k}} \mathcal{W}.$$

In particular, to get rid of the dilaton shift (114) we introduce

$$(173) \qquad \tilde{\tau}_{l+\frac{1}{2}}(\mathbf{t}) = \tau_{l+\frac{1}{2}} \left(\mathbf{t} + \frac{2}{\hbar} \delta_{k,1} \right),$$

which corresponds to the point of the point of Sato Grassmannian

$$(174) \qquad \widetilde{\mathcal{W}}_{l+\frac{1}{2}} = e^{\frac{2\lambda}{\hbar}} \mathcal{W}_{l+\frac{1}{2}}.$$

Let us show that

$$(175) \qquad \lambda^{2k-2-l} \in \widetilde{\mathcal{W}}_{l+\frac{1}{2}} \quad \text{for} \quad l \geq k > 0.$$

Indeed, from (172) it follows that

$$(176) \qquad e^{\frac{2\lambda}{\hbar}} \sum_{j=1}^{\infty} \alpha_j \Phi_j^{(l+\frac{1}{2})}(\lambda) \in \widetilde{\mathcal{W}}_{l+\frac{1}{2}}$$

for any constants α_j. In particular, from (21) and (76) it follows that if we choose these constants such that $\sum_{j=1}^{\infty} \alpha_j \hbar^j t^{1-j} = \exp(t^{-1})$, then (176) reduces to

$$(177) \qquad \lambda^{l+1} \int_{\gamma} e^{\tilde{\lambda}^2 t} \frac{dt}{t^{-l+\frac{1}{2}}} \in \widetilde{\mathcal{W}}_{l+\frac{1}{2}}.$$

Since

$$(178) \qquad \int_{\gamma} e^{\tilde{\lambda}^2 t} \frac{dt}{t^{-l+\frac{1}{2}}} \sim \lambda^{-2l-1}$$

it proves (175) for $k = 1$. To prove (175) for $l \geq k > 1$ one have just to choose other values for the constants α_j.

Thus

$$(179) \qquad \widetilde{\mathcal{W}}_{l+\frac{1}{2}} = \left\langle \lambda^{-l}, \lambda^{2-l}, \lambda^{4-l}, \ldots, \lambda^{l-4}, \lambda^{l-2}, \lambda^{l}, \lambda^{l+1}, \ldots \right\rangle$$

It does not belong to the big cell of the Sato Grassmannian, and the KdV tau-function is (up to a constant factor) the Schur function, corresponding to the triangular Young tableau with

$$(180) \qquad \lambda(l) = (l, l-1, l-2, \ldots, 1).$$

These KdV tau-functions were described already in [51]. The constant C_l in (164) can be easily found from the comparison of the r.h.s. and the l.h.s. for $\mathbf{t} = 0$. Namely,

$$(181) \qquad s_\lambda(\mathbf{t})\Big|_{t_k=0, k>1} = s_\lambda(t_k = \delta_{k,1}) \, t_1^{|\lambda|},$$

thus

$$(182) \qquad C_l = \frac{\hbar^{\frac{l(l+1)}{2}}}{(-2)^{\frac{l(l+1)}{2}} s_{\lambda(l)}(t_k = \delta_{k,1})} = \frac{(-\hbar)^{\frac{l(l+1)}{2}}}{2^{l^2}} \prod_{k=1}^{l} \frac{(2l-2k+1)!}{(l-k)!}.$$

$\square$

We have a corollary

Lemma 3.10. *The tau-function of the KdV hierarchy given by the Schur function*

$$(183) \qquad \tau(\mathbf{t}) = s_{\lambda(l)}(\mathbf{t}),$$

where the partition is given by (180), is uniquely (up to normalization) specified by the Virasoro constraints

$$(184) \qquad \widehat{\mathcal{L}}_m \, \tau(\mathbf{t}) = 0, \quad m \geq 0,$$

where

$$(185) \qquad \widehat{\mathcal{L}}_m = \frac{1}{2} \sum_{k=0}^{\infty} (2k+1)\tilde{t}_{2k+1} \frac{\partial}{\partial t_{2k+2m+1}}$$
$$+ \frac{1}{4} \sum_{a+b=m-1} \frac{\partial^2}{\partial t_{2a+1}\partial t_{2b+1}} - \frac{l(l+1)}{4} \delta_{m,0}.$$

Polynomially of the tau-function (164) means that in this case the expansion (102) is finite, and

$$(186) \qquad \left(\widehat{W}_{l+\frac{1}{2}}\right)^{\frac{l(l+1)}{2}+1} \cdot 1 = 0.$$

Also, since

$$(187) \qquad B_k\left(l + \frac{1}{2}\right) = 0, \quad k > |l|,$$

the terms $\tau_N^{(k)}$, which are polynomials in N, are indeed divisive by $B_l(N)$ for $k \le \frac{l(l+1)}{2}$.

Thus, it is natural to express the free energy in terms of only $B_m(N)$ and T_k. Namely, for

$$(188) \qquad \tau_N(\hbar^{-1}\mathbf{t}, \hbar) = \exp\left(\mathcal{F}(\mathbf{t}, N, \hbar)\right)$$

where we do not introduce the variable S, (111), we have

$$(189) \qquad \mathcal{F}(\mathbf{t}, N, \hbar) = \frac{4N^2 - 1}{8} \log\left(1 - \frac{t_1}{2}\right) + \sum_{k=2} \hbar^{2g-2} F_g(\mathbf{T}, N)$$

where

$$(190) \qquad F_g(\mathbf{T}, N) = \sum_{k=2}^{g} B_k(N) F_{g,k}(\mathbf{T}).$$

Polynomials $F_{g,k}$ can be found using the cut-and-join operator or from (164). In Appendix D we give expressions for $g \le 6$.

4. Concluding remarks

In this paper we have investigated the generalized BGW tau-function. Obtained results are not only interesting for the matrix model theory, but also should help to identify the generalized BGW tau-function with a generating function of some enumerative geometry invariants. A natural candidate would be a version of open $r = -2$ spin intersection numbers. However, probably this interpretation is too naive. One of the reasons is that the introduction of the variable N, which should add boundaries to the theory, is not accompanied by new variables for the descendants on the boundary

(which appears in the Kontsevich-Penner model and constitute a second infinite set of times of the KP tau-function).

The results also should help to develop the theory of the Givental decomposition. The cut-and-join representation (in particular, its free field version) should allow us to represent the decomposition formulas purely in terms of simple exponential operators. Of course, the same analysis can be applied to other antipolynomial generalized Kontsevich models.

This paper contains all necessary prerequisites for construction of the Chekhov–Eynard–Orantin topological recursion for the generalized BGW model, namely, the quantum and classical spectral curves, rational parametrization, wave function, one and two point correlation functions in genus zero and loop equations. It would be interesting to compere our results with the contour integral expressions for the n-point (all-genera) correlation functions obtained in [52] and with the recursion relations for the KdV hierarchy correlation functions from [53]. It is also interesting to find the compact expressions for the higher genera contributions to the free energy. Some compact expressions for the higher genera contributions to the free energy in terms of the moments are given in [54], but their conclusions about the relation of this model with the Kontsevich-Witten tau-function and the structure of the Virasoro constraints look to be not completely consistent with our results. These topics are beyond the scope of the present paper and will be considered later.

Acknowledgments

Appendix A. Free energy of BGW tau-function

$$\text{(A1)} \qquad \mathcal{F}_2 = \frac{9}{128}\, T_3$$

$$\text{(A2)} \qquad \mathcal{F}_3 = \frac{567}{1024}\,{T_3}^2 + \frac{225}{1024}\,T_5$$

$$\text{(A3)} \qquad \mathcal{F}_4 = \frac{64989}{4096}\,{T_3}^3 + \frac{388125}{32768}\,T_5 T_3 + \frac{55125}{32768}\,T_7$$

$$\text{(A4)} \qquad \begin{aligned} \mathcal{F}_5 &= \frac{70864875}{65536}\,T_5 {T_3}^2 + \frac{14123025}{65536}\,T_7 T_3 + \frac{6251175}{262144}\,T_9 \\ &\quad + \frac{130301217}{131072}\,{T_3}^4 + \frac{28252125}{262144}\,{T_5}^2 \end{aligned}$$

$$\text{(A5)} \quad \begin{aligned} \mathcal{F}_6 &= \frac{286765250859}{2621440}\,{T_3}^5 + \frac{2269176525}{4194304}\,T_{11} + \frac{37656646875}{1048576}\,T_3 {T_5}^2 \\ &\quad + \frac{18826455375}{524288}\,T_7 {T_3}^2 + \frac{25035955875}{4194304}\,T_3 T_9 \\ &\quad + \frac{12519714375}{2097152}\,T_7 T_5 + \frac{81770259375}{524288}\,T_5 {T_3}^3 \end{aligned}$$

$$\text{(A6)} \qquad \begin{aligned} \mathcal{F}_7 &= \frac{27537582342375}{16777216}\,T_9 {T_3}^2 + \frac{7852650127875}{33554432}\,T_9 T_5 \\ &\quad + \frac{206914899886875}{16777216}\,{T_3}^2 {T_5}^2 + \frac{34484117212125}{4194304}\,T_7 {T_3}^3 \\ &\quad + \frac{34539827452875}{1048576}\,T_5 {T_3}^4 + \frac{9180336943125}{16777216}\,{T_5}^3 \\ &\quad + \frac{1963178035875}{16777216}\,{T_7}^2 + \frac{19600404065991}{1048576}\,{T_3}^6 \\ &\quad + \frac{602628451425}{33554432}\,T_{13} + \frac{13769702800875}{4194304}\,T_7 T_5 T_3 \\ &\quad + \frac{3925999556325}{16777216}\,T_{11} T_3 \end{aligned}$$

$$\text{(A7)} \qquad \begin{aligned} \mathcal{F}_8 &= \frac{169591162989488625}{67108864}\,T_7 {T_3}^4 + \frac{26488676216338875}{2147483648}\,T_{11} T_5 \\ &\quad + \frac{26487328325483025}{2147483648}\,T_{13} T_3 + \frac{26488802802632625}{2147483648}\,T_9 T_7 \end{aligned}$$

$$+ \frac{75275622313203375}{134217728} T_9 T_3{}^3 + \frac{6632707287504375}{67108864} T_3 T_7{}^2$$
$$+ \frac{339191251470703125}{67108864} T_5{}^2 T_3{}^3 + \frac{645210015875843625}{67108864} T_5 T_3{}^5$$
$$+ \frac{13265904719221875}{134217728} T_7 T_5{}^2 + \frac{13264818560895825}{134217728} T_{11} T_3{}^2$$
$$+ \frac{75280511033859375}{134217728} T_5{}^3 T_3 + \frac{538246474955839917}{117440512} T_3{}^7$$
$$+ \frac{1762688220418125}{2147483648} T_{15} + \frac{26530744498689375}{134217728} T_9 T_5 T_3$$
$$+ \frac{112917280552295625}{67108864} T_5 T_3{}^2 T_7$$

(A8)
$$\mathcal{F}_9 = \frac{77431411127351806875}{1073741824} T_7{}^2 T_3{}^2 + \frac{258385134479852784375}{536870912} T_5{}^3 T_3{}^2$$
$$+ \frac{271565647390264136 0625}{1073741824} T_5{}^2 T_3{}^4 + \frac{258375760190755743375}{1073741824} T_9 T_3{}^4$$
$$+ \frac{32565363218292436875}{4294967296} T_9 T_5{}^2 + \frac{1806818862379174875}{2147483648} T_{11} T_7$$
$$+ \frac{51619703670742474575}{1073741824} T_{11} T_3{}^3 + \frac{3613628364405184125}{4294967296} T_{13} T_5$$
$$+ \frac{8141354220180706875}{1073741824} T_7{}^2 T_5 + \frac{16281810076944587175}{2147483648} T_{13} T_3{}^2$$
$$+ \frac{18067554259285781 25}{2147483648} T_{15} T_3 + \frac{849028159501396875}{17179869184} T_{17}$$
$$+ \frac{30972522027732283 6875}{2147483648} T_5 T_3{}^2 T_9 + \frac{16282283116759356375}{1073741824} T_9 T_3 T_7$$
$$+ \frac{16282250837254450125}{1073741824} T_{11} T_3 T_5 + \frac{15486601372568112 9375}{1073741824} T_5{}^2 T_3 T_7$$
$$+ \frac{129190312318002901875}{134217728} T_5 T_3{}^3 T_7 + \frac{103246107731883140625}{8589934592} T_5{}^4$$
$$+ \frac{7227277861688852625}{17179869184} T_9{}^2 + \frac{996625993639547388375}{268435456} T_5 T_3{}^6$$
$$+ \frac{271561567871335604625}{268435456} T_7 T_3{}^5$$

Appendix B. Coefficients of generalized BGW tau-function

(B1)
$$\tau_N^{(1)} = \frac{B_1(N)}{2^4 \, 1!} \, t_1$$

$$\text{(B2)} \qquad \tau_N^{(2)} = \frac{B_2(N)}{2^7 \, 2!} t_1^2$$

$$\text{(B3)} \qquad \tau_N^{(3)} = \frac{B_2(N)}{2^{11} \, 3!} \left(24 \, t_3 - 4 \, t_1^3 N^2 + 17 \, t_1^3\right)$$

$$\text{(B4)} \qquad \tau_N^{(4)} = \frac{B_3(N)}{2^{16} \, 4!} \left(96 \, t_3 - 4 \, t_1^3 N^2 + 17 \, t_1^3\right) t_1$$

$$\text{(B5)} \qquad \tau_N^{(5)} = \frac{B_3(N)}{2^{20} \, 5!} \big(16 \, t_1^5 N^4 - 200 \, t_1^5 N^2 - 960 \, t_1^2 t_3 N^2 \\ + 3840 \, t_5 + 7920 \, t_1^2 t_3 + 561 \, t_1^5\big)$$

$$\text{(B6)} \qquad \tau_N^{(6)} = \frac{B_3(N)}{2^{24} \, 6!} \big(7680 \, t_1^3 t_3 N^4 - 64 \, N^6 t_1^6 + 1456 \, t_1^6 N^4 \\ - 142080 \, t_1^3 t_3 N^2 + 23001 \, t_1^6 - 92160 \, t_1 t_5 N^2 \\ - 10444 \, t_1^6 N^2 - 23040 \, t_3^2 N^2 + 649440 \, t_1^3 t_3 \\ + 944640 \, t_1 t_5 + 466560 \, t_3^2\big)$$

$$\text{(B7)} \qquad \tau_N^{(7)} = \frac{B_4(N)}{2^{28} \, 7!} \big(1612800 \, t_7 - 10444 \, t_1^7 N^2 - 64 \, N^6 t_1^7 + 23001 \, t_1^7 \\ + 1456 \, t_1^7 N^4 + 13440 \, t_1^4 t_3 N^4 - 248640 \, t_1^4 t_3 N^2 \\ + 1136520 \, t_1^4 t_3 - 322560 \, t_1^2 t_5 N^2 + 3306240 \, t_1^2 t_5 \\ + 3265920 \, t_1 t_3^2 - 161280 \, t_1 t_3^2 N^2\big)$$

$$\text{(B8)} \qquad \tau_N^{(8)} = \frac{B_4(N)}{2^{32} \, 8!} \big(1311057 \, t_1^8 - 687312 \, t_1^8 N^2 + 124768 \, t_1^8 N^4 \\ + 256 \, N^8 t_1^8 - 86016 \, t_1^5 N^6 t_3 + 2817024 \, t_1^5 t_3 N^4 \\ - 29949696 \, t_1^5 t_3 N^2 + 103650624 \, t_1^5 t_3 + 502548480 \, t_1^3 t_5 \\ + 3440640 \, t_1^3 t_5 N^4 - 84295680 \, t_1^3 t_5 N^2 + 2580480 \, t_1^2 t_3^2 N^4 \\ + 744629760 \, t_1^2 t_3^2 - 89026560 \, t_1^2 t_3^2 N^2 - 51609600 \, t_1 t_7 N^2 \\ + 735436800 \, t_1 t_7 + 727695360 \, t_5 t_3 \\ - 20643840 \, t_5 t_3 N^2 - 9472 \, N^6 t_1^8\big)$$

(B9) $\quad \tau_N^{(9)} = \dfrac{B_4(N)}{2^{36}\,9!}\big(105345515520\,t_9 - 28333670400\,t_1{}^2 t_7 N^2$

$\qquad + 928972800\,t_1{}^2 t_7 N^4 - 26295736320\,t_1{}^3 t_3{}^2 N^2$

$\qquad + 1571512320\,t_1{}^3 t_3{}^2 N^4 - 30965760\,t_1{}^3 t_3{}^2 N^6$

$\qquad - 1024\,N^{10}t_1{}^9 + 61931520\,t_3{}^3 N^4 + 1261854720\,t_1{}^4 t_5 N^4$

$\qquad - 30965760\,t_1{}^4 t_5 N^6 + 743178240\,t_1 t_5 t_3 N^4$

$\qquad + 454358016\,t_1{}^6 t_3 N^4 - 25288704\,t_1{}^6 N^6 t_3$

$\qquad + 516096\,N^8 t_1{}^6 t_3 + 425701785600\,t_1 t_5 t_3$

$\qquad - 38273679360\,t_1 t_5 t_3 N^2 - 3541999104\,t_1{}^6 t_3 N^2$

$\qquad + 215115264000\,t_1{}^2 t_7 + 10859168\,t_1{}^9 N^4 - 1114752\,N^6 t_1{}^9$

$\qquad + 10105935840\,t_1{}^6 t_3 + 73497715200\,t_1{}^4 t_5$

$\qquad - 49919508\,t_1{}^9 N^2 + 145202803200\,t_1{}^3 t_3{}^2$

$\qquad - 3622993920\,t_3{}^3 N^2 + 54528\,N^8 t_1{}^9 - 5202247680\,t_9 N^2$

$\qquad + 85218705\,t_1{}^9 + 70265180160\,t_3{}^3 - 16851179520\,t_1{}^4 t_5 N^2\big)$

(B10) $\quad \tau_N^{(10)} = \dfrac{B_4(N)}{2^{40}\,10!}\big(1036733644800\,t_1{}^2 t_5 t_3 N^4 - 22483928678400\,t_1{}^2 t_5 t_3 N^2$

$\qquad - 14863564800\,t_1{}^2 t_5 t_3 N^6 - 9762729984000\,t_1{}^3 t_7 N^2$

$\qquad + 603832320000\,t_1{}^3 t_7 N^4 - 12386304000\,t_1{}^3 t_7 N^6$

$\qquad - 6250999910400\,t_1{}^4 t_3{}^2 N^2 + 549758361600\,t_1{}^4 t_3{}^2 N^4$

$\qquad - 21366374400\,t_1{}^4 t_3{}^2 N^6 + 309657600\,t_1{}^4 N^8 t_3{}^2$

$\qquad - 3048253931520\,t_1{}^5 t_5 N^2 + 319040225280\,t_1{}^5 t_5 N^4$

$\qquad - 14615838720\,t_1{}^5 t_5 N^6 + 247726080\,t_1{}^5 N^8 t_5$

$\qquad - 427128111360\,t_1{}^7 t_3 N^2 + 67623045120\,t_1{}^7 t_3 N^4$

$\qquad - 5233582080\,t_1{}^7 N^6 t_3 + 198328320\,N^8 t_1{}^7 t_3$

$\qquad - 2949120\,N^{10}t_1{}^7 t_3 + 155381151744000\,t_1{}^2 t_5 t_3 + 4096\,N^{12}t_1{}^{10}$

$\qquad - 5455392768000\,t_1 t_3{}^3 N^2 + 190129766400\,t_1 t_3{}^3 N^4$

$\qquad - 2477260800\,t_1 t_3{}^3 N^6 - 8011461427200\,t_1 t_9 N^2$

$\qquad + 208089907200\,t_1 t_9 N^4 + 76202709811200\,t_7 t_3$

$\qquad - 5179952332800\,t_7 t_3 N^2 + 74317824000\,t_7 t_3 N^4$

$\qquad + 38097174528000\,t_5{}^2 + 76902226329600\,t_1 t_9$

$\qquad + 52344714240000\,t_1{}^3 t_7 + 10730666419200\,t_1{}^5 t_5$

$$+\, 992397296\, t_1{}^{10} N^4 - 124813568\, N^6 t_1{}^{10} + 1053904737600\, t_1{}^{7} t_3$$
$$+\, 8439552\, N^8 t_1{}^{10} - 3984998904\, t_1{}^{10} N^2 + 26499511584000\, t_1{}^{4} t_3{}^{2}$$
$$+\, 51293581516800\, t_1 t_3{}^{3} - 2571396710400\, t_5{}^{2} N^2 + 29727129600\, t_5{}^{2} N^4$$
$$+\, 6220965465\, t_1{}^{10} - 292864\, N^{10} t_1{}^{10} \big)$$

Appendix C. Free energy of generalized BGW model

(C1)
$$\tilde{\mathcal{F}}_0^{(1)} = 0$$

(C2)
$$\tilde{\mathcal{F}}_0^{(2)} = \frac{1}{2^3} T_3$$

(C3)
$$\tilde{\mathcal{F}}_0^{(3)} = \frac{1}{2^5 \cdot 2} \left(2^2 \cdot 3\, T_3{}^{2} + 2^2\, T_5 \right)$$

(C4)
$$\tilde{\mathcal{F}}_0^{(4)} = \frac{1}{2^7 \cdot 3} \left(3^3 \cdot 5\, T_5 T_3 + 2^3 \cdot 3^3\, T_3{}^{3} + 3 \cdot 5\, T_7 \right)$$

(C5)
$$\tilde{\mathcal{F}}_0^{(5)} = \frac{1}{2^9 \cdot 4} \big(2^5 \cdot 3^3 \cdot 5\, T_3{}^{2} T_5 + 2^5 \cdot 3 \cdot 7\, T_7 T_3$$
$$+\, 2^4 \cdot 3^3 \cdot 11\, T_3{}^{4} + 2^3 \cdot 3^2 \cdot 5\, T_5{}^{2} + 2^3 \cdot 7\, T_9 \big)$$

(C6)
$$\tilde{\mathcal{F}}_0^{(6)} = \frac{1}{2^{11} \cdot 5} \big(2 \cdot 3^2 \cdot 5^2 \cdot 7\, T_9 T_3 + 2^2 \cdot 5^3 \cdot 7\, T_7 T_5 + 2^4 \cdot 3^2 \cdot 5^2 \cdot 7\, T_3{}^{2} T_7$$
$$+\, 2^4 \cdot 3^3 \cdot 5^2 \cdot 13\, T_5 T_3{}^{3} + 2^3 \cdot 3^3 \cdot 5^3\, T_5{}^{2} T_3$$
$$+\, 2^4 \cdot 3^4 \cdot 7 \cdot 13\, T_3{}^{5} + 2 \cdot 3 \cdot 5 \cdot 7\, T_{11} \big)$$

(C7)
$$\tilde{\mathcal{F}}_0^{(7)} = \frac{1}{2^{13} \cdot 6} \big(2^6 \cdot 3^3 \cdot 5^2 \cdot 7\, T_5 T_3 T_7 + 2^4 \cdot 3^6 \cdot 5^3\, T_3{}^{2} T_5{}^{2}$$
$$+\, 2^4 \cdot 3^5 \cdot 5 \cdot 7\, T_3{}^{2} T_9 + 2^3 \cdot 3^4 \cdot 5^2\, T_9 T_5$$
$$+\, 2^6 \cdot 3^4 \cdot 5^2 \cdot 7\, T_3{}^{3} T_7 + 2^8 \cdot 3^6 \cdot 5^2\, T_5 T_3{}^{4}$$
$$+\, 2^4 \cdot 3^4 \cdot 11\, T_{11} T_3 + 2^4 \cdot 3^3 \cdot 5^3\, T_5{}^{3}$$
$$+\, 2^4 \cdot 3 \cdot 5^2 \cdot 7\, T_7{}^{2} + 2^8 \cdot 3^6 \cdot 17\, T_3{}^{6} + 2^3 \cdot 3^2 \cdot 11\, T_{13} \big)$$

(C8) $\tilde{\mathcal{F}}_0^{(8)} = \dfrac{1}{2^{15} \cdot 7} \big(2^7 \cdot 3^7 \cdot 17 \cdot 19 \, T_3{}^7 + 2^4 \cdot 3^4 \cdot 5^2 \cdot 7^2 \, T_5 T_3 T_9$

$+\, 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 17 \, T_3{}^2 T_7 T_5 + 3^2 \cdot 7^2 \cdot 11 \cdot 13 \, T_{13} T_3$

$+\, 2^5 \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 17 \, T_3{}^4 T_7 + 2^4 \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 17 \, T_5{}^3 T_3$

$+\, 3^3 \cdot 5 \cdot 7^2 \cdot 11 \, T_{11} T_5 + 2^4 \cdot 3^4 \cdot 7^2 \cdot 11 \, T_{11} T_3{}^2$

$+\, 2^4 \cdot 3^2 \cdot 5^3 \cdot 7^2 \, T_7 T_5{}^2 + 2^5 \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 17 \, T_3{}^3 T_5{}^2$

$+\, 2^5 \cdot 3 \cdot 5^2 \cdot 7^3 \, T_7{}^2 T_3 + 3^2 \cdot 5^2 \cdot 7^3 \, T_9 T_7$

$+\, 2^4 \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 17 \, T_3{}^3 T_9 + 2^5 \cdot 3^7 \cdot 7 \cdot 17 \cdot 19 \, T_5 T_3{}^5$

$+\, 3 \cdot 7 \cdot 11 \cdot 13 \, T_{15} \big)$

(C9) $$\tilde{\mathcal{F}}_1^{(1)} = \frac{5}{16} \, T_3$$

(C10) $$\tilde{\mathcal{F}}_1^{(2)} = \frac{93}{64} \, T_3{}^2 + \frac{35}{64} \, T_5$$

(C11) $$\tilde{\mathcal{F}}_1^{(3)} = \frac{75}{8} \, T_3{}^3 + \frac{825}{128} \, T_5 T_3 + \frac{105}{128} \, T_7$$

(C12) $$\tilde{\mathcal{F}}_1^{(4)} = \frac{6225}{1024} \, T_5{}^2 + \frac{35397}{512} \, T_3{}^4 + \frac{17415}{256} \, T_3{}^2 T_5$$
$$+ \frac{3045}{256} \, T_7 T_3 + \frac{1155}{1024} \, T_9$$

(C13) $$\tilde{\mathcal{F}}_1^{(5)} = \frac{140373}{256} \, T_3{}^5 + \frac{40005}{2048} \, T_9 T_3 + \frac{35595}{256} \, T_3{}^2 T_7 + \frac{20825}{1024} \, T_7 T_5$$
$$+ \frac{73125}{512} \, T_5{}^2 T_3 + \frac{178875}{256} \, T_3{}^3 T_5 + \frac{3003}{2048} \, T_{11}$$

(C14) $\tilde{\mathcal{F}}_1^{(6)} = \dfrac{64925}{4096} \, T_7{}^2 + \dfrac{1165509}{256} \, T_3{}^6 + \dfrac{373875}{4096} \, T_5{}^3 + \dfrac{1036665}{4096} \, T_9 T_3{}^2$

$+ \dfrac{542325}{1024} \, T_5 T_3 T_7 + \dfrac{15015}{8192} \, T_{13} + \dfrac{1552635}{1024} \, T_7 T_3{}^3 + \dfrac{256725}{8192} \, T_9 T_5$

$+ \dfrac{121275}{4096} \, T_{11} T_3 + \dfrac{9605925}{4096} \, T_5{}^2 T_3{}^2 + \dfrac{1819665}{256} \, T_3{}^4 T_5$

$$(C15) \quad \tilde{\mathcal{F}}_1^{(7)} = \frac{69903081}{1792} T_3{}^7 + \frac{513135}{32} T_7 T_3{}^4 + \frac{921375}{1024} T_5 T_3 T_9$$
$$+ \frac{9520875}{1024} T_5 T_3{}^2 T_7 + \frac{466725}{1024} T_7{}^2 T_3 + \frac{6041385}{2048} T_9 T_3{}^3$$
$$+ \frac{693693}{16384} T_{13} T_3 + \frac{744975}{16384} T_{11} T_5 + \frac{866943}{2048} T_3{}^2 T_{11}$$
$$+ \frac{33969375}{1024} T_5{}^2 T_3{}^3 + \frac{9218205}{128} T_3{}^5 T_5 + \frac{3290625}{1024} T_5{}^3 T_3$$
$$+ \frac{760725}{16384} T_9 T_7 + \frac{485625}{1024} T_5{}^2 T_7 + \frac{36465}{16384} T_{15}$$

$$(C16) \qquad \tilde{\mathcal{F}}_2^{(1)} = \frac{259}{256} T_5 + \frac{657}{256} T_3{}^2$$

$$(C17) \qquad \tilde{\mathcal{F}}_2^{(2)} = \frac{6201}{128} T_3{}^3 + \frac{36015}{1024} T_5 T_3 + \frac{4935}{1024} T_7$$

$$(C18) \qquad \tilde{\mathcal{F}}_2^{(3)} = \frac{74529}{512} T_7 T_3 + \frac{397035}{512} T_3{}^2 T_5 + \frac{765693}{1024} T_3{}^4$$
$$+ \frac{30723}{2048} T_9 + \frac{149985}{2048} T_5{}^2$$

$$(C19) \qquad \tilde{\mathcal{F}}_2^{(4)} = \frac{7390845}{16384} T_9 T_3 + \frac{28598265}{2048} T_3{}^3 T_5 + \frac{6086745}{2048} T_3{}^2 T_7$$
$$+ \frac{12284325}{4096} T_5{}^2 T_3 + \frac{3744825}{8192} T_7 T_5 + \frac{106851717}{10240} T_3{}^5$$
$$+ \frac{603603}{16384} T_{11}$$

$$(C20) \quad \tilde{\mathcal{F}}_2^{(5)} = \frac{9707635}{16384} T_7{}^2 + \frac{51448425}{16384} T_5{}^3 + \frac{140271507}{1024} T_3{}^6$$
$$+ \frac{229158315}{1024} T_3{}^4 T_5 + \frac{1265323275}{16384} T_5{}^2 T_3{}^2 + \frac{38701215}{32768} T_9 T_5$$
$$+ \frac{149588775}{16384} T_9 T_3{}^2 + \frac{208346985}{4096} T_7 T_3{}^3 + \frac{18927909}{16384} T_{11} T_3$$
$$+ \frac{76102635}{4096} T_5 T_3 T_7 + \frac{2543541}{32768} T_{13}$$

(C21) $\displaystyle \tilde{\mathcal{F}}_2^{(6)} = \frac{441784935}{256} T_3{}^7 + \frac{3217790205}{4096} T_7 T_3{}^4 + \frac{50498175}{2048} T_7{}^2 T_3$

$\displaystyle + \frac{634027905}{4096} T_9 T_3{}^3 + \frac{338585247}{131072} T_{13} T_3 + \frac{349538805}{131072} T_{11} T_5$

$\displaystyle + \frac{97956243}{4096} T_3{}^2 T_{11} + \frac{3266594325}{2048} T_5{}^2 T_3{}^3$

$\displaystyle + \frac{13610282013}{4096} T_3{}^5 T_5 + \frac{1317331125}{8192} T_5{}^3 T_3$

$\displaystyle + \frac{351871695}{131072} T_9 T_7 + \frac{205991625}{8192} T_5{}^2 T_7 + \frac{402291225}{8192} T_5 T_3 T_9$

$\displaystyle + \frac{971414325}{2048} T_5 T_3{}^2 T_7 + \frac{19246227}{131072} T_{15}$

(C22) $\displaystyle \tilde{\mathcal{F}}_3^{(1)} = 75\, T_3{}^3 + \frac{114225}{2048} T_5 T_3 + \frac{16145}{2048} T_7$

(C23) $\displaystyle \tilde{\mathcal{F}}_3^{(2)} = \frac{1399965}{2048} T_7 T_3 + \frac{7170255}{2048} T_3{}^2 T_5 + \frac{13407093}{4096} T_3{}^4$

$\displaystyle + \frac{604835}{8192} T_9 + \frac{2804625}{8192} T_5{}^2$

(C24) $\displaystyle \tilde{\mathcal{F}}_3^{(3)} = \frac{75131595}{16384} T_9 T_3 + \frac{265649625}{2048} T_3{}^3 T_5 + \frac{58951305}{2048} T_3{}^2 T_7$

$\displaystyle + \frac{118222875}{4096} T_5{}^2 T_3 + \frac{37711975}{8192} T_7 T_5 + \frac{192117987}{2048} T_3{}^5$

$\displaystyle + \frac{6483477}{16384} T_{11}$

(C25) $\displaystyle \tilde{\mathcal{F}}_3^{(4)} = \frac{711506425}{65536} T_7{}^2 + \frac{5317009425}{16384} T_5 T_3 T_7 + \frac{201052995}{131072} T_{13}$

$\displaystyle + \frac{3562329375}{65536} T_5{}^3 + \frac{8767078173}{4096} T_3{}^6 + \frac{14797367145}{4096} T_3{}^4 T_5$

$\displaystyle + \frac{84554265825}{65536} T_5{}^2 T_3{}^2 + \frac{2843426025}{131072} T_9 T_5$

$\displaystyle + \frac{10571621445}{65536} T_9 T_3{}^2 + \frac{14033060055}{16384} T_7 T_3{}^3$

$\displaystyle + \frac{1411098975}{65536} T_{11} T_3$

(C26) $$\tilde{\mathcal{F}}_3^{(5)} = \frac{1211928290217}{28672} T_3{}^7 + \frac{84821501835}{4096} T_7 T_3{}^4$$
$$+ \frac{11533969125}{16384} T_7{}^2 T_3 + \frac{140161531545}{32768} T_9 T_3{}^3$$
$$+ \frac{20835465651}{262144} T_{13} T_3 + \frac{21099663585}{262144} T_{11} T_5$$
$$+ \frac{22803571791}{32768} T_3{}^2 T_{11} + \frac{682677669375}{16384} T_5{}^2 T_3{}^3$$
$$+ \frac{344109377925}{4096} T_3{}^5 T_5 + \frac{71155771875}{16384} T_5{}^3 T_3$$
$$+ \frac{21140849835}{262144} T_9 T_7 + \frac{11630143875}{16384} T_5{}^2 T_7$$
$$+ \frac{23039048025}{16384} T_5 T_3 T_9 + \frac{211982823675}{16384} T_5 T_3{}^2 T_7$$
$$+ \frac{1256693295}{262144} T_{15}$$

(C27) $$\tilde{\mathcal{F}}_4^{(1)} = \frac{16776921}{16384} T_7 T_3 + \frac{84428595}{16384} T_3{}^2 T_5 + \frac{155619117}{32768} T_3{}^4$$
$$+ \frac{7400547}{65536} T_9 + \frac{33567585}{65536} T_5{}^2$$

(C28) $$\tilde{\mathcal{F}}_4^{(2)} = \frac{5177752965}{262144} T_9 T_3 + \frac{61659550053}{163840} T_3{}^5 + \frac{461311851}{262144} T_{11}$$
$$+ \frac{17389528605}{32768} T_3{}^3 T_5 + \frac{3952037565}{32768} T_3{}^2 T_7$$
$$+ \frac{7910056125}{65536} T_5{}^2 T_3 + \frac{2591640625}{131072} T_7 T_5$$

(C29) $$\tilde{\mathcal{F}}_4^{(3)} = \frac{25761027005}{262144} T_7{}^2 + \frac{124408920975}{262144} T_5{}^3 + \frac{284951872593}{16384} T_3{}^6$$
$$+ \frac{491651138745}{16384} T_3{}^4 T_5 + \frac{2877392953125}{262144} T_5{}^2 T_3{}^2$$
$$+ \frac{103019276745}{524288} T_9 T_5 + \frac{371988099945}{262144} T_9 T_3{}^2$$
$$+ \frac{478941497055}{65536} T_7 T_3{}^3 + \frac{51393633291}{262144} T_{11} T_3$$
$$+ \frac{186325900005}{65536} T_5 T_3 T_7 + \frac{7612223619}{524288} T_{13}$$

$$
\text{(C30)} \qquad \tilde{\mathcal{F}}_4^{(4)} = \frac{19375419429891}{32768} T_3{}^7 + \frac{39967361286615}{131072} T_7 T_3{}^4
$$
$$
+ \frac{1441907245275}{131072} T_7{}^2 T_3 + \frac{17044272862155}{262144} T_9 T_3{}^3
$$
$$
+ \frac{5457509668611}{4194304} T_{13} T_3 + \frac{5481470419425}{4194304} T_{11} T_5
$$
$$
+ \frac{2872551695013}{262144} T_3{}^2 T_{11} + \frac{80106396840225}{131072} T_5{}^2 T_3{}^3
$$
$$
+ \frac{157614682018959}{131072} T_3{}^5 T_5 + \frac{17136214183125}{262144} T_5{}^3 T_3
$$
$$
+ \frac{5484436521795}{4194304} T_9 T_7 + \frac{2892529880625}{262144} T_5{}^2 T_7
$$
$$
+ \frac{5765593974525}{262144} T_5 T_3 T_9 + \frac{25639814092725}{131072} T_5 T_3{}^2 T_7
$$
$$
+ \frac{343519458711}{4194304} T_{15}
$$

$$
\text{(C31)} \qquad \tilde{\mathcal{F}}_5^{(1)} = \frac{14960246805}{524288} T_9 T_3 + \frac{49062715875}{65536} T_3{}^3 T_5
$$
$$
+ \frac{11274351795}{65536} T_3{}^2 T_7 + \frac{22552975125}{131072} T_5{}^2 T_3
$$
$$
+ \frac{7482105225}{262144} T_7 T_5 + \frac{34468789653}{65536} T_3{}^5
$$
$$
+ \frac{1352576043}{524288} T_{11}
$$

$$
\text{(C32)} \qquad \tilde{\mathcal{F}}_5^{(2)} = \frac{418486752775}{1048576} T_7{}^2 + \frac{1979457545625}{1048576} T_5{}^3
$$
$$
+ \frac{4336231722327}{65536} T_3{}^6 + \frac{7582592016315}{65536} T_3{}^4 T_5
$$
$$
+ \frac{45038717337375}{1048576} T_5{}^2 T_3{}^2 + \frac{1673842629375}{2097152} T_9 T_5
$$
$$
+ \frac{5932794319515}{1048576} T_9 T_3{}^2 + \frac{7503673495185}{262144} T_7 T_3{}^3
$$
$$
+ \frac{836393983425}{1048576} T_{11} T_3 + \frac{2967884007375}{262144} T_5 T_3 T_7
$$
$$
+ \frac{126762200565}{2097152} T_{13}
$$

$$\text{(C33)} \quad \tilde{\mathcal{F}}_5^{(3)} = \frac{295477654037589}{65536} T_3{}^7 + \frac{78832872717795}{32768} T_7 T_3{}^4$$
$$+ \frac{23702887069575}{262144} T_7{}^2 T_3 + \frac{274556802988755}{524288} T_9 T_3{}^3$$
$$+ \frac{46148268185019}{4194304} T_{13} T_3 + \frac{46210569510105}{4194304} T_{11} T_5$$
$$+ \frac{47349163650789}{524288} T_3{}^2 T_{11} + \frac{1262180334313125}{262144} T_5{}^2 T_3{}^3$$
$$+ \frac{152599213285155}{16384} T_3{}^5 T_5 + \frac{137510048829375}{262144} T_5{}^3 T_3$$
$$+ \frac{46217234012355}{4194304} T_9 T_7 + \frac{23725543608375}{262144} T_5{}^2 T_7$$
$$+ \frac{47401263300825}{262144} T_5 T_3 T_9 + \frac{412202168102025}{262144} T_5 T_3{}^2 T_7$$
$$+ \frac{2989207836615}{4194304} T_{15}$$

$$\text{(C34)} \quad \tilde{\mathcal{F}}_6^{(1)} = \frac{2356605625185}{4194304} T_7{}^2 + \frac{11038896821475}{4194304} T_5{}^3$$
$$+ \frac{23660311883769}{262144} T_3{}^6 + \frac{41645338352865}{262144} T_3{}^4 T_5$$
$$+ \frac{249159522789225}{4194304} T_5{}^2 T_3{}^2 + \frac{9426278461365}{8388608} T_9 T_5$$
$$+ \frac{33108811628445}{4194304} T_9 T_3{}^2 + \frac{41522675471235}{1048576} T_7 T_3{}^3$$
$$+ \frac{4712392198503}{4194304} T_{11} T_3 + \frac{16556502355785}{1048576} T_5 T_3 T_7$$
$$+ \frac{721976952807}{8388608} T_{13}$$

$$\text{(C35)} \quad \tilde{\mathcal{F}}_6^{(2)} = \frac{15185336065870311}{917504} T_3{}^7 + \frac{9454513725058185}{1048576} T_7 T_3{}^4$$
$$+ \frac{11387713000125}{32768} T_7{}^2 T_3 + \frac{1041752546225055}{524288} T_9 T_3{}^3$$
$$+ \frac{1442169677387439}{33554432} T_{13} T_3 + \frac{1442721973704165}{33554432} T_{11} T_5$$
$$+ \frac{182141818558623}{524288} T_3{}^2 T_{11} + \frac{1182045374663625}{65536} T_5{}^2 T_3{}^3$$

$$+ \frac{36199921141511001}{1048576} T_3{}^5 T_5 + \frac{4169029399318125}{2097152} T_5{}^3 T_3$$

$$+ \frac{1442776147146495}{33554432} T_9 T_7 + \frac{729014470760625}{2097152} T_5{}^2 T_7$$

$$+ \frac{1457590887545625}{2097152} T_5 T_3 T_9 + \frac{781513453641375}{131072} T_5 T_3{}^2 T_7$$

$$+ \frac{95041284259779}{33554432} T_{15}$$

$$\text{(C36)} \quad \tilde{\mathcal{F}}_7^{(1)} = \frac{23275794518914797}{1048576} T_3{}^7 + \frac{12810322271682495}{1048576} T_7 T_3{}^4$$

$$+ \frac{1999590270994575}{4194304} T_7{}^2 T_3 + \frac{22719742197138315}{8388608} T_9 T_3{}^3$$

$$+ \frac{3987196321745637}{67108864} T_{13} T_3 + \frac{3987584386351335}{67108864} T_{11} T_5$$

$$+ \frac{3998836418130477}{8388608} T_3{}^2 T_{11} + \frac{102487735548028125}{4194304} T_5{}^2 T_3{}^3$$

$$+ \frac{48784570534530045}{1048576} T_3{}^5 T_5 + \frac{11361282083218125}{4194304} T_5{}^3 T_3$$

$$+ \frac{3987621333347085}{67108864} T_9 T_7 + \frac{1999731386512125}{4194304} T_5{}^2 T_7$$

$$+ \frac{3999155802479775}{4194304} T_5 T_3 T_9 + \frac{34081835456653425}{4194304} T_5 T_3{}^2 T_7$$

$$+ \frac{264952094603625}{67108864} T_{15}$$

Appendix D. Free energy of generalized BGW model as a linear combination of $B_k(N)$

$$\text{(D1)} \qquad \mathcal{F}_N^{(2)} = \frac{1}{2^7} B_2(N) T_3$$

$$\text{(D2)} \qquad \mathcal{F}_N^{(3)} = \frac{1}{2^{10}} B_3(N) \left(T_5 + 3 T_3^2 \right) - \frac{3}{2^8} B_2(N) T_3^2$$

$$\text{(D3)} \qquad \mathcal{F}_N^{(4)} = \frac{1}{2^{15}} B_4(N) \left(5 T_7 + 3^2 \cdot 5 T_3 T_5 + 2^3 \cdot 3^2 T_3^3 \right)$$

$$- \frac{3}{2^{10}} B_3(N) \left(5 T_3 T_5 + 13 T_3^3 \right) + \frac{3}{2^7} B_2(N) T_3^3$$

(D4) $\mathcal{F}_N^{(5)} = \dfrac{1}{2^{18}} B_5(N) \big(7\,T_9 + 3^2 \cdot 5\,T_5{}^2 + 2^2 \cdot 3^3 \cdot 5\,T_3{}^2 T_5$

$$+ 2^2 \cdot 3 \cdot 7\,T_7 T_3 + 2 \cdot 3^3 \cdot 11\,T_3{}^4\big)$$

$$- \dfrac{3}{2^{14}} B_4(N)\left(5^2\,T_5{}^2 + 5 \cdot 7\,T_7 T_3 + 3^4 \cdot 5\,T_3{}^2 T_5 + 3^4 \cdot 7\,T_3{}^4\right)$$

$$+ \dfrac{3^2}{2^{11}} B_3(N)\left(5 T_5^2 + 3^2 \cdot 13 T_3^4 + 2^2 \cdot 3 \cdot 5 T_3^2 T_5\right) - \dfrac{3^3}{2^9} B_2(N) T_3^4$$

(D5) $\mathcal{F}_N^{(6)} = \dfrac{1}{2^{22} \cdot 5} B_6(N) \big(3 \cdot 5 \cdot 7\,T_{11} + 3^2 \cdot 5^2 \cdot 7\,T_9 T_3 + 2^3 \cdot 3^4 \cdot 7 \cdot 13\,T_3{}^5$

$$+ 2^3 \cdot 3^2 \cdot 5^2 \cdot 7\,T_3{}^2 T_7 + 2^2 \cdot 3^3 \cdot 5^3\,T_5{}^2 T_3$$

$$+ 2 \cdot 5^3 \cdot 7\,T_7 T_5 + 2^3 \cdot 3^3 \cdot 5^2 \cdot 13\,T_5 T_3{}^3\big)$$

$$- \dfrac{3}{2^{17} \cdot 5} B_5(N) \big(5^3 \cdot 7\,T_7 T_5 + 3^2 \cdot 5^3 \cdot 7\,T_3{}^2 T_7 + 3 \cdot 5^2 \cdot 7\,T_9 T_3$$

$$+ 3^3 \cdot 5^2 \cdot 89\,T_5 T_3{}^3 + 2 \cdot 3 \cdot 5^3 \cdot 13\,T_5{}^2 T_3$$

$$+ 2 \cdot 3^3 \cdot 1087\,T_3{}^5\big)$$

$$+ \dfrac{3^2}{2^{15} \cdot 5} B_4(N)\big(5^3 \cdot 7\,T_7 T_5 + 2 \cdot 3 \cdot 5^3 \cdot 7\,T_3{}^2 T_7$$

$$+ 2^2 \cdot 3^3 \cdot 5^4\,T_5 T_3{}^3 + 2^2 \cdot 3 \cdot 5^3 \cdot 7\,T_5{}^2 T_3$$

$$+ 3^4 \cdot 31^2\,T_3{}^5\big)$$

$$- \dfrac{3^3}{2^{10} \cdot 5} B_3(N)\left(5^3 \cdot 7 T_3^2 T_5 + 5^3 T_5^2 + 2 \cdot 3 \cdot 229 T_3^5\right) T_3$$

$$+ \dfrac{3^4}{2^7 \cdot 5} B_2(N) T_3^5$$

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